JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **217,** 555–578 (1998)
ARTICLE NO. AY975734

Analytic and Asymptotic Properties of Generalized Linnik Probability Densities

M. Burak Erdoğan

Department of Mathematics, Bilkent University, 06533, Bilkent, Ankara, Turkey

I. V. Ostrovskii

Department of Mathematics, Bilkent University, 06533, Bilkent, Ankara, Turkey, and Institute of Low Temperature Physics and Engineering, Kharkov, 310164, Ukraine

Submitted by N. H. Bingham

Received August 23, 1996

1. INTRODUCTION

In 1953, Linnik [1] proved that the function

$$\varphi_\alpha(t) = 1/(1 + |t|^\alpha), \qquad \alpha \in (0, 2)$$

is a characteristic function of a symmetric probability density p_α. Since then, the family of symmetric Linnik densities $\{p_\alpha\colon \alpha \in (0,2)\}$ had several probabilistic applications (see, e.g., [2–7]). Analytic and asymptotic properties of the densities p_α were studied in [8].

In 1984, Klebanov, Maniya, and Melamed [9] introduced the concept of geometric strict stability and proved that the family of geometrically strictly stable densities coincides with the family of densities with characteristic functions

$$\varphi_{\alpha,\theta}(t) = 1/(1 + e^{-i\theta \operatorname{sgn} t}|t|^\alpha),$$
$$\alpha \in (0,2),\ |\theta| \le \min(\pi\alpha/2, \pi - \pi\alpha/2).$$

0022-247X/98 $25.00

This family is wider than that of symmetric Linnik densities. Analytic and asymptotic properties of these densities for $\theta \neq 0$ were studied in [10, 11]. For $|\theta| = \min(\pi\alpha/2, \pi - \pi\alpha/2)$, these densities appeared in the papers by Laha [12] and Pillai [13].

In 1992, Pakes [14] showed that, in some characterization problems of mathematical statistics, the probability distributions with characteristic functions

$$\varphi_{\alpha,\theta,\nu}(t) = \frac{1}{\left(1 + e^{-i\theta \operatorname{sgn} t}|t|^{\alpha}\right)^{\nu}},$$

$$\alpha \in (0,2), |\theta| \leq \min(\pi\alpha/2, \pi - \pi\alpha/2), \nu > 0, \quad (1)$$

play an important role. Therefore, the problem of the study of analytic and asymptotic properties of the distributions with characteristic function (1) seems to be of interest.

Set

$$EPD := \{(\alpha, \theta, \nu) \colon \alpha \in (0,2), |\theta| \leq \min(\pi\alpha/2, \pi - \pi\alpha/2), \nu > 0\}.$$

It turns out (Theorem 1 below) that, for any $(\alpha, \theta, \nu) \in EPD$, the distribution with characteristic function $\varphi_{\alpha,\theta,\nu}$ is absolutely continuous. We denote its density by $p_{\alpha,\theta,\nu}$. Evidently, for $\nu = 1$ and $\theta = 0$, $p_{\alpha,\theta,\nu}$ coincides with p_{α}. Hence, $p_{\alpha,\theta,\nu}$ may be viewed as a generalization of the symmetric Linnik densities p_{α}. We call $p_{\alpha,\theta,\nu}$ the generalized Linnik density.

The aim of our work is to study analytic and asymptotic properties of the generalized Linnik densities $p_{\alpha,\theta,\nu}$ *for any* $(\alpha, \theta, \nu) \in EPD$ and to obtain corresponding generalizations of results of [8, 11]. However, the main methods of [8, 11] are not applicable to this aim. The fact is that these methods are based on the idea that in the case $\nu = 1$, the representation of $p_{\alpha,\theta,\nu}$ by the formula (6) below makes it possible to reduce the problem to the study of some Cauchy type integral. Such a reduction seems to be quite impossible in the general case $\nu > 0$. In this work, we use a different method. It is prompted by an idea used in [15] to study multidimensional generalizations of Linnik densities. One of the advantages of this method is that the proof of our results is much shorter than the proof of less general results of [8, 11].

2. STATEMENT OF RESULTS

We begin with the absolute continuity of the distributions with characteristic function (1). The following theorem is an analogue of Theorem 13.1 of [8], Theorem 7.1 of [11], and Theorem 3 of [15].

THEOREM 1. *For any* $(\alpha, \theta, \nu) \in EPD$, *the probability distribution with the characteristic function* $\varphi_{\alpha,\theta,\nu}$ *defined by* (1) *is absolutely continuous and*

$$p_{\alpha,\theta,\nu}(\pm x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^{\pm}_{\alpha,\theta,\nu}(z)\, x^{z-1}\, dz,$$
$$-\alpha < c < \min(\alpha\nu, 1),\ x > 0, \quad (2)$$

where

$$f^{\pm}_{\alpha,\theta,\nu}(z) = \frac{1}{\alpha\Gamma(\nu)}\,\frac{\sin(\pi z/2 \pm \theta z/\alpha)}{\sin \pi z}\,\frac{\Gamma(z/\alpha)\Gamma(\nu - z/\alpha)}{\Gamma(z)}. \quad (3)$$

Both functions $f^{+}_{\alpha,\theta,\nu}$ and $f^{-}_{\alpha,\theta,\nu}$ in Theorem 1 are analytic outside of the set

$$\{-q\alpha\}_{q=1}^{\infty} \cup \{\alpha(\nu+q)\}_{q=0}^{\infty} \cup \{q\}_{q=1}^{\infty}. \quad (4)$$

Moreover, in any set $\{z\colon |\mathrm{Re}\, z| < H, |\mathrm{Im}\, z| > \varepsilon\}$, the following bound holds

$$\left|f^{\pm}_{\alpha,\theta,\nu}(z)\right| \le \exp\{-C_{\pm}(\alpha,\theta)|\mathrm{Im}\, z|\}, \quad (5)$$

where $C_{\pm}(\alpha, \theta)$ is a positive constant. Since the functions $f^{+}_{\alpha,\theta,\nu}$ and $f^{-}_{\alpha,\theta,\nu}$ are analytic in $\{z\colon -\alpha < \mathrm{Re}\, z < \min(\alpha\nu, 1)\}$, the integral in (2) does not depend on c, under the restrictions mentioned in (2).

Denote the subset $\{(\alpha,\theta,\nu) \in EPD\colon \theta \ge 0\}$ of EPD by EPD^{+}. Note that Eqs. (2) and (3) make it obvious that $p_{\alpha,\theta,\nu}(x) = p_{\alpha,-\theta,\nu}(-x)$ for any $(\alpha,\theta,\nu) \in EPD$. Thus, without loss of generality, we can restrict our study of $p_{\alpha,\theta,\nu}$ to $(\alpha,\theta,\nu) \in EPD^{+}$.

To study the analytic properties and non-symmetry of $p_{\alpha,\theta,\nu}$ for $\theta \ne 0$, we shall need the following representation of $p_{\alpha,\theta,\nu}$. The representation was proved by Linnik [1] in the case $\theta = 0$, $\nu = 1$ and by the first author [11] in the case $\theta \ne 0$, $\nu = 1$. Denote by EPD^{*} the subset $\{(\alpha,\theta,\nu) \in EPD^{+}\colon \theta < \pi - \pi\alpha/2\}$ of EPD^{+}. In what follows (except the proof of Theorem 1 in Section 3), *we will consider the case* $(\alpha,\theta,\nu) \in EPD^{*}$ *only*, and will not consider the case $(\alpha,\theta,\nu) \in EPD^{+} \setminus EPD^{*} = \{(\alpha,\theta,\nu) \in EPD^{+}\colon \theta = \pi - \pi\alpha/2\}$. We are going to consider the latter case in a separate paper.

THEOREM 2. *The density* $p_{\alpha,\theta,\nu}$ *is representable in the form*

$$p_{\alpha,\theta,\nu}(\pm x) = \frac{1}{\pi}\,\mathrm{Im}\int_{0}^{\infty} \frac{e^{-yx}\, dy}{\left(1 + e^{\mp i\theta - i\pi\alpha/2} y^{\alpha}\right)^{\nu}}, \qquad x > 0. \quad (6)$$

Recall that a function f defined on an interval $I \subset \mathbb{R}$ is called *completely monotonic* if it is infinitely differentiable on I and, moreover, $(-1)^k f^{(k)}(x) \geq 0$ for any $x \in I$ and any $k = 0, 1, \dots$.

