Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 901 (2015) 354–381

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Instantons on Calabi–Yau cones

Marcus Sperling

Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany

Received 13 May 2015; received in revised form 3 August 2015; accepted 27 October 2015

Available online 3 November 2015

Editor: Herman Verlinde

Abstract

The Hermitian Yang–Mills equations on certain vector bundles over Calabi–Yau cones can be reduced to a set of matrix equations; in fact, these are Nahm-type equations. The latter can be analysed further by generalising arguments of Donaldson and Kronheimer used in the study of the original Nahm equations. Starting from certain equivariant connections, we show that the full set of instanton equations reduce, with a unique gauge transformation, to the holomorphicity condition alone.

1. Introduction

Instantons have proven to be interesting both for mathematicians and physicists. Starting from the seminal work [1] by Donaldson, anti-self-dual Yang–Mills connections provided a new topological invariant for four-manifolds. However, the moduli spaces of higher-dimensional instantons are still not fully understood.

From a physics perspective, instantons describe non-perturbative Yang–Mills configurations in various settings [2–4]. Focusing, for example, on heterotic string theory and compactifications thereof, the notion of instantons appears naturally in the so-called BPS-equations. In the simplest case, it is necessary to specify a 6-dimensional Calabi–Yau manifold as well as a Hermitian Yang–Mills (HYM) instanton on a gauge bundle over that manifold [5]. However, due to the appearance of phenomenologically problematic moduli, it is physically desirable to relax the strict Calabi–Yau condition to one of more general SU(3)-manifolds (SU(3)-structures with intrinsic

E-mail address: marcus.sperling@itp.uni-hannover.de.

http://dx.doi.org/10.1016/j.nuclphysb.2015.10.014

torsion), for which the instanton notion needs to be adjusted. Examples of those are nearly Kähler and half-flat manifolds. For further details on so-called flux compactifications see for instance [6,7].

Recently, the study of Sasaki–Einstein manifolds [8–12] has led to infinitely many explicit metrics on (non-compact) Calabi–Yau cones. Since there are no explicit Ricci-flat metrics known on compact Calabi–Yau manifolds, metric cones over Sasaki–Einstein spaces provide a testing ground for Calabi–Yau compactifications.

Previously, instantons have been discussed on certain cone constructions starting from a G-manifold, i.e. a manifold that admits a G-structure [13–21]. There on the instanton equations have been reduced to a set of matrix equations.[1] The aim of this paper is to discuss the resulting matrix equations on Calabi–Yau cones over a generic Sasaki–Einstein manifold M^{2n+1}, which carries an $\mathrm{SU}(n)$-structure. In particular, the HINP matrix equations conceptually comprise three types of equations: (i) the so-called equivariance condition, (ii) the holomorphicity condition, and (iii) a stability-like condition. Starting form solutions to (i), i.e. decomposing the matrices into irreducible representations of $\mathfrak{su}(n)$, we show that it suffices to solve (ii) for certain boundary conditions, because (iii) then follows by a unique gauge transformation. The arguments presented are a generalisation of [22–24].

Instantons on Calabi–Yau cones and their resolutions have also been studied in [25,26] and, for the particular orbifolds $\mathbb{C}^n/\mathbb{Z}_n$, in [27]. However, their setting and ansatz are different: on the one hand, [25,26] considered instantons on the *tangent bundle* of a $(2n+2)$-dimensional Calabi–Yau cone whose structure was largely determined by the $2n$-dimensional Einstein–Kähler manifold underlying the Sasaki–Einstein manifold in between. The ansatz for the connection was adapted to the isometry of the Calabi–Yau cone, and the “seed” was the spin connection in the Einstein–Kähler space, which is an instanton. On the other hand, certain gauge backgrounds for heterotic compactifications were constructed in [27] by extending a flat connection on $\mathbb{C}P^{n-1}$ to $\mathrm{U}(1)$ and $\mathrm{U}(n-1)$-valued instanton connections on the orbifolds. In contrast, the approach of [19], which is further discussed here, can conceptually take *any* instanton on the Sasaki–Einstein manifold as a starting point, and the bundle does not need to be the tangent bundle anymore.

This paper is organised as follows: the relevant details on Sasaki–Einstein manifolds and the Calabi–Yau cone over it are briefly summarised in Section 2. In addition, the notion of Hermitian Yang–Mills instantons is recalled. The main body is Section 3 where we firstly recapitulate the ansatz for the connection that reduces the HYM equations to matrix equations. The subsequent paragraphs consider the geometry, symmetries and solutions to these equations. Section 4 concludes.

2. Preliminaries

2.1. Sasaki–Einstein manifolds

Sasakian geometry can be understood as odd-dimensional analogue of Kähler geometry; in particular, an odd-dimensional manifold M^{2n+1} with a Sasakian structure is naturally sandwiched between two different types of Kähler geometry in the neighbouring dimensions $2n$ and $2n+2$.

[1] These matrix equations were first introduced in [19] as a generalisation of the results in [18]. We will refer to these equations as *Harland–Ivanova–Nölle–Popov* (HINP) matrix equations.

Following [28], a Sasakian manifold M^{2n+1} carries a Sasakian structure comprised of the quadruplet $\mathcal{S} = (\xi, \eta, \Phi, g)$, wherein ξ is the Reeb vector field, η the dual contact form, $\Phi \in \mathrm{End}(TM^{2n+1})$ a tensor, and g a Riemannian metric. The defining property for $(M^{2n+1}, \mathcal{S})$ to be Sasakian is that the metric cone $(C(M^{2n+1}), \widehat{g}) = (\mathbb{R}^+ \times M^{2n+1}, \mathrm{d}r^2 + r^2 g)$ is Kähler, i.e. the holonomy group of the Levi-Civita connection on the cone is $\mathrm{U}(n+1)$. The (compatible) complex structure J_c on the cone acts via $J_c(r\partial_r) = \xi$ and $J_c(X) = \Phi(X) - \eta(X) r\partial_r$ for any vector field X on M^{2n+1}. The corresponding Kähler 2-form is $\frac{1}{2}\mathrm{d}(r^2\eta)$.

Moreover, considering the contact subbundle $\mathcal{D} = \ker(\eta) \subset TM^{2n+1}$ one has a complex structure defined by restriction $J_t = \Phi|_{\mathcal{D}}$ and a symplectic structure $\mathrm{d}\eta$. Hence, $(\mathcal{D}, J_t, \mathrm{d}\eta)$ defines the transverse Kähler structure [28].

A *Sasaki–Einstein manifold* is Sasakian and Einstein simultaneously; thus, the defining property is that the metric cone is Calabi–Yau, i.e. the holonomy group on the cone is reduced to $\mathrm{SU}(n+1)$.

For the purposes of this paper, it is convenient to understand a Sasaki–Einstein manifold M^{2n+1} in terms of an $\mathrm{SU}(n)$-structure. For this, one has the 1-form η and the 2-form ω, which are related via $\mathrm{d}\eta = -2\omega$. One can always choose a co-frame $\{e^\mu\} = (e^a, e^{2n+1})$, with $\mu = 1, 2, \dots, 2n+1$ and $a = 1, 2, \dots, 2n$, such that these forms are locally given by

$$\eta = e^{2n+1} \quad \text{and} \quad \omega = e^1 \wedge e^2 + e^3 \wedge e^4 + \ldots + e^{2n-1} \wedge e^{2n} \equiv \frac{1}{2}\omega_{ab} e^{ab} \tag{2.1}$$

and that the metric is

$$g = \delta_{\mu\nu} e^\mu \otimes e^\nu = \delta_{ab} e^a \otimes e^b + \eta \otimes \eta \,. \tag{2.2}$$

Moreover, there exists a canonical connection Γ^P on TM^{2n+1} which is metric compatible, is an instanton with respect to the $\mathrm{SU}(n)$-structure, and has non-vanishing torsion.[2] The torsion components are given by [18]

$$T^{2n+1}_{a\,b} = -2\omega_{ab} \quad \text{and} \quad T^{b}_{a\,2n+1} = \frac{n+1}{n}\omega_{ab} \,. \tag{2.3}$$

2.2. Calabi–Yau metric cone

First of all, recall the basic properties of a Calabi–Yau manifold M^{2n+2}: as a Calabi–Yau space is Kähler, one has the Kähler form, which is an exact $(1,1)$-form on M^{2n+2}. In addition, the Calabi–Yau condition enforces the canonical bundle to be trivial, i.e. $K_{M^{2n+2}} = \Lambda^{(n+1,0)} T^* M^{2n+2} \cong \mathbb{C} \times M^{2n+2}$. Thus, there exists a nowhere vanishing section in $K_{M^{2n+2}}$ which translates into a $(n+1,0)$-form on M^{2n+2}.

The metric on the metric cone $(C(M^{2n+1}), \widehat{g})$ is defined as

$$\widehat{g} = \mathrm{d}r^2 + r^2 g = e^{2t}\left(\mathrm{d}t^2 + \delta_{\mu\nu} e^\mu \otimes e^\nu\right) = e^{2t}\widetilde{g} \,, \tag{2.4}$$

where the last equality employs a conformal rescaling $r = e^t$ from the metric cone with cone coordinate $r \in \mathbb{R}^+$ to the cylinder $(\mathrm{Cyl}(M^{2n+1}), \widetilde{g}) = (\mathbb{R} \times M^{2n+1}, \mathrm{d}t^2 + g)$ with coordinate $t \in \mathbb{R}$. Also, we identify $\mathrm{d}t = e^{2n+2}$ and extend the index range $\hat{\mu} = 1, 2, \dots, 2n+1, 2n+2$. The Kähler form $\widehat{\omega}$ on the cone is

[2] The torsion components can be related to the components of the 3-form $P = \eta \wedge \omega$; hence, the name Γ^P. However, the torsion is not completely antisymmetric itself.

$$\widehat{\omega} = r^2\omega + r\eta \wedge \mathrm{d}r = e^{2t}\,(\omega + \eta \wedge \mathrm{d}t) = e^{2t}\widetilde{\omega}\,, \tag{2.5}$$

which is again related to the Kähler form $\widetilde{\omega}$ on the cylinder. Next, we introduce a complexified basis on the cotangent bundle of $\mathrm{Cyl}(M^{2n+1})$ as follows

$$\theta^j = \mathrm{i}e^{2j-1} + e^{2j} \quad \text{and} \quad \bar{\theta}^j = -\mathrm{i}e^{2j-1} + e^{2j} \qquad \text{for} \quad j = 1, 2, \dots, n+1\,, \tag{2.6}$$

such that the metric and Kähler form read

$$\widetilde{g} = \frac{1}{2}\sum_{j=1}^{n+1}\left(\theta^j \otimes \bar{\theta}^j + \bar{\theta}^j \otimes \theta^j\right) \quad \text{and} \quad \widetilde{\omega} = -\frac{\mathrm{i}}{2}\sum_{j=1}^{n+1}\theta^j \wedge \bar{\theta}^j\,. \tag{2.7}$$

The compatible complex structure J acts via $J\theta^j = \mathrm{i}\theta^j$ and $J\bar{\theta}^j = -\mathrm{i}\bar{\theta}^j$, such that the compatibility relation is $\widetilde{\omega}(\cdot,\cdot) = \widetilde{g}(\cdot, J\cdot)$.

Let us compare the choice (2.6) with the "canonical choice" $\theta^j_{\mathrm{can}} = e^{2j-1} + \mathrm{i}e^{2j}$ and the canonical complex structure $J_{\mathrm{can}}\theta^j_{\mathrm{can}} = \mathrm{i}\theta^j_{\mathrm{can}}$. The conventions used here correspond to $J = -J_{\mathrm{can}}$ such that the $(1,0)$ and $(0,1)$-forms are interchanged, which implies that $\widetilde{\omega}(\cdot,\cdot) = \widetilde{g}(J_{\mathrm{can}}\cdot,\cdot) = -\widetilde{g}(\cdot, J_{\mathrm{can}}\cdot) = \widetilde{g}(\cdot, J\cdot)$ is consistent with the above. The reasons for this choice are that we desire a resemblance to the treatment of [22–24], while at the same time we treat $\mathrm{d}t$ as the $(2n+2)$th basis 1-form instead of the 0th.

2.3. Hermitian Yang–Mills instantons

For the later analysis, the geometric properties of the space of connections and the HYM instanton moduli space over a Kähler manifold are recalled. This brief account is inspired from [29,30].

2.3.1. Space of connections

Let M^{2n} be a (closed) Kähler manifold of $\dim_{\mathbb{C}}(M) = n$ and $\mathcal{G}$ a compact matrix Lie group. Let $P(M^{2n}, \mathcal{G})$ be a $\mathcal{G}$-principal bundle over M^{2n}, $\mathcal{A}$ a connection 1-form and $\mathcal{F}_{\mathcal{A}} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ the curvature.

Let $\mathrm{Int}(P) := P \times_{\mathcal{G}} \mathcal{G}$ be the group bundle (where $\mathcal{G}$ acts via the internal automorphism $h \mapsto ghg^{-1}$), let $\mathrm{Ad}(P) := P \times_{\mathcal{G}} \mathfrak{g}$ be the Lie algebra bundle (where $\mathcal{G}$ acts on $\mathfrak{g}$ via the adjoint action), and $E := P \times_{\mathcal{G}} F$ be an associated vector bundle (where the vector space F, the typical fibre, carries a $\mathcal{G}$-representation).

