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On connected Boolean functions

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Abstract

A Boolean function is called (co-)connected if the subgraph of the Boolean hypercube induced by its (false) true points is connected; it is called strongly connected if it is both connected and co-connected. The concept of (co-)geodetic Boolean functions is defined in a similar way by requiring that at least one of the shortest paths connecting two (false) true points should consist only of (false) true points. This concept is further strengthened to that of convexity where every shortest path connecting two points of the same kind should consist of points of the same kind. This paper studies the relationships between these properties and the DNF representations of the associated Boolean functions. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A Boolean function can be viewed as a partition of the vertex set of the Boolen hypercube into a subset of “true” points and a subset of “false” points. This paper studies classes of Boolean functions for which the induced subgraph of true points and/or the induced subgraph of false points possess some special connectivity properties. For each of the classes studied in this paper, we shall concentrate our efforts on their recognition, and on their structural description.

We shall generally assume the knowledge of disjunctive normal form (DNF) representations of the functions studied, although in some of the cases we also assume the knowledge of their prime implicants.

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We shall first study Boolean functions for which the subgraphs induced by their true points, or their false points, or both, are connected. We shall call these functions *connected*, *co-connected* and *strongly connected*, respectively. In the following part of the paper, we strengthen these conditions by requiring that at least one of the shortest paths connecting two true (false) points belongs to the subgraph induced by the true (false) points; such Boolean functions are called *geodetic* (*co-geodetic*). The strongest connectivity condition studied in this paper, *convexity* (*co-convexity*), requires that every shortest path connecting any pair of true (false) points belong to the subgraph induced by true (false) points. The definitions of *strongly geodetic* and *strongly convex* functions are analogous to that of strongly connected ones.

Boolean functions having connectivity properties similar to those described above play an important role in problems appearing in various areas including in particular discrete optimization, machine learning, automated reasoning, etc.

After a brief introduction of basic concepts and notations, we study in Section 3 connected Boolean functions. This section starts with a simple graph theoretic characterization of functions whose set of true points induces a connected subgraph. This characterization uses the knowledge of all the prime implicants and is followed by a second one using an arbitrary DNF representation of the function. The section concludes by showing that recognizing the connectedness of the set of false points is CoNP-hard.

Section 4 studies various types of geodetic functions. After showing that their recognition is CoNP-hard, we show that the recognition of those functions whose true points are geodetically connected can be reduced to the solution of a satisfiability problem. The classes of monotonically non-decreasing (positive), monotonically non-increasing (negative), and more generally unate functions are all shown to have the property that both their true and false points are geodetically connected.

We introduce the class of *concordant* Boolean functions, generalizing the unate ones and having the property that any two of their prime implicants "conflict" in at most one variable. It is shown that concordant DNFs represent geodetic functions. Moreover, concordant functions possess the strong geodetic property that both the true and the false points are geodetically connected.

Section 5 deals with *k-convex* functions, i.e. functions with the property that all the shortest paths linking two true points at Hamming distance at most k consist only of true points; *co-k-convex* and *strongly-k-convex* functions are defined in a similar way. The first part deals with the characterization of k-convex functions and studies convexity properties of connected functions, while the second part examines problems related to their recognition.

After establishing a set of functional inequalities characterizing k-convexity, we show that the set of true points of a k-convex function in the Boolean hypercube consists of subcubes of true points at "large" Hamming distances from each other. We define for every $k = 2,\dots,n$, the *k-convex hull* of an arbitrary Boolean function as the (unique) minimal k-convex majorant of the function, and provide a polynomial algorithm for its construction.

We continue by characterizing strongly convex functions and show that any strongly convex function effectively depends on at most one variable and that consequently the number of strongly convex functions on n variables is at most $2n+2$.

We later examine convexity properties of connected Boolean functions and show that there are only two kinds of connected functions: those which are k-convex for every $k=2,\dots,n$ and those which are not k-convex for any k; analogous properties are established for co-connected and strongly connected functions.

The second part of Section 5 studies the recognition problem of various convexity properties and shows that in general these problems are computationally difficult. However, we show that for classes of functions for which a family of associated satisfiability problems can all be solved polynomially, the recognition problems become tractable; this class includes Horn, quadratic, renamable Horn and q-Horn functions.

2. Basic concepts

We assume that the reader is familiar with the basic concepts of Boolean algebra, and we only introduce here the notions that we explicitly use in this paper.

A *Boolean function* f of n variables $x_1,\dots,x_n$ is a mapping $\{0,1\}^n \to \{0,1\} = B$, where B^n is commonly referred to as the *Boolean hypercube*. The variables $x_1,\dots,x_n$ and their complements $\bar{x}_1,\dots,\bar{x}_n$ are called *literals*. We shall sometimes denote x by x^1, and $\bar{x}$ by x^0. For two Boolean functions f and g we write $f \leqslant g$ iff for every 0–1 vector $\boldsymbol{x}$, $f(x_1,\dots,x_n)=1$ implies $g(x_1,\dots,x_n)=1$.

A Boolean function $f(x_1,\dots,x_n)$ *depends effectively* on variable x_i iff there exist values x_j^* $(j=1,\dots,i-1,i+1,\dots,n)$ for which

$$f(x_1^*,\dots,x_{i-1}^*,x_i,x_{i+1}^*,\dots,x_n^*) \neq f(x_1^*,\dots,x_{i-1}^*,\bar{x}_i,x_{i+1}^*,\dots,x_n^*).$$

The *dual* of a Boolean function $f(\boldsymbol{x})$ is defined as

$$f^d(\boldsymbol{x}) = \bar{f}(\bar{\boldsymbol{x}}),$$

where $\bar{\boldsymbol{x}} = (\bar{x}_1,\bar{x}_2,\dots,\bar{x}_n)$ is the *complement* of $\boldsymbol{x}$, and where $\bar{f}(\boldsymbol{y})=1$ if and only if $f(\boldsymbol{y})=0$ for each 0–1 point $\boldsymbol{y}$.

The *true* (*false*) *set* of a function f, denoted by $\mathscr{T}_f$ $(\mathscr{F}_f)$, is the collection of the true (false) points of f, i.e.

$$\mathscr{T}_f = \{\boldsymbol{x} \in \{0,1\}^n : f(\boldsymbol{x})=1\} \quad \text{and} \quad \mathscr{F}_f = \{\boldsymbol{x} \in \{0,1\}^n : f(\boldsymbol{x})=0\}.$$

A *term*, or an *elementary conjunction*, is a constant or a conjunction of literals of the form

$$\prod_{i\in P} x_i \prod_{i\in N} \bar{x}_i,$$

where P and N are disjoint subsets of $\{1,\dots,n\}$. The *degree* of a term is the number of literals in it. We shall say that a term T *absorbs* another term T', iff $T \vee T' = T$, i.e. iff $T \geqslant T'$ (e.g. the term $x\bar{y}$ absorbs the term $x\bar{y}z$). A term T *covers* a 0–1 point $\boldsymbol{x}^*$

iff $T(\mathbf{x}^*)=1$. A term T is called an *implicant* of a function f iff $T \leqslant f$. An implicant T of a function is called *prime* iff there is no distinct implicant T' absorbing T.

A *disjunctive normal form* (DNF) is a disjunction of terms. It is well known that every Boolean function can be represented by a DNF, and that this representation may not be unique. A DNF representing a function f is called *prime* iff each term of the DNF is a prime implicant of the function. On the other hand, a DNF representing a function is called *irredundant* iff eliminating any one of the terms results in a DNF which does not represent the same function. The *length* of a DNF is the number of literals in it.

Two terms are said to *conflict* in the variable x_i if x_i is a literal in one of them and $\bar{x}_i$ is a literal in the other. If two terms conflict in exactly one variable, i.e., they have the form x_iP and $\bar{x}_iQ$ and the elementary conjunctions P and Q have no conflict, then their *consensus* is defined to be the term PQ. The *consensus method* applied to an arbitrary DNF Φ performs the following operations as many times as possible:

1. *Consensus*: If there exist two terms of Φ having a consensus T which is not absorbed by any term of Φ then replace the DNF Φ by the DNF $\Phi \vee T$;
2. *Absorption*: if a term T of Φ absorbs a term T' of Φ, delete T'.

It is easy to notice that all the DNFs produced at every step of the consensus method represent the same function as the original DNF. The following result plays a central role in the theory and applications of Boolean functions [1,7]:

Proposition 2.1 (Blake [1], Quine [7]). *The consensus method applied to an arbitrary DNF of a Boolean function f results in the DNF which is the disjunction of all the prime implicants of f.*

A classical hard problem concerning Boolean formulae is the *satisfiability problem* (SAT) (see e.g. [4]). When working with DNFs, this problem can be formulated as follows: given as input a DNF Φ, is there an assignment *satisfying* Φ, i.e. is there a point $\mathbf{x}^* \in \{0,1\}^n$ such that $\Phi(\mathbf{x}^*)=0$?

Throughout the text, the following notation will be used to represent terms:

Definition 2.2. If $S=\{i_1,\ldots,i_{|S|}\}\subseteq\{1,\ldots,n\}$, and $\boldsymbol{\alpha}_S=(\alpha_{i_1},\ldots,\alpha_{i_{|S|}})\in\{0,1\}^{|S|}$ is an "assignment" of 0–1 values to the variables x_i $(i \in S)$, then the term $X^{\boldsymbol{\alpha}_S}$ *associated to* $\boldsymbol{\alpha}_S$ is the conjunction $\prod_{i\in S} x_i^{\alpha_i}$; if $S=\emptyset$, we define $X^{\emptyset}=1$. Similarly, $X_S^{\bar{\boldsymbol{\alpha}}}$ is the term associated to the assignment $\bar{\boldsymbol{\alpha}}_S$.

For example, if $S=\{1,3,5\}$ and $\alpha_1=1, \alpha_3=1, \alpha_5=0$, then the term corresponding to the above assignment is $x_1x_3\bar{x}_5$.

We shall frequently use in this paper the concepts of *restriction* and *projection* of a DNF; the definitions are given below.

Definition 2.3. The *restriction* of $\Phi(x_1,\ldots,x_n)$ to $S\subseteq\{1,\ldots,n\}$ is the DNF Φ^S obtained by eliminating from Φ all the variables x_i with $i \notin S$. If all the variables of a term of Φ are not in S, then the restriction is the constant 1. For completeness, the restriction to the empty set is defined to be 1 and any restriction of a constant is the constant itself.

