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Constructing Convex Directions for Stable Polynomials

A. Bülent Özgüler

***Abstract*—New constructions of convex directions for Hurwitz-stable polynomials are obtained. The technique is based on interpretations of the phase-derivative conditions in terms of the sensitivity of the root-locus associated with the even and odd parts of a polynomial.**

***Index Terms*—Convex direction, polynomials, robust control, stability.**

I. INTRODUCTION

A polynomial $p(s)$ is called a **convex direction** (for all Hurwitz stable polynomials of degree n) if for any Hurwitz stable polynomial $q(s)$ the implication

$$\begin{aligned} &q + p \text{ is Hurwitz and } \deg(q+\lambda p) = n \quad \forall \lambda \in [0,1] \\ &\quad \Rightarrow q + \lambda p \text{ is Hurwitz} \quad \forall \lambda \in (0,1) \end{aligned}$$

holds. Rantzer in [12] has shown that a polynomial $p(s)$ is a convex direction if and only if it satisfies the **phase growth condition** [12], [1]

$$\psi_p'(\omega) \le \left| \frac{\sin(2\psi_p(\omega))}{2\omega} \right| \quad \forall \omega > 0 \tag{1}$$

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The author is with the Department of Electrical and Electronics Engineering, Bilkent University, Bilkent, 06533 Ankara, Turkey (e-mail: ozguler@ee.bilkent.edu.tr).

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whenever $\psi_p(\omega) := \arg p(j\omega) \neq 0$. Condition (1) is in a sense a complement of the phase increasing property of Hurwitz stable polynomials: For a Hurwitz stable polynomial $q(s)$ the rate of change of the argument satisfies

$$\psi_q'(\omega) \ge \left| \frac{\sin(2\psi_q(\omega))}{2\omega} \right| \quad \forall \omega > 0 \tag{2}$$

where the inequality is strict if $\deg q(s) \ge 2$. Property (2) also given in [12] seems to be known in network theory as pointed out by [2] (see also [7] for a proof based on Hermite-Biehler theorem and [8] for related growth conditions). The phase growth condition directly gives that: i) anti-Hurwitz polynomials; ii) polynomials of degree one; iii) even polynomials; iv) odd polynomials; and v) any multiple of polynomials from i)–iv) (taken one from each set) are examples of convex directions for the entire set of Hurwitz polynomials.

There are various reasons for studying the phase growth condition and convex directions in more depth. First, verifying the condition requires checking the nonnegativity of a nonlinear function of frequency at all frequencies, limiting the verification to graphical methods. Second, there has been little success in enlarging the class of **Rantzer polynomials** (i.e., convex directions) given in i)–v); see [3] and [9]. Third, the nature of Rantzer polynomials needs to be understood at least as well as the nature of Hurwitz stable or anti-Hurwitz polynomials. Finally, progress on *local* convex directions seems to require a better understanding of the phase growth condition of Rantzer and the related phase conditions as pointed out in [5]. The local problems are still not sufficiently investigated despite the existence of some geometric criteria such as in [5] or a combinatorial check as in [11].

Below, we examine the phase growth condition from a new perspective. The point of departure is a new interpretation of the phase increasing property of a Hurwitz polynomial in terms of the "sensitivity" of some component root-loci associated with the even and odd parts of the polynomial. This clarifies the exact relation of the phase increasing property to the property of Hurwitz stability. The phase growth condition of Rantzer is then restated in terms of the sensitivity of the component root-loci in Lemma 2. It is then shown in Theorem 1 that the real negative roots of the even and odd parts of a Rantzer polynomial must be interlacing with odd multiplicities. This is a "positive pair" type of property. Finally, various techniques of construction of convex directions are obtained. Corollaries 1 and 2 show that a new convex direction can be obtained from a given convex direction by the addition of a real negative zero or a complex pair of zeros to its even (or, odd) part provided the sensitivity of the component root-loci are bounded from below at certain frequencies. The results demonstrate that the sensitivities of component root-loci are basic tools in characterizing Hurwitz stability as well as the property of being a convex direction.

II. HURWITZ STABLE POLYNOMIALS

Let $\mathbf{R}[s]$ denote the set of polynomials in s with coefficients in the field of real numbers $\mathbf{R}$. Given $q \in \mathbf{R}[s]$, **the even–odd parts** (h, g) **of** $q(s)$ are the unique polynomials $h, g \in \mathbf{R}[u]$ such that $q(s) = h(s^2) + sg(s^2)$. The even–odd parts of a polynomial and the real and imaginary parts of $q(j\omega)$, $\tilde{h}(\omega) := \mathrm{Re}\{q(j\omega)\}$ and $\tilde{g}(\omega) := \mathrm{Im}\{q(j\omega)\}$, are related by $\tilde{h}(\omega) = h(-\omega^2), \tilde{g}(\omega) = \omega g(-\omega^2)$. Note that $q(s)$ is an even (respectively, odd) polynomial of s if and only if $g = 0$ (respectively, $h = 0$). A necessary and sufficient condition for the Hurwitz stability of q in terms of its even–odd parts (h, g) is known as the Hermite-Biehler theorem which is based on the following definition.

