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Reducts of random hypergraphs [1]

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Abstract

For each $k \geqslant 1$, let Γ_k be the countable universal homogeneous k-hypergraph. In this paper, we shall classify the closed permutation groups G such that $Aut(\Gamma_k) \leqslant G \leqslant Sym(\Gamma_k)$. In particular, we shall show that there exist only finitely many such groups G for each $k \geqslant 1$. We shall also show that each of the associated reducts of Γ_k is homogeneous with respect to a finite relational language.

1. Introduction

Let $\mathcal{M}$ be a countable ω-categorical structure. The structure $\mathcal{N}$ for the language L is defined to be a *reduct* of $\mathcal{M}$ if

(1) $\mathcal{N}$ has the same underlying set as $\mathcal{M}$;

(2) for each $R \in L$, $R^{\mathcal{N}}$ is definable without parameters in $\mathcal{M}$.

Thus if $\mathcal{N}$ is a reduct of $\mathcal{M}$, then $Aut(\mathcal{N})$ is a closed permutation group such that $Aut(\mathcal{M}) \leqslant Aut(\mathcal{N}) \leqslant Sym(\mathcal{M})$. (Here $Sym(\mathcal{M})$ denotes the group of all permutations of the underlying set of $\mathcal{M}$.) Conversely, if G is a closed permutation group such that $Aut(\mathcal{M}) \leqslant G \leqslant Sym(\mathcal{M})$, then there exists a structure $\mathcal{N}$ for a suitably chosen language L such that $\mathcal{N}$ is a reduct of $\mathcal{M}$ and $G = Aut(\mathcal{N})$. (For example, see [7].) Two reducts $\mathcal{N}_1$, $\mathcal{N}_2$ of $\mathcal{M}$ are said to be *equivalent* if and only if each is a reduct of the other. This occurs if and only if $Aut(\mathcal{N}_1) = Aut(\mathcal{N}_2)$. Thus the problem of classifying the reducts of $\mathcal{M}$ up to equivalence is the same as that of classifying the closed permutation groups G such that $Aut(\mathcal{M}) \leqslant G \leqslant Sym(\mathcal{M})$.

There are currently very few ω-categorical structures $\mathcal{M}$ for which the reducts of $\mathcal{M}$ have been explicitly classified. The classification problem seems most manageable

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in the case when $\mathcal{M}$ is also ω-stable. In this case, it is possible to make use of the powerful techniques developed in [5]. We shall illustrate this method with two examples. The first example is extremely simple, and can be dealt with by an easy permutation group theoretic argument. In contrast, it appears to be difficult to find a purely permutation group theoretic argument to deal with the second example. (We have included the first example only because this result will be needed later in the paper.) In both of the following examples, R and deg denote Morley rank and degree respectively.

Example 1.1. Let $\mathcal{M} = \langle M; A_0 \rangle$, where A_0 is a 0-definable subset such that both A_0 and $A_1 = M \setminus A_0$ are infinite. Then $Aut(\mathcal{M}) = Sym(A_0) \times Sym(A_1)$. Furthermore, $R(\mathcal{M}) = 1$ and $\deg(\mathcal{M}) = 2$.

Suppose that G is a closed permutation group such that $Aut(\mathcal{M}) < G \leqslant Sym(\mathcal{M})$. Then G acts transitively on M. Let $\mathcal{N}$ be a reduct of $\mathcal{M}$ such that $G = Aut(\mathcal{N})$. Then $\mathcal{N}$ is also ω-stable and ω-categorical; and $R(\mathcal{N}) = 1$, $\deg(\mathcal{N}) \leqslant 2$. If $\deg(\mathcal{N}) = 1$, then $\mathcal{N}$ is strictly minimal and it follows that $G = Sym(\mathcal{M})$. So suppose that $\deg(\mathcal{N}) = 2$. By the Finite Equivalence Relation Theorem, there exists a nontrivial 0-definable equivalence relation E on $\mathcal{N}$. It is clear that the E-classes must be A_0 and A_1. Thus, in this case,

$$G = \{\pi \in Sym(\mathcal{M}) \mid \text{There exists } i \in \{0,1\} \text{ such that } \pi[A_0] = A_i\}.$$

Hence there are exactly three closed permutation groups G such that $Aut(\mathcal{M}) \leqslant G \leqslant Sym(\mathcal{M})$.

Example 1.2. Let $2 \leqslant k \in \omega$, and let $[\mathbb{N}]^k$ be the set of k-subsets of $\mathbb{N}$. Consider the graph $\Gamma = \langle [\mathbb{N}]^k; \sim \rangle$, where $A \sim B$ if and only if $|A \cap B| = k-1$. Then Γ is a totally categorical structure; and the automorphism group of Γ is $Sym(\mathbb{N})$ acting in the natural way.

Suppose that G is a closed permutation group such that $Aut(\Gamma) < G \leqslant Sym(\Gamma)$. Let $\mathcal{N}$ be a reduct of Γ such that $G = Aut(\mathcal{N})$. Then $\mathcal{N}$ is ω-stable and ω-categorical. Since $Aut(\Gamma)$ acts primitively on Γ, $Aut(\mathcal{N})$ also acts primitively on $\mathcal{N}$. By the Coordinatization Theorem [5], $\mathcal{N}$ is isomorphic to a grassmannian over a strictly rank 1 set. Thus there exists a transitive strictly rank 1 set S and a finite algebraically closed subset $X \subset S$ such that $\mathcal{N} \simeq Gr(X,S)$, where $Gr(X,S)$ is the set of all subsets of S which are conjugate to X under the action of $Aut(S)$. Suppose that $\deg(S) = d > 1$. Then there exists a subgroup $H < G$ of finite index such that H acts imprimitively on $\mathcal{N}$. Notice that $[Aut(\Gamma) : H \cap Aut(\Gamma)]$ is also finite. Since $Aut(\Gamma) = Sym(\mathbb{N})$ has no proper subgroups of finite index, it follows that $Aut(\Gamma) \leqslant H$. But this contradicts the fact that $Aut(\Gamma)$ acts primitively on Γ. Thus S is strictly minimal.

For each $n \geqslant k$, consider the subset $\Gamma_n = [n]^k$ of Γ. Since Γ_n is algebraically closed in Γ, it is also algebraically closed in $\mathcal{N}$. Hence Γ_n is a finite homogeneous substructure of $\mathcal{N}$. Since $G \neq Aut(\Gamma)$, for all sufficiently large n, the setwise stabiliser of Γ_n in G induces a group P_n of permutations on Γ_n such that

$$Sym(n) = Aut(\Gamma_n) < P_n \leqslant Sym(\Gamma_n).$$

But if $n > \max\{2k, 6\}$, this implies that $Alt(\Gamma_n) \leqslant P_n$ (see [10]). It follows that G is highly transitive, and hence $G = Sym(\Gamma)$.

In [9], the notion of a smoothly approximated structure was introduced, and the primitive smoothly approximated structures were classified. Cherlin and Hrushovski have extended much of the theory of ω-stable ω-categorical structures to the more general class of smoothly approximated structures (see [8]). It is natural to ask whether the reducts of the smoothly approximated structures can be classified. Unfortunately, there is no reason why a reduct of a smoothly approximated structure should also be smoothly approximated. And, indeed, Evans [6] has found an example which shows that the class of smoothly approximated structures is *not* closed under taking reducts. This suggests the following problem.

Question 1.3. If $\mathcal{M}$ is a smoothly approximated structure, under what conditions are all of its reducts also smoothly approximated?

In [12], I studied the reducts of the random graph $\Gamma = \langle V; E \rangle$; i.e. the countable universal homogeneous graph. The starting point of this work was [4], in which Cameron found the following three examples of closed permutation groups G such that $Aut(\Gamma) < G < Sym(\Gamma)$.

Example 1.4. Let $\overline{\Gamma} = \langle V; \overline{E} \rangle$ be the complementary graph of Γ; i.e. $\overline{E} = [V]^2 \setminus E$. Then clearly $\Gamma \simeq \overline{\Gamma}$. Hence if $D(\Gamma)$ is the closed subgroup of $Sym(\Gamma)$ consisting of all isomorphisms and anti-isomorphisms of Γ, then $[D(\Gamma) : Aut(\Gamma)] = 2$. (Note that $D(\Gamma)$ preserves the parity of edges in every 4-subset of Γ.)

Example 1.5. If X, Y are graphs and $A \subseteq X$, then a bijection $\pi : X \to Y$ is a *switch with respect to* A if

(1) π preserves the adjacency relation for pairs of vertices in A and for pairs of vertices in $X \setminus A$; and
(2) π does *not* preserve the adjacency relation for pairs of vertices of the form $\{a, b\}$, where $a \in A$ and $b \in X \setminus A$.

(Of course, this means that π is also a switch with respect to $X \setminus A$.) Notice that if $\pi : X \to Y$ is a switch with respect to $A \subseteq X$ and $\phi : Y \to Z$ is a switch with respect to $B \subseteq Y$, then $\phi \circ \pi : X \to Z$ is a switch with respect to $A \triangle \pi^{-1}[B] \subseteq X$. (Here $\triangle$ denotes the symmetric difference.) In particular, the set

$$S(\Gamma) = \{\pi \in Sym(\Gamma) \mid \pi \text{ is a switch with respect to some } A \subseteq \Gamma\}$$

is a subgroup of $Sym(\Gamma)$ such that $Aut(\Gamma) \leqslant S(\Gamma) \leqslant Sym(\Gamma)$.

First we shall check that $Aut(\Gamma) < S(\Gamma)$. Let A be any subset of Γ. Define the graph $sw_A(\Gamma) = \langle V; E_A \rangle$ by

$$E_A = (E \cap [A]^2) \cup (E \cap [\Gamma \backslash A]^2) \cup \{\{a,b\} \mid a \in A,\, b \in \Gamma \backslash A,\, \{a,b\} \notin E\}.$$

Then it is easily seen that if A is any finite nonempty subset of Γ, then $sw_A(\Gamma) \simeq \Gamma$. Let $\pi \in Sym(\Gamma) \backslash Aut(\Gamma)$ be an isomorphism from Γ onto $sw_A(\Gamma)$ such that $\pi[A] = A$. Then $\pi : \Gamma \to \Gamma$ is a switch with respect to A.

Next we shall check that $S(\Gamma) \neq Sym(\Gamma)$. To see this, simply note that if $\pi \in S(\Gamma)$ then π preserves the parity of edges in every 3-subset of Γ. In fact, it can be shown that

$$S(\Gamma) = \{\pi \in Sym(\Gamma) \mid \pi \text{ preserves the parity of edges in every 3-subset of } \Gamma\}.$$

Hence $S(\Gamma)$ is a proper closed subgroup of $Sym(\Gamma)$.

Notice that $S(\Gamma)$ is generated as a topological group by

(1) $Aut(\Gamma)$, together with

(2) the set of all $\pi \in S(\Gamma)$ such that π is a switch with respect to $\{v\}$ for some $v \in \Gamma$.

This set of permutations does *not* generate $S(\Gamma)$ as a group, since there exist infinite subsets A of Γ such that $\Gamma \backslash A$ is also infinite and $sw_A(\Gamma) \simeq \Gamma$.

Example 1.6. Let $B(\Gamma) = \langle S(\Gamma), D(\Gamma) \rangle$. Then $B(\Gamma)$ is a proper closed subgroup of $Sym(\Gamma)$. (Note that $B(\Gamma)$ preserves the parity of edges in every 5-subset of Γ.)

In [12], I proved that Cameron's three examples are the only closed permutation groups G such that $Aut(\Gamma) < G < Sym(\Gamma)$. My published proof of this result used a mixture of combinatorics and group theory; and it seems to be very difficult to adapt its method to deal with the reducts of other homogeneous structures. Later I discovered a purely combinatorial proof. Using this new approach, my student James Bennett ([2]) managed to classify the reducts of the countable universal homogeneous tournament, as well as various other binary homogeneous structures. In this paper, I will use this purely combinatorial approach to classify the reducts of the countable universal homogeneous k-graphs for all $k \geqslant 1$. (When $k > 2$, a k-graph is usually called a *uniform hypergraph.*)

Definition 1.7. If $k \geqslant 1$, then a k-graph is a structure of the form $\langle V; E \rangle$, where $E \subseteq [V]^k$.

Definition 1.8. For each $k \geqslant 1$, Γ_k will denote the countable universal homogeneous k-graph. Γ_k is also called the random k-graph.

Thus Γ_1 is just a countably infinite set, equipped with a 0-definable subset E such that both E and $\Gamma_1 \backslash E$ are infinite; and Γ_2 is the random graph Γ. For each $k \geqslant 2$, Γ_k is the unique countable k-graph which satisfies the following property:

- Suppose that H is a finite subset of Γ_k and that $E \subseteq [H]^{k-1}$. Then there exists a vertex $v \in \Gamma_k \backslash H$ such that for each $S \in [H]^{k-1}$, $S \cup \{v\}$ is a k-edge of Γ_k if and only if $S \in E$.

Cameron's examples of closed groups G such that $Aut(\Gamma_2) < G < Sym(\Gamma_2)$ can easily be generalised to Γ_k for arbitrary $k \geqslant 1$.

Definition 1.9. Let X, Y be k-graphs and let $A \in [X]^i$ for some $0 \leqslant i \leqslant k-1$. The bijection $\pi : X \to Y$ is a *switch with respect to* A if for all $B \in [X]^k$, $\pi \restriction B$ is an isomorphism if and only if $A \nsubseteq B$.

Example 1.10. If $\pi : X \to Y$ is a switch with respect to $\emptyset$, then π is an anti-isomorphism.