Definition 2.4. Let $S=\{i_1,\dots,i_{|S|}\}\subseteq\{1,\dots,n\}$ and let $\boldsymbol{\alpha}=(\alpha_{i_1},\dots,\alpha_{i_{|S|}})\in\{0,1\}^{|S|}$. The *projection* of a DNF $\Phi(x_1,\dots,x_n)$ on $(S,\boldsymbol{\alpha})$ is the DNF $\Phi_{(S,\boldsymbol{\alpha})}$ obtained from Φ by the substitutions $x_i=\alpha_i$ for all $i\in S$.

For example, the projection of $x_1x_2\vee\bar{x}_2x_3$ on $(S,\boldsymbol{\alpha})$ where $S=\{2\}$ and $\boldsymbol{\alpha}=(0)$ is the DNF x_3, while the restriction of the same DNF to $\overline{S}=\{1,3\}$ is the DNF $x_1\vee x_3$. The relationship between the previous two DNFs can be generalized. Let $\overline{S}=\{1,\dots,n\}\backslash S$. Then, for any DNF Φ,

$$\Phi_{(S,\boldsymbol{\alpha})}\leqslant\Phi^{\overline{S}}$$

for any $\boldsymbol{\alpha}\in\{0,1\}^{|S|}$.

We shall make use later in the paper of two classes of DNFs for which a family of SAT problems can be solved polynomially.

Definition 2.5. A class of DNFs has the *extended SAT tractability* property if there exists a polynomial time SAT solving algorithm which can decide whether $\Phi'=0$ has a solution for any projection Φ' of a DNF in this class.

Definition 2.6. A class of DNFs has the *strong SAT tractability* property if there exists a polynomial time algorithm which can decide whether $\Phi'=0$ has a solution for any DNF Φ' obtained from a DNF in this class by repeated applications of projections and restrictions.

Some well-known types of DNFs having both of the above properties are the quadratic, Horn, renamable Horn and q-Horn DNFs (see [2,3] for definitions).

Throughout this paper, we shall use the *Hamming metric* which defines the distance between two Boolean vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ as the number of components in which they differ:

$$d(\boldsymbol{x},\boldsymbol{y})=|\{i\colon x_i\neq y_i \quad i\in[1\dots n]\}|.$$

Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are called *neighbors* iff $d(\boldsymbol{x},\boldsymbol{y})=1$. The point $\boldsymbol{y}$ is *between* $\boldsymbol{x}$ and $\boldsymbol{z}$ iff $d(\boldsymbol{x},\boldsymbol{y})+d(\boldsymbol{y},\boldsymbol{z})=d(\boldsymbol{x},\boldsymbol{z})$. A sequence of points $\boldsymbol{x}_1,\dots,\boldsymbol{x}_k$ is called a *path* from $\boldsymbol{x}_1$ to $\boldsymbol{x}_k$ iff any two consecutive points in this sequence are neighbors. A *shortest path* between $\boldsymbol{x}$ and $\boldsymbol{y}$ is a path of length $d(\boldsymbol{x},\boldsymbol{y})$. A *true* (*false*) path is a path consisting only of true (false) points of a Boolean function.

We say that two true (false) points are

- *connected* iff there exists a true (false) path linking them;
- *geodetically connected* iff there exists a shortest true (false) path linking them;
- *convexly connected* iff all the shortest paths connecting them are true (false).

In this paper we shall examine functions in which the sets of true and false points satisfy certain connectedness conditions. More precisely, we shall analyze the classes of *convex*, *geodetic* and *connected* functions in which every pair of true (false) points is, respectively, convexly connected, geodetically connected or just connected. Each of the following sections will be devoted to one of these classes.

Definition 2.7. The *membership problem* for a class of Boolean functions consists in deciding whether a Boolean function given as a DNF does or does not belong to the class.

Hegedüs and Megiddo [6] investigate the complexity of the membership problem for very general classes of Boolean functions having the so called "projection property".

Definition 2.8. A class of Boolean functions $\mathscr{C}$ has the *projection property* if and only if

1. $\mathscr{C}$ is *closed under projection*, i.e. for any function $f \in \mathscr{C}$, fixing some variables of f to 0 or 1 results in a function in $\mathscr{C}$;
2. the constant 1 function belongs to $\mathscr{C}$;
3. there exists at least one function not in $\mathscr{C}$.

The following result was proven in [6] and will be used in this paper.

Theorem 2.9. *If $\mathscr{C}$ is any class of Boolean functions having the projection property, then the membership problem for $\mathscr{C}$ is CoNP-hard.*

3. Connected functions

In this section we shall study Boolean functions whose true points induce a connected subgraph of the hypercube, and prove that if DNF representations of the functions are known, then these functions can be recognized in polynomial time. On the other hand, we shall show that the recognition of those Boolean functions whose false points induce connected subgraphs is CoNP-hard; a similar statement is proven for recognizing functions whose sets of true and false points both induce connected subgraphs.

Definition 3.1. A Boolean function f is called

- *connected* iff any pair of true points is connected;
- *co-connected* iff any pair of false points is connected, i.e. $\bar{f}$ is connected;
- *strongly connected* iff it is both connected and co-connected.

A useful concept is that of the *conflict graph* G_Φ of a DNF Φ. The vertices of G_Φ are the terms of Φ, and the edges link pairs of conflicting terms. Since a Boolean

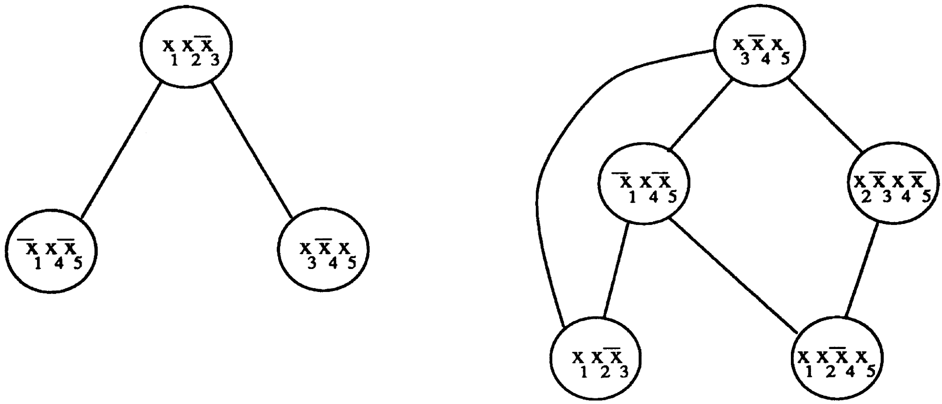

Fig. 1. G^*_Φ and G_f.

function may have many DNF representations, the concept of a conflict graph of a function needs to be clarified. We shall define here the *conflict graph* G_f of a Boolean function f as being the conflict graph associated with that special DNF which consists of the disjunction of all the prime implicants of f. The complement of the conflict graph of f is denoted by $\overline{G}_f$.

In this paper, we shall also consider another graph associated to a DNF Φ, called the *concordance graph* G^*_Φ. Its vertices correspond to the terms of Φ, and there exists an edge between two terms iff they conflict in at most one literal. It is easy to observe that for any DNF Φ, the complement of G_Φ is a subgraph of G^*_Φ.

Example 3.2. Consider the DNF $\Phi = x_1x_2\bar{x}_3 \vee \bar{x}_1x_4\bar{x}_5 \vee x_3\bar{x}_4x_5$ representing the function f. Applying the consensus method to this DNF we obtain a representaion of the function consisting of all its prime implicants:

$$f = x_1x_2\bar{x}_3 \vee \bar{x}_1x_4\bar{x}_5 \vee x_3\bar{x}_4x_5 \vee x_2\bar{x}_3x_4\bar{x}_5 \vee x_1x_2\bar{x}_4x_5.$$

The concordance graph G^*_Φ and the conflict graph G_f are shown in Fig. 1, respectively.

Theorem 3.3. *A Boolean function f is connected if and only if the complement of the conflict graph, $\overline{G}_f$ is connected.*

Proof. We shall first show that if f is connected then $\overline{G}_f$ is connected. Let P and Q be two arbitrary prime implicants of f. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be any two true points covered by P and Q respectively. Since f is connected, there exists a true path between $\boldsymbol{p} = \boldsymbol{t}_0$ and $\boldsymbol{q} = \boldsymbol{t}_{k+1}$, say $(\boldsymbol{p}, \boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_k, \boldsymbol{q})$. Let $\boldsymbol{t}_{i_1}$ be the last true point on this path which is covered by P. Let T_1 be a prime implicant covering $\boldsymbol{t}_{i_1}$ and $\boldsymbol{t}_{i_1+1}$. Now, P and T_1 cannot be in conflict since they both cover the true point $\boldsymbol{t}_{i_1}$. Hence, there exists in $\overline{G}_f$ an edge between P and T_1. Continuing inductively in this fashion leads to a path between P and Q.

Conversely, we shall show that if the complement of a function's conflict graph is connected, then the function is connected. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be two arbitrary true points covered by prime implicants P and Q respectively, and let $(P, T_1, \ldots, T_k, Q)$ be a path in $\overline{G}_f$ connecting P and Q. Each consecutive pair in this path is conflict free. Therefore, P and T_1 both cover a true point, say, $\boldsymbol{t}_{i_1}$ and there exists a true path from $\boldsymbol{p}$ to $\boldsymbol{t}_{i_1}$. Continuing inductively in this way, we can construct a true path connecting $\boldsymbol{p}$ and $\boldsymbol{q}$. (In the case where $\boldsymbol{p}$ and $\boldsymbol{q}$ are covered by the same prime implicant, a true path can easily be constructed). □

The characterization of connected functions given in Theorem 3.3 requires the knowledge of all the prime implicants of the function. However, the recognition of connected functions is a much easier task, as shown by the following.

Theorem 3.4. *A Boolean function f is connected if and only if the concordance graph G_Φ^* associated to any DNF representation Φ of f is connected.*

Proof. The proof follows closely that of Theorem 3.3. Let P and Q be two terms conflicting in at least two literals in an arbitrary DNF representation Φ of f. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be two true points covered by the implicants P and Q, respectively. Since f is connected, there exists a true path between $\boldsymbol{p} = \boldsymbol{t}_0$ and $\boldsymbol{q} = \boldsymbol{t}_{k+1}$, say $(\boldsymbol{p}, \boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_k, \boldsymbol{q})$. Let $\boldsymbol{t}_{i_1}$ be the last of the true points on this path which is covered by P. Let T_1 be a term of Φ covering $\boldsymbol{t}_{i_1+1}$. Now, P and T_1 can conflict in at most one literal, and hence they must have a common edge in G_Φ^*. Continuing inductively in this way, we build a path from P to Q in G_Φ^*.

