Theory and Methodology

A general theory on spectral properties of state-homogeneous finite-state quasi-birth–death processes

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Abstract

In this paper a spectral theory pertaining to Quasi-Birth–Death Processes (QBDs) is presented. The QBD, which is a generalization of the birth–death process, is a powerful tool that can be utilized in modeling many stochastic phenomena. Our theory is based on the application of a matrix polynomial method to obtain the steady-state probabilities in state-homogeneous finite-state QBDs. The method is based on finding the eigenvalue–eigenvector pairs that solve a matrix polynomial equation. Since the computational effort in the solution procedure is independent of the cardinality of the counting set, it has an immediate advantage over other solution procedures. We present and prove different properties relating the quantities that arise in the solution procedure. By also compiling and formalizing the previously known properties, we present a formal unified theory on the spectral properties of QBDs, which furnishes a formal framework to embody much of the previous work. This framework carries the prospect of furthering our understanding of the behavior the modeled systems manifest. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Markov processes; Quasi-birth–death processes; Queuing; Matrix-polynomials; Spectral analysis; Jordan canonical forms

1. Introduction and past work

In this paper, we present a general theory for the spectral properties of state-homogeneous finite-state Quasi-Birth–Death Processes (QBDs). The results presented here are the main theoretical contribution of our work that investigates the matrix polynomial approach to QBDs. A follow-up work will demonstrate how the spectral theory depicted here can be applied to specific instances of QBDs. This application will concentrate on models of production lines, which have been studied extensively by many scholars due to the importance of the subject [1]. The suitability of QBDs to model production lines is illustrated in another paper by the same authors [3].

The purpose of this work is to depict the general structure of the state-homogeneous finite-state QBDs, which are quite pervasive in the domain of stochastic modeling. The study of QBDs was

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initiated by Evans [2] and the Ph.D. thesis of Wallace [10]. Wallace was the one to coin the term quasi-birth–death processes.

The most detailed discussion of QBDs is done by Neuts [8]. In his book, Neuts studies infinite-state QBDs using matrix geometric invariant vectors. His methodology is based on the fact that subvectors of the steady-state probability vector are related to one another in a matrix geometric fashion. The matrix-geometric rate matrix is obtained as the minimal positive solution of a non-linear matrix equation. A generalization of this approach to multiple boundaries is by Hajek [5]. New research on this line concentrates on the computation of this matrix in an efficient fashion. The work of Latouche and Ramaswami [6] is epitomic for the latest developments in the field.

Another approach is to attack the special structure manifested by the QBDs directly. Ye and Li [11] compute the steady-state probabilities for finite-state QBDs by using reduction methods on the full Markov transition matrix.

Although much previous research has been done on the subject of QBDs, matrix polynomial approaches have only been recently applied. Consequently, the spectral properties of these processes, which are of fundamental importance in the application of matrix polynomial methods, have not been thoroughly investigated up to this point. This paper develops the theory, which is to constitute the general foundation for the application of matrix polynomial methods (see Fig. 1).

In this work, we deal with finite-state QBDs. This class of QBDs has two boundaries and due to their finite-state nature steady-state probabilities always exist. Furthermore, the technique that we are employing can be also applied to the analysis of the infinite case, as it can be seen in the work of

Mitrani and Chakka [7]. The spectral theory that is presented here is just as relevant for the non-finite QBDs. Yet, some minor adjustments would be needed for this case.

As stated in the work of Yeralan and Muth [12], a state-homogenous finite-state QBD gives rise to an infinitesimal generator having a block tridiagonal structure.

$$Q = \begin{bmatrix} B_0 & C & & & & & \\ A & B & C & & & & \\ & A & B & C & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & A & B & C \\ & & & & & A & B_M \end{bmatrix}. \qquad (1)$$

The matrix Q, the infinitesimal generator, carries all the information needed to characterize a Markov process. Using this matrix, one can evaluate different quantities of interest for a given process. One point of interest is its evolution in time, referred to as the transient behavior. The transition rate matrix along with an initial condition vector is sufficient to determine the transient behavior. Yet, although one may have all the information needed in hand, the actual evaluation of the transient behavior may be computationally intractable, or just too cumbersome due to the large size of the matrix Q.

For most analysis, one is interested in the steady-state behavior of the process. Since the transient behavior is being overlooked, this kind of analysis is more easily done. All one has to do to obtain the steady-state probabilities of an irreducible Markov process is to find the null-space of the matrix Q and then choose the only element of the null-space having its components adding to
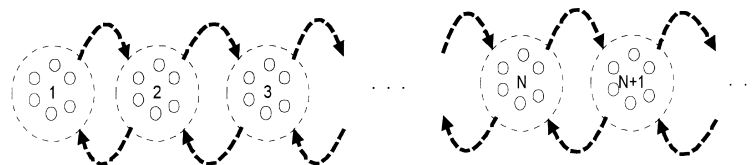

Fig. 1. Representation of a QBD (the dashed lines represent a collection of transitions originating from a state in one group and terminate in a state in another group).

one. For any irreducible Markov process, there is always a unique element satisfying this property.

In the case of state-homogeneous finite-state QBDs, which belong to the general category of Markov processes, some additional structure that is prone to further exploitation is present. Most visibly, the rate matrix has a block tridiagonal structure. Thus a method needs to be devised in order to take advantage of this structure. And here we argue that the application of matrix polynomial technique is quite propitious. This application was first proposed by Tan and Yeralan [9,13].

2. The matrix polynomial solution procedure

All quantities of interest for the evaluation of steady-state probabilities of Markov process is within the infinitesimal, Q. The matrix Q for the QBDs has the block tridiagonal form as depicted in the Eq. (1). The steady-state probability vector, $\underline{\pi}$ towards which our effort is geared – is known to satisfy the following equations:

$$\underline{\pi}Q = \underline{0}, \tag{2}$$

$$\underline{\pi}\underline{1}^{\mathrm{T}} = 1, \tag{3}$$

where Q is the transition rate matrix, $\underline{1} = (1, 1, \ldots, 1)$, and $\underline{0} = (0, 0, \ldots, 0)$. As we have already stated this probability vector is unique provided that Q is irreducible. Any vector g satisfying the Eq. (2) is called a non-normalized steady-state probability vector. Let $(\underline{g_0}, \underline{g_1}, \ldots, \underline{g_M})$ be a partitioning of the vector $\underline{g}$ in such a way that the size of each partition matches the dimension of the corresponding partition in Q as manifested in Eq. (1). Thus we can state

$$\underline{g}\,Q = \underline{0}. \tag{4}$$

Then using the partition notation we can rewrite Eq. (4).

$$\underline{g_0}B_0 + \underline{g_1}A = \underline{0}, \tag{5}$$

$$\underline{g_{i-1}}C + \underline{g_i}B + \underline{g_{i+1}}A = \underline{0}$$
$$\text{for } i = 1, 2, \ldots, M-1, \tag{6}$$

$$\underline{g_{M-1}}C + \underline{g_M}B_M = \underline{0}. \tag{7}$$

Eqs. (5) and (7) are named as the boundary equations and Eq. (6) as the interior equation. The interior equation is actually repeated $M-1$ times and can actually be classified as a difference equation. We propose to exploit this structure of the interior equation by applying the matrix polynomial solution procedure.