Remark 1.11. Suppose that $A \in [\Gamma_k]^i$ for some $0 \leqslant i \leqslant k-1$. Let E be the set of k-edges of Γ_k. Define the k-graph $sw_A(\Gamma_k) = \langle \Gamma_k; E_A \rangle$ by

$$E_A = \{B \in E \mid A \nsubseteq B\} \cup \{B \in [\Gamma_k]^k \backslash E \mid A \subseteq B\}.$$

Then it is easily seen that $sw_A(\Gamma_k) \simeq \Gamma_k$. Let $\pi \in Sym(\Gamma_k) \backslash Aut(\Gamma_k)$ be an isomorphism from Γ_k onto $sw_A(\Gamma_k)$ such that $\pi[A] = A$. Then $\pi : \Gamma_k \to \Gamma_k$ is a switch with respect to A.

Definition 1.12. If $X \subseteq \{0, 1, \ldots, k-1\}$, then $S_X(\Gamma_k)$ is the closed subgroup of $Sym(\Gamma_k)$ generated as a topological group by

(1) $Aut(\Gamma_k)$, together with

(2) the set of all $\pi \in Sym(\Gamma_k)$ such that there exists an $i \in X$ and a subset $A \in [\Gamma_k]^i$ such that π is a switch with respect to A.

Example 1.13. Thus $S_\emptyset(\Gamma_k) = Aut(\Gamma_k)$; and $S_{\{0\}}(\Gamma_k)$ is the group of all isomorphisms and anti-isomorphisms of Γ_k.

Definition 1.14. When $X = \{0, 1, \ldots, k-1\}$, we write $B(\Gamma_k) = S_X(\Gamma_k)$.

The following theorem is the main result of this paper.

Theorem 1.15 (The Classification Theorem). *If G is a closed permutation group such that $Aut(\Gamma_k) \leqslant G < Sym(\Gamma_k)$, then there exists a subset $X \subseteq \{0, 1, \ldots, k-1\}$ such that $G = S_X(\Gamma_k)$.*

We shall study the main properties of the groups $S_X(\Gamma_k)$ and the associated reducts in Section 2. In this section, we shall just show that $B(\Gamma_k) \neq Sym(\Gamma_k)$; and obtain a useful characterisation of the elements of $B(\Gamma_k)$. The following easy observation will be used repeatedly. Throughout this paper, $m \equiv n$ means that $m - n$ is divisible by 2.

Lemma 1.16. *If $n \geqslant k$, then $S_X(\Gamma_k)$ preserves the parity of k-edges in every n-subset of Γ_k if and only if n satisfies the following condition.*

$$(1.16)^k_X \qquad \binom{n-i}{k-i} \equiv 0 \quad \textit{for all } i \in X.$$

Proof. Let $g \in S_X(\Gamma_k)$ be a switch with respect to the set $A \in [\Gamma_k]^i$, where $i \in X$; and let $H \in [\Gamma_k]^n$. If $A \nsubseteq H$, then $g \restriction H$ is an isomorphism. If $A \subseteq H$ and $B \in [H]^k$, then $g \restriction B$ is *not* an isomorphism if and only if B is one of the $\binom{n-i}{k-i}$ k-subsets of H such that $A \subseteq B$. Hence $g \restriction H$ preserves the parity of k-edges in H if and only if $\binom{n-i}{k-i} \equiv 0$. The result follows. $\square$

Theorem 1.17. *There exists an integer $n > k$ such that $B(\Gamma_k)$ preserves the parity of k-edges in every n-subset of Γ_k. In particular, $B(\Gamma_k) \neq Sym(\Gamma_k)$.*

Proof. By Lemma 1.16, it is enough to show that there exists an integer $n > k$ such that

$$\binom{n-i}{k-i} = \frac{(n-i)\cdots(n-k+1)}{(k-i)!} \equiv 0$$

for all $0 \leqslant i \leqslant k-1$. Clearly we can obtain such an integer n by letting $n-k+1$ be a sufficiently high power of 2. $\square$

Definition 1.18. Let $k \geqslant 1$. An integer $n > k$ is *$B(\Gamma_k)$-good* if $\binom{n-i}{k-i} \equiv 0$ for all $0 \leqslant i \leqslant k-1$.

The following observation will be useful in many inductive arguments.

Lemma 1.19. *Let $k \geqslant 2$. If n is $B(\Gamma_k)$-good, then $n-1$ is $B(\Gamma_{k-1})$-good.*

Proof. For each $0 \leqslant i \leqslant k-2$, $\binom{n-1-i}{k-1-i} = \binom{n-(i+1)}{k-(i+1)} \equiv 0$. $\square$

There is also a converse result to Theorem 1.17 which characterises the elements of $B(\Gamma_k)$ as parity-preserving maps.

Theorem 1.20. *Suppose that $\pi \in Sym(\Gamma_k)$ and that there exists a $B(\Gamma_k)$-good integer n such that π preserves the parity of k-edges in every n-subset of Γ_k. Then $\pi \in B(\Gamma_k)$.*

Clearly Theorem 1.20 is an immediate consequence of the following finite version.

Proposition 1.21. *For each $B(\Gamma_k)$-good integer n, there exists an integer $g(k,n)$ with the following property. Suppose that H is a finite k-subgraph of Γ_k such that $|H| \geqslant g(k,n)$; and that $\psi : H \to \Gamma_k$ is an injection such that ψ preserves the parity of k-edges in every n-subset of H. Then there exists an element $\theta \in B(\Gamma_k)$ such that $\theta \restriction H = \psi$.*

Before proving Proposition 1.21, we need to introduce some important notation.

Notation 1.22. Let $k \geqslant 2$. Suppose that $F \in [\Gamma_k]^{<\omega}$ and that $v \in \Gamma_k \setminus F$. Then the $(k-1)$-graph induced on F by v is defined to be $F_v = \langle F; E \rangle$, where $B \in E$ if and only if $B \cup \{v\}$ is a k-edge of Γ_k.

Now suppose that $\phi : F \cup \{v\} \to \Gamma_k$ is an injection. Regard $\phi[F]_{\phi(v)}$ as a $(k-1)$-subgraph of Γ_{k-1}. Then ϕ induces an injection

$$\phi_v : F_v \to \Gamma_{k-1}.$$

(Of course, this map is only defined up to an embedding of $\phi[F]_{\phi(v)}$ into Γ_{k-1}. But this is good enough for our purposes.)

Proof of Proposition 1.21. We shall argue by induction on $k \geqslant 1$. First suppose that $k = 1$. Remember that Γ_1 is just a countably infinite set equipped with a 0-definable subset E such that both E and $\Gamma_1 \setminus E$ are infinite. Let $E_0 = E$ and $E_1 = \Gamma_1 \setminus E$. Then

$$B(\Gamma_1) = \{\pi \in Sym(\Gamma_1) \mid \pi[E_0] = E_i \text{ for some } i \in \{0,1\}\}.$$

It is easily checked that we can take $g(1,n) = n+1$ for each $B(\Gamma_1)$-good integer n.

Now suppose that the result holds for some $k \geqslant 1$. Let n be a $B(\Gamma_{k+1})$-good integer. Suppose that H is an extremely large finite $(k+1)$-subgraph of Γ_{k+1}, and that $\psi : H \to \Gamma_{k+1}$ is an injection such that ψ preserves the parity of $(k+1)$-edges in every n-subset of H. By Ramsey's Theorem, there exists a large subset R of H such that $\psi \restriction R$ is either an isomorphism or an anti-isomorphism. In particular, $\psi \restriction R$ is induced by an element of $B(\Gamma_{k+1})$. Now suppose that $R \subseteq S \subset H$ and that there exists $\theta \in B(\Gamma_{k+1})$ such that $\theta \restriction S = \psi \restriction S$. Let $v \in H \setminus S$ and let $\phi = \theta^{-1} \circ \psi \restriction S \cup \{v\}$. Note that if $B \in [S \cup \{v\}]^{k+1}$ and $\phi \restriction B$ is *not* an isomorphism, then $v \in B$. Let S_v be the k-graph induced on S by v, and let $\phi_v : S_v \to \Gamma_k$ be the map induced by ϕ. Since ϕ preserves the parity of $(k+1)$-edges in every n-subset of $S \cup \{v\}$, it follows that ϕ_v preserves the parity of k-edges in every $(n-1)$-subset of S_v. By Lemma 1.19, $n-1$ is $B(\Gamma_k)$-good. We can suppose that $|S_v| \geqslant g(k, n-1)$. Hence ϕ_v is induced by an element of $B(\Gamma_k)$. It follows that ϕ is induced by an element of $B(\Gamma_{k+1})$. (For example, suppose that ϕ_v is induced by a switch with respect to some subset A of S_v. Then ϕ is induced by a switch with respect to $A \cup \{v\}$.) Hence $\psi \restriction S \cup \{v\}$ is also induced by an element of $B(\Gamma_{k+1})$. Continuing in this manner, we see that there exists an element $\theta \in B(\Gamma_{k+1})$ such that $\theta \restriction H = \psi$. $\square$

This paper is organised as follows. In Section 2, we shall examine the groups $S_X(\Gamma_k)$ and the corresponding reducts of Γ_k. In particular, we shall show that each of these reducts of Γ_k is homogeneous with respect to a finite relational language.

In Section 5, we shall prove that if G is a closed permutation group such that $Aut(\Gamma_k) \leqslant G \leqslant B(\Gamma_k)$, then there exists a subset $X \subseteq \{0, 1, \ldots, k-1\}$ such that $G = S_X(\Gamma_k)$. In Section 6, we shall prove that if G is a closed permutation group such that $Aut(\Gamma_k) \leqslant G < Sym(\Gamma_k)$, then $G \leqslant B(\Gamma_k)$.

As was mentioned earlier, our approach is purely combinatorial. This means that we actually prove a more general classification theorem: namely, we shall classify the nontrivial *pseudo-reducts* of Γ_k. This notion will be introduced in Section 4.

In Section 3, we shall introduce the notion of the *strong finite submodel property*; and we shall prove that it is possible to express $\Gamma_k = \bigcup_{n \in \mathbb{N}} R_n$ as the union of a chain of finite k-subgraphs R_n such that $|R_{n+1}| = |R_n| + 1$ and each R_n is random. This result will be used in the combinatorial analysis of Sections 5 and 6.

2. Some properties of the reducts

In this section, we shall take a closer look at the closed subgroups $S_X(\Gamma_k)$ and the corresponding reducts $\Gamma_k(X)$ of Γ_k. (For the sake of definiteness, we take $\Gamma_k(X)$ to be the structure for the canonical language. See [5, Section 7].) In particular, we shall show that each of the reducts $\Gamma_k(X)$ is homogeneous with respect to a suitably chosen finite relational language. But first we shall consider the problem of deciding when $S_X(\Gamma_k) = S_Y(\Gamma_k)$ for subsets $X, Y \subseteq \{0, 1, \dots, k-1\}$.

Example 2.1. $S_{\{1\}}(\Gamma_3) = S_{\{0,1\}}(\Gamma_3)$.

This can be proved as follows. Clearly $S_{\{1\}}(\Gamma_3) \leqslant S_{\{0,1\}}(\Gamma_3)$. Hence it suffices to prove that if R is a finite subgraph of Γ_3, then there exists $\phi \in S_{\{1\}}(\Gamma_3)$ such that $\phi \restriction R$ is an anti-isomorphism. Let $R = \{v_1, \dots, v_t\}$. Define elements $\phi_i \in S_{\{1\}}(\Gamma_3)$ for $0 \leqslant i \leqslant t$ inductively as follows.

$$\phi_0 = id,$$
$$\phi_{i+1} = \pi_{i+1} \circ \phi_i, \qquad \text{where } \pi_{i+1} \text{ is a switch with respect to } \phi_i(v_{i+1}).$$

Since each $B \in [R]^3$ contains an odd number of vertices, it follows that $\phi_t \restriction R$ is an anti-isomorphism.

Definition 2.2. If $X \subseteq \{0, 1, \dots, k-1\}$, let $c\ell_k(X)$ be the largest subset $Y \subseteq \{0, 1, \dots, k-1\}$ such that $S_X(\Gamma_k) = S_Y(\Gamma_k)$.

The following result shows that if $\ell \in c\ell_k(X) \setminus X$ for some $X \subseteq \{0, 1, \dots, k-1\}$, then it is essentially for the same reason that $0 \in c\ell_3(\{1\})$ in Example 2.1.

Theorem 2.3. *If $X \subseteq \{0, 1, \dots, k-1\}$ and $0 \leqslant \ell \leqslant k-1$, then the following are equivalent.*

(1) *$\ell \in c\ell_k(X)$.*

(2) *For all $n > k$, if $\binom{n-t}{k-t} \equiv 0$ for all $t \in X$, then $\binom{n-\ell}{k-\ell} \equiv 0$.*

(3) *There exists $t \in X$ such that $t \geqslant \ell$ and $\binom{k-\ell}{t-\ell} \equiv 1$.*

We shall make use of the following lemma.

Lemma 2.4. *If r, s, m are integers such that $s \geqslant 2$ and $m = 2^r - 1 > s$, then*
(1) $\binom{m}{s} \equiv 1$;
(2) $\binom{m-i}{s-i} \equiv \binom{s}{i}$ *for all* $1 \leqslant i \leqslant s-1$.

Proof. It is well-known that $\binom{2^r}{j} \equiv 0$ for all $0 < j < 2^r$. Hence, using the recursion formula

$$\binom{2^r - 1}{j+1} = \binom{2^r}{j+1} - \binom{2^r - 1}{j},$$

we see that $\binom{2^r-1}{j} \equiv 1$ for all $0 \leqslant j \leqslant 2^r - 1$. In particular, $\binom{m}{s} \equiv 1$. Now suppose that $1 \leqslant i \leqslant s-1$. Since $\binom{m}{s} \equiv \binom{m}{i} \equiv 1$, the formula

$$\binom{m-i}{s-i} = \frac{\binom{s}{i}\binom{m}{s}}{\binom{m}{i}}$$

implies that $\binom{m-i}{s-i} \equiv \binom{s}{i}$. $\square$

Proof of Theorem 2.3. (1) $\Rightarrow$ (2). Suppose that $n > k$ and that $\binom{n-t}{k-t} \equiv 0$ for all $t \in X$. By Lemma 1.16, $S_X(\Gamma_k)$ preserves the parity of k-edges in every n-subset of Γ_k. Since $\ell \in c\ell_k(X)$, a switch with respect to an ℓ-set must also preserve the parity of k-edges in every n-subset of Γ_k. Hence $\binom{n-\ell}{k-\ell} \equiv 0$.