Conversely, let $\boldsymbol{p}$ and $\boldsymbol{q}$ be two arbitrary true points covered by terms P and Q of Φ. Let $(P, T_1, \ldots, T_k, Q)$ be a path joining P and Q in G_Φ^*. Each consecutive pair in this path has at most one conflict. Therefore, P and T_1 either both cover the same true point, or there exist two neighboring true points $\boldsymbol{p}_i$ and $\boldsymbol{t}_{i_1}$, such that $\boldsymbol{p}_i$ is covered by P and $\boldsymbol{t}_{i_1}$ is covered by T_1. Since $\boldsymbol{p}$ and $\boldsymbol{p}_i$ are covered by P, there exists a true path from $\boldsymbol{p}$ to $\boldsymbol{p}_i$, and hence from $\boldsymbol{p}$ to $\boldsymbol{t}_{i_1}$. In this fashion, we construct a true path connecting $\boldsymbol{p}$ and $\boldsymbol{q}$. □

The following statement follows immediately from Theorem 3.4.

Corollary 3.5. *Given any DNF representation Φ of a function f, the property of connectedness can be checked in time linear in the size of the DNF.*

However, this is not the case for co-connectedness, or for strong connectedness. In fact, we have the following negative result.

Theorem 3.6. *The membership problems for the following two classes of Boolean functions are both CoNP-hard*:

(i) *co-connected,*

(ii) *strongly connected.*

Proof. (i) We will show this by reducing SAT to the recognition of co-connectedness. Given Φ as an input to SAT, let us introduce two additional variables x_{n+1} and x_{n+2} and consider the function $g = x_{n+1}\bar{x}_{n+2} \vee \bar{x}_{n+1}x_{n+2} \vee \Phi$, where Φ does not depend on x_{n+1} or x_{n+2}. We shall show that g is co-connected if and only if $\Phi \equiv 1$. Assume to the contrary that z^* is a solution to $\Phi = 0$. Then $(z^*,0,0)$ and $(z^*,1,1)$ are two false points of g with no false path between them. Similarly, if g is not co-connected, then $\Phi = 0$ must have a solution, otherwise g would trivially be strongly connected.

(ii) The same construction as above proves this part of the statement. □

We remark that the recognition of strong connectedness and of co-connectedness are equivalent. Indeed, if f given by a DNF representation Ψ is strongly connected, then f is also co-connected. If f is not strongly connected, then using Theorem 3.4 one can check in polynomial time whether f is connected or not. If f is connected, then f is not co-connected. If, however, f is not connected, consider the function $g = x_{n+1} \vee \Psi$ where Ψ does not depend on x_{n+1}. Obviously, g is connected since the associated concordance graph is connected, due to the fact that the vertex corresponding to the term x_{n+1} is connected to all the other vertices of the graph. On the other hand, g is co-connected if and only if f is co-connected. Indeed, g is co-connected if and only if $\bar{g} = \bar{x}_{n+1}\bar{f}$ is connected, i.e. if and only if $\bar{f}$ is connected.

4. Geodetic functions

In this section we shall examine Boolean functions having the property that their set of true points, or their set of false points, or both, are geodetically connected.

Definition 4.1. A Boolean function f is called

- *geodetic* iff any pair of true points is geodetically connected;
- *co-geodetic* iff any pair of false points is geodetically connected, i.e. iff $\bar{f}$ is geodetic;
- *strongly geodetic* iff it is both geodetic and co-geodetic.

Lemma 4.2. *A Boolean function f is geodetic (co-geodetic) if and only if there exists a true (false) point strictly between any two non-neighboring true (false) points of f.*

Proof. Indeed, if there exist two true points $\boldsymbol{p}$ and $\boldsymbol{q}$ with no true point between them, f cannot be geodetic. Conversely, let $\boldsymbol{p}$ and $\boldsymbol{q}$ be the closest two true points with no shortest true path linking them, and $\boldsymbol{r}$ be a true point strictly between $\boldsymbol{p}$ and $\boldsymbol{q}$, i.e. $d(\boldsymbol{p},\boldsymbol{r}) + d(\boldsymbol{r},\boldsymbol{q}) = d(\boldsymbol{p},\boldsymbol{q})$. Since $d(\boldsymbol{p},\boldsymbol{r}) < d(\boldsymbol{p},\boldsymbol{q})$, there exists a shortest true path, say $\tilde{p}$ from $\boldsymbol{p}$ to $\boldsymbol{r}$. Similarly, there exists a shortest true path from $\boldsymbol{r}$ to $\boldsymbol{q}$, say $\tilde{q}$. It follows then that $\tilde{p}$ followed by $\tilde{q}$ is a shortest true path from $\boldsymbol{p}$ to $\boldsymbol{q}$. Symmetrical arguments will prove the characterization for co-geodetic functions. □

Theorem 4.3. *The respective membership problems for the following classes of Boolean functions are all CoNP-hard*:
(i) *geodetic*,
(ii) *co-geodetic*,
(iii) *strongly geodetic.*

Proof. It simply follows from Theorem 2.9 since each of the specified classes of Boolean functions has the projection property. It is enough to show this property for the class of geodetic Boolean functions. Let $f(x_1,\dots,x_n)$ be a geodetic Boolean function. Using the Shannon expansion on x_1, we can write f as $x_1 g \vee \bar{x}_1 h$, where the functions g and h are defined on the variables $x_2,\dots,x_n$. Then both g and h must also be geodetic functions. Indeed, assume that $\boldsymbol{y}$ and $\boldsymbol{z}$ are two true points of g with no true point between them. Then, $(1,\boldsymbol{y})$ and $(1,\boldsymbol{z})$ are true points of f which are not geodetically connected. Similar arguments show that h is also geodetic. □

Lemma 4.4. *Let f be a Boolean function which is not geodetic. Let $\boldsymbol{x}^* = (\boldsymbol{\alpha}_A, \boldsymbol{\alpha}_B)$ and $\boldsymbol{y}^* = (\bar{\boldsymbol{\alpha}}_A, \boldsymbol{\alpha}_B)$ be any two non-neighboring true points of f* (i.e. $|A| \geqslant 2$) *with no true point in between. If T and S are implicants of f such that T covers $\boldsymbol{x}^*$ and S covers $\boldsymbol{y}^*$, then every $x_i^{\alpha_i}$ $(i \in A)$ belongs to T, and every $x_i^{\bar{\alpha}_i}$ $(i \in A)$ belongs to S.*

Proof. Assume T to be a term of Φ covering $\boldsymbol{x}^*$, and not containing, say, x^{α_1}. Let $\boldsymbol{\alpha}_A = (\alpha_1, \boldsymbol{\alpha}_{A'})$. Then, $\boldsymbol{z} = (\bar{\alpha}_1, \boldsymbol{\alpha}_{A'}, \boldsymbol{\alpha}_B)$ is a true point between $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$. □

Definition 4.5. The distance between two terms T and S of a DNF, denoted by $d(T,S)$ is the number of literals T and S conflict in.

We shall show now a way of determining whether a DNF represents a geodetic function by solving a satisfiability problem.

Consider an arbitrary DNF representation Φ of a Boolean function f, and let T and T' be two terms of Φ at distance at least two such that

$$T = X^{\boldsymbol{\alpha}_A} X^{\boldsymbol{\alpha}_B} X^{\boldsymbol{\alpha}_C} \quad \text{and} \quad T' = X^{\bar{\boldsymbol{\alpha}}_A} X^{\boldsymbol{\alpha}_B} X^{\boldsymbol{\alpha}_D},$$

where A,B,C and D are disjoint subsets of $\{1,\dots,n\}$, let $A = \{a_1,\dots,a_k\}$, and let E be the set $\{1,\dots,n\} \backslash A \cup B \cup C \cup D$.

Let us define for every $i = 1,\dots,k-1$ the function

$$\Psi_i(\Phi,T,T') = \delta_i^E,$$

where the DNF δ_i^E is the restriction of δ_i to set E and where δ_i is defined as the projection of Φ on $(S,\boldsymbol{\alpha})$ for $S = \{B \cup C \cup D \cup a_i \cup a_{i+1}\}$ and $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_B, \boldsymbol{\alpha}_C, \boldsymbol{\alpha}_D, \alpha_{a_i}, \bar{\alpha}_{a_{i+1}})$. Similarly, let

$$\Psi_k(\Phi,T,T') = \delta_k^E,$$

where δ_k is the projection of Φ on $(S,\boldsymbol{\alpha})$ for $S = \{B \cup C \cup D \cup a_k \cup a_1\}$ and $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_B, \boldsymbol{\alpha}_C, \boldsymbol{\alpha}_D, \alpha_{a_k}, \bar{\alpha}_{a_1})$.

Theorem 4.6. *Let Φ be an arbitrary DNF of the Boolean function f. The Boolean function f is geodetic if and only if for every pair of terms (T,T') such that $d(T,T')\geqslant 2$,*

$$\bigvee_{i=1}^{d(T,T')} \Psi_i(\Phi,T,T') \equiv 1. \tag{1}$$

Proof. Assume that f is geodetic, but that there exist two terms T and T' of a DNF representation of f, say Φ, for which