Assume that this matrix difference equation has solutions of the form

$$\underline{g_i} = \lambda^i \underline{e} \quad \text{for} \quad i = 0, 1, 2, \ldots, M \tag{8}$$

where λ is scalar and $\underline{e}$ is a vector of the same dimension as $\underline{g_i}$. In order to find all solutions of this family we substitute the proposed solution into the Eq. (6) and thereby obtain

$$\lambda^{i-1}\underline{e}C + \lambda^i\underline{e}B + \underline{e}A = \underline{0}. \tag{9}$$

For the values of λ different then zero this equation simplifies to

$$\lambda^2\underline{e}A + \lambda\underline{e}B + eC = \underline{0}. \tag{10}$$

Now, we define the matrix polynomial $(L\lambda)$ in order to formalize our problem within the framework of matrix polynomial theory:

$$L(\lambda) = \lambda^2 A + \lambda B + C. \tag{11}$$

Consequently, our goal – finding all vectors $\underline{g}$ satisfying Eq. (6) – can be reformulated as finding the eigenvalue–eigenvector pairs that belongs to the matrix polynomial $(L\lambda)$. We can also express the Eq. (10) using the new notation as

$$\underline{e}L(\lambda) = \underline{0}. \tag{12}$$

The set of all eigenvalue-eigenvector pairs $(\lambda_i, \underline{e_i})$ of $(L\lambda)$ yields us a set of linearly independent solution vectors which spans the general solution space of the Eq. (6). This is due to Gohberg et al. [4, Theorem 8.3, pp. 225]. The eigenvalues are the roots of the characteristic polynomial $\det((L\lambda)) = 0$. One can readily observe this characteristic equation is of degree $2n$ and thereby has $2n$ solutions where n is the size of one dimension for each partition of Q. Since each eigenvalue is an element of the extended complex numbers set – the extension is for the

introduction of ∞ to the complex numbers set – we can propose such an ordering of the eigenvalues:

$$\lambda_1, \ldots, \lambda_z = 0,$$

$$\lambda_{Z+1}, \ldots, \lambda_{Z+N} \text{ finite and non-zero},$$

$$\lambda_{Z+N+1}, \ldots, \lambda_{2n} = \infty,$$

where Z is the number of zero eigenvalues and N is the number of finite and non-zero eigenvalues. Muller's method is appropriate to obtain the characteristic equation, which is of at most degree $2n$, and subsequently to find each eigenvalue. One should be warned that in this scheme, under the presence of multiplicity in eigenvalues, each instance of a multiple eigenvalue is counted as a separate one. Now we can observe that Z is the dimension of the null space of C; and $2n - Z - N$ is the dimension of the null space of A. By the same token, the eigenvectors belonging to the null eigenvalues are the vectors spanning the null space of C; and those belonging to the infinite eigenvalues are the vectors spanning the null space of A.

When we express the general solution using the eigenvalue–eigenvector pairs we get

$$\underline{g}_i = \sum_{j=Z+1}^{Z+N} w_j \lambda_j^i \underline{e}_j \quad \text{for} \quad i = 1, 2, \ldots, M - 1. \tag{13}$$

The equations for $\underline{g}_0$ and $\underline{g}_M$ have to be expressed separately. The reason for this is that the components of the solution corresponding to the eigenvalues at 0 contribute only to $\underline{g}_0$; and similarly the components corresponding to the eigenvalues at the infinity contribute only to $\underline{g}_M$:

$$\underline{g}_0 = \sum_{j=1}^{N+Z} w_j \underline{e}_j, \tag{14}$$

$$\underline{g}_M = \sum_{j=N}^{N+Z} w_j \lambda_j^M \underline{e}_j + \sum_{j=N+Z+1}^{2n} w_j \underline{e}_j. \tag{15}$$

In all these equations $w\lambda_j$ is the weight associated with component of the solution corresponding to

the jth eigenvalue–eigenvector pair. By changing the weights we can obtain any particular solution.

But all these expressions actually are only correct under the assumption that for each eigenvalue with multiplicity higher than one, there are as many linearly independent genuine eigenvectors belonging to that eigenvalue as the multiplicity of the eigenvector. When this is not the case, the generalized eigenvectors need to be introduced along with Jordan canonical forms.

In order to express the solution within the framework of Jordan pairs, let d be the number of distinct finite eigenvalues of $L(\lambda)$ and consider an ordering of the distinct eigenvalues from 1 to d. Then we can define a finite Jordan pair (X_F, J_F) of $L(\lambda)$ as

$$X_F = [X(\lambda_1), X(\lambda_2), X(\lambda_3), \ldots, X(\lambda_d)],$$
$$J_F = \text{diag}[J(\lambda_1), J(\lambda_2), J(\lambda_3), \ldots, J(\lambda_d)],$$

where $(X(\lambda_j), J(\lambda_j))$ is a Jordan pair for every finite eigenvalue, λ_j, of $(L\lambda)$. In order to express the general solution of Eq. (12), we also need the Jordan pair that belongs to the eigenvalue at the infinity. (X_∞, J_∞) is how this pair is noted and this pair is called the infinite Jordan pair of $L(\lambda)$. One should notice that all the elements of J_∞ are either zero or one, the diagonal elements that would be the eigenvalues in a typical matrix in Jordan form are all one. Then the general solution in its correct form can be expressed as

$$\underline{g}_i = W J_F^i X_F \quad \text{for} \quad i = 0, 1, 2, \ldots, M - 1, \tag{16}$$

$$\underline{g}_M = W J_F^M X_F + W_\infty J_\infty X_\infty, \tag{17}$$

where W is the a row vector with $Z + N$ elements having as component the weights corresponding to the part of the solution due to finite eigenvalues; and W_∞ is a row vector with $2n - Z - N$ elements having as component the weights corresponding to the part of the solution due to the eigenvalue at infinity.

With the Eqs. (16) and (17), we have the correct version of the general solution to the matrix difference equation, which is Eq. (5). That means any solution can be expressed as a special case of this one just by adjusting the weights.

Yet, this closed form is not always useful for our purposes. Thus, we will soon present the general solution in a more explicit form, which is instrumental in showing the properties of the QBDs. Some new notation is needed for this endeavor.

Let ρ be the number of the finite genuine eigenvalues or in other words the number of Jordan canonical blocks in the matrix J_F. Let ρ_∞ be the number of genuine eigenvalues at infinity that is the number of Jordan canonical blocks in the matrix J_∞. Let the series $(\lambda_j / j = 1, 2, \ldots, \rho + \rho_\infty)$ be a non-decreasing sequence of those eigenvalues with the eigenvectors at infinity at the end of the sequence.

$$\underline{g_i} = \sum_{j=1}^{\rho} w_j \big[J(\lambda_j) \big]^i X(\lambda_j)$$
$$+ \delta(M, i) \sum_{j=\rho}^{\rho+\rho_\infty} w_j J_\infty(\lambda_j) X_\infty(\lambda_j)$$
$$\text{for } i = 0, 1, 2, \ldots, M, \tag{18}$$

where w_j is a row vector of the size of Jordan block belonging to λ_j and $\delta(i, j)$ be the indicator function.

It is possible to further expand this expression, since we know the closed-form expression for the powers of the Jordan canonical forms:

$$\underline{g_i} = \sum_{j=1}^{\rho} \sum_{k=1}^{s(\lambda_j)} w_{(j,k)} \sum_{l=\max(i-k+1,0)}^{i} \binom{i}{l} \lambda_j^l \underline{e_{(j,k-i+l)}}$$
$$+ \delta(M, i) \sum_{j=\rho}^{\rho+\rho_\infty} \sum_{k=1}^{s(\lambda_j)} w_{(j,k)} \sum_{l=\max(i-k+1,0)}^{i} \binom{i}{l} \underline{e_{(j,k-i+l)}}$$
$$\text{for } i = 1, 2, \ldots, M, \tag{19}$$

where $s(\lambda_j)$ is the length of the generalized eigenvector cycle, or in other words, the size of the Jordan canonical form corresponding to the eigenvector, λ_j; $\overline{e_{(j,k)}}$ is the kth generalized eigenvector of the generalized eigenvector cycle corresponding to the eigenvector, (λ_j); and $w_{(j,k)}$ is the weight that corresponds to the kth generalized eigenvector of the generalized eigenvector cycle corresponding to the eigenvector, (λ_j). One should observe in the Eq. (19) that the terms that belong to the free variable $w_{(j,k)}$ not only include the generalized eigenvector corresponding to this term, but also generalized eigenvectors that are preceding it in the same generalized eigenvector cycle. This actually foreshadows some properties that are going be presented in the next section.

The general solution presented has $2n$ variables that can be set freely. Using the physics jargon, it has $2n$ degrees of freedom. This freedom is needed since the particular solution that is of interest needs to satisfy the Eqs. (5) and (7) which constitute a set of linearly independent equation system with rank $2n - 1$, along with the normalizing condition. The fact that the rank is $2n - 1$ is quite critical since if the rank were $2n$, the only possible solution of the system, which is homogeneous, would be the trivial solution, being the zero vector. Moreover, since the zero vector solution would always be in contradiction with the normalizing condition, the system would be without a solution. Yet, just as we would expect, the system of equations has a linear dependence. This dependence is also going to be shown in conjunction with the spectral properties of the process in the next section.