Denote the space of all connections on P by $\mathbb{A}(P)$ and note that all associated bundles E inherit their space of connections $\mathbb{A}(E)$ from P. On $\mathbb{A}(P)$ there is a natural action of the gauge group $\widehat{\mathcal{G}}$, i.e. the set of automorphisms on P which are trivial on the base. With

$$\widehat{\mathcal{G}} = \Gamma(M^{2n}, \mathrm{Int}(P)) \tag{2.8}$$

one has an identification with the space of global sections of the group bundle. The action is realised via

$$\mathcal{A} \to \mathcal{A}^g = \mathrm{Ad}(g^{-1})\mathcal{A} + g^{-1}\mathrm{d}g \qquad \text{for} \quad g \in \Gamma(M^{2n}, \mathrm{Int}(P))\,. \tag{2.9}$$

The Lie algebra of the gauge group is then given as

$$\widehat{\mathfrak{g}} = \Gamma(M^{2n}, \mathrm{Ad}(P))\,, \tag{2.10}$$

and the infinitesimal gauge transformations are given by

$$\mathcal{A} \mapsto \delta\mathcal{A} = \mathrm{d}_{\mathcal{A}}\chi := \mathrm{d}\chi + [\mathcal{A}, \chi] \qquad \text{for} \quad \chi \in \Gamma(M^{2n}, \mathrm{Ad}(P)) \,. \tag{2.11}$$

Since $\mathbb{A}(P)$ is an affine space, the tangent space $T_{\mathcal{A}}\mathbb{A}$ for any $\mathcal{A} \in \mathbb{A}(P)$ is canonically identified with $\Omega^1(M^{2n}, \mathrm{Ad}(P))$. Further, assuming $\mathcal{G} \hookrightarrow \mathrm{U}(N)$ for some $N \in \mathbb{N}$, implies that the trace is an Ad-invariant inner product. Hence, a metric on $\mathbb{A}(P)$ is defined via

$$\boldsymbol{g}_{|\mathcal{A}}(X_1, X_2) := \int_{M^{2n}} \mathrm{tr}\,(X_1 \wedge \star X_2) \qquad \text{for} \quad X_1, X_2 \in T_{\mathcal{A}}\mathbb{A} \,, \tag{2.12}$$

with $\star$ the Hodge-dual on M^{2n}. Moreover, the space $\mathbb{A}(P)$ allows for a symplectic structure

$$\boldsymbol{\omega}_{|\mathcal{A}}(X_1, X_2) := \int_{M^{2n}} \mathrm{tr}\,(X_1 \wedge X_2) \wedge \frac{\omega^{n-1}}{(n-1)!} \qquad \text{for} \quad X_1, X_2 \in T_{\mathcal{A}}\mathbb{A} \,. \tag{2.13a}$$

Since $\boldsymbol{\omega}$ is completely base-point independent (on $\mathbb{A}$), $\boldsymbol{\omega}$ is in fact a symplectic form. In addition, one can check that $X \wedge \frac{\omega^{n-1}}{(n-1)!} = \star J(X)$ holds for any $X \in T_{\mathcal{A}}\mathbb{A}$, where J, the (canonical) complex structure of M^{2n}, acts only the 1-form part of X. This allows to reformulate the symplectic structure as

$$\boldsymbol{\omega}_{|\mathcal{A}}(X_1, X_2) = \int_{M^{2n}} \mathrm{tr}\,(X_1 \wedge \star J(X_2)) \qquad \text{for} \quad X_1, X_2 \in T_{\mathcal{A}}\mathbb{A} \,. \tag{2.13b}$$

Moreover, it implies that $\boldsymbol{\omega}$ is non-degenerate as $\boldsymbol{\omega}_{|\mathcal{A}}(X_1, X_2) = \boldsymbol{g}_{|\mathcal{A}}(X_1, J(X_2))$ holds for any X_1, X_2 and any $\mathcal{A}$. Consequently, $(\mathbb{A}, \boldsymbol{g}, \boldsymbol{\omega})$ is an infinite-dimensional Riemannian, symplectic manifold, which is equipped with compatible $\widehat{\mathcal{G}}$-action.

2.3.2. Holomorphic structure

Next, consider the restriction to connections on $E \xrightarrow{\simeq F} M$ which satisfy the so-called holomorphicity condition

$$\mathcal{F}^{2,0}_{\mathcal{A}} = 0 \quad \text{and} \quad \mathcal{F}^{0,2}_{\mathcal{A}} = 0 \,. \tag{2.14}$$

It is well-known that this condition is equivalent to the existence of a holomorphic structure on E, i.e. a Cauchy–Riemann operator $\bar{\partial}_E := \bar{\partial} + A^{0,1}$ that satisfies the Leibniz-rule as well as $\bar{\partial}_E \circ \bar{\partial}_E = 0$. Thus, having a $\mathcal{G}$-bundle with a holomorphic connection induces a holomorphic $\mathcal{G}^{\mathbb{C}}$-bundle. If M^{2n} is also Calabi–Yau, then the condition (2.14) is equivalent to $\Omega \wedge \mathcal{F}_{\mathcal{A}} = 0$, where Ω is a holomorphic $(n, 0)$-form.

Define the subspace of holomorphic connections as

$$\mathbb{A}^{1,1} = \left\{ \mathcal{A} \in \mathbb{A}(E) : \mathcal{F}^{0,2}_{\mathcal{A}} = -\left(\mathcal{F}^{2,0}_{\mathcal{A}}\right)^{\dagger} = 0 \right\} \subset \mathbb{A}(E) \,. \tag{2.15}$$

This definition employs the underlying complex structure on M^{2n}. Moreover, one can show that $\mathbb{A}^{1,1}$ is an infinite-dimensional Kähler space, i.e. $\boldsymbol{g}$ is a Hermitian metric and the symplectic form $\boldsymbol{\omega}$ is Kähler. We note that these objects descend from $\mathbb{A}$ to $\mathbb{A}^{1,1}$ simply by restriction. The compatible complex structure $\boldsymbol{J}$ (with $\boldsymbol{\omega}(\cdot,\cdot) = \boldsymbol{g}(\boldsymbol{J}\cdot,\cdot)$) can be read off from (2.12) and (2.13) to be

$$\boldsymbol{J}_{|\mathcal{A}}(X) = -J(X) \qquad \text{for} \quad X \in T_{\mathcal{A}}\mathbb{A} \,, \tag{2.16}$$

i.e. it is base point independent.

2.3.3. Moment map

The space $\mathbb{A}^{1,1}$ inherits the $\widehat{\mathcal{G}}$-action from $\mathbb{A}$ and since it has a symplectic form, i.e. the Kähler form, one can introduce a moment map

$$\begin{aligned} \mu : \mathbb{A}^{1,1} &\to \widehat{\mathfrak{g}}^* \cong \Omega^{2n}(M^{2n}, \mathrm{Ad}(P)) \\ \mathcal{A} &\mapsto \mathcal{F}_{\mathcal{A}} \wedge \frac{\omega^{n-1}}{(n-1)!} \, . \end{aligned} \tag{2.17}$$

We see that μ is $\widehat{\mathcal{G}}$-equivariant by construction. Nonetheless, for this to be a moment map of the $\widehat{\mathcal{G}}$-action, one needs to verify the defining property

$$(\phi, \mathrm{D}\mu_{|\mathcal{A}})(\psi) = \iota_{\phi^\natural}\boldsymbol{\omega}_{|\mathcal{A}}(\psi) \, , \tag{2.18}$$

where $\phi \in \Gamma(M^{2n}, \mathrm{Ad}(P))$ an element of the gauge Lie algebra, $\phi^\natural$ be the corresponding vector field on $\mathbb{A}^{1,1}$ and $\psi \in \Omega^1(M^{2n}, \mathrm{Ad}(P))$ a tangent vector at the base point $\mathcal{A}$. Moreover, the duality pairing $(\cdot,\cdot)$ of $\widehat{\mathfrak{g}}$ and its dual is defined via the integral over M^{2n} and the invariant product on $\mathfrak{g}$. Generalising the arguments from [29], one can prove that μ is indeed a moment map for the $\widehat{\mathcal{G}}$-action on $\mathbb{A}^{1,1}$. Firstly, in the definition of μ only $\mathcal{F}_{\mathcal{A}}$ is base point dependent, and a standard computation gives $\mathcal{F}_{\mathcal{A}+t\,\psi} = \mathcal{F}_{\mathcal{A}} + t\,\mathrm{d}_{\mathcal{A}}\psi + \frac{1}{2}t^2\,\psi \wedge \psi$ so that $\mathrm{D}\mathcal{F}_{|\mathcal{A}}(\psi) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{\mathcal{A}+t\,\psi}\right)_{|t=0} = \mathrm{d}_{\mathcal{A}}\psi$. Thus the left-hand side of (2.18) is $(\phi, \mathrm{D}\mu_{|\mathcal{A}})(\psi) = \int_M \mathrm{tr}\big((\mathrm{d}_{\mathcal{A}}\psi) \wedge \phi\big) \wedge \frac{\omega^{n-1}}{(n-1)!}$. Secondly, the vector field $\phi^\natural$ can be read off from (2.11) to be $\phi^\natural_{|\mathcal{A}} = \mathrm{d}_{\mathcal{A}}\phi \in \Omega^1(M, \mathrm{Ad}(P))$. Hence the right-hand side is $\iota_{\phi^\natural}\boldsymbol{\omega}_{|\mathcal{A}}(\psi) = \int_M \mathrm{tr}\big((\mathrm{d}_{\mathcal{A}}\phi) \wedge \psi\big) \wedge \frac{\omega^{n-1}}{(n-1)!}$. But from $\int_M \mathrm{d}\left(\mathrm{tr}\,(\psi \wedge \phi) \wedge \frac{\omega^{n-1}}{(n-1)!}\right) = 0$ by Stokes' theorem[3] and $\mathrm{d}\omega = 0$ one has $\int_M \mathrm{tr}\big((\mathrm{d}_{\mathcal{A}}\psi) \wedge \phi\big) \wedge \frac{\omega^{n-1}}{(n-1)!} = -\int_M \mathrm{tr}\left(\psi \wedge (\mathrm{d}_{\mathcal{A}}\phi)\right) \wedge \frac{\omega^{n-1}}{(n-1)!}$, and therefore the relation (2.18) holds, i.e. μ is a moment map of the $\widehat{\mathcal{G}}$-action on $\mathbb{A}^{1,1}$.

However, one can equally well use the dual map defined by

$$\begin{aligned} \mu^* : \mathbb{A}^{1,1} &\to \widehat{\mathfrak{g}} = \Omega^0(M^{2n}, \mathrm{Ad}(P)) \\ \mathcal{A} &\mapsto \omega \lrcorner \mathcal{F}_{\mathcal{A}} \, , \end{aligned} \tag{2.19}$$

which is equivalent to μ of (2.17) due to

$$\mathcal{F}_{\mathcal{A}} \wedge \omega^{n-1} = \frac{1}{n}(\omega \lrcorner \mathcal{F}_{\mathcal{A}})\omega^n \, . \tag{2.20}$$

Thus, we will no longer explicitly distinguish between μ and μ^*.

For $\Xi \in \mathrm{Centre}(\widehat{\mathfrak{g}})$, we know $\mu^{-1}(\Xi) \subset \mathbb{A}^{1,1}$ defines a sub-manifold which allows for a $\widehat{\mathcal{G}}$-action. The quotient

$$\mathbb{A}^{1,1}/\!/\widehat{\mathcal{G}} \equiv \mu^{-1}(\Xi)/\widehat{\mathcal{G}} \tag{2.21}$$

is well-defined and, moreover, is a Kähler manifold, as the Kähler form and the complex structure descend from $\mathbb{A}^{1,1}$.

We recognise the zero-level set as the *Hermitian Yang–Mills* moduli space. In other words, the HYM equations consist of the holomorphicity conditions (2.14) together with the so-called stability condition

[3] For the non-compact Calabi–Yau cone of this paper, the boundary term arising by Stokes' theorem will be cancelled be restriction to *framed* gauge transformations. See Section 3.3.

$$\mu(\mathcal{F}_{\mathcal{A}}) = \mathcal{F}_{\mathcal{A}} \wedge \frac{\omega^{n-1}}{(n-1)!} = 0 = \mu^*(\mathcal{F}_{\mathcal{A}}) = \omega \lrcorner \mathcal{F}_{\mathcal{A}} \,. \tag{2.22}$$

By well-known theorems [31–33], a holomorphic vector bundle admits a solution to the HYM equations if and only if these bundles are (poly-)stable in the algebraic geometry sense.

2.3.4. Complex group action

As the $\widehat{\mathcal{G}}$-action on $\mathbb{A}^{1,1}$ preserves the Kähler structure, one can extend to an $\widehat{\mathcal{G}}^{\mathbb{C}}$-action on $\mathbb{A}^{1,1}$. In other words, the holomorphicity conditions $\mathcal{F}^{0,2}_{\mathcal{A}} = 0$ are invariant under the action of the complex gauge group

$$\widehat{\mathcal{G}}^{\mathbb{C}} = \widehat{\mathcal{G}} \otimes \mathbb{C} \,. \tag{2.23}$$

Let $\mathcal{A} \in \mathbb{A}^{1,1}$, then the orbit $\widehat{\mathcal{G}}^{\mathbb{C}}_{\mathcal{A}}$ of the $\widehat{\mathcal{G}}^{\mathbb{C}}$-action is

$$\widehat{\mathcal{G}}^{\mathbb{C}}_{\mathcal{A}} = \left\{ \mathcal{A}' \in \mathbb{A}^{1,1} \,\middle|\, \exists q \in \widehat{\mathcal{G}}^{\mathbb{C}} : \mathcal{A}' = \mathcal{A}^q \right\} . \tag{2.24}$$

A point $\mathcal{A} \in \mathbb{A}^{1,1}$ is called *stable* if $\widehat{\mathcal{G}}^{\mathbb{C}}_{\mathcal{A}} \cap \mu^{-1}(\Xi) \neq \emptyset$, and we denote by $\mathbb{A}^{1,1}_{st}(\Xi) \subset \mathbb{A}^{1,1}$ the set of all stable points (for a given Ξ). Then, a well-known result (see for example [34]) is

$$\mathbb{A}^{1,1} // \widehat{\mathcal{G}} \equiv \mu^{-1}(\Xi)/\widehat{\mathcal{G}} \cong \mathbb{A}^{1,1}_{st}(\Xi)/\widehat{\mathcal{G}}^{\mathbb{C}} \,. \tag{2.25}$$

2.3.5. Remark

A peculiarity arises for holomorphic bundles E over a compact Kähler manifold M^{2n} with non-empty boundary [35]. Due to the prescription of boundary conditions, the stability condition is automatically satisfied for a unitary connection whose curvature is of type $(1,1)$. Hence, all points in $\mathbb{A}^{1,1}$ are stable in this case.

In the following we will consider the HYM equations (2.14) and (2.22) on the non-compact Calabi–Yau cones. For these, the holomorphicity conditions still imply the existence of a holomorphic structure; while the notion of stability is not applicable anymore. Nonetheless, we will continue referring to $\omega \lrcorner \mathcal{F}_{\mathcal{A}} = 0$ as *stability-like* condition.

3. Equivariant instantons

The main focus of this paper lies on the description of the instantons on certain vector bundles E. However, instead of generic connections the set-up will be restricted to connections that arise from an instanton on the Sasaki–Einstein space M^{2n+1} by an extension $X \in \Omega^1(\mathrm{Cyl}(M^{2n+1}); \mathrm{End}(E))$. This extension has to satisfy a certain invariance condition.

The arguments presented in what follows are a generalisation of [22–24]: i.e. we generalise from spherically symmetric instantons on vector bundles over $C(S^3) \cong \mathbb{R}^4 \backslash \{0\}$ with an SU(2)-structure to SU(n+1)-equivariant instantons on vector bundles over $C(M^{2n+1})$ with an SU(n+1)-structure, where M^{2n+1} is an arbitrary Sasaki–Einstein manifold. Analogous to Donaldson and Kronheimer, it will be necessary to consider boundary conditions for the components of the connection 1-form, i.e. for the Yang–Mills fields.

3.1. Ansatz

Let us recall the ansatz presented in [19] and explicitly discussed in [21]. Start from any Sasaki–Einstein manifold M^{2n+1}, i.e. the manifold carries an SU(n)-structure together with a canonical connection Γ^P on the tangent bundle. The metric cone is Calabi–Yau with holonomy SU(n+1), i.e. an integrable SU(n+1)-structure. By conformal equivalence one can consider Cyl(M^{2n+1}).

Consider a complex vector bundle $E \to \mathrm{Cyl}(M^{2n+1})$ of rank p which has structure group SU(n+1); in particular, that is a Hermitian vector bundle where $\mathcal{F}^\dagger = -\mathcal{F}$ and $\mathrm{tr}(\mathcal{F}) = 0$ hold for the curvature $\mathcal{F}$ of a compatible connection. (In the compact case, one would have a vanishing first Chern class.) For example, the (holomorphic) tangent bundle of the Calabi–Yau cone is such a bundle, but one does not have to restrict to this case.

