

## A mixture representation of the Linnik distribution

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Received April 1994; revised October 1994

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### Abstract

Linnik distribution with the characteristic function

$$\varphi_x(t) = 1/(1 + |t|^\alpha), \quad 0 < \alpha < 2,$$

is shown to possess the following property.

Let  $X_\alpha, X_\beta$  be random variables possessing the Linnik distribution with parameters  $\alpha$  and  $\beta$  respectively ( $0 < \alpha < \beta \leq 2$ ). Denote by  $Y_{\alpha\beta}$  an independent of  $X_\beta$  non-negative random variable with the density

$$g(s; \alpha, \beta) = \left( \frac{\beta}{\pi} \sin \frac{\pi\alpha}{\beta} \right) \frac{s^{\alpha-1}}{1 + s^{2\alpha} + 2s^\alpha \cos \frac{\pi\alpha}{\beta}}, \quad 0 < s < \infty.$$

Then

$$X_\alpha \doteq X_\beta Y_{\alpha\beta},$$

where  $\doteq$  denotes the equality in the sense of distributions.

Infinite divisibility of mixtures of Linnik distributions with respect to the parameter  $\alpha$  and scale is obtained as a corollary.

*AMS 1980 Subject Classification:* Primary 62H05, 60E10; Secondary 33A40

*Keywords:* Characteristic function; Mixture; Infinite divisibility; Contour integration

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### 1. Introduction and statement of the theorem

Recently, the Linnik distribution – originally introduced by Ju.V. Linnik in 1953 (Linnik, 1963) – has attracted attention of a number of researchers (see e.g. Arnold, 1973; Devroye, 1986, 1990; Anderson, 1992;

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Anderson et al., 1993; Devroye, 1993). Although the characteristic function of this distribution is of a simple form, a general expression of the distribution is not easily attainable.<sup>1</sup> In this connection, any properties of the distribution such as a mixture representation which facilitates generation of Linnik random variables ought to be of interest. One such property is proved in the present note.

Recall that the generic definition of a Linnik random variable is given in terms of the characteristic function

$$\varphi_\alpha(t) = 1/(1 + |t|^\alpha), \quad 0 < \alpha < 2.$$

We shall denote the corresponding density by  $p_\alpha(x)$ . This density can be viewed as a generalization of the well-known Laplace (double exponential) density  $p_2(x) = e^{-|x|}/2$  for the case  $\alpha = 2$  (see, e.g. Johnson and Kotz, 1970). The main result of the paper is the following theorem.

**Theorem.** For any  $0 < \alpha < \beta \leq 2$ , the following equality is valid

$$\varphi_\alpha(t) = \int_0^\infty \varphi_\beta(t/s)g(s; \alpha, \beta) ds, \quad -\infty < t < \infty, \quad (1)$$

where

$$g(s; \alpha, \beta) = \left( \frac{\beta}{\pi} \sin \frac{\pi\alpha}{\beta} \right) \frac{s^{\alpha-1}}{1 + s^{2\alpha} + 2s^\alpha \cos \frac{\pi\alpha}{\beta}}.$$

Noting that the equality (1) is equivalent to the following

$$p_\alpha(x) = \int_0^\infty p_\beta(sx)g(s; \alpha, \beta) ds, \quad -\infty < x < \infty,$$

and taking into account that  $g(s; \alpha, \beta)$  is a genuine density function, we arrive at the representation of the form

$$X_\alpha \doteq X_\beta Y_{\alpha\beta},$$

as stipulated in the abstract. This representation allows us to generate Linnik variables of different parameters starting from convenient base, e.g. from the Laplace distribution corresponding to  $\beta = 2$ . That is, our theorem yields immediately the following corollary.

**Corollary 1.** For any  $\alpha \in (0, 2)$ , Linnik distribution with characteristic function  $\varphi_\alpha(t)$  is a scale mixture of Laplace distributions with characteristic functions  $\varphi_2(t/s) = s^2/(s^2 + t^2)$ ,  $0 < s < \infty$ .