Let us illustrate the technique with a numerical example. Let the submatrices in the infinitesimal generator be of size 2×2, and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -2.01 & 0.01 \\ 0.1 & -1.1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B_0 = \begin{bmatrix} -1.01 & 0.01 \\ 0.1 & -1.1 \end{bmatrix},$$
$$B_M = \begin{bmatrix} -1.01 & 0.01 \\ 0.1 & -0.1 \end{bmatrix}, \quad M = 10.$$

The characteristic equation for the QBD is

$$\det(L(x)) = -1.1x^3 + 3.2x^2 - 3.11x + 1.$$

Thence, the four eigenvalues for the QBD are

$$x_1 = 0.8553, \quad x_2 = 1.0628, \quad x_3 = 1, \quad x_4 = \infty.$$

We also need the eigenvectors corresponding to these eigenvalues in order to construct the general solution and the boundary conditions. These eigenvectors along with their corresponding components in the general solution are

$$\underline{e_1} = [-0.9897 \quad 0.1432],$$
$$\underline{Com_1^i} = (0.8853)^i[-0.9897 \quad 0.1432],$$
$$\underline{e_2} = [0.9980 \quad 0.0627],$$
$$\underline{Com_2^i} = (1.0628)^i[0.9980 \quad 0.0627],$$
$$\underline{e_3} = [0.9950 \quad 0.0995],$$
$$\underline{Com_3^i} = [0.9950 \quad 0.0995],$$
$$\underline{e_4} = [0 \quad 1],$$
$$\underline{Com_4} = [0 \quad 1].$$

Thus, the general solution is

$$\underline{g_i} = \sum_{j=1}^{2} w_j \underline{Com_{(j,1)}^i} + \delta(0,i)w_3 \underline{Com_3}$$
$$+ \delta(M,i)w_4 \underline{Com_4}.$$

Using the general solution, one can generate the boundary equations and organize them in matrix notation:

$$\begin{bmatrix} w_{(1,1)} & w_{(2,1)} & w_{(3,1)} & w_{(3,1)} \end{bmatrix}$$
$$\begin{bmatrix} 0.1674 & -0.1674 & -0.0300 & 0.0300 \\ 0.0590 & -0.0590 & -0.1154 & 0.1154 \\ 0 & -0.0995 & 0 & 0.0995 \\ 0 & 0 & 0.1000 & -0.1000 \end{bmatrix} = \underline{0}.$$

One can observe that the boundary equations form a homogeneous system. For a non-trivial solution to exist, the rank of this system should be at most three. Indeed, this is always the case since they are boundary equations. The solution of the system is actually the nullspace of the boundary matrix. For the presented system, the nullspace is

$$\begin{bmatrix} w_{(1,1)} & w_{(2,1)} & w_{(3,1)} & w_{(3,1)} \end{bmatrix}$$
$$= [-0.2365 \quad 0.6708 \quad 0 \quad 0.7029].$$

After having obtained the weights of the components in the solution, one can generate the solution. Then by normalizing the solution, one can finally reach the steady-state probabilities pertaining to the QBD (see Table 1).

3. Spectral properties of state-homogeneous finite-state quasi-birth–death processes

In this section we are going to delve in the spectral properties state-homogeneous finite-state QBDs manifest. This section will directly be founded on the solution procedure developed in Section 2. We begin this treatment of the subject by presenting two readily justifiable assumptions.

Assumption 1. The process is ergodic. That is the transition rate matrix Q is irreducible.

Assumption 2. The matrix $(A + B + C)$ is an irreducible transition rate matrix.

Assumption 1 is a standard assumption in the analysis of Markov processes, since if this assumption did not hold, one could always decompose the process into decoupled subprocesses, and apply the same analysis on these subprocesses.

Assumption 2 makes certain that interior equations of the process do not yield to any kind of decoupling. The fact that $(A + B + C)$ is a transition rate matrix actually needs to be proven. This can be done by making use of Property 2 that will be presented later and the fact that the only negative entries are the diagonal entries of the matrix.

Let the state space of the process be the Kroenicker product of the counting set $C = \{e \in N/0$

Table 1
Steady-state probabilities for a given QBD computed employing the matrix polynomial solution procedure[a]

	0	1	2	3	4	5	6	7	8	9	10
1	0.0851	0.0833	0.0821	0.0812	0.0808	0.0806	0.0808	0.0812	0.0818	0.0826	0.0836
2	0.0008	0.0013	0.0018	0.0022	0.0026	0.0030	0.0033	0.0036	0.0038	0.0041	0.0705

[a] The state space is the Kroenicker product of $\{0,1,\ldots,10\}$ (counting set) and $\{1,2\}$ (internal state space whose cardinality is the dimension of submatrices). The columns correspond to the counting set level (the state groups shown in Fig. 1) and the rows correspond to the state index for a given counting set level (the number of a given state in the aforementioned group).

$\leqslant e \leqslant M\}$ and the internal state space whose cardinality is the size of all the aforementioned submatrices. It is known that only the diagonal element of B are negative and all the other element of matrices $A, B,$ and C are non-negative. Since all coupling between the internal states manifest themselves as positive entries in the submatrices A, B and C these couplings would all be preserved in the stochastic matrix, $A + B + C$. Thus if the mentioned matrix is not an irreducible transition rate matrix, there would be some kind of decoupling in the internal-state-space. That means from a certain group of elements of the space, there would be no transition to another group.

It is also possible to show this decoupling on the elements of the solution procedure. If $(A + B + C)$ is not an irreducible transition rate matrix then the matrix polynomial $L(\lambda)$ would have eigenvalue and eigenvector pairs can be partitioned among the subprocesses. Thus, one could actually work separately with these subprocesses.

One should also make a note that the Assumption 1 does not imply Assumption 2. Even when $A + B + C$ corresponds to a reducible Markov chain, the full chain may be irreducible due to a coupling at the boundary states – the states corresponding to 0 an M in the counting set, C. When this one is the case, one can work with two separate processes up to the application of boundary processes, and consequently reduce the computational effort that needs to be exerted.

Property 1. $A\underline{1}^{\mathrm{T}} \neq \underline{0}^{\mathrm{T}}$ *and* $C\underline{1}^{\mathrm{T}} \neq \underline{0}^{\mathrm{T}}$.

Since both A and C consist of non-negative elements the sum of row elements can be zero only if all the elements of corresponding rows are zero. This means for the Property 1 not to hold, either A or C would have to be a zero matrix which would in turn render Q a reducible transition rate matrix. Since this would contradict Assumption 1, Property 1 always holds.

Property 2.

$$A\underline{1}^{\mathrm{T}} + B\underline{1}^{\mathrm{T}} + C\underline{1}^{\mathrm{T}} = \underline{0}^{\mathrm{T}}. \tag{20}$$

This property directly follows the fact that Q is a transition rate matrix. It is known that the elements in each row of Q has to add to zero, or $Q\underline{1}^{\mathrm{T}} = \underline{0}^{\mathrm{T}}$. When the Q is written in terms of the submatrices that belong to it, one obtains

$$\begin{bmatrix} B_0\underline{1}^{\mathrm{T}} + C\underline{1}^{\mathrm{T}} \\ A\underline{1}^{\mathrm{T}} + B\underline{1}^{\mathrm{T}} + C\underline{1}^{\mathrm{T}} \\ \cdot \\ \cdot \\ \cdot \\ A\underline{1}^{\mathrm{T}} + B\underline{1}^{\mathrm{T}} + C\underline{1}^{\mathrm{T}} \\ A\underline{1}^{\mathrm{T}} + B_M\underline{1}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \underline{0}^{\mathrm{T}} \\ \underline{0}^{\mathrm{T}} \\ \cdot \\ \cdot \\ \cdot \\ \underline{0}^{\mathrm{T}} \\ \underline{0}^{\mathrm{T}} \end{bmatrix}. \tag{21}$$

The interior rows all yield Property 2. Moreover, two more relations are obtained from the boundary rows:

$$B_0\underline{1}^{\mathrm{T}} + C\underline{1}^{\mathrm{T}} = \underline{0}^{\mathrm{T}}, \tag{22}$$

$$A\underline{1}^{\mathrm{T}} + B_M\underline{1}^{\mathrm{T}} = \underline{0}^{\mathrm{T}}. \tag{23}$$

Property 3. *One is always an eigenvalue of* $L(\lambda)$.

