

Thus, we conclude that the optimization problem is equivalent to

$$\min_{\mu \in \mathcal{R}^+} \mu$$

such that

$$\begin{aligned} \langle X - Y, Z_r \rangle &= \langle T_{11}, Z_r \rangle, \quad \forall r = 1, 2, \dots, N \\ \sum_{j=1}^n \sum_{k=0}^T |F_{\alpha(ii)}(x_{ij}(k) + y_{ij}(k))| V_{\alpha(jj)} &\leq \mu \quad \forall \alpha, \\ \forall i &= 1, 2, \dots, m, \text{ and some } T \\ x_{ij} \geq 0, y_{ij} \geq 0, \mu \geq 0 \quad \forall i, j, k. &\quad \square \end{aligned}$$

It is clear that the infimum values at each consecutive application of Steps 2) and 3) will be monotonically nonincreasing and bounded below by zero. Thus the iteration converges. Whenever a desirable robustness level is achieved (as indicated by the value of the infimum at that step), the iteration procedure can be terminated at Step 3).

Note that the above optimization problem is nonconvex. Thus there is no guarantee that the iteration converges to the global minimum or even to a local minimum as it may get stuck at a saddle point.

## V. CONCLUSION

We have applied the Hadamard-weighting approach in [9] to the  $\ell^1$ -optimization case. The results developed in this paper allow one to design compensators which satisfy closed-loop decoupling specifications. Compensators which robustly decouple the system could also be designed using the procedure developed in this paper. These results provide new tools for control system designers to meet decoupling requirements in the presence of uncertainties.

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## Robust Adaptive Sampled-Data Control of a Class of Systems Under Structured Nonlinear Perturbations

Ogan Ocah and M. Erol Sezer

**Abstract**— A robust adaptive sampled-data feedback stabilization scheme is presented for a class of systems with nonlinear additive perturbations. The proposed controller generates a control input by using high-gain static or dynamic feedback from nonuniform sampled values of the output. A simple adaptation rule adjusts the gain and the sampling period of the controller.

**Index Terms**— Adaptive control, output feedback, robust control, sampled-data system.

## I. INTRODUCTION

High-gain feedback is a standard control technique for robust stabilization of systems in the presence of modeling uncertainties (see, for example, [1]–[7], in some of which the problem is considered in the framework of decentralized control). In the case of a single-input/single-output (SISO) system, design of such a controller requires that the system have stable zeros and its relative degree, the sign of its high-frequency gain, and the bounds of the system parameters or perturbations be known. Similar information is needed for multi-input/multi-output (MIMO) systems. It has been shown in [8] that for systems with relative degree one, robust stability can be achieved without the need to know the bounds of the perturbations by tuning the gain parameter adaptively. In [9], a similar result has been obtained for systems with higher relative degree, where an adaptation mechanism is employed to increment the gain parameter stepwise at discrete instants.

In this paper we focus on the same problem for the case where the controllers are allowed to operate on sampled values of the output only, rather than continuous-time measurements. The main difficulty arises from the fact that the sampling process changes the structure of the uncertainty, that is, any uncertainty in the continuous-time system is exponentiated in its discrete model after sampling. This makes a simple and useful characterization of permissible uncertainty structures very difficult. In [10], a sampled-data state-feedback controller was proposed for robust stabilization of systems under time-varying additive perturbations of a certain class. The controller, which simulates high-gain continuous-time feedback in the absence of perturbations, guarantees stability for a sufficiently small sampling period which depends on the bounds of perturbations. In [11], a simpler controller was proposed, together with an adaptation rule for the sampling period, which eliminates the need for *a priori* knowledge of the perturbation bounds. In this paper we extend the result of [10] and [11] to the case where perturbations are nonlinear and time-varying, and sampled measurements of the output rather than state are available for feedback. The controller we propose consists of a high-gain static or discrete dynamic feedback followed by an arbitrary generalized hold function. We first show that the proposed controller achieves robust stability for sufficiently small sampling periods and then present a simple adaptation mechanism

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which decreases the sampling period slowly until it is small enough. In this scheme, the sampling period has a double role: it also determines the controller gain.

## II. SYSTEM AND CONTROLLER STRUCTURE

We consider a SISO system  $\mathcal{S}$  described as

$$\begin{aligned} \mathcal{S}: \dot{x}_p(t) &= A_p x_p(t) + b_p u(t) + e_p[t, x_p(t)] \\ y(t) &= c_p^T x_p(t) \end{aligned} \quad (1)$$

where  $x_p(t) \in \mathcal{R}^n$  is the state,  $u(t), y(t) \in \mathcal{R}$  are the input and output of  $\mathcal{S}$ , respectively, and  $A_p, b_p$ , and  $c_p$  are constant matrices of appropriate dimensions.  $e_p[t, x_p(t)]$  in (1) stands for additive nonlinear perturbations to a linear, nominal system represented by the triple  $(A_p, b_p, c_p^T)$ .

We would like to stabilize  $\mathcal{S}$  using a discrete-time feedback controller operating on the sampled output values  $\{y(t_k)\}$ , where  $t_k$  are the sampling instants. For this we make the following assumptions concerning the nominal system and the perturbations.