$$\bigvee_{i=1}^{d(T,T')} \Psi_i(\Phi,T,T') = 0 \tag{2}$$

has a solution. Without loss of generality, let (A,B,C,D,E) be the partition of variables induced by T and T' where the variables in the set E do not appear in T or T', and let $d(T,T')=k\geqslant 2$. If (2) is satisfiable, we can denote a solution of it by $\boldsymbol{\alpha}_E^*$, since all the variables in B,C and D are fixed, and all the variables in A are either fixed or eliminated. Consider now $\boldsymbol{x}=(\boldsymbol{\alpha}_A,\boldsymbol{\alpha}_B,\boldsymbol{\alpha}_C,\boldsymbol{\alpha}_D,\boldsymbol{\alpha}_E^*)$ and $\boldsymbol{y}=(\overline{\boldsymbol{\alpha}}_A,\boldsymbol{\alpha}_B,\boldsymbol{\alpha}_C,\boldsymbol{\alpha}_D,\boldsymbol{\alpha}_E^*)$, and notice that both of them are true points of f. If there exists a true point, say $\boldsymbol{z}$, strictly between $\boldsymbol{x}$ and $\boldsymbol{y}$, then it will give the value 1 to one of the projected DNFs in contradiction with the fact that $\boldsymbol{\alpha}_E^*$ is a solution of the equation $\bigvee_{i=1}^{k} \Psi_i(\Phi,T,T')=0$. Conversely, assume Eq. (1) to be satisfied, but f to have two non-neighboring true points say $\boldsymbol{x}$ and $\boldsymbol{y}$, with no true point between them. Then there exists a term T taking the value 1 in $\boldsymbol{x}$ and another term T' taking the value 1 in $\boldsymbol{y}$. Indeed, if a specific term of Φ, say T_i, covers both $\boldsymbol{x}$ and $\boldsymbol{y}$, then any point between $\boldsymbol{x}$ and $\boldsymbol{y}$ will also be covered by T_i. Let A denote the set of variables where $\boldsymbol{x}$ and $\boldsymbol{y}$ conflict. Since $\boldsymbol{x}$ and $\boldsymbol{y}$ are not neighbors, $|A|\geqslant 2$. Lemma 4.4 shows that T and T' must conflict only in the variables of A. Without loss of generality, let $T=X^{\boldsymbol{\alpha}_A}X^{\boldsymbol{\alpha}_B}X^{\boldsymbol{\alpha}_C}$ and $T'=X^{\overline{\boldsymbol{\alpha}}_A}X^{\boldsymbol{\alpha}_B}X^{\boldsymbol{\alpha}_D}$, and let $\boldsymbol{\alpha}_E$ be the common part of $\boldsymbol{x}$ and $\boldsymbol{y}$ on set $E=\{1,\dots,n\}\backslash A\cup B\cup C\cup D$. Clearly, $\Phi(\boldsymbol{\alpha}'_A,\boldsymbol{\alpha}_B,\boldsymbol{\alpha}_C,\boldsymbol{\alpha}_D,\boldsymbol{\alpha}_E)=0$ must hold for all $\boldsymbol{\alpha}'_A\neq\boldsymbol{\alpha}_A$ and $\boldsymbol{\alpha}'_A\neq\overline{\boldsymbol{\alpha}}_A$, since otherwise $\boldsymbol{x}$ and $\boldsymbol{y}$ will have a true point between them. However, in this case $\bigvee_{i=1}^{k} \Psi_i(\Phi,T,T')$ will take the value zero in $\boldsymbol{\alpha}_E$. Indeed, if one of the terms of $\bigvee_{i=1}^{k} \Psi_i(\Phi,T,T')$, say S, that belongs to the decomposition $\Psi_j(\Phi,T,T')$ for some $j\in[1,\dots,k]$ takes the value one in $\boldsymbol{\alpha}_E$, then we can find an assignment of variables in A, say, $\boldsymbol{\alpha}'_A$ such that $(\boldsymbol{\alpha}'_A,\boldsymbol{\alpha}_B,\boldsymbol{\alpha}_C,\boldsymbol{\alpha}_D,\boldsymbol{\alpha}_E)$ is a true point of the original DNF Φ. Since by our assumption $\alpha'_{A_j}=\alpha_{A_j}$ and $\alpha'_{A_{j+1}}=\bar{\alpha}_{A_{j+1}}$, it follows that $\boldsymbol{\alpha}'_A\neq\boldsymbol{\alpha}_A$ and $\boldsymbol{\alpha}'_A\neq\overline{\boldsymbol{\alpha}}_A$. However, this implies that $(\boldsymbol{\alpha}'_A,\boldsymbol{\alpha}_B,\boldsymbol{\alpha}_C,\boldsymbol{\alpha}_D,\boldsymbol{\alpha}_E)$ is a true point which is between $\boldsymbol{x}$ and $\boldsymbol{y}$. Contradiction. □

Example 4.7. Consider the function

$$\Phi = x_1x_2x_3\bar{x}_7 \vee \bar{x}_1\bar{x}_4x_5 \vee \bar{x}_2x_3\bar{x}_4\bar{x}_6 \vee x_3x_4x_5\bar{x}_7 \vee \bar{x}_3\bar{x}_4\bar{x}_6\bar{x}_7 \tag{3}$$

and consider the pair $T=x_3x_4x_5\bar{x}_7$ and $T'=\bar{x}_3\bar{x}_4\bar{x}_6\bar{x}_7$. These two terms induce the sets $A=\{3,4\}$, $B=\{7\}$, $C=\{5\}$, $D=\{6\}$ and $E=\{1,2\}$. To obtain $\Psi_1(\Phi,T,T')$, we assign the following values to the variables. We let $x_3=1$, $x_4=0$, $x_5=1$, $x_6=0$, and $x_7 = 0$ in Φ and restrict the resulting DNF to the set $\{1,2\}$. In other words, $\Psi_1(\Phi,T,T') = x_1x_2 \vee \bar{x}_1 \vee \bar{x}_2$, showing that $\Psi_1(\Phi,T,T') \equiv 1$. Hence, the condition of Theorem 4.6 holds for this pair of terms. Since, all the other terms have distances at most 1 from each other, the conditon in Theorem 4.6 is satisfied, showing that (3) defines a geodetic function.

Corollary 4.8. *For any DNF having the strong SAT tractability property, it can be checked in polynomial time whether the function represented by this DNF is geodetic.*

We shall focus now on some classes of Boolean functions properly included in the class of strongly geodetic functions. We start by giving some well-known definitions.

Definition 4.9. A term is called *positive* iff it contains only uncomplemented variables. A DNF is called *positive* iff it contains only positive terms. A Boolean function is called *positive* iff it has at least one positive DNF representation.

Definition 4.10. A DNF Φ is called *unate* iff each variable appears either as complemented or as uncomplemented in Φ. A Boolean function is called *unate* iff it has a unate DNF representation.

Theorem 4.11. *Unate functions are strongly geodetic.*

Proof. Since a renaming of variables does not affect any connectivity properties of a Boolean function, without loss of generality, we shall only show that positive functions are strongly geodetic. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two arbitrary true points of a positive Boolean function f and let i be a component such that $x_i = 0$ and $y_i = 1$. Then, the point $\boldsymbol{x} \vee \boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is the ith unit vector, is a true point of f which is between $\boldsymbol{x}$ and $\boldsymbol{y}$. Similarly, if $\boldsymbol{x}$ and $\boldsymbol{y}$ are two non-neighboring false points of f, and if $\boldsymbol{x}$ and $\boldsymbol{y}$ differ in the ith component (say $x_i = 0$, $y_i = 1$), then the point $\boldsymbol{z}$ which differs from $\boldsymbol{y}$ only in the ith component is a false point between $\boldsymbol{x}$ and $\boldsymbol{y}$. □

Remark 4.12. Unate functions are strongly geodetic, however strongly geodetic functions are not necessarily unate. Indeed, consider for $n \geqslant 4$, the function having as true points only $(1,1,\ldots,1)$, $(0,0,\ldots,0)$ and a shortest true path connecting them. This function is not unate although it is strongly geodetic.

Definition 4.13. A DNF is called *Horn* iff every term contains at most one complemented variable. A Boolean function is called *Horn* iff it has at least one Horn DNF representation.

Definition 4.14. A DNF $\Phi(x_1,\ldots,x_n)$ is called *renamable Horn* iff there exists a set $S \subseteq \{1,\ldots,n\}$ such that $\Phi(\boldsymbol{x}^S)$ is Horn; here $\boldsymbol{x}^S = (x_1^S,\ldots,x_n^S)$ and

$$x_i^S = \begin{cases} \bar{x}_i & \text{if } i \in S, \\ x_i & \text{if } i \notin S. \end{cases}$$

A Boolean function is called *renamable Horn* iff it has at least one renamable Horn DNF representation.

Clearly, renamable Horn functions generalize unate functions. However, renamable Horn functions do not have the strong geodeticity property as do the unate functions.

Remark 4.15. Not all Horn functions are strongly geodetic, e.g.

$$x_1\bar{x}_2 \vee \bar{x}_1 x_2$$

is neither geodetic nor co-geodetic.

Unate functions have the strong property that no two of their prime implicants can conflict, i.e. the distance between any two prime implicants is 0. On the other hand, two prime implicants of a renamable Horn function can be at most at distance 2. It is natural, therefore, to explore the class of functions for which the distance between prime implicants is at most 1.

Definition 4.16. We shall call a DNF *concordant* iff any two terms of it conflict in at most one variable. A Boolean function f is called *concordant* iff any two prime implicants of f conflict in at most one variable.

Remark 4.17. An irredundant DNF may have the property that any two terms conflict in at most one literal but the function itself is not concordant. Indeed, the function f defined by

$$x_1\bar{x}_2 \vee x_2\bar{x}_3 \vee \bar{x}_1 x_3$$

is not concordant, since it has a prime implicant $x_1\bar{x}_3$ (not appearing in the DNF above) which conflicts in two variables with the prime implicant $\bar{x}_1 x_3$.

Lemma 4.18. *If a Boolean function f can be represented by a concordant DNF, then f is geodetic.*

Proof. We shall prove this statement by contradiction. Let Φ be a concordant DNF representing f, let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two true points of f at minimum distance which are not geodetically connected. In other words, $d(\boldsymbol{x},\boldsymbol{y})\geqslant 2$, and every point between $\boldsymbol{x}$ and $\boldsymbol{y}$ is false. In view of Lemma 4.4, we see that any two terms of Φ covering $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, must conflict in at least two literals, in contradiction with the concordance of Φ. □

Theorem 4.19. *Concordant functions are strongly geodetic.*

Proof. It follows from Lemma 4.18 that concordant functions are geodetic.

Let further, $\boldsymbol{x}$ and $\boldsymbol{y}$ be two closest false points of a concordant function f, which conflict in $k\geqslant 2$ variables and are not geodetically connected. In other words, assume that every point strictly between $\boldsymbol{x}$ and $\boldsymbol{y}$ is true. Without loss of generality, assume $\boldsymbol{x}=(0,\dots,0,1,\dots,1)$ and $\boldsymbol{y}=(1,\dots 1,1,\dots 1)$. Then, for any $i \neq j$ $(i,j \in [1\dots k])$ both $x_i\bar{x}_jx_{k+1}\dots x_n$ and $\bar{x}_ix_jx_{k+1}\dots x_n$ are implicants of f. This property is no longer true if either x_i or x_j is removed, in contradiction with the fact that f is concordant. It follows that f is also co-geodetic. □

Remark 4.20. There exist strongly geodetic functions which are not concordant, e.g.

$$h_1 = x_1x_2x_3 \vee x_1x_2\bar{x}_4 \vee x_1\bar{x}_3\bar{x}_4 \vee \bar{x}_2\bar{x}_3\bar{x}_4.$$

This function has as its true points $(1,1,1,1)$, $(0,0,0,0)$ and a shortest true path connecting them, and it is obviously strongly connected, however, the prime implicants $x_1x_2x_3$ and $\bar{x}_2\bar{x}_3\bar{x}_4$ conflict in 2 variables.

It is important to note that concordant functions generalize the class of unate functions. Concordant functions also include a special class of renamable Horn functions called *acyclic renamable Horn* functions introduced in [5].