We recall that the connection 1-forms are $\mathfrak{su}(n{+}1)$-valued 1-forms on Cyl(M^{2n+1}) for any connection $\mathcal{A}$ on E. The ansatz for a connection is

$$\mathcal{A} = \widehat{\Gamma}^P + X \tag{3.1a}$$

where $\widehat{\Gamma}^P$ is the lifted $\mathfrak{su}(n)$-valued connection on E obtained from Γ^P, i.e. one essentially has to change the representation on the fibres. Moreover, on a patch $\mathcal{U} \subset \mathrm{Cyl}(M^{2n+1})$ with the co-frame $\{e^{\hat{\mu}}\}$ we employ the local description

$$X_{|\mathcal{U}} = X_\mu \otimes e^\mu + X_{2n+2} \otimes e^{2n+2} \,, \tag{3.1b}$$

where $X_{\hat{\mu}|x} \in \mathrm{End}(\mathbb{C}^p)$ for $x \in \mathcal{U}$. Usually X_{2n+2} is eliminated by a suitable gauge transformation, but there is no harm in not doing so.

The ansatz (3.1) is a generic connection in the sense that the $X_{\hat{\mu}}$ are base-point dependent, skew-Hermitian, traceless matrices with nontrivial transformation behaviour under change of trivialisation. Hence, *any* connection $\mathcal{A}$ on E can be reached starting from $\widehat{\Gamma}^P$.

Since SU(n) is a closed subgroup of SU($n+1$), one can choose an SU(n)-invariant decomposition

$$\begin{aligned}
\mathfrak{su}(n+1) = \mathfrak{su}(n) \oplus \mathfrak{m} \qquad \text{with} \quad \mathfrak{su}(n+1) &= \mathrm{span}\left\{ I_A \,\middle|\, A = 1, \dots, (n+1)^2 - 1 \right\} , \\
\mathfrak{su}(n) &= \mathrm{span}\left\{ I_\alpha \,\middle|\, \alpha = 2n+2, \dots, (n+1)^2 \right\} , \\
\mathfrak{m} &= \mathrm{span}\left\{ I_\mu \,\middle|\, \mu = 1, \dots, 2n+1 \right\} ,
\end{aligned} \tag{3.2}$$

and denote by $\widehat{I_A}$ the generators in a representation on the fibres $E_x \cong \mathbb{C}^p$. By the invariant splitting, one has the following commutation relations:

$$\left[\widehat{I_\alpha}, \widehat{I_\beta}\right] = f_{\alpha\beta}^{\ \ \gamma} \widehat{I_\gamma} \,, \quad \left[\widehat{I_\alpha}, \widehat{I_\mu}\right] = f_{\alpha\mu}^{\ \ \nu} \widehat{I_\nu} \,, \quad \left[\widehat{I_\mu}, \widehat{I_\nu}\right] = f_{\mu\nu}^{\ \ \alpha} \widehat{I_\alpha} + f_{\mu\nu}^{\ \ \sigma} \widehat{I_\sigma} \,, \tag{3.3}$$

for $\alpha, \beta, \gamma = 2n+2, \dots, (n+1)^2$ and $\mu, \nu, \sigma = 1, \dots, 2n+1$. A suitable choice of these structure constants can be found in [18–21].

Next, we simplify the ansatz by demanding $X_{\hat{\mu}} = X_{\hat{\mu}}(t)$; i.e. not all connections $\mathcal{A}$ on E can be reached anymore. Moreover, this demand is only valid in any trivialisation if the following conditions hold (see [21] for further details)

$$\left[\widehat{I_\alpha}, X_\mu\right] = f_{\alpha\mu}^{\ \ \nu} X_\nu \quad \text{and} \quad \left[\widehat{I_\alpha}, X_{2n+2}\right] = 0 \qquad \text{for} \quad \mu, \nu = 1, \dots, 2n+1 \,. \tag{3.4}$$

The $(f_{\alpha\mu}^{\ \ \nu})$ can be interpreted as the matrix elements $(\rho_*(I_\alpha))_\mu^{\ \nu}$ of a (suitably chosen) representation ρ of SU(n) on the typical fibre of TM^{2n+1}. The representation theoretic content of (3.4) is that the matrix-valued functions $X_{\hat{\mu}}$ have to transform in a representation of $\mathfrak{su}(n)$.

Computing the curvature $\mathcal{F}_\mathcal{A}$ for the ansatz (3.1) together with the *equivariance condition* (3.4) then yields

$$
\begin{aligned}
\mathcal{F}_{\mathcal{A}} = \mathcal{F}_{\widehat{\Gamma}^P} &+ \frac{1}{2}\left([X_a, X_b] + T^{2n+1}_{a\,b} X_{2n+1}\right) e^a \wedge e^b \\
&+ \left(\left[X_a, X_{2n+1}\right] + T^{b}_{a\,2n+1} X_b\right) e^a \wedge e^{2n+1} \\
&+ \left(\left[X_a, X_{2n+2}\right] - \tfrac{\mathrm{d}}{\mathrm{d}t} X_a\right) e^a \wedge e^{2n+2} \\
&+ \left(\left[X_{2n+1}, X_{2n+2}\right] - \tfrac{\mathrm{d}}{\mathrm{d}t} X_{2n+1}\right) e^{2n+1} \wedge e^{2n+2} ,
\end{aligned}
\tag{3.5}
$$

with $\mathcal{F}_{\widehat{\Gamma}^P}$ is the curvature of $\widehat{\Gamma}^P$, and $a, b = 1, \dots, 2n$. The HYM instanton equations (2.14) and (2.22) reduce for the ansatz to a set of matrix equations for the $X_{\hat{\mu}}$, which are given in [19] (note that $X_{2n+2} = 0$ for this case). Moreover, $\mathcal{F}_{\widehat{\Gamma}^P}$ already satisfies the HYM equations, as the $\widehat{\Gamma}^P$ is the lift of an $\mathrm{SU}(n)$-instanton and the corresponding $\mathrm{SU}(n)$-principal bundle is a subbundle in the $\mathrm{SU}(n+1)$-principal bundle associated to E.

3.1.1. Matrix equations: real basis

For completeness, the resulting matrix HINP-equations in the real basis $\{e^{\hat{\mu}}\}$ are the holomorphicity conditions

$$\left[X_{2j-1}, X_{2k-1}\right] - \left[X_{2j}, X_{2k}\right] = 0 , \tag{3.6a}$$

$$\left[X_{2j-1}, X_{2k}\right] + \left[X_{2j}, X_{2k-1}\right] = 0 , \tag{3.6b}$$

$$\left[X_{2j-1}, X_{2n+2}\right] + \left[X_{2j}, X_{2n+1}\right] = \tfrac{\mathrm{d}}{\mathrm{d}t} X_{2j-1} + \tfrac{n+1}{n} X_{2j-1} , \tag{3.6c}$$

$$\left[X_{2j}, X_{2n+2}\right] - \left[X_{2j-1}, X_{2n+1}\right] = \tfrac{\mathrm{d}}{\mathrm{d}t} X_{2j} + \tfrac{n+1}{n} X_{2j} , \tag{3.6d}$$

for $j, k = 1, \dots, n$ and the stability-like condition

$$\tfrac{\mathrm{d}}{\mathrm{d}t} X_{2n+1} + 2n X_{2n+1} = \sum_{k=1}^{n+1} \left[X_{2k-1}, X_{2k}\right] . \tag{3.6e}$$

3.1.2. Matrix equations: complex basis

For the intents and purposes here, it is more convenient to switch to the complex basis $\{\theta^j, \bar{\theta}^j\}$ defined in (2.6) and introduce

$$Y_j := \frac{1}{2}\left(X_{2j} - \mathrm{i} X_{2j-1}\right) \quad \text{and} \quad Y_{\bar{j}} := \frac{1}{2}\left(X_{2j} + \mathrm{i} X_{2j-1}\right) \quad \text{for} \quad j = 1, 2, \dots, n+1 . \tag{3.7}$$

Hence, $Y_{\bar{j}} = -(Y_j)^\dagger$ since $X_{\hat{\mu}}(t) \in \mathfrak{su}(n+1)$ for all $t \in \mathbb{R}$. For the $Y_j : \mathbb{R} \to \mathrm{End}(\mathbb{C}^p)$ one finds the holomorphicity conditions

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Y_j + \tfrac{n+1}{n} Y_j = 2\left[Y_j, Y_{n+1}\right] \quad \text{and} \quad \left[Y_j, Y_k\right] = 0 \qquad \text{for} \quad j, k = 1, \dots, n , \tag{3.8a}$$

and the adjoint equations thereof. The stability-like condition reads

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(Y_{n+1} + Y_{n+1}^\dagger\right) + 2n\left(Y_{n+1} + Y_{n+1}^\dagger\right) + 2\sum_{j=1}^{n+1}\left[Y_j, Y_j^\dagger\right] = 0 . \tag{3.8b}$$

The equivariance conditions for the complex matrices are

$$\left[\widehat{I}_\alpha, Y_j\right] = -\mathrm{i} f_{\alpha 2j-1}^{\;\;2j} Y_j \quad \text{and} \quad \left[\widehat{I}_\alpha, Y_{n+1}\right] = 0 , \tag{3.9}$$

for $j, = 1, \ldots, n$. For these calculations we have used the choice of structure constants $f_{\alpha\mu}^{\ \ \nu} = 0$ if μ or $\nu = 2n+1$ and $f_{\alpha a}^{\ \ b} \propto \omega_{ab}$, see for instance [19–21].

3.1.3. *Change of trivialisation*

The remaining nontrivial effects of a change of trivialisation of the bundle E over $\mathrm{Cyl}(M^{2n+1})$ are given by the set of functions $\{g(t) |\, g : \mathbb{R} \to \mathrm{SU}(p)\}$ that act as

$$X_\mu \mapsto \mathrm{Ad}(g) X_\mu \qquad \text{for} \quad \mu = 1, \ldots, 2n+1 \quad \text{and} \quad X_{2n+2} \mapsto \mathrm{Ad}(g) X_{2n+2} - \left(\tfrac{\mathrm{d}}{\mathrm{d}t} g\right) g^{-1} , \tag{3.10}$$

which follows from $\mathcal{A} \mapsto \mathcal{A}^g = \mathrm{Ad}(g)\mathcal{A} - (\mathrm{d}g)g^{-1}$ and $g = g(t)$.[4] Due to their adjoint transformation behaviour, the X_μ are sometimes called *Higgs fields*, for example in quiver gauge theories. The inhomogeneous transformation of X_{2n+2} is crucial to be able to "gauge away" this connection component. Furthermore, these gauge transformations (and their complexification) will be used to study the solutions of the matrix equations.

3.1.4. *Yang–Mills with torsion*

The instanton equations (on the cone *and* the cylinder) are equivalently given by

$$\star \mathcal{F}_{\mathcal{A}} = - \frac{\omega^{n-1}}{(n-1)!} \wedge \mathcal{F}_{\mathcal{A}} , \tag{3.11}$$

where ω is the corresponding $(1,1)$-form ($\mathrm{d}\omega = 0$ on the cone, but $\mathrm{d}\omega \neq 0$ on the cylinder). An immediate consequence is that the instanton equation for the integrable $\mathrm{SU}(n+1)$-structure implies the Yang–Mills equations, while this is not true for the $\mathrm{SU}(n+1)$-structure with torsion. In detail

$$\text{cone:} \quad (3.11) \Rightarrow \mathrm{d}_{\mathcal{A}} \star \mathcal{F}_{\mathcal{A}} = 0 \qquad \text{Yang–Mills} , \tag{3.12a}$$

$$\text{cylinder:} \quad (3.11) \Rightarrow \mathrm{d}_{\mathcal{A}} \star \mathcal{F}_{\mathcal{A}} + \frac{\omega^{n-2}}{(n-2)!} \wedge \mathrm{d}\omega \wedge \mathcal{F}_{\mathcal{A}} = 0 \qquad \text{Yang–Mills with torsion} . \tag{3.12b}$$

These *torsionful Yang–Mills* equations (3.12b), which arise in the context of non-integrable G-structures (with intrinsic torsion), have been studied in the literature before [13–15,36–38]. In particular, the torsion term does not automatically vanish on instantons because $\mathrm{d}\omega$ contains $(2,1)$ and $(1,2)$-forms. This is, for instance, in contrast to the nearly Kähler case discussed in [39], in which nearly Kähler instantons were found to satisfy the ordinary Yang–Mills equations.

It is known that the appropriate functional for the torsionful Yang–Mills equations comprises the ordinary Yang–Mills functional plus an additional Chern–Simons term

$$S_{\mathrm{YM+T}}(\mathcal{A}) = \int\limits_{\mathrm{Cyl}(M^{2n+1})} \mathrm{tr}\left(\mathcal{F}_{\mathcal{A}} \wedge \star \mathcal{F}_{\mathcal{A}}\right) + \frac{\omega^{n-1}}{(n-1)!} \wedge \mathrm{tr}\left(\mathcal{F}_{\mathcal{A}} \wedge \mathcal{F}_{\mathcal{A}}\right) , \tag{3.13}$$

which is a gauge-invariant functional. The properties of $S_{\mathrm{YM+T}}$ are the following: firstly and most importantly, instanton connections satisfying (3.11) have $S_{\mathrm{YM+T}}(\mathcal{A}) = 0$, i.e. the *action is finite*.

[4] We have simply replaced g in (2.9) by g^{-1}.

Secondly, the stationary points of (3.13) are the vanishing locus of the torsionful Yang–Mills equations (up to boundary terms). For this, we use $\mathcal{F}_{\mathcal{A}+z\Psi} = \mathcal{F}_{\mathcal{A}} + z\mathrm{d}_{\mathcal{A}}\Psi + \frac{1}{2}z^2\Psi \wedge \Psi$ for any $\Psi \in T_{\mathcal{A}}\mathbb{A}(E)$ and compute the variation

$$
\begin{aligned}
\delta S_{\mathrm{YM+T}}(\mathcal{A}) &:= \frac{\mathrm{d}}{\mathrm{d}z} S_{\mathrm{YM}}(\mathcal{A}+z\Psi)\Big|_{z=0} \\
&= \int\limits_{\mathrm{Cyl}(M^{2n+1})} 2\,\mathrm{tr}\,(\mathrm{d}_{\mathcal{A}}\Psi \wedge \star\mathcal{F}_{\mathcal{A}}) + 2\frac{\omega^{n-1}}{(n-1)!} \wedge \mathrm{tr}\,(\mathcal{F}_{\mathcal{A}} \wedge \mathrm{d}_{\mathcal{A}}\Psi) \\
&= 2 \int\limits_{\mathrm{Cyl}(M^{2n+1})} \mathrm{tr}\left(\Psi \wedge \left(\mathrm{d}_{\mathcal{A}} \star \mathcal{F}_{\mathcal{A}} + \frac{\omega^{n-2}}{(n-2)!} \wedge \mathrm{d}\omega \wedge \mathcal{F}_{\mathcal{A}}\right)\right) \\
&\quad + 2 \int\limits_{\mathrm{Cyl}(M^{2n+1})} \mathrm{d}\,\mathrm{tr}\left(\Psi \wedge \left(\star\mathcal{F}_{\mathcal{A}} + \frac{\omega^{n-1}}{(n-1)!} \wedge \mathcal{F}_{\mathcal{A}}\right)\right) . \qquad (3.14)
\end{aligned}
$$

The boundary term would vanish for closed manifolds. In our case, if one assumes M^{2n+1} to be closed, the vanishing of the boundary term requires certain assumptions on the fall-off rate of $\mathcal{F}_{\mathcal{A}}$ for $t \to \pm\infty$. Moreover, it is interesting to observe that the boundary term in (3.14) vanishes for instanton configurations.