- 1)  $(A_p, b_p, c_p^T)$  is controllable and observable.
- 2)  $(A_p, b_p, c_p^T)$  has stable zeros, that is, with  $h(s) = c_p^T (sI - A_p)^{-1} b_p = p_0 p(s)/q(s)$ , the set of zeros of the numerator polynomial  $p(s) = s^{n_0} + p_1 s^{n_0-1} + \dots + p_{n_0}$  is included in the open left-half complex plane.
- 3) The high-frequency gain  $p_0$  and the relative degree  $n_1 = n - n_0$  of  $h(s)$  above are known.
- 4) The perturbations are of the form

$$e_p(t, x) = b_p g(t, x) + h(t, y)$$

where  $g$  and  $h$  satisfy for all  $t, y \in \mathcal{R}, x \in \mathcal{R}^n$

$$\begin{aligned} \|g(t, x)\| &\leq \alpha_g \|x\| \\ \|h(t, y)\| &\leq \alpha_h |y| \end{aligned} \quad (2)$$

for some (unknown) constants  $\alpha_g, \alpha_h > 0$ .

Our choice of a stabilizing sampled-data controller is based on a special internal structure of the system  $\mathcal{S}$  described by the following result of [12].

*Lemma 1:* Under Assumptions 1)–3), there exists a nonsingular matrix  $M$  such that

$$\begin{aligned} M^{-1} A_p M &= \begin{bmatrix} A_0 & d_{01} c_1^T \\ b_1 d_{10}^T & A_1 + b_1 d_{11}^T \end{bmatrix} \\ M^{-1} b_p &= p_0 \begin{bmatrix} 0 \\ b_1 \end{bmatrix} \\ c_p^T M &= [0 \quad c_1^T] \end{aligned} \quad (3)$$

where  $A_0 \in \mathcal{R}^{n_0 \times n_0}$  is a stable matrix whose eigenvalues are the zeros of  $p(s)$  defined in Assumption 2) above;  $A_1 \in \mathcal{R}^{n_1 \times n_1}$ ,  $b_1 \in \mathcal{R}^{n_1}$ , and  $c_1 \in \mathcal{R}^{n_1}$  have the structures

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \\ b_1 &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ c_1^T &= [1 \quad 0 \quad \dots \quad 0] \end{aligned} \quad (4)$$

and  $d_{01}, d_{10}$ , and  $d_{11}$  are constant vectors of appropriate dimensions.

We note that without any restrictions on  $\alpha_g$  and  $\alpha_h$  in (2), Assumption 2) is necessary in order to guarantee stabilizability of

$\mathcal{S}$ . This follows from the fact that a choice of the perturbations as  $g(t, x) = 0, h(t, y) = -p_0^{-1} A_p^m b_p y$  results in a system having uncontrollable modes at the zeros of  $p(s)$  as can easily be shown by using Lemma 1.

We now let

$$\begin{aligned} x(t) &= M^{-1} x_p(t) \\ &= [x_0^T(t) \quad x_1^T(t)]^T \end{aligned}$$

where  $M$  is as in Lemma 1, and  $x_0 \in \mathcal{R}^{n_0}$  and  $x_1 \in \mathcal{R}^{n_1}$  correspond to  $A_0$  and  $A_1$  in (3). Define the sampling periods as  $T_k = t_{k+1} - t_k$ , and consider a further transformation of the state as

$$\begin{aligned} x_k(s) &= D_k^{-1} x(t_k + sT_k) \\ &= \begin{bmatrix} x_{0k}(s) \\ x_{1k}(s) \end{bmatrix} \\ &= \begin{bmatrix} x_0(t_k + sT_k) \\ D_{1k}^{-1} x_1(t_k + sT_k) \end{bmatrix} \end{aligned} \quad (5)$$

for  $0 \leq s < 1$ , where

$$\begin{aligned} D_k &= \text{diag} \{I_{n_0}, D_{1k}\}, \\ D_{1k} &= \text{diag} \{T_k^{n_1-1}, \dots, T_k, 1\}. \end{aligned}$$

On noting from (4) that

$$\begin{aligned} D_{1k}^{-1} A_1 D_{1k} &= T_k^{-1} A_1 \\ c_1^T D_{1k} &= T_k^{n_1-1} c_1^T \\ D_{1k}^{-1} b_1 &= b_1 \end{aligned}$$

the dynamic behavior of  $\mathcal{S}$  over the  $k$ th sampling period  $[t_k, t_{k+1})$  can be described by

$$\begin{aligned} \mathcal{S}: \dot{x}_{0k}(s) &= T_k A_0 x_{0k}(s) + T_k e_{0k}[s, x_k(s)] \\ \dot{x}_{1k}(s) &= A_1 x_{1k}(s) + T_k e_{1k}[s, x_k(s)] + p_0 T_k b_1 u_k(s) \\ y_k(s) &= T_k^{n_1-1} c_1^T x_{1k}(s) \end{aligned} \quad (6)$$

where  $u_k(s) = u(t_k + sT_k), y_k(s) = y(t_k + sT_k)$ , and

$$\begin{aligned} e_{0k}(s, x_k) &= T_k^{n_1-1} d_{01} c_1^T x_{1k} \\ &\quad + h_0(t_k + sT_k, T_k^{n_1-1} c_1^T x_{1k}) \\ e_{1k}(s, x_k) &= p_0 b_1 [d_{10}^T x_{0k} + d_{11}^T D_{1k} x_{1k} \\ &\quad + g(t_k + sT_k, M D_k x_k)] \\ &\quad + D_{1k}^{-1} h_1(t_k + sT_k, T_k^{n_1-1} c_1^T x_{1k}) \end{aligned} \quad (7)$$

with  $M^{-1} h(t, y) = [h_0^T(t, y) \quad h_1^T(t, y)]^T$ . From (7) it follows that for  $0 < T_k \leq 1$

$$\begin{aligned} \|e_{0k}(s, x_k)\| &\leq \alpha_{01} \|x_{1k}\| \\ \|e_{1k}(s, x_k)\| &\leq \alpha_{10} \|x_{0k}\| + \alpha_{11} \|x_{1k}\| \end{aligned} \quad (8)$$

for some constants  $\alpha_{01}, \alpha_{10}, \alpha_{11} > 0$ , which depend on the system parameters  $A_p, b_p, c_p$  and the perturbation bounds  $\alpha_g, \alpha_h$  in (2).

