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Theory and Methodology

Continuation method for nonlinear complementarity problems via normal maps

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Abstract

In a recent paper by Chen and Mangasarian (C. Chen, O.L. Mangasarian, A class of smoothing functions for nonlinear and mixed complementarity problems, Computational Optimization and Applications 2 (1996), 97–138) a class of parametric smoothing functions has been proposed to approximate the plus function present in many optimization and complementarity related problems. This paper uses these smoothing functions to approximate the normal map formulation of nonlinear complementarity problems (NCP). Properties of the smoothing function are investigated based on the density functions that defines the smooth approximations. A continuation method is then proposed to solve the NCPs arising from the approximations. Sufficient conditions are provided to guarantee the boundedness of the solution trajectory. Furthermore, the structure of the subproblems arising in the proposed continuation method is analyzed for different choices of smoothing functions. Computational results of the continuation method are reported.

Keywords: Nonlinear complementarity problem; Continuation method; Normal map

1. Introduction

Given a mapping $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, the nonlinear complementarity problem (NCP) with respect to $\mathbf{f}$, $NCP[\mathbf{f}]$, is to find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{x} \geqslant \mathbf{0}, \quad \mathbf{f}(\mathbf{x}) \geqslant \mathbf{0}, \quad \text{and} \quad \mathbf{x}^{\mathrm{T}}\mathbf{f}(\mathbf{x}) = 0.$$

It is well known that $NCP[\mathbf{f}]$ can be reformulated as a system of nonsmooth equations by using either the "min" map or normal map approach (see e.g. [1]). In either formulation, the plus function $\mathbf{z}_+ = \max\{\mathbf{0}, \mathbf{z}\}$ is involved, where the "max" is taken component-wise. The nonsmooth equation arising from the "min" formulation is given by

$$\min\{\mathbf{x}, \mathbf{f}(\mathbf{x})\} = \mathbf{x} - (\mathbf{x} - \mathbf{f}(\mathbf{x}))_+ = \mathbf{0}, \tag{1}$$

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and $\mathbf{x}$ solves $NCP[\mathbf{f}]$ if and only if it solves the above equation.

In a recent paper, Chen and Mangasarian [2] proposed a class of parametric smooth functions, called plus-smooth functions, to approximate the plus function. The plus-smooth function is obtained by twice integrating a parameterized probability density function d. With the help of the plus-smooth function, the nonsmooth "min" formulation is approximated as a system of smooth nonlinear equations. An approximate solution can be obtained by solving these equations. Chen and Mangasarian's numerical experiments indicate that the smoothing approach is very effective and efficient. Chen and Harker [3] refined the plus-smooth function proposed by Chen and Mangasarian, which allows them to establish the existence, uniqueness, and properties of the solution trajectory of the parametric approximations to the NCP.

The normal map formulation relates $NCP[\mathbf{f}]$ to the following system of nonsmooth equations, called Normal Map Equation (NME):

$$\mathbf{f}(\mathbf{z}_+) + \mathbf{z}_- = \mathbf{0}, \tag{2}$$

where $\mathbf{z}_- = \min\{\mathbf{0}, \mathbf{z}\}$. It is well known that $\mathbf{x} = \mathbf{z}_+$ solves $NCP[\mathbf{f}]$ if $\mathbf{z}$ solves the NME, and $\mathbf{z} = \mathbf{x} - \mathbf{f}(\mathbf{x})$ solves the NME if $\mathbf{x}$ solves $NCP[\mathbf{f}]$. Unlike the "min" formulation, the normal map formulation only requires that $\mathbf{f}$ be defined on $\mathbb{R}^n_+$, instead of $\mathbb{R}^n$.

This paper applies the plus-smooth function to approximate both $\mathbf{z}_+$ and $\mathbf{z}_-$ in the NME, and proposes a continuation method to solve $NCP[\mathbf{f}]$. Section 2 studies the properties of the plus-smooth function. In particular, the plus-smooth function is classified by whether it is derived from a density function with finite or infinite support. Section 3 analyzes the smooth approximation to the NME. Sufficient conditions are provided to guarantee the boundedness and the monotonicity of the solution trajectory for the continuation method, respectively. Section 4 investigates the structure of the subproblems arising in a Newton corrector-based continuation method. It is shown that the plus-smooth function with finite support results in subproblems of reduced dimension, an advantage shared by many B-differentiable approaches to solve complementarity related problems (see, for example, [4,5]). Section 5 reports our numerical experiments of the continuation method using plus-smooth functions with finite and infinite support.

The following notation will be used throughout the paper. All vectors (vector functions) are column vectors (vector functions) and are denoted by boldface letters. $\mathbf{0}$ and $\mathbf{1}$ represent vectors of appropriate dimension with all components equal to 0 and 1, respectively. $\mathbb{R}^n$, $\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$ denote, respectively, n dimensional Euclidean space, the nonnegative orthant of $\mathbb{R}^n$, and the strictly positive orthant of $\mathbb{R}^n$.

2. Smooth approximation of z_+

As shown in Section 1, many complementarity related problems can be reformulated as a system of equations. The plus function z_+ is involved in many of these formulations. However, since the plus function is nonsmooth, many of the resulting equations are also nonsmooth. Thus, the problems defined by these nonsmooth equations cannot be solved directly by the traditional techniques for smooth equations. To overcome the difficulty, Chen and Mangasarian [2] introduced a class of plus-smooth functions $p(z,u)$ that approximate the plus function z_+ by twice integrating a probability density function with parameter $0 \leqslant u < \infty$. More specifically, the plus-smooth function is given by

$$p(z,u) = \int_{-\infty}^{z}\int_{-\infty}^{t} \frac{1}{u} d\left(\frac{x}{u}\right) \mathrm{d}x\, \mathrm{d}t, \tag{3}$$

where $d(x)$ is a probability density function satisfying certain technical assumptions. Clearly, as u approaches zero, the probability density function $(1/u)d(x/u)$ approaches the delta function with all the probability mass concentrated at origin; double integration $p(z,u)$ then approaches the plus function z_+. In this regard, $p(z,u)$ can be considered as a natural approximation of the plus function z. Indeed, it has been shown by

Chen and Harker [3] that any smooth approximation of the plus function that leads to a "well-behaved" continuation path for NCPs must be a double integration of a probability density function.

The smooth approximation $p(z,u)$ preserves many structural properties of the plus function z_+, which will be explored in the sequel. Our characterization is based on whether the plus-smooth function p has a finite or an infinite support. The function p is said to have a finite (infinite) support if the probability density function d it derives from has a finite (infinite) support. A probability density function d is supported on range $[a,b]$ if $d(x)>0$ for all $x\in[a,b]$ and $d(x)=0$ otherwise. If both a and b are finite numbers, we say that d has a finite support; otherwise, d has an infinite support. Some of the following assumptions on the probability density function d will be used to characterize the plus-smooth function p:

(C1) $d(x)$ is symmetric and piecewise continuous with finite number of pieces;
(C2) $E(|x|)<\infty$;
(C3) $\lim_{x\to\infty} x^3 d(x)<\infty$;
(C4) $d(x)$ has a finite support on $[-s,s]$ for some $0<s<\infty$;
(C5) $d(x)$ has an infinite support.

The symmetry assumption in (C1) is made only for the convenience of presentation and is not essential for the results in this paper. To date, all plus-smooth functions that have been proposed and implemented derive from symmetric probability density functions. Assumption (C3) requires that both tails of d are thin enough, a property to be used to establish the boundedness of solution. Clearly, Assumption (C4) implies (C3), which in turn implies (C2).

The following results characterize the plus-smooth function $p(z,u)$ under various assumptions of the probability density function d:

Proposition 1. *Let $p(z,u)$ be defined by (3) with $u>0$ and the probability density function d satisfy Assumptions (C1) and (C2).*

1. *$p(z,u)$ is continuously differentiable, nondecreasing, and convex.*
2. *$0\leqslant p(z,u)-z_+\leqslant Du$ for all z, where $D>0$ is a fixed constant depending on d.*
3. *$\lim_{z\to-\infty} p(z,u)=0$ and $\lim_{z\to\infty} p(z,u)/z=1$ for all $u>0$.*
4. *$0\leqslant p'(z,u)\leqslant 1$ and $p'(-z,u)=1-p'(z,u)$, where $p'(z,u)=\mathrm{d}p(z,u)/\mathrm{d}z$.*
5. *Equation $p(z,u)=b$ has a unique solution for all $u\geqslant 0$ and $b>0$.*
6. *If, in addition, d satisfies (C5), then $p(z,u)$ is strictly increasing and convex, $0<p'(z,u)<1$, and $z_+<p(z,u)$.*
7. *If, in addition, d satisfies (C4), then $p(z,u)$ is strictly increasing and convex in $(-su,su)$, $z_+<p(z,u)$ and $0<p'(z,u)<1$ for all $z\in(-su,su)$, and $p(z,u)=z_+$ otherwise.*
8. *If, in addition, d satisfies (C3), then $p(z,u)p(-z,u)<\infty$ for all z.*

Proof. The proof of Results (1) and (2), and the first two parts of Result (6) have been shown in [4], and Result (3) and the last part of Result (6) have been shown in [3]. Result (4) and the last part of Result (7) are true by the definition of $p(z,u)$ and the fact that $(1/u)d(x/u)$ is a probability density function.

