The structure of invariants in conformal mechanics

Tigran Hakobyan [a], David Karakhanyan [b], Olaf Lechtenfeld [c,*]

[a] *Yerevan State University, 1 Alex Manoogian, 0025 Yerevan, Armenia*
[b] *Yerevan Physics Institute, 2 Alikhanyan br., 0036 Yerevan, Armenia*
[c] *Leibniz Universität Hannover, Institut für Theoretische Physik, Appelstr. 2, 30167 Hannover, Germany*

Abstract

We investigate the integrals of motion of general conformal mechanical systems with and without confining harmonic potential as well as of the related angular subsystems, by employing the $sl(2, \mathbb{R})$ algebra and its representations. In particular, via the tensor product of two representations we construct new integrals of motion from old ones, both in the classical and in the quantum case. Furthermore, the temporally periodic observables (including the integrals) of the angular subsystem are explicitly related to those of the full system in a confining harmonic potential. The techniques are illustrated for the rational Calogero models and their angular subsystems, where they generalize known methods for obtaining conserved charges beyond the Liouville ones.

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1. Introduction

Arguably the most important one-dimensional multi-particle system is defined by the inverse-square two-body interaction potential, which is based on the $A_{N-1} \oplus A_1$ root system. It has been introduced by Calogero four decades ago and is integrable both with and without a confining harmonic potential [1,2]:

$$H_\omega = \frac{1}{2} \sum_{i=1}^{N} \left(p_i^2 + \omega^2 q_i^2 \right) + \sum_{i<j} \frac{g^2}{(q_i - q_j)^2}, \tag{1.1}$$

$$H_0 = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \sum_{i<j} \frac{g^2}{(q_i - q_j)^2}. \tag{1.2}$$

This system continues to attract much interest due to its rich internal structure and numerous applications. So far, various integrable extensions have been constructed and studied, in particular, for trigonometric potentials [3], for particles with spins [4], for supersymmetric systems [5] and for other root systems [6].

Such systems exhibit a rich spectrum of physical properties: fractional statistics [7], Laughlin-type wavefunctions [8] and a resonance valence-bond ground state for a related spin chain [9]. Calogero models appear in many areas of physics and mathematics, like black holes [10], quantum hydrodynamics, or orthogonal polynomials.

The rational Calogero models are maximally superintegrable, i.e. they possess $N-1$ additional integrals of motion. For the classical Hamiltonian without confining harmonic potential, these have been constructed explicitly by Wojciechowski [11]. Later this construction was extended to the quantum case [12,13] and to the inclusion of a confining harmonic potential, where oscillatory behavior with commensurate frequencies implies the superintegrability [14–16]. This property has been established also for the hyperbolic Calogero model [17] and the relativistic extension known as the rational Ruijsenaars–Schneider model [18,19].

An important feature of rational Calogero models is the dynamical conformal $sl(2,\mathbb{R})$ symmetry,

$$\{H_0, D\} = 2H_0, \qquad \{K, D\} = -2K, \qquad \{H_0, K\} = D, \tag{1.3}$$

generated by the Hamiltonian (1.2) together with the dilatation and conformal boost generators [20]

$$D = \sum_{i=1}^{N} p_i q_i, \qquad K = \frac{1}{2} \sum_{i=1}^{N} q_i^2. \tag{1.4}$$

Many properties of these systems, like superintegrability [16,17], equivalence to a free-particle system [21,22], or the existence of action-angle variables [23] are simple consequences of the conformal symmetry. The Casimir element of this algebra,

$$\mathcal{I} = 4H_0 K - D^2, \tag{1.5}$$

coincides with the angular part of the Calogero model and is an integral of motion of both Hamiltonians (1.1) and (1.2). It does not belong to the usual system of Liouville integrals, but its commutator with them produces all additional integrals of motion responsible for the superintegrability. The angular part (1.5) can be considered as a separate (super)integrable system describing a particle moving on the $(N-1)$-dimensional sphere, which has been defined and studied in a number of recent papers [24–28].