Note that the transformation leading to (6) is the same as the lifting operation considered in [13], except that nonuniform sampling is used in (5).

We generate the control input to  $\mathcal{S}$  by a discrete-time dynamic feedback controller followed by a generalized hold function as

$$\begin{aligned} \mathcal{C}: x_c(t_{k+1}) &= A_c x_c(t_k) + T_k^{1-n_1} b_c y(t_k) \\ w(t_k) &= c_c^T x_c(t_k) + T_k^{1-n_1} y(t_k) \\ u_k(s) &= p_0^{-1} T_k^{-1} \psi(s) w(t_k), \quad 0 \leq s < 1 \end{aligned} \quad (9)$$

where  $x_c \in \mathcal{R}^{n_c}$  is the state and  $w \in \mathcal{R}$  is the output of  $\mathcal{C}$ , and  $\psi: [0, 1) \rightarrow \mathcal{R}$  is a bounded hold function. In the case of static output feedback, the controller in (9) reduces to

$$u_k(s) = p_0^{-1} T_k^{-n_1} \psi(s) y(t_k), \quad 0 \leq s < 1. \quad (10)$$

The system  $\mathcal{S}$  in (6) and the controller  $\mathcal{C}$  in (9) form a closed-loop hybrid system  $\hat{\mathcal{S}} = (\mathcal{S}, \mathcal{C})$ . The open-loop solutions of  $\hat{\mathcal{S}}$  are given for  $0 \leq s < 1$  as

$$\begin{aligned} x_{0k}(s) &= e^{T_k A_0 s} x_{0k}(0) + \xi_{0k}[s, x_{0k}(0), x_{1k}(0), w(t_k)] \\ x_{1k}(s) &= e^{A_1 s} x_{1k}(0) + \xi_{1k}[s, x_{0k}(0), x_{1k}(0), w(t_k)] \\ &\quad + \Gamma_1(\psi)w(t_k) \end{aligned} \quad (11)$$

where

$$\begin{aligned} \xi_{0k}[s, x_{0k}(0), x_{1k}(0), w(t_k)] &= T_k \int_0^s e^{T_k A_0 (s-\tau)} e_{0k}[\tau, x_{1k}(\tau)] d\tau \\ \xi_{1k}[s, x_{0k}(0), x_{1k}(0), w(t_k)] &= T_k \int_0^s e^{A_1 (s-\tau)} e_{1k}[\tau, x_{0k}(\tau), x_{1k}(\tau)] d\tau \end{aligned} \quad (12)$$

and

$$\Gamma_1(\psi) = \int_0^1 e^{A_1(1-\tau)} b_1 \psi(\tau) d\tau. \quad (13)$$

Defining the discrete-time signals

$$\begin{aligned} \hat{x}_0(k) &= x_{0k}(0) \\ \hat{x}_1(k) &= [x_{1k}^T(0) \quad x_c^T(t_k)]^T \end{aligned}$$

and using (9) and (11), the dynamic behavior of  $\hat{\mathcal{S}}$  at the sampling instants is described by a discrete-time system

$$\begin{aligned} \mathcal{D}: \hat{x}_0(k+1) &= \hat{\Phi}_0(k) \hat{x}_0(k) + \hat{\xi}_0[k, \hat{x}_0(k), \hat{x}_1(k)] \\ \hat{x}_1(k+1) &= \hat{\Phi}_1 \hat{x}_1(k) + \hat{\xi}_1[k, \hat{x}_0(k), \hat{x}_1(k)] \end{aligned} \quad (14)$$

where

$$\begin{aligned} \hat{\Phi}_0(k) &= e^{T_k A_0} \\ \hat{\Phi}_1 &= \begin{bmatrix} \Phi_1 + \Gamma_1(\psi)c_1^T & \Gamma_1(\psi)c_c^T \\ b_c c_1^T & A_c \end{bmatrix} \end{aligned} \quad (15)$$

with  $\Phi_1 = e^{A_1}$ . In the case of static output feedback as in (10),  $\hat{\Phi}_1$  in (15) reduces to

$$\hat{\Phi}_1 = \Phi_1 + \Gamma_1(\psi)c_1^T. \quad (16)$$

The terms  $\hat{\xi}$  in (14) are due to the perturbations  $\xi_{0k}$  and  $\xi_{1k}$  in (11). The following lemma gives bounds on  $\hat{\xi}$ , which will be the key to stabilization of the discrete model  $\mathcal{D}$ .