We now prove Result (5). By Results (2) and (3), we have

$$\lim_{z\to-\infty} p(z,u)=0,\quad \lim_{z\to+\infty} p(z,u)\to+\infty.$$

It follows that equation $p(z,u)=b$ has at least one solution for all $u\geqslant 0$ and $b>0$. Suppose on the contrary that the equation $p(z,u)=b$ has two solutions $z_1<z_2$. By Result (3), there exists a $z_0<z_1$ such that $p(z_0,u)=b_0<b$. Clearly, $p(z_1,u)$ is strictly greater than the linear interpolation of $p(z_0,u)$ and $p(z_2,u)$ at z_1. However, this contradicts the fact that $p(z,u)$ is a convex function. Thus, Result (5) is true.

For the first two parts of Result (7), notice that if the probability density function $d(x)$ is supported on $[-s,s]$, then the function $(1/u)d(x/u)$ has a support on $[-us,us]$. The remaining proof is almost identical to that of Result (6).

It remains to show Result (8). Since d is assumed to be symmetric, it suffices to show that $\lim_{z\to\infty} p(z,u)p(-z,u)<\infty$. Indeed,

$$\lim_{z\to\infty} p(z,u)p(-z,u) = \lim_{z\to\infty} \frac{p(-z,u)}{1/p(z,u)}$$
$$= \lim_{z\to\infty} \frac{\int_{-\infty}^{-z} \frac{1}{u} d\left(\frac{x}{u}\right) \mathrm{d}x}{\int_{-\infty}^{z} \frac{1}{u} d\left(\frac{x}{u}\right) \mathrm{d}x} p^2(z,u)$$
$$= \lim_{z\to\infty} \frac{\int_{-\infty}^{-z} \frac{1}{u} d\left(\frac{x}{u}\right) \mathrm{d}x}{z^{-2}}$$
$$= \lim_{z\to\infty} \frac{1}{2} u^{-2} z^3 \,\mathrm{d}(-z/u)$$
$$= \lim_{x\to\infty} \frac{1}{2} u x^3 \,\mathrm{d}(x)$$
$$< \infty.$$

l'Hospital's rule is used to obtain the second and fourth equalities. The third equality is true since d is a probability density function, and $\lim_{z\to\infty} p(z,u)/z = 1$ by Result (3). □

Below are several examples of the plus-smooth function derived from a probability function with an infinite support. In all these examples, the function d satisfies Assumptions (C1), (C2), (C3), and (C5).

Example 1. Neural network plus-smoothing function (Chen and Mangasarian [2]):

$$d(x) = \mathrm{e}^{-x}/(1+\mathrm{e}^{-x})^2,$$
$$p(z,u) = z + u \log(1+\mathrm{e}^{-z/u}).$$

Example 2. Interior point plus-smooth function (Chen and Harker [6], Kanzow [7], Smale [8]):

$$d(x) = 2/(x^2+4)^{1.5}, \quad p(z,u) = (z+\sqrt{z^2+4u})/2.$$

Notice that the above probability density function d is a scaled variant of the t distribution with parameter $n = 2$. It has been shown [8] that the central path of many interior point algorithms can be characterized as the solution of the parametric smooth equations obtained by applying this plus-smooth function to the "min" formulation of $NCP[\mathbf{f}]$ (1).

Example 3. Normal plus-smooth function:

$$d(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2},$$

no closed form expression for $p(z,u)$.

Although it is possible to derive a plus-smooth function from many probability density functions with finite support, the following seems to be a natural choice in that it satisfies Assumptions (C1)–(C4).

Example 4. Uniform plus-smooth function (Zang [9])

$$d(x) = \begin{cases} 1 & \text{if } -1/2 \leqslant x \leqslant 1/2, \\ 0 & \text{otherwise,} \end{cases}$$

$$p(z,u) = \begin{cases} 0 & \text{if } z < -u/2, \\ (z+u/2)^2/2u & \text{if } |z| \leqslant u/2, \\ z & \text{if } z > u/2. \end{cases}$$

3. Application to the nonlinear complementarity problem

In this section, the plus-smooth functions introduced in the previous section are applied to the NME (2), the normal map formulation of $NCP[\mathbf{f}]$. As a result, the NME is approximated by a series of parametric smooth equations. The properties of the solution path(s), consisting of solutions of the smooth equations, are investigated. Sufficient conditions are provided to ensure the existence and the monotonicity of the solution path. Some of the results have been previously established in literature (see [10]), but only for a specific choice of the plus-smooth function, such as the interior point plus-smooth function. In the remainder of this paper, the probability density function d that defines the plus-smooth function is assumed to satisfy at least Assumptions (C1) and (C2).

3.1. Smooth approximation of NME

As mentioned in Section 1, $\mathbf{x} = \mathbf{z}_+$ solves $NCP[\mathbf{f}]$ if $\mathbf{z}$ solves the NME

$$\mathbf{f}(\mathbf{z}_+) + \mathbf{z}_- = \mathbf{0}.$$

Since $\mathbf{z}_+$ and $\mathbf{z}_-$ can be approximated by $p(z,u)$ and $-p(-z,u)$, respectively, the NME can be approximated by the following parametric equations, called the Smooth Normal Map Equations (SNME):

$$\mathbf{h}(\mathbf{z},u) = (1-u)\mathbf{f}(\mathbf{p}(\mathbf{z},u\mathbf{a})) - \mathbf{p}(-\mathbf{z},u\mathbf{a}) + u\mathbf{b} = \mathbf{0}, \tag{4}$$

where $\mathbf{a} \in \mathbb{R}^n_+$ and $\mathbf{b} \in \mathbb{R}^n_{++}$ are fixed parameters and $\mathbf{p}(\mathbf{z},u\mathbf{a})$ and $\mathbf{p}(-\mathbf{z},u\mathbf{a})$ are column vectors with components $p(z_i,ua_i)$ and $p(-z_i,ua_i)$, $i=1,\ldots,n$, respectively. If the vector $\mathbf{a}$ is strictly positive (i.e., $\mathbf{a} > \mathbf{0}$), $\mathbf{h}(\mathbf{z},u)$ is C^1 by Result (1) of Proposition 1. If some of the components of $\mathbf{a} \geqslant \mathbf{0}$ are equal to zero, $\mathbf{h}(\mathbf{z},u)$ is in general piecewise C^1.

At $u=1$, the SNME reduces to

$$\mathbf{h}(\mathbf{z},1) = -\mathbf{p}(-\mathbf{z},\mathbf{a}) + \mathbf{b} = \mathbf{0}.$$

By Result (5) of Proposition 1, it has a unique solution. On the other hand, at $u=0$, the SNME reduces to the NME

$$\mathbf{h}(\mathbf{z},0) = \mathbf{f}(\mathbf{z}_+) + \mathbf{z}_- = \mathbf{0}.$$

Therefore, if there exists a path from the unique solution at $u=1$ to a solution at $u=0$, we could apply standard homotopy techniques to find the solution of the NME and thus, a solution of $NCP[\mathbf{f}]$.

3.2. Existence of solution paths

We establish the existence of solution by following the techniques developed by Kojima et al. [10]. Let S be the set of all solutions of the SNME; i.e.,

$$S = \{(\mathbf{z},u) \in \mathbb{R}^n \times (0,1]: \mathbf{h}(\mathbf{z},u) = \mathbf{0}\}.$$

Notice that the solution set S is a function of chosen vectors $\mathbf{a}$ and $\mathbf{b}$, which are dropped in the remaining part of the paper for simplicity. Define the solution path T as the connected components of S emanating from the unique solution of $\mathbf{h}(\mathbf{z},1) = \mathbf{0}$. Let

$$\mathbf{g}(\mathbf{z},u) = [(1-u)\mathbf{f}(\mathbf{p}(\mathbf{z},u\mathbf{a})) - \mathbf{p}(-\mathbf{z},u\mathbf{a})]/u.$$

Then, we can rewrite the SNME as

$$\mathbf{g}(\mathbf{z},u) = -\mathbf{b}.$$

Clearly, the mapping $\mathbf{g}: \mathbb{R}^n \times (0,1] \to \mathbb{R}^n$ is C^1 if $\mathbf{a} > \mathbf{0}$ and piecewise C^1 if some of the components of $\mathbf{a} \geqslant \mathbf{0}$ are equal to zero. Based on the results on regular values of a piecewise C^1-mapping (see e.g. [10], Theorem 3.1), we have:

1. Almost every $-\mathbf{b} < \mathbf{0}$ is a regular value of the piecewise C^1-mapping $\mathbf{g}$.
2. If $-\mathbf{b} < \mathbf{0}$ is a regular value of the piecewise C^1-mapping $\mathbf{g}$, then S is a disjoint union of smooth 1-dimensional manifolds. Specifically, its connected component T forms a piecewise smooth trajectory (or a smooth trajectory when $\mathbf{a} > \mathbf{0}$) such that either $\|\mathbf{z}\|$ tends to infinity or u tends to 0 along the trajectory T.