The following theorem related to analytic properties of $p_{\alpha,\theta,\nu}$ was proved in the case $\theta = 0$, $\nu = 1$ in [8], in the case $\theta \neq 0$, $\nu = 1$ in [11]; part (ii) was also proved in [10] under some additional restrictions. Let $[x]^*$ denote the greatest integer strictly less than x.

THEOREM 3. (i) *If* $\alpha\nu \leq 2$, $\theta \leq \min(\pi\alpha/2, \pi/\nu - \pi\alpha/2)$, *then both of the functions* $p_{\alpha,\theta,\nu}(\pm x)$ *are completely monotonic on* $(0,\infty)$. *This assertion ceases to be true for other values of* $(\alpha,\theta,\nu) \in EPD^*$. *Nevertheless, for any* $(\alpha,\theta,\nu) \in EPD^*$, *both of the functions* $p_{\alpha,\theta,\nu}(\pm x)$ *are differences of two completely monotonic functions on* $(0,\infty)$.

(ii) *Suppose* $\sin(\nu(\theta \pm \pi\alpha/2)) \neq 0$. *Then, for* $k = [\alpha\nu]^*, [\alpha\nu]^* + 1, \dots,$

$$(-1)^{k+[\nu(\alpha/2+\theta/\pi)]^*} \lim_{x\to 0^+} p^{(k)}_{\alpha,\theta,\nu}(x) = +\infty, \tag{7}$$

$$(-1)^{[\nu(\alpha/2-\theta/\pi)]^*} \lim_{x\to 0^-} p^{(k)}_{\alpha,\theta,\nu}(x) = +\infty. \tag{8}$$

If $\theta = \pi\alpha/2$, *then* $p_{\alpha,\theta,\nu}(x) = 0$ *for* $x < 0$.

If $\alpha\nu > 1$, *then the function* $p_{\alpha,\theta,\nu}$ *is* $[\alpha\nu]^* - 1$ *times continuously differentiable on* $\mathbb{R}$ *and, moreover,*

$$p^{(k)}_{\alpha,\theta,\nu}(0) = \frac{\sin[(\pi/2 - \theta/\alpha)(k+1)]}{\pi\alpha} B((k+1)/\alpha, \nu - (k+1)/\alpha)$$

for $k = 0, 1, \dots, [\alpha\nu]^* - 1$.

Recall that a probability density is called *unimodal with mode a* if it is non-decreasing on $(-\infty, a)$ and is non-increasing on (a, ∞). In the case $\nu = 1$, the following theorem was proved independently by Kozubowski [10] and by the first author [11]. It is worthwhile to mention that the proof of our theorem is straightforward and based on a different idea than that used in [10] where a difficult Yamazato unimodality theorem [16] was utilized.

THEOREM 4. *All generalized Linnik densities are unimodal. Moreover, for* $|\theta| \leq \max(0, \pi/\nu - \pi\alpha/2)$ *the mode is zero and for* $\theta > \max(0, \pi/\nu - \pi\alpha/2)$ *the mode is positive.*

The following theorem measures the non-symmetry of $p_{\alpha,\theta,\nu}$.

THEOREM 5. (i) *If* $\alpha\nu \le 1$, *then*

$$p_{\alpha,\theta,\nu}(x) \ge p_{\alpha,\theta,\nu}(-x), \qquad x > 0.$$

(ii) *As a function of* θ, $0 \le \theta \le \min(\pi\alpha/2, \pi/(2\nu) - \pi\alpha/2)$, $p_{\alpha,\theta,\nu}(x)$ *increases and* $p_{\alpha,\theta,\nu}(-x)$ *decreases for any fixed* $x > 0$ *and* α, ν *such that* $\alpha\nu \le 1$.

The following theorem characterizes the asymptotic behaviour of $p_{\alpha,\theta,\nu}$ at ∞. It was proved in [8] in the case $\theta = 0$, $\nu = 1$ and proved in [10, 11] in the case $\theta \ne 0$, $\nu = 1$.

THEOREM 6. *For any* $(\alpha, \theta, \nu) \in \operatorname{int}(EPD)$ *and for any* $n = 1, 2, 3, \ldots,$

$$p_{\alpha,\theta,\nu}(\pm x) = \frac{1}{2\Gamma(\nu)} \sum_{q=1}^{n} \frac{(-1)^q \sin(\pi\alpha q/2 \pm \theta q)\Gamma(\nu + q)}{\sin(\pi q\alpha)\Gamma(-q\alpha)\Gamma(1+q)} |x|^{-1-q\alpha} + o\left(|x|^{-1-n\alpha}\right), \tag{9}$$

as $|x| \to \infty$.

COROLLARY 1. *For any* $(\alpha, \theta, \nu) \in \operatorname{int}(EPD)$,

$$p_{\alpha,\theta,\nu}(\pm x) = \frac{\nu}{2\pi} \sin(\pi\alpha/2 \pm \theta)\Gamma(1+\alpha)|x|^{-1-\alpha} + o\left(|x|^{-1-\alpha}\right), \qquad |x| \to \infty. \tag{10}$$

COROLLARY 2. *For any* $(\alpha, \theta, \nu) \in \operatorname{int}(EPD)$,

$$\lim_{x \to +\infty} \frac{p_{\alpha,\theta,\nu}(x)}{p_{\alpha,\theta,\nu}(-x)} = \frac{\sin(\pi\alpha/2 + \theta)}{\sin(\pi\alpha/2 - \theta)}.$$

COROLLARY 3. *For any* $(\alpha, \theta_1, \nu_1), (\alpha, \theta_2, \nu_2) \in \operatorname{int}(EPD)$,

$$\lim_{x \to +\infty} \frac{p_{\alpha,\theta_1,\nu_1}(x)}{p_{\alpha,\theta_2,\nu_2}(x)} = \frac{\nu_1 \sin(\pi\alpha/2 + \theta_1)}{\nu_2 \sin(\pi\alpha/2 + \theta_2)}.$$

Note that Theorem 6 remains valid for $(\alpha, \theta, \nu) \in \partial(EPD)$ as well, but for $(\alpha, \theta, \nu) \in \partial(EPD)$, the sum in the right hand side of (9) vanishes therefore (9) does not give satisfactory information related to the behaviour of $p_{\alpha,\theta,\nu}$ at ∞ (except that it decreases faster than $|x|^{-K}$ for any K).

The following phenomenon was discovered in [8]: the structure of series representations of the symmetric Linnik density p_α depends on the arithmetical nature of α. This phenomenon is valid for $p_{\alpha,\theta,\nu}$, $\theta \neq 0$, $\nu = 1$, as shown in [11]. We will show that this phenomenon remains in force in the general case of $p_{\alpha,\theta,\nu}(x)$ taking into account the arithmetical nature of *two* parameters α, ν.

Theorems 7, 8, 10, 11 below are generalizations of the corresponding ones of [8, 11]. Theorem 9 is a generalization of corresponding ones of [11, 15].

THEOREM 7. *Suppose one of the following conditions is satisfied*:

(i) $\alpha \notin \mathbb{Q}$ *and* $\nu \in \mathbb{Q}$;

(ii) $\alpha \in \mathbb{Q}$ *and* $\nu \notin \mathbb{Q}$;

(iii) $\alpha \notin \mathbb{Q}$, $\nu \notin \mathbb{Q}$, *and* $\nu \notin \{a = (2q+1)/\alpha - p; q \in \mathbb{Z}^+, p \in \mathbb{N}\}$;

(iv) $\alpha \in \mathbb{Q}$ *and* $\nu \in \mathbb{Q}$ *where* α, ν *are representable in the form* $\alpha = m/n$, $\nu = k/l$ *where* m, n *and* k, l *are relatively prime integers and* l *does not divide* m.

Then

$$p_{\alpha,\theta,\nu}(\pm x) = \frac{1}{x\alpha\Gamma(\nu)} \lim_{s\to\infty} \times \left\{ \alpha \sum_{q=0}^{s} \frac{(-1)^q \sin((\nu+q)(\pi\alpha/2 \pm \theta))\Gamma(\nu+q)}{\sin(\pi\alpha(\nu+q))\Gamma(\alpha(\nu+q))\Gamma(1+q)} x^{(\nu+q)} + \sum_{1\le q\le(\nu+s+1/2)\alpha} \frac{(-1)^{q+1}\sin(\pi q/2 \pm \theta q/\alpha)\Gamma(q/\alpha)}{\sin(\pi(\nu-q/\alpha))\Gamma(q)\Gamma(1-\nu+q/\alpha)} x^q \right\}, \quad x > 0. \tag{11}$$

This limit is uniform with respect to x on every compact subset of $\mathbb{R}^+$.

