ELSEVIER
15 October 1995
Optics Communications 120 (1995) 166–170
OPTICS COMMUNICATIONS

Non-orthogonal domains in phase space of quantum optics and their relation to fractional Fourier transforms

Orhan Aytür, Haldun M. Ozaktas
Department of Electrical Engineering, Bilkent University, TR-06533 Bilkent, Ankara, Turkey

Received 20 January 1995

Abstract

It is customary to define a phase space such that position and momentum are mutually orthogonal coordinates. Associated with these coordinates, or domains, are the position and momentum operators. Representations of the state vector in these coordinates are related by the Fourier transformation. We consider a continuum of "fractional" domains making arbitrary angles with the position and momentum domains. Representations in these domains are related by the fractional Fourier transformation. We derive transformation, commutation, and uncertainty relations between coordinate multiplication, differentiation, translation, and phase shift operators making arbitrary angles with each other. These results have a simple geometric interpretation in phase space and applications in quantum optics.

The fractional Fourier transform has already found applications in many areas, such as the solution of some differential equations [1,2], the study of optical beam propagation and laser resonators [3], the modeling and description of first-order optical systems [4–8], and most notably signal processing [1,2,9,10]. The fractional transform can be realized optically [4,11–14,7,15], as well as digitally.

The theory of the fractional transform is not widely known among physicists. Despite this, some authors in quantum optics have used concepts and results inherently similar to those found in the theory of the fractional transform, without being aware of the larger framework in which these concepts are situated [16–20]. This is because the fractional transform naturally asserts itself in this area. Only very recently have some authors in these areas became aware of the fractional Fourier transform [21,22]. We believe that the fractional Fourier transform and associated concepts will serve as useful tools in quantum optics, as well as other ares of physics. In this letter we consolidate the essentials of this body of knowledge in the language and notation of quantum mechanics, and derive important results such as the commutation and uncertainty relations between two arbitrary domains.

In the quantum theory of light, the radiation field is expressed in terms of photon annihilation and creation operators ($\hat{a}$ and $\hat{a}^\dagger$). The electric and magnetic fields associated with one mode of the radiation field, as well as their quadrature components ($\hat{a}_1 = (\hat{a}+\hat{a}^\dagger)/2$ and $\hat{a}_2 = (\hat{a}-\hat{a}^\dagger)/2\mathrm{i}$) are analogous to the momentum and position ($\hat{p}$ and $\hat{q}$) of a one dimensional harmonic oscillator. It is well known that for any state vector $|\psi\rangle$, there is a Fourier transform relation between the position representation $\psi_q(q) \equiv \langle q|\psi\rangle$ and the momentum representation $\psi_p(p) \equiv \langle p|\psi\rangle$. In this Letter, we explore other representations that are related by the more general fractional Fourier transform, and investigate the operators associated with these representations.

SSDI 0030-4018(95)00452-1

Let $\mathcal{F}$ denote the Fourier transform operation so that $\psi_1(\,) = \mathcal{F}[\psi_0(\,)]$ is the Fourier transform of the function $\psi_0(\,)$. The ath order fractional Fourier transform operation is denoted as $\mathcal{F}^a$ so that $\psi_a(\,) = \mathcal{F}^a[\psi_0(\,)]$ is the ath order fractional Fourier transform of the function $\psi_0(\,)$. $\mathcal{F}^1$ corresponds to the ordinary Fourier operation $\mathcal{F}$, and $\mathcal{F}^0$ is the identity operation. $\mathcal{F}^2$ corresponds to the parity operation so that $\mathcal{F}^4$ is also equivalent to the identity operation. We also have $\mathcal{F}^{a_1}\mathcal{F}^{a_2} = \mathcal{F}^{a_1+a_2}$. The fractional Fourier transform can be defined for $0 < |a| < 2$ as [9]

$$\psi_a(x) = \int_{-\infty}^{\infty} B_a(x,x')\psi_0(x')\,\mathrm{d}x', \qquad B_a(x,x') = A_\phi \exp\left[\mathrm{i}\left(\frac{x^2}{2\tan\phi} - \frac{xx'}{\sin\phi} + \frac{x'^2}{2\tan\phi}\right)\right],$$
$$A_\phi = (2\pi|\sin\phi|)^{-1/2}\exp[-\mathrm{i}\pi\,\mathrm{sgn}(\sin\phi)/4 + \mathrm{i}\phi/2], \tag{1}$$

where $\phi = a\pi/2$. The kernel $B_a(x,x')$ approaches $\delta(x-x')$ or $\delta(x+x')$ when a approaches 0 or 2, respectively.

