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Do CAPM results hold in a dynamic economy? A numerical analysis

Levent Akdeniz[a], W. Davis Dechert[b,*]

[a] *Faculty of Business Administration, Bilkent University, Ankara, Turkey*
[b] *Department of Economics, University of Houston, Houston, TX, USA*

Abstract

In this research we use the projection method (reported by Judd) to find numerical solutions to the Euler equations of a stochastic dynamic growth model. The model that we solve is Brock's asset pricing model for a variety of parameterizations of the production functions. Using simulated data from the model, conjectures (which are not analytically tractable) can be verified. We show that the market portfolio is mean-variance efficient in this dynamic context. We also show a result that is not available from the static CAPM theory: the efficient frontier shifts up and down over the business cycle.

Keywords: Computational economics; Projection methods; Asset pricing models; Stochastic growth models
JEL classification: C63, D90, G12

In this paper we present a method for solving the multifirm stochastic growth model of Brock (1979). After obtaining a solution to the growth model, we derive a solution to the asset pricing model of Brock (1982) using the duality between the two models. Brock's asset pricing model forms an intertemporal general equilibrium theory of capital asset pricing, thus with this solution we can analyze a number of financial issues in a dynamic, general equilibrium framework.

In general, there are no closed form solutions for stochastic growth models, except for the specific case of logarithmic utility and Cobb–Douglas production

* Corresponding author. Email: wdechert@facstaff.wisc.edu

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functions with 100% discounting.[1] The recent improvements in computer hardware and computational methods have enabled economists to study these models. As a result more and more economists have been using computational methods for solving dynamic economic models over the last two decades. A description and comparison of some of the various methods can be found in Taylor and Uhlig (1990) and Danthine and Donaldson (1995). An important observation pointed out by Judd (1995) is that computational methods provide a strong complement to economic theory for those models which are not analytically tractable.

Brock's model has been used and cited in the literature over the past two decades. However, some researchers have only used the specification mentioned above which is characterized by a linear investment function. Others starting with Kydland and Prescott (1982) have used a quadratic approximation to the value function which also results in a linear policy function. As a result, these studies fail to capture the wide variety of implications that are available from Brock's model.

In this study we use numerical techniques to obtain solutions for Brock's model for any type of utility and production functions. Using the projection method described in Judd (1992), we solve the stochastic growth model for the optimal investment functions. After solving for the optimal investment functions, we then solve the asset pricing model for the asset pricing functions, profits and returns.

There are a wide variety of applications of asset pricing models. In this study we examine to what extent the classical Capital Asset Pricing Model (CAPM) results hold in a dynamic framework. Although the CAPM has come under attack ever since its inception, it has nevertheless been the main model of asset pricing in modern finance theory. From our point of view there are two problems with the CAPM: (1) it is a static model, and (2) the sources of uncertainty are arbitrary random variables. However, in the Brock (1979, 1982) models (which are inherently dynamic), the shocks directly affect the production processes, and hence asset returns are linked to the underlying sources of production and uncertainty. Solutions of the dynamic model give us a framework within which to check the validity of some of the CAPM results. For example, does the return on the market portfolio lie on the mean-variance efficient set in each period?[2]

By studying which of the CAPM results hold in a dynamic context, we get a better understanding of the strengths and weaknesses of the model. As we report in Section 5, there are some results that are essentially dynamic in character, and are not available from the classical CAPM.

In Sections 2 and 3 we outline the stochastic growth model and the corresponding asset pricing model of Brock (1979, 1982). In Section 4, we describe

[1] Benhabib and Rustichini (1994) have found closed form solutions for a more general class of stochastic growth models.

[2] Roll (1977) argues that the only legitimate test of the CAPM is whether or not the market portfolio is mean-variance efficient.

the projection method of Judd (1992) and illustrate how to solve for the optimal investment functions. We also show how to numerically derive the asset pricing functions given the optimal investment functions. In Section 5 we present results for some parameterizations of the model. In Section 6 we discuss some of the possible extensions of the model and other applications of this research.

1. The growth model

In this section we introduce the growth model as in Brock (1979). Here, we heavily borrow from Brock and recapitulate the essential elements of the model. There is a single consumer with a per period utility function, u. There are n production processes with production functions g_i and depreciation rates, δ_i. Define the net production functions

$$f_i(x_{it}, \xi_t) = g(x_{it}, \xi_t) + (1 - \delta_i)x_{it}.$$

Then the planner's problem is to solve

$$\max_{\{c_t, x_{it}\}} \quad \mathsf{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], \tag{1.1}$$

$$\text{s.t.} \quad \boldsymbol{x}_t = \sum_{i=1}^{N} x_{it}, \tag{1.2}$$

$$y_{t+1} = \sum_{i=1}^{N} f(x_{it}, \xi_t), \tag{1.3}$$

$$c_t + \boldsymbol{x}_t = y_t, \tag{1.4}$$

$$c_t, x_{it} \geq 0, \tag{1.5}$$

$$y_0 \text{ historically given}, \tag{1.6}$$

where E is the mathematical expectation, β the discount factor on future utility, u the utility function of consumption, c_t the consumption at date t, $\boldsymbol{x}_t$ the capital stock at date t, y_t the output at date t, f_i the production function of process i, x_{it} the capital allocated to process i at date t, δ_i the depreciation rate for capital in process i, ξ_t the random shock.

For a description and interpretation of the model see Brock (1982). The main assumptions for this model are

(A1) the functions u and f_i are concave, increasing, twice continuously differentiable, and satisfy the Inada conditions,

(A2) the stochastic process is independent and identically distributed,

(A3) the maximization problem has a unique optimal solution.

The first-order conditions[3] for the intertemporal maximization are

$$u'(c_{t-1}) = \beta \mathsf{E}_{t-1}[u'(c_t) f_i'(x_{it}, \xi_i)] \tag{1.7}$$

and the transversality condition is

$$\lim_{t \to \infty} \beta^t \mathsf{E}_0 [u'(c_t) \boldsymbol{x}_t] = 0. \tag{1.8}$$

Eq. (1.7) is the one that is used below to derive a numerical solution to the growth model. Since the problem given by Eqs. (1.1)–(1.6) is time-stationary the optimal levels of c_t, x_t, and $\boldsymbol{x}_{it}$ are functions of the output level y_t, and can be written as

$$c_t = g(y_t), \qquad \boldsymbol{x}_t = h(y_t), \qquad x_{it} = h_i(y_t). \tag{1.9}$$

In Section 3 we describe how to use Judd's (1992) technique to solve for the investment functions, h_i. The first two functions in Eq. (1.9) can be expressed in terms of these investment functions:

$$h(y) = \sum_{i-1}^{N} h_i(y), \qquad c(y) = y - h(y).$$

2. An asset pricing model

The asset pricing model in Brock (1982) is much like the Lucas (1978) model. The difference between the two models is that former includes production. By incorporating the shocks in with the production processes, Brock's model has the sources of uncertainty in the asset prices directly tied to economic fluctuations in output levels (and hence in profits).