The rational Calogero models are integrable members of the more general class of *conformal* mechanical systems, whose action is invariant under the conformal transformations (1.3) and which were first introduced in [29]. As a recent application, such systems can describe particle dynamics near the horizon of an extremal black hole [30–32]. A lot of what is derived in this paper also applies to any conformal mechanics system.

The main goal of the current article is to establish relations between the integrals of motion and related oscillating quantities of (a) general conformal mechanics, (b) the related angular subsystem, and (c) conformal mechanics in an external harmonic potential, employing the properties of $sl(2, \mathbb{R})$ and its representations. The article is structured in the following way.

In Section 2, we describe the $sl(2, \mathbb{R})$ representation content of integrals of motion in conformal classical mechanics and employ the conformal algebra to expand the system of conserved charges. Any such integral of motion with a definite value of the conformal spin is, by definition, a highest-weight state and thus generates a representation of the conformal algebra. The descendant states in this representation are not conserved, but can be used to construct additional integrals [11] which, of course, are highest weights in another representation. The tensor product of two representations is a convenient way to generate new ones. Thus, given two integrals of conformal classical mechanics, we may decompose the tensor product of their conformal representations into irreducible pieces and pick from each of these the highest state, which will yield a new conserved charge. In this way, extending a method applied in [11,13], we express new integrals of conformal classical mechanics in terms of descendants of two given integrals of motion. The simplest application to the rational Calogero model recovers Wojciechowski's construction [11]. From the standard tensor product, the descendant states are combined symmetrically, i.e. the corresponding phase-space functions are pointwise multiplied. When instead one combines them under the Poisson bracket, it seems that merely the known Liouville integrals are reproduced, but this option needs a more detailed study.

In Section 3, the aforementioned construction of additional integrals of motion is extended to the quantum case. The new integrals appear in symmetrized products of operator-valued conformal representations, which in the semiclassical limit reduce to pointwise products.

Section 4 is devoted to the integrals of motion for conformal mechanics in a confined harmonic potential, e.g. (1.1), and to those for the related angular mechanics (1.5). Recently, it has been shown that the spectrum and eigenstates of these two systems are closely related for the quantum Calogero model [28]. Here we study this relation at the level of more general conformal mechanical systems. Starting with one or more integrals of motion for some unconfined conformal mechanics, e.g. (1.2), a system of oscillating observables is constructed for the corresponding angular mechanics. The frequencies of these observables are integer multiples of the basic frequency, the latter being the square root $\sqrt{\mathcal{I}}$ of the angular Hamiltonian [24,26]. These observables can easily be combined to products with vanishing frequency, giving rise to further integrals of motion. Similarly, for the model with a confining harmonic potential, oscillating observables and integrals of motion are derived from conserved charges of the unconfined system, but now the basic frequency is the frequency ω of the confining potential [13]. In both cases we employ appropriate conformal rotation operators. For the angular system, in addition a noncanonical special conformal transformation inverting the radial coordinate is involved. As a result, we have found the exact relation between the oscillating observables of confined conformal mechanics and of the related angular system.

Section 5 revisits the matrix-model construction of the Calogero Hamiltonians H_0 and H_ω. We consider the additional integrals described in Section 2 and generated from the standard Liouville integrals of the Calogero Hamiltonian H_0. Again based on these Liouville integrals, we construct the oscillating observables of the confined system H_ω treated in Section 4 and describe them in terms of oscillating matrices. We then prove that the Poisson action of the angular Hamiltonian $\mathcal{I}$ on the standard Liouville integrals of H_ω produces $N - 1$ additional integrals, which combine with the N Liouville integrals to a complete and independent system. This generalizes a similar property for the unconfined Calogero Hamiltonian H_0 [24,26].

2. $sl(2, \mathbb{R})$ structure of the integrals of motion: classical case

For any function f on phase space, define the associated Hamiltonian vector field by the Poisson bracket action

$$\hat{f} = \{f, \cdot\}. \tag{2.1}$$

The assignment $f \to \hat{f}$ is a Lie algebra homomorphism, the constants on the phase space form its kernel. For an interaction potential V, the vector fields

$$\hat{H}_0 = \sum_i \left(p_i \frac{\partial}{\partial q_i} - \frac{\partial V}{\partial q_i} \frac{\partial}{\partial p_i} \right), \qquad \hat{K} = -\sum_i q_i \frac{\partial}{\partial p_i},$$

$$\hat{D} = \sum_i \left(q_i \frac{\partial}{\partial q_i} - p_i \frac{\partial}{\partial p_i} \right) \tag{2.2}$$

satisfy the $sl(2, \mathbb{R})$ algebra (1.3), and the vector field of the Casimir element $\hat{\mathcal{I}}$, of course, commutes with them.