Summarizing the above discussion, we have the following result on the existence of solution.

Theorem 1. *Let* $\mathbf{a} \geqslant \mathbf{0}$ *be fixed. Then for almost every* $\mathbf{b} > \mathbf{0}$, *the set* T *forms a trajectory, a 1-dimensional manifold which is homeomorphic to* $(0,1]$, *such that*

$$T = \{(\xi(t),\tau(t)):\ 0 < t \leqslant 1\}$$

and $\lim_{t\to 0}\tau(t) = 0$ *whenever* T *is bounded. Here,* $\xi:\ (0,1] \to \mathbb{R}^n, \tau(t):\ (0,1] \to (0,1]$ *are piecewise* C^1*-mappings or* C^1*-mappings when* $\mathbf{a} > \mathbf{0}$.

The significance of the above result has been discussed in [10]; namely, the set T generically forms a smooth or piecewise smooth trajectory. Furthermore, if the trajectory T is bounded, there exists at least one limit point as u tends to zero along the trajectory, and every limit point is a solution of the NME. Sufficient conditions to ensure the boundedness of the solution set S and, therefore, the trajectory T are discussed next.

3.3. Boundedness of trajectory T

In this section, we show that the solution set S is bounded if $\mathbf{f}$ is a monotone function and $NCP[\mathbf{f}]$ has a strictly positive feasible point, or if $\mathbf{f}$ is an R_0-function (to be defined).

Let D be a nonempty subset of $\mathbb{R}^n$. A continuous mapping $\mathbf{f}: D \to \mathbb{R}^n$ is said to be *monotone* over D if

$$[\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})]^{\mathrm{T}}(\mathbf{x} - \mathbf{y}) \geqslant 0 \quad \text{for all } \mathbf{x}, \mathbf{y} \in D.$$

Proposition 2. *Let* $\mathbf{a} \geqslant \mathbf{0}$ *and the plus-smooth function* p *satisfy Assumption (C3). If* $\mathbf{f}$ *is monotone over* $\mathbb{R}^n_+$ *and NCP*[$\mathbf{f}$] *has a strictly positive feasible point, then the solution set* S, *and therefore the trajectory* T, *is bounded.*

Proof. Let $\mathbf{z}$ be any point in the set S for some $u \in (0, 1]$. We need to show that $\|\mathbf{z}\|$ is bounded. Denote $\mathbf{x} = \mathbf{p}(\mathbf{z}, u\mathbf{a}) \geqslant \mathbf{0}$ and $\mathbf{y} = \mathbf{p}(-\mathbf{z}, u\mathbf{a}) \geqslant \mathbf{0}$. By Result (2) of Proposition 1,

$$|z_i| = (z_i)_+ - (z_i)_- \leqslant p(z_i, ua_i) + p(-z_i, ua_i).$$

Thus, it suffices to show that $\mathbf{1}^{\mathrm{T}}\mathbf{x} + \mathbf{1}^{\mathrm{T}}\mathbf{y}$ is bounded. By assumption, there exists a point $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) > \mathbf{0}$ such that $\tilde{\mathbf{y}} = \mathbf{f}(\tilde{\mathbf{x}})$. Define the positive numbers ϵ and ω by

$$\epsilon = \min\{b_i, \tilde{x}_i, \tilde{y}_i : i = 1, \ldots, n\},$$
$$\omega = \max\{b_i, \tilde{x}_i, \tilde{y}_i : i = 1, \ldots, n\}.$$

It has been shown by Kojima et al. ([10], p. 953) that if $\mathbf{f}$ is a monotone function, then

$$\mathbf{1}^{\mathrm{T}}\mathbf{x} + \mathbf{1}^{\mathrm{T}}\mathbf{y} \leqslant [\mathbf{x}^{\mathrm{T}}\mathbf{y} + n\omega^2]/\epsilon.$$

Since $\mathbf{x}^{\mathrm{T}}\mathbf{y} = \mathbf{p}(\mathbf{z}, u\mathbf{a})^{\mathrm{T}}\mathbf{p}(-\mathbf{z}, u\mathbf{a})$ is bounded by Assumption (C3) and Result (8) of Proposition 1, $\mathbf{1}^{\mathrm{T}}\mathbf{x} + \mathbf{1}^{\mathrm{T}}\mathbf{y}$ is bounded and the result follows. □

We now present another condition introduced in [3] for S to be bounded. Let D be a nonempty subset of $\mathbb{R}^n$. A continuous mapping $\mathbf{f}: D \to \mathbb{R}^n$ is said to be a R_0-function over the set D if for any sequence $\{\mathbf{x}^k\} \in D$ satisfying $\|\mathbf{x}^k\| \to \infty$ and

$$\liminf_{k\to\infty} \frac{\min_i x_i^k}{\|\mathbf{x}^k\|} \geqslant 0,$$

$$\liminf_{k\to\infty} \frac{\min_i f_i(\mathbf{x}^k)}{\|\mathbf{x}^k\|} \geqslant 0,$$

there exists an index j such that $x_j^k \to \infty$ and $f_j(x^k) \to \infty$.

Proposition 3. *Let* $\mathbf{a} \geqslant \mathbf{0}$. *If* $\mathbf{f}$ *is a* R_0*-function over* $\mathbb{R}^n_+$, *then the solution set* S, *and therefore the trajectory* T, *is bounded.*

Proof. Suppose on the contrary that S is unbounded. Then there exists an unbounded sequence $\{\mathbf{z}^k\}$ such that $\mathbf{h}(\mathbf{z}^k, u^k) = \mathbf{0}$ with $u^k \in [0, 1]$. Denote

$$\mathbf{x}^k = \mathbf{p}(\mathbf{z}^k, u^k\mathbf{a}) \geqslant \mathbf{0} \quad \text{and} \quad \mathbf{y}^k = \mathbf{p}(-\mathbf{z}^k, u^k\mathbf{a}) \geqslant \mathbf{0}.$$

Then

$$(1 - u^k)\mathbf{f}(\mathbf{x}^k) - \mathbf{y}^k + u^k\mathbf{b} = \mathbf{0}. \tag{5}$$

We claim that sequence $\{\mathbf{x}^k\}$ must also be unbounded. Otherwise, to satisfy Eq. (5), the sequence $\{\mathbf{y}^k\}$ must be bounded, and therefore, $\{\mathbf{z}^k\}$ is bounded since $|z_i^k| \leqslant x_i^k + y_i^k$ from the proof of Proposition 2. However, this contradicts the assumption that sequence $\{\mathbf{z}^k\}$ is unbounded. We now consider the limiting behavior of unbounded sequence $\{\mathbf{x}^k\}$ and the associated sequence $\{\mathbf{f}(\mathbf{x}^k)\}$. For any index i:

- if $\{z_i^k\}$ is bounded, then $\{x_i^k\}$ and $\{y_i^k\}$ are bounded by definition, and $\{f_i(\mathbf{x}^k)\}$ bounded by Eq. (5);
- if $z_i^k \to \infty$, then $x_i^k \to \infty$, $y_i^k \to 0$ by Result (3) of Proposition 1, and $f_i(\mathbf{x}^k)$ is bounded by Eq. (5);
- if $z_i^k \to -\infty$, then, by the same argument, $x_i^k \to 0$, $y_i^k \to \infty$, and $f_i(\mathbf{x}^k) \to \infty$.