The following corollary of Theorem 7 characterizes the asymptotic behaviour of $p_{\alpha,\theta,\nu}$ at 0.

COROLLARY 4. *Under the conditions of Theorem* 7,

$$p_{\alpha,\theta,\nu}(\pm x) = \begin{cases} \dfrac{\sin(\nu(\pi\alpha/2 \pm \theta))}{\alpha \sin(\pi\alpha\nu)\Gamma(\alpha\nu)} x^{\alpha(\nu-1)}(1+o(1)) & \text{for } \alpha\nu < 1, \\ \dfrac{\sin(\pi/2 \pm \theta/\alpha)\Gamma(1/\alpha)}{\alpha\Gamma(\nu)\sin(\pi(\nu-1/\alpha))\Gamma(1-\nu+1/\alpha)}(1+o(1)) & \\ \quad \text{for } \alpha\nu > 1, & \end{cases}$$

as $x \to 0^+$.

It is natural to ask whether the limits of each of the two sums in the right hand side of (11) exist. Similarly to [8, 11, 15], it is the case for an uncountable dense subset of *EPD*. To describe this subset we need Liouville numbers. Recall that an irrational number l is called a Liouville number if, for any $r = 2, 3, 4, \dots,$ there exist $p, q \in \mathbb{Z}$, $q \geq 2$, such that

$$0 < \left| l - \frac{p}{q} \right| < \frac{1}{q^r}.$$

We denote the set of all Liouville numbers by L. By the famous Liouville theorem (see, e.g., [17, p. 7]), all numbers in L are transcendental. Moreover [17, p. 8], the set L has the Lebesgue measure zero.

THEOREM 8. *Suppose one of the following conditions is satisfied*:

(i) $\alpha \notin \mathbb{Q} \cup L$ *and* $\nu \in \mathbb{Q}$;

(ii) $\alpha \in \mathbb{Q}$ *and* $\nu \notin \mathbb{Q} \cup L$;

(iii) $\alpha \in \mathbb{Q}$ *and* $\nu \in \mathbb{Q}$ *where* α, ν *are representable in the form* $\alpha = m/n$, $\nu = k/l$ *where* m, n *and* k, l *are relatively prime integers and* l *does not divide* m.

Then

$$p_{\alpha,\theta,\nu}(\pm x) = \frac{1}{\Gamma(\nu)} \sum_{q=0}^{\infty} \frac{(-1)^q \sin((\nu+q)(\pi\alpha/2 \pm \theta))\Gamma(\nu+q)}{\sin(\pi\alpha(\nu+q))\Gamma(\alpha(\nu+q))\Gamma(1+q)} x^{\alpha(\nu+q)-1} + \frac{1}{\alpha\Gamma(\nu)} \sum_{q=1}^{\infty} \frac{(-1)^{q+1} \sin(\pi q/2 \pm \theta q/\alpha)\Gamma(q/\alpha)}{\sin(\pi(\nu - q/\alpha))\Gamma(q)\Gamma(1-\nu+q/\alpha)} x^{q-1}, \quad x > 0, \qquad (12)$$

where both of the series in the right hand side converge absolutely and uniformly with respect to x *on every bounded subset of* $\mathbb{R}^+$.

Nevertheless, the absolute and uniform convergence of both of the series in (12) is not valid for all (α, θ, ν) satisfying conditions of Theorem 7.

THEOREM 9. *Both of the series in the right hand side of* (12) *diverge on an uncountable dense subset of EPD*.

Now, we shall consider the case when none of the conditions in the statement of Theorem 7 is satisfied. It suffices to consider the following two cases:

(i) $\alpha \notin \mathbb{Q}$, $\nu = q^*/\alpha - p^*$ for some $q^* \in \mathbb{Z}^+$, $p^* \in \mathbb{N}$

(ii) $\alpha = m/n \in \mathbb{Q}$, $\nu = k/l \in \mathbb{Q}$, and l divides m.

Note that the Diophantine equation

$$q = \alpha(\nu + p), \qquad q \in \mathbb{Z}^+, p \in \mathbb{N} \tag{13}$$

has a unique solution (q^*, p^*) in the case (i), and it has infinitely many solutions in the case (ii). Denote by $D_{\alpha,\nu}$, the set of all solutions (q, p) in the case (ii) and set

$$P_{\alpha,\nu} = \{p: (q,p) \in D_{\alpha,\nu}, p \geq 0, q \geq 1\}$$
$$Q_{\alpha,\nu} = \{q: (q,p) \in D_{\alpha,\nu}, p \geq 0, q \geq 1\}.$$

THEOREM 10. *If the condition* (i) *is satisfied, then*

$$\begin{aligned}
&p_{\alpha,\theta,\nu}(\pm x) \\
&\quad= \frac{1}{x\alpha\Gamma(\nu)} \lim_{s\to\infty} \\
&\qquad\times\Bigg\{\alpha \sum_{q=0,\, q\neq p^*}^{s} \frac{(-1)^q \sin((\nu+q)(\pi\alpha/2 \pm \theta))\Gamma(\nu+q)}{\sin(\pi\alpha(\nu+q))\Gamma(\alpha(\nu+q))\Gamma(1+q)} x^{\alpha(\nu+q)} \\
&\quad+ \sum_{1\leq q\leq(\nu+s+1/2)\alpha,\, q\neq q^*} \frac{(-1)^{q+1}\sin(\pi q/2 \pm \theta q/\alpha)\Gamma(q/\alpha)}{\sin(\pi(\nu - q/\alpha))\Gamma(q)\Gamma(1-\nu+q/\alpha)} x^q\Bigg\} \\
&\qquad+ \frac{(-1)^{p^*+q^*} x^{q^*-1}}{\pi\Gamma(\nu)\Gamma(1+p^*)}\Bigg(\left(\frac{\pi}{2} \pm \frac{\theta}{\alpha}\right)\cos q^*\left(\frac{\pi}{2} \pm \frac{\theta}{\alpha}\right)\Gamma\left(\frac{q^*}{\alpha}\right) \\
&\qquad\qquad+ \frac{1}{\alpha}\Gamma'\left(\frac{q^*}{\alpha}\right)\sin q^*\left(\frac{\pi}{2} \pm \frac{\theta}{\alpha}\right)\Bigg) \\
&\qquad+ \frac{(-1)^{p^*+q^*} x^{q^*-1}\Gamma(q^*/\alpha)\sin q^*(\pi/2 \pm \theta/\alpha)}{\pi\Gamma(\nu)\Gamma(1+p^*)} \\
&\qquad\quad\times\left(\log x - \frac{\Gamma'(1+p^*)}{\Gamma(1+p^*)}\right),
\end{aligned} \tag{14}$$

where q^*, p^* *in the equation are connected by the formula* (13). *The limit in the right hand side of the equation is uniform with respect to x on every compact subset of* $\mathbb{R}^+$.

THEOREM 11. *If the condition* (ii) *is satisfied, then*

$$
\begin{aligned}
p_{\alpha,\theta,\nu}(\pm x) &= \frac{1}{\Gamma(\nu)} \sum_{q=0,\, q\notin P_{\alpha,\nu}}^{\infty} \frac{(-1)^q \sin((\nu+q)(\pi\alpha/2 \pm \theta))\Gamma(\nu+q)}{\sin(\pi\alpha(\nu+q))\Gamma(\alpha(\nu+q))\Gamma(1+q)} x^{\alpha(\nu+q)-1} \\
&\quad + \frac{1}{\alpha\Gamma(\nu)} \sum_{q=1,\, q\notin Q_{\alpha,\nu}}^{\infty} \frac{(-1)^{q+1} \sin(\pi q/2 \pm \theta q/\alpha)\Gamma(q/\alpha)}{\sin(\pi(\nu - q/\alpha))\Gamma(q)\Gamma(1-\nu+q/\alpha)} x^{q-1} \\
&\quad + \sum_{q\in Q_{\alpha,\nu}} \frac{(-1)^{p+q} x^{q-1}}{\pi\Gamma(\nu)\Gamma(1+p)} \left(\left(\frac{\pi}{2} \pm \frac{\theta}{\alpha}\right) \cos q\left(\frac{\pi}{2} \pm \frac{\theta}{\alpha}\right)\Gamma\left(\frac{q}{\alpha}\right) \right. \\
&\qquad\qquad \left. + \frac{1}{\alpha}\Gamma'\left(\frac{q}{\alpha}\right)\sin q\left(\frac{\pi}{2} \pm \frac{\theta}{\alpha}\right)\right) \\
&\quad + \sum_{q\in Q_{\alpha,\nu}} \frac{(-1)^{p+q} x^{q-1}\Gamma(q/\alpha)\sin q(\pi/2 \pm \theta/\alpha)}{\pi\Gamma(\nu)\Gamma(1+p)} \\
&\qquad \times \left(\log x - \frac{\Gamma'(1+p)}{\Gamma(1+p)}\right), \qquad (15)
\end{aligned}
$$

where q, p in the third and fourth sums of the equation are connected by the formula (13). *Both of the series in the right hand side converge absolutely and uniformly with respect to x on every bounded subset of* $\mathbb{R}^+$.