A pair of polynomials $(h(u), g(u))$ is said to be a ***positive pair*** [4, Section XV, 14] if $h(0)g(0) > 0$, the roots $\{u_i\}$ of $h(u)$ and $\{v_j\}$ of

$g(u)$ are all real, negative, simple, and with $k := \deg h$ and $l := \deg g$ either i) or ii) holds:

i) $k = l$ and $0 > u_1 > v_1 > u_2 > v_2 > \cdots > u_k > v_l$;
ii) $k = l + 1$ and $0 > u_1 > v_1 > u_2 > v_2 > \cdots > v_l > u_k$.

*The **Hermite-Biehler theorem**,* [4], *states*: A polynomial $q(s)$ is Hurwitz stable if and only if its even–odd parts $(h(u), g(u))$ form a positive pair.

The "root interlacing conditions" i) and ii) can be replaced by positivity of certain polynomials of u. Consider the polynomials

$$V_q(u) := h'(u)g(u) - h(u)g'(u)$$
$$V_{sq}(u) := h(u)g(u) - u[h'(u)g(u) - h(u)g'(u)].$$

Lemma 1: Let $h, g \in \mathbf{R}[u]$ be coprime with $\deg h = \deg g \geq 1$ or with $\deg h = \deg g + 1 \geq 1$. Then, (h, g) is a positive pair if and only if: i) all roots of h and g are real and negative, ii) $V_q(u) > 0\ \forall u < 0$, and iii) $V_{sq}(u) > 0\ \forall u < 0$.

We note that the necessity of the conditions ii) and iii) of Lemma 1 is essentially known since they are closely related to the phase increasing property as elaborated in Remark 3. What may be new is that the addition of i), together with the degree requirements, makes the conditions sufficient. A proof of Lemma 1 is given in [10].

Remark 1: Variation on the Statement of Lemma 1: Conditions ii) and iii) are equivalent to

$$V_q(u) > 0 \quad \forall u < 0 \text{ such that } h(u)g(u) \geq 0$$
$$V_{sq}(u) > 0 \quad \forall u < 0 \text{ such that } h(u)g(u) < 0. \tag{3}$$

To see that (3) implies i) and ii), note that if $h(u)g(u) \geq 0$, then $V_{sq}(u) = h(u)g(u) - uV_q(u) > 0$ so that $V_{sq}(u) > 0$ for all $u < 0$. Also if $h(u)g(u) < 0$, then $-uV_q(u) = V_{sq}(u) - h(u)g(u) > 0$ yielding $V_q(u) > 0$ for all $u < 0$. △

Remark 2: Root-Loci Interpretation: Let us consider $\phi(K, u) := h(u) + Kg(u)$ and $\psi(K, u) := ug(u) + Kh(u)$ for $K \in \mathbf{R}$. The equation $\phi(K, u) = 0$ implicitly defines a function $u(K)$. The *root sensitivity* of $\phi(K, u)$ (see, e.g., [6]) is defined by $|K|(du/dK)$, and gives a measure of the variations in the root location of $\phi(K, u)$ with respect to percentage variations in K. The root sensitivities of $\phi(K, u)$ and $\psi(K, u)$, respectively, are easily computed to be

$$S_q(u) := \frac{h(u)g(u)}{V_q(u)}, \quad S_{sq}(u) := \frac{ug(u)h(u)}{V_{sq}(u)}. \tag{4}$$

Suppose all roots of h and g are real and negative. If (h, g) is a positive pair, then a plot of the root-loci of $\phi(K, u)$ and $\psi(K, u)$ for $K \geq 0$ shows that the roots remain on the negative real axes and move to the left with increasing K. This implies that the sensitivities (4) are positive for all u in the root-loci. Conversely, if the sensitivities (4) are positive for all u in the root-loci, then for $K \geq 0$ roots of $\phi(K, u)$ and $\psi(K, u)$ are contained in the negative real axis and they all move to the left with increasing K. This is the case only if the roots of h and g are interlacing as required in the definition of a positive pair. Note that this argument which establishes the equivalence between positivity of root sensitivities, or (3), and positive-pairness is a sketch of an alternative proof of Lemma 1. See Fig. 1 (where the dashed curves show the polynomials $h(u), g(u)$, and $ug(u)$ with the horizontal axes taken to be u). △