It follows that the sequence $\{\mathbf{x}^k\}$ satisfies the assumptions of an R_0-function

$$\liminf_{k\to\infty} \frac{\min_i x_i^k}{\|\mathbf{x}^k\|} \geqslant 0,$$

$$\liminf_{k\to\infty} \frac{\min_i f_i(\mathbf{x}^k)}{\|\mathbf{x}^k\|} \geqslant 0.$$

By the definition of R_0-function, there exists an index j such that $x_j^k \to \infty$ and $f_j(\mathbf{x}^k) \to \infty$. However, $x_j^k \to \infty$ implies $z_j^k \to \infty$, $f_j(\mathbf{x}^k) \to \infty$ implies $y_j^k \to \infty$, and therefore, $z_j^k \to -\infty$. This leads to a contradiction. □

3.4. Existence of monotone trajectory T

In general, the trajectory T defined in the previous two subsections is not necessarily monotone with respect to the parameter u; i.e., to follow the trajectory from the starting point at $u = 1$ to a solution at $u = 0$, u does not always decrease monotonically. More sophisticated techniques (see e.g. [11]) are needed to trace such a trajectory. However, u is decreased monotonically in most implementations of smoothing methods and interior point methods for complementarity related problems. This subsection provides a set of sufficient conditions under which there exists a unique monotone trajectory T. The following definitions are needed. Let D be a nonempty subset of $\mathbb{R}^n$. The mapping $\mathbf{f} : D \to \mathbb{R}^n$ is said to be a

1. P_0-function over the set D if

$$\max_{1 \leqslant i \leqslant n, x_i \neq y_i} [f_i(\mathbf{x}) - f_i(\mathbf{y})](x_i - y_i) \geqslant 0$$

for all $\mathbf{x}, \mathbf{y} \in D$, $\mathbf{x} \neq \mathbf{y}$,

2. P-function over the set D if

$$\max_{1 \leqslant i \leqslant n} [f_i(\mathbf{x}) - f_i(\mathbf{y})](x_i - y_i) > 0$$

for all $\mathbf{x}, \mathbf{y} \in D$, $\mathbf{x} \neq \mathbf{y}$,

3. uniform P-function over the set D if, for some $\gamma > 0$,

$$\max_{1 \leqslant i \leqslant n} [f_i(\mathbf{x}) - f_i(\mathbf{y})](x_i - y_i) \geqslant \gamma \|\mathbf{x} - \mathbf{y}\|^2$$

for all $\mathbf{x}, \mathbf{y} \in D$.

It is well known that the P_0-property is implied by both the P-property and monotonicity; the P-property is, in turn, implied by the uniform P-property. In addition, it has been shown [3] that the R_0-property introduced in the previous section is also implied by the uniform P-property. The above definitions are also closely related to the concept of P_0 and P matrices: A matrix A is said to be a P_0-matrix (P-matrix) if all of its principle minors are non-negative (positive). It is well known that the Jacobian of a P_0-function (uniform P-function) at any point is a P_0-matrix (P-matrix). The uniqueness of solution to the SNME is studied first.

Proposition 4. *Let* $\mathbf{a} > 0$ *and the smooth function* p *have an infinite support (Assumption C5). If* $\mathbf{f}$ *is a* P_0*-function over* R^n_+*, then the SNME has at most one solution for each* $u \in (0, 1]$.

Proof. The result is true for $u = 1$, as indicated in Section 3.1. Thus, consider the case where $u < 1$ and suppose on the contrary that the equation $\mathbf{h}(\mathbf{z}, u) = \mathbf{0}$ has two different solutions $\mathbf{z}^1 \neq \mathbf{z}^2$ for some $u \in (0, 1)$. Denote $\mathbf{x}^k = \mathbf{p}(\mathbf{z}^k, u\mathbf{a})$ and $\mathbf{y}^k = \mathbf{p}(-\mathbf{z}^k, u\mathbf{a})$ for $k = 1, 2$. Since p has an infinite support, it is strictly increasing and convex by Result (6) of Proposition 1. Thus, $\mathbf{x}^1 \neq \mathbf{x}^2$ and $\mathbf{y}^1 \neq \mathbf{y}^2$. In addition, we have

$$\mathbf{f}(\mathbf{x}^k) = \frac{1}{1-u}\mathbf{y}^k + \frac{u}{1-u}\mathbf{b} \quad \text{for} \quad k = 1, 2.$$

Since $\mathbf{f}$ is a P_0-function, there exists an index i such that

$$x_i^1 \neq x_i^2 \quad \text{and} \quad (f_i(\mathbf{x}^1) - f_i(\mathbf{x}^2))(x_i^1 - x_i^2) \geqslant 0,$$

or equivalently,

$$x_i^1 \neq x_i^2 \quad \text{and} \quad (y_i^1 - y_i^2)(x_i^1 - x_i^2) \geqslant 0. \tag{6}$$

Without loss of generality, one may assume $x_i^1 > x_i^2$. Then, the inequality in (6) implies that $y_i^1 \geqslant y_i^2$. However, by Result (6) of Proposition 1, $x_i^1 > x_i^2$ implies $z_i^1 > z_i^2$, $y_i^1 \geqslant y_i^2$ implies $z_i^1 \leqslant z_i^2$. This leads to a contradiction. □

Similar uniqueness result can be established for the SNME when $\mathbf{a} \geqslant 0$ has zero components, or the plus-smooth function involved has a finite support.

Proposition 5. *Let* $\mathbf{a} \geqslant 0$ *and the plus-smooth function* p *have a finite support (Assumption C4). If* $\mathbf{f}$ *is a* P*-function, then the SNME has at most one solution for all* $u \in [0, 1]$.

Proof. The proof is a slight modification of the proof of Proposition 4. Suppose on the contrary that the equation $\mathbf{h}(\mathbf{z}, u) = \mathbf{0}$ has two different solutions $\mathbf{z}^1 \neq \mathbf{z}^2$ for some $u \in [0, 1)$. Let $\mathbf{x}^k$ and $\mathbf{y}^k$ be defined as in the proof of Proposition 4. We claim that $\mathbf{x}^1 \neq \mathbf{x}^2$. Otherwise,

$$\mathbf{y}^1 = \mathbf{y}^2 = (1-u)\mathbf{f}(\mathbf{x}^k) + u\mathbf{b},$$

which implies that $\mathbf{z}^1 = \mathbf{z}^2$. However, this contradicts the assumption. Since $\mathbf{f}$ is a P-function and $\mathbf{x}^1 \neq \mathbf{x}^2$, by the same argument as in the proof of Proposition 4, there exists an index i such that

$$x_i^1 \neq x_i^2 \quad \text{and} \quad (y_i^1 - y_i^2)(x_i^1 - x_i^2) > 0. \tag{7}$$

Without loss of generality, one may assume that $x_i^1 > x_i^2$. Then, the inequality in (7) implies that $y_i^1 > y_i^2$. However, for any plus-smooth function, $x_i^1 > x_i^2$ implies $z_i^1 > z_i^2, y_i^1 > y_i^2$ implies $z_i^1 < z_i^2$. This leads to a contradiction. □

We now provide a set of sufficient conditions to guarantee the existence of a monotone trajectory T for plus-smooth functions with finite or infinite supports.

(A1) $\mathbf{a} > 0$, p satisfies (C5), and $\mathbf{f}$ is both a P_0 and R_0-function;

(A2) $\mathbf{a} > 0$, p satisfies (C3) and (C5), $\mathbf{f}$ is a monotone function over $\mathbb{R}^n_+$, and the $NCP[\mathbf{f}]$ has a strictly positive feasible point;

(A3) $\mathbf{a} \geqslant 0$, p satisfies either (C4) or (C5), and $\mathbf{f}$ is a uniform P-function.

Theorem 2. *If one of the three assumptions (A1)–(A3) is satisfied, the following statements are true:*
1. *For each $u \in (0, 1]$, the SNME has a unique solution $\mathbf{z}(u)$; hence, the trajectory can be rewritten as $T = \{(\mathbf{z}(u), u): u \in (0, 1]\}$, which is monotone with respect to u.*
2. *The trajectory T is bounded; hence, there is at least one limit point of $\mathbf{z}(u)$ as $u \to 0$.*
3. *Let $\mathbf{z}^*$ be any limit point of $\mathbf{z}(u)$ as $u \to 0$. Then $\mathbf{z}^*_+$ is a solution of the NME. In particular, the solution is unique under assumption (A3).*

Proof. To prove the first result, in view of Propositions 4 and 5, it suffices to show the existence of a solution. Recall that the SNME $\mathbf{h}(\mathbf{z}, u) = \mathbf{0}$ has a unique solution for $u = 1$. Let $0 \leqslant \bar{u} \leqslant 1$ be the infimum of u's such that the SNME has a solution for every $u \in [\bar{u}, 1]$. Then there exists a sequence $\{(\mathbf{z}^k, u^k)\}$ such that $\mathbf{z}^k$ solves the SNME with parameter u^k and $\lim_{k\to\infty} u^k = \bar{u}$. Propositions 2 and 3 ensure that the sequence $\{\mathbf{z}^k\}$ is bounded and therefore has a limit point under all three assumptions (A1)–(A3). Let $\bar{\mathbf{z}} \in \mathbb{R}^n$ be a limit point of $\{z^k\}$. Since function $\mathbf{h}$ is continuous with respect to $\mathbf{z}$, it must happen that $\mathbf{h}(\bar{\mathbf{z}}, \bar{u}) = \mathbf{0}$. Hence, if $\bar{u} = 0$, the desired result follows. Assume on the contrary that $\bar{u} > 0$. Under any of the assumptions (A1)–(A3), the generalized Jacobian $\partial\mathbf{h}(\mathbf{z}, u)$ is nonsingular (see Propositions 6 and 7 in Section 4) for all $\mathbf{z} \in \mathbb{R}^n$ and $u > 0$. By the generalized implicit function theorem of Clarke ([12], p. 256), the SNME has a unique solution for every u sufficiently close to $\bar{u}$. However, this contradicts $\bar{u} > 0$. Therefore, the SNME has a unique solution for all $u \in (0, 1]$, and the trajectory T is monotone. The second result is a direct consequence of the first result and the fact that T is bounded. To prove the last result, let $\mathbf{z}^*$ be any limit point of $\mathbf{z}(u)$ as $u \to 0$. By the continuity of the mapping $\mathbf{h}$, we have $\mathbf{h}(\mathbf{z}^*, 0) = \mathbf{0}$ or $\mathbf{f}(\mathbf{z}^*_+) + \mathbf{z}^*_- = \mathbf{0}$. Hence $\mathbf{z}^*$ is a solution of the NME. Finally, it is well known that if $\mathbf{f}$ is a uniform P-function then the solution is unique. □

As a corollary to the above result, we have also established the existence of solutions for the NCP with a P_0- and R_0-function. In addition, if the trajectory T is monotone, the implementation of a continuation method could be simplified: a simple lift step on u can be used as a predictor.

