jet INTL JOURNAL OF ELECTRONICS AND TELECOMMUNICATIONS, 2010, VOL. 56, NO. 4, PP. 357–366
Manuscript received September 24, 2009; revised October 2010. DOI: 10.2478/v10177-010-0047-7

# On Noise Analysis of Oscillators Based on Statistical Mechanics

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***Abstract*—In this paper a new approach of thermal noise analysis of electronic oscillators is presented. Although nonlinear electronic oscillators are one of the most essential subcircuits in electronic systems typical design concepts for these oscillators are based on ideas of linear circuits. Because the functionality of oscillators depends on nonlinearities, advanced design methods are developed where nonlinearities are an integral part. Since low voltage oscillator concepts have to be developed in modern IC technologies there is a need to include at least thermal noise aspects into the design flow. For this reason we developed new physical descriptions of thermal noise in electronic oscillators where we use ideas from nonequilibrium statistical mechanics as well as the Langevin approach. We illustrate our concepts by some examples.**

***Keywords*—Nonlinear oscillators, thermal noise, statistical mechanics.**

## I. Introduction

ANALOGUE electronic oscillators are essential components of communication and computer systems such that systematic design methods are needed in order to adapt the circuit parameters to the prescribed specifications using some circuit simulation cycles. Especially if we use microelectronic technologies in GHz era, high costs can arise. In oscillator circuits we have to expect two difficulties:

1) The circuit functionality is nonlinear that is nonlinear modelling concepts are needed where additional parasitic perturbations occur and frequency coupling is a characteristic phenomenon [1].
2) Different kinds of noise (incl. thermal noise) appear but in contrast to linear circuits in nonlinear circuits the stochastic moments of the probability distribution are coupled [2].

Furthermore nonlinear oscillators are dissipative systems with limit cycles and therefore these circuits are not near an equilibrium point but far from an equilibrium and bifurcations under the influence of noise can occur [3]. Unfortunately almost no nonlinear differential equations exist where analytical solutions can be used such that perturbation methods have to be applied. In section 4 we consider a whole class of equations where the limit cycle can be formulated explicitly.

Noise in self-sustained oscillators will spread the $\delta$-function spectrum of an ideal oscillator into a finite width, which is also known as Lorentzian spectrum. Self-sustained oscillators differ from ordinary nonlinear systems since the nonlinearities cannot be regarded to be small and therefore neglected or linearized. With a classical, quasilinear treatment of noisy, self-sustained oscillators the spectrum of the oscillator would consist of a $\delta$-function plus a background, which is not satisfactory. Despite this fact, many common oscillator design techniques are based on linear approaches.

One of the first articles about noise in oscillators was written by Leeson [4] in the year 1966. After this approach the line broadening of the spectrum is formed by a linear filter arrangement which yields a spectrum in a Lorentzian shape. One year later Lax [5] showed a more elegant derivation, after which the line broadening can be obtained by introducing the thermal noise in the nonlinear dynamics of the oscillator. Therefore the Lorentzian shape is reasoned by the thermal noise itself and doesn't have to be reproduced by a linear filter.

It should be mentioned that several concepts of noise analysis of oscillators are developed but these authors consider oscillators mainly from a mathematical point of view and use concepts from the mathematical theory of stochastic dynamical systems; see e. g. Kärtner [6], Demir et al. [7] and Hajimiri et al. [8]. A more recent approach in the same direction can be found in Hong et al. [9] although the title of the paper suggests a physical approach. In contrast to these papers we developed a physical based concept of thermal noise analysis for oscillators. Although we already published a new approach for thermal noise analysis of nonlinear circuit (see e. g. Weiss, Mathis [10]) it is only suitable for nonlinear circuits near the thermal equilibrium. It is well-known that circuits and systems where a limit cycle arises are in a state far from equilibrium. Therefore we present a new approach for thermal noise analysis for circuit in this regime.

## II. Early Concepts of Brownian Movement and Noise

Since the early days of electronic circuits its noise properties became a main subject of research in physics and engineering. Probably Schottky [11] was the first who considered noise aspects of electronic devices (resistors and tubes) in order to find out limits of signal transmission in tube amplifiers. Following Einstein's modeling of the Brownian motion Schottky identified the quantized matter and especially the charge into electrons as fundamental reason behind the fluctuations of voltages and currents in electronic arrangements. In his first studies Schottky was mainly interested in the so-called shot-noise, however, he observed also other noise aspects including the thermal noise. Although Schottky suggested an interesting mathematical technique for noise analysis he

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failed because of a mistake in his calculations. It was noticed and corrected by Johnson [12]. Additionally to the shot-noise effect Schottky and others consider [13] also the so-called flicker noise effect - 1/f-noise - but thermal noise was not discussed until Nyquist [14] and Johnson [15], [16] published some fundamental results with respect to this kind of noise. Nyquist presented a so-called "Gedanken" experiment where he derived his well-known noise formula for linear Ohmian resistors. For his consideration Nyquist used ideas from equilibrium thermodynamics where the arrangement consisting of resistors and a transmission line was constructed by means of the measurement arrangement of Johnson.

Even though these discussions of noise in electronic circuits applied ideas from Brownian motion both, the technical and the physical discipline, were studied in a separate manner for a long time because of the different motivations. Maybe a remark of Einstein was the reason that thermal fluctuations of electrical currents became not essential subject in physics at that time. In a paper [17] from 1906 he mentioned this effect for the first time but he considered these fluctuations as "uncontrollable" and consequently further studies about this subject as "useless". Later on G. de Haas-Lorentz the daughter of H. A. Lorentz showed in her dissertation [18] that a modification of a technique of Einstein can be used to analyze the noise behavior of linear circuits consisting of resistors, capacitors and inductors whereby she was able to derive a certain case of Nyquists's formula; see also Matare [19]. Unfortunately the results of de Haas-Lorentz became widely unknown so her approach was reinvented by Nyquist more than ten years later. At the same time Kolmogorov [20] developed a new mathematical concept of the stochastic processes where he used measure theory and previous research of Markov, Khintchine, Wiener, Levy and others. Later on Khintchine [21] and Wiener [22] presented a correlation theory of stationary stochastic processes which became the basis of the noise theory of linear time-invariant systems.