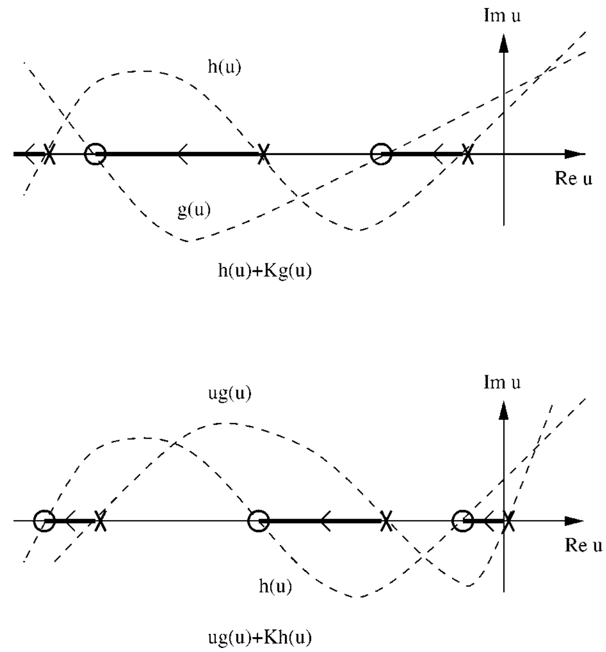

Fig. 1. Root-loci with $K > 0$ for Remark 2.

Remark 3: Phase Increasing Property: Suppose $\deg q(s) \geq 2$. Using the following relations between $V_q(u)$, $V_{sq}(u)$, and $\psi_q(\omega) := \arg(q(j\omega))$:

$$\begin{aligned}\psi_q'(\omega) &= \frac{h(u)g(u) - 2u[h'(u)g(u) - h(u)g'(u)]}{h(u)^2 - ug(u)^2}\\ &= \frac{V_{sq}(u) - uV_q(u)}{h(u)^2 - ug(u)^2}, \quad \frac{\sin(2\psi_q(\omega))}{2\omega}\\ &= \frac{h(u)g(u)}{h(u)^2 - ug(u)^2}\end{aligned} \tag{5}$$

where $u = -\omega^2$, it is easy to see that (3) holds if and only if (2) holds with strict inequality whenever $\psi_q(\omega) \neq 0$. Hence, $q(s)$ is Hurwitz stable if and only if: i) all roots of $h(u)$ and $g(u)$ are real and negative and ii) the inequality (2) holds with strict inequality at all $\omega > 0$. If $q(s) = h + sg$ has degree one, then by direct computation (2) holds with equality and $V_q(u) = 0$, $V_{sq}(u) = hg$. △

III. Convex Directions

Let $p(s) = f(s^2) + se(s^2)$ and consider the conditions

$$V_p(u) \leq 0 \quad \forall u < 0 : f(u)e(u) \geq 0$$
$$V_{sp}(u) \leq 0 \quad \forall u < 0 : f(u)e(u) < 0. \tag{6}$$

In terms of the sensitivities of the root-loci of $\phi(-K, u) = f(u) - Ke(u)$ and $\psi(-K, u) = ue(u) - Kf(u)$ for $K \geq 0$, condition (6) becomes

$$S_p(u) \leq 0 \quad \forall u < 0 : f(u)e(u) \geq 0$$
$$S_{sp}(u) \leq 0 \quad \forall u < 0 : f(u)e(u) < 0. \tag{7}$$

We can hence state the following.

Lemma 2: A polynomial $p(s)$ is a convex direction if and only if one of the equivalent conditions (1), (6), or (7) holds.

Proof: If $\deg p(s) \geq 2$, then the result follows by the identities (5), where p, f, e replaces q, h, g, respectively, employed in (1). If

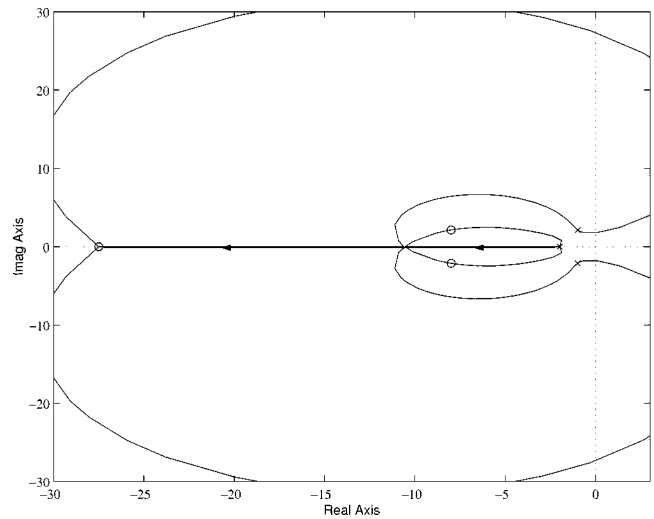

Fig. 2. Root-locus of $f(u) - Ke(u)$.

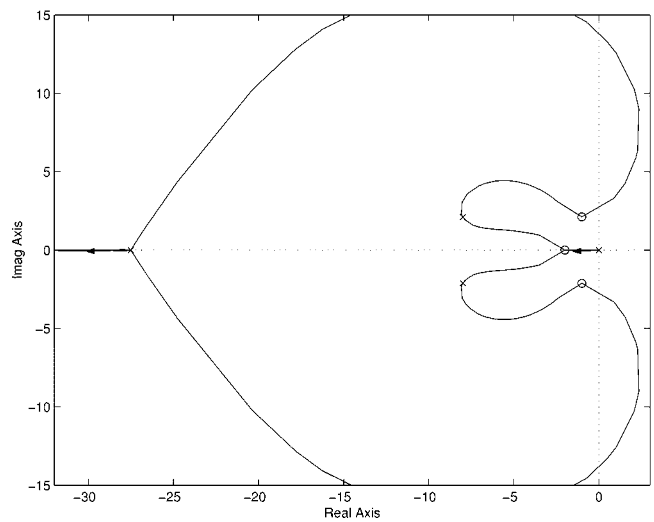

Fig. 3. Root-locus of $ue(u) - Kf(u)$.