Note that the $\log x$ term in Eqs. (14), (15) disappears when $\theta = \pm\pi\alpha/2$.

It is possible to obtain asymptotic formulas describing the behaviour of $p_{\alpha,\theta,\nu}$ as $x \to 0+$ from Theorems 10 and 11. We present the following two examples:

EXAMPLE 1 (Theorem 10). $\alpha = 1/(\pi+1)$, $\nu = \pi$, $\theta = 1/(2\pi+2)$.

$$
p_{\alpha,\theta,\nu}(\pm x) = \frac{\sin(\pi(\pi \pm 1)/(2\pi+2))}{\sin(\pi^2/(\pi+1))\Gamma(\pi/(\pi+1))} x^{-1/(\pi+1)} + O\left(\log\frac{1}{x}\right), \qquad x \to 0+.
$$

EXAMPLE 2 (Theorem 11). $\alpha = 2/3$, $\nu = 3/2$, $\theta = 1$.

$$
p_{\alpha,\theta,\nu}(\pm x) = \frac{\cos(3/2)}{\pi} \log\frac{1}{x} + O(1), \qquad x \to 0+.
$$

3. REPRESENTATION OF $p_{\alpha,\theta,\nu}$ BY A CONTOUR INTEGRAL

Proof of Theorem 1. Let us define the function $p_{\alpha,\theta,\nu}$ by formulas (2), (3). Since $\varphi_{\alpha,\theta,\nu}$ is a characteristic function [14], it suffices to prove that $p_{\alpha,\theta,\nu}$ belongs to $L_1(\mathbb{R})$ and its Fourier transform coincides with $\varphi_{\alpha,\theta,\nu}$.

If we take c positive in (2), then estimating the integral with the help of the bound (5), we have

$$p_{\alpha,\theta,\nu}(x) = O\big(|x|^{c-1}\big), \qquad |x| \to 0.$$

If we take c negative, then we have

$$p_{\alpha,\theta,\nu}(x) = O\big(|x|^{-|c|-1}\big), \qquad |x| \to \infty.$$

Thus $p_{\alpha,\theta,\nu} \in L_1(\mathbb{R})$.

Now, let us prove that the Fourier transform of $p_{\alpha,\theta,\nu}$ coincides with $\varphi_{\alpha,\theta,\nu}$. It suffices to prove this for the case $t > 0$ since the case $t < 0$ can be reduced to the first one by using the equalities $\varphi_{\alpha,\theta,\nu}(t) = \varphi_{\alpha,-\theta,\nu}(-t)$, $p_{\alpha,\theta,\nu}(x) = p_{\alpha,-\theta,\nu}(-x)$. We have

$$\begin{aligned} I(t) &:= \int_{-\infty}^{\infty} p_{\alpha,\theta,\nu}(x) e^{itx}\, dx \\ &= \left(\int_0^1 + \int_1^\infty\right) p_{\alpha,\theta,\nu}(-x) e^{-itx}\, dx + \left(\int_0^1 + \int_1^\infty\right) p_{\alpha,\theta,\nu}(x) e^{itx}\, dx. \end{aligned}$$

Let $0 < \varepsilon < \min(\alpha, \alpha\beta, 1)$. Using the equality (2), we obtain

$$\begin{aligned} I(t) = {} & \frac{1}{2\pi i}\int_0^1 e^{-itx}\, dx \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} f^-_{\alpha,\theta,\nu}(z) x^{z-1}\, dz \\ & + \frac{1}{2\pi i}\int_1^\infty e^{-itx}\, dx \int_{-\varepsilon - i\infty}^{-\varepsilon + i\infty} f^-_{\alpha,\theta,\nu}(z) x^{z-1}\, dz \\ & + \frac{1}{2\pi i}\int_0^1 e^{itx}\, dx \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} f^+_{\alpha,\theta,\nu}(z) x^{z-1}\, dz \\ & + \frac{1}{2\pi i}\int_1^\infty e^{itx}\, dx \int_{-\varepsilon - i\infty}^{-\varepsilon + i\infty} f^+_{\alpha,\theta,\nu}(z) x^{z-1}\, dz. \end{aligned}$$

In all integrals in the right hand side, we can change the order of integration by using (5) and Fubini's theorem. Thus

$$\begin{aligned} I(t) = & \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} f^{-}_{\alpha,\theta,\nu}(z)\,dz\int_0^1 e^{-itx}x^{z-1}\,dx \\ & + \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} f^{-}_{\alpha,\theta,\nu}(z)\,dz\int_1^\infty e^{-itx}x^{z-1}\,dx \\ & + \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} f^{+}_{\alpha,\theta,\nu}(z)\,dz\int_0^1 e^{itx}x^{z-1}\,dx \\ & + \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} f^{+}_{\alpha,\theta,\nu}(z)\,dz\int_1^\infty e^{itx}x^{z-1}\,dx. \end{aligned} \tag{16}$$

Both of the integrals $\int_1^\infty e^{-itx}x^{z-1}\,dx$ and $\int_1^\infty e^{itx}x^{z-1}\,dx$ converge uniformly on any compact set lying in $\{z\colon \operatorname{Re} z < 1\}$. Using the equality

$$\int_1^\infty e^{\pm itx}x^{z-1}\,dx = \mp\frac{e^{\pm it}}{it} \mp \frac{z-1}{it}\int_1^\infty e^{\pm itx}x^{z-2}\,dx = O(|z|), \qquad |z|\to\infty,$$

the integrations in the second and fourth integrals in the right hand side of (16) can be translated from $\{z\colon \operatorname{Re} z = -\varepsilon\}$ to $\{\operatorname{Re} z = \varepsilon\}$. Therefore, the equality (16) can be rewritten in the form

$$\begin{aligned} I(t) = & \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} f^{-}_{\alpha,\theta,\nu}(z)\,dz\int_0^\infty e^{-itx}x^{z-1}\,dx \\ & + \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} f^{+}_{\alpha,\theta,\nu}(z)\,dz\int_0^\infty e^{itx}x^{z-1}\,dx. \end{aligned} \tag{17}$$

Using the equalities

$$\int_0^\infty e^{\pm itx}x^{z-1}\,dx = t^{-z}\Gamma(z)e^{\pm i\pi z/2}, \qquad 0 < \operatorname{Re} z < 1,$$

(3) and (17), we obtain

$$I(t) = \frac{1}{2\pi i\alpha\Gamma(\nu)}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \Gamma(z/\alpha)\Gamma(\nu - z/\alpha)e^{iz\theta/\alpha}t^{-z}\,dz. \tag{18}$$

If $0 < t < 1$, then, evaluating the integral with the same integrand as in (18) along the boundary of the region $\{z\colon |z| < (n+1/2)\alpha, \operatorname{Re} z < \varepsilon\}$ by

Cauchy's residue theorem, and then letting $n \to \infty$, we obtain

$$I(t) = \frac{1}{\alpha\Gamma(\nu)} \sum_{q=0}^{\infty} \operatorname{Res}_{z=-q\alpha}\big(\Gamma(z/\alpha)\Gamma(\nu - z/\alpha)e^{iz\theta/\alpha}t^{-z}\big)$$

$$= \frac{1}{\Gamma(\nu)} \sum_{q=0}^{\infty} \frac{(-1)^q\Gamma(\nu+q)}{\Gamma(1+q)} e^{-i\theta q}t^{\alpha q}$$

$$= 1/(1 + e^{-i\theta}t^{\alpha})^{\nu} = \varphi_{\alpha,\theta,\nu}(t).$$

If $t > 1$, then evaluating the integral with the same integrand as in (18) along the boundary of the region $\{z: |z| < (\nu + n + 1/2)\alpha, \operatorname{Re} z > \varepsilon\}$, and then letting $n \to \infty$, we obtain

$$I(t) = \frac{-1}{\alpha\Gamma(\nu)} \sum_{q=0}^{\infty} \operatorname{Res}_{z=(\nu+q)\alpha}\big(\Gamma(z/\alpha)\Gamma(\nu - z/\alpha)e^{iz\theta/\alpha}t^{-z}\big)$$

$$= \frac{1}{\Gamma(\nu)} \sum_{q=0}^{\infty} \frac{(-1)^q\Gamma(\nu+q)}{\Gamma(1+q)} e^{i\theta(\nu+q)}t^{-\alpha(\nu+q)}$$

$$= 1/(1 + e^{-i\theta}t^{\alpha})^{\nu} = \varphi_{\alpha,\theta,\nu}(t). \quad \blacksquare$$