By Property 2 $(A + B + C)\underline{1}^{\mathrm{T}} = \underline{0}^{\mathrm{T}}$ This can also be written as $L(1)\underline{1}^{\mathrm{T}} = \underline{0}^{\mathrm{T}}$. This means one is an eigenvalue of $L\lambda$ and $\underline{1}^{\mathrm{T}}$ is a right-eigenvector corresponding to it.

Property 4.

$$\underline{g}_i C\underline{1}^{\mathrm{T}} = \underline{g}_{i+1} A\underline{1}^{\mathrm{T}} \quad for \; i = 0, 1, 2, \ldots, M - 1. \tag{24}$$

This equation is called the balance of flow equations for the QBDs. It is a generalization of the balance of flow equations for normal birth-death processes. This equation simply states the fact that in the equilibrium, the rate at which the probability of being in the states corresponding to the ith element of the counting set is transformed to the probability of being in the states corresponding to the $(i + 1)$th element of the counting set, is equal to the rate the reverse occurs. This transformation that conserves the quantities can be likened to a flow, when the liquid metaphor is used for the probabilities.

This property is proven by mathematical induction. By multiplying both sides of the Eq. (5)

with the additive vector, $\underline{1}^{\mathrm{T}}$, from the right, one obtains $\underline{g}_0 B_0 \underline{1}^{\mathrm{T}} + \underline{g}_1 A \underline{1}^{\mathrm{T}} = 0$. Using (22) along this one, one reaches

$$\underline{g}_0 C \underline{1}^{\mathrm{T}} = \underline{g}_1 A \underline{1}^{\mathrm{T}}. \tag{25}$$

Thereby the property holds for $i = 0$. Now one should assume that the property holds for $i = k$ in order to use induction argument, i.e.,

$$\underline{g}_k C \underline{1}^{\mathrm{T}} = \underline{g}_{k+1} A \underline{1}^{\mathrm{T}}. \tag{26}$$

When Eq. (20) is used to substitute $C\underline{1}^{\mathrm{T}}$ in Eq. (6) for $i = k + 1$, the following is obtained:

$$\underline{g}_k C \underline{1}^{\mathrm{T}} - \underline{g}_{k+1}(A\underline{1}^{\mathrm{T}} + C\underline{1}^{\mathrm{T}}) + \underline{g}_{k+2} A \underline{1}^{\mathrm{T}} = 0.$$

Substituting (26) in this expression yields

$$\underline{g}_{k+1} C \underline{1}^{\mathrm{T}} = \underline{g}_{k+2} A \underline{1}^{\mathrm{T}}.$$

Thereby the identity holds for $i = k + 1$ given that it holds for $i = k$. Now by using the induction argument we prove that Property 4 is always true.

Theorem 1.

$$\lambda_j \underline{e}_{(j,1)} A \underline{1}^{\mathrm{T}} = \lambda \underline{e}_{(j,1)} C \underline{1}^{\mathrm{T}} \tag{27}$$

for all eigenvalues λ_j different from one.

Eq. (27) is referred as a Balance Equation in Component Form (BECF) by Tan who first proposed and proved it. Theorem 1 states that BECF hold for all eigenvalue–eigenvector pairs satisfying the matrix polynomial Eq. (12) except for those with eigenvalue one. The case of those with eigenvalue one will be investigated separately. Also, one should notice that Theorem 1 does not state anything about the generalized eigenvectors that may be present in the solution due to multiplicity in the roots of characteristic equation.

Proof. Let $\underline{e}_{(j,1)}$ be an eigenvalue–eigenvector pair as defined for the Eq. (19). This eigenvalue, λ_j, may or may not correspond to a generalized eigenvector cycle. In either case the first element of

the cycle, the genuine eigenvector exists. Since this pair is a solution of the matrix polynomial Eq. (10), one can write

$$\lambda_j^2 \underline{e}_{(j,1)} A + \lambda_j \underline{e}_{(j,1)} B + \underline{e}_{(j,1)} C = \underline{0}. \tag{28}$$

Postmultiplying both side of the equation by $\underline{1}$ and then by using Property 2 one gets

$$\lambda_j^2 \underline{e}_{(j,1)} A \underline{1}^{\mathrm{T}} - \lambda_j \underline{e}_{(j,1)}(A\underline{1}^{\mathrm{T}} + C\underline{1}^{\mathrm{T}}) + \underline{e}_{(j,1)} C\underline{1}^{\mathrm{T}} = 0, \text{ or}$$
$$\lambda_j(\lambda_j - 1)\underline{e}_{(j,1)} A\underline{1}^{\mathrm{T}} - (\lambda_j - 1)\underline{e}_{(j,1)} C\underline{1}^{\mathrm{T}} = 0, \text{ or}$$
$$(\lambda_j - 1)\left(\lambda_j \underline{e}_{(j,1)} A\underline{1}^{\mathrm{T}} - \underline{e}_{(j,1)} C\underline{1}^{\mathrm{T}}\right) = 0. \tag{29}$$

Thus, under the condition, λ_j is different than one, (27) holds. This is exactly what was to be proven. $\quad\square$

Theorem 2.

$$\underline{e}_{(j,k)}(\lambda_j A - C)\underline{1}^{\mathrm{T}} = -\underline{e}_{(j,k-1)} A\underline{1}^{\mathrm{T}} \tag{30}$$

for all eigenvalues λ_j different from one and $k = 2, 3, \ldots, s(\lambda_j)$ where $s(\lambda_j)$ is the length of the generalized eigenvector cycle, or in other words, the size of the Jordan canonical form corresponding to the eigenvector, λ_j.

We refer to Eq. (30) as the Raw Balance Equations in the Component Form (RBECF). These equations are to be used for the generalized eigenvectors. For the first element of the cycle, which is the genuine eigenvector, BECF are used. But as one can observe for the case of generalized eigenvectors, the relation is a little bit more complicated. Each expression depends on the previous element of the cycle. Yet, we are going to prove that this expression still yields a balance component by component. The impurity introduced by the left-hand side of (30) is actually essential for this occurrence since in the case of generalized eigenvectors, each term of the general solution starting with the coefficient corresponding to a certain generalized eigenvector also includes the previous eigenvectors of the same cycle. This fact is well manifested in Eq. (19).

Proof. We are going to prove this theorem using the induction method. First we have to show that the theorem is correct for $k = 2$ which is the initial point for which the theorem is valid. The defining equation for the second eigenvector of a cycle is

$$\underline{e}_{(j,1)} \frac{\mathrm{d}L(\lambda)}{\mathrm{d}\lambda} + \underline{e}_{(j,2)} L(\lambda) = \underline{0}. \tag{31}$$

Substituting $L(\lambda)$ from (11) one obtains

$$\underline{e}_{(j,1)}(2\lambda_j A + B) + \underline{e}_{(j,2)}(\lambda_j^2 A + \lambda_j B + C) = \underline{0}. \tag{32}$$

Postmultiplying this equation with the $\underline{1}^{\mathrm{T}}$ vector and then making use of Property 1 yields

$$\underline{e}_{(j,1)}(2\lambda_j A - A - C)\underline{1}^{\mathrm{T}}$$
$$+ \underline{e}_{(j,2)}(\lambda_j^2 A - \lambda_j(A + C) + C)\underline{1}^{\mathrm{T}} = 0.$$

Since the eigenvalue, λ_j, is different from one by hypothesis, one can use Theorem 1 and obtain

$$(\lambda_j - 1)\left[\underline{e}_{(j,1)} A\underline{1}^{\mathrm{T}} + \underline{e}_{(j,2)}(\lambda_j A - C)\underline{1}^{\mathrm{T}}\right] = 0, \text{ or}$$

$$\underline{e}_{(j,2)}(\lambda_j A - C)\underline{1}^{\mathrm{T}} = -\underline{e}_{(j,1)} A\underline{1}^{\mathrm{T}}. \tag{33}$$

Thereby it is shown that the Theorem 2 is correct for $k = 2$.