The functions $\psi_a(\,)$ for different values of a may be considered as different representations of the state vector $|\psi\rangle$ for a single particle in one dimension. In particular, $\psi_0(x_0) = \langle x_0|\psi\rangle$ may be interpreted as the position representation of the state vector, where $|x_0\rangle$ is an eigenket of the position operator $\hat{x}_0$ with eigenvalue x_0. Likewise, $\psi_1(x_1) = \langle x_1|\psi\rangle$ is interpreted as the momentum (spatial frequency) representation of the state vector, where $|x_1\rangle$ is an eigenket of the momentum operator $\hat{x}_1$ with eigenvalue x_1. ($\hat{x}_0$ and $\hat{x}_1$ are commonly known as $\hat{q}$ and $\hat{p}$.) In the language of signal processing, $\psi_0(x_0)$ and $\psi_1(x_1)$ are referred to as the representation of the signal (state vector) in the space and spatial frequency domains, respectively. Generalizing this terminology, we may refer to the x_a axis for each a as the ath fractional Fourier domain.

There is nothing special about the $a = 0$ representation, it merely corresponds to the choice of origin of the parameter a. We can transform from the x_a representation to the $x_{a'}$ representation by taking a fractional Fourier transform [9]

$$\psi_{a'}(x_{a'}) = \int B_{(a'-a)}(x_{a'},x_a)\psi_a(x_a)\,\mathrm{d}x_a. \tag{2}$$

Since $\psi_a(x_a) = \langle x_a|\psi\rangle$ and from completeness

$$\langle x_{a'}|\psi\rangle = \int \langle x_{a'}|x_a\rangle\langle x_a|\psi\rangle\,\mathrm{d}x_a, \tag{3}$$

we observe $B_{(a'-a)}(x_{a'},x_a) = \langle x_{a'}|x_a\rangle$. Associated with the x_a representation for each value of a, an operator $\hat{x}_a$ is defined such that $\langle x_a|\hat{x}_a|\psi\rangle = x_a\langle x_a|\psi\rangle$. We now derive several relations between these operators.

The x_a representation of the ket vector $\hat{x}_{a+1}|\psi\rangle$ is given by

$$\langle x_a|\hat{x}_{a+1}|\psi\rangle = \frac{1}{\mathrm{i}}\frac{d}{\mathrm{d}x_a}\langle x_a|\psi\rangle. \tag{4}$$

This is nothing but the well known property of Fourier transforms stating that multiplication in the Fourier domain corresponds to differentiation in the space domain. The effect of $\hat{x}_a$ in any other arbitrary domain $x_{a'}$ may be expressed as

$$\hat{x}_a = \hat{x}_{a'}\cos(\phi'-\phi) - \hat{x}_{a'+1}\sin(\phi'-\phi). \tag{5}$$

This is derived in a straightforward manner from the fractional Fourier transform property [2,12]

$$\mathcal{F}^{a'-a}[x_a\psi_a(x_a)] = x_{a'}\{\mathcal{F}^{a'-a}[\psi_a(x_a)]\}\cos(\phi'-\phi) - \frac{1}{\mathrm{i}}\frac{d}{\mathrm{d}x_{a'}}\{\mathcal{F}^{a'-a}[\psi_a(x_a)]\}\sin(\phi'-\phi), \tag{6}$$

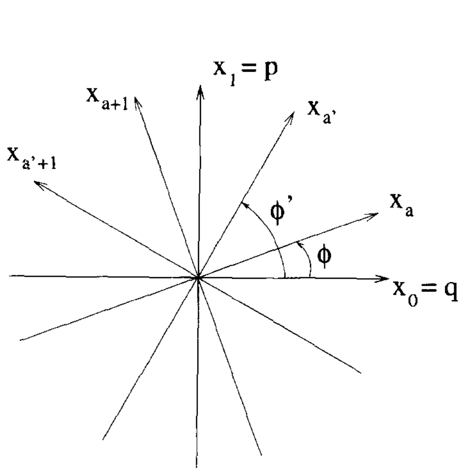

Fig. 1. Schematic.