There is one representative consumer whose preferences are given in Eq. (1.1). There are N different firms. Firms rent capital from the consumers at the rate r_{it} to maximize their profits:

$$\pi_{i,t+1} = f_i(x_{it}, \xi_t) - r_{it} x_{it}. \tag{2.1}$$

Each firm rents capital given ξ_t. Here r_{it} denotes the interest rate on capital in industry i at date t and is determined within the model. Asset shares are normalized so that there is one perfectly divisible equity share for each firm. Ownership of the share in firm i at date t entitles the consumer to the firms profits at date t. It is also assumed (as in Lucas, 1978) that the optimum levels of asset prices, capital, consumption and output form a rational expectations equilibrium.

[3] We assume that $x_{it} > 0$ wp1. If not then the Kuhn–Tucker-type conditions in Brock (1979) must be used.

2.1. The model

The representative consumer solves the following problem:

$$\max \quad \mathsf{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], \tag{2.2}$$

$$\text{s.t.} \quad c_t + x_t + P_t Z_t \leq \pi_t Z_{t-1} + P_t Z_{t-1} + \sum_{i=1}^{N} r_{i,t-1} x_{i,t-1}, \tag{2.3}$$

$$c_t, Z_t, x_{it} \geq 0, \tag{2.4}$$

$$r_{it} = f_i'(x_{it}, \xi_t), \tag{2.5}$$

$$\pi_{it} = f_i(x_{i,t-1}, \xi_{t-1}) - f_i'(x_{i,t-1}, \xi_{t-1}) x_{i,t-1}, \tag{2.6}$$

where P_{it} is the price of one share of firm i at date t, Z_{it} the number of shares of firm i owned by the consumer at date t, π_{it} the profits of firm i at date t.

The details of the model are in Brock (1982). The first-order conditions are

$$P_{it} u'(c_t) = \beta \mathsf{E}_t[u'(c_{t+1})(\pi_{i,t+1} + P_{i,t+1})] \tag{2.7}$$

and

$$u'(c_t) = \beta \mathsf{E}_t[u'(c_{t+1}) f_i'(x_{i,t+1}, \xi_{t+1})] \tag{2.8}$$

from which we get the prices for the assets. In addition, the transversality conditions

$$\lim_{t \to \infty} \beta^t \mathsf{E}_0 \left[u'(c_t) \sum_i P_{it} Z_{it} \right] = 0, \tag{2.9}$$

$$\lim_{t \to \infty} \beta^t \mathsf{E}_0 [u'(c_t) x_t] = 0. \tag{2.10}$$

are needed to fully characterize the optimum. Brock (1979) shows that there is a duality between the growth model (1.1)–(1.6) and the asset pricing model (2.2)–(2.6), and that the solution to the growth model is also the solution to the asset pricing model. Once the growth model is solved, the asset pricing functions can be solved for by Eq. (2.7). As for the transversality condition, Judd (1992) points out that it implies that we are looking for the bounded solution[4] to the growth model.

Since (in equilibrium) there is one share of each asset, the value weighted market portfolio is

$$M_t = \sum_{i=1}^{n} P_{it},$$

[4] The optimal solution remains in a bounded interval: $0 < a < y_t < b < \infty$ for all t.

and the dividends (profits) are

$$\pi_t = \sum_{i=1}^{n} \pi_{it}.$$

Define the return on each asset by

$$R_{it} = \frac{P_{i,t+1} + \pi_{i,t+1}}{P_{it}},$$

and the return on the market portfolio by

$$R_{Mt} = \frac{M_{t+1} + \pi_{t+1}}{M_t}.$$

From the first-order condition (2.7), the return on each asset satisfies

$$u'(c_t) = \beta \mathsf{E}[u'(c_{t+1})R_{it}]$$

which is the efficiency condition from the growth model. By summing Eq. (2.7), we get that the return on the market portfolio satisfies

$$u'(c_t) = \beta \mathsf{E}[u'(c_{t+1})R_{Mt}]$$

and so it too is efficient. (This is one of the hypotheses of the CAPM, which in this model is a consequence of the optimizing behavior of the consumer.)
Now define the profit, consumption and output functions by

$$\pi_i(y,\xi) = f_i(h_i(y),\xi) - h_i(y)f_i'(h_i(y),\xi), \tag{2.11}$$

$$c(y) = y - \sum_{i=1}^{n} h_i(y), \tag{2.12}$$

$$Y(y,\xi) = \sum_{i=1}^{n} f_i(h_i(y),\xi), \tag{2.13}$$

and the asset pricing functions by

$$P_i(y)u'(c(y)) = \beta \mathsf{E}[u'(c(Y(y,\xi)))(P_i(Y(y,\xi)) + \pi_i(y,\xi))]. \tag{2.14}$$

Once we have the pricing functions we next define the return functions

$$R_i(y) = \frac{P_i(Y(y,\xi)) + \pi_i(y,\xi)}{P_i(y)},$$

$$R_M(y) = \frac{\sum_{i=1}^{n} [P_i(Y(y,\xi)) + \pi_i(y,\xi)]}{\sum_{i=1}^{n} P_i(y)}.$$

The mean functions are

$$\mu_i(y) = \mathsf{E}[R_i(y,\xi)],$$
$$\mu_M(y) = \mathsf{E}[R_M(y,\xi)],$$

and the functions for the variances

$$\sigma_i^2(y) = \mathsf{var}[R_i(y,\xi)],$$
$$\sigma_M^2(y) = \mathsf{var}[R_M(y,\xi)].$$

In the standard CAPM risk is measured by β. As we see above, the returns and their variances are functions of the output level, y, and so we define short turn βs by

$$\beta_i(y) = \frac{\mathsf{cov}(R_i(y,\xi), R_M(y,\xi))}{\sigma_M^2(y)}, \tag{2.15}$$

and long run βs as the average over the limiting distribution of y:

$$\beta_i = \int \beta_i(y)\,\mathrm{d}F_Y(y). \tag{2.16}$$

Based on the argument in Brock and Mirman (1972), Brock (1979) shows that this limiting distribution exists.