Then in order to use the induction argument one has to assume that the theorem is valid up to $k = x$ and prove that this one ensures the validness of the theorem for $k = x + 1$, i.e.,

$$\underline{e}_{(j,x)}(\lambda_j A - C)\underline{1}^{\mathrm{T}} = -\underline{e}_{(j,x-1)} A\underline{1}^{\mathrm{T}}. \tag{34}$$

The defining equation for the $(x + 1)$th eigenvector of a cycle is

$$\sum_{y=0}^{x} \frac{1}{y!} \underline{e}_{(j,x+1-y)} \frac{\mathrm{d}^y L(\lambda)}{\mathrm{d}\lambda^y} = \underline{0}. \tag{35}$$

One can expand this expression by substituting $L(\lambda)$ from (11). The following expression is valid $x + 1 \geqslant 3$. Yet, since our induction starts at $x = 2$, we can use it freely for our purposes.

$$\underline{e}_{(j,x-1)}(A) + \underline{e}_{(j,x)}(2\lambda_j A + B)$$
$$+ \underline{e}_{(j,x+1)}(\lambda_j^2 A + \lambda_j B + C) = \underline{0}. \tag{36}$$

Postmultiplying this equation with the $\underline{1}^{\mathrm{T}}$ vector and then making use of Property 1 yields

$$\underline{e}_{(j,x-1)} A\underline{1}^{\mathrm{T}} + \underline{e}_{(j,x)}(2\lambda_j A - A - C)\underline{1}^{\mathrm{T}}$$
$$+ \underline{e}_{(j,x+1)}(\lambda_j^2 A - \lambda_j(A + C) + C)\underline{1}^{\mathrm{T}} = \underline{0}.$$

Applying the induction hypothesis, Eq. (34), one obtains

$$(\lambda_j - 1)\left(\underline{e}_{(j,x)} A\underline{1}^{\mathrm{T}} + \underline{e}_{(j,x+1)}(\lambda_j A - C)\underline{1}^{\mathrm{T}}\right) = 0, \text{ or}$$

$$\underline{e}_{(j,x+1)}(\lambda_j A - C)\underline{1}^{\mathrm{T}} = -\underline{e}_{(j,x)} A\underline{1}^{\mathrm{T}}.$$

Thereby, we have shown that Theorem 2 is valid for $k = x + 1$ given that it is valid for $k = x$. This along with the validity at $k = 2$ is enough for the induction argument. Thus by induction, Theorem 2 is proven. $\square$

Now we would like to show that the balance of flow equations that holds for the general solution, as expressed by Property 4, can also be formulated for each component of the solution that corresponds to a non-zero eigenvalue. Although the idea is similar, the expression will be somewhat different due to non-trivial dependence of the solution components on the counting set.

Here, the word *component* is used for each expression that corresponds to a term starting with a $w_{(j,k)}$ in the Eq. (19). Thus, each component is a part of the solution that corresponds to a given genuine or generalized eigenvector. At total there are $2n$ components. This property is important since it is essential for the freedom of setting each $w_{(j,k)}$ arbitrarily, or more correctly, independently from the internal equations.

Before proceeding to the proof of Theorem 3, we have to formally define the components. One should note that each component is an ordered collection of M vectors, one for each element of the counting space. Let $\underline{Com}_{(j,k)}^i$ be the ith element of the collection that is the component that corresponds to kth generalized eigenvector of the jth eigenvalue, defined as

$$\underline{Com}^i_{(j,k)} = \sum_{l=\max(i-k+1,0)}^{i} \binom{i}{l} \lambda_j^l \underline{e}_{(j,k-i+l)}. \tag{37}$$

Then the component, $Com_{(i,j)}$ can be defined as

$$\underline{Com}_{(j,k)} = \left(\underline{Com}^1_{(j,k)}, \underline{Com}^2_{(j,k)}, \ldots, \underline{Com}^M_{(j,k)} \right) \tag{38}$$

by concatenating the elements of the collection according to the order.

This notation is also used for the eigenvalues at the infinity. In this case, they are defined as

$$\underline{Com}_{(j,k)} = \left(\underline{0}, \underline{0}, \ldots, \underline{Com}^M_{(j,k)} \right), \quad \text{where}$$

$$\underline{Com}^M_{(j,k)} = \sum_{l=\max(M-k+1,0)}^{M} \binom{M}{l} \underline{e}_{(j,k-i+l)} \tag{39}$$

After defining the entire notation needed, the theorem can now be stated.

Theorem 3.

$$\underline{Com}^i_{(j,k)} C\underline{1}^{\mathrm{T}} = \underline{Com}^{i+1}_{(j,k)} A\underline{1}^{\mathrm{T}} \tag{40}$$

for all (j,k) pair corresponding to an generalized eigenvector for which $\lambda_j \neq 1$ and for $i = 0, 1, 3, \ldots, M - 1$.

We refer to Eq. (40) as the Generalized Balance Equations in Component Form (GBECF). We can readily notice that when $k = 1$, these equations are the same as (27) which we have stated as Theorem 1. Thus, Theorem 1 is a special case of the Theorem 3. But this theorem is correct also for the generalized eigenvectors. This fact justifies name selected for the theorem.

Before attempting the proof of this theorem one more step needs to be taken. Now we are going to present a Lemma that we are going to use in proving Theorem 3. Before presenting the lemma, let us define

$$q_x = \sum_{l=x}^{i} \lambda_j^l \underline{e}_{(j,k-i+l)} \left[\lambda_j \binom{i+1}{l+1} A\underline{1}^{\mathrm{T}} - \binom{i}{l} C\underline{1}^{\mathrm{T}} \right]. \tag{41}$$

Lemma 1.

(i) $i - k + 1 \geqslant 0 \Rightarrow \underline{Com}^{i+1}_{(j,k)} A\underline{1}^{\mathrm{T}} - \underline{Com}^i_{(j,k)} C\underline{1}^{\mathrm{T}}$
$= q_{i-k+1}.$

(ii) $i - k + 1 < 0 \Rightarrow \underline{Com}^{i+1}_{(j,k)} A\underline{1}^{\mathrm{T}} - \underline{Com}^i_{(j,k)} C\underline{1}^{\mathrm{T}}$
$= q_0 + \underline{e}_{(j,k-i-1)} A\underline{1}^{\mathrm{T}}.$

Proof. Both parts of Lemma 1 can be easily shown by substituting Eq. (37) and by changing the summation variable. □

Lemma 2.

$$q_x = -\binom{i}{x} \lambda_j^x \underline{e}_{(j,k+x-i-1)} A\underline{1}^{\mathrm{T}} \tag{42}$$

for $\max(i - k + 2, 0) \leqslant x \leqslant i$.

Proof. We prove this theorem by induction. The induction starts at point $x = i$ and ends at point $x = i - k + 2$. Thus the induction is done by decreasing the variable, x, one by one. First one has to show that the theorem is valid at the initial point. By the definition of q_x,

$$q_i = \lambda_j^i \underline{e}_{(j,k)} \left(\lambda_j A\underline{1}^{\mathrm{T}} - C\underline{1}^{\mathrm{T}} \right).$$

When Theorem 2 is applied, one obtains

$$q_i = -\lambda_j^i \underline{e}_{(k-1)} A\underline{1}^{\mathrm{T}}. \tag{43}$$

Thereby the theorem is valid at $x = i$.

In order to apply the induction method, we assume that the theorem is valid from $x = i$ to $x = z$. At this point, we have

$$q_z = -\binom{i}{z} \lambda_j^z \underline{e}_{(k+z-i-1)} A\underline{1}^{\mathrm{T}}. \tag{44}$$

We know that

$$q_{z-1} = q_z + \lambda_j^{z-1} \underline{e}_{(j,k+z-i-1)} \left[\lambda_j \binom{i+1}{z} A\underline{1}^{\mathrm{T}} - \binom{i}{z-1} C\underline{1}^{\mathrm{T}} \right]. \tag{45}$$

Moreover, by algebraic manipulation, one can easily show that

$$\binom{i+1}{z} = \binom{i}{z} + \binom{i}{z-1}. \tag{46}$$

By substituting (44) and (46) in (45) one obtains

$$q_{z-1} = \binom{i}{z-1} \lambda_j^{z-1} \underline{e_{(j,k+z-i-1)}} \left[\lambda_j A\underline{1}^{\mathrm{T}} - C\underline{1}^{\mathrm{T}} \right].$$

Since $k + z - i - 1 \leqslant 2$ we can make use of the Theorem 2 and thereby reach

$$q_{z-1} = -\binom{i}{z-1} \lambda_j^{z-1} \underline{e_{(k+z-i-2)}} A\underline{1}^{\mathrm{T}}.$$

We observe that the theorem is valid for $x = z - 1$. Thus the proof by induction is complete. $\square$

Proof of Theorem 3. In order to prove Theorem 3 two separate cases need to be investigated.