In contrast to the linear Brownian movement and noise processes in linear time-invariant systems first discussions about nonlinear cases were presented by Kramers [23] and later on by MacDonald [24], Stratonovich [25] and van Kampen [26]. However the problem of noise analysis in nonlinear systems is much more involved than in linear cases. From a physical point of view we have to distinguish external and internal noise effects. Whereas external noise can be analyzed by using standard mathematical techniques that is transformation of stochastic processes (see e.g. Stratonovich [25]), internal noise has to be studied by special mathematical techniques with a useful physical interpretation. Even in the case of weak nonlinear systems close to a thermal equilibrium a corresponding theory was completed at the end of the last century by van Kampen [27], Stratonovich [28] and many others; it is denoted as nonlinear non-equilibrium statistical thermodynamics. Based on this theory Weiss and Mathis [29], [30] developed a noise theory for nonlinear reciprocal electrical circuits near a thermal equilibrium. Unfortunately there are many nonlinear systems do not work near to a thermal equilibrium state but far-from-equilibrium. For example, electrical systems driven by energy sources as well as nonlinear oscillations or - in mathematical terms - limit cycles are working in non-equilibrium states. These dynamics have to be interpreted as so-called dissipative structures in the sense of non-equilibrium thermodynamics which became also a central subject of Haken's synergetics [31]. One of the most interesting properties of electronic oscillators is phase noise and the close related line-broadening effect in the frequency domain where many applications in RF CMOS circuit design exist; a recent discussion and further references as well as measurement results can be found e. g. in Magierowski and Zukotynski [28].

In the following sections we will discuss the fundamental aspects of nonlinear non-equilibrium statistical thermodynamics near an equilibrium state and in a far-from-equilibrium state. We emphasize such methods that can be applied to nonlinear electrical and electronic circuits.

## III. Noise in Nonlinear Circuits Near a Thermal Equilibrium State

### A. The Langevin and the Fokker-Planck Equation

In statistical mechanics Boltzmann introduced a (scalar) density function depending of positions and momentums and derived a corresponding equation of motion. The deterministic behaviour of a system can be described by a set of differential equations or state space equations

$$\frac{dx}{dt} = F(x), \tag{1}$$

where $F\colon R^n \to R^n$. If we are interested in the dynamics of a suitable class of density functions $f\colon R^n \to R$ a corresponding evolution equation can be formulated using an associated Frobenius-Perron-Operator $P^t$.

$$f(x,t) = P^t\{f(x)\}. \tag{2}$$

$P^t$ is the solution operator of the generalized Liouville equation

$$\frac{\partial f}{\partial t} = -div(fF) = -\sum_{i=1}^{n} \frac{\partial (fF_i)}{\partial x_i}. \tag{3}$$

The Langevin approach of stochastic systems starts with a deterministic description and an additional stochastic process $\xi$.

$$\frac{dx}{dt} = F(x) + \sigma(x)\xi, \tag{4}$$

where the coefficient $\sigma(x)$ characterizes the coupling of the deterministic system and the noise sources. The first term on the right should be interpreted as dissipation whereas the second term corresponds to fluctuations.

Using the concept of stochastic differential equations $\xi$ has to be a generalized white noise process since the solution $x$ should be a Gaussian stationary stochastic process. However in order to solve these equations a more generalized concept of integration is needed. Essentially there are two concepts of stochastic integration which are due to Ito and Stratonovich, respectively, and associated types of stochastic differential equations

$$dx = F(x)dt + \sigma(x)dw, \tag{5}$$

where $w$ is the so-called Wiener process. Both concepts are mathematically equivalent to a partial differential equation for the probability density $f$ of Fokker-Planck type which generalizes in some sense the concept of the generalized Liouville equation (3) (see Arnold [32] section 4.2)

$$\frac{\partial f}{\partial t} = -\sum_{i=1}^{n} \frac{\partial (f F_i)}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial^2 (\sigma^2 f)}{\partial x_i \partial x_j}. \tag{6}$$

In the case of linear stochastic differential equations - the original subject of Langevin - there is no difference between Ito's and Stratonovich's type; see e. g. van Kampen [33]. Unfortunately stochastic differential equations (of Ito or Stratonovich type) are consistent only from a mathematical but not from a physical point of view if we consider nonlinear Langevin equations. The reason is that each type corresponds to a certain interpretation rule; otherwise its meaning is not well defined. It is interesting to see that for nonlinear Langevin equations in contrast to linear ones the deterministic equation (without noise) does not correspond to the averaged equation (see van Kampen's paper [26] for further details)

$$\left\langle \frac{dx}{dt} \right\rangle = \frac{d\langle x \rangle}{dt} = \langle F(x) \rangle + \langle \sigma(x)\xi \rangle. \tag{7}$$

Even if $\sigma(x)$ is constant we note that the function $F(x)$ and the statistical average operator $\langle . \rangle$ do not commute in every case. Only if $\langle F(x) \rangle = F(\langle x \rangle)$ is valid, that is the linear case, the averaged equation for the first moment $\langle x \rangle$ of $x$ is structural identical with the deterministic equation $\dot{x} = F(x)$. Using an argument from the perturbation theory we find out that there is coupling of the first moment of the stochastic process with its higher moments. Therefore it is not clear why the dissipation term should be identical to the vector field of the deterministic equation; see van Kampen [26], [34]. In order to obtain a sound description of physical systems additional considerations are needed. In the next section we will discuss some ideas in this direction with respect to nonlinear and noisy electronic circuits.

### *B. The Stratonovich Approach and Nonlinear Reciprocal Circuits*

It was discussed in the previous section that in the nonlinear Langevin approach deterministic differential equations are extended by a stochastic part with an additive white noise process. Although the stochastic part can be cancelled if the coupling coefficient $\sigma(x)$ between the deterministic system and the white noise process will be set to zero, the equivalence of the deterministic equation and the differential equation for the first moment of the describing quantity get lost. The reason is simply that in nonlinear stochastic systems a coupling of moments occurs which is similar to the coupling of frequencies in nonlinear deterministic systems (e.g. distortion in almost nonlinear amplifiers or nonlinear oscillators). Therefore the mathematical procedure of the Langevin approach for nonlinear deterministic systems can be justified but there are problems with its physical interpretations. Following van Kampen [27] and Stratonovich [28] a physical reasonable approach for a large class of nonlinear noisy electrical circuits was presented by Weiss and Mathis [29], [30], [35]. In the following we will give a short survey of the main ideas of this approach.