which in turn is derived directly from Eq. (2). Substituting $a \to a+1$ in Eq. (5) (or by using a property analogous to Eq. (6) for $\mathcal{F}^{a'-a}[d\psi_a(x_a)/\mathrm{d}x_a]$) we obtain an equation for $\hat{x}_{a+1}$, which when combined with Eq. (5) gives

$$\begin{bmatrix} \hat{x}_a \\ \hat{x}_{a+1} \end{bmatrix} = \begin{bmatrix} \cos(\phi'-\phi) & -\sin(\phi'-\phi) \\ \sin(\phi'-\phi) & \cos(\phi'-\phi) \end{bmatrix} \begin{bmatrix} \hat{x}_{a'} \\ \hat{x}_{a'+1} \end{bmatrix}. \tag{7}$$

This equation suggests an analogy with basis vectors in $\mathbf{R}^2$. Referring to Fig. 1, we see that the angle $\phi = a\pi/2$ may be interpreted as the angle the ath domain makes with the 0th (position or space) domain and $(\phi'-\phi) = (a'-a)\pi/2$ may be interpreted as the angle between the ath and a'th domains. In particular, domains whose indices differ by unity are orthogonal. (This is consistent with the customary definition of phase space such that q and p are mutually orthogonal coordinates.)

The $\hat{x}_a$ operator, which simply multiplies $\psi_a(x_a)$ by x_a in the x_a domain, results in a linear combination of multiplication by $x_{a'}$ and differentiation with respect to $x_{a'}$ in the $x_{a'}$ domain. The $\hat{x}_{a+1}$ operator, which differentiates with respect to x_a in the x_a domain, results in a similar linear combination. The coefficients of these linear combinations are cosines and sines corresponding to the "projection" of these operators on the $x_{a'}$ domain.

It is also possible to express $\hat{x}_a$ in terms of *any* two operators $\hat{x}_{a'}$ and $\hat{x}_{a''}$, provided the latter two are not collinear, and express any function of an arbitrary number of operators $F(\hat{x}_{b'}, \hat{x}_{b''}, \cdots)$ in the form $F'(\hat{x}_{a'}, \hat{x}_{a''})$. Furthermore, it is possible to define the space spanned by any two noncollinear operators $\hat{x}_a$ and $\hat{x}_{a'}$, and operations such as inner products and norms. Although we do not present these extensions here, we will further discuss the above geometrical interpretation at the end of this paper in conjunction with Wigner distributions.

The commutator between two arbitrary operators $\hat{x}_a$ and $\hat{x}_{a'}$

$$[\hat{x}_a, \hat{x}_{a'}] = \mathrm{i}\sin(\phi'-\phi) \tag{8}$$

is derived using Eqs. (4), (7) and the well known commutator $[\hat{x}_0, \hat{x}_1] = [\hat{q}, \hat{p}] = \mathrm{i}$. This commutation relation between two nonorthogonal domains implies the uncertainty relationship

$$\langle\Delta\hat{x}_a^2\rangle\langle\Delta\hat{x}_{a'}^2\rangle \geq \tfrac{1}{4}\sin^2(\phi'-\phi) \tag{9}$$

between representations in these two domains. Here $\langle\Delta\hat{x}_a^2\rangle = \langle\hat{x}_a^2\rangle - \langle\hat{x}_a\rangle^2$ is the variance of $\hat{x}_a$. The existence of such an uncertainty relationship was previously suggested in a signal processing context [9].

We now define the operator $\exp(\mathrm{i}\xi\hat{x}_a)$. Its effect in the x_a domain is given simply by $\langle x_a|\exp(\mathrm{i}\xi\hat{x}_a)|\psi\rangle = \exp(\mathrm{i}\xi x_a)\langle x_a|\psi\rangle$. On the other hand, the effect of the $\exp(\mathrm{i}\xi\hat{x}_{a+1})$ operator in the same domain is given by

$$\langle x_a|\exp(\mathrm{i}\xi\hat{x}_{a+1})|\psi\rangle = \langle x_a+\xi|\psi\rangle = \psi_a(x_a+\xi). \tag{10}$$

This is just the Fourier transform property stating that multiplication by a phase factor in one domain corresponds to translation in the orthogonal domain. Using Eq. (7), the effect of $\exp(\mathrm{i}\xi\hat{x}_a)$ in the $x_{a'}$ domain can be obtained as

$$\begin{aligned}\langle x_{a'}|\exp(\mathrm{i}\xi\hat{x}_a)|\psi\rangle &= \exp[-\mathrm{i}\xi^2\sin(\phi'-\phi)\cos(\phi'-\phi)/2]\\ &\quad\times\langle x_{a'}|\exp[\mathrm{i}\xi\hat{x}_{a'}\cos(\phi'-\phi)]\exp[-\mathrm{i}\xi\hat{x}_{a'+1}\sin(\phi'-\phi)]|\psi\rangle\\ &= \exp[\mathrm{i}\xi^2\sin(\phi'-\phi)\cos(\phi'-\phi)/2]\\ &\quad\times\langle x_{a'}|\exp[-\mathrm{i}\xi\hat{x}_{a'+1}\sin(\phi'-\phi)]\exp[\mathrm{i}\xi\hat{x}_{a'}\cos(\phi'-\phi)]|\psi\rangle,\end{aligned} \tag{11}$$