*Lemma 2:* Suppose that the sampling periods satisfy

$$\begin{aligned} T_{k+1} &\leq T_k \leq 1 \\ \left( \frac{T_k}{T_{k+1}} \right)^{n_1-1} &\leq 1 + T_k. \end{aligned} \quad (17)$$

Then the perturbation terms  $\hat{\xi}$  in (14) are bounded as

$$\begin{aligned} \|\hat{\xi}_0(k, \hat{x}_0, \hat{x}_1)\| &\leq T_k^2 \hat{\beta}_{00} \|\hat{x}_0\| + T_k \hat{\beta}_{01} \|\hat{x}_1\| \\ \|\hat{\xi}_1(k, \hat{x}_0, \hat{x}_1)\| &\leq T_k \hat{\beta}_{10} \|\hat{x}_0\| + T_k \hat{\beta}_{11} \|\hat{x}_1\| \end{aligned} \quad (18)$$

for some constants  $\hat{\beta}$ 's which depend on the nominal system parameters and the perturbation bounds.

*Proof:* See the Appendix.  $\square$

In the next section, we investigate stabilizability of  $\mathcal{D}$  by a suitable choice of the discrete controller parameters ( $A_c, b_c, c_c$ ) and the generalized hold function  $\psi$  in (9) and the sampling periods  $T_k$ .

### III. STABILIZATION OF THE DISCRETE MODEL

We first note that due to the special structures of  $A_1, b_1$ , and  $c_1$ , the pair  $(\Phi_1, c_1^T)$  is observable, and the pairs  $(A_1, b_1)$  and  $(\Phi_1, \Gamma_1)$  are controllable, where  $\Gamma_1 = \Gamma_1(1)$ .

First, consider the case where static output feedback is used so that  $\hat{\Phi}_1$  is as given in (16). Observability of the pair  $(\Phi_1, c_1^T)$  implies that there exists  $\Psi_1 \in \mathcal{R}^m$  such that  $\Phi_1 + \Psi_1 c_1^T$  has a desired spectrum. On the other hand, controllability of the pair  $(A_1, b_1)$  implies that for any  $\Psi_1, \psi(s)$  in (9) can be chosen to satisfy  $\Gamma_1(\psi) = \Psi_1$ . As a result,  $\psi(s)$  can be chosen to assign any stable spectrum to  $\hat{\Phi}_1 = \Phi_1 + \Gamma_1(\psi)c_1^T = \Phi_1 + \Psi_1 c_1^T$ . Next, consider the case where  $\psi(s) = \psi_c$  (a constant, corresponding to a zero-order hold). Then from (13) we have  $\Gamma_1(\psi) = \Gamma_1 \psi_c$ , and from (15)

$$\hat{\Phi}_1 = \begin{bmatrix} \Phi_1 + \Gamma_1 \psi_c c_1^T & \Gamma_1 \psi_c c_c^T \\ b_c c_1^T & A_c \end{bmatrix}.$$

Note that  $\hat{\Phi}_1$  represents the system matrix of a hypothetical system consisting of a discrete plant  $(\Phi_1, \Gamma_1, c_1^T)$  and a discrete dynamic output feedback compensator  $(A_c, b_c, \psi_c c_c^T, \psi_c)$ . Since the plant  $(\Phi_1, \Gamma_1, c_1^T)$  is controllable and observable, the compensator  $(A_c, b_c, \psi_c c_c^T, \psi_c)$  with  $n_c \geq n_1 - 1$  can be chosen to result in a  $\hat{\Phi}_1$  with a desired spectrum [14]. A wide choice of  $\psi(s)$  and the controller exists between the two extreme cases.

Suppose that the generalized hold function  $\psi(s)$  and the discrete feedback controller  $\mathcal{C}$  are designed to have a Schur-stable  $\hat{\Phi}_1$ . Since  $A_0$  is Hurwitz-stable by assumption, there exist positive definite matrices  $P_0$  and  $P_1$  satisfying

$$\begin{aligned} A_0^T P_0 + P_0 A_0 &= -I \\ \hat{\Phi}_1^T P_1 \hat{\Phi}_1 - P_1 &= -I. \end{aligned}$$

Let

$$v(\hat{x}_0, \hat{x}_1) = \hat{x}_0^T P_0 \hat{x}_0 + \hat{x}_1^T P_1 \hat{x}_1 \quad (19)$$

be a candidate for a Lyapunov function for the system  $\mathcal{D}$  in (14). Noting that

$$\begin{aligned} \hat{\Phi}_0^T(k) P_0 \hat{\Phi}_0(k) - P_0 &= \int_0^{T_k} \frac{d}{dt} (e^{A_0^T t} P_0 e^{A_0 t}) dt \\ &= - \int_0^{T_k} e^{A_0^T t} e^{A_0 t} dt \end{aligned}$$

so that  $\|\hat{\Phi}_0^T(k) P_0 \hat{\Phi}_0(k) - P_0\| \leq -\gamma_0 T_k$  for some  $\gamma_0 > 0$ , and using (18), the difference of  $v$  along the solutions of  $\mathcal{D}$  can be computed and bounded for  $T_k$  satisfying (17) as

$$\begin{aligned} \Delta v(k) &\leq -\gamma_0 T_k \|\hat{x}_0\|^2 - \|\hat{x}_1\|^2 + 2\hat{\xi}_0^T P_0 \hat{\Phi}_0 \hat{x}_0 \\ &\quad + 2\hat{\xi}_1^T P_1 \hat{\Phi}_1 \hat{x}_1 + \hat{\xi}_0^T P_0 \hat{\xi}_0 + \hat{\xi}_1^T P_1 \hat{\xi}_1 \\ &\leq -T_k \nu^T(k) W(T_k) \nu(k) \end{aligned}$$