Proof of Theorem 2. It suffices to prove that if we define $p_{\alpha,\theta,\nu}$ by (6), then its Fourier Transform coincides with $\varphi_{\alpha,\theta,\nu}(t)$. Set

$$H(t) := \int_{-\infty}^{\infty} e^{itx} p_{\alpha,\theta,\nu}(x)\, dx.$$

Let us substitute $p_{\alpha,\theta,\nu}(x)$ from (6). Changing the order of integrals by Fubini's Theorem, we derive

$$H(t) = \frac{1}{\pi} \operatorname{Im}\left\{ \int_0^{\infty} \frac{y\, dy}{(1 + e^{-i\theta - i\pi\alpha/2}y^{\alpha})^{\nu}(y^2 + t^2)} \right\}$$

$$+ \frac{it}{\pi} \operatorname{Im}\left\{ \int_0^{\infty} \frac{dy}{(1 + e^{-i\theta - i\pi\alpha/2}y^{\alpha})^{\nu}(y^2 + t^2)} \right\}$$

$$+ \frac{1}{\pi} \operatorname{Im}\left\{ \int_0^{\infty} \frac{y\, dy}{(1 + e^{i\theta - i\pi\alpha/2}y^{\alpha})^{\nu}(y^2 + t^2)} \right\}$$

$$- \frac{it}{\pi} \operatorname{Im}\left\{ \int_0^{\infty} \frac{dy}{(1 + e^{i\theta - i\pi\alpha/2}y^{\alpha})^{\nu}(y^2 + t^2)} \right\}$$

$$=: \frac{1}{\pi}\big[\operatorname{Im} A + it \operatorname{Im} B + \operatorname{Im} C - it \operatorname{Im} D\big]. \tag{19}$$

In the complex y-plane, we consider the region

$$G_R = \{y = \xi + i\eta : |y| < R, \eta > 0\}, \qquad R > |t|$$

and define the branch of the multivalued function y^α as

$$y^\alpha = |y|^\alpha e^{i\alpha \arg y}, \qquad 0 \le \arg y \le \pi.$$

The integrands of A and C are analytic in the closure of G_R except the simple pole at $y = i|t|$. By Cauchy's residue theorem we have

$$\oint_{\partial G_R} \frac{y\,dy}{\left(1 + e^{-i\theta - i\pi\alpha/2} y^\alpha\right)^\nu \left(y^2 + t^2\right)} = 2\pi i \operatorname{Re} s_{i|t|} = \frac{\pi i}{\left(1 + e^{-i\theta}|t|^\alpha\right)^\nu}.$$

Letting $R \to \infty$, and using the notation A and C, we obtain

$$\operatorname{Im} A + \operatorname{Im} C = \pi \operatorname{Re} \frac{1}{\left(1 + e^{-i\theta}|t|^\alpha\right)^\nu}. \tag{20}$$

We have in the similar way

$$\operatorname{Im} B - \operatorname{Im} D = \frac{\pi}{|t|} \operatorname{Im} \frac{1}{\left(1 + e^{-i\theta}|t|^\alpha\right)^\nu}. \tag{21}$$

Substituting (20) and (21) into (19), we have

$$\begin{aligned} H(t) &= \operatorname{Re} \frac{1}{\left(1 + e^{-i\theta}|t|^\alpha\right)^\nu} + i \operatorname{sgn} t \operatorname{Im} \frac{1}{\left(1 + e^{-i\theta}|t|^\alpha\right)^\nu} \\ &= \frac{1}{\left(1 + e^{-i\theta \operatorname{sgn} t}|t|^\alpha\right)^\nu} = \varphi_{\alpha,\theta,\nu}(t). \quad \blacksquare \end{aligned}$$

4. SOME PROPERTIES OF $p_{\alpha,\theta,\nu}(x)$

Proof of Theorem 3. (i) Equation (6) is equivalent to

$$p_{\alpha,\theta,\nu}(\pm x) = \frac{1}{\pi} \int_0^\infty \frac{e^{-yx} \sin\left(\nu \arg\left(1 + e^{\pm i\theta + i\pi\alpha/2} y^\alpha\right)\right) dy}{\left|1 + e^{\mp i\theta - i\pi\alpha/2} y^\alpha\right|^\nu}, \qquad x > 0. \tag{22}$$

Hence

$$p_{\alpha,\theta,\nu}(\pm x) = \int_0^\infty e^{-yx} g^{1,\pm}_{\alpha,\theta,\nu}(y)\,dy - \int_0^\infty e^{-yx} g^{2,\pm}_{\alpha,\theta,\nu}(y)\,dy, \qquad x > 0, \tag{23}$$

where $g^{1,\pm}_{\alpha,\theta,\nu}(y)$, $g^{2,\pm}_{\alpha,\theta,\nu}(y)$ are non-negative, bounded functions. Evidently both of the integrals in the right hand side of Eq. (23) are infinitely differentiable and both are completely monotonic for $x > 0$.

Note that, for $|\theta| \le \pi\alpha/2$, $\arg(1 + e^{\pm i\theta + i\pi\alpha/2} y^{\alpha})$ is an increasing function of $y > 0$. It takes value 0 for $y = 0$ and takes value $\pm\theta + \pi\alpha/2$ for $y = \infty$. Therefore, if $\alpha\nu \le 2$, $\theta \le \min(\pi\alpha/2, \pi/\nu - \pi\alpha/2)$, then the function $\sin(\nu \arg(1 + e^{\pm i\theta + i\pi\alpha/2} y^{\alpha}))$ is non-negative, which completes the proof.

(ii) Taking the kth derivative of both sides of Eq. (23), and then applying the monotone convergence theorem, we obtain

$$\lim_{x\to 0^+} p^{(k)}_{\alpha,\theta,\nu}(x) = \frac{(-1)^{(k)}}{\pi} \int_0^{\infty} \frac{y^k \sin\big(\nu \arg\big(1 + e^{i\theta + i\pi\alpha/2} y^{\alpha}\big)\big)\, dy}{|1 + e^{-i\theta - i\pi\alpha/2} y^{\alpha}|^{\nu}},$$

$$\lim_{x\to 0^-} p^{(k)}_{\alpha,\theta,\nu}(x) = \frac{1}{\pi} \int_0^{\infty} \frac{y^k \sin\big(\nu \arg\big(1 + e^{-i\theta + i\pi\alpha/2} y^{\alpha}\big)\big)\, dy}{|1 + e^{i\theta - i\pi\alpha/2} y^{\alpha}|^{\nu}}.$$

Evidently, the integrals in the right hand side are divergent for $k = [\alpha\nu]^*, [\alpha\nu]^* + 1, \ldots,$ and convergent for $k = 0, 1, \ldots, [\alpha\nu]^* - 1$.

In the first case, we have

$$(-1)^{k + [\nu(\alpha/2 + \theta/\pi)]^*} \lim_{x\to 0^+} p^{(k)}_{\alpha,\theta,\nu}(x)$$
$$= \frac{(-1)^{[\nu(\alpha/2+\theta/\pi)]^*}}{\pi} \left\{ \int_0^c + \int_c^{\infty} \right\} \frac{y^k \sin\big(\nu \arg\big(1 + e^{i\theta + i\pi\alpha/2} y^{\alpha}\big)\big)\, dy}{|1 + e^{-i\theta - i\pi\alpha/2} y^{\alpha}|^{\nu}},$$

where c is the last point where $\sin(\nu \arg(1 + e^{i\theta + i\pi\alpha/2} y^{\alpha}))$ changes its sign. The first integral in the right hand side is convergent and finite and the second one is equal to $+\infty$, which proves (7). The proof of (8) is similar. The validity of the assertion that if $\theta = \pi\alpha/2$, then $p_{\alpha,\theta,\nu}(x) = 0$ for $x < 0$ is obvious.