$p(s) = f + se$ has degree one, then $V_p(s) = 0$, $V_{sp}(s) = fe$ for all $u < 0$ so that (6) is satisfied. Condition (1) is also satisfied with equality for all $\omega > 0$. ■

The root sensitivity condition (7) allows an immediate identification of a Rantzer polynomial from the root-loci of $h(u) - Kg(u)$ and $ug(u) - Kh(u)$. The conditions mean that the values of the real negative roots of $\phi(-K, u)$ and $\psi(-K, u)$ do not increase with increasing $K \geq 0$. Hence, once the root-loci of $h(u) - Kg(u)$ and $ug(u) - Kh(u)$ are plotted for $K \geq 0$ (with arrows pointing from poles to zeros), all arrows on the negative real axis point to the left if and only if $p(s) = f(s^2) + se(s^2)$ is a Rantzer polynomial. The root-loci in Figs. 2 and 3 indicate that $f(s^2) + se(s^2)$ is a Rantzer polynomial.

This root sensitivity interpretation of a Rantzer polynomial indicates a "positive-pairlike" property which is made precise in Theorem 1 below. It is easy to verify by Lemma 2 that $r(s) = n(s^2) + sm(s^2)$ is a Rantzer polynomial if and only if $\bar{r}(s) = \bar{n}(s^2) + s\bar{m}(s^2)$ is, where $n = d\bar{n}$, $m = d\bar{m}$ and d is a greatest common factor of (n, m) over $\mathbf{R}[u]$. The assumption that (n, m) is coprime in Theorem 1 is hence without loss of generality.

Theorem 1: Let $r(s) = n(s^2) + sm(s^2)$ be a Rantzer polynomial with coprime even–odd parts (n, m). Then, all real negative roots, if any, of $n(u)$ and $m(u)$ have odd multiplicities and are interlacing.

Proof: Let $u_1 < 0$ be a root of $n(u)$ with multiplicity $k \geq 1$. Let $f(u) = n(u)/(u - u_1)^k$ and $e(u) = m(u)$ so that $r(s) = (u - u_1)^k f(u) + se(u)$. Suppose that k is even. By Lemma 2 and (6) applied to r, the following implications hold for $u < 0$:

$$
\begin{aligned}
&u - u_1 \geq 0\\
&f(u)e(u) \geq 0 \Rightarrow kf(u)e(u) + (u - u_1)V_p(u) \leq 0\\
&u - u_1 \leq 0\\
&f(u)e(u) \geq 0 \Rightarrow kf(u)e(u) + (u - u_1)V_p(u) \geq 0\\
&u - u_1 > 0\\
&f(u)e(u) < 0 \Rightarrow -kuf(u)e(u) + (u - u_1)V_{sp}(u) \leq 0\\
&u - u_1 < 0\\
&f(u)e(u) < 0 \Rightarrow -kuf(u)e(u) + (u - u_1)V_{sp}(u) \geq 0.
\end{aligned}
$$

Since (n, m) is coprime, we have $f(u_1)e(u_1) \neq 0$ and hence for $\epsilon > 0$ sufficiently small $f(u_1 \pm \epsilon)e(u_1 \pm \epsilon) \neq 0$ and has the same sign as $f(u_1)e(u_1)$. Suppose $f(u_1)e(u_1) < 0$ and consider $u = u_1 - \epsilon$. We have $u - u_1 < 0$ and $f(u)e(u) < 0$ so that according to the fourth implication, we must have $-kuf(u)e(u) - \epsilon V_{sp}(u) \geq 0$. This inequality holds for sufficiently small ϵ only if $f(u)e(u) \geq 0$ which contradicts our assumption that $f(u)e(u) < 0$. Suppose $f(u_1)e(u_1) > 0$ and consider $u = u_1$. We have $u - u_1 = 0$ and $f(u)e(u) > 0$ so that according to the first implication $f(u)e(u) \leq 0$, giving a contradiction. Therefore, k must be odd. This shows that any real negative root of $n(u)$ has odd multiplicity. In a similar manner, it is shown that any root $v_1 < 0$ of $m(u)$ has odd multiplicity as well.