where  $\nu(k) = [\|\hat{x}_0(k)\| \quad \|\hat{x}_1(k)\|]^T$

$$W(T_k) = \begin{bmatrix} \gamma_0 - T_k q_{00}(T_k) & -q_{01}(T_k) \\ -q_{01}(T_k) & T_k^{-1} - q_{11}(T_k) \end{bmatrix}$$

and  $q$ 's are polynomials in  $T_k$  of degree at most 2 with nonnegative coefficients independent of  $T_k$ . Thus, there exists a sufficiently small  $T^* \leq 1$  such that provided  $T_k \leq T^*$  in addition to (17), we have

$$\Delta v(k) \leq -\sigma T_k v(k) \quad (20)$$

for some  $\sigma > 0$ . This shows that  $\mathcal{D}$  in (14) can be made exponentially stable.

From the proof of Lemma 2 in the Appendix, it follows that the open-loop solutions  $x_k(s)$  in (11) of  $\mathcal{S}$  are bounded for a bounded input sequence  $\{w(t_k)\}$ . Hence, if the discrete-time system  $\mathcal{D}$  in (14)

is asymptotically stable, and if  $T_k$  are also bounded from below so that  $t_k = t_0 + \sum_{j=1}^k T_j \rightarrow \infty$  as  $k \rightarrow \infty$ , then the closed-loop sampled-data system  $\hat{S}$  is also asymptotically stable (in the continuous sense). We summarize the above results as a theorem.

*Theorem 1:* Suppose the controller parameters  $(A_c, b_c, c_c)$  and the generalized hold function  $\psi$  in (9) are chosen to have  $\hat{\Phi}_1$  in (9) exponentially stable and that the sampling periods  $T_k$  satisfy (17) and (20). Then the closed-loop discrete system  $\mathcal{D}$  in (14) is exponentially stable. If, in addition,  $T_k \geq T^*$ ,  $k = 0, 1, \dots$  for some  $T^* > 0$ , then the closed-loop sampled-data system  $\hat{S}$  is exponentially stable.

From the development leading to Theorem 1, we observe that the choice of  $\mathcal{C}$  is independent of the system parameters and perturbation bounds except  $n_1$  and  $p_0$ . However, the sampling intervals  $T_k$  should be smaller than a critical value  $T^*$ , which depends on the nominal system parameters and the perturbation bounds. To eliminate the need to know these bounds, we propose in the next section an adaptation mechanism which decreases the values of  $T_k$  slowly until it is small enough to stabilize the system.

#### IV. ADAPTATION OF THE SAMPLING INTERVALS

We employ a simple adaptation rule for the sampling intervals

$$T_{k+1}^{-1} = T_k^{-1} + T_k S_k \quad (21)$$

where  $T_0 \leq 1$  is arbitrary, and

$$S_k = \min \{c_0, \alpha_y |y(t_k)|^2 + \alpha_c \|x_c(t_k)\|^2\}$$

with  $c_0 = 2^{1/(n_1-1)} - 1$ , and  $\alpha_y, \alpha_c > 0$  being arbitrary numbers. This choice guarantees that  $\{T_k\}$  satisfies the inequalities in (17).

Two cases are possible.

*Case I:*  $T_k \leq T^*$  for some  $k^* \geq 0$ . Then  $\mathcal{D}$  is exponentially stable. Also, noting that  $S_k \leq \eta v(k)$  for some  $\eta > 0$  where  $v$  is the Lyapunov function in (19), from (20) and (21) we have  $T_{k+1}^{-1} \leq T_k^{-1} - (\eta/\sigma)\Delta v(k)$  for  $k \geq k^*$  so that

$$T_k^{-1} \leq T_{k^*}^{-1} + \frac{\eta}{\sigma} v(k^*).$$

Thus  $\lim_{k \rightarrow \infty} T_k^{-1} = T_{k^*}^{-1}$  exists, and Theorem 1 guarantees stability of the closed-loop sampled-data system  $\hat{S}$ .

*Case II:*  $T_k > T^*$  for all  $k \geq 0$ . In this case, since  $\{T_k\}$  is nonincreasing,  $\lim_{k \rightarrow \infty} T_k = T_\infty < \infty$  exists. Then from (21) we have

$$\begin{aligned} T^* \sum_{k=0}^{\infty} S_k &< \sum_{k=0}^{\infty} T_k S_k \\ &= T_\infty^{-1} - T_0^{-1} \\ &< \infty \end{aligned} \quad (22)$$

which implies that the set  $\mathcal{K} = \{k | S_k = c_0\} = \{k | \alpha_y |y(t_k)|^2 + \alpha_c \|x_c(t_k)\|^2 \geq c_0\}$  is finite. Since solutions of  $\mathcal{D}$  cannot escape infinity in finite steps,  $y(t_k)$  and  $x_c(t_k)$  are bounded on  $\mathcal{K}$ . Then, (22) further implies that  $\sum_{k=0}^{\infty} (\alpha_y |y(t_k)|^2 + \alpha_c \|x_c(t_k)\|^2) < \infty$ . Thus  $\lim_{k \rightarrow \infty} y(t_k) = 0$  and  $\lim_{k \rightarrow \infty} z(t_k) = 0$ , which shows that  $\mathcal{D}$  is stable. However, internal stability of the adaptive closed-loop system  $\hat{S}$  cannot be guaranteed due to a possibility of the existence of hidden oscillations. To avoid the difficulty, we introduce a small randomness in  $T_k$  and assume that  $\lim_{t \rightarrow \infty} y(t) = 0$ . Then,  $\lim_{k \rightarrow \infty} u(t_k) = 0$ , and the fact that  $T_k > T^*$  for all  $k \geq 0$ , together with boundedness of  $\psi$ , imply that  $\lim_{t \rightarrow \infty} u(t) = 0$ . We then complete our analysis with the following result of [15].