In the second case, we have

$$I_{\pm} := \lim_{x\to 0^{\pm}} p^{(k)}_{\alpha,\theta,\nu}(x) = \frac{(\mp 1)^k}{\pi} \operatorname{Im} \int_0^{\infty} \frac{y^k\, dy}{\left(1 + e^{\mp i\theta - i\pi\alpha/2} y^{\alpha}\right)^{\nu}}$$
$$= \frac{(\mp 1)^k}{\pi\alpha} \operatorname{Im} \int_0^{\infty} \frac{u^{-1 + (k+1)/\alpha}\, du}{\left(1 + e^{\mp i\theta - i\pi\alpha/2} u\right)^{\nu}}.$$

Changing the contour of integration in the last integral from $\mathbb{R}^+$ to $\{u = re^{\pm i\theta + i\pi\alpha/2}: r \in \mathbb{R}^+\}$, we have

$$I_{\pm} = \frac{\sin((\pi/2 - \theta/\alpha)(k+1))}{\pi\alpha} \int_0^\infty \frac{r^{-1+(k+1)/\alpha}\, dr}{(1+r)^\nu}$$
$$= \frac{\sin((\pi/2 - \theta/\alpha)(k+1))}{\pi\alpha} B\left(\frac{k+1}{\alpha}, \nu - \frac{k+1}{\alpha}\right). \qquad \blacksquare$$

To prove Theorem 4, we shall use the following theorem.

THEOREM [18, p. 48]. *Let f be a real valued function defined on* $[0, \infty)$ *having finitely many, m say, changes of sign. Assume that the integral*

$$p(x) = \int_0^\infty e^{-xs} f(s)\, ds,$$

converges for any $x > 0$. *Then the number of zeros of the function p does not exceed m.*

Proof of Theorem 4. Firstly we shall prove the unimodality. Note from representation (6) that the probability density $p_{\alpha,\theta,\nu}$ is a continuous function with respect to parameters α, θ, ν, for fixed $x \in \mathbb{R}$ and $(\alpha, \theta, \nu) \in EPD^*$. Consider the following dense subset of EPD^*

$$S = \{(\alpha, \theta, \nu) \in EPD^*: 0 < \theta < \pi\alpha/2, \theta/\pi\alpha \notin \mathbb{Q}\}.$$

Since any probability density which is a limit of unimodal probability densities is unimodal, it suffices to prove that for any $(\alpha, \theta, \nu) \in S$, the density $p_{\alpha,\theta,\nu}$ is unimodal.

Case (i). $\alpha\nu > 2$. By Theorem 3 (ii), for any $(\alpha, \theta, \nu) \in S$ and for any $k = 0, 1, 2, \dots, [\alpha\nu]^* - 1$, we have $p^{(k)}_{\alpha,\theta,\nu}(0) \neq 0$. By the Theorem of [18] mentioned above and the formula (22), for any $k = 0, 1, \dots, [\alpha\nu]^* - 1$, the number of zeros of $p^{(k)}_{\alpha,\theta,\nu}$ does not exceed $[\nu(\pi\alpha/2 + \theta)/\pi]^*$ on $[0, \infty)$ and does not exceed $[\nu(\pi\alpha/2 - \theta)/\pi]^*$ on $(-\infty, 0]$. Thus the total number of zeros of any derivative $p^{(k)}_{\alpha,\theta,\nu}$, $k = 0, 1, \dots, [\alpha\nu]^* - 1$, does not exceed $[\nu(\pi\alpha/2 + \theta)/\pi]^* + [\nu(\pi\alpha/2 - \theta)/\pi]^*$. Evidently, this sum does not exceed $[\alpha\nu]^*$.

Assume, $p_{\alpha,\theta,\nu}$ is not unimodal. Then $p'_{\alpha,\theta,\nu}$ has at least 3 zeros. By Rolle's Theorem, $p''_{\alpha,\theta,\nu}$ has at least 4 zeros. Repeated application of Rolle's Theorem enables us to say that $p^{([\alpha\nu]^*-1)}_{\alpha,\theta,\nu}$ has at least $[\alpha\nu]^* - 1 + 2 = [\alpha\nu]^* + 1$ zeros which is impossible.

Case (ii). $\alpha\nu \leq 2$. By a similar argument, the number of zeros of $p'_{\alpha,\theta,\nu}$ does not exceed $[\nu(\pi\alpha/2 + \theta)/\pi]^* \leq 1$ on $(0, \infty)$ and does not exceed

$[\nu(\pi\alpha/2 - \theta)/\pi]^* = 0$ on $(-\infty, 0)$. Evidently $p'_{\alpha,\theta,\nu}(x)$ is unbounded as $x \to 0\pm$. Since $p'_{\alpha,\theta,\nu}$ has no zeros on $(-\infty, 0)$, we have $\lim_{x\to 0^-} p'_{\alpha,\theta,\nu}(x) = +\infty$. We have two cases: $\lim_{x\to 0^+} p'_{\alpha,\theta,\nu}(x) = +\infty$ and $\lim_{x\to 0^+} p'_{\alpha,\theta,\nu}(x) = -\infty$, but it is easy to see that $p_{\alpha,\theta,\nu}$ is unimodal in both cases.

Now, we shall prove our assertion about the modes. For $\theta = 0$, the function $p_{\alpha,\theta,\nu}$ is symmetric hence the mode is zero. For $\alpha\nu > 2$ and $0 < \theta < \min(\pi\alpha/2, \pi - \pi\alpha/2)$, we have $p'_{\alpha,\theta,\nu}(0) > 0$ by Theorem 3(ii), hence the mode is positive. For $(\alpha, \theta, \nu) \in \mathrm{int}(EPD^*)$ such that $\alpha\nu \le 2$ and $\theta \le \pi/\nu - \pi\alpha/2$, the number of zeros of $p'_{\alpha,\theta,\nu}$ is equal to zero on both $(-\infty, 0), (0, \infty)$ by our argument in case (ii), hence the mode is zero. For $(\alpha, \theta, \nu) \in \mathrm{int}(EPD^*)$ such that $\alpha\nu \le 2$ and $\theta > \pi/\nu - \pi\alpha/2$, we have $\lim_{x\to 0^+} p'_{\alpha,\theta,\nu}(x) = +\infty$ by Theorem 3(ii) hence the mode is positive.

Proof of Theorem 5. (i) Note that, under the conditions mentioned in part (i) of the theorem,

$$|1 + e^{-i\theta - i\pi\alpha/2} y^\alpha| \le |1 + e^{i\theta - i\pi\alpha/2} y^\alpha|,$$

$$\sin\left(\nu \arg\left(1 + e^{i\theta + i\pi\alpha/2} y^\alpha\right)\right) \ge \sin\left(\nu \arg\left(1 + e^{-i\theta + i\pi\alpha/2} y^\alpha\right)\right).$$

The proof follows from those inequalities and Eq. (22).

(ii) It is easy to see that under the conditions mentioned in part (ii) of the theorem, $|1 + e^{-i\theta - i\pi\alpha/2} y^\alpha|$ decreases and $\sin(\nu \arg(1 + e^{i\theta + i\pi\alpha/2} y^\alpha))$ increases as a function of θ. The proof follows from Eq. (22). ∎

5. ASYMPTOTIC BEHAVIOUR AT INFINITY

Proof of Theorem 6. Fix $-\alpha < c < 0$. Denote the boundary of the region $\{z: -(n + 1/2)\alpha < \mathrm{Re}\, z < c, |\mathrm{Im}\, z| < R\}$ by $L_{n,R}$. Applying Cauchy's residue theorem to the integral

$$\oint_{L_{n,R}} f^{\pm}_{\alpha,\theta,\nu}(z) x^{z-1}\, dz, \qquad x > 0,$$

we obtain

$$\oint_{L_{n,R}} f^{\pm}_{\alpha,\theta,\nu}(z)x^{z-1}\,dz$$

$$= \left\{\int_{c-iR}^{c+iR} + \int_{c+iR}^{-(n+1/2)\alpha+iR}\right\} f^{\pm}_{\alpha,\theta,\nu}(z)x^{z-1}\,dz$$

$$+ \left\{\int_{-(n+1/2)\alpha+iR}^{-(n+1/2)\alpha-iR} + \int_{-(n+1/2)\alpha-iR}^{c-iR}\right\} f^{\pm}_{\alpha,\theta,\nu}(z)x^{z-1}\,dz$$

$$= 2\pi i \sum_{q=1}^{n} \operatorname{Res}_{z=-q\alpha}\left(f^{\pm}_{\alpha,\theta,\nu}(z)x^{z-1}\right). \tag{24}$$