Since k is odd, (6) applied to $r(s)$ now gives that for all $u < 0$,

$$
\begin{aligned}
&u - u_1 \geq 0\\
&f(u)e(u) \geq 0 \Rightarrow kf(u)e(u) + (u - u_1)V_p(u) \leq 0\\
&u - u_1 \leq 0\\
&f(u)e(u) \leq 0 \Rightarrow kf(u)e(u) + (u - u_1)V_p(u) \leq 0\\
&u - u_1 > 0\\
&f(u)e(u) < 0 \Rightarrow -kuf(u)e(u) + (u - u_1)V_{sp}(u) \leq 0\\
&u - u_1 < 0\\
&f(u)e(u) > 0 \Rightarrow -kuf(u)e(u) + (u - u_1)V_{sp}(u) \leq 0 \quad (8)
\end{aligned}
$$

for every root u_1 of $n(u)$. If $f(u_1)e(u_1) \geq 0$, then the first implication would give a contradiction at $u = u_1$. Hence, $f(u_1)e(u_1) < 0$ for every root $u_1 < 0$ of $n(u)$ so that $\operatorname{sign} f(u_1) = -\operatorname{sign} e(u_1) = -\operatorname{sign} m(u_1)$. Let k_1 denote the number of distinct real roots of $n(u)$ in the interval $(u_1, 0)$. Since each root has odd multiplicity, it follows that $\operatorname{sign} f(u_1) = (-1)^{k_1}\operatorname{sign} f(0+)$. Thus, $\operatorname{sign} m(u_1) = (-1)^{k_1+1}\operatorname{sign} f(0+) = (-1)^{k_1+1}\operatorname{sign} n(0+)$, where we used $\operatorname{sign} f(0+) = \operatorname{sign} n(0+)$ which is due to $(u - u_1)^k > 0$ at $u = 0$. It follows, by $\operatorname{sign} m(u_1) = (-1)^{k_1+1}\operatorname{sign} n(0+)$, that there is an odd number of distinct real roots of $m(u)$ between every pair of distinct real negative roots of $n(u)$. Let $v_1 < 0$ be a root of $m(u)$ with odd multiplicity $l \geq 1$. Write $r = n(u) + s(u - v_1)^l \bar{m}(u)$ for polynomial $\bar{m} := m/(u - v_1)^l$. Then, condition (6) and a similar reasoning as above gives that there are an odd number of real roots of $n(u)$ between every pair of distinct real negative roots of $m(u)$. Therefore, the real negative roots of n and m are interlacing. ■

The necessary condition given by Theorem 1 need not be sufficient, neither does it seem possible to state a necessary and sufficient condition in terms of the algebraic properties of n and m as the values of the roots do matter.

Example 1: Let $n(u) = (u - u_1)^k$ and $m(u) = (u - v_1)^l$ with $u_1 < v_1 < 0$ and k, l odd integers. It is straightforward to show using Lemma 2 and condition (6) that r is Rantzer if and only if $k > l + 1$ *and* $lu_1 - kv_1 \leq 0$. If, for instance, $l = 1, k = 3, u_1 = -4, v_1 = -1$, then r is Rantzer. •

Given a Rantzer polynomial $p(s) = f(s^2) + se(s^2)$, we now obtain conditions on p, in terms of root sensitivities, under which the composite polynomial $r(s) := f(s^2)h(s^2) + se(s^2)g(s^2)$ is also a Rantzer polynomial for some polynomials $h(u)$ and $g(u)$ having real negative or complex zeros. This will give a construction procedure which starts with a Rantzer polynomial and gives new convex directions of increasing complexity by adding zeros to its even and/or odd parts.

Corollary 1: Let $p(s) = f(s^2) + se(s^2)$ be a Rantzer polynomial with $(f(u), e(u))$ coprime and $\deg p(s) > 1$.

i) There exist an odd integer $k > 0$ and a real number $u_1 < 0$ such that $r(s) = (s^2 - u_1)^k f(s^2) + se(s^2)$ is also a Rantzer polynomial if and only if

$$\min_{u \in \mathcal{U}_+} S_p(u) = \min_{u \in \mathcal{U}_+} \frac{f(u)e(u)}{V_p(u)} \tag{9}$$

exists, where $\mathcal{U}_+ := \{u \le 0 : f(u)e(u) \ge 0\}$.

ii) There exist an odd integer $l > 0$ and a real number $v_1 < 0$ such that $r(s) = f(s^2) + s(s^2 - v_1)^l e(s^2)$ is also a Rantzer polynomial if and only if $\min_{u \in \mathcal{U}_-} S_{sp}(u)$ exists, where $\mathcal{U}_- := \{u \le 0 : f(u)e(u) \le 0\}$.

Proof: i) Let us first note that $V_p(u)$ is not identically zero since if it were, then, by coprimeness of (f, e), both f, e would be nonzero constants and $\deg p(s) = 1$. By Lemma 2 and (6) $r(s)$ is a Rantzer polynomial if and only if (8) above holds.