3.2. *Rewriting the instanton equations*

3.2.1. *Real equations*

Returning to the instanton equations for the X-matrices (3.6), the linear terms can be eliminated via a change of coordinates:

$$X_{2j-1} =: e^{-\frac{n+1}{n}t}\mathcal{X}_{2j-1} , \qquad X_{2j} =: e^{-\frac{n+1}{n}t}\mathcal{X}_{2j} \quad \text{for} \quad j = 1, \dots, n , \qquad (3.15a)$$

$$X_{2n+1} =: e^{-2nt}\mathcal{X}_{2n+1} , \qquad X_{2n+2} =: e^{-2nt}\mathcal{X}_{2n+2} , \qquad (3.15b)$$

$$s = -\frac{1}{2n}e^{-2nt} \in \mathbb{R}^- , \qquad \lambda_n(s) := \left(\frac{-1}{2ns}\right)^{2-\frac{n+1}{n^2}} . \qquad (3.15c)$$

Note that the exponent $2 - \frac{n+1}{n^2}$ vanishes for $n = 1$ and is strictly positive for any $n > 1$. The matrix HINP equations (3.6) read now as follows:

$$
\begin{aligned}
&\left[\mathcal{X}_{2j-1}, \mathcal{X}_{2k-1}\right] - \left[\mathcal{X}_{2j}, \mathcal{X}_{2k}\right] = 0 \quad \text{and} \\
&\left[\mathcal{X}_{2j-1}, \mathcal{X}_{2k}\right] + \left[\mathcal{X}_{2j}, \mathcal{X}_{2k-1}\right] = 0 , \qquad (3.16a) \\
&\left[\mathcal{X}_{2j-1}, \mathcal{X}_{2n+2}\right] + \left[\mathcal{X}_{2j}, \mathcal{X}_{2n+1}\right] = \tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{X}_{2j-1} \quad \text{and} \\
&\left[\mathcal{X}_{2j}, \mathcal{X}_{2n+2}\right] - \left[\mathcal{X}_{2j-1}, \mathcal{X}_{2n+1}\right] = \tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{X}_{2j} , \qquad (3.16b)
\end{aligned}
$$

for $j, k = 1, \dots, n$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{X}_{2n+1} = \lambda_n(s) \sum_{k=1}^{n} \left[\mathcal{X}_{2k-1}, \mathcal{X}_{2k}\right] + \left[\mathcal{X}_{2n+1}, \mathcal{X}_{2n+2}\right] . \qquad (3.16c)$$

3.2.2. Complex equations

Completely analogous, the change of coordinates for the complex equations is performed via

$$Y_j =: e^{-\frac{n+1}{n}t}\mathcal{Y}_j \qquad \text{for} \quad j=1,\dots,n \quad \text{and} \quad Y_{n+1} =: e^{-2nt}\mathcal{Z} \,. \tag{3.17}$$

We will refer to this set of matrices simply by $(\mathcal{Y}, \mathcal{Z})$. In summary, the instanton equations are now comprised by the "complex equations"

$$\left[\mathcal{Y}_j, \mathcal{Y}_k\right] = 0 \quad \text{and} \quad \tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{Y}_j = 2\left[\mathcal{Y}_j, \mathcal{Z}\right] \qquad \text{for} \quad j,k=1,\dots,n \,, \tag{3.18a}$$

and the "real equation"

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{Z}+\mathcal{Z}^\dagger\right) + 2\left[\mathcal{Z}, \mathcal{Z}^\dagger\right] + 2\lambda_n(s)\sum_{j=1}^{n}\left[\mathcal{Y}_j, \mathcal{Y}_j^\dagger\right] = 0 \,. \tag{3.18b}$$

These equations are reminiscent to the considerations of the instantons on $\mathbb{R}^4\backslash\{0\}$ of [22–24], and, in fact, they reduce to the same system for $n = 1$, but in general one a Calabi–Yau 2-fold $\mathbb{C}^2/\Gamma$. To see this, we recall [28] that all 3-dimensional Sasaki–Einstein spaces are given by S^3/Γ, where Γ is a finite subgroup of SU(2) (and commutes with $\mathrm{U}(1) \subset \mathrm{SU}(2)$) which acts freely and isometrically from the left on $S^3 \cong \mathrm{SU}(2)$.

3.2.3. Remarks

The equivariance conditions for the rescaled matrices $\{\mathcal{X}_{\hat{\mu}}\}$ or $(\{\mathcal{Y}_j\}, \mathcal{Z})$ are exactly the same as (3.4) or (3.9), respectively.

Moreover, the rescaling has another salient feature: the matrices $\{\mathcal{X}_{\hat{\mu}}\}$ or $(\{\mathcal{Y}_j\}, \mathcal{Z})$ (as well as their derivatives) are bounded (see for instance [23]); in contrast, the original connection components will develop a pole at the origin $r = 0$. This will become apparent once the boundary conditions are specified. For further details, see Appendix A.1.

In addition, we observe that the exponents on the rescaling (3.15) reflect the torsion components (2.3). The choice of a flat "starting point" $\Gamma = 0$ would lead to Nahm-type equations straight away, but solutions to the resulting matrix equations would not interpolate between any (non-trivial) lifted instanton from M^{2n+1} and instantons on the Calabi–Yau space $C(M^{2n+1})$, cf. [14,19].

3.2.4. Real gauge group

The full set of instanton equations (3.18) is invariant under the action of the gauge group

$$\widehat{\mathcal{G}} := \left\{g(s) | g : \mathbb{R}^- \to \mathrm{U}(p)\right\} \,, \tag{3.19}$$

wherein the action is defined via

$$\mathcal{Y}_j \mapsto \mathcal{Y}_j^g := \mathrm{Ad}(g)\mathcal{Y}_j \qquad \text{for} \quad j=1,\dots,n \,, \tag{3.20a}$$

$$\mathcal{Z} \mapsto \mathcal{Z}^g := \mathrm{Ad}(g)\mathcal{Z} - \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}s}g\right)g^{-1} \,. \tag{3.20b}$$

Note that only the real equation (3.18b) requires $g^{-1} = g^\dagger$ for it to be gauge invariant. Moreover, one can always find a gauge transformation $g \in \widehat{\mathcal{G}}$ such that $\mathcal{Z}^g = (\mathcal{Z}^g)^\dagger$ (Hermitian) or, equivalently, $X^g_{2n+2} = 0$.

In summary, these properties follow from (3.10) as the X-matrices are extensions to a connection. However, the gauge group (3.19) still contains a nontrivial centre $\{g(s)|g(s) =$

$\phi(s)\mathbb{1}_{p\times p}$ with $\phi:\mathbb{R}^-\to \mathrm{U}(1)\}$, such that (3.10) corresponds to the quotient of $\widehat{\mathcal{G}}$ by its centre.

3.2.5. Complex gauge group

Moreover, the complex equations (3.18a) allow for an action of the complexified gauge group

$$\widehat{\mathcal{G}}^{\mathbb{C}} \equiv \left\{ g(s) \middle| g:\mathbb{R}^- \to \mathrm{GL}(p,\mathbb{C}) \right\} , \tag{3.21}$$

given by

$$\mathcal{Y}_k \mapsto \mathrm{Ad}(g)\mathcal{Y}_k \,, \qquad \mathcal{Y}_{\bar{k}} \mapsto \mathrm{Ad}((g^{-1})^\dagger)\mathcal{Y}_{\bar{k}} \,, \qquad \text{for} \quad k=1,\dots,n \,, \tag{3.22a}$$

$$\mathcal{Z} \mapsto \mathrm{Ad}(g)\mathcal{Z} - \tfrac{1}{2}\left(\tfrac{\mathrm{d}}{\mathrm{d}s} g\right) g^{-1} \,, \qquad \bar{\mathcal{Z}} \mapsto \mathrm{Ad}((g^{-1})^\dagger)\bar{\mathcal{Z}} + \tfrac{1}{2}(g^{-1})^\dagger \left(\tfrac{\mathrm{d}}{\mathrm{d}s} g^\dagger\right) . \tag{3.22b}$$

The extension to $\widehat{\mathcal{G}}^{\mathbb{C}}$-invariance for the holomorphicity conditions exemplifies the generic situation discussed in Section 2.3.

3.2.6. Equivariance condition

Actually, one needs to be a bit more careful in considering these equations and their symmetries. Recall that we restrict ourselves to the matrices $X_{\hat{\mu}}$ which satisfy the equivariance conditions (3.4). However, if the equivariance conditions are not invariant under the gauge transformations (3.20), then a solution obtained by gauge transformation might not be equivariant anymore.

The real gauge transformations can be interpreted as change of basis on the fibres $E_x \cong \mathbb{C}^p$ or, more appropriately, change of trivialisation. Since the $\widehat{I_\alpha}$ are representations of the generators I_α on these fibres, the same transformation acts on them as well. In order to preserve the Lie algebra (3.3), all generators have to transform as

$$\widehat{I_A} \mapsto \mathrm{Ad}(g)\widehat{I_A} \qquad \text{for} \quad g\in\widehat{\mathcal{G}} \quad \text{and} \quad A=1,\dots(n+1)^2-1 \,. \tag{3.23}$$

The same transformation behaviour is adopted when passing to the complexified gauge group. This renders $[\widehat{I_\alpha}, X_\mu] = f_{\alpha\mu}^{\ \ \nu} X_\nu$ into a gauge invariant condition for both $\widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}}^{\mathbb{C}}$-transformations; the $\widehat{\mathcal{G}}^{\mathbb{C}}$-invariance follows as (3.9) does not intertwine $\{Y_k\}$ and $\{Y_{\bar{k}}\}$ for any $k=1,\dots,n+1$. Unfortunately, $[\widehat{I_\alpha}, X_{2n+2}]=0$ is not gauge invariant, due the inhomogeneous transformation behaviour (3.10) of X_{2n+2}. However, the way out is that we will only impose this last condition at the very end, i.e. once we have chosen a gauge transformation g such that $X_{2n+2}^g=0$, the last equivariance condition follows trivially.

3.2.7. Boundary conditions

We observe that a trivial solution of (3.16) is

$$\begin{aligned} &\mathcal{X}_{2n+2}(s)=0 \quad \text{and} \quad \mathcal{X}_\mu(s)=T_\mu \\ &\quad \text{with} \quad [T_\mu,T_\nu]=0 \quad \text{for} \quad \mu,\nu=1,\dots,2n+1 \,, \end{aligned} \tag{3.24}$$

where the (constant) T_μ are elements in the Cartan subalgebra of $\mathfrak{su}(p)$; i.e. the (real) $(p-1)$-dimensional space spanned by the diagonal, traceless matrices with purely imaginary values. From the rescaling (3.15) of the $X_{\hat{\mu}}$, it is apparent that these matrices become singular as $r\to 0$

($t \to -\infty$ or $s \to -\infty$). Following [23,40], it is appropriate to choose the boundary conditions for X_μ to be[5]

$$s \to 0: \quad X_\mu(s) \to 0 \quad \text{for} \quad \mu = 1, \ldots, 2n+1 \quad \text{and} \tag{3.25a}$$

$$s \to -\infty: \quad \exists g_0 \in \mathrm{U}(p) \text{ such that } \mathcal{X}_\mu(s) \to \mathrm{Ad}(g_0)T_\mu \quad \text{for} \quad \mu = 1, \ldots, 2n+1\,. \tag{3.25b}$$

One can show [23] that this implies the existence of the limit of $\mathcal{X}_\mu$ for $s \to 0$. Hence, the solutions extend to the interval $(-\infty, 0]$, see also Appendix A.1. Thus, we are led to consider (3.16) for matrices $\mathcal{X}_\mu(s)$ over $(-\infty, 0]$ with one remaining boundary condition:

$$\exists g_0 \in \mathrm{U}(p) \text{ such that } \forall \mu = 1, \ldots, 2n+1: \lim_{s \to -\infty} \mathcal{X}_\mu(s) = \mathrm{Ad}(g_0)T_\mu\,. \tag{3.26}$$

Moreover, since one has first order differential equations it suffices to impose this one boundary condition, here at $s = -\infty$. Thus, the values of $\mathcal{Y}_k$ at $s = 0$ are completely determined by the solution. Following [23], we observe that (3.18a) implies that $\mathcal{Y}_k(s)$ lies entirely in a single adjoint orbit $\mathcal{O}(k)$ of the complex group $\widehat{\mathcal{G}}^{\mathbb{C}}$, for each $k = 1, \ldots, n$. Next, assuming that $\mathcal{T}_k = \frac{1}{2}(T_{2k} - \mathrm{i}T_{2k-1})$ for $k = 1, \ldots, n$ is a *regular tuple* in the Cartan subalgebra of $\widehat{\mathfrak{g}}^{\mathbb{C}}$ in the sense of [23] (that is the joint stabiliser of the T_μ in $\mathrm{SU}(p)$ is the maximal torus), one obtains that $\mathcal{Y}_k(s{=}0) \in \mathcal{O}(k)$, i.e. the values at $s = 0$ are in a conjugacy class of $\mathcal{T}_k$. Moreover, only the conjugacy class has a gauge-invariant meaning.

Nonetheless, the boundary conditions (3.26) clearly show that the original connection (3.1) develops the following poles at the origin $r = 0$ of the Calabi–Yau cone:

$$\begin{aligned} &\lim_{r \to 0} r^{\frac{n+1}{n}} X_a = \mathrm{Ad}(g_0)T_a \quad \text{for} \quad a = 1, \ldots, 2n \quad \text{and} \\ &\lim_{r \to 0} r^{2n} X_{2n+1} = \mathrm{Ad}(g_0)T_{2n+1}\,. \end{aligned} \tag{3.27}$$

Note that the case $n = 1$ is reminiscent to the *instantons with poles* considered in [23].

3.3. Geometric structure

3.3.1. Space of connections under consideration

Consider the space of $\mathfrak{su}(n+1)$-valued connections $\mathbb{A}(E)$ in which any element can be parameterised as in (3.1). Due to the ansatz of Section 3.1, we restrict ourselves to the subspace $\mathbb{A}_{\mathrm{equi}}(E) \subset \mathbb{A}(E)$ of connections which satisfy (3.4). Specialising the considerations of Section 2.3, we will now establish certain (formal) geometric structures.