Utilizing (3), we have

$$\operatorname{Res}_{z=-q\alpha}\left(f^{\pm}_{\alpha,\theta,\nu}(z)x^{z-1}\right) = \frac{(-1)^q \sin(\pi q\alpha/2 \pm q\theta)\Gamma(v+q)}{2\Gamma(\nu)\sin \pi q\alpha\,\Gamma(-q\alpha)\Gamma(1+q)} x^{-q\alpha-1}.$$

The bound (5) implies that the second and fourth integrals in the right hand side of (24) $\to 0$ as $R \to \infty$. Thus, for $x > 0$, we have

$$p_{\alpha,\theta,\nu}(\pm x) = \sum_{q=1}^{n} \operatorname{Res}_{z=-q\alpha}\left(f^{\pm}_{\alpha,\theta,\nu}(z)x^{z-1}\right)$$

$$+ \frac{1}{2\pi i}\int_{-(n+1/2)\alpha-i\infty}^{-(n+1/2)\alpha+i\infty} f^{\pm}_{\alpha,\theta,\nu}(z)x^{z-1}dz. \tag{25}$$

Evidently, the integral in the right hand side of (25) is $o(|x|^{-1-n\alpha})$, which completes the proof. ∎

6. SERIES REPRESENTATIONS OF $p_{\alpha,\theta,\nu}$

Proof of Theorem 7. Note that if one of the conditions in the statement of the theorem is satisfied, then the Diophantine equation (13) has no solution. Thus the sets $\{q\}_{q=1}^{\infty}$, $\{\alpha(\nu+q)\}_{q=0}^{\infty}$ which are the singularities of $f^{+}_{\alpha,\theta,\nu}$ and of $f^{-}_{\alpha,\theta,\nu}$ in $\{z\colon \operatorname{Re} z > 0\}$ are disjoint. Hence, all singularities of $f^{+}_{\alpha,\theta,\nu}$ and of $f^{+}_{\alpha,\theta,\nu}$ in $\{z\colon \operatorname{Re} z > 0\}$ are simple poles.

Fix $0 < c < \min(\alpha\nu, 1)$ and let L_k be the boundary of the region $\{z\colon \operatorname{Re} z > c, |z| < Q_k\}$. The direction on the boundary is counterclockwise. Denote by C_k and D_k the contours $L_k \cap \{z\colon |z| = Q_k\}$ and $L_k \setminus C_k$, respectively. The number Q_k belonging to the interval $(\alpha(\nu+k), \alpha(\nu+k+1))$ is chosen in the following way.

Each of the intervals $(\alpha(\nu+k), \alpha(\nu+k+1))$ contains at most two points of the set $\{q\}_{q=1}^{\infty}$. If it contains none, we set $Q_k = (\nu + k + 1/2)\alpha$. If it contains one, q_k say, we choose Q_k so that the distance from Q_k to the nearest of three points q_k, $\alpha(\nu+k)$, $\alpha(\nu+k+1)$ be not less than $\alpha/4$. If it contains two, q_k and q_{k+1} say, we choose $Q_k = q_k + 1/2$. Note that the distance between Q_k and any pole of $f^{+}_{\alpha,\theta,\nu}$ and of $f^{-}_{\alpha,\theta,\nu}$ is not less than $\alpha/4$ for any $k = 1, 2, \dots$.

Consider the integral

$$\frac{1}{2\pi i}\oint_{L_k} f^{\pm}_{\alpha,\theta,\nu}(z)|x|^{z-1}\,dz.$$

Applying Cauchy's residue theorem, we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{L_k} f^{\pm}_{\alpha,\theta,\nu}(z)|x|^{z-1}\,dz \\
&\quad= \frac{1}{2\pi i}\int_{D_k} f^{\pm}_{\alpha,\theta,\nu}(z)|x|^{z-1}\,dz + \frac{1}{2\pi i}\int_{C_k} f^{\pm}_{\alpha,\theta,\nu}(z)|x|^{z-1}\,dz \\
&\quad= \sum_{q=0}^{k} \operatorname{Res}_{z=\alpha(\nu+q)}\Big(f^{\pm}_{\alpha,\theta,\nu}(z)|x|^{z-1}\Big) \\
&\qquad+ \sum_{1\le q\le \alpha(\nu+k+1/2)} \operatorname{Res}_{z=q}\Big(f^{\pm}_{\alpha,\theta,\nu}(z)|x|^{z-1}\Big). \qquad (26)
\end{aligned}$$

Utilizing the formula (3) for $f^{\pm}_{\alpha,\theta,\nu}$, we obtain

$$\begin{aligned}
&\operatorname{Res}_{z=\alpha(\nu+q)}\Big(f^{\pm}_{\alpha,\theta,\nu}(z)|x|^{z-1}\Big) \\
&\quad= \frac{(-1)^{q+1}\sin((\nu+q)(\pi\alpha/2 \pm \theta))\Gamma(\nu+q)}{\Gamma(\nu)\sin(\pi\alpha(\nu+q))\Gamma(\alpha(\nu+q))\Gamma(1+q)}|x|^{\alpha(\nu+q)-1}, \\
&\operatorname{Res}_{z=q}\Big(f^{\pm}_{\alpha,\theta,\nu}(z)|x|^{z-1}\Big) \\
&\quad= \frac{(-1)^{q}\sin(\pi q/2 \pm \theta q/\alpha)\Gamma(q/\alpha)}{\alpha\Gamma(\nu)\sin(\pi(\nu - q/\alpha))\Gamma(q)\Gamma(1-\nu+q/\alpha)}|x|^{q-1}. \qquad (27)
\end{aligned}$$

To estimate the integral along C_k, we note that for

$$\begin{aligned}
&z \notin B_\alpha \\
&:= \left[\bigcup_{q=-\infty}^{\infty}\{z : |z-q| < \alpha/4\}\right] \cup \left[\bigcup_{q=-\infty}^{\infty}\{z : |z-\alpha(\nu+q)| < \alpha/4\}\right]
\end{aligned}$$

the following bounds are valid,

$$|\sin \pi z| \geq C \exp(\pi |\mathrm{Im}\, z|) \qquad \text{and}$$
$$|\sin \pi(\nu - z/\alpha)| \geq C \exp(\pi |\mathrm{Im}\, z|/\alpha),$$

where C is a positive constant not depending on z.

Note that

$$|\sin(\pi z/2 \pm \theta z/\alpha)| \leq \exp(|(\pi/2 \pm \theta/\alpha)\mathrm{Im}\, z|).$$

Moreover, for any $M > 0$ Stirling's formula yields

$$|1/\Gamma(z)| \leq C_M \exp(\pi |\mathrm{Im}\, z|/2 - M \,\mathrm{Re}\, z) \qquad \text{when } z \in \{z\colon \mathrm{Re}\, z \geq c\},$$

where C_M does not depend on z. Finally, using Stirling's formula once more,

$$\left| \frac{\Gamma(z/\alpha)}{\Gamma(1 - \nu + z/\alpha)} \right| \leq C_1 |z|^{\nu - 1} \qquad \text{when } z \in \{z\colon \mathrm{Re}\, z \geq c\}.$$

Hence in $\{z\colon \mathrm{Re}\, z \geq c, z \notin B_\alpha\}$

$$|f^{\pm}_{\alpha,\theta,\nu}(z)|\, |x|^{z-1} \leq A_M \exp(-\pi |\mathrm{Im}\, z|/2 - M \,\mathrm{Re}\, z + \mathrm{Re}\, z \log|x|) |z|^{\nu - 1},$$

where A_M is a constant not depending on z. In particular, choosing $M = \log D + \pi/2$ where D is a positive constant, we obtain

$$|f^{\pm}_{\alpha,\theta,\nu}(z)|\, |x|^{z-1} \leq A \exp(-\pi |z|/2) |z|^{\nu - 1}, \qquad \text{for } x \in [0, D].$$

Hence, as $k \to \infty$, the integral along C_k tends to zero uniformly with respect to $x \in [0, D]$. This proves the theorem. ∎

Proof of Theorem 8. In virtue of Theorem 7, this theorem will be proved if we show that both of the series in the right hand side of (12) converge absolutely for any x.

If the condition (iii) in the statement of the theorem is satisfied, then obviously $|\sin \pi\alpha(\nu + q)|$ and $|\sin \pi(\nu - q/\alpha)|$ are bounded from below by a positive constant, whence both the series in (12) converge absolutely for any x.

Since the proof in the cases of conditions (i) and (ii) is similar, we will prove only the former case.