Since in the nonlinear extension of the Langevin approach serious problems with its physical interpretation occur one must describe the evolution of the entire system as a stochastic process (see van Kampen [27], p. 184). If it is reasonable to associate the Markov property to the describing variable $x(t)$ then the evolution equation is a so-called master equation having the general form

$$\frac{\partial P(x,t)}{\partial t} = \int \big(W(x|x')P(x',t) - W(x'|x)P(x,t)\big)\, dx', \tag{8}$$

where $P(x,t|x_0,t_0)$ is the transition probability between $t_0$ and $t$, while $W(x|x')\Delta t$ is so small that $P$ does not vary too much, but large enough for the Markov property (see van Kampen [27], p. 184). For solving the integro-differential equation – equivalent with the so-called Chapman-Kolmogorov equation – several approaches are available which result in approximate solutions near the thermodynamic equilibrium. For rather complex systems it is suitable to use Stratonovich's approach that solved the master equation with the so-called Kramers-Moyal expansion and derived the following equation

$$\frac{\partial P(x,t)}{\partial t} = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{\alpha_1 \ldots \alpha_m = 1}^{r} \frac{\partial^m}{\partial x_{\alpha_1} \ldots \partial x_{\alpha_m}} \big[K_{\alpha_1 \ldots \alpha_m}(x) P(x,t)\big]. \tag{9}$$

In order to apply this equation to a noisy physical system the coefficients $K_{\alpha_1 \ldots \alpha_m}(x)$ have to be determined as functions of the deterministic system parameters. Using ideas of Stratonovich a corresponding procedure for nonlinear noisy electrical circuits is presented by Weiss and Mathis [29], [30], [35]. In the following the main steps are described.

It is known from equilibrium thermodynamics that the m-fold moments are proportional to $kT$ where $k$ is Boltzmann's constant and $T$ is the absolute temperature. Stratonovich assumed that the same must hold for (conditional) non-equilibrium moments (in a "certain neighbourhood" of equilibrium) and therefore for the unknown coefficients $K_{\alpha_1 \ldots \alpha_m} \infty (kT)^{m-1}$. Hence Stratonovich used an expansion with respect to the small parameter $kT$. Since the equilibrium case must be enclosed in the non-equilibrium theory, the equilibrium distribution (Gibbs distribution)

$$P_{eq}(x) = C e^{-\Psi(x)/kT}$$

must be a stationary solution of the master equation (and its Kramers-Moyal expansion) where $\Psi$ denotes Helmholtz free energy. Near the equilibrium a similar relation between $P(x)$ and $\Psi$ should be valid. Following Stratonovich we choose

$$P(x) = C e^{-(\Psi(x) - yx)/kT}, \tag{10}$$

where $y = \partial \Psi / \partial x$ denotes the so-called conjugate thermodynamic forces.

The deterministic equations impose additional constraints on the coefficients $K_{\alpha_1 \ldots \alpha_m}$. Finally the Onsager-Casimir reciprocity relations of linear non-equilibrium thermodynamics (see e.g. Casimir [36]) can be used to reduce the number of unknown coefficients. These relations are a consequence of the time reversal symmetry on the microscopic level.

In a series of papers Weiss and Mathis (see e.g. also the monograph of Weiss [35]) adapted Stratonovich's approach to nonlinear noisy electrical circuits. They showed that the so-called topologically complete electrical circuits represent the most comprehensive class of circuits that can be described by Stratonovich's approach. This class of networks was characterized mathematically for the first time by Brayton and Moser [37]

$$L_\rho(i_\rho)\frac{di_\rho}{dt} = \frac{\partial B(i,u)}{\partial i_\rho} \quad (= 1,\ldots,r),$$

$$C_\sigma(i_\sigma)\frac{du_\sigma}{dt} = \frac{\partial B(i,u)}{\partial u_\sigma} \quad (\sigma = 1,\ldots,s), \tag{11}$$

where the so-called (Brayton-Moser) mixed potential $B(i,u) = \Sigma(i) - \Pi(u) + (i, \Gamma u)$ is related to the dissipative parts $\Sigma$ (current potential) and $P$ (voltage potential) as well as $\Gamma$ (interconnection) of a circuit.

If currents $i_\rho$ through inductors and voltages $u_\sigma$ across the capacitors of topologically complete electrical circuits are used as state space variables $x_\alpha$ of the Stratonovich approach different approximations of the master equation with Kramers-Moyal expansion can be derived for circuits of this class. As free energy $\Psi(x)$ the energy relation for all $r$ inductors and $s$ capacitors

$$\Psi(i,u) = \frac{1}{2}\sum_{\rho=1}^{r} L_\rho(i_\rho) i_\rho^2 + \frac{1}{2}\sum_{\sigma=r+1}^{r+s} C_\sigma(u_\sigma) u_\sigma^2 \tag{12}$$

is used. In the case of linear noisy electrical circuits a first order Fokker-Planck type equation for the probability density $P(i,u)$ has been derived by Weiss and Mathis [30]

$$\frac{\partial P(i,u)}{\partial t} = \Lambda_1 P(i,u) \tag{13}$$

where

$$\Lambda_1 = -\sum_{\rho_1,\rho_2} \frac{\sum_{\rho_1,\rho_2}(0)}{L_{\rho_1}} i_{\rho_2}\frac{\partial}{\partial i_{\rho_1}} - \sum_{\rho_1,\rho_2} \frac{\gamma_{\rho_1,\sigma_2-r}}{L_{\rho_1}} u_{\sigma_2}\frac{\partial}{\partial i_{\rho_1}} + $$
$$+ \sum_{\rho_1,\rho_2} \frac{\gamma_{\rho_2,\sigma_1-r}}{C_{\sigma_1}} i_{\rho_2}\frac{\partial}{\partial u_{\sigma_1}} - \sum_{\rho_1,\rho_2} \frac{\Pi_{\sigma_1\sigma_2}(0)}{C_{\sigma_1}} u_{\sigma_2}\frac{\partial}{\partial u_{\sigma_1}} -$$
$$-kT\sum_{\rho_1,\rho_2} \frac{\sum_{\rho_1,\rho_2}(0)}{L_{\rho_1}L_{\rho_2}}\frac{\partial^2}{\partial i_{\rho_1}\partial i_{\rho_2}} - kT\sum_{\rho_1,\rho_2} \frac{\Pi_{\sigma_1,\sigma_2}(0)}{C_{\rho_1}C_{\rho_2}}\frac{\partial^2}{\partial u_{\sigma_1}\partial u_{\sigma_2}} \tag{14}$$