3.3.2. Kähler structure

The first step is to establish a Kähler structure on $\mathbb{A}_{\mathrm{equi}}(E)$. Since $\mathbb{A}_{\mathrm{equi}}(E)$ descends from the space of all connection $\mathbb{A}(E)$, one can simply obtain the geometric structures by restriction. A tangent vector

$$\boldsymbol{y} = \sum_{j=1}^{n+1} \left(\boldsymbol{y}_j \theta^j + \boldsymbol{y}_{\bar{j}} \bar{\theta}^j \right) \tag{3.28}$$

[5] One does not need to worry about X_{2n+2}, as it can always be gauged away.

at a point $\mathcal{A} \in \mathbb{A}_{\text{equi}}(E)$ is defined by the linearisation of (3.8) for paths $\boldsymbol{y}_j : \mathbb{R} \to \mathfrak{su}(p)$. Their gauge transformations are

$$\boldsymbol{y}_j \to \boldsymbol{y}_j^g := \text{Ad}(g)\boldsymbol{y}_j \qquad \text{for} \quad j = 1, \ldots, n+1 \,. \tag{3.29}$$

Taking the generic expressions for the metric (2.12) and the symplectic structure (2.13), we can specialise to the case at hand by transition to the cylinder and neglecting the volume integral of M^{2n+1}. Thus, for a metric on $\mathbb{A}_{\text{equi}}$ we obtain

$$\boldsymbol{g}_{|\mathcal{A}}(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) \equiv 2 \int_{\mathbb{R}} \mathrm{d}t \; e^{2nt} \; \text{tr} \left\{ \sum_{j=1}^{n+1} \left(\boldsymbol{y}_j^{(1)\dagger} \boldsymbol{y}_j^{(2)} + \boldsymbol{y}_j^{(1)} \boldsymbol{y}_j^{(2)\dagger} \right) \right\} . \tag{3.30}$$

Similarly, the symplectic form reads as

$$\boldsymbol{\omega}_{|\mathcal{A}}(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) \equiv -2\mathrm{i} \int_{\mathbb{R}} \mathrm{d}t \; e^{2nt} \; \text{tr} \left\{ \sum_{j=1}^{n+1} \left(\boldsymbol{y}_j^{(1)\dagger} \boldsymbol{y}_j^{(2)} - \boldsymbol{y}_j^{(1)} \boldsymbol{y}_j^{(2)\dagger} \right) \right\} . \tag{3.31}$$

Moreover, a complex structure $\boldsymbol{J}$ on $\mathbb{A}(E)_{\text{equiv}}$ has been given in (2.16). Keeping in mind that (2.6) implies $J = -J_{\text{can}}$, we obtain

$$\boldsymbol{J}_{|\mathcal{A}}(\boldsymbol{y}) = J(\boldsymbol{y}) = \mathrm{i} \sum_{j=1}^{n+1} \left(\boldsymbol{y}_j \theta^j - \boldsymbol{y}_{\bar{j}} \bar{\theta}^j \right) \tag{3.32}$$

As before, the symplectic form $\boldsymbol{\omega}$ and the metric $\boldsymbol{g}$ are compatible, i.e. $\boldsymbol{g}(\boldsymbol{J}\cdot, \cdot) = \boldsymbol{\omega}(\cdot, \cdot)$. We note that both structures are gauge-invariant by construction.

3.3.3. *Moment map*

The subspace of holomorphic connections $\mathbb{A}^{1,1}_{\text{equi}}(E) \subset \mathbb{A}_{\text{equi}}(E)$ is defined by the condition (3.18a). This condition only restricts the allowed endmorphism-valued 1-forms, because $\widehat{\Gamma}^P$ is already a $(1,1)$-type connection, since it is an HYM-instanton. Again, the metric $\boldsymbol{g}$ and Kähler form $\boldsymbol{\omega}$ descend to $\mathbb{A}^{1,1}_{\text{equi}}(E)$ from the corresponding objects on $\mathbb{A}_{\text{equi}}(E)$. Moreover, on the Kähler space $\mathbb{A}^{1,1}_{\text{equi}}(E)$, one defines a moment map

$$\begin{aligned} \mu: \qquad & \mathbb{A}^{1,1}_{\text{equi}}(E) \to \widehat{\mathfrak{g}}_0 = \text{Lie}(\widehat{\mathcal{G}}_0) \\ & (\mathcal{Y}, \mathcal{Z}) \mapsto \mathrm{i} \left(\frac{\mathrm{d}}{\mathrm{d}s} \left(\mathcal{Z} + \mathcal{Z}^\dagger \right) + 2 \left[\mathcal{Z}, \mathcal{Z}^\dagger \right] + 2\, \lambda_n(s) \sum_{k=1}^{n} \left[\mathcal{Y}_k, \mathcal{Y}_k^\dagger \right] \right) , \end{aligned} \tag{3.33}$$

where $\widehat{\mathcal{G}}_0$ is the corresponding *framed* gauge group. That is

$$\widehat{\mathcal{G}}_0 := \left\{ g(s) | g : \mathbb{R}^- \to \text{U}(p) \,, \; \lim_{s \to 0} g(s) = \lim_{s \to -\infty} g(s) = 1 \right\} . \tag{3.34}$$

It is an important realisation that on the non-compact Calabi–Yau cone (and the conformally equivalent cylinder) one has to compensate the appearing boundary terms in Stokes' theorem by the transition to the framed gauge transformations. The details of the proof that (3.33) satisfies conditions (2.18) are given in the Appendix A.2. Here, we just note that the map (3.33) maps the matrices $(\mathcal{Y}, \mathcal{Z})$ into the correct space: the factor of i renders the expression anti-hermitian; while the boundary conditions (3.25) together with the gauge choice $\mathcal{Z} = -\mathcal{Z}^\dagger$ yield the vanishing of $\mu(\mathcal{Y}, \mathcal{Z})$ at $s \to 0$ and $s \to -\infty$.

The part of instanton moduli space that is connected with the lift $\widehat{\Gamma}^P$ (in the sense of our ansatz (3.1)) is then readily obtained by the Kähler quotient

$$\mathcal{M}_{\Gamma^P} = \mu^{-1}(0)/\widehat{\mathcal{G}}_0 \, . \tag{3.35}$$

3.3.4. Stable points

Alternatively, one can describe this part of the moduli space via the stable points

$$\mathbb{A}^{1,1}_{st}(E) \equiv \left\{ \widehat{\Gamma}^P + X \in \mathbb{A}^{1,1}(E) \middle| \, (\widehat{\mathcal{G}}_0^{\mathbb{C}})_{(\mathcal{Y},\mathcal{Z})} \cap \mu^{-1}(0) \neq \emptyset \right\} \, , \tag{3.36}$$

where the tuple $(\mathcal{Y}, \mathcal{Z})$ is obtained from X via complex linear combinations and rescaling as before. The moduli space arises then by taking the $\widehat{\mathcal{G}}_0^{\mathbb{C}}$-quotient

$$\mathbb{A}^{1,1}_{st}(E)/\widehat{\mathcal{G}}_0^{\mathbb{C}} \cong \mathcal{M}_{\Gamma^P} \, . \tag{3.37}$$

We argue in the next couple of paragraphs that it suffices to solve the complex equations (3.18a), because the solution to the real equation (3.18b) follows from a framed complex gauge transformation. More precisely: for every point in $\mathbb{A}^{1,1}_{\mathrm{equi}}(E)$ there exists a unique point in the complex gauge orbit such that the real equation is satisfied. In other words, every point in $\mathbb{A}^{1,1}_{\mathrm{equi}}(E)$ is stable.

3.4. Solutions to matrix equations

3.4.1. Solutions to complex equation

In the spirit of [22], one can also understand the complex equations as being locally trivial. That is, take (3.22) and demand the gauge transformed $\mathcal{Z}$ to be zero

$$\mathcal{Z}^g = \mathrm{Ad}(g)\mathcal{Z} - \tfrac{1}{2}\left(\tfrac{\mathrm{d}}{\mathrm{d}s} g\right) g^{-1} \stackrel{!}{=} 0 \qquad \Rightarrow \qquad \mathcal{Z} = \tfrac{1}{2} g^{-1} \tfrac{\mathrm{d}}{\mathrm{d}s} g \, . \tag{3.38}$$

From the holomorphicity equations (3.18a) one obtains

$$\tfrac{\mathrm{d}}{\mathrm{d}s} \mathcal{Y}_k^g = 0 \quad \text{and} \quad \mathcal{Y}_k^g = \mathrm{Ad}(g_0)\mathcal{T}_k \;\; \text{with} \quad \left[\mathcal{T}_j, \mathcal{T}_k\right] = 0 \, , \tag{3.39}$$

for $j, k = 1, \ldots, n$ and g_0 is a *constant* gauge transformation.[6] Consequently, the general local solution of the complex equations (3.18a) is

$$\mathcal{Y}_k = \mathrm{Ad}(g^{-1})\mathcal{T}_k \;\; \text{with} \quad \left[\mathcal{T}_j, \mathcal{T}_k\right] = 0 \quad \text{and} \quad \mathcal{Z} = \frac{1}{2} g^{-1} \frac{\mathrm{d}}{\mathrm{d}s} g \, , \tag{3.40}$$

for *any* $g \in \widehat{\mathcal{G}}^{\mathbb{C}}$. A solution to the commutator constraint is choosing $\mathcal{T}_k$ for $k = 1, \ldots, n$ as elements of the Cartan subalgebra of the Lie algebra $\mathfrak{gl}(p, \mathbb{C})$, which are all diagonal (complex) $p \times p$ matrices.

[6] This g_0 can also be gauge away to 1.

3.4.2. Solution to the real equation

In any case, one can in principle solve the complex equations; now, the real equation (3.18b) needs to be solved as well. Following the ideas of [22], the considerations are split in two steps: (i) a variational description and (ii) a differential inequality. We provide the details of (i) in this paragraph, while we postpone the details of (ii) to the Appendix A.4. Let us recall that the complete set of instanton equations is gauge-invariant under $\widehat{\mathcal{G}}$. Thus, define for each $g \in \widehat{\mathcal{G}}^{\mathbb{C}}$ the map

$$h = h(g) = g^{\dagger} g : \mathbb{R}^{-} \to \mathrm{GL}(p, \mathbb{C})/\mathrm{U}(p)\ . \tag{3.41}$$

The quotient $\mathrm{GL}(p, \mathbb{C})/\mathrm{U}(p)$ can be identified with the set of positive, self-adjoint $p \times p$ matrices. Then, fix a tuple $(\mathcal{Y}, \mathcal{Z})$ and define the functional $\mathcal{L}_{\epsilon}[g]$ for g

$$\mathcal{L}_{\epsilon}[g] = \tfrac{1}{2} \int\limits_{-\frac{1}{\epsilon}}^{-\epsilon} \mathrm{d}s\ \mathrm{tr} \left(\left| \mathcal{Z}^{g} + (\mathcal{Z}^{\dagger})^{g} \right|^{2} + 2\lambda_{n}(s) \sum_{k=1}^{n} \left| \mathcal{Y}_{k}^{g} \right|^{2} \right) \qquad \text{for} \quad 0 < \epsilon < 1\ , \tag{3.42}$$

where $(\mathcal{Y}^{g}, \mathcal{Z}^{g})$ denotes the gauge-transformed tuple. For the variation of (3.42) it suffices to consider variations with $\delta g = \delta g^{\dagger}$ around $g = 1$, but of course $\delta g \neq 0$. Then the gauge transformations (3.22) imply

$$\delta \mathcal{Z} = \left[\delta g, \mathcal{Z} \right] - \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} \delta g \quad \text{and} \quad \delta \mathcal{Y}_{k} = \left[\delta g, \mathcal{Y}_{k} \right] \qquad \text{for} \quad k = 1, \ldots, n\ . \tag{3.43}$$

The variation then leads to

$$\delta_{g} \mathcal{L}_{\epsilon} = -\mathrm{i} \int\limits_{-\frac{1}{\epsilon}}^{-\epsilon} \mathrm{d}s\ \mathrm{tr} \left\{ \mu(\mathcal{Y}, \mathcal{Z})\ \delta g \right\}\ , \tag{3.44}$$

i.e. critical points of (3.42) are precisely the zero-level set of the moment map. Next, we take the solution (3.40) and insert it as a starting point for $\mathcal{L}_{\epsilon}$. Thus, one obtains a functional for h

$$\begin{aligned} \mathcal{L}_{\epsilon}[h] &= \tfrac{1}{2} \int\limits_{-\frac{1}{\epsilon}}^{-\epsilon} \mathrm{d}s\ \left\{ \tfrac{1}{4} \mathrm{tr} \left(h^{-1} \frac{\mathrm{d}h}{\mathrm{d}s} \right)^{2} + 2\lambda_{n}(s) \sum_{k=1}^{n} \mathrm{tr} \left(h \mathcal{T}_{k} h^{-1} \mathcal{T}_{k}^{\dagger} \right) \right\} \\ &= \tfrac{1}{2} \int\limits_{-\frac{1}{\epsilon}}^{-\epsilon} \mathrm{d}s\ \left\{ \tfrac{1}{4} \mathrm{tr} \left(h^{-1} \frac{\mathrm{d}h}{\mathrm{d}s} \right)^{2} + V \right\}\ . \end{aligned} \tag{3.45}$$

Following [22], the potential $V(h) = 2\lambda_{n}(s) \sum_{k=1}^{n} \mathrm{tr} \left(h \mathcal{T}_{k} h^{-1} \mathcal{T}_{k}^{\dagger} \right)$ is positive,[7] implying that for *any* boundary values $h_{-}, h_{+} \in \mathrm{GL}(p, \mathbb{C})/\mathrm{U}(p)$ there exists a continuous path[8]

$$h : \left[-\tfrac{1}{\epsilon}, -\epsilon \right] \to \mathrm{GL}(p, \mathbb{C})/\mathrm{U}(p) \qquad \text{with} \quad h(-\tfrac{1}{\epsilon}) = h_{-} \quad \text{and} \quad h(-\epsilon) = h_{+}\ , \tag{3.46}$$

[7] Note that $\lambda_{n}(s)$ is strictly positive and smooth on $\left(-\frac{1}{\epsilon}, -\epsilon \right)$ for any $0 < \epsilon < 1$.

[8] See for instance the note under [22, Corollary 2.13]: One knows that $\mathrm{GL}(p, \mathbb{C})/\mathrm{U}(p)$ satisfies all necessary conditions for the existence of a unique stationary path between any two points.

which is smooth in $I_\epsilon = \left(-\frac{1}{\epsilon}, -\epsilon\right)$ and minimising the functional. Hence, for any choice of gauge transformation g such that $g^\dagger g = h$ one has that $(\{\mathcal{T}_k\}_{k=1,\dots,n}, 0)^g = (\{\mathrm{Ad}(g)\mathcal{T}_k\}_{k=1,\dots,n}, -\frac{1}{2}(\frac{\mathrm{d}}{\mathrm{d}s} g)g^{-1})$ satisfies the real equation in I_ϵ for any $0 < \epsilon < 1$. From now on, we restrict the attention to $h_+ = h_- = 1$, i.e. h is framed.