(2)$\Rightarrow$(3). Suppose that (2) holds, but that $\binom{k-\ell}{t-\ell} \equiv 0$ for all $\ell \leqslant t \in X$. In particular, we must have that $\ell \notin X$. Let n be a natural number such that $n - \ell = 2^r - 1$ for some very large integer r. Then for all $t \in X$ such that $t < \ell$, we have that

$$\binom{n-t}{k-t} = \frac{(n-t)\cdots(n-[\ell-1])\cdots(n-k+1)}{(k-t)!} \equiv 0.$$

Now suppose that $t \in X$ satisfies $\ell < t \leqslant k-1$. Then $s = k - \ell \geqslant 2$ and $2^r - 1 > s$. Hence, by Lemma 2.4, $\binom{n-\ell}{k-\ell} \equiv 1$ and

$$\binom{n-t}{k-t} = \binom{(n-\ell)-(t-\ell)}{(k-\ell)-(t-\ell)} \equiv \binom{k-\ell}{t-\ell} \equiv 0.$$

Thus $\binom{n-t}{k-t} \equiv 0$ for all $t \in X$ and $\binom{n-\ell}{k-\ell} \equiv 1$. But this contradicts the assumption that (2) holds.

(3) $\Rightarrow$ (1). This is just a slight generalisation of the argument of Example 2.1. Let R be a finite k-subgraph of Γ_k and let $A \in [R]^\ell$. We must find an element $\phi \in S_X(\Gamma_k)$ such that $\phi \restriction R$ is a switch with respect to A. Let $t \in X$ be such that $t \geqslant \ell$ and $\binom{k-\ell}{t-\ell} \equiv 1$. Let $\{B_i \mid 1 \leqslant i \leqslant r\}$ be an enumeration of the subsets $B \in [R]^t$ such that $A \subset B$. Define elements $\phi_i \in S_X(\Gamma_k)$ for $0 \leqslant i \leqslant r$ as follows:

$$\phi_0 = id,$$
$$\phi_{i+1} = \pi_{i+1} \circ \phi_i, \quad \text{where } \pi_{i+1} \text{ is a switch with respect to } \phi_i[B_i].$$

For each $D \in [R]^k$ with $A \subset D$, there exist $\binom{k-\ell}{t-\ell}$ sets $B \in [R]^t$ such that $A \subset B \subset D$. It follows that $\phi_r \upharpoonright R$ is a switch with respect to A. $\square$

Next we shall show that each of the groups $S_X(\Gamma_k)$ can be characterised as the subgroup of $Sym(\Gamma_k)$ which preserves the parity of k-edges in a suitably chosen collection of finite subsets of Γ_k. Then we shall use this result to find a finite relational language L_X^k such that the reduct $\Gamma_k(X)$ is a homogeneous L_X^k-structure.

Theorem 2.5. *For each $X \subseteq \{0,1,\ldots,k-1\}$, there exists a finite subset $\Phi_X^k \subset \mathbb{N} \setminus k$ such that*

$$S_X(\Gamma_k) = \left\{ g \in Sym(\Gamma_k) \;\middle|\; \begin{array}{l} \text{For all } n \in \Phi_X^k,\ g \text{ preserves the parity} \\ \text{of } k\text{-edges in every } n\text{-subset of } \Gamma_k \end{array} \right\}.$$

Theorem 2.5 is an immediate consequence of the following finite version.

Theorem 2.6. *For each $X \subseteq \{0,1,\ldots,k-1\}$, there exists a finite subset $\Phi_X^k \subset \mathbb{N} \setminus k$ and an integer N such that the following conditions are satisfied.*

(1) *For all $n \in \Phi_X^k$, n satisfies condition $(1.16)_X^k$; i.e. $\binom{n-i}{k-i} \equiv 0$ for all $i \in X$.*

(2) *Suppose that H is a finite k-subgraph of Γ_k such that $|H| \geqslant N$; and that $\psi : H \to \Gamma_k$ is an injection such that ψ preserves the parity of k-edges in every n-subset of H for all $n \in \Phi_X^k$. Then there exists an element $g \in S_X(\Gamma_k)$ such that $g \upharpoonright H = \psi$.*

Proof. We argue by induction on $k \geqslant 1$. The result is easily seen to be true when $k = 1$. (If $X = \emptyset$, then we can take $N = 1$ and $\Phi_X^1 = \{1\}$. If $X = \{0\}$, then we can take $N = 1$ and $\Phi_X^1 = \{2\}$.) So suppose that the result holds for $k-1$, where $k-1 \geqslant 1$.

Case 1: Suppose that $\binom{k}{i} \equiv 0$ for all $i \in X$. By Lemma 2.4, there exists an integer $a > k$ such that $\binom{a}{k} \equiv 1$ and $\binom{a-i}{k-i} \equiv 0$ for all $i \in X$. In particular, a satisfies $(1.16)_X^k$. Let $Y = \{i-1 \mid i \in X\}$. (Notice that $0 \notin X$, since $\binom{k}{0} = 1$.) Let $D = \{m+1 \mid m \in \Phi_Y^{k-1}\}$. We claim that if N is a sufficiently large integer, then $\Phi_X^k = D \cup \{a\}$ and N satisfy our requirements.

First we check that (1) holds. So suppose that $n = m + 1 \in D$. Then m satisfies $(1.16)_Y^{k-1}$. Hence if $i \in X$, then

$$\binom{n-i}{k-i} = \binom{m-(i-1)}{(k-1)-(i-1)} \equiv 0.$$

Thus n satisfies $(1.16)_X^k$.

Now we check that (2) holds. So suppose that H is a finite k-subgraph of Γ_k such that $|H| \geqslant N$; and that $\psi : H \to \Gamma_k$ is an injection such that ψ preserves the parity of k-edges in every n-subset of H for all $n \in \Phi_X^k$. By Ramsey's Theorem, there exists a *large* k-subgraph $S \subset H$ such that $\psi \upharpoonright S$ is either an isomorphism or an anti-isomorphism. Since $|S| > a$ and $\binom{a}{k} \equiv 1$, it follows that $\psi \upharpoonright S$ must be an isomorphism. Now suppose that T is a k-subgraph of H such that $S \subseteq T \subset H$ and such that $\psi \upharpoonright T$ is induced by an element $h \in S_X(\Gamma_k)$. Let $v \in H \setminus T$. Let $\phi = h^{-1} \circ \psi$, so that $\phi \restriction T = id$. Let T_v be the $(k-1)$-graph induced on T by v, and let $\phi_v : T_v \to \Gamma_{k-1}$ be the map induced by $\phi \restriction T \cup \{v\}$. Then ϕ_v preserves the parity of $(k-1)$-edges in every m-subset of T_v for all $m \in \Phi_Y^{k-1}$. Since $|T_v|$ is large, there exists an element $\theta \in S_Y(\Gamma_{k-1})$ such that $\theta \restriction T_v = \phi_v$. It follows that $\phi \restriction T \cup \{v\}$ is induced by an element of $S_X(\Gamma_k)$. Hence $\psi \restriction T \cup \{v\}$ is also induced by an element of $S_X(\Gamma_k)$. Continuing in this fashion, we see that there exists $g \in S_X(\Gamma_k)$ such that $g \restriction H = \psi$.

Case 2: Suppose that there exists $i \in X$ such that $\binom{k}{i} \equiv 1$. By the implication (3) $\Rightarrow$ (1) of Theorem 2.3, $0 \in c\ell_k(X)$ and so $S_X(\Gamma_k)$ contains the set of anti-isomorphisms of Γ_k. Let $Y = \{i - 1 \mid 0 < i \in X\}$. We claim that if N is a sufficiently large integer, then $\Phi_X^k = \{m + 1 \mid m \in \Phi_Y^{k-1}\}$ and N satisfy our requirements. Clearly (1) holds. The main point is to check that (2) holds. So suppose that H is a finite k-subgraph of Γ_k such that $|H| \geqslant N$; and that $\psi : H \to \Gamma_k$ is an injection such that ψ preserves the parity of k-edges in every n-subset of H for all $n \in \Phi_X^k$. By Ramsey's Theorem, there exists a *large* k-subgraph $S \subset H$ such that $\psi \restriction S$ is either an isomorphism or an anti-isomorphism. Since $S_X(\Gamma_k)$ contains both the isomorphisms and the anti-isomorphisms of Γ_k, $\psi \restriction S$ is induced by an element $h \in S_X(\Gamma_k)$. Arguing as in Case 1, we see that there exists $g \in S_X(\Gamma_k)$ such that $g \restriction H = \psi$. $\square$

Theorem 2.7. *For each $X \subseteq \{0, 1, \ldots, k-1\}$, the reduct $\Gamma_k(X)$ is homogeneous with respect to a finite relational language.*

Proof. Let $\Phi_X^k \subset \mathbb{N}$ and $N \in \mathbb{N}$ be as in Theorem 2.6. Let $t = \max(\Phi_X^k \cup \{N\})$. For each orbit Δ of $S_X(\Gamma_k)$ on $[\Gamma_k(X)]^\ell$ for $1 \leqslant \ell \leqslant t$, let R_Δ be a corresponding ℓ-ary relation symbol; and let L_X^k be the resulting finite relational language. Then Theorem 2.6 implies that $\Gamma_k(X)$ is a homogeneous L_X^k-structure. $\square$

3. The strong finite submodel property

In this section, we shall introduce the notion of the strong finite submodel property (sfsp), and prove that Γ_k has the sfsp. This result will be used in Sections 5 and 6.

Definition 3.1. A countable structure $\mathscr{M}$ has the *strong finite submodel property* (sfsp) if it is possible to express $\mathscr{M} = \bigcup_{n \in \mathbb{N}} M_n$ as the union of an increasing chain of substructures M_n such that

(1) $|M_n| = n$ for each $n \in \mathbb{N}$; and
(2) for each sentence σ such that $\mathscr{M} \models \sigma$, there exists an integer N_σ such that $M_n \models \sigma$ for all $n \geqslant N_\sigma$.

Theorem 3.2. *For each $k \geqslant 1$, Γ_k has the sfsp.*

My original proof of this result was very long and involved. But then Jeff Kahn pointed out that it is an easy consequence of the Borel–Cantelli Lemma.

Definition 3.3. If $\langle A_n \mid n \in \mathbb{N}\rangle$ is a sequence of events in a probability space, then

$$\{A_n \text{ i.o.}\} = \bigcap_{n\in\mathbb{N}} \left[\bigcup_{n\leqslant k\in\mathbb{N}} A_k \right]$$

is the event that consists of the realisation of infinitely many of the A_n.

Theorem 3.4 (The Borel–Cantelli Lemma). *Let $\langle A_n \mid n \in \mathbb{N}\rangle$ be a sequence of events in a probability space. If $\sum_{n=0}^{\infty} P(A_n) < \infty$, then $P(\{A_n \text{ i.o.}\}) = 0$.*

Proof. For example, see [3]. □

Proof of Theorem 3.2. It is easily seen that Γ_1 has the sfsp. So from now on, we shall suppose that $k \geqslant 2$. Remember that the theory $\mathrm{Th}(\Gamma_k)$ of the random k-graph is axiomatised by the set $\{\phi_\ell^k \mid \ell \in \mathbb{N}\}$ of extension axioms, where ϕ_ℓ^k is the sentence which says the following:

- For each ℓ-set C and each $E \subseteq [C]^{k-1}$, there exists a vertex $v \notin C$ such that for each $S \in [C]^{k-1}$, $S \cup \{v\}$ is a k-edge if and only if $S \in E$.

Let Ω be the probability space of all k-graphs of the form $\langle \omega; E\rangle$, where each set $A \in [\omega]^k$ is a k-edge independently with probability $\frac{1}{2}$. For each $\ell < n \in \mathbb{N}$, let $B_{\ell,n}$ be the event that

$$\langle n; E \cap [n]^k \rangle \not\models \phi_\ell^k.$$

Then clearly

$$P(B_{\ell,n}) \leqslant \binom{n}{\ell} 2^{\binom{\ell}{k-1}} \left(1 - \left(\frac{1}{2}\right)^{\binom{\ell}{k-1}}\right)^{n-\ell}.$$

Let $f : \omega \to \omega$ be a slow-growing nondecreasing function and let $A_n = B_{f(n),n}$. By choosing f appropriately, we can ensure that $\sum_{n=0}^{\infty} P(A_n) < \infty$, and hence that $P(\{A_n \text{ i.o.}\}) = 0$. Thus there exists a k-graph $\langle \omega; E\rangle$ and an integer N such that $\langle n; E \cap [n]^k\rangle \models \phi_{f(n)}^k$ for all $n \geqslant N$. Clearly $\langle \omega; E\rangle \simeq \Gamma_k$, and so Γ_k has the sfsp. This completes the proof of Theorem 3.2. □

4. Pseudo-reducts

In Sections 5 and 6, we shall classify the reducts of Γ_k for all $k \geqslant 1$. Our approach will be purely combinatorial, and no essential use will be made of the fact that we are dealing with a group of permutations of Γ_k. The combinatorial content of this paper is best stated in terms of the more general notion of a *pseudo-reduct*.

Definition 4.1. Let $\mathscr{M}$ be a countable structure. Then a *pseudo-reduct* of $\mathscr{M}$ is a set $\mathscr{F}$ of functions which satisfies the following conditions.

(1) If $\pi \in \mathscr{F}$, then $\operatorname{dom}\pi \in [\mathscr{M}]^{<\omega}$ and $\pi : \operatorname{dom}\pi \to \mathscr{M}$ is an injection.