and $\sum_{\rho_1,\rho_2} \equiv \partial^2\Sigma/\partial i_{\rho_1}\partial i_{\rho_2}$, etc. It can be shown by inspection that

$$P_{eq}(i,u) = Ce^{-\Psi(i,u)/kT} \tag{15}$$

is a stationary solution of the first order equation, where $\Psi(i,u)$ is the free energy of an electrical circuit. Furthermore this is a partial differential equation for the probability density $P(i,u)$ of a Fokker-Planck type and therefore a set of mathematical equivalent stochastic differential equations (with Stratonovich's interpretation rule) exists ($\rho = 1,\ldots,r$, $\sigma = 1,\ldots,s$)

$$di_\rho = \Big(\sum_{\rho_2} \frac{\sum_{\rho_1,\rho_2}(0)}{L_\rho} i_{\rho_2} + \sum_{\sigma} \frac{\gamma_{\rho,\sigma-r}}{L_\rho} u_\sigma\Big)dt +$$
$$+ \sum_{k=r+s+t}^{r+s+t} \pm\sqrt{\frac{2kTR_k}{L_\rho^2}}dw_k \tag{16}$$

$$du_\sigma = \Big(\sum_{\sigma_2} \frac{\Pi_{\sigma_1,\sigma_2}(0)}{C_\sigma} u_{\sigma_2} - \sum_{\rho} \frac{\gamma_{\rho,\sigma-r}}{C_\sigma} i_\rho\Big)dt +$$
$$+ \sum_{l=r+s+t+1}^{r+s+t+u} \pm\sqrt{\frac{2kTG_l}{C_\sigma^2}}dw_l \tag{17}$$

These relationships correspond to multidimensional Nyquist formulas for linear noisy electrical circuits. If we consider nonlinear noisy circuits in the linear-quadratic series approximation of their nonlinearities, compared to the first order equation an additional term occurs in the evolution equation of the probability density

$$\frac{\partial P(i,u)}{\partial t} = \Lambda_1 P(i,u) + \Lambda_2 P(i,u) \tag{18}$$

where $\Lambda_2$ is a third order differential operator; see Weiss and Mathis [2]. Since the resulting partial differential equation for the probability density function is more general than a Fokker-Planck equation a corresponding stochastic differential equation for the state space variables $i$ and $u$ not exists and the Gaussian white noise source model approach fails. But even if no third order derivatives occur just like in circuits without inductors additional deterministic (drift) terms ($dt$-part of stochastic differential equations) arise. If these terms are omitted we get classical nonlinear Fokker-Planck equations but then the Brillouin paradoxon can be constructed. This means that the description does not satisfy the second law of thermodynamics which is acceptable at most as approximation under certain conditions (see also Wyatt and Coram [38]) It was emphasized by Weiss and Mathis that their noise theory for nonlinear electrical circuits can be used to explain and to quantify the restrictions of the classical noise source picture. Applying this framework Weiss and Mathis [39] were able to show that the thermal noise spectra of most of all semiconductor devices can be reconstructed by using its nonlinear characteristics. The known thermal noise spectra are derived by microscopic balance calculations; e. g. van der Ziel [40].

## IV. Noise in Nonlinear Systems Far-From-Equilibrium

### A. Oscillatory Circuits and Canonical Dissipative Systems

In section 1 it was already discussed that some oscillations of a nonlinear system are related to so-called limit cycles. Such solutions of nonlinear autonomous differential equations

$$\dot{x} = f(x), \quad f\colon R^n \to R^n \tag{19}$$

exist if periodic solutions $x_L(t)$ are isolated where each solution $x(t)$ in the "neighbourhood" of $x_L(t)$ converge to it asymptotically. Unfortunately it is not easy to find out whether a differential equation possesses limit cycle solutions. Only for $n = 2$ rather simple criteria exist but for $n > 2$ a bifurcation approach has to be used Jordan and Smith [41].

If we would like to study nonlinear differential equations which possess a limit cycle under the influence of noise, the situation is even more complicate. Since in linear dynamical systems the so-called Langevin approach is successful where a white noise term is added and stochastic differential equations (SDE) arise, the situation is not so simple in nonlinear cases. It is already known since around 1960 (see e.g. van Kampen [26]) that an additive white noise term leads to physical inconsistencies in nonlinear dynamical systems. But even if we ignore this problem, methods from statistical mechanics, that is especially the ensemble approach, cannot be applied because a Hamilton description is needed. Since limit cycles in nonlinear systems can be interpreted as special cases of so-called dissipative structures, a Hamilton description where the energy is preserved cannot exist. However there is a certain class of nonlinear differential equation which can be described as generalized Hamilton systems; this class of systems is called canonical-dissipative systems (CD systems). Although these systems were developed in physics many years ago, they are unknown to many researchers and therefore we give a short overview about this concept.

The CD systems are based on the classical Hamilton description for energy preserving systems

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{20}$$

where $H$ is the Hamiltonian function depending on the state space coordinates $q$ and $p$ and the partial differential operators correspond to gradient operators. The energy of such a system is preserved and $H(q(t), p(t)) = E$ defines an energy surface. The energy $E$ is fixed by the prescribed initial conditions $q(t_0)$ and $p(t_0)$ in the initial time $t_0$ and corresponding solutions of the above Hamilton equations which define trajectories on the energy surface.