*Lemma 3:* Under Assumptions 1)–4), if  $\lim_{t \rightarrow \infty} u(t) = \lim_{t \rightarrow \infty} y(t) = 0$  for the system  $\mathcal{S}$  in (1), then  $\lim_{t \rightarrow \infty} x(t) = 0$ .

In conclusion, if the closed-loop adaptive sampled-data systems has no hidden oscillations in the output, then it is stable in the continuous sense for the case  $T_k > T^*$  too.

#### V. EXAMPLE

To illustrate our results we consider the equation of a damped inverted pendulum

$$\ddot{\theta} = u - c_1 \sin \theta + c_2 \dot{\theta}$$

where  $\theta$  is the clockwise angular displacement from the vertical,  $u$  is the normalized control torque, and the parameters  $c_1, c_2 \geq 0$  are determined by the damping coefficient, mass, and the length of the pendulum. With  $x_1 = \theta, x_2 = \dot{\theta}$ , we get the state equations

$$\begin{aligned} \mathcal{S}: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ &+ \begin{bmatrix} 0 \\ c_1 \sin x_1 - c_2 x_2 \end{bmatrix} \\ y &= x_1 \end{aligned}$$

where the terms containing  $c_1$  and  $c_2$  are treated as perturbations. The nominal system with  $h(s) = 1/s^2$  has high-frequency gain  $p_0 = 1$  and relative degree  $n_1 = 2$ . Assumptions 1)–4) are satisfied, and the nominal system matrices are already in the form in Lemma 1, with  $A_0$  nonexisting.

We first consider a dynamic controller followed by a zero-order hold. After few trials, we choose the controller parameters as  $a_c = -0.15, b_c = -0.75, c_c = 1, \psi(s) = \psi_c = -0.5$ , which results in a nominal discrete model having the poles at  $z_{1,2} = 0.8 \mp j0.4$  and  $z_3 = 0$ . We choose the adaptation rule for the sampling periods as  $T_{k+1}^{-1} = T_k^{-1} + \min\{1, 5[y^2(t_k) + x_c^2(k)]\}$ . The simulation results corresponding to arbitrarily selected system parameters  $c_1 = c_2 = 1$  and the initial conditions  $x_1(0) = x_2(0) = T_0 = 0.5$  are shown in Fig. 1, which are obtained by Runge–Kutta method with a step size of  $0.01T_k$  for the  $k$ th sampling period. It is observed that the controller stabilizes the system with a reasonable control input and with the sampling period converging to a not too small steady-state value.

Next, we consider a static controller as in (10). A choice of  $\psi(s) = -19.2s + 55.2s^2 - 36s^3$  results in a  $\hat{\Phi}_1$  having the same nonzero eigenvalues as  $\hat{\Phi}_1$  above does. Since  $\psi(0) = \psi(1) = 0$ , this choice of  $\psi$  also guarantees a continuous input  $u(t)$  independent of  $y(t_k)$ . The simulation results for the same system parameters and initial conditions as before and with the adaptation rule  $T_{k+1}^{-1} = T_k^{-1} + \min\{1, 5y^2(t_k)\}$  indicate that stability is achieved without the sampling periods getting too small; however, the input is highly oscillatory. This is a further verification of the observation in [16], where it was argued that generalized hold functions result in poor intersample behavior.

We note that in both of the above simulations, adaptation of the sampling period was necessary. In both cases, a fixed sampling period at  $T_k = T_0 = 0.5$  resulted in an unstable closed-loop system. By trial, the critical value of a fixed sampling period that resulted in a stable system was found to be about  $T_k \approx 0.4$  (for the chosen initial conditions). However, in both cases, the adaptation rule decreased  $T_k$  to a steady-state value about half of this critical value. This observation suggests that the adaptation rule can be modified to allow for an increase in the sampling period after the system is taken under control. With this in mind, we changed the adaptation rule to increase  $T_k$  slightly whenever the decrease in the previous step is smaller than a certain percent. Fig. 2 shows the variation of  $T_k$  with the same dynamic discrete controller considered above and the modified adaptation rule for two different initial values. In both cases the system was stable with responses almost identical to those in Fig. 1.

#### VI. CONCLUDING REMARKS

We would like to discuss few points about our results.

As explained in Section III, the design of the controller parameters  $(A_c, b_c, c_c^T)$ , and  $\psi$  in (9) is independent of the choice of the