3. Numerical solution of the model

Except for a very special case of the utility and production functions, there is no closed-form solution for the optimal investment functions. We must use numerical techniques instead, in order to analyze the properties of the solutions to the asset pricing model. We have chosen to use the projection method in Judd (1992). Other methods are discussed in the special volume of the Journal of Business & Economic Statistics (January 1990). One advantage of Judd's method is that it is fast, and that the entire investment functions can be estimated. A second advantage is that the solutions are extremely accurate.[5]

Rather than solving for a specific solution to the Euler equation (1.7), this method takes advantage of the time stationarity of the solutions and solves for the optimal investment functions, $h_i(y)$. In order to do this, the optimal policy functions are represented by Chebyshev sums:[6]

$$h(y,\boldsymbol{a}_i) = \sum_{j=1}^{n} a_{ij}\Psi_j(y), \tag{3.1}$$

[5] As Judd points out, this methods yields results that are often accurate to within \$1 in \$1,000,000. See Judd (1992) for a comparison with other methods.

[6] See Rivlin (1990) for a description of approximation methods using Chebyshev polynomials.

where Ψ_j is the $j-1$ Chebyshev polynomial which has been shifted to the estimation range $[y_{\min}, y_{\max}]$

$$\Psi_j(y) = T_{j-1}\left(2\frac{y - y_{\min}}{y_{\max} - y_{\min}} - 1\right), \tag{3.2}$$

and n is the number of coefficients used. Using the fact that along an optimal solution

$$x_{i,t+1} = h_i(y_t),$$

and

$$\begin{aligned} x_{i,t+2} &= h_i(y_{t+1}) \\ &= h_i\left(\sum_{i=1}^{N} f_i(h_i(y_t), \xi_t)\right), \end{aligned}$$

the residual functions are defined from the Euler equation as

$$\begin{aligned} \Re_i(y, \boldsymbol{a}) = u'&\left(y - \sum_j h(y, \boldsymbol{a}_j)\right) \\ &-\beta \mathrm{E}\left[u'\left(\sum_j \left[f_j(h(y, \boldsymbol{a}_j), \xi) - h\left(\sum_k f_k(h(y, \boldsymbol{a}_k), \xi), \boldsymbol{a}_j\right)\right]\right)\right. \\ &\left.\times f_i'(h(y, \boldsymbol{a}_i), \xi)\right]. \end{aligned} \tag{3.3}$$

In the numerical simulations we use a discrete probability space for the random shocks. In the text we will use the expectation notation instead of writing out the sums explicitly. When evaluated at the optimal policy functions, these residual functions are zero for all values of y. To solve for the optimal values of the coefficients of the Chebyshev sums, take a discrete set of y's that correspond to the zeros of a Chebyshev polynomial of order m, greater than the number of coefficients. The projection[7] of these residual functions are

$$\mathfrak{P}_{ij}(\boldsymbol{a}) = \sum_{k=1}^{m} \Re_i(y_k, \boldsymbol{a})\Psi_j(y_k), \tag{3.4}$$

[7] See Fletcher (1984) for the use of the Galerkin projection method in numerical analysis.

and we seek the values of the coefficients for which these projections are (simultaneously) zero. Numerically, this can be done with a Newton–Raphson routine.[8]

Once we have solved for the optimal investment functions we can solve Eq. (2.7) for the asset pricing functions. It turns out that this can be reduced to solving a set of linear equations. Using the notation of Eqs. (2.11)–(2.13), define $G_i(y) = P_i(y)u'(c(y))$ and

$$b_i(y) = \sum_s u'(c(Y(y, \xi_s)))\pi_i(y, \xi_s)q_s,$$

where a discrete random variable is used with $\mathsf{P}\{\xi = \xi_s\} = q_s$. (For programing purposes, a discrete random variable must be used.) We seek a solution to

$$\beta^{-1}G_i(y) = \sum_s G_i(Y(y, \xi_s))q_s + b_i(y), \tag{3.5}$$

for the functions G_i. One way to do this is to use Chebyshev approximations to G_i and to solve for the coefficients of the approximating polynomials. Let

$$G_i(y) = \sum_{j=1}^{n} c_{ij}\Psi_j(y),$$

where Ψ_j is defined in Eq. (3.2). Let $y_1, \ldots, y_m$ be the zeros of Ψ_m where $m > n+1$. Then we seek to find $c_{i1}, \ldots, c_{in}$ such that for $k = 1, \ldots, m$

$$\begin{aligned}\beta^{-1}\sum_{j=1}^{n} c_{ij}\Psi_j(y_k) &= \sum_s \sum_{j=0}^{n} c_{ij}\Psi_j((Y(y_k, \xi_j))q_s + b_i(y_k) \\ &= \sum_{j=1}^{n}\left(\sum_s \Psi_j((Y(y_k, \xi_s))q_s\right)c_{ij} + b_i(y_k),\end{aligned}$$

which is linear in the coefficients. However, there are $m > n$ equations here, and so, in general, we cannot expect them all to hold. So, as the solution take the coefficients that minimize the sum of the least-square errors between the left- and right-hand sides of the equations. For notational purposes define $\boldsymbol{T}$ to be the $m \times n$ matrix whose k, j element is $\Psi_j(y_k)$, and define $\boldsymbol{M}$ to be the $m \times n$ matrix whose k, j element is $\sum_s \Psi_i(Y(y_k, \xi_s))q_s$. Also, let $\boldsymbol{b}_i$ be the vector of elements of $b_i(y_k)$. Then in matrix notation we need to find $\boldsymbol{a}_i$ that minimizes

$$[(\beta^{-1}\boldsymbol{T} - \boldsymbol{M})\boldsymbol{a}_i - \boldsymbol{b}_i]'[(\beta^{-1}\boldsymbol{T} - \boldsymbol{M})\boldsymbol{a}_i - \boldsymbol{b}_i].$$

[8] We used a C language routine in Press et al. (1992).