4. Subproblems arising from a newton corrector step

From the discussion of previous section, $\mathbf{z}(u)$, the solution of the SNME $\mathbf{h}(\mathbf{z}, u) = \mathbf{0}$, in general, forms a trajectory with a unique starting point at $u = 1$ and an ending point at a solution of the NME. Therefore, a continuation method can be designed to follow the trajectory and locate a solution of the NME. Most continuation methods perform predictor and corrector steps alternatively. The predictor step starts from a point close to the trajectory and moves along the trajectory approximately. Depending on the property of the trajectory, one may choose to use a more complicated Euler predictor, or use a simple lift predictor by monotonically reducing u. The corrector step brings back the new point to a neighborhood of the trajectory in order to prepare for the next predictor step. Newton's method is often used as a corrector in many continuation methods. We will study the subproblems of the Newton corrector arising from the solution of the SNME in this section.

Let $\mathbf{a} \geqslant \mathbf{0}$ and $0 < u \leqslant 1$. As discussed in Section 3.1, $\mathbf{h}(\mathbf{z}, u)$ is C^1 if $\mathbf{a} > \mathbf{0}$ and piecewise C^1 if some components of $\mathbf{a}$ are equal to zero. At a given point $\mathbf{z}$, the Newton direction $\mathbf{d}$ is obtained by solving the following generalized Newton equations:

$$\mathbf{V}\mathbf{d} + \mathbf{h}(\mathbf{z}, u) = \mathbf{0},$$

where $\mathbf{V} \in \partial\mathbf{h}(\mathbf{z}, u)$, with $\partial\mathbf{h}(\mathbf{z}, u)$ being the generalized Jacobian (see Clarke [12]) of $\mathbf{h}$ defined at $(\mathbf{z}, u)$. The method and its convergence properties has been studied by Qi [13].

Although it is in general difficult to represent a generalized Jacobian explicitly, it can be done for the SNME due to its special structure:

$$\begin{aligned}\partial\mathbf{h}(\mathbf{z}, u) = \{&(1-u)\nabla\mathbf{f}(\mathbf{p}(\mathbf{z}, u\mathbf{a}))\mathbf{D} + \mathbf{I} - \mathbf{D} \mid \mathbf{D} \\ &= \operatorname{diag}\{D_i\}, \\ D_i &= p'(z_i, ua_i) \quad \text{if } a_i > 0, \\ D_i &= 1 \quad \text{if } a_i = 0, z_i > 0, \\ D_i &\in [0,1] \quad \text{if } a_i = 0, z_i = 0, \\ D_i &= 0 \quad \text{if } a_i = 0, z_i < 0\}.\end{aligned}$$

The following two lemmas are introduced for establishing nonsingularity of generalized Jacobian $\partial\mathbf{h}(\mathbf{z}, u)$.

Lemma 1. *Let* $\mathbf{D}_i$, $i = 1, \ldots, 3$, *be positive diagonal matrices of appropriate dimensions as defined in matrix* $\mathbf{M}'$ *below*:

$$\mathbf{M}' = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12}\mathbf{D}_1 & \mathbf{0} \\ \mathbf{M}_{21} & \mathbf{M}_{22}\mathbf{D}_1 + \mathbf{D}_2 & \mathbf{0} \\ \mathbf{M}_{31} & \mathbf{M}_{32}\mathbf{D}_1 & \mathbf{D}_3 \end{pmatrix}.$$

$\mathbf{M}'$ *is nonsingular if* $\mathbf{M}_{11}$ *is nonsingular and* $\mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{M}_{12}$ *is a* P_0*-matrix.*

Proof. Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)^{\mathrm{T}}$ be any vector such that $\mathbf{M}'\mathbf{x} = \mathbf{0}$. Since $\mathbf{M}_{11}$ is nonsingular, we have, after some algebraic calculations,

$$[(\mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{M}_{12})\mathbf{D}_1 + \mathbf{D}_2]\mathbf{x}_2 = \mathbf{0}.$$

By assumption, $\mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{M}_{12}$ is a P_0-matrix. It follows that the whole matrix in front of $\mathbf{x}_2$ is a P-matrix and thus, is nonsingular. This implies $\mathbf{x}_2 = \mathbf{0}$. Following similar calculations, we have

$$\mathbf{x}_1 = -\mathbf{M}_{11}^{-1}\mathbf{M}_{12}\mathbf{D}_1\mathbf{x}_2 = \mathbf{0},$$
$$\mathbf{x}_3 = -\mathbf{D}_3^{-1}(\mathbf{M}_{31}\mathbf{x}_1 + \mathbf{M}_{23}\mathbf{D}_1\mathbf{x}_2) = \mathbf{0}.$$

Therefore, $\mathbf{x} = \mathbf{0}$ and $\mathbf{M}'$ is nonsingular by definition. □

Lemma 2. *Let* $\mathbf{M}$ *be a P-matrix. Then* $\mathbf{MD} + \mathbf{I} - \mathbf{D}$ *is nonsingular for all diagonal matrix* $\mathbf{D}$ *such that* $0 \leqslant D_i \leqslant 1$ *for all i.*

Proof. Let $\mathbf{D}$ be any diagonal matrix such that $0 \leqslant D_i \leqslant 1$ for all i. Define the following index set associated with $\mathbf{D}$:

$$\alpha = \{i: D_i = 1\}, \quad \beta = \{i: 0 < D_i < 1\},$$
$$\gamma = \{i: D_i = 0\}.$$

Then, the matrix $\mathbf{MD} + \mathbf{I} - \mathbf{D}$ can be simplified as

$$\begin{pmatrix} \mathbf{M}_{\alpha\alpha} & \mathbf{M}_{\alpha\beta}\mathbf{D}_\beta & \mathbf{0}_\alpha \\ \mathbf{M}_{\beta\alpha} & \mathbf{M}_{\beta\beta}\mathbf{D}_\beta + \mathbf{I}_\beta - \mathbf{D}_\beta & \mathbf{0}_\beta \\ \mathbf{M}_{\gamma\alpha} & \mathbf{M}_{\gamma\beta}\mathbf{D}_\beta & \mathbf{I}_\gamma \end{pmatrix}.$$

It is well known that any submatrix of a P-matrix is also a P-matrix, and any Schur-complement of a P-matrix is also a P-matrix. Thus, $\mathbf{M}_{\alpha\alpha}$ is nonsingular and $\mathbf{M}_{\beta\beta} - \mathbf{M}_{\beta\alpha}\mathbf{M}_{\alpha\alpha}^{-1}\mathbf{M}_{\alpha\beta}$ is a P-matrix, since $\mathbf{M}$ is a P-matrix. By Lemma 1, the whole matrix is nonsingular. □

The following result provides a sufficient condition for all $\mathbf{V} \in \partial\mathbf{h}(\mathbf{z}, u)$ to be nonsingular.

Proposition 6. *Let* $\mathbf{a} \geqslant \mathbf{0}$. *Then all* $\mathbf{V} \in \partial\mathbf{h}(\mathbf{z}, u)$ *are nonsingular at* $(\mathbf{z}, u) \in \mathbb{R}^n \times [0, 1]$ *if* $\mathbf{f}$ *is a uniform P-function over* R^n_+.

Proof. Since $\mathbf{f}$ is a uniform P-function over $\mathbb{R}^n_+$, $\nabla\mathbf{f}(\mathbf{p}(\mathbf{z}, u\mathbf{a}))$ is a P-matrix for all $\mathbf{z} \in \mathbb{R}^n$. The result then follows from the representation of $\partial\mathbf{h}(\mathbf{z}, u)$ and Lemma 2. □

The nonsingularity condition can be weakened if $\mathbf{a}$ is chosen to be strictly positive. In this case, $\mathbf{h}(\mathbf{z}, u)$ is differentiable for all $u > 0$ and the generalized Jacobian $\partial\mathbf{h}(\mathbf{z}, u)$ reduces to the regular Jacobian $\nabla\mathbf{h}(\mathbf{z}, u)$. If the plus-smooth function p has an infinite support, the Jacobian of $\mathbf{h}$ is given by

$$\nabla\mathbf{h}(\mathbf{z},u) = (1-u)\nabla\mathbf{f}(\mathbf{p}(\mathbf{z},u\mathbf{a}))\mathrm{diag}\{p'(z_i,ua_i)\} + \mathrm{diag}\{p'(-z_i,ua_i)\}.$$

The Newton corrector subproblem is obviously a system of linear equations of full dimension n. The nonsingularity condition is given as follows:

Proposition 7. *Let* $\mathbf{a} > \mathbf{0}$ *and the plus-smooth function* p *has an infinite support. Then* $\nabla\mathbf{h}(\mathbf{z},u)$ *is nonsingular at* $(\mathbf{z},u) \in \mathbb{R}^n \times (0,1]$ *if* $\mathbf{f}$ *is a* P_0-*function over* $\mathbb{R}^n_+$.