To a Horn DNF Φ, a directed graph $G(\Phi)$ was associated in [5] in the following way. The vertices of $G(\Phi)$ correspond to the variables in Φ. Two vertices u and v are linked by an arc $u \rightarrow v$ if and only if there is a term in Φ containing both u and $\bar{v}$. It has been stated in [5] that it is easy to check whether a given Horn DNF represents an acyclic function. Indeed, it is sufficient to transform the given DNF to a prime irredundant one (in time quadratic in the length of the given DNF), and then check in linear time whether the graph of the resulting DNF has an oriented cycle (see [5]).

While the class of concordant functions includes the class of acyclic Horn functions, it neither contains (e.g. $x_1\bar{x}_2 \vee x_2\bar{x}_3 \vee \bar{x}_1x_3$), nor is it contained in (e.g. $x_1x_2 \vee \bar{x}_1\bar{x}_3$) the class of Horn functions.

Remark 4.21. The following example shows that a concordant function is not necessarily renamable Horn:

$$x_1x_6\bar{x}_7 \vee x_2x_7\bar{x}_8 \vee x_3x_6x_8 \vee x_4\bar{x}_6x_8 \vee x_5\bar{x}_6\bar{x}_7 \vee x_1x_2\bar{x}_8 \vee x_1x_4\bar{x}_7 \vee x_1x_5\bar{x}_7 \vee x_2x_3x_6$$
$$\vee x_2x_4x_7 \vee x_2x_5\bar{x}_6 \vee x_3x_4x_8 \vee x_3x_5\bar{x}_6 \vee x_3x_5x_8 \vee x_2x_3x_5 \vee x_1x_2x_4.$$

The above DNF lists all the prime implicants of the function. No two of them conflict in more than one variable. On the other hand, none of the first five prime implicants are redundant, and the variables cannot be renamed in such a way that the disjunction of the first five terms is Horn.

Since any prime DNF representation of an acyclic Horn function is concordant, it follows from Lemma 4.18 that acyclic Horn functions are geodetic. Since acyclic Horn functions are a proper subclass of concordant functions, Theorem 4.19 shows that they are also strongly geodetic.

Remark 4.22. The following function is both strongly geodetic and Horn, but is not concordant:

$$x_1\bar{x}_2x_3 \vee x_1\bar{x}_2x_4 \vee x_1\bar{x}_3x_4 \vee x_2\bar{x}_3x_4.$$

This function is obtained from h_1 by renaming the variables x_2 and x_4.

Remark 4.23. A Horn function which is also concordant is not necessarily acyclic as the following example shows:

$$h_2 = x_1x_2\bar{x}_3 \vee x_3x_4\bar{x}_5 \vee \bar{x}_2x_5x_6 \vee x_1x_4\bar{x}_5 \vee \bar{x}_2x_4x_6 \vee x_1\bar{x}_3x_6.$$

Indeed, $G(h_2)$ has the oriented cycle $x_2 \to x_3 \to x_5 \to x_2$.

Remark 4.24. The dual of a concordant function is not necessarily concordant:

$$h_3 = x_1\bar{x}_2 \vee x_1\bar{x}_3 \vee x_1\bar{x}_4 \vee x_2\bar{x}_3 \vee x_2\bar{x}_4 \vee x_3\bar{x}_4.$$

Indeed, $h_3^d = h_1$ which can be seen not to be concordant.

Remark 4.25. It follows by duality from Theorem 4.19 that duals of concordant functions are also strongly geodetic. On the other hand, strongly geodetic functions do not have to be concordant or have concordant duals, as shown by

$$h_4 = x_1x_2x_3 \vee x_1\bar{x}_3\bar{x}_4 \vee \bar{x}_1\bar{x}_2\bar{x}_3 \vee \bar{x}_1x_3x_4 \vee x_1x_2\bar{x}_4 \vee x_2x_3x_4 \vee \bar{x}_2\bar{x}_3\bar{x}_4 \vee \bar{x}_1\bar{x}_2x_4$$

and its dual

$$h_4^d = x_1\bar{x}_2x_3 \vee \bar{x}_1x_2\bar{x}_3 \vee x_1\bar{x}_2x_4 \vee \bar{x}_1x_2\bar{x}_4 \vee x_1\bar{x}_3x_4 \vee \bar{x}_1x_3\bar{x}_4 \vee x_2\bar{x}_3x_4 \vee \bar{x}_2x_3\bar{x}_4.$$

Remark 4.26. It was established above that the following implications hold between the properties studied in this section:

- unate $\Rightarrow$ acyclic renamable Horn,
- acyclic renamable Horn $\Rightarrow$ concordant,
- concordant $\Rightarrow$ strongly geodetic.

Moreover, none of the implications above is an equivalence.

Contrary to the case of a Horn DNF, the well-known satisfiability problem is not tractable for concordant DNFs. In fact we have the following.

Theorem 4.27. *Let Φ be a concordant DNF. Then, it is NP-hard to decide whether $\Phi = 0$ has a solution.*

Proof. We shall show that we can associate in polynomial time to an arbitrary DNF Ψ a concordant DNF Φ such that Ψ is satisfiable if and only if Φ is satisfiable.

Indeed, let Ψ be defined on $\{x_1,\dots,x_n\}$ and let us introduce for each variable x_i in Ψ two additional variables y_i and z_i in Φ. In this way, the DNF Φ will have $3n$ variables. Let us also replace in Ψ each occurrence of $\bar{x}_i$ by y_i, and let us denote the resulting DNF by Δ. Now, Δ consists only of positive terms. In order to guarantee that in every solution of $\Delta=0$ the variables $\bar{x}_i$ and y_i are equal and to assure that the new DNF is concordant, we shall introduce one additional variable z_i for every i. Consider now the DNF

$$\Phi(x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,x_n)=\Delta(x_1,\dots,x_n,y_1,\dots,y_n)\vee\bigvee_{i=1}^{n}(\bar{x}_i\bar{y}_i\vee y_i\bar{z}_i\vee z_ix_i).$$

Clearly, Φ is a concordant DNF, since the only possibility for two conflicts to occur in it is between a term $\bar{x}_i\bar{y}_i$ and a term of Δ. However, by construction, no term of Δ contains both x_i and y_i, since otherwise the corresponding term of Ψ would contain both x_i and $\bar{x}_i$. It is obvious that Ψ is satisfiable if and only if the concordant DNF Φ is. Indeed, if $\Psi(\mathbf{x}^*)=0$ then $\Phi(\mathbf{x}^*,\mathbf{y}^*,\mathbf{z}^*)=0$ where $y_i^*=z_i^*=\bar{x}_i^*$ for each i. Similarly, if $\Phi(\mathbf{x}',\mathbf{y}',\mathbf{z}')=0$ then $\Psi(\mathbf{x}')=0$ as well. □

5. Convex functions

This section studies functions with the property that all the shortest paths between any pair of true points consist only of true points, and the functions for which analogous conditions are imposed on the set of false points, or on both the sets of true and of false points. The first part of this section deals with the characterization and with convexity properties of connected functions, while the second part deals with the recognition of such functions.

5.1. Characterization

The extremely powerful requirement of convexity puts a severe limitation on the number of functions with this property. In order to provide more flexibility, we introduce the following relaxation of the definition.

Definition 5.1. For any integer $k\in\{2,\dots,n\}$, a Boolean function f is called

- *k-convex* iff any pair of true points at distance at most k is convexly connected;
- *co-k-convex* iff any pair of false points at distance at most k is convexly connected, i.e. $\bar{f}$ is k-convex;
- *strongly k-convex* iff it is both k-convex and co-k-convex.

Example 5.2. The function $\bar{x}_1\bar{x}_2\bar{x}_3\vee x_1x_2x_3$ is 2-convex, and its negation $x_1\bar{x}_2\vee x_2\bar{x}_3\vee\bar{x}_1x_3$ is co-2-convex.

For the important case of $k=n$, the following functional characterizations hold.

Theorem 5.3. *A Boolean function* $f(x_1,\dots,x_n)$ *is*

(i) *n-convex (convex) if and only if for any* $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in B^n$

$$f(\boldsymbol{xy} \vee \boldsymbol{xz} \vee \boldsymbol{yz}) \geqslant f(\boldsymbol{x})f(\boldsymbol{y}) \vee f(\boldsymbol{x})f(\boldsymbol{z}) \vee f(\boldsymbol{y})f(\boldsymbol{z}),$$

(ii) *co-n-convex (co-convex) if and only if for any* $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in B^n$

$$f(\boldsymbol{xy} \vee \boldsymbol{xz} \vee \boldsymbol{yz}) \leqslant f(\boldsymbol{x})f(\boldsymbol{y}) \vee f(\boldsymbol{x})f(\boldsymbol{z}) \vee f(\boldsymbol{y})f(\boldsymbol{z}),$$

(iii) *strongly n-convex (strongly convex) if and only if for any* $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in B^n$

$$f(\boldsymbol{xy} \vee \boldsymbol{xz} \vee \boldsymbol{yz}) = f(\boldsymbol{x})f(\boldsymbol{y}) \vee f(\boldsymbol{x})f(\boldsymbol{z}) \vee f(\boldsymbol{y})f(\boldsymbol{z}).$$

Proof. (i) Assume f is n-convex, and there exist $\boldsymbol{x}^*$, $\boldsymbol{y}^*$ and $\boldsymbol{z}^*$ such that $f(\boldsymbol{x}^*) = f(\boldsymbol{y}^*) = 1$ but $f(\boldsymbol{x}^*\boldsymbol{y}^* \vee \boldsymbol{x}^*\boldsymbol{z}^* \vee \boldsymbol{y}^*\boldsymbol{z}^*) = 0$. Then, $\boldsymbol{z}^* \neq \boldsymbol{x}^*$, $\boldsymbol{y}^*$ and the point $\boldsymbol{x}^*\boldsymbol{y}^* \vee \boldsymbol{x}^*\boldsymbol{z}^* \vee \boldsymbol{y}^*\boldsymbol{z}^*$ is a false point between the two true points $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$. Conversely, if the inequality is satisfied, there cannot be a false point between two true points.

(ii) Follows by complementation.