By Steutel's theorem (Steutel, 1970), any scale mixture of Laplace distributions is infinitely divisible. Therefore, Corollary 1 yields infinite divisibility of Linnik distributions. This fact is not new and was proved in Devroye (1990). However, noting that any mixture of scale mixtures of Laplace distributions is again a scale mixture of Laplace distributions, we obtain a stronger result: any mixture of Linnik distributions is infinitely divisible. More precisely, the following result is valid.

**Corollary 2.** Let  $P$  be a probability measure on the half-strip  $S = \{(\alpha, s): 0 < \alpha \leq 2, 0 \leq s < \infty\}$ . Then the distribution with the characteristic function

$$\varphi(t) = \int_S \varphi_\alpha(st)P(d\alpha ds)$$

is infinitely divisible.

<sup>1</sup> See Hayfavi, A., S. Kotz and I.V. Ostrovskii (1994), Analytic and asymptotic properties of Linnik's probability densities, *C.R. Acad. Sci. Paris, Série I*, **319**, 985–990.

## 2. Proof of the theorem

Note, that the equality (1) is equivalent to the following one:

$$\frac{1}{1+t^\alpha} = \int_0^\infty \frac{s^\beta}{s^\beta + t^\beta} g(s; \alpha, \beta) ds, \quad 0 < \alpha < \beta \leq 2, \quad t \geq 0.$$

To prove the latter, we denote

$$I = \frac{\pi}{\beta \sin \frac{\pi\alpha}{\beta}} \int_0^\infty \frac{s^\beta}{s^\beta + t^\beta} g(s; \alpha, \beta) ds = \int_0^\infty \frac{s^{\beta+\alpha-1} ds}{(s^\beta + t^\beta)(1 + s^{2\alpha} + 2s^\alpha \cos \frac{\pi\alpha}{\beta})}.$$

It is sufficient to establish that

$$I = \frac{\pi}{\beta \sin \frac{\pi\alpha}{\beta}} \cdot \frac{1}{1+t^\alpha}. \tag{2}$$

Transforming the integral  $I$  by means of the change of variables  $\tau = t^\beta s^{-\beta}$ , we have

$$I = \frac{t^\alpha}{\beta} \int_0^\infty \frac{\tau^{(\alpha/\beta)-1} d\tau}{(1+\tau)(\tau^{2\alpha/\beta} + t^{2\alpha} + 2\tau^{\alpha/\beta} t^\alpha \cos \frac{\pi\alpha}{\beta})}.$$

Utilizing the identity

$$\frac{1}{\tau^{2\alpha/\beta} + t^{2\alpha} + 2\tau^{\alpha/\beta} t^\alpha \cos \frac{\pi\alpha}{\beta}} = \frac{1}{t^\alpha 2i \sin \frac{\pi\alpha}{\beta}} \left\{ \frac{1}{\tau^{\alpha/\beta} + t^\alpha e^{-i\pi\alpha/\beta}} - \frac{1}{\tau^{\alpha/\beta} + t^\alpha e^{i\pi\alpha/\beta}} \right\},$$

we have

$$I = \frac{1}{2i\beta \sin \frac{\pi\alpha}{\beta}} \left\{ \int_0^\infty \frac{\tau^{(\alpha/\beta)-1} d\tau}{(1+\tau)(\tau^{\alpha/\beta} + t^\alpha e^{-i\pi\alpha/\beta})} - \int_0^\infty \frac{\tau^{(\alpha/\beta)-1} d\tau}{(1+\tau)(\tau^{\alpha/\beta} + t^\alpha e^{i\pi\alpha/\beta})} \right\}. \tag{3}$$

Consider the function

$$q(\tau) = \frac{1}{(1+\tau)(\tau^{\alpha/\beta} + t^\alpha e^{i\pi\alpha/\beta})}, \tag{4}$$

in the complex  $\tau$ -plane cut along the positive ray. Define the branch of  $\tau^{\alpha/\beta}$  in (4) by the conditions

$$\tau^{\alpha/\beta} = r^{\alpha/\beta} e^{i\varphi\alpha/\beta}, \quad \tau = re^{i\varphi}, \quad 0 < \varphi < 2\pi.$$