Proof. From Result (6) of Proposition 1, $0 < p'(z_i,ua_i) < 1$ for all $z_i \in R$ and $u > 0$. Since $\mathbf{f}$ is a P_0-function over $\mathbb{R}^n_+$, $\nabla\mathbf{f}(\mathbf{p}(\mathbf{z},u\mathbf{a}))$ is a P_0-matrix for all $\mathbf{z} \in \mathbb{R}^n$. It follows that $\nabla\mathbf{h}(\mathbf{z},u)$ is a P-matrix and therefore, is nonsingular. □

Now consider the case where the plus-smooth function p has a finite support and $\mathbf{a} > \mathbf{0}$. Without loss of generality, assume that the support of the plus-smooth function is on $[-1,1]$; i.e., $s = 1$. To study the structure of the Newton corrector subproblem, we define the following index sets at a given point $(\mathbf{z},u) \in R^n \times (0,1]$:

$$\alpha(\mathbf{z},u) = \{i: z_i \geqslant ua_i\},$$
$$\beta(\mathbf{z},u) = \{i: -ua_i < z_i < ua_i\},$$
$$\gamma(\mathbf{z},u) = \{i: z_i \leqslant -ua_i\}.$$

For simplicity, the argument $(\mathbf{z},u)$ of all index sets will be dropped. By definition and Result (7) of Proposition 1,

$$p'(z_i,ua_i) = 1 \quad \text{for all } i \in \alpha,$$
$$0 < p'(z_i,ua_i) < 1 \quad \text{for all } i \in \beta,$$
$$p'(z_i,ua_i) = 0 \quad \text{for all } i \in \gamma.$$

Let $\mathbf{P}'_\beta = \mathrm{diag}\{p'(z_i,ua_i), i \in \beta\}$ and denote $\nabla\mathbf{f}_{st}$ as a submatrix with elements $\partial f_i(\mathbf{p}(\mathbf{z},u\mathbf{a}))/\partial z_j$, $i \in s$ and $j \in t$, and $s,t \subseteq \alpha\cup\beta\cup\gamma$. Then, after some algebraic manipulations, we have

$$\nabla\mathbf{h}(\mathbf{z},\mathbf{u}) = (1-u)\begin{pmatrix} \nabla\mathbf{f}_{\alpha\alpha} & \nabla\mathbf{f}_{\alpha\beta}\mathbf{P}'_\beta & \mathbf{0}_\alpha \\ \nabla\mathbf{f}_{\beta\alpha} & \nabla\mathbf{f}_{\beta\beta}\mathbf{P}'_\beta + (\mathbf{I}_\beta - \mathbf{P}'_\beta)/(1-u) & \mathbf{0}_\beta \\ \nabla\mathbf{f}_{\gamma\alpha} & \nabla\mathbf{f}_{\gamma\beta}\mathbf{P}'_\beta & \mathbf{I}_\gamma/(\mathbf{1}-u) \end{pmatrix}.$$

The next result provides a condition for the Jacobian to be nonsingular.

Proposition 8. *Let* $\mathbf{a} > \mathbf{0}$ *and the plus-smooth function* p *have a finite support. Then* $\nabla\mathbf{h}(\mathbf{z},u)$ *is nonsingular at* $(\mathbf{z},u) \in \mathbb{R}^n \times (0,1]$ *if*

1. $\nabla\mathbf{f}_{\alpha\alpha}$ *is nonsingular, and*
2. *the Schur-complement*

$$\nabla\mathbf{f}_{\alpha\cup\beta,\alpha\cup\beta}/\nabla\mathbf{f}_{\alpha\alpha} = \nabla\mathbf{f}_{\beta\beta} - \nabla\mathbf{f}_{\beta\alpha}\nabla\mathbf{f}_{\alpha\alpha}^{-1}\nabla\mathbf{f}_{\alpha\beta}$$

is a $\mathbf{P}_0$-matrix.

Proof. The result follows directly from Lemma 1. □

Based on the structure of the above Jacobian, it is clear that the Newton corrector subproblem of a continuation method using a plus-smooth function with a finite support is a system of linear *equations* of *reduced* dimension $|\alpha\cup\beta|$. It compares favorably to the continuation method using a plus-smooth function with an infinite support, where linear equations of *full* dimension need to be solved as a Newton corrector subproblem. It also compares favorably to B-differentiable equation approaches as proposed by Pang [4] and Harker and Xiao [5], where a *mixed LCP* of *reduced* dimension is solved at each iteration. In addition, if a solution of the NME is nondegenerate and if a continuation method converges to the solution, the correct active set α and γ are identified in a finite number of steps.

Proposition 9. *Let* $\mathbf{z}^*$ *be a solution of the NME and that* $z_i^* \neq 0$ *for all* i. *If the plus-smooth function* p *has a finite support, then there exists a* $\bar{u} > 0$ *such that* $\mathbf{h}(\mathbf{z}^*,u\mathbf{a}) = \mathbf{0}$ *for all* $0 \leqslant u \leqslant \bar{u}$.

Proof. By Result (7) of Proposition 1, for any $z \neq 0$, there exist a $\bar{u} > 0$ such that $p(z,u) = z_+$ and $p(-z,u) = -z_-$ for all $0 \leqslant u \leqslant \bar{u}$. The result then follows directly. □

As a result, a continuation method using a plus-smooth function with a finite support converges to a nondegenerate solution of the NME in finite steps (if the Newton corrector is used). More detailed comparisons with these two methods are summarized in the following table.

	Continuation method with $s=\infty$	Continuation method with $s<\infty$	B-differentiable method
Subproblems	Linear equations	Linear equations	Mixed LCP
Dimension of subproblems	Same as **z**	Reduced size	Reduced size
Assumption on solvability	$\nabla \mathbf{f}$ P_0-matrix	$\mathbf{f}_{\alpha\alpha}$ nonsingular, $\nabla \mathbf{f}_{\alpha\cup\beta,\alpha\cup\beta}/\nabla \mathbf{f}_{\alpha\alpha}$ P_0-matrix	$\mathbf{f}_{\alpha'\alpha'}$ nonsingular, $\nabla \mathbf{f}_{\alpha'\cup\beta',\alpha'\cup\beta'}/\nabla \mathbf{f}_{\alpha}$ P-matrix
Global convergence	With probability 1	With probability 1	Regularity assumptions
Convergence for LCP	Infinite sequence	Finite for nondegenerate solution	Finite

In the above table, $\alpha'(\mathbf{z}) = \alpha(\mathbf{z}, 0)$ and $\beta'(\mathbf{z}) = \beta(\mathbf{z}, 0)$ for all $\mathbf{z} \in \mathbb{R}^n$.

5. Proposed algorithm and computational results

In this section, we describe a simple version of a continuation method for solving the SNME and describe preliminary computational tests of the method. We used (a) the interior point plus-smooth function given in Example 2, and (b) the uniform plus-smooth function given in Example 4, both in Section 2. The algorithm monotonically reduces u at each iteration (predictor step) and uses a damped Newton method as a corrector step. A nonmonotone line search procedure [14] is chosen instead of the traditional Armijo line search for use in this corrector step. Our experience indicates that the nonmonotone line search may significantly reduce the number of function evaluations and improve convergence. For each corrector step, define the merit function $\theta(\mathbf{z}, u) \equiv \|\mathbf{h}(\mathbf{z}, u)\|_2^2$. The nonmonotone line search consists of finding the smallest $i^k = 0, 1, 2, \ldots$ and thus, $\alpha^k = 2^{-i^k}$, such that

$$\theta(\mathbf{z}^k + \alpha^k \mathbf{d}^k, u^k) \leqslant \mathscr{W} + \beta\alpha^k \nabla\theta(z^k, u^k)^{\mathrm{T}} \mathbf{d}^k, \tag{8}$$

where $\mathscr{W}$ is any value satisfying

$$\theta(\mathbf{z}^k, u^k) \leqslant \mathscr{W} \leqslant \max_{j=0,1,\ldots,M^k} \theta(\mathbf{z}^{k-j}, u^{k-j}), \tag{9}$$

and M^k is a nonnegative integer bounded above for any k. We assume that k is large enough so as to guarantee the occurrence of positive indices in (9). Our implementation of the nonmonotone search is as follows:

1. Set $\mathscr{W} = \theta(\mathbf{z}^0, u^0)$ at the beginning of the algorithm,
2. keep the value of $\mathscr{W}$ fixed as long as

$$\theta(\mathbf{z}^k, u^k) \leqslant \min_{j=0,1,\ldots,5} \theta(\mathbf{z}^{k-j}, u^{k-j}), \tag{10}$$

3. If (10) is not satisfied at iteration k, set $\mathscr{W} = \theta(\mathbf{z}^k, u^k)$.

We refer the reader to [14,15] for a more detailed description of the nonmonotone line search. The continuation method is now described in detail:

Continuation method:

Step 0: Set $k = 0$. Choose $\mathbf{z}^k$, u^k, and $\mathbf{a}, \mathbf{b} > \mathbf{0}$.