(iii) Follows by combining the first two inequalities. □

Lemma 5.4. *If* $\boldsymbol{\beta} = (\beta_1,\dots,\beta_r,\beta_{r+1},\dots,\beta_n)$ *and* $(\beta_1,\dots,\beta_{i-1},\bar{\beta}_i,\beta_{i+1},\dots,\beta_r,\beta_{r+1},\dots,\beta_n)$ $(i = 1,\dots,r)$ *are* $r+1$ *true points of a 2-convex function f, then* $\prod_{j=r+1}^{n} x_j^{\beta_j}$ *is an implicant of f.*

Proof. Without loss of generality we may assume that the $r+1$ true vectors are $\mathbf{0} = (0,\dots,0)$ and the first r unit vectors $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_r\}$. In order to prove that every vector $\boldsymbol{x} = (x_1,\dots,x_r,0,\dots,0)$ is a true point of f, we shall use induction on the number t of ones among the first r components of $\boldsymbol{x}$. The statement is valid for $t \leqslant 1$. For $t = 2$, the point $\boldsymbol{x}$ is between two true vectors $(\boldsymbol{e}_i,0,\dots,0)$ and $(\boldsymbol{e}_j,0,\dots,0)$, implying the validity of the statement. Assuming the statement to hold for all $t \leqslant k$, and noticing that $\boldsymbol{x}$ is between $\boldsymbol{x}^i = (x_1,\dots,x_{i-1},\bar{x}_i,x_{i+1},\dots,x_k,0,\dots,0)$ and $\boldsymbol{x}^j = (x_1,\dots,x_{j-1},\bar{x}_j,x_{j+1},\dots,x_k,0,\dots,0)$ where both x_i and x_j are one, we conclude that $\boldsymbol{x}$ is a true point. □

Definition 5.5. The *generalized consensus* of two elementary conjunctions $S = X^{\boldsymbol{\alpha}_A}X^{\boldsymbol{\alpha}_B}X^{\boldsymbol{\alpha}_C}$ and $T = X^{\bar{\boldsymbol{\alpha}}_A}X^{\boldsymbol{\alpha}_B}X^{\boldsymbol{\alpha}_D}$ is the elementary conjunction

$$\langle S,T\rangle = X^{\boldsymbol{\alpha}_B}X^{\boldsymbol{\alpha}_C}X^{\boldsymbol{\alpha}_D}.$$

Lemma 5.6. *Let* T_1 *and* T_2 *be two implicants of a k-convex function* f *such that* $d(T_1,T_2) \leqslant k$. *Then the generalized consensus* $\langle T_1,T_2\rangle$ *is an implicant of f.*

Proof. Let $T_1 = X^{\boldsymbol{\alpha}_A}X^{\boldsymbol{\alpha}_B}X^{\boldsymbol{\alpha}_C}$ and $T_2 = X^{\bar{\boldsymbol{\alpha}}_A}X^{\boldsymbol{\alpha}_B}X^{\boldsymbol{\alpha}_D}$. Then $|A| \leqslant k$. Let E be the set $\{1,\dots,n\} \backslash (A \cup B \cup C \cup D)$ and $A = \{a_1,\dots,a_k\}$. We shall represent an arbitrary vector $\boldsymbol{\beta}$ in $\{0,1\}^n$ as $(\boldsymbol{\beta}_A,\boldsymbol{\beta}_B,\boldsymbol{\beta}_C,\boldsymbol{\beta}_D,\boldsymbol{\beta}_E)$, where $\boldsymbol{\beta}_A$ is the subvector of components corresponding to the set A, etc. Let us now consider the vectors $\boldsymbol{x} = (\boldsymbol{\alpha}_A,\boldsymbol{\alpha}_B,\boldsymbol{\alpha}_C,*,*)$ and $\boldsymbol{y} = (\bar{\boldsymbol{\alpha}}_A,\boldsymbol{\alpha}_B,*,\boldsymbol{\alpha}_D,*)$, where the $*'$s represent arbitrary 0–1 vectors of appropriate sizes.

Then, any vector of the form $\boldsymbol{x}$ or $\boldsymbol{y}$ is a true vector of f. In particular, for any vector $\boldsymbol{\alpha}_E$, the vectors $\boldsymbol{x}^* = (\boldsymbol{\alpha}_A, \boldsymbol{\alpha}_B, \boldsymbol{\alpha}_C, \boldsymbol{\alpha}_D, \boldsymbol{\alpha}_E)$ and $\boldsymbol{y}^* = (\overline{\boldsymbol{\alpha}}_A, \boldsymbol{\alpha}_B, \boldsymbol{\alpha}_C, \boldsymbol{\alpha}_D, \boldsymbol{\alpha}_E)$ are true vectors. Since $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are at distance k, any point between them has the form $(*, \boldsymbol{\alpha}_B, \boldsymbol{\alpha}_C, \boldsymbol{\alpha}_D, \boldsymbol{\alpha}_E)$ and by our hypothesis it is also a true point. Since $\boldsymbol{\alpha}_E$ is arbitrary, $T_3 = X_B^{\boldsymbol{\alpha}_B} X_C^{\boldsymbol{\alpha}_C} X_D^{\boldsymbol{\alpha}_D}$ must be an implicant of f. □

Lemma 5.7. *If a function f can be represented by a DNF Φ so that for any two terms T_1 and T_2 of Φ their generalized consensus $\langle T_1, T_2\rangle$ is an implicant of f whenever $d(T_1, T_2) \leqslant k$, then the function f is k-convex.*

Proof. We shall show that any two true points of f at distance at most k are convexly connected. If there exists a term of Φ covering both points, then they are obviously convexly connected. Otherwise, one of the points is covered by a term T_1 of Φ, while the other one is covered by another term T_2 of Φ. Since the two points are at distance at most k, $d(T_1, T_2) \leqslant k$, and therefore $T_3 = \langle T_1, T_2\rangle$ is an implicant of f. It is easy to see that T_3 covers both points, and they are therefore convexly connected. □

Theorem 5.8. *For any $k \geqslant 2$, a Boolean function f is k-convex if and only if any two prime implicants of f conflict in at least $k+1$ literals.*

Proof. We shall prove the sufficiency of the condition by contradiction. Let us assume that $T_1 = X^{\boldsymbol{\alpha}_A} X^{\boldsymbol{\alpha}_B} X^{\boldsymbol{\alpha}_C}$ and $T_2 = X^{\overline{\boldsymbol{\alpha}}_A} X^{\boldsymbol{\alpha}_B} X^{\boldsymbol{\alpha}_D}$ are two prime implicants of f conflicting in at most k literals. Then, it follows from Lemma 5.6 that $T_3 = X^{\boldsymbol{\alpha}_B} X^{\boldsymbol{\alpha}_C} X^{\boldsymbol{\alpha}_D}$ must be an implicant of f. Let $\boldsymbol{p} = (\boldsymbol{\alpha}_A, \boldsymbol{\alpha}_B, \boldsymbol{\alpha}_C, \boldsymbol{\alpha}_D, \boldsymbol{\alpha}_E)$ and $\boldsymbol{q} = (\overline{\boldsymbol{\alpha}}_A, \boldsymbol{\alpha}_B, \boldsymbol{\alpha}_C, \boldsymbol{\alpha}_D, \boldsymbol{\alpha}_E)$ and let us consider a point $\hat{\boldsymbol{p}}$ differing from $\boldsymbol{p}$ only in the kth component of $\boldsymbol{\alpha}_A$. Since $\hat{\boldsymbol{p}}$ is covered by T_3, it is a true point. Let us also construct a point $\hat{\boldsymbol{q}}_l$ differing from $\boldsymbol{q}$ only in the component l of $\boldsymbol{\alpha}_C$. Since $\hat{\boldsymbol{q}}_l$ is covered by T_2, it is also a true point. Obviously, $d(\hat{\boldsymbol{p}}, \hat{\boldsymbol{q}}_l) = k$, implying that any point between $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}_l$ for any l is true. For every $i \in \{1, \ldots, k-1\}$, the vector $\boldsymbol{s}_i$ which differs from $\hat{\boldsymbol{p}}$ only in the ith component of $\boldsymbol{\alpha}_A$ is between $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}_l$, and is therefore a true point. Similarly, the point $\boldsymbol{r}_j$ differing from $\hat{\boldsymbol{p}}$ in the jth component of $\boldsymbol{\alpha}_C$ is also between $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}_j$, and is therefore also a true point. We have constructed in this way $k - 1 + |C|$ true neighbors of $\hat{\boldsymbol{p}}$. All these points have the same value in the kth component α_k of A, and in all the components of B, D and E. It follows now from Lemma 5.4 that $X^{\overline{\boldsymbol{\alpha}}_k} X^{\boldsymbol{\alpha}_B} X^{\boldsymbol{\alpha}_D} X^{\boldsymbol{\alpha}_E}$ is an implicant. This fact remains true for every choice of $\boldsymbol{\alpha}_E$, showing that $I = X^{\overline{\boldsymbol{\alpha}}_k} X^{\boldsymbol{\alpha}_B} X^{\boldsymbol{\alpha}_D}$ is an implicant. However, I absorbs T_2, proving that contrary to our assumption, T_2 is not prime.

Conversely, if any two prime implicants of f conflict in at least $k+1$ literals, there does not exist a true point covered by two different prime implicants. Hence, two true points covered by different prime implicants are at a distance of at least $k+1$. Therefore, any two true points at distance at most k must be covered by the same (unique) prime implicant. Any shortest path connecting the two points will consist only of points covered by the same prime implicant, and will therefore consist only of true points. □

Theorem 5.8 and Proposition 2.1 imply the following statements.

Remark 5.9. A k-convex Boolean function has a unique irredundant prime DNF representation.

Remark 5.10. A k-convex function having an implicant of degree at most k is an elementary conjunction, and does not depend effectively on any variable not included in this conjunction.

With increasing values of k, the statement of Theorem 5.8 gets stronger. In particular, an n-convex function consists of a single prime implicant. An $(n-1)$-convex function is either an elementary conjunction, or has the form $x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n} \vee x_1^{\bar{\alpha}_1}x_2^{\bar{\alpha}_2}\dots x_n^{\bar{\alpha}_n}$. It follows by negation that co-n-convex functions are linear, while co-$(n-1)$-convex functions have the form $\bigvee_{i=1}^{n-1}(x_i^{\alpha_i}x_{i+1}^{\bar{\alpha}_{i+1}} \vee x_n^{\alpha_n}x_1^{\bar{\alpha}_1})$.

It is well known that given an arbitrary DNF of a positive Boolean function, we can obtain a positive DNF of it by simply "erasing" all the complemented variables from the given DNF. Obviously, the polynomiality of this algorithm is a consequence of the a-priori knowledge of the positivity of the function.

In order to obtain a similarly efficient method for finding the prime implicants of a Boolean function which is assumed to be known to be k-convex, we shall need the following operation.

Definition 5.11. The *convex hull* of two elementary conjunctions $S = X^{\alpha_A}X^{\alpha_B}X^{\alpha_C}$ and $T = X^{\bar{\alpha}_A}X^{\alpha_B}X^{\alpha_D}$ is the elementary conjunction

$$[S,T] = X^{\alpha_B}.$$

Note that when $B = \emptyset$, the convex hull is simply the constant 1. Obviously, $S \vee T \leqslant [S,T]$.