The solution can be found be letting $\boldsymbol{A} = \beta^{-1}\boldsymbol{T} - \boldsymbol{M}$. From least-squares principles, the solution is

$$\boldsymbol{a}_i = (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{b}_i.$$

In the appendix it is shown that Eq. (3.5) defines a contraction mapping of modulus β on the space of bounded continuous functions, and so the procedure outlined above is well-defined and has a unique solution.

4. A numerical example

In this section, we present a solution to the model for one set of parameter values in order to demonstrate the numerical solution in detail. It is a common practice in this literature to calibrate the model so that the model of the economy displays certain properties in common with the actual economies. In this study, we use a CRRA utility function,

$$u(c) = \frac{c^{\gamma}}{\gamma},$$

where γ is the utility curvature. Campbell and Cochrane (1994) estimate that a CRRA utility function with a utility curvature parameter of -1.37 matches the postwar US data, and so we use $\gamma = -1.37$ for the value of the utility curvature parameter. In keeping with the common practice in this literature, we chose the value of the discount parameter, β, to be 0.97. On the production side, firms are characterized by the Cobb–Douglas production functions:

$$f(x, \xi) = \theta(\xi)x^{\alpha(\xi)} + (1 - \delta)x,$$

where ξ is the shock parameter in the production function. We choose the values of α and θ at random,[9] and pick the value of δ to correspond to values that agree with aggregate data. By using shocks to both the magnitude of the functions as well as to the technical coefficients, both output levels and elasticities are subject to random shocks. We solve the model for four equally likely states of uncertainty. The parameters for the firms are reported in Table 1. As can be seen from the table, fairly substantial variations in the parameter values are used.

Even though the estimation problem appears to be a straightforward computational exercise, there are a number of difficulties in actually implementing a procedure that converges to the desired policy functions. For example, for small

[9] Since these variables are based on the micro characteristics of various sectors of an economy, we would need detailed empirical studies on firms in order to calibrate them to the US economy. It would also require knowledge of how shocks affect firms, both individually and cross sectionally.

Table 1
Production function parameters

State	α	θ	δ
Firm 1			
1	0.78	0.23	0.16
2	0.52	0.19	0.16
3	0.35	0.17	0.16
4	0.24	0.14	0.16
Firm 2			
1	0.80	0.20	0.17
2	0.54	0.24	0.17
3	0.51	0.23	0.17
4	0.76	0.14	0.17
Firm 3			
1	0.26	0.15	0.18
2	0.62	0.25	0.18
3	0.30	0.25	0.18
4	0.72	0.19	0.18
Firm 4			
1	0.82	0.14	0.19
2	0.34	0.18	0.19
3	0.87	0.15	0.19
4	0.39	0.21	0.19
Firm 5			
1	0.74	0.20	0.20
2	0.35	0.13	0.20
3	0.10	0.28	0.20
4	0.56	0.10	0.20

values of inputs or consumption, the marginal products or marginal utility become so large that the computational procedure becomes unstable. Also, when the estimation interval, $[y_{\min}, y_{\max}]$, is large relative to the domain of the ergodic distribution of y_t, the accuracy of the estimation is poor. To overcome these problems, in the former case we used a quadratic approximation to both the production and utility functions at low levels of inputs and consumption;[10] in the latter case the estimation interval was reduced to be slightly larger than the range of the ergodic distribution of output, y_t.[11]

[10] At the solution, these approximations have no effect since the optimal levels of inputs and consumption are above the cutoff values. See Judd (1992, p. 430 footnote 14) for further details on this technique.

[11] The range of the ergodic distribution was found by simulating the model for 1500 repetitions.

able 2
hebyshev coefficients

	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
ptimal investment functions					
$_{i,1}$	2.123962 e−01	3.093897 e−01	2.130184 e−01	1.302617 e−01	1.351617 e−01
$_{i,2}$	1.268130 e−01	2.274925 e−01	1.153519 e−01	7.099912 e−02	6.890228 e−02
$_{i,3}$	1.074281 e−03	9.829991 e−03	−1.844800 e−03	−7.469934 e−04	2.810106 e−03
$_{i,4}$	−7.605716 e−05	−1.051157 e−03	4.938602 e−05	6.107679 e−05	1.807051 e−04
$_{i,5}$	1.425141 e−05	1.547462 e−04	5.300107 e−06	−7.790038 e−06	−9.447078 e−06
$_{i,6}$	−2.898387 e−06	−2.715017 e−05	−2.688727 e−06	1.098002−e06	−1.777999−e06
$_{i,7}$	9.435150 e−07	5.152574 e−06	6.433982 e−07	6.613542 e−08	4.470539 e−07
$_{i,8}$	8.067908 e−08	−1.311474 e−06	−3.594632 e−07	1.170111 e−07	−4.736169 e−07
sset pricing functions					
$_{i,1}$	4.875885 c+01	3.470750 e+01	6.000923 e+01	3.255002 e+01	3.471903 e+01
$_{i,2}$	−6.370022 e+00	−1.883479 e+00	−8.508065 e+00	−4.480307 e+00	−4.396293 e+00
$_{i,3}$	1.212786 e+00	2.315946 e−01	1.667265 e+00	8.618664 e−01	7.731952 e−01
$_{i,4}$	6.852115 e−02	7.256082 e−03	9.947493 e−02	4.917147 e−02	3.871699 e−02
$_{i,5}$	4.674486 e−03	3.132451 e−04	7.140142 e−03	3.377241 e−03	2.425267 e−03
$_{i,6}$	3.415680 e−04	1.599470 e−05	5.459517 e−04	2.478660 e−04	1.709297 e−04
$_{i,7}$	−9.794481 e−05	−3.212621 e−06	−1.596897 e−04	−7.130021 e−05	−4.823995 e−05
$_{i,8}$	−7.123452 e−06	−7.887880 e−08	−1.194525 e−05	−5.216887 e−06	−3.421859 e−06

Table 3
Residual errors

Output level	Firm	Error
	1	−3.711881 e−05
	2	−4.100356 e−05
1.035451	3	−3.350386 e−05
	4	−3.446781 e−05
	5	−3.975074 e−05
	1	−4.205853 e−04
	2	−4.298913 e−04
1.166399	3	−4.188296 e−04
	4	−4.157242 e−04
	5	−4.326260 e−04
	1	1.020113 e−05
	2	8.776299 e−06
1.297347	3	1.201577 e−05
	4	1.133725 e−05
	5	9.576837 e−06
	1	2.854980 e−04
	2	2.915838 e−04
1.428296	3	2.845688 e−04
	4	2.823997 e−04
	5	2.935146 e−04
	1	2.258665 e−05
	2	2.219140 e−05
1.559244	3	2.349294 e−05
	4	2.303016 e−05
	5	2.270281 e−05

The following values are used for this example:

Estimation interval:	[0.55,1.91].
Observed range of output:	[0.57,1.9].
Quadratic approximation cutoff:	0.025.
Number of Chebyshev coefficients:	8.
Number of zeros for the approximations:	18.