Consider now the general conformal mechanical system with the Hamiltonian H_0 obeying the symmetry relations (1.3). First, this implies that the Casimir element (1.5) is an integral of motion of the system with the zero conformal dimension. Next, suppose that the Hamiltonian apart from itself and (1.5) possesses other integrals of motion. Here we plan to study in detail the interrelation of the conformal symmetry and these integrals.

Note that any constant of motion is a highest-weight vector of the conformal algebra (1.3), since it is annihilated by the Hamiltonian. Without any restriction, one can choose it to have a certain conformal dimension (spin),

$$\hat{S}_+ I_s = 0, \qquad \hat{S}_z I_s = s I_s, \tag{2.3}$$

where we have introduced more conventional notations for the raising, lowering and diagonal generators of the conformal group:

$$S_+ = H_0, \qquad S_- = -K, \qquad S_z = -\frac{1}{2}D, \tag{2.4}$$

$$[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z, \qquad [\hat{S}_z, \hat{S}_\pm] = \pm\hat{S}_\pm. \tag{2.5}$$

The covariant basis for the conformal algebra reads

$$S_{x,y} = \frac{1}{2}(S_+ \pm S_-), \tag{2.6}$$

$$\{S_x, S_y\} = -S_z, \qquad \{S_y, S_z\} = -S_x, \qquad \{S_z, S_x\} = S_y, \tag{2.7}$$

$$\mathbf{S}^2 = \frac{1}{4}\mathcal{I} = \sum_{\alpha=x,y,z} S_\alpha S^\alpha = -S_x^2 + S_y^2 - S_z^2, \tag{2.8}$$

where the indices are raised and lowered by the metric $g_{\alpha\beta} = \text{diag}(-1, 1, -1)$. The Casimir element $\hat{\mathbf{S}}^2$ of the related vector field algebra

$$[\hat{S}_x, \hat{S}_y] = -\hat{S}_z, \qquad [\hat{S}_y, \hat{S}_z] = -\hat{S}_x, \qquad [\hat{S}_z, \hat{S}_x] = \hat{S}_y \tag{2.9}$$

equals $-s(s+1)$ times the identity on an $sl(2,\mathbb{R})$ representation of spin s as given in (2.3). It is important to distinguish between the square of the conformal spin (2.9), as a second-order differential operator, and the vector field $\hat{\mathcal{I}}$ generated by the Casimir invariant (2.8), as a first-order operator:

$$\hat{\mathbf{S}}^2 = \sum_\alpha \hat{S}_\alpha \hat{S}^\alpha, \qquad \hat{\mathcal{I}} = \{\mathcal{I}, \cdot\} = 8\sum_\alpha S_\alpha \hat{S}^\alpha. \tag{2.10}$$

Note that $\hat{\mathcal{I}}$ does not preserve representations but acts as an intertwiner between them.

The descendants

$$I_{s,k} = (\hat{S}_-)^k I_s \quad \text{for } k = 0, 1, 2, \ldots \tag{2.11}$$

form the basic states of the spin-s representation of the conformal algebra (2.9). For generic real values of s, there is an infinity of them. For non-negative integer or half-integer values of s however, the state $I_{s,2s+1}$ either vanishes or it is another integral of motion for the conformal mechanics Hamiltonian.

If $I_{s,2s+1} = 0$, then we deal with a finite-dimensional irreducible representation. This includes the rational Calogero model, whose Liouville constants of motion are polynomials of order $(2s)$ in the momenta. Their multiplets are nonunitary and similar to the spin-s representations of $su(2)$.