The uniqueness of the solution h on each interval I_ϵ and the existence of the limit h_∞ for $\epsilon \to 0$ follows from the aforementioned differential inequality similar to [22] and the discussion of [23, Lemma 3.17]. The details are presented in Appendix A.4. The relevant (framed) gauge transformation is then simply given by $g = \sqrt{h_\infty}$.[9]

However, we need to emphasise two crucial points. Firstly, the construction of a solution for the limit $\epsilon \to 0$ relies manifestly on the use of the boundary conditions (3.26), and the fact that these give rise to a (constant) solution of *both* the complex equations *and* the real equation. Secondly, the corresponding complex gauge transformation $g = g(h_\infty)$ is only determined up to unitary gauge transformations, i.e. it is not unique. This ambiguity in the choice of g can be removed, when we recall that a $\widehat{\mathcal{G}}$ gauge transformation suffices to eliminate X_{2n+2}. Hence, one can demand that the gauge-transformed system $(\mathcal{Y}^g, \mathcal{Z}^g)$ of a solution $(\mathcal{Y}, \mathcal{Z})$ satisfies $\mathcal{Z}^g = (\mathcal{Z}^g)^\dagger$. This fixes $g = g(h)$ uniquely, see also Appendix A.4.4 for further details.

3.4.3. Result

In summary, it is sufficient to search for solutions $(\mathcal{Y}', \mathcal{Z}')$ of the complex equations (3.18a) on the interval $(-\infty, 0]$ such that the boundary conditions (3.26) are satisfied. Then one has the existence of a unique complex gauge transformation g such that

(i) $(\mathcal{Y}, \mathcal{Z}) = (\mathcal{Y}', \mathcal{Z}')^g$ satisfies (3.18b),
(ii) $\mathcal{Z}$ is Hermitian (i.e. $\mathcal{X}_{2n+2} = 0$) and
(iii) g is bounded and framed.

In other words, it suffices to solve the complex equations subject to some boundary conditions and the real equation will be satisfied automatically.

Moreover, the above indicates that any point in $\mathbb{A}^{1,1}_{\mathrm{equi}}$ is stable, which we recall to be exactly the condition that every complex gauge orbit intersects $\mu^{-1}(0)$. We believe that this circumstance holds because we restricted ourselves to the space of equivariant connections. The benefit is then, that one, in principle, only has to show the solvability of the holomorphicity conditions in order to solve the instanton (matrix) equations. Nevertheless, one still has to find an ansatz that satisfies the equivariance conditions (3.4).

3.5. Further directions

Before concluding we can further exploit the results collected so far as well as illustrate another viewpoint of the HINP matrix equations.

---

[9] We use the *unique* principal root of the positive Hermitian matrix h, which is a continuous operation. Consequently, the framing of h implies the framing of g.

3.5.1. Relation to coadjoint orbits

Let us denote by $\mathcal{M}_n(E)$ the moduli space of solutions to the complex and real equations satisfying the boundary conditions (3.26) together with the equivariance condition. From the considerations above, we can establish the following map[10]

$$\begin{aligned} &\mathcal{M}_n(E) \to \mathcal{O}_{\mathcal{T}_1} \times \cdots \times \mathcal{O}_{\mathcal{T}_n} \\ &(\mathcal{Y}, \mathcal{Z}) \mapsto (\mathcal{Y}_1(0), \ldots, \mathcal{Y}_n(0)) \end{aligned} \tag{3.47}$$

where $\mathcal{O}_{\mathcal{T}_k}$ denotes the adjoint orbit of $\mathcal{T}_k$ in $\mathfrak{gl}(p, \mathbb{C})$. Analogous to [23], this map is a bijection due to the construction of the local solution (3.40) and the uniqueness of the corresponding solution of the real and complex equations. Moreover, one knows that the orbit of an element $\mathcal{T}_k$ of the Cartan subalgebra is of the form $\mathrm{GL}(p, \mathbb{C})/\mathrm{Stab}(\mathcal{T}_k)$. The product of coadjoint orbits in (3.47) is a complex symplectic manifold of complex dimension $n \dim(\mathrm{GL}(p, \mathbb{C})) - \sum_{j=1}^{n} \dim(\mathrm{Stab}(\mathcal{T}_j))$. Each orbit is equipped with the so-called Kirillov–Kostant–Souriau symplectic form and the product thereof gives the symplectic structure on the total space. In addition, the bijection above preserves the holomorphic symplectic structure.

3.5.2. Relation to quiver representations

The HINP matrix equations can be seen to define quiver representations, depending on the chosen $\mathrm{SU}(n+1)$-representation on the typical fibre $\mathbb{C}^p$. Then, by the employed ansatz, we decompose this representation with respect to $\mathrm{SU}(n)$ into

$$\mathbb{C}^p \Big|_{\mathrm{SU}(n)} = \bigoplus_{w \in J} \mathbb{C}^{n_w} , \tag{3.48}$$

where $\mathbb{C}^{n_w}$ carries a n_w-dimensional irreducible $\mathrm{SU}(n)$-representation. More explicitly, w should be understood as pair of labels: let ϕ label the irreducible $\mathrm{SU}(n)$-representations and recall that the centraliser of $\mathrm{SU}(n)$ inside $\mathrm{SU}(n+1)$ is a $\mathrm{U}(1)$. Then each representation space $\mathbb{C}^{n_w}$ carries also a $\mathrm{U}(1)$-representation characterised by a "charge" q. Therefore, the decomposition is labelled by pairs $w = (\phi, q)$.

As a consequence, the equivariance condition (3.4) dictates the decomposition of the X_μ-matrices into homomorphisms

$$X_\mu = \bigoplus_{w, w' \in J} (X_\mu)_{w,w'} \qquad \text{with} \quad (X_\mu)_{w,w'} \in \mathrm{Hom}\left(\mathbb{C}^{n_w}, \mathbb{C}^{n_{w'}}\right) . \tag{3.49}$$

The quiver representation then arises as follows: the set Q_0 of vertices is the set $\{\mathbb{C}^{n_w} | w \in J\}$ of vector spaces and the set Q_1 of arrows is given by the non-vanishing homomorphisms $\{(X_\mu)_{w,w'} | w, w' \in J \, , \ \mu = 1, \ldots, 2n+1\}$.

The instanton equations (or HINP equations) then lead to relations on the quiver representation. Examples for the arising quiver diagrams as well as their relations for the case $n = 1$ and $M^3 = S^3$ can be found in [41] and for $n = 2$ and $M^5 = S^5$ in [42]. To study the representations of a quiver one would rather use the constructions of [41,42], instead of the ansatz employed here. Because once the bundle E and the action of $\mathrm{SU}(n+1)$ on the fibres is chosen, there is no freedom to change the quiver representation anymore.

10 I thank Richard Szabo for pointing this out to me.

4. Conclusions

It is known that the instanton moduli space over a Kähler manifold is a Kähler space. Therefore, also the moduli space of certain invariant connections should inherit this property. The overall situation remains unknown.

In the ansatz presented here, we restricted ourselves to a subset of all possible connections by, firstly, imposing an equivariance condition and simplifying to t-dependence only and by, secondly, fixing an instanton Γ^P as a starting point. Hence, by this construction one can only reach a particular part of the full instanton moduli space by the solutions of the HINP matrix equations.

The arguments presented in this paper show that the reduced HINP matrix equations can be treated similarly to the Nahm-equations of SU(2) monopoles. As a consequence, one gains local solvability of the holomorphicity conditions together with the fact that any solution can be uniquely gauge-transformed into a solution of the stability-like condition. Moreover, the structure of the (framed) moduli space shares, at least locally, all features of a Kähler space due to the Kähler quotient construction or the GIT quotient.

It is of interest to extend the ansatz presented here from cones to their smooth resolutions as in [19,26], and to consider quiver gauge theories which can be associated to Calabi–Yau cones along the lines of [41,42].

Acknowledgements

I would like to thank Jakob Geipel, Lutz Habermann, Olaf Lechtenfeld and Alexander Popov for discussions and useful comments. This work was done within the framework of the DFG project LE 838/13. Moreover, I am grateful to the Research Training Group GRK 1463 "Analysis, Geometry and String Theory" for support.

Appendix A. Details

In this appendix, we provide the proofs of the statements made in Sections 3.2–3.4. Although the steps are similar to those performed in [22–24], we believe that these are necessary because the HINP equations are generalisations of the Nahm equations.

A.1. Boundedness of rescaled matrices

Recall the boundary conditions (3.26) for the original matrices

$$t \to +\infty : \qquad X_\mu \to 0 \,, \tag{A.1a}$$

$$t \to -\infty : \qquad e^{\frac{n+1}{n}t} X_a \to \mathrm{Ad}(g_0) T_a \quad \text{and} \quad e^{2nt} X_{2n+1} \to \mathrm{Ad}(g_0) T_{2n+1} \,. \tag{A.1b}$$

Evaluating the asymptotic behaviour for $t \to +\infty$ of (3.6), one finds the leading behaviour of (the real and imaginary part) of each matrix element to be

$$\frac{\mathrm{d}}{\mathrm{d}t}(X_a)_{AB} + \frac{n+1}{n}(X_a)_{AB} \simeq 0 \qquad \to \qquad (X_a)_{AB} \sim e^{-\frac{n+1}{n}t} \text{ as } t \to \infty \,, \tag{A.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(X_{2n+1})_{AB} + 2n(X_{2n+1})_{AB} \simeq 0 \qquad \to \qquad (X_{2n+1})_{AB} \sim e^{-2nt} \text{ as } t \to \infty \,, \tag{A.2b}$$

because the commutator terms vanish faster than linear order. These results imply the following:

(i) The rescaled matrices $\mathcal{X}_\mu$ of (3.15) are bounded for $s \to 0$.

(ii) The commutators $e^{\frac{n+1}{n}t}[X_a, X_{2n+1}]$ are integrable over $(0, \infty)$.

(iii) The derivatives $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{n+1}{n}t} X_a\right)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2nt} X_{2n+1}\right)$ are integrable, which follows by the use of the equations (3.6).

In conclusion, the $\mathcal{X}_{\hat{\mu}}$ as well as their derivatives are bounded.

A.2. Well-defined moment map

We need to prove (2.18) for μ defined in (3.33); recall that $\mu(\mathcal{A}) := \mathcal{F}_\mathcal{A} \wedge \frac{\widehat{\omega}^{n-1}}{(n-1)!}$ and we identified μ^* with μ. Moreover, it is crucial to use the closed Kähler 2-form from the cone, i.e. $\widehat{\omega} = e^{2t}\widetilde{\omega}$ on the cylinder. We will work with the original connection components Y_k defined in (3.7).

For the left-hand-side we proceed as follows: Let $\phi \in \widehat{\mathfrak{g}}_0$ and $\Psi = \Psi_k\theta^k - \Psi_k^\dagger\bar{\theta}^k$ be a tangent vector at $\mathcal{A}$. The duality pairing of Lie- and dual Lie-algebra is realised by the integration over the cylinder and the subsequent invariant product on $\mathfrak{u}(p)$.

$$(\phi, \mathrm{D}\mu_{|\mathcal{A}})\Psi = \int\limits_{\mathrm{Cyl}(M^{2n+1})} \mathrm{tr}\left\{\phi \frac{\mathrm{d}}{\mathrm{d}z}\mathcal{F}_{\mathcal{A}+z\Psi}\big|_{z=0}\right\} \wedge \frac{\widehat{\omega}^n}{n!} \tag{A.3a}$$

$$= \int\limits_{\mathbb{R}} \mathrm{d}t\; e^{2nt}\; \mathrm{tr}\left\{\phi \cdot \mathrm{i}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\Psi_{n+1} + \Psi_{n+1}^\dagger) + 2n(\Psi_{n+1} + \Psi_{n+1}^\dagger) + 2\sum_{k=1}^{n+1}\left(\left[\Psi_k, Y_k^\dagger\right] + \left[Y_k, \Psi_k^\dagger\right]\right)\right]\right\} \cdot \int\limits_{M^{2n+1}} \mathrm{vol}\,. \tag{A.3b}$$

Hence, for the dual moment map one can neglect the volume integral over M^{2n+1} and the dual pairing is defined via the first integral over t.

The compute the right-hand-side of (2.18) we need to take a step back and derive the symplectic form on $\mathbb{A}$ from (2.13) as follows

$$\boldsymbol{\omega}_{|\mathcal{A}}(\Psi, \Xi) = -\int\limits_{\mathrm{Cyl}(M^{2n+1})} \mathrm{tr}\,(\Psi \wedge \Xi) \wedge \frac{\widehat{\omega}^n}{n!} \tag{A.4a}$$

$$= -2\mathrm{i}\int\limits_{\mathbb{R}} \mathrm{d}t\; e^{2nt}\; \mathrm{tr}\sum_{k=1}^{n+1}\left\{\Psi_k^\dagger\Xi_k - \Psi_k\Xi_k^\dagger\right\} \cdot \int\limits_{M^{2n+1}} \mathrm{vol}\,. \tag{A.4b}$$

Again, we can drop the volume of the Sasaki–Einstein space. Next, we need the infinitesimal gauge transformation generated by an (framed) Lie-algebra element ϕ. From (3.20) we obtain

$$\phi^\# = \frac{\mathrm{d}}{\mathrm{d}z} Y_j^{g=\exp(z\phi)}\Big|_{z=0} = \begin{cases} \left[\phi, Y_j\right], & j = 1, \dots, n \\ \left[\phi, Y_{n+1}\right] - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi, & j = n+1\,, \end{cases} \tag{A.5}$$

which then brings us to

$$
\begin{aligned}
\iota_{\phi^\#}\boldsymbol{\omega}_{|\mathcal{A}}(\Psi) &= -2\mathrm{i}\int_{\mathbb{R}} \mathrm{d}t\ e^{2nt}\ \mathrm{tr}\Bigg\{ \sum_{k=1}^{n} \Big\{ [\phi, Y_k]^\dagger\, \Psi_k - [\phi, Y_k]\, \Psi_k^\dagger \Big\} \\
&\quad + \left(\Big[\phi, Y_{n+1}\Big] - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi \right)^\dagger \Psi_{n+1} - \left(\Big[\phi, Y_{n+1}\Big] - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\phi \right) \Psi_{n+1}^\dagger \Bigg\} \qquad \text{(A.6a)} \\
&= \int_{\mathbb{R}} \mathrm{d}t\ e^{2nt}\ \mathrm{tr}\Bigg\{ \phi \cdot \mathrm{i}\Bigg[\frac{\mathrm{d}}{\mathrm{d}t}(\Psi_{n+1} + \Psi_{n+1}^\dagger) + 2n(\Psi_{n+1} + \Psi_{n+1}^\dagger) \\
&\quad + 2\sum_{k=1}^{n+1}\left(\Big[\Psi_k, Y_k^\dagger\Big] + \Big[Y_k, \Psi_k^\dagger\Big] \right) \Bigg]\Bigg\} - \mathrm{i}\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}\left\{ e^{2nt}\mathrm{tr}\phi(\Psi_{n+1} + \Psi_{n+1}^\dagger)\right\} . \qquad \text{(A.6b)}
\end{aligned}
$$

A close inspection of the boundary term reveals that

$$
\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}t}\left\{ e^{2nt}\ \mathrm{tr}\Big(\phi(\Psi_{n+1} + \Psi_{n+1}^\dagger)\Big)\right\} = e^{2nt}\ \mathrm{tr}\Big(\phi(\Psi_{n+1} + \Psi_{n+1}^\dagger)\Big)\Bigg|_{t\to-\infty}^{t\to+\infty} \qquad \text{(A.7)}
$$

vanishes provided $\lim_{t\to\pm\infty}\phi(t) = 0$ i.e. the map defined in (3.33) is a moment map for the action of the *framed gauge group* $\widehat{\mathcal{G}}_0 = \{g(t) | g : \mathbb{R} \to \mathrm{U}(p), \text{s.t. } \lim_{t\to\pm\infty} g(t) = 1\}$.