The coefficients of the optimal policy functions and the asset pricing functions are reported in Table 2. According to the Chebyshev approximation theorem, the coefficients should eventually decrease with the degree of the polynomial, which is the case for our example. The residual errors in Table 3 are normalized by the consumption level. As Judd (1992) points out, this gives a economic measure of the accuracy of the numerical solution. An error of 10^{-6}

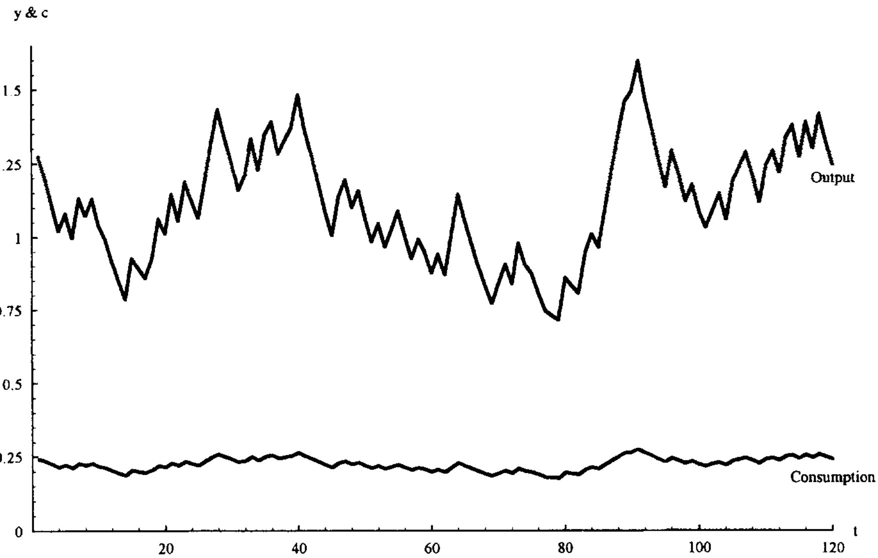

Fig. 1.

implies "... that agents made only a $1.00 mistake for every $1,000,000.00 they spent."[12]

The optimal policy functions and the optimal consumption function are graphed in Fig. 3. In the logarithmic utility function case with an isoelastic production function and no uncertainty, the optimal policy function (and hence the consumption function) is known to be linear. In our example, there appears to be a slight departure from linearity in the policy and consumption functions.[13]

We also simulated the economy with 120 realizations of the random shocks. Figs. 1 and 2 depict the results. Fig. 1 is the output levels for 120 successive periods. Although output fluctuates substantially, the pattern of consumption is smooth over all periods, which is consistent with observed data. In other words, this model replicates the typical pattern of widely fluctuating output levels and relatively constant levels of consumption over time. The corresponding prices and return on the assets and the return on the market portfolio are in Fig. 2. The (marginal utility weighted) returns for the 5 firms fluctuate around $\beta^{-1} \cong 1.03$, which is a consequence of the first-order condition (2.7).

[12] Judd (1992), pp. 438.

[13] At this stage, we do not have a means of directly estimating the accuracy of the policy function coefficients. We do report below a measure of accuracy in finding the zeros of the residual function (3.3), which shows how close we come to accurately estimating the Euler equation. As Judd (1995) points out, one of the goals for future research in the area of computational economics is to develop measures of accuracy for simulation models.

Table 4
Production function parameters

State	α	θ	δ
Firm 1			
1	0.67	0.15	0.16
2	0.86	0.17	0.16
3	0.21	0.15	0.16
4	0.74	0.21	0.16
Firm 2			
1	0.25	0.12	0.17
2	0.40	0.19	0.17
3	0.45	0.15	0.17
4	0.69	0.20	0.17
Firm 3			
1	0.74	0.19	0.18
2	0.39	0.19	0.18
3	0.21	0.16	0.18
4	0.62	0.11	0.18
Firm 4			
1	0.36	0.22	0.19
2	0.28	0.23	0.19
3	0.45	0.22	0.19
4	0.41	0.15	0.19
Firm 5			
1	0.67	0.11	0.20
2	0.35	0.20	0.20
3	0.81	0.18	0.20
4	0.26	0.22	0.20

One application of this study to financial economics is to analyze which results from the CAPM still hold in a dynamic general equilibrium framework. More specifically, we examine whether or not the return on the market portfolio lies on the mean-variance efficient frontier in each period. To do so, we calculate and plot the mean-variance efficient frontier along with the returns on the firms and the market portfolio for each period and verify if the return on the market portfolio lies on the frontier. We have done this for 50 randomly chosen different sets of production function parameters. In Figs. 4–6 we plot the efficient frontier for three levels of output [14] because in the dynamic model profits and returns fluctuate with the level of output. The graphs in Fig. 4 are based on the parameterization reported in Table 1. The parameters for Figs. 5 and 6 are reported in Tables 4 and 5, respectively.

[14] The levels are mean output, one standard deviation below mean output and one standard deviation above mean output.

Table 5
Production function parameters

State	α	θ	δ
Firm 1			
1	0.68	0.20	0.16
2	0.85	0.12	0.16
3	0.54	0.12	0.16
4	0.45	0.19	0.16
Firm 2			
1	0.22	0.10	0.17
2	0.37	0.18	0.17
3	0.20	0.23	0.17
4	0.59	0.12	0.17
Firm 3			
1	0.28	0.15	0.18
2	0.73	0.17	0.18
3	0.25	0.23	0.18
4	0.65	0.17	0.18
Firm 4			
1	0.83	0.15	0.19
2	0.67	0.16	0.19
3	0.68	0.14	0.19
4	0.72	0.26	0.19
Firm 5			
1	0.32	0.18	0.20
2	0.24	0.24	0.20
3	0.62	0.18	0.20
4	0.66	0.24	0.20

An interesting feature in these figures is that the market portfolio is always on the opportunity set. While this is a *conclusion* of the CAPM, there is no guarantee that it also holds in the dynamic model. Nevertheless, based on the 50 different parameterizations, we have found that this property holds in the dynamic model. Judd (1995) argues that one of the contributions of computational methods is to 'prove' theorems numerically. If a property holds for a large number of randomly chosen parameterizations, then we can have a high degree of confidence that the property is true.