(i) *Case of* $i - k + 1 \geqslant 0$

By Lemma 1, we know that

$$\underline{Com_{(j,k)}^{i+1}} A\underline{1}^{\mathrm{T}} - \underline{Com_{(j,k)}^{i}} C\underline{1}^{\mathrm{T}} = q_{i-k+1}.$$

Moreover from Lemma 2 we have

$$q_{i-k+2} = -\binom{i}{k-2} \lambda_j^{i-k+2} \underline{e_{(j,1)}} A\underline{1}^{\mathrm{T}}, \text{ or}$$

$$q_{i-k+1} = -\binom{i}{k-2} \lambda_j^{i-k+2} \underline{e_{(j,1)}} A\underline{1}^{\mathrm{T}}$$
$$+ \lambda_j^{i-k+1} \underline{e_{(j,1)}} \left[\lambda_j \binom{i+1}{k-1} A\underline{1}^{\mathrm{T}} \right.$$
$$\left. - \binom{i}{k-1} C\underline{1}^{\mathrm{T}} \right].$$

Using Eq. (46) one obtains

$$q_{i-k+1} = \binom{i}{k-1} \lambda_j^{i-k+1} \underline{e_{(j,1)}} \left[\lambda_j A\underline{1}^{\mathrm{T}} - C\underline{1}^{\mathrm{T}} \right].$$

Applying Theorem 1 yields $q_{i-k+1} = 0$, or,

$$\underline{Com_{(j,k)}^{i}} C\underline{1}^{\mathrm{T}} = \underline{Com_{(j,k)}^{i+1}} A\underline{1}^{\mathrm{T}}.$$

Thereby the theorem is proven for the first case.

(ii) *Case of* $i - k - 1 < 0$

By Lemma 1, we know that

$$\underline{Com_{(j,k)}^{i+1}} A\underline{1}^{\mathrm{T}} - \underline{Com_{(j,k)}^{i}} C\underline{1}^{\mathrm{T}} = q_0 + \underline{e_{(j,k-i-1)}} A\underline{1}^{\mathrm{T}}.$$

Moreover from Lemma 2 we have

$$q_0 = -\underline{e_{(j,k-i-1)}} A\underline{1}^{\mathrm{T}}.$$

This translates to

$$\underline{Com_{(j,k)}^{i}} C\underline{1}^{\mathrm{T}} = \underline{Com_{(j,k)}^{i+1}} A\underline{1}^{\mathrm{T}}.$$

Thereby the theorem is proven for the second case and the proof is complete. $\square$

Theorem 4. *There is one Jordan block that corresponds to the roots of the characteristic equation at one. That is, all roots at one give rise to a single cycle of generalized eigenvectors.*

Proof. If there existed more than one Jordan block that corresponds to the eigenvector, $\lambda_j = 1$, then the nullspace of $(A + B + C)^{\mathrm{T}}$ would be greater than two. That is, $A + B + C$ would be reducible. Since this would contradict Assumption 2, the theorem is proven by contradiction. $\square$

Theorem 5. *The eigenvalue $\lambda_j = 1$ has multiplicity greater than two if and only if $\underline{e_{(j,1)}} A\underline{1}^{\mathrm{T}} = \underline{e_{(j,1)}} C\underline{1}^{\mathrm{T}}$.*

We refer to Theorem 5 as the Non-Balance at Unity Theorem (NBUT). This is the equivalent of BECF of Theorem 1 for the case of the eigenvalue at one. Yet, we observe that when the multiplicity of the eigenvalue at one is one, there is no balance of flow in component form for the component that corresponds to this eigenvalue.

Proof. We already showed that there is a single generalized eigenvector cycle corresponding to the eigenvalue at one. This means that the multiplicity of the eigenvalue at one is equal to the number of the elements of the mentioned cycle.

Now let us assume that the multiplicity of the eigenvalue at one is greater than one. Consequently, we know that Eq. (32), the defining equation for the second generalized eigenvector, is valid for this eigenvalue, i.e.,

$$\underline{e_{(j,1)}}(2A + B) + \underline{e_{(j,2)}}(A + B + C) = \underline{0}. \tag{47}$$

Postmultiplying both sides of the equation with $\underline{1}^T$ and using Property 2 yields

$$\underline{e}_{(j,1)}A\underline{1}^T = \underline{e}_{(j,1)}C\underline{1}^T. \tag{48}$$

Thus the theorem is shown in forward direction.

The proof in the reverse direction is a little more involved. For this we have to assume that

$$\underline{e}_{(j,1)}A\underline{1}^T = \underline{e}_{(j,1)}C\underline{1}^T$$

and show that there is always some vector $\underline{e}_{(j,2)}$ that would satisfy the defining equation for the second eigenvector which happens to be Eq. (47). In other words we have to show that we can always find a vector that is transformed to $-\underline{e}_{(j,1)}(2A + B)$ by the transformation $(A + B + C)$.

We can show that $-\underline{e}_{(j,1)}(2A + B)$ is orthogonal to $\underline{1}$ by multiplying the two vectors and making use of Property 2 and Eq. (48) which is in our hypothesis, i.e.,

$$-\underline{e}_{(j,1)}(2A + B)\underline{1}^T = -\underline{e}_{(j,1)}(A - C)\underline{1}^T = 0.$$

At this point we are going to present some properties of the transformation $(A + B + C)$. It is known that the rank of $(A + B + C)$ is $n - 1$ where n yields the dimension of $(A + B + C)$ by the expression $n \times n$. This is due to Assumption 2 that states that $(A + B + C)$ is an irreducible transition rate matrix. Consequently, S_R, the row space of $(A + B + C)$, is $n - 1$ dimensional. Moreover for any vector $\underline{x}$, one can write

$$\underline{x}(A + B + C)\underline{1}^T = 0,$$

which means $\underline{1}$ is also orthogonal to the row space of $(A + B + C)$. Thus a direct sum of span of $\underline{1}$ and S_R would be the entire space of n dimensional row vectors.

We can now conclude that $-\underline{e}_{(j,1)}(2A + B)$ that is orthogonal to $\underline{1}$ has to be an element of S_R, the row space of $(A + B + C)$. This means that we can always find a vector $\underline{e}_{(j,2)}$ satisfying Eq. (47). Thus the multiplicity of the eigenvalue at one has to be greater than two. □

Theorem 6. *For $m \geqslant 3$, the eigenvalue, $\lambda_j = 1$, has multiplicity greater than m if and only if*

$$\underline{e}_{(j,1)}A\underline{1}^T = \underline{e}_{(j,1)}C\underline{1}^T$$

and

$$\underline{e}_{(j,k)}(A - C)\underline{1}^T = -\underline{e}_{(j,k-1)}(A)\underline{1}^T$$

for $2 \leqslant k \leqslant m - 1$.

We refer to Theorem 6 as the Deficient Raw Balance at Unity Theorem (DRBUT). This is the equivalent of RBECF of Theorem 2 for the case of the eigenvalue at one. Yet, we observe for this case, the equation that would cause the balance of flow in component form for the *components* corresponding to the last element of the cycle of eigenvectors that belongs to the eigenvector at one, never holds. For the previous elements of the same cycle, equalities, which will cause the balance of flow in component form for the *components* corresponding to these elements, hold. The equivalance between the balance of flow equations and these equalities will be shown by Theorem 7.

Proof. The proof of the theorem will be based on induction. First we will show that the theorem is correct for $m = 3$. One should note that the induction argument should be used to prove a double implication.