A.3. Notation

We need to introduce some notation, which is relevant for the proofs later.

A.3.1. $\partial, \bar{\partial}$-operators

Following [22], we define the following ∂, $\bar{\partial}$-operators on $\mathbb{C}^p$-valued functions f on $\mathbb{R}^-$

$$
\mathrm{d}_{\mathcal{Z}} f = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} f + \mathcal{Z} f\ , \qquad \bar{\mathrm{d}}_{\mathcal{Z}} f = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} f - \mathcal{Z}^\dagger f\ , \qquad \text{(A.8a)}
$$

$$
\mathrm{d}_j f = \mathcal{Y}_j f\ , \qquad \bar{\mathrm{d}}_j f = -\mathcal{Y}_j^\dagger f\ , \qquad \text{(A.8b)}
$$

and on matrix-valued functions γ on $\mathbb{R}^-$

$$
\mathrm{d}_{\mathcal{Z}}\gamma = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\gamma + \big[\mathcal{Z}, \gamma\big]\ , \qquad \bar{\mathrm{d}}_{\mathcal{Z}}\gamma = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\gamma - \Big[\mathcal{Z}^\dagger, \gamma\Big]\ , \qquad \text{(A.8c)}
$$

$$
\mathrm{d}_j\gamma = \big[\mathcal{Y}_j, \gamma\big]\ , \qquad \bar{\mathrm{d}}_j\gamma = -\Big[\mathcal{Y}_j^\dagger, \gamma\Big]\ . \qquad \text{(A.8d)}
$$

These operators will give rise to the $\bar{\partial}$-operators associated to the connection $\mathcal{A}$. For that we take the covariant derivative $\mathrm{d}_{\mathcal{A}} = \mathrm{d} + \widehat{\Gamma}^P + Y_j\theta^j + Y_{\bar{j}}\bar{\theta}^j$ and define $\bar{\partial}_{\mathcal{A}} = \bar{\partial} + (\widehat{\Gamma}^P)^{(0,1)} + Y_{\bar{j}}\bar{\theta}^j$. Hence, the above definitions are understood as components of $\bar{\partial}_{\mathcal{A}}$. However, our notation and conventions differ slightly from [22] in the sense that we work with the equivalent $\partial_{\mathcal{A}}$-operator. In detail, the cone direction s in [22] is considered as 0th coordinate such that the *canonical* complex structure is defined via the choice of $(1,0)$-forms $\mathrm{d}s + \mathrm{i}e^1$ and $e^2 + \mathrm{i}e^3$ ($\{e^p,\ p = 1,2,3\}$ a co-frame on $\mathbb{R}^3$). In contrast, we designated the cone coordinate as e^{2n+2} and choose the $(1,0)$-forms as in (2.6) in order to avoid unnecessary factors of i. With respect to the canonical choice $e^{2j-1} + \mathrm{i}e^{2j}$ our complex structure is simply $J = -J_{\mathrm{can}}$, implying that we interchanged $(1,0)$ and $(0,1)$-forms. Consequently, we consider the $\partial_{\mathcal{A}}$-operator.

A.3.2. Gauge transformations

For the ∂-operators the action of the complex automorphisms is defined via

$$\mathrm{d}_j^g := g \circ \mathrm{d}_j \circ g^{-1} \quad \text{and} \quad \mathrm{d}_{\mathcal{Z}}^g := g \circ \mathrm{d}_{\mathcal{Z}} \circ g^{-1} . \tag{A.9}$$

From these definitions, we obtain

$$g^{-1}\mathrm{d}_{\mathcal{Z}}^g g = \mathrm{d}_{\mathcal{Z}} , \qquad g^{-1}\bar{\mathrm{d}}_{\mathcal{Z}}^g g = \bar{\mathrm{d}}_{\mathcal{Z}} + h^{-1}\bar{\mathrm{d}}_{\mathcal{Z}} h , \tag{A.10a}$$

$$g^{-1}\mathrm{d}_j^g g = \mathrm{d}_j , \qquad g^{-1}\bar{\mathrm{d}}_j^g g = \bar{\mathrm{d}}_j + h^{-1}\bar{\mathrm{d}}_j h , \tag{A.10b}$$

for $h := g^\dagger g$.

A.3.3. Complex equations

The complex equations it holds

$$\left[\mathrm{d}_j, \mathrm{d}_k\right] = 0 \qquad \Leftrightarrow \qquad \left[\mathcal{Y}_j, \mathcal{Y}_k\right] = 0 , \tag{A.11a}$$

$$\left[\mathrm{d}_{\mathcal{Z}}, \mathrm{d}_j\right] = 0 \qquad \Leftrightarrow \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{Y}_j = \left[\mathcal{Y}_j, \mathcal{Z}\right] , \tag{A.11b}$$

where the right-hand-side is understood as acting on $\mathbb{C}^p$- or matrix-valued functions.

For the integrability of $\partial_{\mathcal{A}}$, i.e. $\partial_{\mathcal{A}}^2 = 0$, we need besides (A.11) also $\partial_{\widehat{\Gamma}^P}^2 = 0$ and (3.9) to hold. Fortunately, $\widehat{\Gamma}^P$ is an HYM-instantons and, thus, defines an integrable ∂-operator. Moreover, by construction we restricted to matrix-valued functions $\mathcal{Y}_j$ and $\mathcal{Z}$ that satisfy the equivariance. In summary, the complex equations are the integrability conditions for $\partial_{\mathcal{A}}$.

A.3.4. Real equation

Recall the definition (3.33) of the moment map $\mu(\mathcal{Y}, \mathcal{Z})$. The expression is identical to the action of the operator[11]

$$\Upsilon(\mathcal{Y}, \mathcal{Z}) := 2\left(\left[\bar{\mathrm{d}}_{\mathcal{Z}}, \mathrm{d}_{\mathcal{Z}}\right] + \lambda_n(s)\sum_{j=1}^n \left[\bar{\mathrm{d}}_j, \mathrm{d}_j\right]\right) \tag{A.12}$$

in the usual sense. This operator behaves under complex gauge transformations as follows

$$g^{-1}\left(\Upsilon(\mathcal{Y}^g, \mathcal{Z}^g)\right) g = \Upsilon(\mathcal{Y}, \mathcal{Z}) - 2\left(\mathrm{d}_{\mathcal{Z}}(h^{-1}\bar{\mathrm{d}}_{\mathcal{Z}}h) + \lambda_n(s)\sum_{j=1}^n \mathrm{d}_j(h^{-1}\bar{\mathrm{d}}_j h)\right) . \tag{A.13}$$

A.4. Adaptation of proofs

A.4.1. Differential inequality

Let $\{\kappa_i\}_{i=1,\dots,p}$ be the *positive* eigenvalues (still functions of s) of h on I_ϵ. Define

$$\Phi(h) := \ln\left(\max_{i=1,\dots,p} \kappa_i\right) , \tag{A.14}$$

which is well-defined. The claim is that the inequalities

[11] This object is analogous to $\widehat{F}$ of [22, eq. (1.10)].

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h) \geq -2\left(\|\Upsilon(\mathcal{Y},\mathcal{Z})\| + \|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\|\right) , \tag{A.15a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h^{-1}) \geq -2\left(\|\Upsilon(\mathcal{Y},\mathcal{Z})\| + \|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\|\right) \tag{A.15b}$$

hold in a weak sense.

Proof. Following [22], it is sufficient to consider the case where all eigenvalues of h are distinct for each s. Further, by a unitary gauge transformation one finds in each $\mathrm{GL}(p,\mathbb{C})/\mathrm{U}(p)$-equivalence class an element g (which corresponds to a given h) such that

$$g = \mathrm{diag}(e^{t_1},\ldots,e^{t_p}) \qquad \text{with} \quad t_1(s) > t_2(s) > \ldots > t_p(s) \quad \forall s \in I_\epsilon \, . \tag{A.16}$$

Hence, one obtains $h = \mathrm{diag}(e^{2t_1},\ldots,e^{2t_p})$ and $h^{-1} = \mathrm{diag}(e^{-2t_1},\ldots,e^{-2t_p})$ such that $\Phi(h) = 2t_1$ and $\Phi(h^{-1}) = -2t_p$. Next, we compute

$$\bar{\mathrm{d}}_{\mathcal{Z}} h = \mathrm{diag}(e^{2t_j}\tfrac{\mathrm{d}}{\mathrm{d}s}t_j) - \left[\mathcal{Z}^\dagger, h\right] , \tag{A.17a}$$

$$h^{-1}\bar{\mathrm{d}}_{\mathcal{Z}} h = \mathrm{diag}(\tfrac{\mathrm{d}}{\mathrm{d}s}t_j) + \mathcal{Z}^\dagger - h^{-1}\mathcal{Z}^\dagger h , \tag{A.17b}$$

$$\begin{aligned}\mathrm{d}_{\mathcal{Z}}(h^{-1}\bar{\mathrm{d}}_{\mathcal{Z}} h) &= \mathrm{diag}\left(\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}t_j\right) + \left[\mathcal{Z}, \mathrm{diag}(\frac{\mathrm{d}}{\mathrm{d}s}t_j)\right] \\ &\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{Z}^\dagger - h^{-1}\mathcal{Z}^\dagger h\right) + \left[\mathcal{Z}, \mathcal{Z}^\dagger - h^{-1}\mathcal{Z}^\dagger h\right] .\end{aligned} \tag{A.17c}$$

Now, we consider the diagonal elements

$$\begin{aligned}&\left(\mathrm{d}_{\mathcal{Z}}(h^{-1}\bar{\mathrm{d}}_{\mathcal{Z}} h)\right)_{(a,a)} \\ &\quad = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}t_a + \sum_{b\neq a}|\mathcal{Z}_{ab}|^2\left\{\left(1-e^{2(t_a-t_b)}\right) - \left(1-e^{-2(t_a-t_b)}\right)\right\} ,\end{aligned} \tag{A.18}$$

where we used

$$\left(\left[\mathcal{Z}, \mathrm{diag}(\frac{\mathrm{d}}{\mathrm{d}s}t_j)\right]\right)_{(a,a)} = 0 \quad \text{and} \quad \left(\mathcal{Z}^\dagger - h^{-1}\mathcal{Z}^\dagger h\right)_{(a,a)} = 0 \, . \tag{A.19}$$

Similarly, one derives

$$\left(\mathrm{d}_j(h^{-1}\bar{\mathrm{d}}_j h)\right)_{(a,a)} = \sum_{b\neq a}|(\mathcal{Y}_j)_{ab}|^2\left\{\left(1-e^{2(t_a-t_b)}\right) - \left(1-e^{-2(t_a-t_b)}\right)\right\} . \tag{A.20}$$

Then, one proceeds

$$\begin{aligned}&\left(\Upsilon(\mathcal{Y},\mathcal{Z}) - \Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\right)_{(a,a)} \\ &\quad = \left(\Upsilon(\mathcal{Y},\mathcal{Z}) - g^{-1}\left(\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\right)g\right)_{(a,a)} \\ &\quad = 2\left(\mathrm{d}_{\mathcal{Z}}(h^{-1}\bar{\mathrm{d}}_{\mathcal{Z}} h) + \lambda_n(s)\sum_{j=1}^n \mathrm{d}_j(h^{-1}\bar{\mathrm{d}}_j h)\right)_{(a,a)} \\ &\quad = \frac{\mathrm{d}^2}{\mathrm{d}s^2}t_a + 2\sum_{b\neq a}\left(|\mathcal{Z}_{ab}|^2 + \lambda_n(s)\sum_{j=1}^n|(\mathcal{Y}_j)_{ab}|^2\right)\end{aligned}$$

$$\times\left\{\left(1-e^{2(t_a-t_b)}\right)-\left(1-e^{-2(t_a-t_b)}\right)\right\} \tag{A.21}$$

To get the estimate for $\Phi(h)=2t_1$ take $a=1$ and use $\left\{\left(1-e^{2(t_1-t_b)}\right)-\left(1-e^{-2(t_1-t_b)}\right)\right\}<0$ as $t_1>t_b$ for all $b>1$. Then

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s^2}t_1 &\geq -\left(\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)-\Upsilon(\mathcal{Y},\mathcal{Z})\right)_{(1,1)} \geq -\left(|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)_{(1,1)}|+|\Upsilon(\mathcal{Y},\mathcal{Z})_{(1,1)}|\right)\\ &\geq -\left(\|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\|+\|\Upsilon(\mathcal{Y},\mathcal{Z})\|\right)\\ &\Rightarrow \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h)\geq -2\left(\|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\|+\|\Upsilon(\mathcal{Y},\mathcal{Z})\|\right)\end{aligned} \tag{A.22}$$

Similarly, the estimate for $\Phi(h^{-1})$ is obtained by taking $a=p$ and $\{(1-e^{2(t_p-t_b)})-(1-e^{-2(t_p-t_b)})\}>0$ for all $b<p$. Then

$$\begin{aligned}&\left(\Upsilon(\mathcal{Y},\mathcal{Z})-\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\right)_{(p,p)} \geq \frac{\mathrm{d}^2}{\mathrm{d}s^2}t_p \quad \Rightarrow\\ &\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h^{-1})\geq -2\left(\|\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\|+\|\Upsilon(\mathcal{Y},\mathcal{Z})\|\right)\end{aligned} \tag{A.23}$$

Thus, the claim (A.15) holds. □

A.4.2. Uniqueness

Suppose that $(\mathcal{Y},\mathcal{Z})$ is a solution to the complex equations on I_ϵ. Let us assume that we have two complex gauge transformations g_1 and g_2 such that

(i) $\mu(\mathcal{Y}^{g_1},\mathcal{Z}^{g_1})=0$ and $\mu(\mathcal{Y}^{g_2},\mathcal{Z}^{g_2})=0$ in I_ϵ
(ii) $h_1=g_1^\dagger g_1$ and $h_2=g_2^\dagger g_2$ satisfying $h_1|_{\partial I_\epsilon}=h_2|_{\partial I_\epsilon}$.

Then $h_1=h_2$ in I_ϵ.