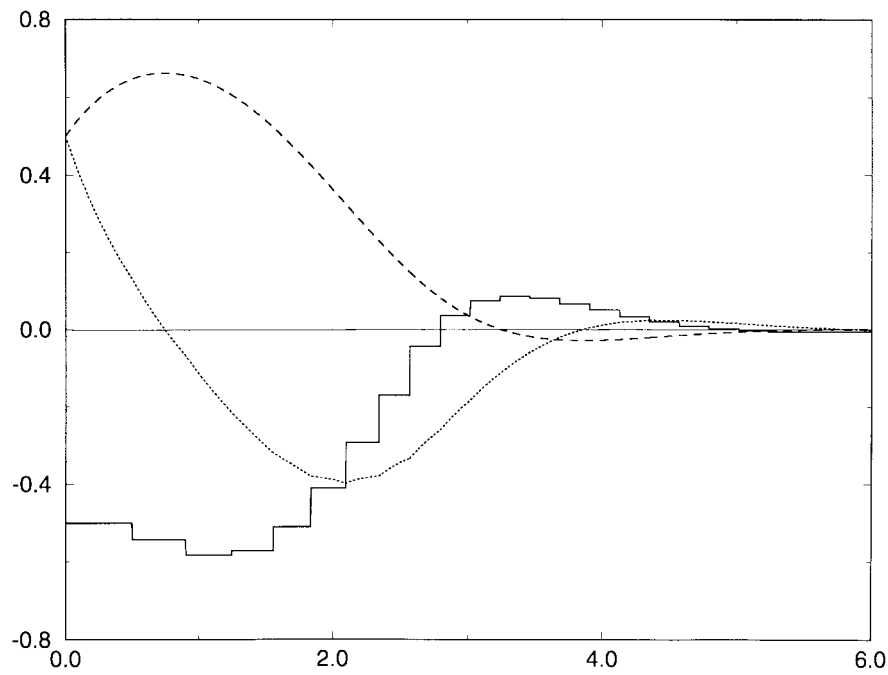


Fig. 1. States and input of the inverted pendulum with dynamic compensator and zero-order hold ( $x_1$ : dashed,  $x_2$ : dotted,  $u/2$ : solid).

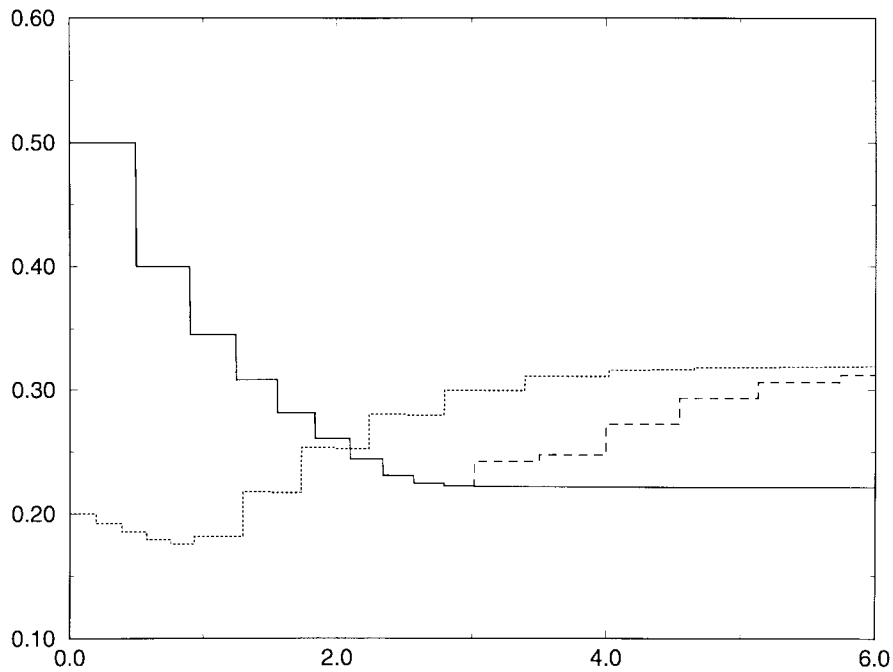


Fig. 2. Sampling period with standard (solid) and modified (dashed and dotted) adaptations.

sampling period  $T_k$ . In the case of dynamic controller with zero-order hold, the discrete model of the nominal system is treated (after inclusion of the  $d_{01}$ ,  $d_{10}$ , and  $d_{11}$  terms in the corresponding perturbation terms) as having the pulse transfer function  $H(z) = [d_0(z)/d_0(z)]H_1(z)$ , where  $d_0(z) = \det(zI - e^{A_0})$  corresponds to the uncontrollable and unobservable part of the system represented by  $A_0$ , and  $H_1(z) = c_1^T(zI - \Phi_1)^{-1}\Gamma_1 = (1 - z^{-1})\mathcal{Z}\{1/s^{n_1+1}\}$ , where the  $\mathcal{Z}$ -transform is taken with unity sampling period, describes the high-frequency behavior. The design of the controller parameters then reduces to finding  $H_c(z) = \psi_c[1 + c_c^T(zI - A_c)^{-1}b_c]$  such that the closed-loop pulse transfer function  $\hat{H}(z) =$

$[1 - H_1(z)H_c(z)]^{-1}H_1(z)$  has desired (stable) poles. The actual controller of (9) is obtained by a simple scaling with  $T_k$ .

A closer look at the development leading to Theorem 1 reveals that for Case I considered in Section IV

$$v(k + K) \leq v(k) \prod_{l=0}^{K-1} (1 - \sigma T_{k+l})$$

for any  $k \geq k^*$ ,  $K > 0$ , where  $\sigma$  is as in (20). This shows that the sampled-data closed-loop system has an equivalent continuous-time

degree of stability

$$\rho_k = \lim_{K \rightarrow \infty} - \frac{\sum_{l=0}^{K-1} \ln(1 - \sigma T_{k+l})}{2 \left\{ \sum_{l=0}^{K-1} T_{k+l} \right\}}.$$

In particular, in steady-state when  $T_k \approx T_\infty$ ,  $\rho_\infty \approx -\sigma/2$ , consistent with the expected behavior of the closed-loop system.