Another feature in Figs. 4–6 is that the frontier shifts down for higher levels of output. This reflects the fact that in the dynamic model asset profits and returns vary with the output level, which is not a part of the CAPM.[15]

[15] Another result that we can see from these graphs is that there is apparent spanning in this model with 4 states of uncertainty and 5 randomly chosen firms.

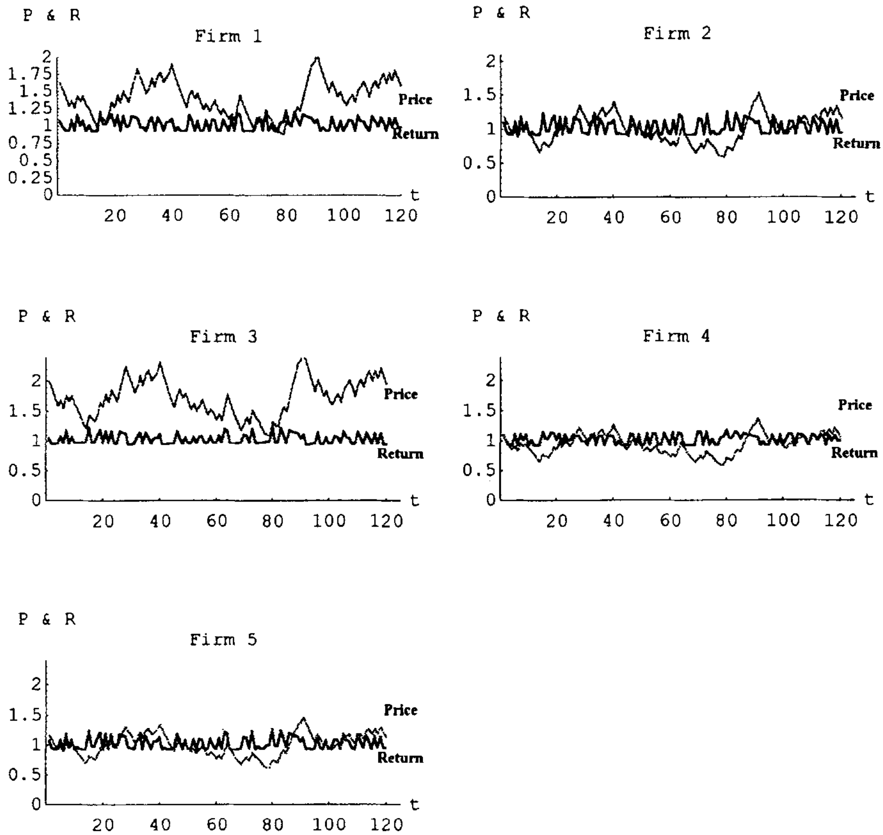

Fig. 2.

5. Conclusion

There are a number of issues in financial economics that can be addressed with the solution to the dynamic model presented in this paper. In the previous section, we demonstrated one of the CAPM results that seems to hold for this model. Another is the linearity and the slope of the security market line. We can also use this model to examine some of the issues raised by Fama and French (1993) who conclude that an asset's beta is not the only measure of risk. For example large and small firms can be modeled, and the resulting simulated data studied for the impact that firm size has on the asset return. Similarly, price to earnings ratios can also be calculated to study their effect on asset returns.

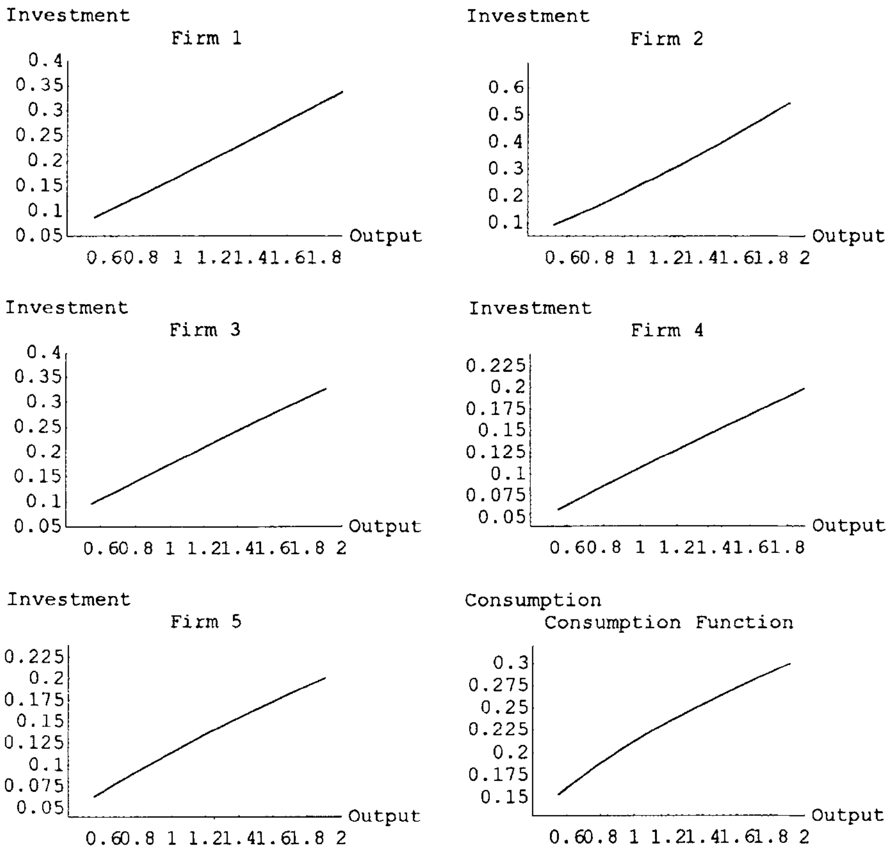

Fig. 3. Optimal policy functions.

There are several extensions of this model that would allow other questions to be addressed. Prescott (1982) conjectures "... the principal source of risk is the business cycle. If this is correct, it surely will be necessary to introduce the labor supply decision ..." By adding labor in the production process and leisure in the utility function,[16] the model can be used to analyze business cycles.