Let us assume that the multiplicity of the eigenvalue at one is greater than three. Consequently, we can use Theorem 5 that asserts that

$$\underline{e}_{(j,1)}A\underline{1}^T = \underline{e}_{(j,1)}C\underline{1}^T.$$

Moreover we can use Eq. (36), the defining equation for $(x + 1)$th eigenvector of the cycle, for the third eigenvector, and obtain

$$\underline{e}_{(j,1)}(A) + \underline{e}_{(j,2)}(2A + B) + \underline{e}_{(j,3)}(A + B + C) = \underline{0}. \tag{49}$$

Postmultiplying both sides of the equation with $\underline{1}^T$ and using Property 2 yields

$$\underline{e}_{(j,2)}(A - C)\underline{1}^T = -\underline{e}_{(j,1)}(A)\underline{1}^T. \tag{50}$$

Thus, the theorem is proven in forward direction for $m = 3$.

Now we present the proof in the reverse direction for this case. For this endeavor, we have to assume that Eqs. (50) and (48) hold and show that there is always some vector $e_{(j,3)}$ that would satisfy the defining equation for the third eigenvector which happens to be Eq. (49). In other words we have to show that we can always find a vector that is transformed to

$$-\left(\underline{e_{(j,1)}}(A) + \underline{e_{(j,2)}}(2A + B)\right)$$

by the transformation $(A + B + C)$.

We can show that

$$-\left(\underline{e_{(j,1)}}(A) + \underline{e_{(j,2)}}(2A + B)\right)$$

is orthogonal to $\underline{1}$ by multiplying the two vectors and making use of Property 2 and Eq. (50) which is in our hypothesis, i.e.,

$$-\left(\underline{e_{(j,1)}}(A) + \underline{e_{(j,2)}}(2A + B)\right)\underline{1}^{\mathrm{T}}$$
$$= -\left(\underline{e_{(j,1)}}(A) + \underline{e_{(j,2)}}(A - C)\right)\underline{1}^{\mathrm{T}} = 0.$$

In the proof of the Theorem 5 we had shown that the entire space of n dimensional row vectors can be expressed as a direct sum of the span of $\underline{1}$ and S_R, the row space of $(A + B + C)$. Thereby we can conclude that

$$-\left(\underline{e_{(j,1)}}(A) + \underline{e_{(j,2)}}(2A + B)\right)$$

which is orthogonal to $\underline{1}$ has to be an element of S_R. This means that we can always find a vector $e_{(j,3)}$ satisfying Eq. (49). Thus the multiplicity of the eigenvalue at one has to be greater than two, which means that the theorem is valid in reverse direction for $m = 3$.

Now we assume that the theorem is correct for $m = x$, in order to apply the inductive method. We want to prove the theorem in forward direction for $m = x + 1$. If the eigenvalue at one has multiplicity greater than $x + 1$, then, since the multiplicity is greater than x, by the inductive hypothesis, we have

$$\underline{e_{(j,1)}}A\underline{1}^{\mathrm{T}} = \underline{e_{(j,1)}}C\underline{1}^{\mathrm{T}}$$

and

$$\underline{e_{(j,k)}}(A - C)\underline{1}^{T} = -\underline{e_{(j,k-1)}}(A)\underline{1}^{T}$$

for $2 \leqslant k \leqslant x - 1$. Moreover, since the multiplicity is greater than $x + 1$ the defining equation for $(x + 1)$th eigenvector of the cycle, Eq. (36), also holds. Postmultiplying both sides of the equation with $\underline{1}^{\mathrm{T}}$ and using Property 2 yields

$$\underline{e_{(j,x)}}(A - C)\underline{1}^{\mathrm{T}} = -\underline{e_{(j,x-1)}}(A)\underline{1}^{\mathrm{T}}. \qquad (51)$$

Thus, the theorem is proven in forward direction for $m = x + 1$.

The proof in the reverse direction for $m = x + 1$ is similar to the proof in the case of $m = 3$. We show that we can always find a vector that is transformed to

$$-\left(\underline{e_{(j,x-1)}}(A) + \underline{e_{(j,x)}}(2A + B)\right)$$

by the transformation $(A + B + C)$. We do this with exactly the same methodology that has been applied to the case of $m = 3$. Thus the proof of Theorem 6 by induction is completed. $\square$

Theorem 7. *For $\lambda_j = 1$, the following statements are true*:

$$\underline{Com^{i}_{(j,k)}}C\underline{1}^{\mathrm{T}} = \underline{Com^{i+1}_{(j,k)}}A\underline{1}^{\mathrm{T}} \quad \text{for}$$
$$i = 0, 1, 2, \ldots, M - 1 \text{ and}$$
$$k = 1, 2, \ldots, s(\lambda_j) - 1,$$

$$\underline{Com^{i}_{(j,s(\lambda_j))}}C\underline{1}^{\mathrm{T}} \neq \underline{Com^{i+1}_{(j,s(\lambda_j))}}A\underline{1}^{\mathrm{T}} \quad \text{for}$$
$$i = 0, 1, 2, \ldots, M - 1.$$

We refer to Theorem 7 as the Deficient Generalized Balance at Unity theorem (DGBUT). This is the equivalent of GBECF of Theorem 3 for the case of the eigenvalue at one. Yet, we observe for this case, there is no balance of flow in component form for the *components* corresponding to the last element of the cycle of eigenvectors that belongs to the eigenvector at one. For the previous elements of the same cycle, the balance of flow in compo-

nent form for the components corresponding to these elements exists. One should notice that for $s(\lambda_j) = 2$, Theorem 7 is equivalent to Theorem 4. Thus Theorem 7 is a generalization of the Theorem 4 for the generalized eigenvector cycles of any length. This fact justifies the name selected for the theorem.

Proof. Note that up to $k = s(\lambda j) - 1$, Theorem 6 states exactly that the results that were valid for eigenvalues that are not at one, are valid for the case of the eigenvalue at one. When $\lambda_j = 1$ is substituted in the Eq. (27) of the Theorem 1 and in the Eq. (30) of the Theorem 2, we obtain the expressions of the Theorem 6. This means that the proof of Theorem 3 which is based on the Eqs. (27) and (30) would also hold for the case of the eigenvalue at one up to $k = s(\lambda) - 1$, the component corresponding to the eigenvector preceding the last eigenvector of the cycle. Since this result of Theorem 3 is identical with part i of this theorem, part (i) of the theorem is shown by following the steps of the proof of Theorem 3. These steps are not replicated here since the isomorphism is obvious at this point.

We now investigate the case of $k = s(\lambda_j)$. We can observe that we should have

$$\underline{e}_{(j,s(\lambda_j))}(A - C)\underline{1}^{\mathrm{T}} \neq -\underline{e}_{(j,s(\lambda_j)-1)}(A)\underline{1}^{\mathrm{T}}, \tag{52}$$

because if this were not the case, by Theorem 6 the multiplicity of the eigenvalue at one would have to be greater than $s(\lambda_j)$. Since this would be a contradiction, we should have Eq. (51). Yet, we know that if we had

$$\underline{e}_{(j,s(\lambda_j))}(A - C)\underline{1}^{\mathrm{T}} = -\underline{e}_{(j,s(\lambda_j)-1)}(A)\underline{1}^{\mathrm{T}}, \tag{53}$$

by the argument that we have done for the proof of part (i), we could show that

$$\underline{Com}_{(j,s(\lambda_j))}^{i+1}A\underline{1}^{\mathrm{T}} - \underline{Com}_{(j,s(\lambda_j))}^{i}C\underline{1}^{\mathrm{T}} = 0. \tag{54}$$

In the proof of this identity, $\underline{e}_{(j,s(\lambda_j))}(A - C)\underline{1}^{\mathrm{T}}$ would be the first element of the summation that would add up to zero if (53) were to hold. Using this argument, we can observe that

$$\underline{Com}_{(j,s(\lambda_j))}^{i+1}A\underline{1}^{\mathrm{T}} - \underline{Com}_{(j,s(\lambda_j))}^{i}C\underline{1}^{\mathrm{T}}$$
$$= \underline{e}_{(j,s(\lambda_j))}(A - C)\underline{1}^{\mathrm{T}} + \underline{e}_{(j,s(\lambda_j)-1)}A\underline{1}^{\mathrm{T}}. \tag{55}$$

By Eq. (51) the right-hand side of Eq. (55) is different from zero. Thus we have shown that Eq. (55) does not hold. And this completes the proof of part (ii).