Step 1: If termination criteria are satisfied, stop.

Step 2: If $\nabla\mathbf{h}(\mathbf{z}^k, u^k)$ is nonsingular, solve for $\mathbf{d^k}$ in

$$\mathbf{h}(\mathbf{z}^k, u^k) + \nabla\mathbf{h}(\mathbf{z}^k, u^k)\mathbf{d}^k = \mathbf{0}.$$

Otherwise, let $\mathbf{d}^k = -\mathbf{h}(\mathbf{z}^k, u^k)$.

Step 3: Compute a step length α^k using the nonmonotone line search (8)–(9).

Step 4: Set

$$\mathbf{z}^{k+1} = \mathbf{z}^k + \alpha^k \mathbf{d}^k,$$

$$u^{k+1} = u^k \beta,$$

$$k \leftarrow k + 1.$$

Go to Step 1.

In the implementation of the above algorithm, we choose $\beta \in (0, 1)$, $\mathbf{a} = \mathbf{1}$ and $\mathbf{b} = \mathbf{1}$. For comparison with other methods, the stopping criterion in Step 1 of the algorithm is implemented as follows:

1. Set $\mathbf{x}^k = \mathbf{p}(\mathbf{z}^k, u^k)$,
2. Check whether

$$\|[-\mathbf{x}^k, -\mathbf{f}(\mathbf{x}^k), \mathbf{vec}\{x_i^k f_i(\mathbf{x}^k)\}]_+\| \leqslant 10^{-6},$$

where the $_+$ operator is applied componentwise, and $\mathbf{vec}\{x_i^k f_i(\mathbf{x}^k)\} \in \mathbb{R}^n$ with components $x_i^k f_i(\mathbf{x}^k)$ for all $i = 1, \ldots, n$.

It is well known that the above norm is an error bound for several classes of NCPs, and therefore can be used as a termination criterion. The algorithm was implemented in Fortran 77 and tested on a 60 MHz Micro SPARC with 1 GB main memory running Unix. The implementation exploits sparsity of the matrices $\nabla \mathbf{h}$ by using a sparse linear system solver from the Harwell Subroutine Library. We report the results of two experiments below. The first experiment was conducted using a uniform reduction scheme for u. That is, we reduce u by the same factor after each nonmonotone search. The second experiment consists of using a hybrid strategy: reduce u slowly at the beginning, then reduce u rapidly as the algorithm progresses. We have used twenty-seven test problems, from De Luca et al. [15], and Ferris and Rutherford [16] as the basis for these tests. Next, we describe briefly the test problem characteristics.

5.1. Test problems

Our brief descriptions of test problems have been summarized from the papers [15] and [16]. The interested reader is referred to these sources for additional information.

Kojima–Shindo: This is a 4-variable problem. F is not a P_0-function. The starting points are: (a) $(0,0,0,0)$, (b) $(1,1,1,1)$.

Kojima–Josephy: This is a 4-variable problem. F is not P_0. The starting points are: (a) $(0,0,0,0)$, (b) $(1,1,1,1)$.

HS34: This problem represents the Karush–Kuhn–Tucker first-order optimality (KKT) conditions of a nonlinear program. F is monotone on the positive orthant but not even P_0 on R^n; $n=8$. The starting point is (a) $(0,1.05,2.9,0,0,0,0,0)$.

HS35: This problem represents the KKT conditions for the 35th problem in [17]. F is linear and monotone but not strictly monotone; $n=4$. The starting point is (a) $(0.5,0.5,0.5,0)$.

HS66: This problem represents the KKT conditions for the 66th problem in [17]. F is monotone on the positive orthant but not even P_0 on $\mathbb{R}^n$; $n=8$. The starting point is (a) $(0,1.05,2.9,0,0,0,0,0)$.

HS76: This problem represents the KKT conditions for the 76th problem in [17]. F is linear and monotone but not strictly monotone. The dimension is 7. Starting point is (a) $(0.5,0.5,0.5,0.5,0,0,0)$.

Watson3: This is a linear complementarity problem with $F(x)=Mx+q$. M is not even semimonotone. This is known to be a hard problem. In fact, De Luca et al. [15] indicate that none of the standard algebraic techniques can solve this problem. We choose $q=(-1,0,\dots,0)$ as in [15]; $n=10$. The starting point is (a) $(0,0,\dots,0)$.

Watson4: This problem represents the KKT conditions for a convex programming problem involving exponentials. F is monotone on the positive orthant but not even P_0 on $\mathbb{R}^n$; $n=5$. The starting point is (a) $(0,0,\dots,0)$.

Nash–Cournot: F is not twice continuously differentiable. F is a P-function on the strictly positive orthant; $n=10$. Starting points are (a) $(10,10,\dots,10)$, (b) $(1,1,\dots,1)$.

Modified Mathiesen: F is not defined everywhere and does not belong to any known class of functions; $n=4$. Starting points are (a) $(1,1,1,1)$, (b) $(10,10,10,10)$.

Spatial: This is a spatial equilibrium model. F is a P-function; $n=42$. Starting points are (a) $(0,0,\dots,0)$, (b) $(1,1,\dots,1)$.

Traffic: This is a traffic equilibrium problem; $n=50$. The starting point is (a) All the components are 0 except z_1, z_2, z_3, z_{10}, z_{11}, z_{20}, z_{21}, z_{22}, z_{29}, z_{30}, z_{40}, z_{45} which are 1, z_{39}, z_{42}, z_{43}, z_{46} which are 7, z_{41}, z_{47}, z_{48}, z_{50} which are 6, z_{44} and z_{49} which are 10.

Bertsekas: Bertsekas problem posed in the form of NCP; $n=15$. Starting point is (a) $(1,1,\dots,1)$.

Colvdual: Dual of Colville nonlinear programming example posed in the form of NCP; $n=20$. Starting point is (a) $(1,1,\dots,1)$.

Colvncp: Colville nonlinear programming example posed in the form of NCP; $n=15$. Starting point is (a) $(0,0,\dots,0)$.

Ehl-kost: Elasto-hydrodynamic equilibrium problem; $n=101$. Starting points are (a) $(0,0,\dots,0)$, and (b) $(1,1,\dots,1)$.

Explcp: A sample linear complementarity problem; $n=16$. Starting points are (a) $(1,1,\dots,1)$, and (b) $(0,0,\dots,0)$.

Gafni: This problem is based on a traffic assignment problem; $n=5$. Starting point is (a) $(1,1,\dots,1)$.

Hanskoop: Hansen–Koopmans invariant capital stock problem; $n=14$. Starting point is (a) $(1,1,\dots,1)$.

Hydroc06: Distillation problem; $n=29$. Starting point is (a) $(1,1,\dots,1)$.

Hydroc20: Distillation problem; $n=99$. Starting point is (a) $(1,1,\dots,1)$.

Methan08: Distillation problem; $n=31$. Starting point is (a) $(1,1,\dots,1)$.

Pies: Energy equilibrium problem; $n=42$. Starting point is (a) $(0,0,\dots,0)$.

Powell: Powell's nonlinear programming example posed in the form of NCP; $n=16$. Starting point is (a) $(1,1,\dots,1)$.

Scarfanum: Walrasian problem; $n=14$. Starting point is (a) $(0,0,\dots,0)$.

Scarfbnum: Walrasian problem; $n=40$. Starting points are (a) $(0,0,\dots,0)$, and (b) $(1,1,\dots,1)$.

Sppe: Spatial price equilibrium example; $n=30$. Starting points are (a) $(0,0,\dots,0)$, and (b) $(1,1,\dots,1)$.