Necessity: If $r(s)$ is Rantzer and $f(u)e(u) > 0$ at some $u < u_1$, then $V_p(u) \le 0$ as p is Rantzer. The fourth implication in (8) gives that $-kuf(u)e(u) + (u - u_1)V_{sp}(u) \le 0$ or $-[u_1 + (k-1)u]f(u)e(u) - u(u - u_1)V_p(u) \le 0$. The quantity on the left-hand side in this last inequality is, however, positive and a contradiction is obtained. Hence, $f(u)e(u) \le 0$ for all $u < u_1$. By Theorem 1, any possible root of $f(u)e(u)$ at some $u < u_1$ should have odd multiplicity so that $f(u - \epsilon)e(u - \epsilon)$ and $f(u + \epsilon)e(u + \epsilon)$ will have opposite signs for small $\epsilon > 0$. Therefore, $f(u)e(u) < 0$ for all $u < u_1$. Since $V_p(u) \le 0$ for all $u < 0$ such that $f(u)e(u) \ge 0$, the first implication in (8) gives that

$$u \ge u_1, \quad f(u)e(u) \ge 0, \quad V_p(u) \ne 0 \Rightarrow \frac{f(u)e(u)}{V_p(u)} \ge \frac{u_1 - u}{k}. \tag{10}$$

We have established $f(u)e(u) < 0 \, \forall u < u_1$ so that $\mathcal{U}_+ = \{u_1 \le u \le 0 : f(u)e(u) \ge 0\}$, on which the right-hand side of (10) is bounded below by the negative number u_1/k. It follows that (9) exists.

Sufficiency: Suppose that the minimum in (9) is equal to (a finite number) c_1. Since $V_p(u) \le 0$ for all $u \in \mathcal{U}_+$, it follows that $c_1 \le 0$. Suppose, by way of contradiction, that $f(u)e(u) > 0$ for $u \to -\infty$. Then the set $\mathcal{U}_+$ contains all sufficiently small negative numbers and is an infinite interval. Hence, $f(u)e(u) > 0$ for $u \to -\infty$ contradicts the hypothesis that the minimum (9) exists. Therefore, $f(u)e(u) < 0$ for $u \to -\infty$, i.e., $\deg[f(u)e(u)]$ is odd. Let us now choose an odd integer k such that $\deg[kf(u)e(u) + uV_p(u)]$ is odd. Since, $\deg[f(u)e(u)] \ge \deg uV_p(u)$, such a k always exists. Let $c_2 < 0$ be such that

$$kf(u)e(u) + uV_p(u) < 0 \quad \forall u < c_2. \tag{11}$$

Such a c_2 exists as $kf(u)e(u) + uV_p(u)$ has odd degree. Note that for all $u < c_2$, we also have $f(u)e(u) < 0$ since $f(u)e(u) > 0$ for some $u < c_2$ would give that $V_p(u) \le 0$ and hence $kf(u)e(u) + uV_p(u) > 0$ contradicting (11). Define $u_1 := kc_1 + c_2 < 0$. We now show that $r(s) = (u - u_1)^k f(u) + se(u)$ is Rantzer by verifying the implications (8). By the fact that $f(u)e(u) < 0$ for all $u < u_1 < c_2$, the fourth implication in (8) trivially holds. The third implication also holds since $p(s)$ is a Rantzer polynomial by hypothesis. The second implication holds by (11) and by the equality $-u[kf(u)e(u) + (u - u_1)V_p(u)] = -ku_1 f(u)e(u) - (u - u_1)[kf(u)e(u) + uV_p(u)]$, where the right-hand side is nonpositive for all $u \le u_1$ such that $f(u)e(u) \le 0$. Finally, to see that the first implication in (8) also holds, note that $\mathcal{U}_+ = \{u \le 0 : u \ge c_2, f(u)e(u) \ge 0\}$ by (11). Hence, for all $u \in \mathcal{U}_+$ we have $kf(u)e(u) + (u - u_1)V_p(u) = kV_p(u)((u - u_1/k) + (f(u)e(u)/(V_p(u))) \le kV_p(u)((u - u_1/k) + \min_{u \in \mathcal{U}_+} f(u)e(u)/V_p(u)) \le kV_p(u)((c_2 - u_1/k) + \min_{u \in \mathcal{U}_+} f(u)e(u)/V_p(u)) = 0$, where the second inequality follows by $u \ge c_2$ and the last equality by the definition of u_1. ii) The proof parallels the proof of i) and is omitted. ■

Whenever $\deg[f(u)e(u)]$ is even, $\mathcal{U}_+$ is an infinite interval and the minimum (9) does not exist. The following example shows that the minimum (9) may also fail to exist when $f(u)e(u)/V_p(u)$ has a pole of even multiplicity in $\mathcal{U}_+$.

Example 2: Consider $p(s) = f(s^2) + se(s^2)$, where $f(u) = -(u + 2.0)^3[(u + 1.0)^2 + 4.5]^2$, $e(u) = (u + 27.5)^3[(u + 8.0)^2 + 4.5]^2$. The root-loci of $f(u) - Ke(u)$ and $ue(u) - Kf(u)$ for $K > 0$ are shown in Figs. 2 and 3. It is clear from the figures that the sensitivities $S_p(u)$ and $S_{sp}(u)$ have correct signs for $f(u)e(u) \ge 0$ and $f(u)e(u) < 0$, respectively, so that $p(s)$ is a Rantzer polynomial. However $\lim_{u \to -10.6} S_p(u) = -\infty$. At $u = -10.6$, three branches of the root-locus of Fig. 2 intersect and $S_p(u)$ has a pole of multiplicity two.