We shall describe now the *k-intersection method* for finding the *k-convex hull* of a DNF. Given any DNF $\Phi = \bigvee_{k=1}^{m} T_k$, if $T_i = X^{\alpha_A}X^{\alpha_B}X^{\alpha_C}$ and $T_j = X^{\bar{\alpha}_A}X^{\alpha_B}X^{\alpha_D}$ are two implicants of Φ such that $d(T_i,T_j) \leqslant k$, replace Φ by

$$\Phi' = [T_i,T_j] \vee \left(\bigvee_{k=1, k\neq i,j}^{m} T_k \right),$$

perform all possible absorptions, and repeat as many times as possible. Let $[\Phi]_k$ be the last DNF obtained after applying the k-intersection method. We call $[\Phi]_k$ the *k-convex hull* of Φ.

We shall show now that the k-convex hull of a DNF is the unique smallest k-convex majorant of the function represented by the given DNF.

Theorem 5.12. *Let a Boolean function f be represented by a DNF Φ. Let $[\Phi]_k$ be the k-convex hull of Φ. Then, the following are all true*:

(i) $[\Phi]_k$ *is a k-convex majorant of* Φ,

(ii) $[\Phi]_k$ *is irredundant and prime,*
(iii) *If* Ψ *is a* k*-convex majorant of* Φ, *then* $[\Phi]_k \leqslant \Psi$,
(iv) $[\Phi]_k$ *is unique,*
(v) *If* f *is* k*-convex for some* $k \geqslant 2$, *then* $f = [\Phi]_2$.

Proof.
(i) Since the convex hull of any two terms absorbs both of the terms, $[\Phi]_k$ is clearly a majorant of Φ. In $[\Phi]_k$ any two terms conflict in at least $k+1$ literals, and hence it follows from Theorem 5.8 that $[\Phi]_k$ is also k-convex.

(ii) By construction, either $[\Phi]_k = 1$, or any two terms of it are at distance $k+1$. The statement simply follows from Proposition 2.1.

(iii) If Ψ is a k-convex majorant of Φ, then $\Psi \geqslant [S,T] \vee \Phi$ for any two terms S,T in Φ such that $d(S,T) \leqslant k$. Now, applying repeatedly the same argument to $[S,T] \vee \Phi$, etc, we end up with the desired result, $\Psi \geqslant [\Phi]_k$.

(iv) Follows from (i) and (iii).

(v) If a function f represented by a DNF Φ is k-convex for some $k \geqslant 2$, then applying the 2-intersection method to Φ, we obtain the unique prime irredundant DNF representation of f. Indeed, if f is k-convex for some $k \geqslant 2$ then it is also 2-convex. Let $T_1 = X^{\alpha_A} X^{\alpha_B} X^{\alpha_C}$ and $T_2 = X^{\overline{\alpha}_A} X^{\alpha_B} X^{\alpha_D}$ be two terms of Φ which conflict in at most 2 literals. It follows from Theorem 5.8 that, any two prime implicants of a 2-convex function must conflict in at least 3 literals. Hence, T_1 and T_2 cannot be prime. Moreover, if T_1 is absorbed by a prime implicant P_1, and T_2 is absorbed by a prime implicant P_2, then P_1 and P_2 cannot possibly conflict in more than 2 literals. Therefore, T_1 and T_2 must be absorbed by the same prime implicant, which can only involve variables appearing in the same form (i.e. complemented or uncomplemented) both in P_1 and P_2. It follows that X^{α_B} must be an implicant. Hence $\Phi = \Phi \vee X^{\alpha_B}$. Applying now repeatedly the 2-intersection method to Φ, we obtain a new DNF Ψ in which any 2 terms conflict in at least 3 literals, or $\Phi \equiv 1$. Hence the regular consensus method will produce no changes in the DNF Ψ, showing clearly that the terms of Ψ are the prime implicants of f. Let k^* be the minimum number of variables in which two prime implicants in Ψ conflict. Then, f is (k^*-1)-convex. □

We shall concentrate now on functions which are strongly k-convex, i.e which are both k-convex and co-k-convex. The following two theorems indicate that this property is very strong, and consequently, the class of functions satisfying it is very small.

It is obvious from the definition that a strongly k-convex function is also strongly $(k-1)$-convex. Surprisingly, the next statement shows that the converse is also true:

Theorem 5.13. *A Boolean function* f *is strongly* n*-convex if and only if it is strongly 2-convex.*

Proof. A strongly n-convex function is also strongly 2-convex.

Conversely, let f be strongly 2-convex, and let for example $\boldsymbol{p}$ and $\boldsymbol{q}$ be two true points which are not convexly connected. Let us further assume that the Hamming distance of $(\boldsymbol{p},\boldsymbol{q})$ is the smallest among all such pairs. Without loss of generality, assume $\boldsymbol{p}=(0,\dots,0,x_{l+1},\dots,x_n)$ and $\boldsymbol{q}=(1,\dots,1,x_{l+1},\dots,x_n)$, and $l\geqslant 3$. Let a shortest path containing a false point between $\boldsymbol{p}$ and $\boldsymbol{q}$ be $(\boldsymbol{p},\boldsymbol{p}\vee\boldsymbol{e}_1,\boldsymbol{p}\vee\boldsymbol{e}_1\vee\boldsymbol{e}_2,\dots,\boldsymbol{p}\vee\boldsymbol{e}_1\vee\cdots\vee\boldsymbol{e}_l,\boldsymbol{q})$. Because of the way $\boldsymbol{p}$ and $\boldsymbol{q}$ were selected, it follows that each intermediate point is false. It is easy to observe that $\boldsymbol{p}\vee\boldsymbol{e}_2$ is also a true point, since otherwise, $\boldsymbol{p}$ being between $\boldsymbol{p}\vee\boldsymbol{e}_1$ and $\boldsymbol{p}\vee\boldsymbol{e}_2$, it would have to be a false point. However, in that case $\boldsymbol{p}\vee\boldsymbol{e}_2$ and $\boldsymbol{q}$ is a pair of true points at a smaller distance, which are not strongly connected. Contradiction. □

Corollary 5.14. *A Boolean function is strongly* 2-*convex if and only if it is either a constant or a variable* x *or* $\bar{x}$.

Proof. The result follows directly from the two facts that an n-convex function consists of a single prime implicant, and that a co-n-convex function is linear. □

Corollary 5.15. *The number of strongly n-convex functions is* $2n+2$.

Among the classes of Boolean functions studied so far, the connected class is the largest whereas the convex class is the smallest. The following result is interesting, since it shows that within the class of connected functions, all the various k-convexity properties are equivalent. Indeed, we have the following.

Theorem 5.16.

(i) *A connected Boolean function* f *is n-convex if and only if it is* 2-*convex.*

(ii) *A co-connected Boolean function* f *is co-n-convex if and only if it is co*-2-*convex.*

Proof. In order to prove the first statement, notice that, since f is 2-convex, it follows from Theorem 5.8 that any two prime implicants of f must conflict in at least 3 literals. However, in this case, the complement $\overline{G}_f$ of the conflict graph is connected (a necessary and sufficient condition for the connectivity of f, according to Theorem 3.3) if and only if f is either a constant or has a single prime implicant, i.e. if and only if f is n-convex. The result on co-connected functions follows similarly. □

Remark 5.17. The analogous result for strongly connected functions is valid but is superceded by Theorem 5.13 which states the fact that a function is strongly k-convex for any $k=2,\dots,n$ if and only if it is strongly 2-convex.

5.2. Recognition

Theorem 5.18. *Given a DNF and a number* $k\in\{2,\dots,n\}$, *it is CoNP-hard to decide whether the Boolean function represented by this DNF is*:

(i) *k-convex*,

(ii) *co-k-convex*,
(iii) *strongly k-convex.*

Proof. This is simply a consequence of Theorem 2.9 (see the argument in the proof of Theorem 4.3). □

Theorem 5.19. *Let Φ be a DNF having the extended SAT tractability property and representing a function f. We can decide in polynomial time whether there exists a $k\geqslant 2$ for which f is k-convex. Moreover, we can determine a k^* with the property that f is k-convex for any $k\leqslant k^*$, and is not k-convex for any $k\geqslant k^*+1$.*

Proof. Since the DNF Φ has the extended SAT tractability property, there exists an algorithm $\mathscr{B}$ that can solve in polynomial time the SAT problem for any projection Φ' of Φ. We shall proceed by finding a prime DNF representation of f. For each term $T_k=\prod_{i\in A_k} x_i^{\alpha_i}$ in Φ, and for each $j\in A_k$, let Φ_j^k be the projection of Φ obtained by the substitutions $x_i=\alpha_i$ for all $i\in A_k\setminus\{j\}$. Let us apply $\mathscr{B}$ to Φ_j^k. If it reports that $\Phi_j^k\equiv 1$, we can conclude that $T_k^j=\Pi_{i\in A_k\setminus\{j\}}x_i^{\alpha_i}$ is an implicant of f. If T_k was not a prime implicant to start with, by dropping literals from T_k one by one, and applying $\mathscr{B}$ we will arrive at a prime implicant absorbing T_k (and possibly other terms of Φ). Proceeding in this fashion, each time replacing the original term with a prime implicant absorbing it and carrying out all possible other absorptions, we will reach a DNF, say, Ψ. Now, if Ψ is not identically 1, each term of Ψ is a prime implicant of f. Consider the number of conflicts between any two prime implicants and let l be the the minimum conflict among all possible pairs. It follows from Theorem 5.8 that, if $l\leqslant 2$, then f is not k-convex for any k. Otherwise, f is $(l-1)$-convex. □

Lemma 5.20. *Let $f(x_1,\dots,x_n)$ be a Boolean function which for some $k\geqslant 2$ is co-k-convex but not co-$(k+1)$-convex. Then, between any two false points of f at distance $k+1$, all the points are true.*

Proof. Let $\boldsymbol{a}\in\{0,1\}^{n-k-1}$, and let $\boldsymbol{x}=(\alpha_1,\dots,\alpha_{k+1},\boldsymbol{a})$ and $\boldsymbol{y}=(\bar{\alpha}_1,\dots,\bar{\alpha}_{k+1},\boldsymbol{a})$ be two false points of f at distance $k+1$. We shall show that any point of the form $\boldsymbol{z}=(*,\dots,*,\boldsymbol{a})$, where the $*$'s represent arbitrary 0-1 values and $\boldsymbol{z}$ differs from both $\boldsymbol{x}$ and $\boldsymbol{y}$, is a true point of f. First we proceed by showing that neither $\boldsymbol{x}$ nor $\boldsymbol{y}$ can have a false neighbor. Indeed, assume to the contrary that $\boldsymbol{u}=(\bar{\alpha}_1,\alpha_2,\dots,\alpha_{k+1},\boldsymbol{a})$ is a false neighbor of $\boldsymbol{x}$. Since f is co-k-convex, and since $d(\boldsymbol{u},\boldsymbol{y})=k$, any point between $\boldsymbol{u}$ and $\boldsymbol{y}$, (i.e. a point of the form $(\bar{\alpha}_1,*,\dots,*,\boldsymbol{a})$) is a false point. In particular, the point $\boldsymbol{v}=(\bar{\alpha},\alpha_2,\bar{\alpha}_3,\dots,\bar{\alpha}_{k+1},\boldsymbol{a})$ is false. However, since $d(\boldsymbol{x},\boldsymbol{v})=k$, any point of the form $(*,\alpha_2,*,\dots,*,\boldsymbol{a})$ must also be a false point. Consider now the false points $(\bar{\alpha}_1,\bar{\alpha}_2,\beta_3,\dots,\beta_{k+1},\boldsymbol{a})$ and $(\alpha_1,\alpha_2,\beta_3,\dots,\beta_{k+1},\boldsymbol{a})$ where the β_i's are arbitrary. Since these points are at distance two, any point between them (i.e. a point of the form $(*,*,\beta_3,\dots,\beta_{k+1},\boldsymbol{a})$) must also be false, and since the β_i's are arbitrary, any point of the form $(*,\dots,*,\boldsymbol{a})$ is a false point. Since $\boldsymbol{x}$ and $\boldsymbol{y}$ were chosen arbitrarily, this contradicts our assumption that f is not co-$(k+1)$-convex. Hence, all the neighbors of $\boldsymbol{x}$, and by symmetry all the neighbors of $\boldsymbol{y}$, are true points.