As for the impact of leverage on the return of an asset, the financial model would have to be expanded to include bonds as well as shares. However, the impact of the risk of bankruptcy would not be captured, since firms with strictly

[16] Also stocks can be added to a two good utility function, which would allow for a comparison between taste shocks and production shocks.

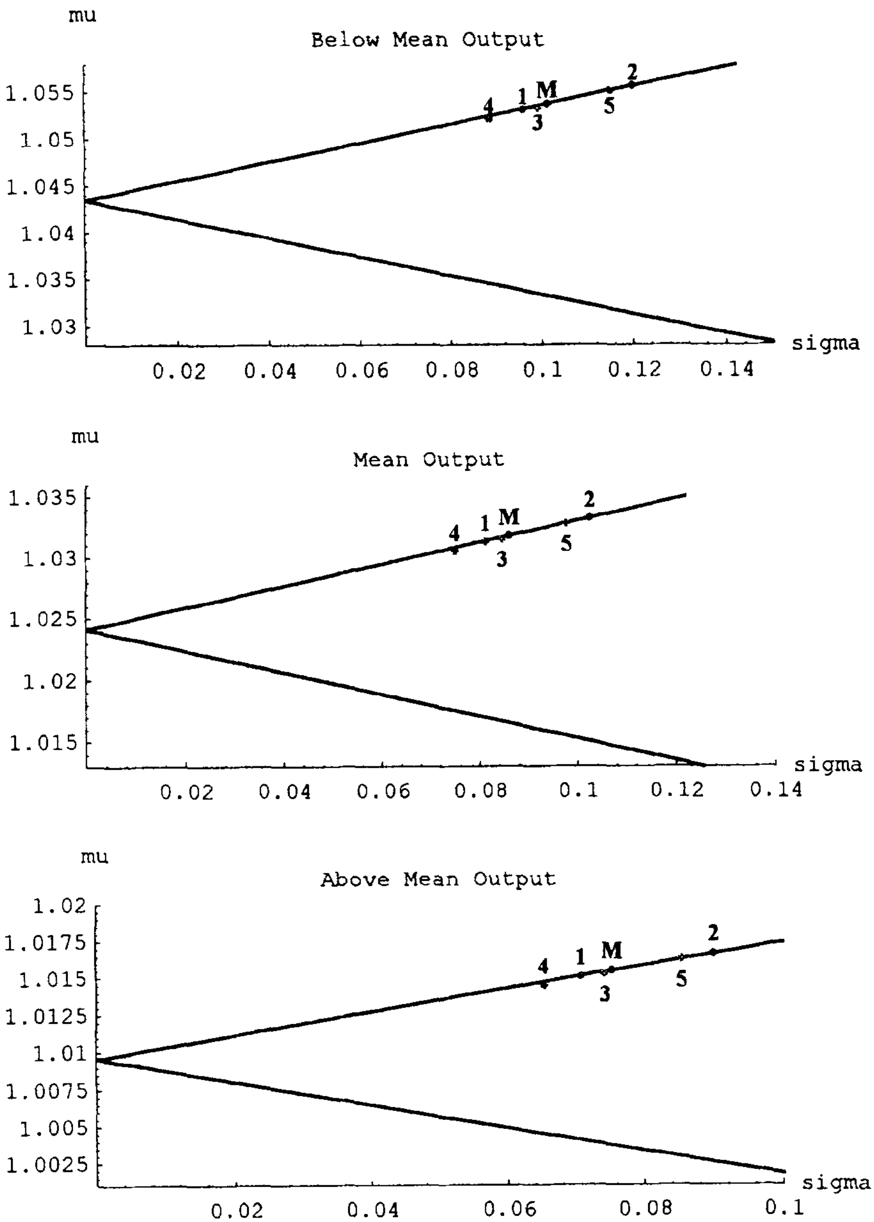

Fig. 4. Investment opportunity set.

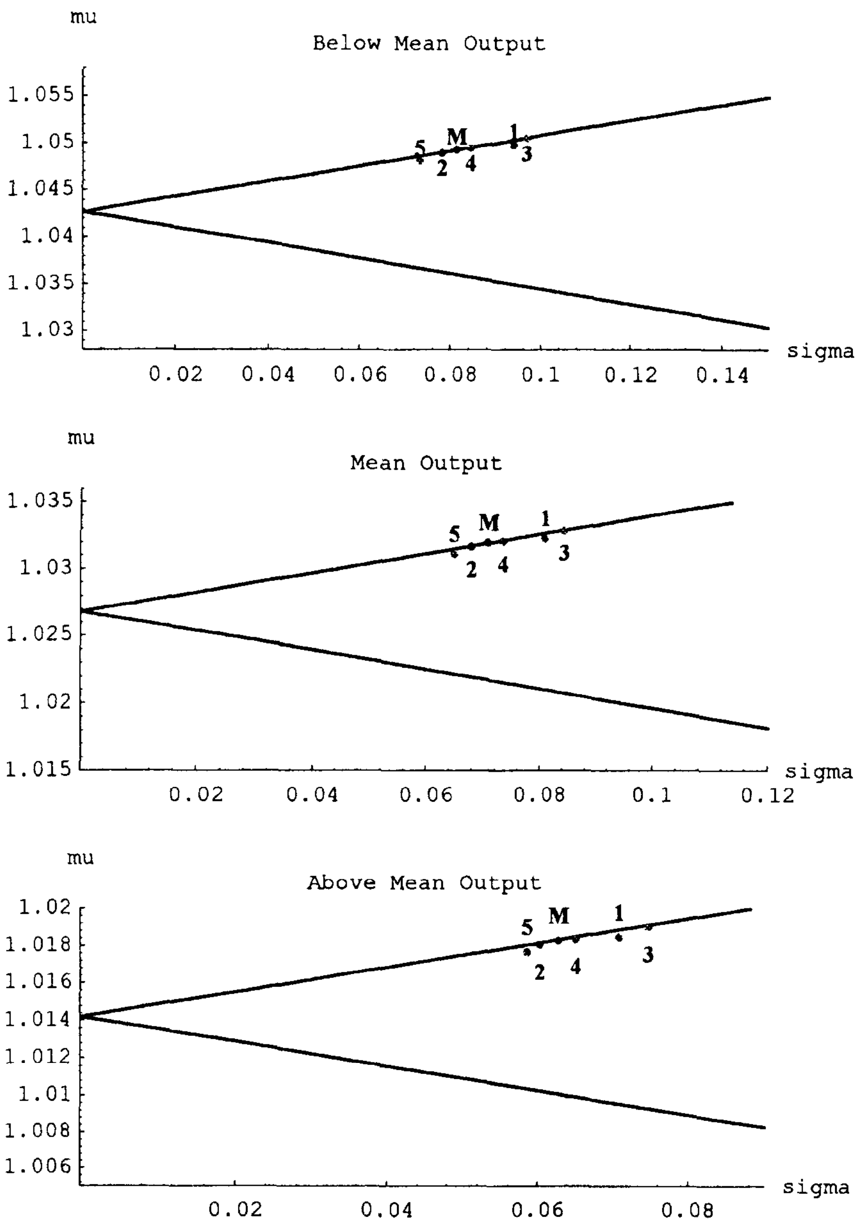

Fig. 5. Investment opportunity set.