(2) If $g \in Aut(\mathcal{M})$ and $X \in [\mathcal{M}]^{<\omega}$, then $g \restriction X \in \mathcal{F}$.
(3) If $\pi \in \mathcal{F}$ and $X \subseteq \operatorname{dom} \pi$, then $\pi \restriction X \in \mathcal{F}$.
(4) If $\pi \in \mathcal{F}$ and $\operatorname{dom} \pi \subseteq Y \in [\mathcal{M}]^{<\omega}$, then there exists $\phi \in \mathcal{F}$ such that $\operatorname{dom} \phi = Y$ and $\phi \restriction \operatorname{dom} \pi = \pi$.
(5) If π, $\phi \in \mathcal{F}$ and $\operatorname{ran} \pi = \operatorname{dom} \phi$, then $\phi \circ \pi \in \mathcal{F}$.

Example 4.2. Let G be any group such that $Aut(\mathcal{M}) \leqslant G \leqslant Sym(\mathcal{M})$. Then

$$\mathcal{F}(G) = \{g \restriction X \mid g \in G,\ X \in [\mathcal{M}]^{<\omega}\}$$

is a pseudo-reduct of $\mathcal{M}$. If $\overline{G}$ is the closure of G in $Sym(\mathcal{M})$, then $\mathcal{F}(G) = \mathcal{F}(\overline{G})$.

Suppose that the pseudo-reduct $\mathcal{F}$ of $\mathcal{M}$ also satisfies the following condition:
- If $\pi \in \mathcal{F}$, then $\pi^{-1} \in \mathcal{F}$.

Then an easy back-and-forth argument shows that there exists a group G such that $Aut(\mathcal{M}) \leqslant G \leqslant Sym(\mathcal{M})$ and $\mathcal{F} = \mathcal{F}(G)$. By choosing the maximal such group, we can suppose that G is a closed permutation group. In an earlier version of this paper, I asked whether every pseudo-reduct of a countable ω-categorical structure $\mathcal{M}$ arises from a closed permutation group in this fashion. However, the referee pointed out that this is not the case. To see this, we shall make use of the following characterisation of the pseudo-reducts of $\mathcal{M}$.

Definition 4.3. Let $\mathcal{M}$ be a countable structure, and let

$$Inj(\mathcal{M}) = \{\phi \mid \phi : \mathcal{M} \to \mathcal{M} \text{ is an injection}\}.$$

Let $\mathcal{I} \subseteq Inj(\mathcal{M})$.
(1) $C(\mathcal{I})$ is the set of all maps $\pi : \mathcal{M} \to \mathcal{M}$ such that $\pi = g_1 \circ \cdots \circ g_n$ for some $g_1, \ldots, g_n \in Aut(\mathcal{M}) \cup \mathcal{I}$.
(2) $\mathcal{F}(\mathcal{I}) = \{\pi \restriction X \mid \pi \in C(\mathcal{I}),\ X \in [\mathcal{M}]^{<\omega}\}$.

Proposition 4.4. *Let $\mathcal{M}$ be a countable structure.*
(1) *If $\mathcal{I} \subseteq Inj(\mathcal{M})$, then $\mathcal{F}(\mathcal{I})$ is a pseudo-reduct of $\mathcal{M}$. (We shall say that $\mathcal{F}(\mathcal{I})$ is the pseudo-reduct of $\mathcal{M}$* generated *by $\mathcal{I}$.)*
(2) *Conversely, if $\mathcal{F}$ is any pseudo-reduct of $\mathcal{M}$, then there exists a subset $\mathcal{I} \subseteq Inj(\mathcal{M})$ such that $\mathcal{F} = \mathcal{F}(\mathcal{I})$.*

Proof. Left to the reader. □

Now we can give the referee's examples of some pseudo-reducts of Γ_k which do not arise from closed permutation groups.

Example 4.5. Let $k \geqslant 1$ and let C be an infinite complete k-subgraph of Γ_k. Let $\phi : \Gamma_k \to C$ be an injection, and let $\mathcal{F}_0 = \mathcal{F}(\{\phi\})$ be the pseudo-reduct generated by $\{\phi\}$. If E is a k-edge of Γ_k, then $\pi[E]$ is also a k-edge for all $\pi \in \mathcal{F}_0$. Hence

$\mathscr{F}_0 \neq \mathscr{F}(Sym(\Gamma_k))$. Suppose that G is a closed group such that $Aut(\Gamma_k) \leqslant G \leqslant Sym(\Gamma_k)$ and $\mathscr{F}_0 = \mathscr{F}(G)$. Let X, Y be finite subsets of Γ_k such that $|X| = |Y|$. Since $\phi[X]$, $\phi[Y] \subset C$, there exist $g, h \in G$ such that $g[X]$, $h[Y] \subset C$. It follows easily that G is highly transitive. But this means that $G = Sym(\Gamma_k)$, which is a contradiction. (We can obtain similar examples by taking $\phi : \Gamma_k \to N$ to be an injection into an infinite null k-subgraph of Γ_k.)

Definition 4.6. Let $k \geqslant 1$ and let $\mathscr{F}$ be a pseudo-reduct of Γ_k. Then $\mathscr{F}$ is said to be a *trivial* pseudo-reduct if either of the following two conditions hold.

(a) For each $X \in [\Gamma_k]^{<\omega}$ such that $|X| \geqslant k$, there exists $\pi \in \mathscr{F}$ such that $\pi[X]$ is a complete k-graph.

(b) For each $X \in [\Gamma_k]^{<\omega}$such that $|X| \geqslant k$, there exists $\pi \in \mathscr{F}$ such that $\pi[X]$ is a null k-graph.

Example 4.7. Let $k \geqslant 1$. Let Γ_k^* be the k-graph obtained from Γ_k by changing exactly one k-nonedge into a k-edge. Then clearly $\Gamma_k^* \simeq \Gamma_k$. Hence there exists $\psi \in Sym(\Gamma_k)$ such that

(1) there exists a k-nonedge A such that $\psi[A]$ is a k-edge; and

(2) $\psi \upharpoonright E$ is an isomorphism for all $E \in [\Gamma_k]^k \setminus \{A\}$.

Let $\mathscr{F}_1 = \mathscr{F}(\{\psi\})$ be the pseudo-reduct generated by $\{\psi\}$. Suppose that X is any finite k-subgraph of Γ_k such that $|X| \geqslant k$, and that $B \in [X]^k$ is a k-nonedge. Then there exists $\theta \in Aut(\Gamma_k)$ such that $\theta[B] = A$. Thus $\sigma = \psi \circ \theta \upharpoonright X \in \mathscr{F}_1$, and $\sigma[X]$ has one less k-nonedge than X. Continuing in this manner, we eventually obtain an element $\pi \in \mathscr{F}_1$ such that $\pi[X]$ is a complete k-graph. Arguing as in Example 4.5, we see that $\mathscr{F}_1 \neq \mathscr{F}(Sym(\Gamma_k))$. In fact, we have the following proper inclusions

$$\mathscr{F}_0 \subsetneq \mathscr{F}_1 \subsetneq \mathscr{F}(Sym(\Gamma_k))$$

of trivial pseudo-reducts.

In Section 6, we shall prove that if $\mathscr{F}$ is a nontrivial pseudo-reduct of Γ_k, then there exists a subset $X \subseteq \{0, 1, \ldots, k-1\}$ such that $\mathscr{F} = \mathscr{F}(S_X(\Gamma_k))$. Clearly this implies Theorem 1.15. We end this section by dealing with the easy case when $k = 1$. We shall make use of the following observation, which will also be used in Section 6.

Lemma 4.8. *Let $k \geqslant 1$ and let $\mathscr{F}$ be a pseudo-reduct of Γ_k. Suppose that for each $X \in [\Gamma_k]^{<\omega}$ such that $|X| \geqslant k$, there exists $\pi \in \mathscr{F}$ such that $\pi[X]$ is either a complete or a null k-graph. Then $\mathscr{F}$ is a trivial pseudo-reduct.*

Proof. Express $\Gamma_k = \bigcup_{n \in \mathbb{N}} X_n$ as the union of an increasing chain of finite k-subgraphs. We can suppose that there is an infinite subset I of $\mathbb{N}$ such that for each $n \in I$, there exists $\pi_n \in \mathscr{F}$ such that $\pi_n[X_n]$ is a complete k-graph. This implies that if X is any finite k-subgraph of Γ_k such that $|X| \geqslant k$, then there exists $\pi \in \mathscr{F}$ such that $\pi[X]$ is a complete k-graph. $\square$

Proposition 4.9. *If $\mathscr{F}$ is a nontrivial pseudo-reduct of Γ_1, then there exists a subset $X \subseteq \{0\}$ such that $\mathscr{F} = \mathscr{F}(S_X(\Gamma_1))$.*

Proof. Let $\mathscr{F}$ be a nontrivial pseudo-reduct of Γ_1. Remember that Γ_1 is just a countably infinite set, equipped with a 0-definable subset E_0 such that both E_0 and $E_1 = \Gamma_1 \setminus E_0$ are infinite. Since $\mathscr{F}$ is nontrivial, there exists $X_0 \in [\Gamma_1]^{<\omega}$ such that $\pi[X_0] \nsubseteq E_0$ and $\pi[X_0] \nsubseteq E_1$ for all $\pi \in \mathscr{F}$. First suppose that there exists a closed permutation group G such that $Aut(\Gamma_1) \leqslant G < Sym(\Gamma_1)$ and $\mathscr{F} = \mathscr{F}(G)$. By Example 1.1, there exists a subset $X \subseteq \{0\}$ such that $G = S_X(\Gamma_1)$.

So we can suppose that there does *not* exist a group G such that $\mathscr{F} = \mathscr{F}(G)$. Hence there exists an element $\psi \in \mathscr{F}$ such that $\psi^{-1} \notin \mathscr{F}$. Using Definition 4.1(4), we can inductively construct an injective function $g : \Gamma_1 \to \Gamma_1$ such that $\psi \subset g$ and $\{g{\restriction}X \mid X \in [\Gamma_1]^{<\omega}\} \subseteq \mathscr{F}$. Since $\mathscr{F}$ is nontrivial, there must exist

(1) an infinite subset $A \subseteq E_0$ and an $i \in \{0,1\}$ such that $g[A] \subseteq E_i$, and

(2) an infinite subset $B \subseteq E_1$ such that $g[B] \subseteq E_{1-i}$.

Since $\psi^{-1} \notin \mathscr{F}$, we must also have that

(3) either there exists $u \in E_0$ such that $g(u) \in E_{1-i}$, or there exists $u \in E_1$ such that $g(u) \in E_i$.

(Suppose not. Then $g : \Gamma_1 \to g[\Gamma_1]$ is either an isomorphism or an anti-isomorphism. But this implies that $\psi^{-1} \in \mathscr{F}$.) Notice that $A \cup B \cup \{u\} \simeq \Gamma_1$. Hence, without loss of generality, we can suppose that $\Gamma_1 = A \cup B \cup \{u\}$, and that $u \in E_0$, $g(u) \in E_{1-i}$. Let

$$P = \{h \in Sym(\Gamma_1) \mid h{\restriction}X \in \mathscr{F} \text{ for all } X \in [\Gamma_1]^{<\omega}\}.$$

Then $g \in P$ and $k^{-1} \circ g \circ k \in P$ for all $k \in Aut(\Gamma_1)$. Also if h_1, $h_2 \in P$ then $h_1 \circ h_2 \in P$. But this easily implies that for each $X \in [\Gamma_1]^{<\omega}$, there exists $h \in P$ such that $h[X] \subset E_{1-i}$. This contradicts the fact that $\mathscr{F}$ is a nontrivial pseudo-reduct. $\square$

5. Analysing parity-preserving maps

In this section, we shall prove the following result.

Theorem 5.1. *Suppose that $\mathscr{F}$ is a pseudo-reduct of Γ_k, and that X is the largest subset of $\{0,1,\ldots,k-1\}$ such that $\mathscr{F}(S_X(\Gamma_k)) \subseteq \mathscr{F}$. Then there exists an integer ℓ such that whenever $f \in \mathscr{F} \cap \mathscr{F}(B(\Gamma_k))$ and* $\operatorname{dom} f \models \phi^k_\ell$, *then $f \in \mathscr{F}(S_X(\Gamma_k))$.*

In the statement of the above theorem, ϕ^k_ℓ is the ℓ^{th} extension axiom in the usual axiomatisation of $Th(\Gamma_k)$ (see Section 3). From now on, fix a pseudo-reduct $\mathscr{F}$ of Γ_k, and let $X \subseteq \{0,1,\ldots,k-1\}$ be the largest subset such that $\mathscr{F}(S_X(\Gamma_k)) \subseteq \mathscr{F}$. Also fix a $B(\Gamma_k)$-good integer n.

Lemma 5.2. *There exists an integer $N > n$ with the following property. Suppose that $f \in \mathscr{F}$ satisfies*

(1) $|\mathrm{dom}\, f| \geqslant N$; *and*

(2) *for all* $A \in [\mathrm{dom}\, f]^N$, $f \upharpoonright A \in \mathscr{F}(S_X(\Gamma_k))$.

Then $f \in \mathscr{F}(S_X(\Gamma_k))$.

Proof. This is an easy consequence of Theorem 2.6. □

Now suppose that $f \in \mathscr{F} \cap \mathscr{F}(B(\Gamma_k))$ and that $\mathrm{dom}\, f \models \phi_\ell^k$ for some extremely large integer ℓ. In particular, $|\mathrm{dom}\, f| > N$, where N is the integer given by Lemma 5.2. Let $T \in [\mathrm{dom}\, f]^N$ be arbitrary. Then it is enough to show that $f \upharpoonright T \in \mathscr{F}(S_X(\Gamma_k))$. To accomplish this, we shall adjust f repeatedly via multiplication by elements of $\mathscr{F}(S_X(\Gamma_k))$ until we eventually obtain an element $h \in \mathscr{F} \cap \mathscr{F}(B(\Gamma_k))$ such that $h \upharpoonright T$ is an isomorphism. Our strategy is based upon the following lemma.