Table 1
Results using the interior point plus-smooth function and the uniform plus-smooth function

Problem	IPPS						UPS(1)						UPS(2)
	u^0	RS	SP	IT	FE	CPU	u^0	RS	SP	IT	FE	CPU	CPU
Kojima–Shindo	1	1	*a*	14	19	0.04	1	1	*a*	8	13	0.02	0.02
	10	1	*b*	15	17	0.04	1	1	*b*	9	14	0.02	0.03
Kojima–Josephy	10	1	*b*	9	15	0.02	1	1	*b*	8	12	0.02	0.02
HS34	1	1	*a*	10	31	0.04	1	2	*a*	23	27	0.06	0.05
HS35	1	1	*a*	8	9	0.02	1	1	*a*	8	10	0.02	0.02
HS66	10	1	*a*	9	24	0.03	1	2	*a*	23	24	0.07	0.05
HS76	10	1	*a*	9	10	0.03	1	1	*a*	8	13	0.02	0.02
Watson3	1	2	*a*	22	30	0.10	10	2	*a*	26	35	0.10	0.08
Watson4	1	2	*a*	39	39	0.09	10	2	*a*	26	26	0.06	0.06
Nash–Cournot	1	1	*a*	9	15	0.05	10	1	*a*	9	10	0.04	0.04
	10	1	*a*	10	13	0.05	10	1	*b*	11	14	0.05	0.05
Mod. Mathiesen	1	2	*a*	22	24	0.05	1	2	*a*	2	5	0.01	0.01
	1	1	*b*	8	19	0.02	10	2	*b*	3	10	0.01	0.01
Spatial eq.	10	2	*a*	27	36	0.66	10	2	*a*	27	58	0.45	0.34
	1	2	*b*	24	31	0.55	10	1	*b*	18	30	0.29	0.21
Traffic eq.	1	1	*a*	15	27	0.44	10	2	*a*	28	87	0.54	0.43
Bertsekas	10	1	*a*	23	56	0.38	1	1	*a*	27	113	0.50	0.41
Colvdual	1	1	*a*	19	83	0.18	10	1	*a*	18	94	0.16	0.11
Colvncp	10	2	*a*	20	139	0.16	10	2	*a*	26	148	0.24	0.17
Ehl-kost	10	2	*a*	42	53	0.56	10	1	*a*	38	82	0.58	0.39
Explcp	10	2	*a*	27	73	0.22	10	2	*a*	37	170	0.24	0.14
	1	1	*b*	8	11	0.07	1	1	*a*	9	89	0.06	0.03
Gafni	1	1	*a*	10	14	0.07	10	2	*a*	9	12	0.07	0.06
Hanskoop	10	2	*a*	38	94	0.58	10	1	*a*	34	102	0.63	0.49
Hydroc06	1	1	*a*	13	23	0.26	1	1	*a*	11	21	0.21	0.17
Hydroc20	10	1	*a*	24	52	0.92	10	1	*a*	18	94	0.91	0.72
Methan08	10	2	*a*	12	22	0.27	1	1	*a*	14	18	0.33	0.26
Pies	10	1	*a*	17	174	0.42	10	2	*a*	14	41	0.43	0.30
Powell	10	2	*a*	16	22	0.23	1	1	*a*	11	31	0.16	0.14
Scarfanum	1	1	*a*	18	45	0.42	1	2	*a*	20	71	0.53	0.33
Scarfbnum	1	1	*a*	23	61	0.53	10	1	*b*	24	95	0.63	0.41
Sppe	10	1	*a*	12	102	0.15	10	1	*a*	12	35	0.12	0.07
	10	1	*b*	9	45	0.11	1	2	*b*	23	75	0.23	0.13

5.2. Experiment 1: Uniform reduction of u

We summarize below in Table 1 our experiments with the interior point plus-smooth function (with infinite support) and the uniform plus-smooth function (with finite support), respectively. The columns under the heading IPPS refer to the interior point plus-smooth function while the columns under UPS refer to the uniform plus-smooth function. For the latter, we use UPS(1) to mean the implementation which does not exploit the block structure of the Jacobian matrices, and UPS(2) the version of the algorithm which takes advantage of the block structure in solving a smaller linear system of equations of dimensions $|\alpha \cup \beta|$. The other column headings are as follows: u^0 denotes the starting value of u, RS refers to the u reduction strategy, SP stands for the starting point in reference to the previous section, IT stands for the number of iterations to reach a solution, and FE represents the number of function evaluations. The run times are given in seconds under the heading CPU. We use two alternative strategies for reducing u:

1. Reduce u by $u \leftarrow u/10$
2. Reduce u by $u \leftarrow u/2$.

These reduction strategies were adopted as the simplest and most robust strategies after some preliminary experimentation with the continuation algorithm. The results reported in Table 1 are based on the best combination of starting point and u-reduction strategies in terms of the resulting performance of the continuation algorithm.

Based on Table 1, the version of the continuation algorithm which uses the interior point plus-smooth function is slightly superior to the version using the uniform plus-smooth function when the best run time performances of the respective codes are compared. In particular, the total run time for IPPS using the minimum times is found to 6.59 seconds, while this statistic is equal to 6.81 seconds for UPS(1), and 5.09 for UPS(2). Comparing the results of UPS(1) and UPS(2), we observe that as expected, the exploitation of the block structure in the Newton system leads to a reduction in the run time even for this set of small test problems. This result shows that the version of the continuation algorithm based on the uniform smooth-plus function is competitive with the version based on the interior point function when a structure exploiting implementation is made.

5.3. Experiment 2: A hybrid reduction scheme for u

We observed in the experiments reported above that some test problems require that u be reduced slowly at the start of the algorithm. However, this slow reduction of u seems to slow down the convergence rate for these problems. Therefore, we tested a hybrid u-reduction scheme where u is reduced slowly at the beginning ($u \leftarrow u/2$) until the complementarity error is under a certain threshold. Once the complementarity error is under the threshold, u is reduced rapidly. We have tested two alternatives at the faster reduction phase:

Strategy 1 $u \leftarrow u/100$,
Strategy 2 $u \leftarrow u/10$.

In Tables 2 and 3, we report the results of this experiment on problems where some improvements were realized. The pair of numbers under the column heading RS refer to the u reduction strategy and the threshold value of the error at which we start to reduce faster, respectively. These threshold values were found after some preliminary experimentation. We report the best computational results we obtained in this experiment. These tests were made in MATLAB. Therefore, we do not report CPU times.We observe that the

Table 2
Results using the interior plus-smooth function and a hybrid u-reduction scheme

Problem	u^0	RS	SP	IT	FE
Mod. Math.	1	(1,1)	*a*	5	7
Watson4	1	(1,10)	*a*	19	19
Nash–Cournot	1	(1,100)	*a*	9	15
Watson3	1	(1,1)	*a*	7	13
Spatial eq.	10	(2,10)	*a*	24	33

Table 3
Results using the uniform plus-smooth function and a hybrid u-reduction scheme

Problem	u^0	RS	SP	IT	FE
Watson4	1	(2,100)	*a*	18	18
Watson3	1	(1,1)	*a*	13	22
Spatial eq.	10	(2,10)	*b*	29	47

gains realized with a hybrid reduction strategy are substantial for problems Watson3, Watson4 and Modified Mathiesen.

All the experiments summarized above were conducted using the starting points given in [15]. Since the algorithm of [15] and the continuation algorithm of this paper are substantially different, more favorable starting points for the continuation method may be sought along with more sophisticated choice mechanisms for u^0. These issues will be studied in the future.

5.4. Comparison to the semismooth equations method

In this section, we reproduce the results of the paper by De Luca, Facchinei and Kanzow [15] where a semismooth equation based approach is developed and tested on the first twelve problems of our test set. The purpose of this section is to compare the two approaches on basis of the number of iterations and the number of function evaluations.

Comparing the above results to the results of Tables 1–4 we find that the normal map approach taken in this paper can be competitive with the semismooth equations approach. However, the results are definitely mixed; some problems favor the approach of the present paper, and some the semismooth equation method in [15]. The conclusion based on the above tests is that the continuation algorithm of this paper seems to be a viable computational alternative to existing algorithms for complementarity problems.

Table 4
Results using the semismooth equations in [15]

Problem	SP	IT	FE
Kojima–Shindo	*a*	13	14
	b	7	12
Kojima–Josephy	*a*	20	26
	b	7	11
HS34	*a*	14	36
HS35	*a*	5	6
HS66	*a*	6	8
HS76	*a*	6	7
Watson3	*a*	7	12
Watson4	*a*	21	22
Nash–Cournot	*a*	7	8
	b	10	11
Mod. Mathiesen	*a*	4	5
	b	5	6
Spatial eq.	*a*	20	23
	b	22	23
Traffic eq.	*a*	13	14

6. Summary and future research

This paper provides, through the use of density functions with finite support, a unification of the nonsmooth and smooth equation approaches for solving complementarity problems using the normal map formulation that has appeared in the literature in recent years. With such density functions, computational methods are developed that have the advantage of solving smooth equations while at the same time being able to solve subproblems of reduced size, as in the B-differentiable approaches. In addition, the computational results reported above show that this method can be competitive, in terms of iterations and function evaluations, with other known solution methods. For large-scale problems, the ability to reduce the size of the subproblems should prove to be a major computational advantage. Future research will be devoted to the testing of the computational performance of this method on large-scale problems in order to ascertain the practical impact of solving smaller subproblems at each iteration.

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