Evidently, for any $q > 1/\alpha$, there is an integer j_q such that

$$|\alpha(\nu + q) - j_q| = |\alpha(k + lq)/l - j_q| < 1/2.$$

Since α is not a Liouville number, there exist an $r \geq 2$ such that

$$\left| \alpha - \frac{j_q l}{k + lq} \right| \geq \frac{1}{(k + lq)^r}.$$

Thus, for any $q > 1/\alpha$, we have

$$\frac{(k + lq)^{1-r}}{l} \leq \left| \alpha(\nu + q) - j_q \right| = \left| \alpha(k + lq)/l - j_q \right| < 1/2.$$

Hence

$$\left| \sin \pi\alpha(\nu + q) \right| = \left| \sin \pi\left(\alpha(\nu + q) - j_q\right) \right| \geq \frac{2}{l}(k + lq)^{1-r}.$$

Therefore, the first of the series in (12) converges absolutely for any x.

Obviously, for any $q > k/l$, there is an integer j_q such that

$$\left| \frac{1}{\alpha} - \frac{k}{ql} - \frac{j_q}{q} \right| \leq \frac{1}{2q}. \tag{28}$$

Hence

$$\frac{k}{ql} + \frac{j_q}{q} \leq \frac{1}{\alpha} + \left| \frac{1}{\alpha} - \frac{k}{ql} - \frac{j_q}{q} \right| < \frac{2}{\alpha}.$$

Thus

$$k + lj_q \leq \frac{2ql}{\alpha}.$$

Note that $k + lj_q \geq 2$. Since $\alpha \notin L$, we have, for some integer $r \geq 2$,

$$\left| \alpha - \frac{ql}{k + lj_q} \right| \geq \left(k + lj_q\right)^{-r}.$$

Multiplying this inequality by $(k + lj_q)/(l\alpha)$ we obtain

$$\left| \frac{k + lj_q}{l} - \frac{q}{\alpha} \right| = \left| \nu - \frac{q}{\alpha} + j_q \right| \geq \frac{\left(k + lj_q\right)^{1-r}}{l\alpha} \geq \frac{2}{\alpha^2}\left(\frac{2l}{\alpha}\right)^{-r} q^{1-r}. \tag{29}$$

The inequalities (28) and (29) yield

$$1/2 > \left| \nu - \frac{q}{\alpha} + j_q \right| \geq \frac{2}{\alpha^2}\left(\frac{2l}{\alpha}\right)^{-r} q^{1-r}$$

for $q > k/l$. Therefore, for such q, we have

$$|\sin \pi(\nu - q/\alpha)| = \left|\sin \pi\left(\nu - q/\alpha + j_q\right)\right| \geq \frac{4}{\alpha^2}\left(\frac{2l}{\alpha}\right)^{-r} q^{1-r}.$$

Thus the second of the series in (12) converges absolutely for any x. ∎

Proof of Theorem 9. We shall construct a subset D of *EPD* which (i) is dense in *EPD*, (ii) is uncountable, and (iii) is such that, for $(\alpha, \theta, \nu) \in D$, both the series in (12) diverge.

Let $\{\sigma_n^l\}_{n=1}^{\infty}$ be a sequence of rapidly increasing integers defined by the equations

$$\sigma_1^l = 1 + 2l, \qquad \sigma_{n+1}^l = (1 + 2l)^{3\sigma_n^l}, \qquad n = 1, 2, \ldots . \tag{30}$$

Denote by Δ_l the set of all $0 - 1$-valued sequences $\{\delta_j\}_{j=1}^{\infty}$ satisfying the conditions:

(i) δ_j is allowed to be equal to 1 only if $j \in \{\sigma_n^l\}_{n=1}^{\infty}$;

(ii) infinitely many δ_j's equal to 1.

Let $\Omega_l = \{y\colon\ y = \sum_{j=1}^{\infty} \delta_j (1 + 2l)^{-j}, \{\delta_j\}_{j=1}^{\infty} \in \Delta_l\}$ and let Λ_l be the subset of $(0, \infty)$ having the form

$$\lambda = \sum_{j=-s}^{t} a_j (1 + 2l)^{-j}$$

for some $s, t \in \mathbb{N}$, $a_j \in \{0, 1, \ldots, 2l\}$. Set

$$E_{k,l} = \left\{\alpha \in (0,2)\colon \alpha k/l = x + y, x \in \Lambda_l, y \in \Omega_l\right\}.$$

Evidently $E_{k,l}$ is uncountable and dense in $(0, 2)$ for any $k, l \in \mathbb{N}$. Set

$$D = \left\{(\alpha, \theta, \nu) \in EPD\colon \nu = k/l \in \mathbb{Q}, \alpha \in E_{k,l}, (\pi\alpha/2 + \theta) \notin L \cup \mathbb{Q}\right\}.$$

It is easy to see that D is uncountable and dense in *EPD*.

It suffices to prove that for any $(\alpha, \theta, \nu) \in D$ the first of series in (12) diverges.

If $\alpha \in E_{k,l}$ then there are integers m, i such that

$$\alpha k/l = \sum_{j=-i}^{m} a_j (1 + 2l)^{-j} + \sum_{j=m+1}^{\infty} \delta_j (1 + 2l)^{-j},$$

where $a_j \in \{0, 1, \dots, 2l\}$ and $\{\delta_j\}_{j=1}^{\infty} \in \Delta_l$. Denote by $\{\eta_n\}_{n=1}^{\infty}$ the subsequence of $\{\sigma_n{}^l\}_{n=1}^{\infty}$ such that

$$\delta_j = 1 \qquad \text{iff } j \in \{\eta_n\}_{n=1}^{\infty}.$$

Then for any $\eta_n > m$, we have

$$\begin{aligned} 0 < \alpha k/l - \left(\sum_{j=-i}^{m} a_j (1+2l)^{-j} + \sum_{j=m+1}^{\eta_n} \delta_j (1+2l)^{-j} \right) \\ = \sum_{j=\eta_{n+1}}^{\infty} \delta_j (1+2l)^{-j} \\ < (1+2l)^{-\eta_{n+1}+1}. \end{aligned}$$

Multiplying this inequality by $(1+2l)^{\eta_n}$, we see that there is an integer p_n such that

$$0 < \alpha k (1+2l)^{\eta_n}/l - p_n < (1+2l)^{\eta_n - \eta_{n+1}+1} < (1+2l)^{-(1/2)\eta_{n+1}}, \tag{31}$$

for sufficiently large n. Consider the terms of the first series in (12) with $q = q_n = ((1+2l)^{\eta_n} - 1)k/l$.

From (31) we obtain

$$\begin{aligned} |\sin \pi\alpha(k/l + q_n)| &= \left|\sin\left(\pi\alpha k (1+2l)^{\eta_n}/l\right)\right| \\ &= \left|\sin\left(\pi\alpha k(1+2l)^{\eta_n}/l - \pi p_n\right)\right| \\ &< \pi (1+2l)^{-(1/2)\eta_{n+1}}. \end{aligned} \tag{32}$$

Since $\pi\alpha/2 + \theta$ is an irrational non-Liouville number, then as in the proof of the previous theorem there is an integer $r \geq 2$ such that

$$\left|\sin((\nu + q_n)(\pi\alpha/2 + \theta))\right| \geq 2(k + lq_n)^{1-r}/l. \tag{33}$$

Hence, for sufficiently large n we have

$$\begin{aligned} &\left| \frac{\sin((\nu+q_n)(\pi\alpha/2+\theta))\Gamma(\nu+q_n)|x|^{\alpha(\nu+q_n)-1}}{\sin(\pi\alpha(\nu+q_n))\Gamma(\alpha(\nu+q_n))\Gamma(1+q_n)} \right| \\ &\quad \geq \frac{2}{\pi l}(k+lq_n)^{1-r}|x|^{\alpha(\nu+q_n)-1}(1+2l)^{(1/2)\eta_{n+1}} \\ &\quad\quad \times (1+2l)^{(1+q_n)^2}(1+2l)^{\alpha^2(\nu+q_n)^2}. \end{aligned} \tag{34}$$

Since $\{\eta_n\}_{n=1}^{\infty}$ is a subsequence of $\{\sigma_n^l\}_{n=1}^{\infty}$, then

$$\eta_{n+1} \geq (1 + 2l)^{3\eta_n} = (1 + lq_n/k)^3.$$

Hence from (34) the series diverges. ∎

We omit the proofs of Theorem 10 and Theorem 11, since they are similar to that of Theorem 7. The only difference is connected with double poles of $f_{\alpha,\theta,\nu}^{\pm}$ and a more complicated form of residues.

ACKNOWLEDGMENT

The authors thank the anonymous referee for suggestions which improved very much the presentation of this paper.

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