Lemma 5.3. *Suppose that* $h \in \mathscr{F}(B(\Gamma_k))$ *and that the following conditions hold.*

(1) $U, T \subseteq \mathrm{dom}\, h$ *are disjoint subsets, and* $|U| \geqslant n - k$.

(2) *If* $B \in [T \cup U]^k \setminus [T]^k$, *then* $h \upharpoonright B$ *is an isomorphism.*

Then $h \upharpoonright T$ *is an isomorphism.*

Proof. Suppose that there exists $C \in [T]^k$ such that $h \upharpoonright C$ is *not* an isomorphism. Let $D \in [U]^{n-k}$ and consider $h \upharpoonright C \cup D$. Since $h \upharpoonright B$ is an isomorphism for all $B \in [C \cup D]^k \setminus \{C\}$, h fails to preserve the parity of k-edges in the n-set $C \cup D$. But this contradicts the fact that $B(\Gamma_k)$ preserves the parity of k-edges in every n-subset of Γ_k. □

We shall make use of the following characterisation of X.

Lemma 5.4. *There exists a finite* k*-subgraph* H *of* Γ_k *with the following property. For each* $0 \leqslant i \leqslant k-1$, $i \in X$ *if and only if there exists a set* $A \in [H]^i$ *and an element* $f \in \mathscr{F}$ *such that*

(1) $\mathrm{dom}\, f = H$, *and*

(2) $f : H \to \Gamma_k$ *is a switch with respect to* A.

Proof. Suppose that $0 \leqslant i \leqslant k-1$ and that $i \notin X$. Then there exists a finite k-subgraph H_i of Γ_k and an i-subset $A_i \in [H_i]^i$ such that

- if $f \in \mathscr{F}$ with $\mathrm{dom}\, f = H_i$, then $f : H_i \to \Gamma_k$ is *not* a switch with respect to A_i.

Let σ_i be the sentence

$$(\forall a_1 \ldots a_i)\left(\exists b_1 \ldots b_{|H_i|}\right) \Psi(a_1, \ldots, a_i, b_1, \ldots, b_{|H_i|})$$

which says the following:

($\dagger$)$_i$ For every i-subset $A = \{a_j \mid 1 \leqslant j \leqslant i\}$, there exists an $|H_i|$-subset $B = \{b_j \mid 1 \leqslant j \leqslant |H_i|\}$ such that

(a) $A \subset B$, and

(b) there exists an isomorphism $\pi : B \to H_i$ such that $\pi[A] = A_i$.

Let σ be the sentence $\bigwedge_{i \notin X} \sigma_i$. Then $\Gamma_k \models \sigma$. Hence, by Theorem 3.2, there exists a finite k-subgraph H of Γ_k such that $H \models \sigma$. Clearly H satisfies our requirements. □

We shall also make use of the following rather technical notion.

Definition 5.5. Let $m > k$. Suppose that $f \in \mathscr{F}(B(\Gamma_k))$ and that $Z = \operatorname{dom} f$ satisfies $|Z| \geqslant m$. Then an *m-analysis* of f consists of a finite sequence of elements $g_0, g_1, \ldots, g_t$ of $\mathscr{F}(B(\Gamma_k))$ which satisfies the following conditions:

(1) $g_0 = f$.

(2) For each $0 \leqslant j \leqslant t-1$, there exists $Y_j \in [Z]^m$ and an element $\theta_j \in B(\Gamma_k)$ such that

  (a) θ_j is a switch with respect to some i_j-subset A_j of Y_j;

  (b) $\theta_j \upharpoonright Y_j = g_j \circ \cdots \circ g_0 \upharpoonright Y_j$;

  (c) $g_{j+1} = \theta_j^{-1} \upharpoonright \operatorname{ran} g_j \circ \cdots \circ g_0$.

(3) $g_t \circ \cdots \circ g_0 : Z \to \Gamma_k$ is an isomorphic embedding.

As the above notion is quite difficult to understand, we shall now give an idea of the manner in which it will be used. Let H be the finite k-graph given by Lemma 5.4, and let $m = |H|$. Let $f \in \mathscr{F} \cap \mathscr{F}(B(\Gamma_k))$ and suppose that $Z = \operatorname{dom} f$ satisfies $|Z| \geqslant m$. We want to show that $f \in \mathscr{F}(S_X(\Gamma_k))$. Suppose that we can find an m-analysis $g_0, g_1, \ldots, g_t$ of f with the further property that $Y_j \simeq H$ for each $0 \leqslant j \leqslant t-1$. We shall show that $i_j \in X$ and $g_{j+1} \in \mathscr{F}(S_X(\Gamma_k))$ for all $0 \leqslant j \leqslant t-1$. First consider the case when $j = 0$. Then θ_0 is a switch with respect to some i_0-subset A_0 of Y_0, and $g_0 \upharpoonright Y_0 = \theta_0 \upharpoonright Y_0$. Since $Y_0 \simeq H$, Lemma 5.4 yields that $i_0 \in X$. Hence $g_1 = \theta_0^{-1} \upharpoonright \operatorname{ran} g_0 \in \mathscr{F}(S_X(\Gamma_k))$. Now consider the case when $j = 1$. Then θ_1 is a switch with respect to some i_1-subset A_1 of Y_1, and $g_1 \circ g_0 \upharpoonright Y_1 = \theta_1 \upharpoonright Y_1$. Since $g_1 \circ g_0 \in \mathscr{F}$ and $Y_1 \simeq H$, Lemma 5.4 yields that $i_1 \in X$. Hence $g_2 = \theta_1^{-1} \upharpoonright \operatorname{ran} g_1 \circ g_0 \in \mathscr{F}(S_X(\Gamma_k))$. Continuing in this fashion, we obtain that $i_j \in X$ and $g_{j+1} \in \mathscr{F}(S_X(\Gamma_k))$ for all $0 \leqslant j \leqslant t-1$. This implies that $g_{j+1}^{-1} \in \mathscr{F}(S_X(\Gamma_k))$ for all $0 \leqslant j \leqslant t-1$. Finally note that there exists an isomorphic embedding $\pi : Z \to \Gamma_k$ such that

$$f = g_0 = g_1^{-1} \circ \cdots \circ g_t^{-1} \circ \pi.$$

It follows that $f \in \mathscr{F}(S_X(\Gamma_k))$.

The next lemma shows that $f \in \mathscr{F} \cap \mathscr{F}(B(\Gamma_k))$ has an m-analysis if $|\operatorname{dom} f|$ is sufficiently large. (However, it does *not* say that there exists an m-analysis such that $Y_j \simeq H$ for all $0 \leqslant j \leqslant t-1$.)

Lemma 5.6. *For each $m > k$, there exists an integer $s(k,m)$ such that whenever $f \in \mathscr{F}(B(\Gamma_k))$ satisfies $|\operatorname{dom} f| \geqslant s(k,m)$, then there exists an m-analysis of f.*

Proof. We shall argue by induction on $k \geqslant 1$. First suppose that $k = 1$. Remember that Γ_1 consists of a countably infinite set equipped with a partition $E_0 \cup E_1$ into two infinite subsets. Also

$$B(\Gamma_1) = \{\pi \in Sym(\Gamma_1) \mid \pi[E_0] = E_i \text{ for some } i \in \{0,1\}\}.$$

Then we can take $s(1,m) = m$. For suppose that $f \in \mathscr{F}(B(\Gamma_1))$, and that $Z = \operatorname{dom} f$ satisfies $|Z| \geqslant m$. Then $f : Z \to \Gamma_1$ is either an isomorphic or an anti-isomorphic embedding. Let $g_0 = f$. In the former case, g_0 is an m-analysis of f of length $t = 0$. In the latter case, let Y_0 be any m-subset of Z and let $\theta_0 \in B(\Gamma_1)$ be an anti-isomorphism such that $\theta_0 \restriction Y_0 = g_0 \restriction Y_0$. (Thus $i_0 = 0$ and $A_0 = \emptyset$.) Let $g_1 = \theta_0^{-1} \restriction \operatorname{ran} g_0$. Then $g_1 \circ g_0 \restriction Y_0 = id_{Y_0}$, and so $g_1 \circ g_0 : Z \to \Gamma_1$ is an isomorphic embedding. Hence g_0, g_1 is an m-analysis of f of length $t = 1$.

Now suppose that the result holds for some $k \geqslant 1$. Let p be a $B(\Gamma_{k+1})$-good integer. Fix an integer $m > k + 1$. Let $f \in \mathscr{F}(B(\Gamma_{k+1}))$ be such that $Z = \operatorname{dom} f$ is a very large subset of Γ_{k+1}. By Ramsey's Theorem, there exists a large subset R of Z such that $f \restriction R$ is either an isomorphism or an anti-isomorphism. First suppose that $f \restriction R$ is an anti-isomorphism. Choose any m-subset Y_0 of R, and let $\theta_0 \in B(\Gamma_k)$ be an anti-isomorphism such that $\theta_0 \restriction R = f \restriction R$. (So $i_0 = 0$ and $A_0 = \emptyset$.) Let $g_1 = \theta_0^{-1} \restriction \operatorname{ran} f$. Notice that if $f' = g_1 \circ f = g_1 \circ g_0$, then $f' \restriction R$ is the identity isomorphism. Also if $g'_0, \ldots, g'_t$ is an m-analysis of f', then $g_0, g_1, g'_1, \ldots, g'_t$ is an m-analysis of f. To simplify notation, we shall suppose that $f \restriction R$ is an isomorphism. Let $v \in Z \backslash R$ and consider $f \restriction R \cup \{v\}$. Note that if $E \in [R \cup \{v\}]^{k+1}$ and $f \restriction E$ is *not* an isomorphism, then $v \in E$. Let R_v be the k-graph induced on R by v and let $f_v : R_v \to \Gamma_k$ be the map induced by $f \restriction R \cup \{v\}$. Since f preserves the parity of $(k + 1)$-edges in every p-subset of $R \cup \{v\}$, it follows that f_v preserves the parity of k-edges in every $(p-1)$-subset of R_v. By Lemma 1.19, $p-1$ is $B(\Gamma_k)$-good. Hence Proposition 1.21 yields that $f_v \in \mathscr{F}(B(\Gamma_k))$. We can suppose that $|R_v| \geqslant s(k, m-1)$. Hence there exists an $(m-1)$-analysis $\tilde{g}_0, \ldots, \tilde{g}_t$ of f_v. Let $\tilde{Y}_j \in [R_v]^{m-1}$, $\tilde{A}_j \subseteq \tilde{Y}_j$ and $\tilde{\theta}_j \in B(\Gamma_k)$, $0 \leqslant j \leqslant t-1$, be as in Definition 5.5. Then clearly $f \restriction \tilde{Y}_0 \cup \{v\}$ is induced by a switch $\theta_0 \in B(\Gamma_{k+1})$ with respect to $\tilde{A}_0 \cup \{v\}$. Let $g_1 = \theta_0^{-1} \restriction \operatorname{ran}(f \restriction R \cup \{v\})$. Continuing in this manner, we can convert $\tilde{g}_0, \ldots, \tilde{g}_t$ into an m-analysis $g_0, \ldots, g_t$ of $f \restriction R \cup \{v\}$. Let $\theta_0, \ldots, \theta_{t-1}$ be the corresponding sequence of elements of $B(\Gamma_{k+1})$. By clauses 2(c) and 3 of Definition 5.5, $\theta_{t-1}^{-1} \circ \cdots \circ \theta_0^{-1} \circ f \restriction R \cup \{v\}$ is an isomorphism. Let $R' = R \cup \{v\}$ and suppose that $w \in Z \backslash R'$. Let $h = \theta_{t-1}^{-1} \circ \cdots \circ \theta_0^{-1} \circ f$ and consider $h \restriction R' \cup \{w\}$. Then $h \restriction R'$ is an isomorphism. Arguing as above, there exists an m-analysis $g'_0, \ldots, g'_{t'}$ of $h \restriction R' \cup \{w\}$. By combining the m-analysis $g_0, \ldots, g_t$ of $f \restriction R'$ and the m-analysis $g'_0, \ldots, g'_{t'}$ of $h \restriction R' \cup \{w\}$, we shall obtain an m-analysis of $f \restriction R' \cup \{w\}$. First we shall extend the domains of $g_0, \ldots, g_t$ so that they can form an initial segment of our m-analysis of $f \restriction R' \cup \{w\}$. Define g_j^*, $0 \leqslant j \leqslant t$, inductively by

(1) $g_0^* = f \restriction R' \cup \{w\}$

(2) $g_{j+1}^* = \theta_j^{-1} \restriction \operatorname{ran} g_j^* \circ \cdots \circ g_0^*$.

Then $g_t^* \circ \cdots \circ g_0^* = h \restriction R' \cup \{w\}$. It follows that $g_0^*, \ldots, g_t^*$, $g'_1, \ldots, g'_{t'}$ is an m-analysis of $f \restriction R' \cup \{w\}$. Continuing in this fashion, we eventually obtain an m-analysis of f. The result follows. $\square$

We shall also make use of the following generalisation of Ramsey's Theorem, which is due independently to Abramson and Harrington [1] and Nešetřil and Rödl [11].

Definition 5.7. A *system of colors* of length n, $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_n)$ is an $(n+1)$-sequence of finite nonempty sets. An *$\boldsymbol{\alpha}$-colored set* consists of a finite ordered set X and a function $\tau : [X]^{\leqslant n} \to \alpha_0 \cup \cdots \cup \alpha_n$ such that $\tau(A) \in \alpha_k$ for each $A \in [X]^k$. For each $A \in [X]^{\leqslant n}$, $\tau(A)$ is called the color of A.

An *α-pattern* is an α-colored set whose underlying ordered set is an integer. This integer is called the *length of the pattern*. Each α-colored set is isomorphic to a unique α-pattern.