Remark 4: In i), the choice of the multiplicity k of the introduced zero u_1 is restricted only by the condition "$\deg[kf(u)e(u) + uV_p(u)]$ is odd" and is otherwise free. Similarly, in ii), the choice of l is only subject to "$\deg[lf(u)e(u) + V_{sp}(u)]$ is even". △

Corollary 2: Let $p(s) = f(s^2) + se(s^2)$ be a Rantzer polynomial with $(f(u), e(u))$ coprime and $\deg p(s) > 1$.

i) There exist real numbers a, b such that $r(s) = [(s^2 + a)^2 + b^2]f(s^2) + se(s^2)$ is also a Rantzer polynomial if and only if

$$\min_{u \in \mathcal{U}_- \cap (-\infty, u_0]} S_{sp}(u), \qquad \min_{u \in \mathcal{U}_+ \cap [u_0, \infty)} S_p(u) \tag{12}$$

exist for some real number u_0.

ii) There exist real numbers c, d such that $r(s) = f(s^2) + s[(s^2 + c)^2 + d^2]e(s^2)$ is also a Rantzer polynomial if and only if $\min_{u \in \mathcal{U}_+ \cap (-\infty, u_0]} S_p(u)$, $\min_{u \in \mathcal{U}_- \cap [u_0, \infty)} S_{sp}(u)$ exist for some $u_0 \le 0$.

Proof: We prove i) only as the proof of ii) is similar. By Lemma 1 and the condition (6) applied to r, it is straightforward to see that $r(s)$ is Rantzer if and only if

$$\frac{V_p(u)}{f(u)e(u)} \le -\frac{2(u + a)}{(u + a)^2 + b^2}, \quad \forall u < 0 : f(u)e(u) \ge 0$$
$$\frac{V_{sp}(u)}{uf(u)e(u)} \le \frac{2(u + a)}{(u + a)^2 + b^2}, \quad \forall u < 0 : f(u)e(u) < 0.$$

Since $p(s)$ is Rantzer, the first inequality is satisfied for all $u \le -a, u < 0, f(u)e(u) \ge 0$ and the second inequality is satisfied for all $u \ge -a, u < 0, f(u)e(u) < 0$. Hence, $r(s)$ is Rantzer if and only if

$$\frac{V_p(u)}{f(u)e(u)} \le -\left|\frac{2(u + a)}{(u + a)^2 + b^2}\right|, \quad \forall u < 0 : u > -a, \quad f(u)e(u) \ge 0$$
$$\frac{V_{sp}(u)}{uf(u)e(u)} \le -\left|\frac{2(u + a)}{(u + a)^2 + b^2}\right|, \quad \forall u < 0 : u < -a, \quad f(u)e(u) < 0. \tag{13}$$

Necessity: Let r be Rantzer so that (13) holds. We first show that the second condition in (13) implies $\lim_{u \to -\infty} f(u)e(u) > 0$, i.e., $\deg[f(u)e(u)]$ is even. In fact, if $\lim_{u \to -\infty} f(u)e(u) < 0$, then since $\deg, V_{sp}(u) < \deg[uf(u)e(u)]$, the left-hand side is asymptotically u^{-k} for some $k \ge 1$ whereas the right-hand side is $2u^{-1}$ as $u \to -\infty$. It follows that the second condition will fail for sufficiently small u unless $\lim_{u \to -\infty} f(u)e(u) > 0$. Hence, $\mathcal{U}_- \cap (-\infty, u_0]$ is a union of closed intervals for any u_0. Let $T(u) := -|(2(u + a))/((u + a)^2 + b^2)|$. Note that $T(-a) = 0$, $T(u) < 0$ for all $u \in (-\infty, -a) \cup$

$(-a, \infty)$, and $T(u) \geq -1/|b|$ for all u with equality holding at the local minima $u = -a - b$ and $u = -a + b$. It follows by (13) and by the characteristics of $T(u)$ for $u \leq 0$ that the rational functions $S_p^{-1} = V_p/fe$ and $S_{sp}^{-1} = V_{sp}/ufe$ can have at most one zero at $u = -a$. Moreover, $S_p^{-1}(-a) = 0$ and $S_{sp}^{-1}(-a) = 0$ are not both possible in view of the identity $S_{sp}^{-1} + S_p^{-1} = u^{-1}$ which follows by $V_{sp} = fe - uV_p$. We now let $\epsilon > 0$ be arbitrarily small and let $u_0 = -a + \epsilon$ if $S_p^{-1}(-a) = 0$, $u_0 = -a - \epsilon$ if $S_{sp}^{-1}(-a) = 0$, and $u_0 = 0$ (or any other number) if $S_p^{-1}(-a) \neq 0$ and $S_{sp}^{-1}(-a) \neq 0$. It follows that the minima in (12) exist (or the set over which the minimum is taken is empty).