#### APPENDIX PROOF OF LEMMA 2

We first find suitable bounds for the  $\xi$  terms in (12). For this purpose, we define

$$E_k(s, x_k) = \begin{bmatrix} T_k A_0 x_{0k} + T_k e_{0k}(s, x_{1k}) \\ A_{11} x_{1k} + T_k e_{1k}(s, x_{0k}, x_{1k}) \end{bmatrix}$$

$$F(s) = \begin{bmatrix} 0 \\ b_1 \psi(s) \end{bmatrix}.$$

Then, with  $u(s)$  as in (9), (6) can be written in a compact form as

$$\dot{x}_k(s) = E_k(s, x_k) + F(s)w(t_k), \quad 0 \leq s < 1. \quad (23)$$

The solution of (23) is given by

$$x_k(s) = x_k(0) + \int_0^s \{E_k[\tau, x_k(\tau)] + F(\tau)w(t_k)\} d\tau. \quad (24)$$

Using the bounds in (8) and boundedness of  $\psi(s)$ , we obtain from (24)

$$\|x_k(s)\| \leq \|x_k(0)\| + \int_0^s [\alpha_E \|x_k(\tau)\| + \alpha_F |w(t_k)|] d\tau \quad (25)$$

for some constants  $\alpha_E, \alpha_F > 0$ . Using a variation of the Bellman–Gronwall lemma [17], (25) implies that

$$\|x_k(s)\| \leq e^{\alpha_E s} \|x_k(0)\| + \int_0^s \alpha_F e^{\alpha_E(s-\tau)} |w(t_k)| d\tau$$

$$\leq \alpha_x [\|x_{0k}(0)\| + \|x_{1k}(0)\|] + \alpha_w |w(t_k)| \quad (26)$$

for  $0 \leq s < 1$ , where  $\alpha_x, \alpha_w > 0$  are constants. Taking the norm of  $\xi_{1k}$  in (12), and using (8) and (26),  $\xi_{1k}$  is easily bounded as

$$\|\xi_{1k}(s, x_0, x_1, w)\| \leq T_k \beta_{10} \|x_0\| + T_k \beta_{11} \|x_1\|$$

$$+ T_k \beta_{12} |w|. \quad (27)$$

Now, using (27), we can bound  $x_{1k}$  in (11) as

$$\|x_{1k}(s)\| \leq T_k \gamma_{10} \|x_{0k}(0)\| + \gamma_{11} \|x_{1k}(0)\| + \gamma_{12} |w(t_k)|. \quad (28)$$

Finally, taking norm of  $\xi_{0k}$  in (12), and using (8) and (28), we get

$$\|\xi_{0k}(s, x_0, x_1, w)\| \leq T_k^2 \beta_{00} \|x_0\| + T_k \beta_{01} \|x_1\|$$

$$+ T_k \beta_{02} |w|. \quad (29)$$

Having obtained bounds for  $\xi_{0k}$  and  $\xi_{1k}$ , we now note that by continuity of solutions of  $\mathcal{S}$  and (5) we have

$$x_{0, k+1}(0) = x_0(t_{k+1})$$

$$= x_0(t_k + T_k)$$

$$= x_{0k}(1)$$

$$x_{1, k+1}(0) = D_{1, k+1}^{-1} x_1(t_{k+1})$$

$$= D_{1, k+1}^{-1} x_1(t_k + T_k)$$

$$= D_{1, k+1}^{-1} D_{1k} x_{1k}(1). \quad (30)$$

Also, from (11) we have

$$x_{0k}(1) = e^{T_k A_0} x_{0k}(0) + \xi_{0k}[1, x_{0k}(0), x_{1k}(0), w(t_k)]$$

$$x_{1k}(1) = \Phi_1 x_{1k}(0) + \xi_{1k}[1, x_{0k}(0), x_{1k}(0), w(t_k)]$$

$$+ \Gamma_1(\psi)w(t_k). \quad (31)$$

Then (14) follows from (7), (9), (30), and (31) with

$$\hat{\xi}_0[k, \hat{x}_0(k), \hat{x}_1(k)] = \xi_{0k}[1, x_{0k}(0), x_{1k}(0), c_1^T x_{1k}(0)$$

$$+ c_c^T x_c(t_k)]$$

$$\hat{\xi}_1[k, \hat{x}_0(k), \hat{x}_1(k)] = \{\bar{\xi}_1^T[k, \hat{x}_0(k), \hat{x}_1(k)] \quad 0^T\}^T \quad (32)$$

where

$$\bar{\xi}_1[k, \hat{x}_0(k), \hat{x}_1(k)]$$

$$= (D_{1, k+1}^{-1} D_{1k} - I)$$

$$\cdot [(\Phi_1 + \Gamma_1 c_1^T) x_{1k}(0) + \Gamma_1 c_c^T x_c(t_k)]$$

$$+ D_{1, k+1}^{-1} D_{1k}$$

$$\cdot \xi_{1k}[1, x_{0k}(0), x_{1k}(0), c_1^T x_{1k}(0) + c_c^T x_c(t_k)]. \quad (33)$$

Since  $\|D_{1, k+1}^{-1} D_{1k} - I\| = (T_k/T_{k+1})^{n_1-1} - 1 \leq T_k$ , for  $T_k$  satisfying (17); (27), (29), (32), and (33) yield the bounds in (18), completing the proof.

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