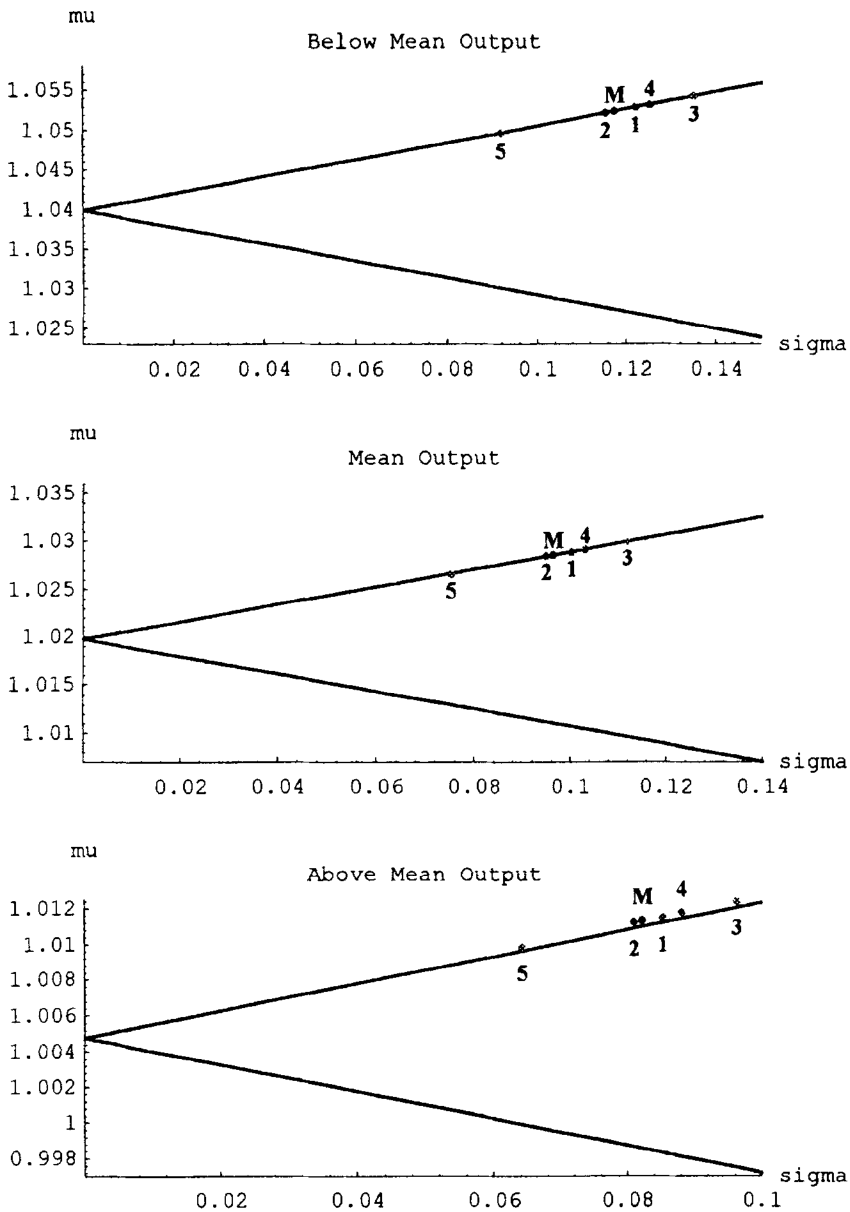

Fig. 6. Investment opportunity set.

concave production functions that satisfy the Inada conditions will never go out of business.

One of the ways in which numerical models are useful in real world applications is to fit the model to real data by a suitable choice of parameters. The model we have presented is unlikely to fit the asset prices of corporate stocks. In Figs. 1 and 2 it can be seen that the swings in output levels and asset prices and returns are far greater than those observed in any real economy. This is due to several factors. In our model the consumers have homogeneous tastes, there is only one good in the economy, and there are no labor supply decisions. Clearly, in order to achieve results that would simulate a real economy, the model must incorporate these features.

Appendix

Theorem. There is a unique solution to the functional equation

$$G(y) = \beta \sum_s G(Y(y, \xi_s))q_s + \beta b(y).$$

Proof. Define the operator $\mathfrak{T}$ on the space of functions $C([y_{\min}, y_{\max}])$ by

$$(\mathfrak{T}v)(y) = \beta \sum_s v(Y(y, \xi_s))q_s + \beta b(y),$$

where for each $s, Y(\cdot, \xi_s)$ is a continuous function mapping $[y_{\min}, y_{\max}]$ to itself and $b \in C([y_{\min}, y_{\max}])$. Then for $v, w \in C([y_{\min}, y_{\max}])$

$$\begin{aligned} |(\mathfrak{T}v)(y) - (\mathfrak{T}w)(y)| &= \beta \left| \sum_s [v(Y(y, \xi_s) - w(Y(y, \xi_s)]q_s \right| \\ &\leq \beta \sum_s |v(Y(y, \xi_s) - w(Y(y, \xi_s)|q_s \\ &\leq \beta \sum_s \|v - w\|q_s = \beta\|v - w\|, \end{aligned}$$

where $\|\cdot\|$ is the sup norm on $C([y_{\min}, y_{\max}])$. Therefore, $\mathfrak{T}$ is a contraction mapping of modulus β, and since $\beta < 1$ there is a unique element, $G \in C([y_{\min}, y_{\max}])$, such that

$$G = \mathfrak{T}G.$$

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