Theorem 5.8. (Abramson and Harrington [1]). *Given $n, e, M \in \mathbb{N}$, a system α of colors of length n and an α-pattern P, there exists an α-pattern Q with the following property. For any α-colored set (X, τ) with α-pattern Q and any function $F : [X]^e \to M$, there exists $Y \subseteq X$ such that $(Y, \tau \restriction Y)$ has α-pattern P and such that for all $A \in [Y]^e$, $F(A)$ depends only on the α-pattern of $(A, \tau \restriction A)$. (We say that Y is F-homogeneous.)*

Proof of Theorem 5.1. Suppose that $f \in \mathscr{F} \cap \mathscr{F}(B(\Gamma_k))$ and that $\operatorname{dom} f \models \phi^k_\ell$ for some extremely large integer ℓ. Let $T \in [\operatorname{dom} f]^N$, where N is the integer given by Lemma 5.2. Then it is enough to show that $f \restriction T \in \mathscr{F}(S_X(\Gamma_k))$. Choose a very large k-subgraph U_0 of $\operatorname{dom} f$ such that the following conditions hold:

(1) $T \cap U_0 = \emptyset$.

(2) $T \cup U_0$ is a "sufficiently random" k-graph; i.e. for some suitably chosen $t < \ell$, $T \cup U_0 \models \phi^k_t$.

(As usual, we will not define t explicitly. As the proof proceeds, the reader will see which extension properties we need.)

Fix an ordering $\prec$ of the vertices of $T \cup U_0$, chosen so that T is an initial segment, and such that $\langle T \cup U_0; \prec \rangle$ is "sufficiently complex". We will not define "sufficiently complex" explicitly, but we will give the reader some idea of what we have in mind. (The notion will become clearer as the proof proceeds.) If U' is a k-subgraph of U_0, then we shall say that $\langle T \cup U'; \prec \rangle$ is "sufficiently complex" if the following conditions hold:

(a) $T \cup U'$ is a "sufficiently random" k-graph.

(b) $\langle T \cup U'; \prec \rangle$ contains various ordered k-subgraphs which will be defined later.

(c) $\langle T \cup U'; \prec \rangle$ is such that we can make a certain number of successive applications of Theorem 5.8. (For example, $\langle T \cup U_0; \prec \rangle$ must be such that we can make $2^{|T|} - 1$ successive applications of Theorem 5.8.)

For a suitable system of colors α, define an α-coloring τ of $[U_0]^{\leqslant k}$ by setting

(3) $\tau(A) = \tau(B)$ iff $|A| = |B|$ and the order-preserving bijection $T \cup A \to T \cup B$ is a k-graph isomorphism.

Thus if $Z \subseteq U_0$, then the α-pattern of $(Z, \tau \restriction Z)$ essentially consists of the type $tp(Z|T)$ of Z over T. Now define the function $F_f : [U_0]^k \to \{0, 1\}$ by

(4) $F_f(C) = 1$ iff $f \restriction C$ is an isomorphism.

Let U_1 be a very large k-subgraph of U_0 such that $\langle T \cup U_1; \prec \rangle$ is still "sufficiently complex". By Theorem 5.8, we can suppose that the α-colored set $(U_1, \tau \restriction U_1)$ is F_f-homogeneous. Finally define a function χ on $[U_1]^k$ such that

(5) $\chi(C) = \chi(D)$ iff the order-preserving bijection $\phi : T \cup C \to T \cup D$ has the property that

$$\phi(A) \text{ is a } k\text{-edge} \Leftrightarrow A \text{ is a } k\text{-edge}$$

for all $A \in [T \cup C]^k \setminus \{C\}$.

Claim 5.9. *If C, $D \in [U_1]^k$ and $\chi(C) = \chi(D)$, then $f \restriction C$ is an isomorphism if and only if $f \restriction D$ is an isomorphism.*

Proof. Suppose that C, D form a counterexample. (Since U_1 is F_f-homogeneous, it follows that the order-preserving map $T \cup C \to T \cup D$ is not an isomorphism. Hence exactly one of C, D is a k-edge.) Without loss of generality, we can suppose that $f \restriction C$ is an isomorphism and that $f \restriction D$ is not an isomorphism. Since $\langle T \cup U_1; \prec \rangle$ is "sufficiently complex", there exist C', $D' \in [U_1]^k$ with the following properties:

(i) $\tau(C) = \tau(C')$ and $\tau(D) = \tau(D')$.

(ii) There exist subsets V, $W \subset U_1$ such that the following conditions are satisfied:

(a) $C' \subset V$, $D' \subset W$ and $|V| = |W|$.

(b) Let $\phi : T \cup V \to T \cup W$ be the order-preserving bijection. Then $\phi[C'] = D'$, and $\phi \restriction E$ is an isomorphism for all $E \in [T \cup V]^k \setminus \{C'\}$.

(c) The integer $|V|$ is $B(\Gamma_k)$-good.

In particular, $\tau(E) = \tau(\phi[E])$ for all $E \in [V]^k \setminus \{C'\}$. Since U_1 is F_f-homogeneous, it follows that

$$f \restriction E \text{ is an isomorphism} \Leftrightarrow f \restriction \phi[E] \text{ is an isomorphism}$$

for all $E \in [V]^k \setminus \{C'\}$. Because $\tau(C') = \tau(C)$ and $\tau(D') = \tau(D)$, we have that $f \restriction C'$ is an isomorphism and that $f \restriction D'$ is not an isomorphism. Let

$$p = \left|\{E \in [V]^k \mid f \restriction E \text{ is not an isomorphism}\}\right|$$

and

$$q = \left|\{E \in [W]^k \mid f \restriction E \text{ is not an isomorphism}\}\right|.$$

Then we have shown that $q = p + 1$. But, by (c), $f \restriction V$ and $f \restriction W$ preserve the parity of k-edges in V, W respectively. Hence we must have that $p \equiv 0$ and $q \equiv 0$; which is a contradiction. $\square$

Claim 5.10. *Suppose that $S_1, S_2 \subseteq U_1$ and that $|S_1| = |S_2|$. Let $\phi : T \cup S_1 \to T \cup S_2$ be the order-preserving bijection. Suppose that*

$$\phi[E] \textit{ is a } k\textit{-edge iff } E \textit{ is a } k\textit{-edge}$$

for all $E \in [T \cup S_1]^k \setminus [S_1]^k$. Then

$$f \restriction E \textit{ is an isomorphism iff } f \restriction \phi[E] \textit{ is an isomorphism}$$

for all $E \in [S_1]^k$.

Proof. Simply note that $\chi(E) = \chi(\phi[E])$ for all $E \in [S_1]^k$. So the result is an immediate consequence of Claim 5.9. □

Claim 5.11. $f \upharpoonright U_1 \in \mathscr{F}(S_X(\Gamma_k))$.

Proof. Let H be the finite k-graph given by Lemma 5.4, and let $m = |H|$. By Lemma 5.6, there exists an m-analysis of $f \upharpoonright U_1$; say $g_0, g_1, \dots, g_t \in \mathscr{F}(B(\Gamma_k))$. Thus for each $0 \leqslant j \leqslant t-1$, there exists $Y_j \in [U_1]^m$ and an element $\theta_j \in B(\Gamma_k)$ such that

(i) $g_0 = f \upharpoonright U_1$;
(ii) θ_j is a switch with respect to some i_j-subset A_j of Y_j;
(iii) $\theta_j \upharpoonright Y_j = g_j \circ \cdots \circ g_0 \upharpoonright Y_j$;
(iv) $g_{j+1} = \theta_j^{-1} \upharpoonright \operatorname{ran} g_j \circ \cdots \circ g_0$;
(v) $g_t \circ \cdots \circ g_0 : U_1 \to \Gamma_k$ is an isomorphic embedding.

If $\{i_0, \dots, i_{t-1}\} \subseteq X$, then we are done. If not, then let j be minimal such that $i_j \notin X$. Thus $g_\ell \in \mathscr{F}(S_X(\Gamma_k))$ for all $1 \leqslant \ell \leqslant j$. Note that $g_j \circ \cdots \circ g_0 \upharpoonright Y_j$ is a switch with respect to the i_j-subset A_j of Y_j. Since $\langle T \cup U_1; \prec \rangle$ is "sufficiently complex", there exists $Y \in [U_1]^m$ such that the following conditions are satisfied:

(a) $Y \simeq H$.
(b) Let $\phi : T \cup Y_j \to T \cup Y$ be the order-preserving bijection. Then $\chi(E) = \chi(\phi[E])$ for all $E \in [Y_j]^k$.

By Claim 5.10, for all $E \in [Y_j]^k$, $f \upharpoonright E$ is an isomorphism iff $f \upharpoonright \phi[E]$ is an isomorphism. We claim that there exist $g_1^*, \dots, g_j^* \in \mathscr{F}(S_X(\Gamma_k))$ such that $g_j^* \circ \cdots \circ g_1^* \circ g_0 \upharpoonright Y$ is a switch with respect to the i_j-subset $\phi[A_j]$ of Y. But then Lemma 5.4 yields that $i_j \in X$. We shall define $g_\ell^* \in \mathscr{F}(S_X(\Gamma_k))$, $1 \leqslant \ell \leqslant j$, inductively so that for all $E \in [Y_j]^k$, $g_\ell \circ \cdots \circ g_1 \circ g_0 \upharpoonright E$ is an isomorphism iff $g_\ell^* \circ \cdots \circ g_1^* \circ g_0 \upharpoonright \phi[E]$ is an isomorphism. In particular, $g_j^* \circ \cdots \circ g_1^* \circ g_0 \upharpoonright Y$ will be a switch with respect to the i_j-subset $\phi[A_j]$ of Y. Suppose that we have defined $g_1^*, \dots, g_{\ell-1}^*$.

Case 1: Suppose that $A_{\ell-1} \nsubseteq Y_j$. Then $g_\ell \in \mathscr{F}(S_X(\Gamma_k))$ restricts to an isomorphism on $g_{\ell-1} \circ \cdots \circ g_1 \circ g_0[Y_j]$. So we can take g_ℓ^* to be the identity map on $g_{\ell-1}^* \circ \cdots \circ g_1^* \circ g_0[U_1]$.

Case 2: Suppose that $A_{\ell-1} \subseteq Y_j$. Then $g_\ell \in \mathscr{F}(S_X(\Gamma_k))$ restricts to a switch with respect to $g_{\ell-1} \circ \cdots \circ g_1 \circ g_0[A_{\ell-1}]$ on $g_{\ell-1} \circ \cdots \circ g_1 \circ g_0[Y_j]$. Let $\theta^* \in S_X(\Gamma_k)$ be a switch with respect to $g_{\ell-1}^* \circ \cdots \circ g_1^* \circ g_0[\phi[A_{\ell-1}]]$. Then we can take $g_\ell^* = \theta^* \upharpoonright g_{\ell-1}^* \circ \cdots \circ g_1^* \circ g_0[U_1]$.

This completes the induction. □

Choose $\theta \in S_X(\Gamma_k)$ such that $\theta \upharpoonright U_1 = f \upharpoonright U_1$, and let $h = \theta^{-1} \circ f \upharpoonright T \cup U_1$. Then $h \upharpoonright E$ is an isomorphism for all $E \in [U_1]^k$. This completes the first stage of the proof.

Choose a vertex $v \in T$ and consider $h \upharpoonright U_1 \cup \{v\}$. Notice that if $E \in [U_1 \cup \{v\}]^k$ is such that $h \upharpoonright E$ is *not* an isomorphism, then $v \in E$. Define the function $F_h : [U_1]^{k-1} \to \{0,1\}$ by

(6) $F_h(C) = 1$ iff $h \upharpoonright C \cup \{v\}$ is an isomorphism.

Let U_2 be a very large k-subgraph of U_1 such that $\langle T \cup U_2; \prec\rangle$ is still "sufficiently complex". By Theorem 5.8, we can suppose that the α-colored set $(U_2, \tau\restriction U_2)$ is F_h-homogeneous.

Claim 5.12. *Suppose that $S_1, S_2 \subseteq U_2$ and that $|S_1| = |S_2|$. Let $\phi : T \cup S_1 \to T \cup S_2$ be the order-preserving bijection. Suppose that*

$$\phi[E] \text{ is a } k\text{-edge iff } E \text{ is a } k\text{-edge}$$

for all $E \in [T \cup S_1]^k \setminus [S_1 \cup \{v\}]^k$. Then

$$h\restriction E \text{ is an isomorphism iff } h\restriction\phi[E] \text{ is an isomorphism}$$

for all $E \in [S_1 \cup \{v\}]^k$.

Proof. Argue as in the proof of Claims 5.9 and 5.10. (Note that we are really only concerned about those $E \in [S_1 \cup \{v\}]^k$ such that $v \in E$. If $v \notin E$, then $h\restriction E$ and $h\restriction\phi[E]$ are both isomorphisms.) □

Claim 5.13. $h\restriction U_2 \cup \{v\} \in \mathscr{F}(S_X(\Gamma_k))$.

Proof. Let $R = (U_2)_v$ be the $(k-1)$-graph induced on U_2 by v, and let $h_v : R \to \Gamma_{k-1}$ be the map induced by $h\restriction U_2 \cup \{v\}$. Then h_v preserves the parity of $(k-1)$-edges in every $(n-1)$-subset of R. By Proposition 1.21, $h_v \in \mathscr{F}(B(\Gamma_{k-1}))$. So, by Lemma 5.6, there exists an $(m-1)$-analysis $\tilde{g}_0, \ldots, \tilde{g}_t$ of h_v. Arguing as in the proof of Lemma 5.6, this yields an m-analysis $g_0, \ldots, g_t$ of $h\restriction U_2 \cup \{v\}$ with the following operty. For each $0 \leqslant j \leqslant t-1$, there exists $Y_j \in [U_2 \cup \{v\}]^m$ and an element $\theta_j \in B(\Gamma_k)$ such that

(i) $v \in Y_j$ for all $0 \leqslant j \leqslant t-1$;
(ii) $g_0 = h\restriction U_2 \cup \{v\}$;
(iii) θ_j is a switch with respect to some i_j-subset A_j of Y_j such that $v \in A_j$;
(iv) $\theta_j\restriction Y_j = g_j \circ \cdots \circ g_0\restriction Y_j$;
(v) $g_{j+1} = \theta_j^{-1}\restriction \operatorname{ran} g_j \circ \cdots \circ g_0$;
(vi) $g_t \circ \cdots \circ g_0 : U_2 \cup \{v\} \to \Gamma_k$ is an isomorphic embedding.