where Glauber's formula [23] $\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp(-[\hat{A},\hat{B}]/2)$ for any two operators $\hat{A}$ and $\hat{B}$ that commute with their commutator has been used. Eq. (11) shows that the $\exp(\mathrm{i}\xi\hat{x}_a)$ operator, which simply results in a phase shift in the x_a domain, results in a translation followed by a phase shift (or a phase shift followed by a translation) in the $x_{a'}$ domain. (The fact that translation and phase shifting do not commute accounts for the additional phase factor coming from Glauber's formula.) By employing the substitution $a \to a+1$, we can obtain a second pair of equations similar to Eq. (11):

$$\begin{aligned}\langle x_{a'}|\exp(\mathrm{i}\xi\hat{x}_{a+1})|\psi\rangle &= \exp[\mathrm{i}\xi^2\sin(\phi'-\phi)\cos(\phi'-\phi)/2]\\ &\quad\times\langle x_{a'}|\exp[\mathrm{i}\xi\hat{x}_{a'}\sin(\phi'-\phi)]\exp[\mathrm{i}\xi\hat{x}_{a'+1}\cos(\phi'-\phi)]|\psi\rangle\\ &= \exp[-\mathrm{i}\xi^2\sin(\phi'-\phi)\cos(\phi'-\phi)/2]\\ &\quad\times\langle x_{a'}|\exp[\mathrm{i}\xi\hat{x}_{a'+1}\cos(\phi'-\phi)]\exp[\mathrm{i}\xi\hat{x}_{a'}\sin(\phi'-\phi)]|\psi\rangle.\end{aligned} \tag{12}$$

Eq. (12) shows that the $\exp(\mathrm{i}\xi\hat{x}_{a+1})$ operator, which simply results in a translation in the x_a domain, results in a translation followed by a phase shift (or a phase shift followed by a translation) in the $x_{a'}$ domain. As with Eq. (11), here also the amount of translation and phase shift is given by cosine and sine multipliers corresponding to the projection of the translation or phase shift on the new set of axes. It is also worth noting that starting from Eqs. (11) and (12) we can obtain formulas similar to Eq. (6) for $\mathcal{F}^{a'-a}[\exp(\mathrm{i}\xi x_a)\psi_a(x_a)]$ and $\mathcal{F}^{a'-a}[\psi_a(x_a-\xi)]$. (These formulas can also be derived directly from Eq. (1) [2,12].)

Everything derived until now strongly supports the analogy depicted in Fig. 1. Finally, we discuss how this is directly related to an important property of fractional Fourier transforms. The Wigner distribution $W_\psi(q,p)$ of $|\psi\rangle$ is given by

$$W_\psi(q,p) = \int_{-\infty}^{\infty} \psi_q(q+q'/2)\psi_q^*(q-q'/2)\exp(-\mathrm{i}pq')\,\mathrm{d}q' \tag{13}$$

where $\psi_q(q) = \langle q|\psi\rangle$. It is possible to relate $\psi_a(x_a) = \langle x_a|\psi\rangle$ to the Wigner distribution by [25,9]

$$\mathcal{R}_\phi[W_\psi(q,p)] = |\psi_a(x_a)|^2 \tag{14}$$

where the Radon transform operation $\mathcal{R}_\phi$ takes the integral projection of the Wigner distribution on an axis making angle ϕ with the $x_0 = q$ axis. Two widely known special cases are

$$\int W(q,p)\,\mathrm{d}p = |\psi_q(q)|^2, \qquad \int W(q,p)\,\mathrm{d}q = |\psi_p(p)|^2. \tag{15, 16}$$

Eq. (14) means that the projection of the Wigner distribution of $|\psi\rangle$ on an axis making angle $\phi = a\pi/2$ with the $x_0 = q$ axis gives the (absolute square) of the representation of $|\psi\rangle$ in the x_a domain. This supports the idea of referring to the axis making an angle ϕ with the x_0 axis as the x_a axis (or the x_a domain), as depicted in Fig. 1 [9].