Theorem 8. *The coefficient in the general solution, corresponding to the last element of the eigenvector cycle that corresponds to, $\lambda_j = 1$, is always zero.*

Proof. The Eq. (19) states the ith element of the solution in an explicit form. Using the notation defined by Eq. (36), we can rewrite it in a more concise form

$$\underline{g_i} = \sum_{j=1}^{\rho}\sum_{k=1}^{s(\lambda_j)} w_{(j,k)}\underline{Com}_{(j,k)}^{i}$$
$$+ \delta(M, i)\sum_{j=\rho}^{\rho+\rho_\infty}\sum_{k=1}^{s(\lambda_j)} w_{(j,k)}\underline{Com}_{(j,k)}^{M}. \tag{56}$$

Now we can substitute this expression into Eq. (24) whose validity is stated by Property 4 and obtain

$$\sum_{j=1}^{\rho}\sum_{k=1}^{s(\lambda_j)} w_{(j,k)}\underline{Com}_{(j,k)}^{i}C\underline{1}^{\mathrm{T}}$$
$$+ \delta(M, i)\sum_{j=\rho}^{\rho+\rho_\infty}\sum_{k=1}^{s(\lambda_j)} w_{(j,k)}\underline{Com}_{(j,k)}^{M}C\underline{1}^{\mathrm{T}}$$
$$= \sum_{j=1}^{\rho}\sum_{k=1}^{s(\lambda_j)} w_{(j,k)}\underline{Com}_{(j,k)}^{i+1}A\underline{1}^{\mathrm{T}}$$
$$+ \delta(M, i+1)\sum_{j=\rho}^{\rho+\rho_\infty}\sum_{k=1}^{s(\lambda_j)} w_{(j,k)}\underline{Com}_{(j,k)}^{M}A\underline{1}^{\mathrm{T}}.$$

This expression is valid for $i = 0, 1, 3, \ldots, M - 1$. When Theorem 3 is applied, all the terms corresponding to eigenvalues different than one cancel out. Thereby, we are left just with the terms belonging to eigenvalue one, which incorporate only a single cycle of eigenvectors.

$$\sum_{k=1}^{s(\lambda_j)} w_{(j,k)}\underline{Com}_{(j,k)}^{i}C\underline{1}^{\mathrm{T}} = \sum_{k=1}^{s(\lambda_j)} w_{(j,k)}\underline{Com}_{(j,k)}^{i+1}A\underline{1}^{\mathrm{T}}. \tag{57}$$

Applying Theorem 7, we observe that all terms that correspond to $k = 1, 2, \lambda, s(\lambda_j) - 1$ cancel out. Thereby Eq. (57) is reduced to

$$w_{(j,s(\lambda_j))}\left(\underline{Com}_{(j,s(\lambda_j))}^{i+1} A \underline{1}^{\mathrm{T}} - \underline{Com}_{(j,s(\lambda_j))}^{i} C \underline{1}^{\mathrm{T}}\right) = 0.$$

Part (ii) of Theorem 7 states that

$$\underline{Com}_{(j,s(\lambda_j))}^{i} C \underline{1}^{\mathrm{T}} \neq \underline{Com}_{(j,s(\lambda_j))}^{i+1} A \underline{1}^{\mathrm{T}}.$$

Thus $w_{(j,s(\lambda_j))} = 0$. □

4. General implications

The quantities of central importance in the matrix polynomial procedure are the eigenvalues of the characteristic equation for a given QBD. In the previous section, we have demonstrated that for each QBD that is modeled appropriately, one could easily associate a characteristic equation. Although the term characteristic equation belongs to the mathematical concept being employed, it is also quite befitting from a modeling perspective. The roots of the characteristic equation, the eigenvalues, determine the behavior of the solution, thereby the steady-state characteristics of the model.

The solution for a given QBD model is always a linear combination of the *components* that we have defined by (37) and (38). Each *component* actually corresponds to a generalized eigenvector of an eigenvalue of the system. The closed-form expression for a component includes the value of the eigenvalue and the elements of the cycle of generalized eigenvectors from the first element to the given eigenvector. The general solution for a given QBD is given in (56).

When we examine this solution structure we see that any *component* is a solution candidate. Furthermore the spectral theory shows us that each of these *components* acts like the solution on their own. That means, the elements forming the solution all have the properties of the full solution. This is demonstrated by the fact that all the *components* have a balance of flow property within themselves. Thus, the balance of flow property for the full solution is not a property that manifests itself only at that level, it is the consequence of the fact that each of the elements that form the solution exhibit it on their own.

Each *component* consists of two building blocks: the eigenvalue and the eigenvectors – just one if the eigenvalue has a simple eigenvector corresponding to it, otherwise the elements of the generalized eigenvector cycle. Yet, the eigenvalue is the more crucial block since it determines the behavior of the *component*. If the eigenvalue's norm is greater than one, the *component* that corresponds to the eigenvalue will become more and more pronounced for larger elements of the counting space. Complementarily, if the eigenvalue's norm is smaller than one, the *component* that corresponds to the eigenvalue will become less and less pronounced for larger elements of the counting space. If the eigenvalue is at one, the component will have equal contribution all over the counting set. Furthermore, if the eigenvalue is complex, one would observe an oscillatory behavior in the *component* corresponding to it.

If we have the eigenvalues of the system at hand we can tell quite a bit about the possible behavior of the system. As in the control theory one could even try to make a root-locus diagram for the eigenvalues for design and sensitivity analysis purposes. If one knows how the system parameters change the eigenvalues of the system, one can use this information to perturb the system towards a desired behavior.

Thus, the eigenvalues determine the conduct of the *components*, which are actually candidate solutions. One can compare them with modes that present themselves in a more established scientific field, electromagnetic theory. The modes are the possible electromagnetic waves that can exist in a given space, and boundary conditions determine what kind of combination of the modes would actually be observed in the space. Here, similarly, *components* are possible solutions of the system, and the boundary equations determine which combination of them is the final solution to the system yielding the steady-state probabilities.

One can imagine that by perturbing the boundary equations, it would be possible to change the weight of the modes and thereby to push the system in a desired direction. This would ipso facto amount to choosing the desirable modes

of the system and making them more pronounced by changing the conduct of the system only at those states corresponding to the boundaries of the counting set.

Yet, one should be careful with the fact that although the modes are possible solutions in an independent fashion, they are still interrelated. For example, Theorem 8 states that the *component* corresponding to the last generalized eigenvector of the eigenvalue at one, has always zero weight, which means that it does not appear in the final solution. Thus, although this *component* is a candidate solution per se, it can never be a part of the solution.

Moreover, one can observe that certain eigenvalues work in groups. Numerical experimentation shows that in certain cases, certain elements of a *component* may be negative, which would not make any sense if it were to be elected to be a solution in isolation. Yet, since it is always balanced with another *component* in an intrinsic fashion, whatever the boundary equations are, never a negative number appears in the final solution.

The intrinsic relation between the *components* is crucial because otherwise the solutions obtained through the matrix polynomial procedure could never be valid. Although it is not quite possible to grasp this interrelation in a theoretical way at this point with the tools we have presented, we are going to demonstrate it on actual examples in a follow-up work.

5. Conclusions

This work formalizes the matrix-polynomial approach for the analysis of QBDs. We provide new properties relating the quantities of interest in the matrix-polynomial solution procedure. Most importantly, we unify the new and the previously stated properties in a formal theory. Certain relations that were previously stated were not rigorously proven due to the lack of a formal framework. Here, we contribute rigorous proofs to the entire known and discovered properties of the state-homogeneous QBDs. Thus, we present a formal framework to embody all the previous work on the subject. Furthermore, this framework carries the prospect of furthering the understanding of the probabilistic behavior of the systems that can be modeled using the QBDs.

After presenting this spectral theory, we have discussed how it can be exploited for a better understanding of any given model of a QBD. But whenever something is put in such general terms it always carries the curse of being vague. This curse will be remedied with the application of the theory on concrete models that will be presented in a follow-up paper.

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