The canonical-dissipative systems are extended Hamiltonian systems since a certain dissipative term is added to the differential equation of $p$, that is, we have (see Ebeling, Sokolov [42])

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} - g(H)\frac{\partial H}{\partial p}, \tag{21}$$

where the "dissipation function" $g$ depends only on the Hamiltonian $H$. In general the energy of a CD system does not conserve the energy because we have:

$$\frac{dH}{dt} = -g(H)\left\|\frac{\partial H}{\partial p}\right\|^2. \tag{22}$$

Only if we consider a trajectory $(q(t), p(t))$ with initial conditions $(q(t_0), p(t_0))$ where $E_0 = H(q(t), p(t))$ is a zero of $g$ the energy is preserved. Discussing $dH/dt$ the energy of the system increases for trajectories where $g < 0$ and decreases if $g > 0$, that is, we have dissipation in the system. Obviously the trajectory leading to $g(E_0) = 0$ is a limit cycle of the CD system. On the other hand this trajectory can be expressed by means of the corresponding level function $\tilde{g}(q, p) = g(H(q, p)) = 0$. A nice example of a CD system is the Rayleigh-van Pol equation

$$\ddot{x} + (x^2 + \dot{x}^2 - 1)\dot{x} + x = 0. \tag{23}$$

This equation can easily be derived from the Hamiltonian function $H$ if we choose $H(q, p) = (q^2 + p^2)/2$ and $g(H) = 2(H - 1/2)$ where $q \equiv x$ and $p \equiv \dot{x}$. The trajectory in state space of the limit cycle is a circle $\tilde{g}(q, p) = q^2 + p^2$ where $q(t) = \cos t$ and $p(t) = \sin t$ parameterize this trajectory.

The concept of CD systems together with some applications in the dynamics of swarms are considered by e. g. Ebeling and Sokolov [42] where also a more general concept of CD systems is presented using other invariants (e. g. total momentum and total angular momentum) of mechanical systems. For our purposes we restrict us to CD systems where only the energy is considered.

We emphasize that the above mentioned Rayleigh-van der Pol equation and some further extensions of this equations and its electrical realizations are discussed by Philipow and Büntig [43]. Although these equations are very nice toys for fundamental studies because at least the limit cycle can be given explicitly the corresponding oscillator circuits were not used in electronic applications until now. However there are mathematical techniques to derive an approximate CD system for an oscillatory system. E. g. with the so-called phase-averaging technique of Klimontovich [44] a corresponding CD system for the van der Pol equation can be derived.

### B. Stochastic Canonical Dissipative Systems

We have shown in the last section that CD systems are very useful for fundamental studies of nonlinear oscillators. Also the stochastic variant of CD systems shares this advantage. Since this class of systems were extended from Hamiltonian systems it is possible to generalize concepts from statistical mechanics for the equilibrium to far-from-the-equilibrium systems. Actually it was shown that the so-called micro-canonical ensemble theory – construction of a corresponding probability density – can be generalized to these systems. The reason behind is that the special of dissipation drive the system to certain subspaces of the energy surface. In many cases the system is ergodic on this surface. Then a non-equilibrium ensemble on a slightly extended energy shell can be defined. However there are fundamental differences between equilibrium and non-equilibrium cases. Most of the typical properties of an equilibrium ensemble are related to the energy which comes from the thermal fluctuation. Especially the mean energy is proportional to the temperature $T$ – characterize the energy in the thermal bath – and also the mean quadratic derivation depends on $T$ and is proportional to $T^2$. In contrast to that, the energy of the nonlinear excitations and the noise energy are decoupled in non-equilibrium, that is, the mean energy is proportional to the properties of the energy source which is nearly independent from the thermal noise level, which is denoted by $D$ in non-equilibrium. As a result the equilibrium canonical distribution function is not compatible to these properties but a distribution similar to the

Gaussian distribution can be constructed; further discussion can be found by Ebeling and Sokolov [42].

Alternatively to the concepts from statistical mechanics we can use a stochastic generalization of the CD systems. In addition to the dissipative term a white noise term, where the coefficient depends only on $H$, is added. Therefore the stochastic CD systems are described by the following Langevin equations (or in mathematical terms a stochastic differential equation (SDE))

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} - g(H)\frac{\partial H}{\partial p} + \sqrt{D(H)}\xi(t) \quad (24)$$

It turns out that the restriction of the coefficients $g$ and $D$ leads also to simplifications. It is known that to each Langevin equation a corresponding Fokker-Planck equation for the probability distribution exists. For the stochastic CD systems we have

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial H}{\partial p_i}\frac{\partial \rho}{\partial q_i} - \sum_i \frac{\partial H}{\partial q_i}\frac{\partial \rho}{\partial p_i} = \\ = \sum_i \frac{\partial}{\partial p_i}\Big[g(H)\frac{\partial H}{\partial p_i}\rho + D(H)\frac{\partial \rho}{\partial p_i}\Big] \quad (25)$$

The stationary distribution $\rho_0$ can be derived exactly (see Ebeling and Sokolov [42])

$$\rho_0(q_i, p_i) = Q^{-1} \exp\Big(-\int_0^H \frac{g(\hat{H})}{D(\hat{H})} d\hat{H}\Big) \quad (26)$$

where $Q$ is the normalizing constant. In the case of a affine function $g(H) = 2(H - E_0)$ (with $E_0 = 1/2$) of the Rayleigh-van der Pol equation we have

$$\rho_0(q_i, p_i) = Q^{-1} \exp\Big(\frac{2H(1-H)}{2D}\Big) \quad (27)$$

where $E_0$ is a property of the energy source. In Fig. 1 a probability density of a 2D system with a limit cycle is shown.

With Ebeling [45] we have to emphasize that CD systems are a class of models, which in reality, strictly according to their definition, practically do not exist. However there are many complex systems and among them systems of central importance which have several properties in common with CD systems as the support of the dynamics with free energy from internal or external reservoirs which drives the system to a state of non-equilibrium in an energy shell different from the state of equilibrium.

Because of the fact that in CD systems the energy of the nonlinear excitations and the noise energy are decoupled in non-equilibrium, a decoupled treatment of amplitude- and phase fluctuations is possible. Therefore the orthogonal treatment of amplitude- and phase fluctuations, which in general cases must be realized by the averaging method by Stratonovich [25] is physically justified for CD systems.