Sufficiency: Suppose (12) both exist for some u_0. Let $a := -u_0$. By the existence of the first minimum in (12), $\deg[f(u)e(u)]$ is even. Hence, there exists sufficiently large $\alpha > 0$ such that $f(u)e(u) > 0$ for all $u < -\alpha$. Also let $m_{sp} := \min_{u \in \mathcal{U}_- \cap [-\alpha, u_0]} S_{sp}(u)$, $m_p := \min_{u \in \mathcal{U}_+ \cap [u_0, \alpha]} S_p(u)$. If $\mathcal{U}_+ \cap [u_0, \alpha] = \emptyset$, then let $m_p = -\infty$. Finally, choose $b > 0$ such that $-b^{-1} > \max\{m_p, m_{sp}\}$. It is now straigtforward to verify that (13) is satisfied and, thus, $r(s)$ is Rantzer. ■

We illustrate the constructions given in the sufficiency parts of the proofs of Corollaries 1 and 2 by the following example.

Example 3: Consider the Rantzer polynomial $p(s) = s^3 + s + 1$ with the even-odd parts $f(u) = 1, e(u) = u + 1$. We employ the procedure in the proof of Corollary 1.i to obtain another Rantzer polynomial by introducing a real negative zero to its even part. We have $V_p(u) = -1$, $S_p(u) = -(u+1)$, and $\mathcal{U}_+ = \{u \in [-1, 0]\}$. The minimum (9) is attained at $u = 0$ and has value -1 so that $c_1 = -1$. The smallest odd integer k for which $kf(u)e(u) + uV_p(u) = k(u+1) + u(-1)$ has odd degree is $k = 3$. With this choice $kf(u)e(u) + uV_p(u) = 2u + 3 < 0$ for all $u < -1.5$ so that we can set $c_2 = -2$. Finally, we set $u_1 = kc_1 + c_2 = -5$. The polynomial $r(s) = (s^2+5)^3 + s(s^2+1)$ is a Rantzer polynomial. Let us now employ the proof of sufficiency of Corollary 2.i to further introduce a pair of complex zeros to the even part of $r(s)$ and obtain yet another Rantzer polynomial. Our initial polynomial now has even-odd parts $f(u) = (u+5)^3, e(u) = u + 1$ and $S_p(u) = (u+1)(u+5)/2(u-1)$, $S_{sp}(u) = -u(u+1)/(u^2 - 8u - 5)$. Let $u_0 = 0.1 > 0$ so that only the first minimum in (12) need be checked. We have $\mathcal{U}_- = \{u \in [-5, -1]\} = \mathcal{U}_- \cap (-\infty, 0.1]$ and the first minimum has value -0.29 attained at $u = -3.96$. We let $a = -0.1$ and proceed with a choice of b. With $\alpha = 5$, $f(u)e(u) > 0\ \forall u < -\alpha$ and we compute $m_{sp} = -0.29$. The choice $b = 4$ satisfies $-1/b > m_{sp}$. Therefore, $r(s) = (s^2+5)^3[(s^2 - 0.1)^2 + 16] + s(s^2+1)$ is a Rantzer polynomial. We remark that both minima in (12) actually exist for any choice of u_0 and a Rantzer polynomial will be obtained for any $a \in \mathbf{R}$ and a corresponding b. •

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Disturbance Decoupled Observer Design for Linear Time-Invariant Systems: A Matrix Pencil Approach

Delin Chu

***Abstract*—In this paper we give a new analysis of the observer design problem for linear time-invariant systems with partly unknown inputs. We use a matrix pencil approach that is based on a condensed form under orthogonal transformations. The solvability conditions that we obtain can be verified and the desired observer can be constructed by a numerically stable method.**

***Index Terms*—Condensed form, disturbance decoupled estimation, observer, orthogonal transformation.**

I. INTRODUCTION

In this paper we study the classical disturbance decoupled observer design problem for linear time-invariant systems of the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Gq, \quad x(t_0) = x^0 \\ y &= Cx + Du, \quad z = Hx, \end{aligned} \tag{1}$$

where

y, u are observations,
z is an estimated output and
x^0 is a given initial value.

The system matrices satisfy $A \in \boldsymbol{R}^{n \times n}$, $B \in \boldsymbol{R}^{n \times m}$, $G \in \boldsymbol{R}^{n \times p}$, $C \in \boldsymbol{R}^{q \times n}$, $D \in \boldsymbol{R}^{q \times m}$, $H \in \boldsymbol{R}^{l \times n}$. The disturbance $q(t)$ may represent noise or just an unknown input to the system.

Consider the construction of an observer of the form

$$\begin{aligned} \dot{w} &= A_c w + Ky + Su, \quad w(t_0) = w^0, \\ \hat{z} &= Fw + Ly + Nu, \end{aligned} \tag{2}$$

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The author is with the Department of Mathematics, National University of Singapore, Lower Kent Ridge, Singapore 119260 (e-mail: matchudl@math.nus.edu.sg).

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