If $\{i_0, \ldots, i_{t-1}\} \subseteq X$, then we are done. If not, let $0 \leqslant j \leqslant t-1$ be minimal such that $i_j \notin X$. Note that $g_j \circ \cdots \circ g_0\restriction Y_j$ is a switch with respect to the i_j-subset A_j of Y_j. Since $\langle T \cup U_2; \prec\rangle$ is "sufficiently complex", there exists $Y \in [U_2 \cup \{v\}]^m$ such that the following conditions are satisfied.

(a) $v \in Y$.
(b) $Y \simeq H$.
(c) Let $\phi : T \cup (Y_j \setminus \{v\}) \to T \cup (Y \setminus \{v\})$ be the order-preserving bijection. Then $\phi[E]$ is a k-edge iff E is a k-edge for all $E \in [T \cup (Y_j \setminus \{v\})]^k \setminus [Y_j]^k$.

Note that $\phi(v) = v$, and so $\phi[Y_j] = Y$. Also Claim 5.12 yields that

$$h\restriction E \text{ is an isomorphism iff } h\restriction\phi[E] \text{ is an isomorphism}$$

for all $E \in [Y_j]^k$. As in the proof of Claim 5.11, this implies that $i_j \in X$. □

Using the fact that $v \in A_j$ for each $0 \leqslant j \leqslant t-1$, we see that there exists $\psi \in S_X(\Gamma_k)$ such that

(7) (a) $\psi \restriction U_2 \cup \{v\} = h \restriction U_2 \cup \{v\}$, and

(b) $\psi \restriction E$ is an isomorphism for all $E \in [T \cup U_2]^k$ such that $v \notin E$.

Let $h' = \psi^{-1} \circ h \restriction T \cup U_2$. Now choose a second vertex $w \in T \setminus \{v\}$. Arguing as above, we find a very large k-subgraph U_3 of U_2 such that $\langle T \cup U_3; \prec \rangle$ is still "sufficiently complex", and an element $\psi' \in S_X(\Gamma_k)$ such that

(8) (a) $\psi' \restriction U_3 \cup \{w\} = h' \restriction U_3 \cup \{w\}$, and

(b) $\psi' \restriction E$ is an isomorphism for all $E \in [T \cup U_3]^k$ such that $w \notin E$.

Here clause 8(b) is of crucial importance. It means that $\psi' \restriction E$ is an isomorphism for all of those k-sets E such that $E \cap T = \emptyset$ or $E \cap T = \{v\}$. Thus when we next adjust h' to $h'' = (\psi')^{-1} \circ h' \restriction T \cup U_3$, we do not spoil the progress which we made with our earlier adjustments. Continuing in this fashion, we can deal successively with the other vertices $z \in T \setminus \{v, w\}$. After this, we consider those k-sets E such that $|E \cap T| = 2$. As above, we can deal successively with each of the 2-subsets of T, without spoiling the progress which we have made for those k-subsets E such that $|E \cap T| \leqslant 1$. Then we can deal successively with each of the 3-subsets of T, and then each of the 4-subsets of T, etc. Eventually we obtain a large subset U^* of U_3 and a map $h^* : T \cup U^* \to \Gamma_k$ such that

(9) (a) there exists $\psi^* \in S_X(\Gamma_k)$ such that $h^* = \psi^* \circ f \restriction T \cup U^*$; and

(b) $h^* \restriction E$ is an isomorphism for all $E \in [T \cup U^*]^k \setminus [T]^k$.

Now Lemma 5.3 implies that $h^* \restriction T$ is an isomorphism. Hence $f \restriction T \in \mathscr{F}(S_X(\Gamma_k))$. This completes the proof of Theorem 5.1. □

We shall end this section with the following easy observation, which will be used in the proof of Theorem 6.1.

Lemma 5.14. *Let $k \geqslant 1$ and let $X \subseteq \{0, 1, \ldots, k-1\}$. Then there exists an integer N_X^k with the following property. Let $f \in \mathscr{F}(S_X(\Gamma_k))$, and suppose that $Z = \operatorname{dom} f$ satisfies $|Z| \geqslant N_X^k$. Suppose further that $T \in [Z]^{<k}$ is a subset such that*

(‡) *if $E \in [Z]^k$ and $f \restriction E$ is not an isomorphism, then $T \subseteq E$.*

Then there exists $\theta \in S_X(\Gamma_k)$ such that

(a) *$\theta \restriction Z = f \restriction Z$; and*

(b) *if $E \in [\Gamma_k]^k$ and $\theta \restriction E$ is not an isomorphism, then $T \subseteq E$.*

Proof. We shall prove the following statement by induction on $k \geqslant 1$.

$(5.14)_k$: For each $X \subseteq \{0, 1, \ldots, k-1\}$, there exists an integer N_X^k with the following property. Let $f \in \mathscr{F}(S_X(\Gamma_k))$, and suppose that $Z = \operatorname{dom} f$ satisfies $|Z| \geqslant N_X^k$. Suppose further that $T \in [Z]^{<k}$ is a subset such that (‡) holds. Then there exist elements $\psi_0, \ldots, \psi_t \in S_X(\Gamma_k)$ such that the following conditions are satisfied:

(1) $\psi_0 = id$.

(2) If $1 \leqslant i \leqslant t-1$, then ψ_i is a switch with respect to some subset A_i such that $\psi_{i-1} \circ \cdots \circ \psi_0[T] \subseteq A_i$.

(3) ψ_t is an isomorphism.

(4) $\psi_t \circ \cdots \circ \psi_0 \restriction Z = f$.

When $k = 1$, we must have that $T = \emptyset$; and so the result is obviously true. Suppose that the result holds for $k-1$, where $k-1 \geqslant 1$. Let $f \in \mathscr{F}(S_X(\Gamma_k))$, and suppose that $Z = \operatorname{dom} f$ is a very large k-subgraph of Γ_k. Suppose that $T \in [Z]^{<k}$ is a subset such that (‡) holds. Clearly we can assume that $T \neq \emptyset$. Let $Y = \{i - 1 \mid 0 < i \in X\}$, and let Φ_X^k, Φ_Y^{k-1} be the finite subsets of $\mathbb{N}$ given by Theorem 2.6. Then f preserves the parity of k-edges in every n-subset of Z for all $n \in \Phi_X^k$. Also $\{m+1 \mid m \in \Phi_Y^{k-1}\} \subseteq \Phi_X^k$. Let $v \in T$ and let $Z' = Z \setminus \{v\}$. Let Z'_v be the $(k-1)$-graph induced on Z' by v, and let $f_v : Z'_v \to \Gamma_{k-1}$ be the map induced by f. Since $f \restriction Z'$ is an isomorphism, it follows that f_v preserves the parity of $(k-1)$-edges in every m-subset of Z'_v for all $m \in \Phi_Y^{k-1}$. By Theorem 2.6, $f_v \in \mathscr{F}(S_Y(\Gamma_{k-1}))$. Notice that if $E' \in [Z'_v]^{k-1}$ and $f_v \restriction E'$ is not an isomorphism, then $T \setminus \{v\} \subseteq E'$. By induction hypothesis, there exists a sequence of elements $\psi'_0, \ldots, \psi'_t \in S_Y(\Gamma_{k-1})$ which satisfies the conclusion of $(5.14)_{k-1}$ with respect to $T \setminus \{v\}$. It is now easy to convert this sequence into a sequence of elements $\psi_0, \ldots, \psi_t \in S_X(\Gamma_k)$ which satisfies the conclusion of $(5.14)_k$ with respect to T. (For example, if ψ'_1 is a switch with respect to A'_1, then ψ_1 is a switch with respect to $A_1 = A'_1 \cup \{v\}$, etc.) □

6. The classification of the pseudo-reducts

In this section, we shall prove the following result.

Theorem 6.1. *If $\mathscr{F}$ is a nontrivial pseudo-reduct of Γ_k, then there exists a subset $X \subseteq \{0, 1, \ldots, k-1\}$ such that $\mathscr{F} = \mathscr{F}(S_X(\Gamma_k))$.*

Proposition 4.9 dealt with the case when $k = 1$. So let $k \geqslant 2$. For the rest of this section, we shall fix some nontrivial pseudo-reduct $\mathscr{F}$ of Γ_k. Let X be the largest subset of $\{0, 1, \ldots, k-1\}$ such that $\mathscr{F}(S_X(\Gamma_k)) \subseteq \mathscr{F}$. Let ℓ be the integer given by Theorem 5.1. Thus whenever $f \in \mathscr{F} \cap \mathscr{F}(B(\Gamma_k))$ and $\operatorname{dom} f \models \phi_\ell^k$, then $f \in \mathscr{F}(S_X(\Gamma_k))$.

Lemma 6.2. *If $\mathscr{F} \subseteq \mathscr{F}(B(\Gamma_k))$, then $\mathscr{F} = \mathscr{F}(S_X(\Gamma_k))$.*

Proof. Let $g \in \mathscr{F}$ be arbitrary. There exists $f \in \mathscr{F}$ such that $g \subseteq f$ and $\operatorname{dom} f \models \phi_\ell^k$. Then $f \in \mathscr{F}(S_X(\Gamma_k))$, and so $g \in \mathscr{F}(S_X(\Gamma_k))$. □

From now on, we shall assume that $\mathscr{F} \nsubseteq \mathscr{F}(B(\Gamma_k))$. Eventually we shall derive a contradiction from this assumption. We shall make use of the following two lemmas, each of which witnesses the fact that $\mathscr{F}$ is a *nontrivial* pseudo-reduct.

Lemma 6.3. *There exists a finite k-subgraph R_0 of Γ_k such that $|R_0| \geqslant k$ and such that for all $\pi \in \mathscr{F}$, if $R_0 \subseteq \operatorname{dom} \pi$ then $\pi[R_0]$ is neither a complete nor a null k-graph.*

Proof. This is an immediate consequence of Lemma 4.8. □

Lemma 6.4. *There exists a finite k-subgraph R_1 of Γ_k such that $|R_1| \geqslant k$ and which satisfies the following property. Suppose that $\pi \in \mathscr{F}$ and that $R_1 \subseteq \operatorname{dom} \pi$. Then there do* not *exist $1 \leqslant i \leqslant k$ and $A \in [R_1]^i$ such that either of the following two clauses holds.*

(i) *For all $E \in [R_1]^k$, $\pi \upharpoonright E$ is not an isomorphism iff E is a k-edge such that $A \subseteq E$.*

(ii) *For all $E \in [R_1]^k$, $\pi \upharpoonright E$ is not an isomorphism iff E is a k-nonedge such that $A \subseteq E$.*

Proof. Let R_0 be the k-graph given by Lemma 6.3, and let $r = |R_0|$. Let $1 \leqslant i \leqslant k$. Then there exist a k-subgraph $Z_i^0 \in [\Gamma_k]^r$ and an i-subset A_i^0 of Z_i^0 such that the following holds:

(1) (a) If $i = k$, then A_k^0 is a k-edge.

(b) Suppose that $\pi \in \mathscr{F}$ and that $Z_i^0 \subseteq \operatorname{dom} \pi$. Then it is *not* the case that for all $E \in [Z_i^0]^k$, $\pi \upharpoonright E$ is not an isomorphism iff E is a k-edge such that $A_i^0 \subseteq E$.

For suppose that no such Z_i^0 and A_i^0 exist. Let $Z \in [\Gamma_k]^r$ be any non-null k-subgraph. Let $A \in [Z]^i$ be arbitrary; subject only to the requirement that if $i = k$, then A is a k-edge. Then there exists $\pi_A \in \mathscr{F}$ such that $\pi_A \upharpoonright Z$ is an isomorphism, *except* on those k-edges $E \in [Z]^k$ such that $A \subseteq E$. So in passing from Z to $\pi_A[Z]$, we have "erased" precisely those k-edges which contain A. Continuing in this fashion, we eventually obtain an element $\pi \in \mathscr{F}$ such that $\pi[Z]$ is a null k-graph. In particular, there exists $\pi \in \mathscr{F}$ such that $\pi[R_0]$ is a null k-graph; which contradicts the choice of R_0.

Similarly there exist a k-graph $Z_i^1 \in [\Gamma_k]^r$ and an i-subset A_i^1 of Z_i^1 such that the following holds:

(2) (a) If $i = k$, then A_k^1 is a k-nonedge.

(b) Suppose that $\pi \in \mathscr{F}$ and that $Z_i^1 \subseteq \operatorname{dom} \pi$. Then it is *not* the case that for all $E \in [Z_i^1]^k$, $\pi \upharpoonright E$ is not an isomorphism iff E is a k-nonedge such that $A_i^1 \subseteq E$.

Note that Γ_k satisfies the following property:

(3) For every $\varepsilon \in \{0, 1\}$, $1 \leqslant i \leqslant k$ and i-subset A, there exists an r-subset Z such that the following clauses hold:

(a) $A \subseteq Z$.

(b) If $i < k$, then there exists an isomorphism $\tau : Z \to Z_i^\varepsilon$ such that $\tau[A] = A_i^\varepsilon$.

(c) If $i = k$, $\varepsilon = 0$ and A is a k-edge, then there exists an isomorphism $\tau : Z \to Z_k^0$ such that $\tau[A] = A_k^0$.