Finally we would like to add some comments about detailed balance in nonlinear oscillatory systems. It was emphasized in section 2.2 that this property is crucial for developing Stratonovich's noise theory of nonlinear systems near the equilibrium. By means of the assumption of detailed balance which is equivalent to reciprocity additional constraints for the coefficients of the Fokker-Planck equation arise. Since oscillatory circuits are not reciprocal systems Stratonovich's approach is not available for this class of systems. However it can be shown that there is a certain class of nonlinear stochastic dynamical systems where detailed balance is fulfilled; see Langley [46]. Moreover San Miguel and Chaturvedi [47] showed that the true implication of detailed balance is not on the existence of a limit-cycle but rather on its physical character. We find that detailed balance implies that the limit cycle has a reversible or conservative character which corresponds to the case limit cycles in CD systems. If detailed balance is absent we may classify the limit cycle as irreversible or dissipative.

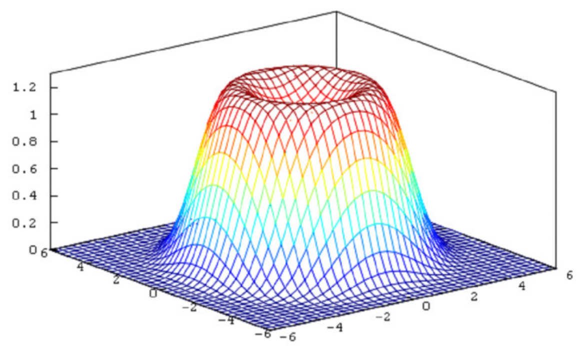


Fig. 1. 2D probability distribution $\rho_0$.

## V. Noise Analysis of Nonlinear Oscillators

Now we discuss the line broadening of nonlinear oscillators disturbed by thermal noise sources described by a Langevin type equation. We concentrate primarily to a specific type of nonlinear oscillators such as the van der Pol oscillator, which is discussed in a variety of works [1], [5], [48]. Self-sustained oscillators differ from ordinary nonlinear systems since the nonlinearities cannot regarded to be small and therefore neglected or linearized. With a classical, quasilinear treatment of noisy, self-sustained oscillators the spectrum of the oscillator would consist of a $\delta$-function plus a background. This is not satisfactory for our purposes. We anticipate that the noise will spread the $\delta$-function spectrum of an ideal oscillator into a finite width, which is also known as Lorentzian spectrum [5]. The reason for the ability to talk about signal plus noise in ordinary nonlinear is that these systems are stable. The stability of the amplitude of a noisy oscillator can be explained by the back-drifting force of the limit cycle. The phase fluctuations instead cannot be regarded to be stable, because there is no cost of energy to pass from one transient solution of the SDE to another.

### A. Evolution of Limit Cycles in a Noisy van der Pol Oscillator

As mentioned above we consider the noisy van der Pol equation from a Langevin point of view such that corresponding

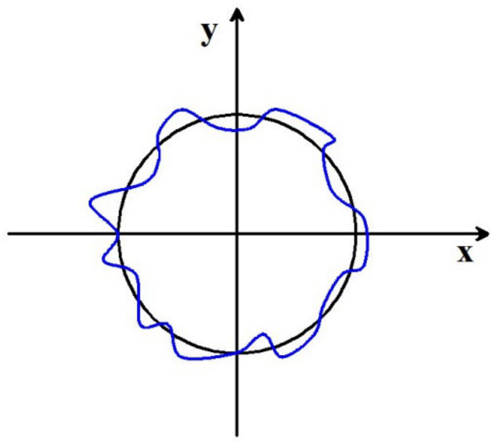


Fig. 2. Phase plane representation of several solutions of DEQN.

SDE has the form

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + \omega_0^2 x = K\xi(t), \tag{28}$$

where $\xi(t)$ is an additive, white gaussian noise connected with the damping factor through the dissipation-fluctuation theorem. To study the evolution of the limit cycle it is useful to look at a van der Pol type equation below

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} + \omega_0^2 x = K\xi(t). \tag{29}$$

The equivalent description with a first order system is

$$\dot{x} = y \tag{30}$$

$$\dot{y} = -\omega_0^2 x - (\alpha + \beta x^2)y + K\xi(t). \tag{31}$$

We assume the nonlinearity of the oscillator to be small, so that the solution of the deterministic differential equation (29) without stochastic process $\xi(t)$ can be modelled as a small perturbation of the orbital linearized solution (see Fig. 2).

Therefore a description of nonlinear systems with limit cycles in polar-coordinates is meaningful.

$$x = R\cos(\omega_0 t + \phi) = R\cos(\theta) \tag{32}$$

$$y = -\omega_0 R\sin(\omega_0 t + \phi) = -\omega_0 R\sin(\theta) \tag{33}$$

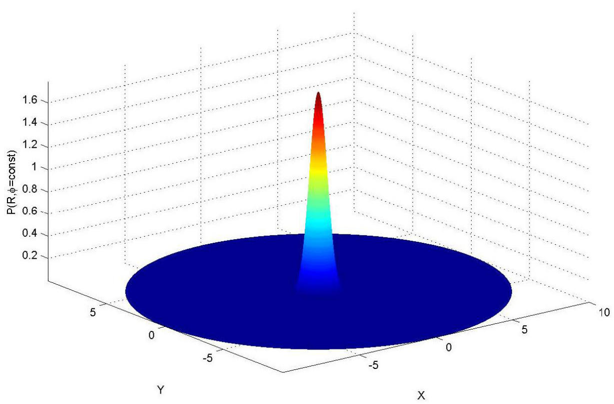


Fig. 3. 3D probability distribution for $\alpha = 0.1$.

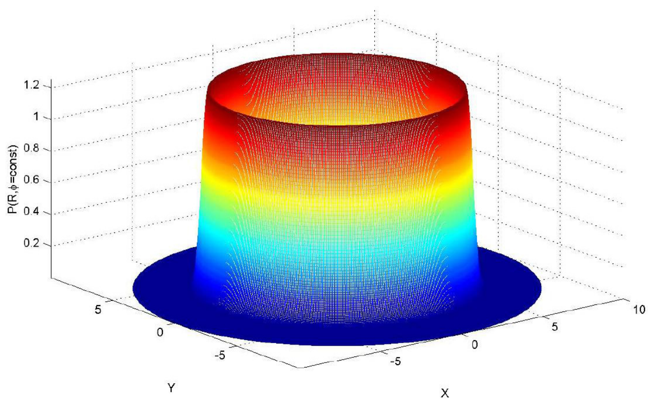


Fig. 4. 3D probability distribution for $\alpha = -0.1$.