Since

$$\left| \frac{\alpha}{\beta}(\pi - \varphi) \right| < \pi$$

when  $0 < \varphi < 2\pi$ , we have

$$\tau^{\alpha/\beta} + t^\alpha e^{i\pi\alpha/\beta} = e^{i\varphi\alpha/\beta} (r^{\alpha/\beta} + t^\alpha e^{i\alpha/\beta(\pi - \varphi)}) \neq 0.$$

Therefore the function  $q(\tau)$  is analytic in the cut out plane. Denote by  $G_{R,\varepsilon}$  the simply connected region

$$G_{R,\varepsilon} = \{ \tau: \varepsilon < |\tau| < R \} \setminus \{ \tau: \varepsilon < \tau < R \}, \quad 0 < \varepsilon < R (> 1)$$

and denote by  $\partial G_{R,\varepsilon}$  its boundary traversed in the direction which leaves  $G_{R,\varepsilon}$  from the left (the line interval  $\{ \tau: \varepsilon < \tau < R \}$  is being traversed twice in the opposite directions). By the Cauchy Residue Theorem, we have

$$\oint_{\partial G_{R,\varepsilon}} q(\tau) d\tau = 2\pi i (\text{Residue of } q(\tau) \text{ at } \tau = -1) = -\frac{2\pi i}{1+t^\alpha}.$$

Taking the limit of the last integral as  $R \rightarrow \infty$ ,  $\varepsilon \rightarrow 0$ , we arrive at

$$\int_0^{\infty} \frac{\tau^{(\alpha/\beta)-1} d\tau}{(1+\tau)(\tau^{\alpha/\beta} + t^\alpha e^{i\pi\alpha/\beta})} - e^{2\pi i[(\alpha/\beta)-1]} \int_0^{\infty} \frac{\tau^{(\alpha/\beta)-1} d\tau}{(1+\tau)(\tau^{\alpha/\beta} e^{2\pi i\alpha/\beta} + t^\alpha e^{i\pi\alpha/\beta})} = -\frac{2\pi i}{1+t^\alpha}.$$

It is evident that this equality can be rewritten in the form

$$\int_0^{\infty} \frac{\tau^{(\alpha/\beta)-1} d\tau}{(1+\tau)(\tau^{\alpha/\beta} + t^\alpha e^{i\pi\alpha/\beta})} - \int_0^{\infty} \frac{\tau^{(\alpha/\beta)-1} d\tau}{(1+\tau)(\tau^{\alpha/\beta} + t^\alpha e^{-i\pi\alpha/\beta})} = -\frac{2\pi i}{1+t^\alpha}.$$

Thus, the difference of the integrals appearing in the braces of (3) has been calculated to be equal to  $2\pi i/(1+t^\alpha)$ . Substituting this value into (3), we arrive at (2). The theorem is thus proved.

## References

- Anderson, D.N. (1992), A multivariate Linnik distribution, *Statist. Probab. Lett.* **14**, 333–336.
- Anderson, D.N. and B.C. Arnold (1993), Linnik distributions and processes, *J. Appl. Probab.* **30**, 330–340.
- Arnold, B.C. (1973), Some characterizations of the exponential distribution by geometric compounding, *SIAM J. Appl. Math.* **24**, 242–244.
- Devroye, L. (1986), *Non-Uniform Random Variable Generation* (Springer, New York).
- Devroye, L. (1990), A note on Linnik's distribution, *Statist. Probab. Lett.* **9**, 305–306.
- Devroye, L. (1993), A triptych of discrete distributions related to stable law, *Statist. Probab. Lett.* **18**, 349–351.
- Johnson, N.L. and S. Kotz (1970), *Distributions in Statistics, Continuous Univariate Distributions, II* (J. Wiley, New York).
- Linnik, Ju.V. (1963), Linear forms and statistical criteria, I, II, Selected Translations in *Math. Statist. Probab.* **3**, 1–90. (Original paper appeared in: *Ukrainskii Mat. Zhournal* **5** (1953) 207–290.)
- Anderson, D.N. and B.C. Arnold (1993), Linnik distributions and processes, *J. Appl. Probab.* **30**, 330–340.
- Steutel, F.W. (1970), *Preservation of Infinite Divisibility under Mixing and Related Topics*, Vol. 33 (Mathematical Center Tracts, Mathematisch Centrum, Amsterdam).