(d) If $i = k$, $\varepsilon = 1$ and A is a k-nonedge, then there exists an isomorphism $\tau : Z \to Z_k^1$ such that $\tau[A] = A_k^1$.

By quantifying out $A = \{a_1, \ldots, a_i\}$ and $Z = \{z_1, \ldots, z_r\}$, we can express (3) by a first-order sentence. Hence, by Theorem 3.2, there exists a finite k-subgraph R_1 of Γ_k which also satisfies (3). Clearly R_1 satisfies our requirements. □

In each of the applications of Theorem 3.2 which we have made so far, we have only used the fact that Γ_k has the usual finite submodel property. We are finally approaching the point in this paper where we will use the *strong* finite submodel property. Using Theorem 3.2, express $\Gamma_k = \bigcup_{j\in\mathbb{N}} H_j$ as the union of an increasing chain of k-subgraphs such that

(1) $|H_j| = j$ for all $j \in \mathbb{N}$; and
(2) for each sentence σ such that $\Gamma_k \models \sigma$, there exists an integer N_σ such that $H_j \models \sigma$ for all $j \geqslant N_\sigma$.

Lemma 6.5. *For each j, $n \in \mathbb{N}$, there exist s, $N \in \mathbb{N}$ such that the following is true for all t with $N \leqslant t \in \mathbb{N}$.*

(6.5) *Suppose that $Z \in [H_t]^j$ and that $Z \simeq H_j$. Then for each $Y \in [H_t]^n$, there exists $M \in [H_t]^s$ such that*
(a) *$Y \cup Z \subseteq M$; and*
(b) *there exists an isomorphism $\tau : M \to H_s$ such that $\tau[Z] = H_j$.*

Proof. There exists an integer s such that for all $Y \in [\Gamma_k]^n$, there exists $\theta \in Aut(\Gamma_k)$ such that $\theta[Y] \subseteq H_s$ and $\theta \upharpoonright H_j = id_{H_j}$. Let $Z \in [\Gamma_k]^j$ satisfy $Z \simeq H_j$, and let $Y \in [\Gamma_k]^n$ be arbitrary. Then there exists $\psi \in Aut(\Gamma_k)$ such that $\psi[Y \cup Z] \subseteq H_s$ and $\psi[Z] = H_j$. Let $M = \psi^{-1}[H_s]$ Then

(a) $Y \cup Z \subseteq M$; and
(b) $\tau = \psi \upharpoonright M$ is an isomorphism from M onto H_s such that $\tau[Z] = H_j$.

Thus (6.5) is true if H_t is replaced by Γ_k. By quantifying out $Z = \{z_1, \dots, z_j\}$, $Y = \{y_1, \dots, y_n\}$ and $M = \{m_1, \dots, m_s\}$, we can express (6.5) as a first-order property σ of H_t. Since $\Gamma_k \models \sigma$, we have that $H_t \models \sigma$ for all $t \geqslant N = N_\sigma$. $\square$

From now on, let n be a fixed $B(\Gamma_k)$-good integer. Let $g \in \mathscr{F} \setminus \mathscr{F}(B(\Gamma_k))$. Then there exists $f \in \mathscr{F}$ such that $g \subseteq f$ and $\operatorname{dom} f = H_d$ for some extremely large integer d. (During the course of the proof, it will become clear how large d should be chosen.) We shall define inductively

(1) a decreasing sequence $d_0 > d_1 > \cdots > d_k$ of integers; and
(2) a sequence of maps $f_0, \dots, f_k \in \mathscr{F} \setminus \mathscr{F}(B(\Gamma_k))$ such that $\operatorname{dom} f_i = H_{d_i}$.

We shall also define an i-subset $\{v_1^i, \dots, v_i^i\}$ of H_{d_i} for $1 \leqslant i \leqslant k$ such that the following condition is satisfied:

$(\ddagger)_i$ If $E \in [H_{d_i}]^k$ and $f_i \upharpoonright E$ is *not* an isomorphism, then $\{v_1^i, \dots, v_i^i\} \subseteq E$.

We begin by setting $d_0 = d$ and $f_0 = f$. Now we shall describe how to find d_1, f_1 and v_1^1. First choose a very large integer c_0 such that $c_0 < d_0$.

Lemma 6.6. *There exists a k-subgraph Z of H_{d_0} such that $Z \simeq H_{c_0}$ and $f_0 \upharpoonright Z$ is either an isomorphism or an anti-isomorphism.*

Proof. By Theorem 5.8, there exists a finite k-graph Q with the following property:

(†) Suppose that $\chi : [Q]^k \to \{0, 1\}$ is any 2-coloring. Then there exists a k-subgraph Z of Q such that

(a) $Z \simeq H_{c_0}$;
(b) if E_1, $E_2 \in [Z]^k$ are k-edges, then $\chi(E_1) = \chi(E_2)$;
(c) if F_1, $F_2 \in [Z]^k$ are k-nonedges, then $\chi(F_1) = \chi(F_2)$.

Since d_0 is extremely large, we can suppose that $Q \subset H_{d_0}$. Define a coloring $\chi : [Q]^k \to \{0,1\}$ by

$$\chi(E) = 1 \quad \text{iff } f_0 \restriction E \text{ is an isomorphism.}$$

Let $Z \subseteq Q$ be a k-subgraph which satisfies the conclusion of (†). Then one of the following conditions must hold.
(i) $f_0 \restriction Z$ is an isomorphism.
(ii) $f_0 \restriction Z$ is an anti-isomorphism.
(iii) $f_0[Z]$ is a complete k-graph.
(iv) $f_0[Z]$ is a null k-graph.

Let R_0 be the k-graph given by Lemma 6.3. Since c_0 is very large, there exists an isomorphic embedding $R_0 \to H_{c_0}$. As $Z \simeq H_{c_0}$, it follows that neither (iii) nor (iv) can hold; and so Z satisfies our requirements. □

Since $f_0 \in \mathscr{F} \setminus \mathscr{F}(B(\Gamma_k))$, Proposition 1.21 implies that there exists an n-subset Y of H_{d_0} such that $f_0 \restriction Y$ does *not* preserve the parity of k-edges in Y. By Lemma 6.5, we can suppose that there exists an integer $s > c_0$ and a k-subgraph $M \in [H_{d_0}]^s$ such that
(a) $Y \cup Z \subseteq M$; and
(b) there exists an isomorphism $\tau : M \to H_s$ such that $\tau[Z] = H_{c_0}$.

For each $c_0 \leqslant m \leqslant s$, let $Z_m = \tau^{-1}[H_m]$. By Lemma 6.6, $f_0 \restriction Z_{c_0} \in \mathscr{F}(B(\Gamma_k))$. Let a be the greatest integer such that $c_0 \leqslant a \leqslant s$ and $f_0 \restriction Z_a \in \mathscr{F}(B(\Gamma_k))$. Since $Y \subseteq Z_s$, we have that $a < s$. Let $Z_{a+1} = Z_a \cup \{w\}$. Since $a \geqslant c_0$, Theorem 5.1 implies that there exists $\theta \in S_X(\Gamma_k))$ such that $\theta \restriction Z_a = f_0 \restriction Z_a$. We now define $d_1 = a+1$, $f_1 = \theta^{-1} \circ f_0 \circ \tau^{-1} \restriction H_{a+1}$ and $v_1^1 = \tau(w)$. By the maximality of a, $f_0 \restriction Z_{a+1} \notin \mathscr{F}(B(\Gamma_k))$. Thus $f_1 \in \mathscr{F} \setminus \mathscr{F}(B(\Gamma_k))$. Clearly condition $(\ddagger)_1$ holds.

Let $1 \leqslant i < k$. Suppose inductively that we have defined d_i, $f_i \in \mathscr{F} \setminus \mathscr{F}(B(\Gamma_k))$ and $\{v_1^i, \ldots, v_i^i\} \subset H_{d_i}$ such that condition $(\ddagger)_i$ holds. Let $T = \{v_1^i, \ldots, v_i^i\}$ and $U = H_{d_i} \setminus T$. Fix an ordering $\prec$ of H_{d_i}, chosen so that T is an initial segment. (As the reader will soon see, the ordering $\prec$ does not play a significant role in this argument.) For a suitable system of colors $\boldsymbol{\alpha}$, define an $\boldsymbol{\alpha}$-coloring η of $[U]^{\leqslant k}$ by setting
(I) $\eta(A) = \eta(B)$ iff $|A| = |B|$ and the order-preserving bijection $T \cup A \to T \cup B$ is a k-graph isomorphism.

Thus if $R \subseteq U$, then the $\boldsymbol{\alpha}$-pattern of $(R, \eta \restriction R)$ essentially consists of the type $tp(R \mid T)$ of R over T. Now define the function $F_i : [U]^{k-i} \to \{0,1\}$ by
(II) $F_i(C) = 1$ iff $f_i \restriction C \cup T$ is an isomorphism.

Let c_i be a large integer such that $c_i < d_i$. Then there exists an $\boldsymbol{\alpha}$-pattern P such that if $(R, \eta \restriction R)$ has $\boldsymbol{\alpha}$-pattern P, then $T \cup R \simeq H_{c_i}$. By Theorem 5.8, there exists a subset U_0 of U such that

(III) (a) $(U_0, \eta \restriction U_0)$ has α-pattern P, and

(b) $(U_0, \eta \restriction U_0)$ is F_i-homogeneous.

Notice that if A, $B \in [U]^{k-i}$, then $\eta(A) = \eta(B)$ iff $T \cup A$ and $T \cup B$ are either both k-edges or both k-nonedges. (Thus the ordering $\prec$ is unimportant from this point onwards.) Let $Z = T \cup U_0$. Then $Z \simeq H_{c_i}$, and one of the following conditions must hold.

(IV) (i) $f_i \restriction Z$ is an isomorphism.

(ii) $f_i \restriction Z$ is a switch with respect to T.

(iii) For all $E \in [Z]^k$, $f_i \restriction E$ is not an isomorphism iff E is a k-edge such that $T \subseteq E$.

(iv) For all $E \in [Z]^k$, $f_i \restriction E$ is not an isomorphism iff E is a k-nonedge such that $T \subseteq E$.

Let R_1 be the k-graph given by Lemma 6.4. Since c_i is large, for each $D \in [H_{c_i}]^i$ there exists an isomorphic embedding $\psi : R_1 \to H_{c_i}$ such that $D \subseteq \psi[R_1]$. As $Z \simeq H_{c_i}$, it follows that neither (iii) nor (iv) can hold. Thus $f_i \restriction Z \in \mathscr{F}(B(\Gamma_k))$.

Since $f_i \in \mathscr{F} \setminus \mathscr{F}(B(\Gamma_k))$, Proposition 1.21 implies that there exists an n-subset Y of H_{d_i} such that $f_i \restriction Y$ does *not* preserve the parity of k-edges in Y. By Lemma 6.5, we can suppose that there exists an integer $s > c_i$ and a k-subgraph $M \in [H_{d_i}]^s$ such that

(V) (a) $Y \cup Z \subseteq M$; and

(b) there exists an isomorphism $\tau : M \to H_s$ such that $\tau[Z] = H_{c_i}$.

For each $c_i \leqslant m \leqslant s$, let $Z_m = \tau^{-1}[H_m]$. Let a be the greatest integer such that $c_i \leqslant a \leqslant s$ and $f_i \restriction Z_a \in \mathscr{F}(B(\Gamma_k))$. Then $a < s$. Let $Z_{a+1} = Z_a \cup \{w\}$. Since $a \geqslant c_i$, Theorem 5.1 implies that $f_i \restriction Z_a \in \mathscr{F}(S_X(\Gamma_k))$. Remember that condition $(\ddagger)_i$ holds. Thus Lemma 5.14 implies that there exists an element $\theta \in S_X(\Gamma_k)$ such that

(VI) (a) $\theta \restriction Z_a = f_i \restriction Z_a$; and

(b) if $E \in [\Gamma_k]^k$ and $\theta \restriction E$ is not an isomorphism, then $T \subseteq E$.

Consider $g_i = \theta^{-1} \circ f_i \restriction Z_{a+1}$.

Claim 6.7. *If $E \in [Z_{a+1}]^k$ and $g_i \restriction E$ is not an isomorphism, then $T \cup \{w\} \subseteq E$.*

Proof. Suppose that $E \in [Z_{a+1}]^k$ and that $g_i \restriction E$ is not an isomorphism. Then clearly $w \in E$. Suppose that $T \not\subseteq E$. By $(\ddagger)_i$, $f_i \restriction E$ is an isomorphism. Condition (VI)(b) implies that if $B \in [\Gamma_k]^k$ and $\theta^{-1} \restriction B$ is not an isomorphism, then $\theta[T] \subseteq B$. But $\theta[T] = f_i[T]$. Hence $\theta^{-1} \restriction f_i[E]$ is an isomorphism, which is a contradiction. $\square$

We now define $d_{i+1} = a + 1$, $f_{i+1} = \theta^{-1} \circ f_i \circ \tau^{-1} \restriction H_{a+1}$ and $\{v_1^{i+1}, \dots, v_{i+1}^{i+1}\} = \tau[T \cup \{w\}]$. Then $f_{i+1} \in \mathscr{F} \setminus \mathscr{F}(B(\Gamma_k))$; and condition $(\ddagger)_{i+1}$ holds. But now consider f_k. Since f_k is not an isomorphism, it follows that $f_k \restriction \{v_1^k, \dots, v_k^k\}$ is not an isomorphism. Thus f_k satisfies the following condition.

(VII) For all $E \in [H_{d_k}]^k$, $f_k \restriction E$ is an isomorphism iff $E \neq \{v_1^k, \dots, v_k^k\}$.

Let R_1 be the k-graph given by Lemma 6.4. Since d_k is large, there exists an isomorphic embedding $\psi : R_1 \to H_{d_k}$ such that $\{v_1^k, \dots, v_k^k\} \subseteq \psi[R_1]$. But now (VII) contradicts Lemma 6.4. This completes the proof of Theorem 6.1. $\square$

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