Let us now pick a vector $\boldsymbol{w}$ between $\boldsymbol{x}$ and $\boldsymbol{y}$, such that, $\boldsymbol{w}$ is not a neighbor of $\boldsymbol{x}$ or $\boldsymbol{y}$. Since $\boldsymbol{w}$ is between $\boldsymbol{x}$ and $\boldsymbol{y}$, $d(\boldsymbol{x},\boldsymbol{w})$ must be strictly less than $k+1$. If $\boldsymbol{w}$ is a false point of f, then any point between $\boldsymbol{x}$ and $\boldsymbol{w}$ must also be false. One of these false points is a neighbor of $\boldsymbol{x}$, in contradiction with what was proven above. Hence, we can finally conclude that any point of the form $\boldsymbol{z}$ is a true point. □

Theorem 5.21. *Let $\Phi = \bigvee_{i=1}^{m} T_i$ be a DNF having the extended SAT tractability property and representing a function $f(x_1,\ldots,x_n)$. Then, for any fixed K, we can decide in polynomial time whether f is co-K-convex.*

Proof. First we shall decide if f is co-2-convex. If not, then f cannot be co-k-convex for any $k \geqslant 2$. If on the other hand f is co-2-convex, we shall proceed in the following way. For any $3 \leqslant k \leqslant K$, after establishing that f is co-$(k-1)$- convex, we shall try to establish whether f is co-k-convex.

In order to find out whether f is co-2-convex, we shall search for two false points of f at distance 2, having at least one true point between them. If such a pair of false points exists, f is not co-2-convex, otherwise it is. For each pair of variables x_i and x_j, let $D_{ij}^{\boldsymbol{\alpha}}$ be the projection of Φ on $((i,j),\boldsymbol{\alpha})$, where $\boldsymbol{\alpha} = (0,0),(0,1),(1,0)$ or $(1,1)$. We shall apply the polynomial time SAT solving algorithm $\mathscr{B}$ to the DNF $D_1 = D_{ij}^{(0,0)} \vee D_{ij}^{(1,1)}$. If $\mathscr{B}$ reports that D_1 is satisfiable, then f has two false points at distance 2, namely,

$$\boldsymbol{x} = (\beta_1,\ldots,\beta_{i-1},0,\beta_{i+1},\ldots,\beta_{j-1},0,\beta_{j+1},\ldots,\beta_n)$$

and

$$\boldsymbol{y} = (\beta_1,\ldots,\beta_{i-1},1,\beta_{i+1},\ldots,\beta_{j-1},1,\beta_{j+1},\ldots,\beta_n),$$

assuming $\boldsymbol{\beta} \in \{0,1\}^{n-2}$ is a solution of D_1. (Note that we do not need to know this satisfying solution explicitly; its existence is sufficient for the proof). We now have to check whether these two points have a true point between them. We shall apply the polynomial time SAT solving algorithm $\mathscr{B}$ to the DNFs $D_2 = D_{ij}^{(0,1)} \vee D_1$ and $D_3 = D_{ij}^{(1,0)} \vee D_1$. Obviously the two false points $\boldsymbol{x}$ and $\boldsymbol{y}$ have a true point in between them if and only if either D_2 or D_3 is not satisfiable. If both are satisfiable, we proceed to find two other false points at distance 2 that might have a true point between. We apply $\mathscr{B}$ to the DNF $D_4 = D_{ij}^{(0,1)} \vee D_{ij}^{(1,0)}$. If $\mathscr{B}$ reports that D_4 is satisfiable, we check if either $D_5 = D_{ij}^{(0,0)} \vee D_4$ or $D_6 = D_{ij}^{(1,1)} \vee D_4$ is satisfiable. If not, then f cannot be co-2-convex. Otherwise, we pick another pair of variables and repeat the whole procedure. In the end, we either find a pair of false points satisfying the hypothesis and conclude that f is not co-2-convex, or we find no such pair and conclude that f is co-2-convex. In the worst case, the number of calls to the algorithm $\mathscr{B}$ is

$6 \times \binom{n}{2}$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Now, assuming we have established that f is co-$(k-1)$-convex, we proceed to check if f is co-k-convex for $3 \leqslant k \leqslant K$. Using Lemma 5.20, if we know that f is co-$(k-1)$-convex, we can decide that f is not co-k-convex if and only if we find two false points at distance k with everything in between as true points. Therefore, we shall search for two false points satisfying the above hypothesis. In order to do this, we fix a subset of variables of cardinality k. Without loss of generality, let $A = \{x_1, \ldots, x_k\}$ be such a set. Furthermore, let $\boldsymbol{\alpha}_i \in \{0,1\}^k$, $i \in [1 \ldots 2^{|A|}]$ be all possible assignments to the set of variables in A i.e. $\boldsymbol{\alpha}_1 = (0, \ldots, 0)$, $\boldsymbol{\alpha}_2 = (0, 0, \ldots, 1)$ and so on $\boldsymbol{\alpha}_{2^{|A|}} = (1, \ldots, 1)$. We say that $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_j$ is an *orthogonal pair* iff $\boldsymbol{\alpha}_i = \overline{\boldsymbol{\alpha}}_j$. For each orthogonal pair $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_j$ (altogether 2^{k-1} of them), let D_i be the projection of Φ on $(A, \boldsymbol{\alpha}_i)$ and D_j be the projection of Φ on $(A, \boldsymbol{\alpha}_j)$. Run $\mathscr{B}$ on $D = D_i \vee D_j$. If D is satisfiable, then there exist two false points say $\boldsymbol{x}$ and $\boldsymbol{y}$ at distance k. Now, for these false points we have to check if everything between them is true. In order to do this, let D_p be the projection of Φ on the assignment $x_p = \alpha_i^p$ and $x_{p+1} = \alpha_j^{p+1}$ for $p \in [1 \ldots k-1]$ and let D_k be the projection of Φ on the assignment $x_k = \alpha_i^k$ and $x_1 = \alpha_j^1$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ have all true points between them, then one of the DNFs $D_q \vee D$ will be identically 1 for some $q \in [1 \ldots k]$. So, we run $\mathscr{B}$ on each one of these DNFs. If any one of them is identically 1, we can stop, knowing that f is not co-k-convex. Otherwise, we try to see if another pair of false points have all true points between them. In other words, we pick another subset of variables of cardinality k and repeat the whole procedure. The maximum number of calls to $\mathscr{B}$ for iteration k is $2^{k-1} \times (k+1) \times \binom{n}{k}$. $\square$

Theorem 5.22. *Let Φ be a DNF having the extended SAT tractability property and representing a function f. For any $k = 2, \ldots, n$, we can decide in polynomial time whethere f is strongly k-convex.*

Proof. We proceed as in the proof of Theorem 5.19 by finding a prime DNF representation of f. Using the result of Corollary 5.14, f is strongly k-convex (for any $k = 2, \ldots, n$) if and only if f is one of the $2n+2$ simple functions. $\square$

6. Concluding remarks

We have seen that the classes of Boolean functions examined in this paper can be represented as in Fig. 2, where all the inclusions are proper.

We have also seen that the DNFs of a Boolean function reflect to a large extent their various connectedness properties. In spite of that, most of the recognition problems of the classes of functions studied in this paper were proven to be CoNP-hard. The only exceptions concern DNFs of connected functions, and DNFs for which some associated SAT problems are polynomially solvable.

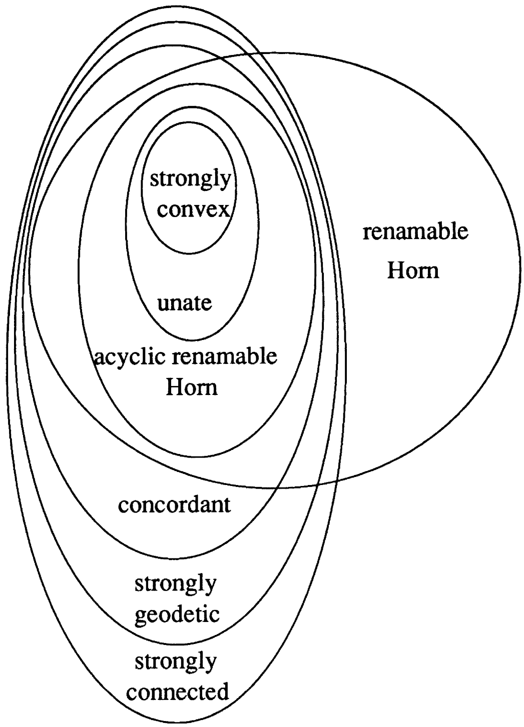

Fig. 2. Relationships between classes of Boolean functions (all inclusions are proper).

An interesting class of Boolean functions introduced in this paper is that of concordant functions, generalizing the class of unate functions. Concordant functions also include the class of acyclic renamable Horn functions and are included in the class of strongly geodetic functions.

It has been shown that the set of the true points of a k-convex function consists of entire subcubes at large Hamming distances from each other. Perhaps not suprisingly, the class of strongly k-convex functions was shown to contain very few functions; in fact, the only strongly k-convex functions are those that depend effectively on at most one variable. Finally, it has been shown that there are only two kinds of connected functions: those which are k-convex for any k, and those which are not k-convex for any k.

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