We now summarize. Two representations which are related through a Fourier transform are orthogonal to each other. The operator $\hat{x}_a$ is orthogonal to the operator $\hat{x}_{a+1}$, or equivalently, the operation of multiplying by x_a is orthogonal to the operation $\mathrm{d}/\mathrm{d}x_a$. Likewise, multiplication by a phase factor is orthogonal to a corresponding translation, and so forth. Two representations that are related through a fractional Fourier transform of order a make an angle $\phi = a\pi/2$ with each other. Coordinate multiplication or differentiation in one of these domains results in a combination of these two operations in the other domain, as given by Eq. (7). Likewise, multiplication by a phase factor or a translation in one of these domains results in a combination of these two operations in the other domain, as given by Eqs. (11) and (12). The weighting factors appearing in these equations are cosines and sines with a direct interpretation as projections. The commutator and uncertainty relation between nonorthogonal domains are also interpreted in terms of this geometric picture.

The Wigner phase space is a particularly useful picture in dealing with coherent and squeezed states in quantum optics [26]. In particular, rotation, translation, and squeeze operators in phase space can be used to generate coherent and squeezed states from the vacuum state. Thus, the theory and tools of fractional Fourier transforms are expected to have applications in this area.

Finally, we note an aveune for future research. In standard texts [23], one usually starts with the commutator as given and then proceeds to derive the (Fourier) relationship between the space and momentum representations. It would be likewise interesting to start from Eq. (8) and derive the transformation relation Eq. (1) among the various domains.

References

[1] V. Namias, J. Inst. Maths. Applics. 25 (1980) 241.
[2] A.C. McBride and F.H. Kerr, IMA J. Appl. Math. 39 (1987) 159.
[3] H.M. Ozaktas and D. Mendlovic, Optics Lett. 19 (1994) 1678.
[4] H.M. Ozaktas and D. Mendlovic, J. Opt. Soc. Am. A 12 (1995) 743.
[5] P. Pellat-Finet, Optics Lett. 19 (1994) 1388.
[6] P. Pellat-Finet and G. Bonnet, Optics Comm. 111 (1994) 141.
[7] L.M. Bernardo and O.D.D. Soares, Optics Comm. 110 (1994) 517.
[8] L.M. Bernardo and O.D.D. Soares. J. Opt. Soc. Am. A 11 (1994) 2622.
[9] H.M. Ozaktas, B. Barshan, D. Mendlovic and L. Onural, J. Opt. Soc. Am. A 11 (1995) 547.
[10] L.B. Almeida, IEEE Trans. Signal Proc. 42 (1994) 3084.
[11] H.M. Ozaktas and D. Mendlovic, Optics Comm. 101 (1993) 163.
[12] D. Mendlovic and H.M. Ozaktas; J. Opt. Soc. Am. A 10 (1993) 1875.
[13] H.M. Ozaktas and D. Mendlovic, J. Opt. Soc. Am. A 10 (1994) 2522.
[14] A.W. Lohmann, J. Opt. Soc. Am. A 10 (1993) 2181.
[15] G.S. Agarwal and R. Simon, Optics Comm. 110 (1994) 23.
[16] W. Vogel and H. Risken, Rev. A 40 (1989) 2847.
[17] B. Yurke, W. Schleich and D.F. Walls, Phys. Rev. A 42 (1990) 1703.
[18] W. Vogel and W. Schleich, Phys. Rev. A 44 (1991) 7642.
[19] D.T. Smithey, M. Beck, M.G. Raymer and A. Faridani, Phys. Rev. Letts. 70 (1993) 1244.
[20] M. Beck, M.G. Raymer, I.A. Walmsley and V. Kong, Optics Lett. 18 (1993) 2041.
[21] M.G. Raymer, M. Beck and D.F. McAlister, Phys. Rev. Let. 72 (1994) 1137.
[22] M.G. Raymer, M. Beck and D.F. McAlister, Spatial and temporal optical field reconstruction using phase-space tomography, in: Quantum Optics VI, eds. J.D. Harvey and D.F. Wall (Springer, Berlin, 1994.
[23] W.H. Louisell, Quantum Statistical Properties of Radiation (Wiley, New York, 1973).
[24] H.O. Bartelt, K.-H. Brenner and A.W. Lohmann, Optics Comm. 32 (1980) 32.
[25] A.W. Lohmann and B.H. Soffer, J. Opt. Soc. Am. A 11 (1994) 1798.
[26] Y.S. Kim and M.E. Noz, Phase Space Picture of Quantum Mechanics (World Scientific, Singapore, 1991).