The amplitude equation can be derived to

$$\dot{R} = -(\alpha + \beta R^2 \cos^2\theta)R\sin^2(\theta) - \frac{K}{\omega_0}\xi(t)\sin(\theta). \tag{34}$$

Because of the fact that the amplitude fluctuations are slow compared to the oscillatory term, the higher harmonics can be neglected. With setting the phase constant to a representative value, one can approximate the radial distribution with the solution of the stationary Fokker-Planck equation [12].

$$P(R, \phi = const., t) = P_0 \exp\left[-\frac{\alpha}{4D}R^2 - \frac{\beta}{32D}R^4\right] \tag{35}$$

The diffusion coefficient $D$ is associated with the random force $F(t) = K/\omega_0\xi(t)\sin\theta$ in the kind $\langle F(t)F(t')\rangle = = 2D\delta(t - t')$, where again the oscillatory terms were neglected. $P_0$ is the normalization coefficient and can be derived with the reciprocal of the area under each distribution.

In Fig. 3 one can see the gaussian distribution around the origin. As the parameter $\alpha$ changes from positive to negative it develops a Gaussian ring shape distribution (see Fig. 4) which leads to a limit cycle.

### *B. Power Spectral Density of a Noisy van der Pol Oscillator*

Now we discuss the line broadening of a noisy van der Pol oscillator with the sde (28). To study the power spectral density, the state variable $x(t)$ can be approximate by neglecting amplitude fluctuations with

$$x(t) = \frac{R}{2}\exp(j(\omega_0 t + \phi)) = x_0 \exp(j(\omega_0 t + \phi)) \tag{36}$$

for further studies. The amplitude and phase equations of the van der Pol oscillator can be determined by

$$\dot{R} = \epsilon(1 - R^2\cos^2(\omega_0 t + \phi))R\sin^2(\omega_0 t + \phi) - \\ -\frac{K}{\omega_0}\xi(t)\sin(\omega_0 t + \phi) \tag{37}$$

and

$$\dot{\phi} = \epsilon(1 - R^2\cos^2(\omega_0 t + \phi))\sin(\omega_0 t + \phi)\cos(\omega_0 t + \phi) - \\ -\frac{K}{R\omega_0}\xi(t)\cos(\omega_0 t + \phi). \tag{38}$$

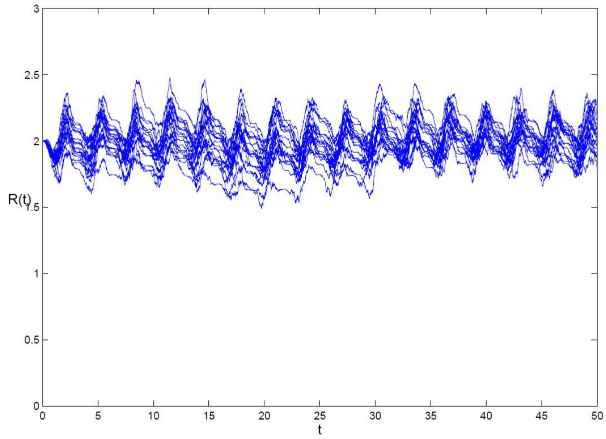


Fig. 5. Weak noise-induced amplitude fluctuations.

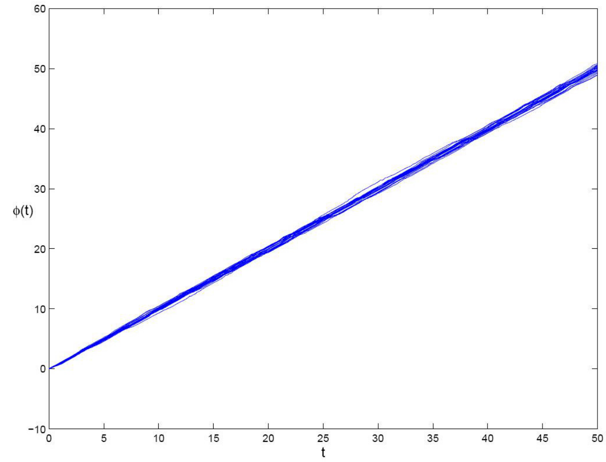


Fig. 7. Weak noise-induced phase fluctuations.

In Fig. 5 and Fig. 6 one can see the amplitude fluctuations of 20 sample trajectories derived numerically by the SDE (37) with the Euler-Maruyama technique [49]. Because of the nonorbital limit cycle of the van der Pol oscillator the amplitude varies around the stationary solution 2. Nevertheless one can see the boundary of amplitude fluctuations by the attracting behavior of the limit cycle. In Fig. 6 certain trajectories concentrate around an amplitude $-2$.

This can be explained by the phase fluctuations shown in Fig. 7 and 8. As supposed phase fluctuations cannot be limited and show a diffusion behavior for increasing noise. Because of the coupling between amplitude and phase fluctuations, some trajectories concentrate around a shifted solution about 180 degree. Neglecting higher harmonics and setting the amplitude constant in equation (38) the SDE of the phase dynamics reduces to

$$\dot{\phi} \approx -\frac{K}{R\omega_0}\xi(t)\cos(\omega_0 t + \phi) = G(t)\cos(\omega_0 t + \phi), \quad (39)$$

where $G(t)$ is linear proportional to the gaussian process $\xi(t)$. With deriving the equivalent Fokker-Planck equation [50] and determining the diffusion-coefficients for a long time interval $t \gg 1$, one can show, that the nonlinear SDE (39) matches with the reduced, linear process (40) [5]. An alternative approach is to determine the averaged amplitude and phase dynamics with the averaging method by Stratonovich [25] and Bogoliubov & Mitropolsky [51]. The averaging results in a decoupling of amplitude and frequency dynamics. With neglecting the amplitude fluctuations, it can be shown that the variance of the phase increases linearly in time which expresses a "simple" diffusion process of $\phi$ and the reduced linear process

$$\frac{d\phi}{dt} = G(t)\cos(\omega_0 t) \quad (40)$$

is applicable [48]. The phase displacement of the linear process can be determined by

$$\phi(t+\tau) - \phi(t) \approx \int_t^\tau G(s)\cos(\omega_0 t)ds. \quad (41)$$