Proof. We can suppose $g_2=1$ such that $h_2=1$ in I_ϵ and ∂I_ϵ. Hence, $g\equiv g_1$ and $h|_{\partial I_\epsilon}=1$. Since $\Upsilon(\mathcal{Y},\mathcal{Z})=0$ and $\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)=0$, we have

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h)=2\frac{\mathrm{d}^2}{\mathrm{d}s^2}t_1\geq 0 \text{ in } I_\epsilon\ ,\ t_1|_{\partial I_\epsilon}=0 \quad \text{and}\\ &\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h^{-1})=-2\frac{\mathrm{d}^2}{\mathrm{d}s^2}t_p\geq 0 \text{ in } I_\epsilon\ ,\ t_p|_{\partial I_\epsilon}=0\,.\end{aligned} \tag{A.24}$$

By (weak) convexity, it follows $t_1\leq 0$ in I_ϵ and $t_p\geq 0$ in I_ϵ, but we now arrive at $0\geq t_1>t_2>\ldots>t_p\geq 0$. Hence, $t_j=0$ in I_ϵ and $h=1$ in I_ϵ (modulo unitary transformations). □

A.4.3. Boundedness

Next, we need to show the boundedness of $\mu(\mathcal{Y},\mathcal{Z})$. The only critical term is $\lambda_n(s)$, which diverges for $s\to 0$. However, it is straight forward to derive the pole structure of the gauge transformed operator Υ to be

$$g^{-1}\left(\Upsilon(\mathcal{Y}^g,\mathcal{Z}^g)\right)g\Big|_{\text{pole}}=\Upsilon(\mathcal{Y},\mathcal{Z})\Big|_{\text{pole}}-2\lambda_n\sum_{j=1}^{n}\mathrm{d}_j\left(h^{-1}\bar{\mathrm{d}}_j h\right)$$

$$= 2\lambda_n \sum_{j=1}^{n} \Big[\mathcal{Y}_j, h^{-1}\mathcal{Y}_j^\dagger h\Big]_{s\to 0} . \tag{A.25}$$

But recall that we will consider framed gauge transformation, i.e. $h = 1$ at the boundaries, and $\mathcal{Y}(s = 0)$ are elements of a Cartan subalgebra. Hence, the potential pole vanishes for any gauge transformation once the correct boundary conditions (3.26) are imposed. Thus, $\mu(\mathcal{Y}, \mathcal{Z})$ is bounded.

A.4.4. Limit $\epsilon \to 0$

Finally, we need to show that the limit $\epsilon \to 0$ exists, for which we follow [23,24]. Let $(\mathcal{Y}, \mathcal{Z})$ be *any* solution of the complex equation, then for each $\epsilon > 0$ there exists a unique *complex* gauge transformation g_ϵ such that $(\mathcal{Y}^{g_\epsilon}, \mathcal{Z}^{g_\epsilon})$ satisfies the real equation in I_ϵ. Associate $h_\epsilon = g_\epsilon^\dagger g_\epsilon$.

We start by constructing a solution $(\widehat{\mathcal{Y}}, \widehat{\mathcal{Z}})$ of the complex equations with the properties

$$(\widehat{\mathcal{Y}}_j, \widehat{\mathcal{Z}})(s) = \begin{cases} (\tau_j, 0) & \text{for } s = -\epsilon \\ (\mathcal{T}_j, \mathcal{T}_{n+1}) & \text{for} -\frac{1}{\epsilon} < s < -1 \end{cases} \tag{A.26}$$

where $(\mathcal{T}_j, \mathcal{T}_{n+1})$ correspond to the complex linear combinations of the T_μ of the boundary condition (3.26), i.e. they lie in a Cartan subalgebra of $\mathfrak{su}(n + 1)$. The τ_j are arbitrary points in the complex orbits $\mathcal{O}(\mathcal{T}_j)$, because we know that the boundary values at $s \to 0$ are in gauge orbits of the $\mathcal{T}_j$.

The existence of such a solution follows from the local triviality of the complex equations. Note that this solution is *constant* in $(-\frac{1}{\epsilon}, -1)$ and $\mu(\widehat{\mathcal{Y}}, \widehat{\mathcal{Z}}) = 0$ for $-\frac{1}{\epsilon} < s < -1$.

The claim then is: Starting from $(\widehat{\mathcal{Y}}, \widehat{\mathcal{Z}})$ as above, for each $\epsilon > 0$ there exists a unique gauge transformation g_ϵ such that

(i) $(\widehat{\mathcal{Y}}^{g_\epsilon}, \widehat{\mathcal{Z}}^{g_\epsilon})$ satisfies the real equation everywhere in I_ϵ,
(ii) $(\widehat{\mathcal{Y}}^{g_\epsilon}, \widehat{\mathcal{Z}}^{g_\epsilon})$ has the correct boundary conditions (3.26),
(iii) $g = 1$ at the boundaries and $\widehat{\mathcal{Z}}^{g_\epsilon}$ is Hermitian,
(iv) $\Phi(h_\epsilon)$, $\Phi(h_\epsilon^{-1})$ are uniformly bounded.

Thus, by the uniform bound, one has the existence a C^∞ limit $h_\infty := \lim_{\epsilon\to 0} h_\epsilon$ such that $g_\infty := \sqrt{h_\infty}$ has all desired properties on the negative half-line.

Proof. The existence and the uniqueness of such a g_ϵ follows from the above. Using the differential inequalities (A.15) and the boundedness of μ we derive at

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(h_\epsilon) \geq \begin{cases} -2\|\Upsilon(\widehat{\mathcal{Y}}, \widehat{\mathcal{Z}})\| \geq -2C & \text{for } -1 < s < -\epsilon \\ 0 & \text{for} -\frac{1}{\epsilon} < s < -1. \end{cases} \tag{A.27}$$

Moreover, since $h_\epsilon = 1$ at ∂I_ϵ, the eigenvalues have to vanish, which implies $\Phi(h_\epsilon) = 0 = \Phi(h_\epsilon^{-1})$ at ∂I_ϵ. Consider the bounded, continuous, non-negative function

$$f_\epsilon(s) = \begin{cases} -C(s+1)(s+\epsilon) & \text{for } -1 < s < -\epsilon \\ 0 & \text{for } -\frac{1}{\epsilon} < s < -1 \end{cases} \tag{A.28}$$

$$\text{with} \quad \frac{\mathrm{d}^2}{\mathrm{d}s^2} f_\epsilon = \begin{cases} -2C & \text{for } -1 < s < -\epsilon \\ 0 & \text{for } -\frac{1}{\epsilon} < s < -1 \end{cases} \tag{A.29}$$

in a weak sense. But then, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\Phi(h_\epsilon) - f_\epsilon\right) \geq 0 \text{ in } I_\epsilon \quad \text{and} \quad \Phi(h_\epsilon) - f_\epsilon = 0 \text{ at } \partial I_\epsilon \ . \tag{A.30}$$

By convexity, $\Phi(h_\epsilon) - f_\epsilon \leq 0$ in I_ϵ, which then implies

$$\Phi(h_\epsilon) = 2t_1 \leq \begin{cases} -C(s+1)(s+\epsilon) \leq -Cs(s+1) & \text{for } -1 < s < -\epsilon \\ 0 & \text{for } -\frac{1}{\epsilon} < s < -1. \end{cases} \tag{A.31}$$

Applying the very same reasoning to $\Phi(h_\epsilon^{-1})$, we obtain $\Phi(h^{-1}) - f_\epsilon \leq 0$ in I_ϵ and thus

$$-\Phi(h_\epsilon^{-1}) = 2t_p \geq \begin{cases} Cs(s+1) & \text{for } -1 < s < -\epsilon \\ 0 & \text{for } -\frac{1}{\epsilon} < s < -1. \end{cases} \tag{A.32}$$

In conclusion, the eigenvalues of h_ϵ are uniformly bounded

$$\frac{1}{2} f \geq t_1 > \ldots > t_p \geq -\frac{1}{2} f \qquad \text{for} \quad f(s) = \begin{cases} -Cs(s+1) & \text{for } -1 < s < -\epsilon \\ 0 & \text{for } -\frac{1}{\epsilon} < s < -1 \end{cases} \tag{A.33}$$

independent of ϵ. This uniform bound leads to the existence of the limit $\epsilon \to 0$ of h_ϵ. $\square$

References

[1] S.K. Donaldson, Self-dual connections and the topology of smooth 4-manifolds, Bull. Am. Math. Soc. (N.S.) 8 (1) (1983) 81–83.

[2] R. Rajaraman, Solitons and Instantons. An Introduction to Solitons and Instantons in Quantum Field Theory, North-Holland, Amsterdam, 1982.

[3] N. Manton, P. Sutcliffe, Topological Solitons, Cambridge University Press, Cambridge, 2004.

[4] E.J. Weinberg, Classical Solutions in Quantum Field Theory, Cambridge University Press, Cambridge, 2012.

[5] P. Candelas, G.T. Horowitz, A. Strominger, E. Witten, Vacuum configurations for superstrings, Nucl. Phys. B 258 (1985) 46–74.

[6] M. Grana, Flux compactifications in string theory: a comprehensive review, Phys. Rep. 423 (2006) 91–158, arXiv:hep-th/0509003.

[7] R. Blumenhagen, B. Kors, D. Lüst, S. Stieberger, Four-dimensional string compactifications with D-branes, orientifolds and fluxes, Phys. Rep. 445 (2007) 1–193, arXiv:hep-th/0610327.

[8] J.P. Gauntlett, D. Martelli, J. Sparks, D. Waldram, Sasaki–Einstein metrics on $S^2 \times S^3$, Adv. Theor. Math. Phys. 8 (2004) 711–734, arXiv:hep-th/0403002.

[9] J.P. Gauntlett, D. Martelli, J.F. Sparks, D. Waldram, A new infinite class of Sasaki–Einstein manifolds, Adv. Theor. Math. Phys. 8 (2006) 987–1000, arXiv:hep-th/0403038.

[10] M. Cvetic, H. Lu, D.N. Page, C. Pope, New Einstein–Sasaki spaces in five and higher dimensions, Phys. Rev. Lett. 95 (2005) 071101, arXiv:hep-th/0504225.

[11] M. Cvetic, H. Lu, D.N. Page, C. Pope, New Einstein–Sasaki and Einstein spaces from Kerr–de Sitter, J. High Energy Phys. 0907 (2009) 082, arXiv:hep-th/0505223.

[12] H. Lu, C. Pope, J.F. Vazquez-Poritz, A new construction of Einstein–Sasaki metrics in $D \geq 7$, Phys. Rev. D 75 (2007) 026005, arXiv:hep-th/0512306.

[13] D. Harland, T.A. Ivanova, O. Lechtenfeld, A.D. Popov, Yang–Mills flows on nearly Kähler manifolds and G_2-instantons, Commun. Math. Phys. 300 (2010) 185–204, arXiv:0909.2730.

[14] D. Harland, A.D. Popov, Yang–Mills fields in flux compactifications on homogeneous manifolds with SU(4)-structure, J. High Energy Phys. 1202 (2012) 107, arXiv:1005.2837.

[15] I. Bauer, T.A. Ivanova, O. Lechtenfeld, F. Lubbe, Yang–Mills instantons and dyons on homogeneous G_2-manifolds, J. High Energy Phys. 1010 (2010) 044, arXiv:1006.2388.

[16] A.S. Haupt, T.A. Ivanova, O. Lechtenfeld, A.D. Popov, Chern–Simons flows on Aloff–Wallach spaces and Spin(7)-instantons, Phys. Rev. D 83 (2011) 105028, arXiv:1104.5231.

[17] K.-P. Gemmer, O. Lechtenfeld, C. Nölle, A.D. Popov, Yang–Mills instantons on cones and sine-cones over nearly Kahler manifolds, J. High Energy Phys. 1109 (2011) 103, arXiv:1108.3951.

[18] D. Harland, C. Nölle, Instantons and Killing spinors, J. High Energy Phys. 1203 (2012) 082, arXiv:1109.3552.

[19] T.A. Ivanova, A.D. Popov, Instantons on special holonomy manifolds, Phys. Rev. D 85 (2012) 105012, arXiv:1203.2657.

[20] S. Bunk, T.A. Ivanova, O. Lechtenfeld, A.D. Popov, M. Sperling, Instantons on sine-cones over Sasakian manifolds, Phys. Rev. D 90 (6) (2014) 065028, arXiv:1407.2948.
[21] S. Bunk, O. Lechtenfeld, A.D. Popov, M. Sperling, Instantons on conical half-flat 6-manifolds, J. High Energy Phys. 1501 (2015) 030, arXiv:1409.0030.
[22] S.K. Donaldson, Nahm's equations and the classification of monopoles, Commun. Math. Phys. 96 (3) (1984) 387–407.
[23] P.B. Kronheimer, A Hyper-Kählerian structure on coadjoint orbits of a semisimple complex group, J. Lond. Math. Soc. 42 (1990) 193–208.
[24] P.B. Kronheimer, Instantons and the geometry of the nilpotent variety, J. Differ. Geom. 32 (2) (1990) 473–490.
[25] F.P. Correia, Hermitian Yang–Mills instantons on Calabi–Yau cones, J. High Energy Phys. 0912 (2009) 004, arXiv:0910.1096.
[26] F.P. Correia, Hermitian Yang–Mills instantons on resolutions of Calabi–Yau cones, J. High Energy Phys. 1102 (2011) 054, arXiv:1009.0526.
[27] S. Groot Nibbelink, M. Trapletti, M. Walter, Resolutions of C^n/Z_n orbifolds, their U(1) bundles, and applications to string model building, J. High Energy Phys. 0703 (2007) 035, arXiv:hep-th/0701227.
[28] C.P. Boyer, K. Galicki, Sasakian Geometry, Oxford University Press, Oxford, 2008.
[29] M. Atiyah, R. Bott, The Yang–Mills equations over Riemann surfaces, Philos. Trans. R. Soc. Lond. A 308 (1982) 523–615.
[30] A. Deser, O. Lechtenfeld, A.D. Popov, Sigma-model limit of Yang–Mills instantons in higher dimensions, Nucl. Phys. B 894 (2015) 361–373, arXiv:1412.4258.
[31] S.K. Donaldson, Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles, Proc. Lond. Math. Soc. (3) 50 (1) (1985) 1–26.
[32] K. Uhlenbeck, S.T. Yau, On the existence of hermitian-Yang–Mills connections in stable vector bundles, Commun. Pure Appl. Math. 39 (1986) S257–S293.
[33] S.K. Donaldson, Infinite determinants, stable bundles and curvature, Duke Math. J. 54 (1987) 231–247.
[34] R.P. Thomas, Notes on GIT and symplectic reduction for bundles and varieties, in: Surveys in Differential Geometry, in: Surv. Differ. Geom., vol. 10, Int. Press, Somerville, MA, 2006, pp. 221–273.
[35] S. Donaldson, Boundary value problems for Yang–Mills fields, J. Geom. Phys. 8 (1992) 89–122.
[36] A.D. Popov, R.J. Szabo, Double quiver gauge theory and nearly Kähler flux compactifications, J. High Energy Phys. 1202 (2012) 033, arXiv:1009.3208.
[37] O. Lechtenfeld, Instantons and Chern–Simons flows in 6, 7 and 8 dimensions, Phys. Part. Nucl. 43 (2012) 569–576, arXiv:1201.6390.
[38] M. Wolf, Contact manifolds, contact instantons, and twistor geometry, J. High Energy Phys. 07 (2012) 074, arXiv:1203.3423.
[39] F. Xu, Geometry of $SU(3)$ manifolds, PhD thesis, 2008.
[40] N. Hitchin, Hyper-Kähler manifolds, Astérisque 206 (1992) 137.
[41] O. Lechtenfeld, A.D. Popov, R.J. Szabo, Sasakian quiver gauge theories and instantons on Calabi–Yau cones, arXiv:1412.4409.
[42] O. Lechtenfeld, A.D. Popov, M. Sperling, R.J. Szabo, Sasakian quiver gauge theories and instantons on cones over lens 5-spaces, Nucl. Phys. B 899 (2015) 848–903.