Assume $x(t)$ to be a stationary process, the autocorrelation

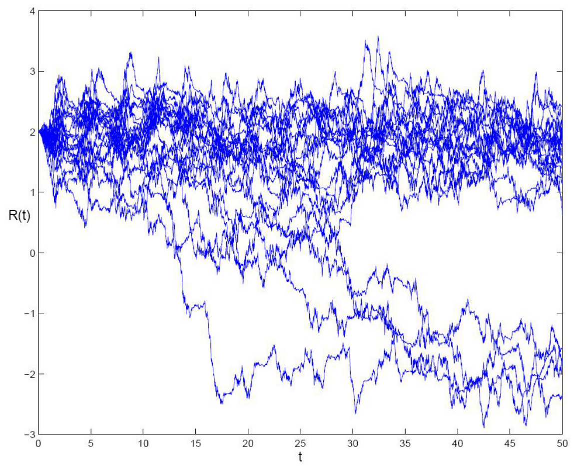


Fig. 6. Strong noise-induced amplitude fluctuations.

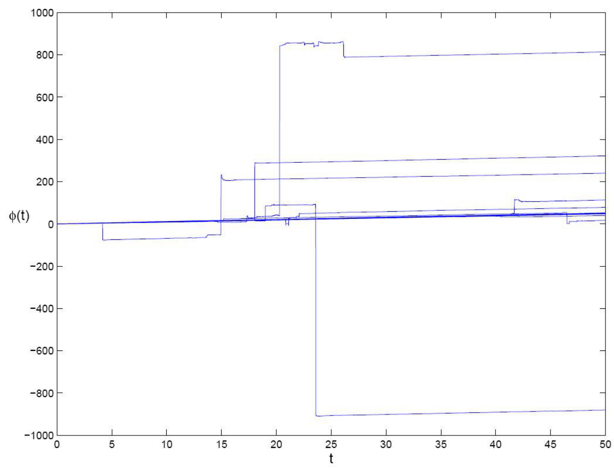


Fig. 8. Strong noise-induced phase fluctuations.

function $R_{xx}(\tau)$ can be derived to

$$R_{xx}(\tau) \approx x_0^2 \exp[j\omega_0\tau]\big\langle \exp\big[j[\phi(t+\tau)-\phi(t)]\big]\big\rangle \quad (42)$$

$$= x_0^2 \exp[-0.5j\langle[\phi(t+\tau)-\phi(t)]^2\rangle]. \quad (43)$$

The last step can be explained by the characteristic function of a gaussian process. The mean square displacement of $\phi$ can be described by

$$\langle[\phi(t+\tau)-\phi(t)]^2\rangle \approx$$

$$\approx \int_t^{t+\tau} ds \int_t^{t+\tau} ds' \cos(\omega_0 s)\cos(\omega_0 s')\langle G(s)G(s')\rangle \quad (44)$$

With the Wiener-Khintchine theorem the autocorrelation function (acf) of $G(s)$ is

$$R_{GG}(s-s') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(j\omega(s-s'))S_{GG}(\omega)d\omega \quad (45)$$

Using the fact that $G(s)$ is an additive, white gaussian noise with the acf $R_{GG}(s-s') = S_{GG}(\omega_0)\delta(s-s')$ inserted in eqn. (44), the mean square displacement of $\phi$ is

$$\langle[\phi(t+\tau)-\phi(t)]^2\rangle =$$

$$= \frac{1}{2}S_{GG}(\omega_0)\Big[\tau + \frac{1}{2\omega_0}2\cos(\omega_0(2t+\tau))\sin(\omega_0\tau)\Big] \quad (46)$$

For $\omega_0 t \gg 1$ the second term in eqn. (46) can be neglected and $\phi$ becomes a diffusion process like

$$\langle[\phi(t+\tau)-\phi(t)]^2\rangle = \frac{1}{2}S_{GG}(\omega_0)|\tau| = W|\tau|. \quad (47)$$

Inserting eqn. (47) in the acf (42) yield the one-sided power spectral density (psd)

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} \exp(-j\omega\tau)R_{xx}(\tau)d\tau \quad (48)$$

$$= x_0^2 \frac{0.5W}{(\omega-\omega_0)^2 + (0.5W)^2}. \quad (49)$$

In Fig. 9 one can see the normalized psd of the noisy van der Pol oscillator for different noise factors K. As assumed the line-broadening in a Lorentzian shape is caused by the thermal noise source itself and not by a linear forming of filter arrangements.

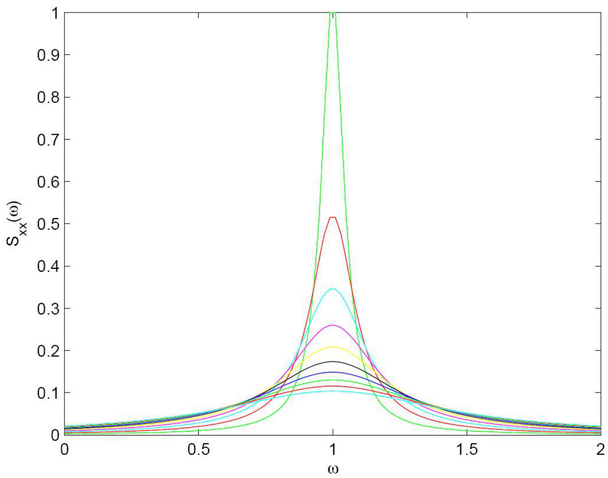

Fig. 9. Normalized psd of the noisy van der Pol oscillator.

## VI. Conclusions

In this article some essential aspects of noise analysis are discussed. It is shown that there is a main difference between systems near the thermal equilibrium and far-from-equilibrium. Although in physics these aspects are studied and discussed since a long time these corresponding results are not used in order to analyse and design electronic oscillators. One of the most interesting properties of electronic oscillators is phase noise and the close related line-broadening effect in the frequency domain where very essential applications in RF CMOS circuit design can be found. Detailed and early discussions about these aspects were published by Lax [5]. In this article we discussed the line broadening of nonlinear oscillators in some details and illustrated this concept by some numerical calculations. But phase noise and line broadening can be studied also by means of the concept discussed in section 3. Therefore it seems that we are at the beginning of new area of noise analysis of nonlinear circuits where also physical concepts from statistical non-equilibrium thermodynamics play an essential role.

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