If $I_{s,2s+1}$ does not vanish, it is another integral of the conformal Hamiltonian, since the spin raising operator $\hat{S}_+$ annihilates it, as is easy to check using the definition (2.3) and commutation relations (2.5). As a highest-weight state, it generates another conformal representation, which forms an invariant subspace. We thus encounter an indecomposable representation, which is reducible but not fully reducible.

The time evolution of the observables (2.11) is given by a kth order polynomial in time [20], since the $(k+1)$th power of the evolution operator $d/dt = \hat{H}_0 = \hat{S}_+$ annihilates it. However, they can be used to construct new integrals of motion different from I_s. This construction is be done in terms of the representation theory of the conformal algebra, as will be described below.

Denote by (s) the $sl(2,\mathbb{R})$ representation (2.11), generated by the integral of motion I_s. For two integrals I_{s_1} and I_{s_2} the products of the corresponding descendant states form the product representation, which decomposes into a direct sum of representations:

$$(s_1) \otimes (s_2) = (s_1 + s_2) \oplus (s_1 + s_2 - 1) \oplus \ldots \oplus (s_1 + s_2 - k) \oplus \ldots. \tag{2.12}$$

For finite dimensional irreducible representations, this series terminates at $(|s_1 - s_2|)$, giving rise to the usual momentum sum rule in quantum mechanics. The highest-weight states of the kth multiplet in the decomposition (2.12) are also integrals of motion with conformal spins $s = s_1 + s_2 - k$, which we denote by $I_s^{(s_1,s_2)}$. They can be calculated using the $sl(2,\mathbb{R})$ Clebsch–Gordan coefficients. However, it is easier to derive them directly with the commutation relation

$$\left[\hat{S}_+, \hat{S}_-^l\right] = l\hat{S}_-^{l-1}(2\hat{S}_z - l + 1) \tag{2.13}$$

and the highest-state conditions (2.3). Choosing a suitable normalization factor, we define the new integrals of motion as

$$I_{s_1+s_2-k}^{(s_1,s_2)} = \sum_{l=0}^k (-1)^l \binom{k}{l} \frac{\Gamma(2s_1 - k + l + 1)\Gamma(2s_2 - l + 1)}{\Gamma(2s_1 - k + 1)\Gamma(2s_2 - k + 1)} I_{s_1,k-l} I_{s_2,l}$$
$$\text{for } k = 0, 1, 2, \ldots. \tag{2.14}$$

For generic real values of s_1 or s_2, we obtain an infinity of these. For integer or half-integer values of the spins however, there appears only a finite number, limited by $k < 2\min(s_1, s_2)$. In multi-particle models, the new integrals $I_s^{(s_1,s_2)}$ are quadratic in the descendants and hence involve a double sum over the particle index, making them composite objects.

Let us consider particular cases of the composite integrals of motion (2.14). The case $k = 0$ is uninteresting, since it merely yields the product $I_{s_1} I_{s_2}$. For $k = 1$ we have

$$I_{s_1+s_2-1}^{(s_1,s_2)} = 2s_2 I_{s_1,1} I_{s_2} - 2s_1 I_{s_1} I_{s_2,1}, \tag{2.15}$$

which is a new integral of motion for any pair I_{s_1} and I_{s_2} [11]. If the first integral is the Hamiltonian, $I_{s_1} = I_1 = S_+$, then we simply obtain the bracket with the Casimir element (2.8),

$$I_s^{(1,s)} = -4s\, S_z I_s - 2S_+ \hat{S}_- I_s = -(4S_z \hat{S}_z + 2S_+ \hat{S}_- + 2S_- \hat{S}_+)I_s$$

$$= 4\sum_\alpha S_\alpha \hat{S}^\alpha I_s = \frac{1}{2}\hat{\mathcal{I}} I_s. \tag{2.16}$$

In the last equation the second relation in (2.10) is used. For the N-particle Calogero system, the Casimir invariant $\mathcal{I}$ of the conformal algebra produces in this way the additional $N - 1$ integrals of motion from the Liouville integrals [24,26]. Another special case is $s_1 = s_2$. Exchanging $l \leftrightarrow k - l$ in the sum (2.14), one concludes that terms with k odd cancel out, leaving only even values for k. More generally, taking n integrals of motion I_{s_i,k_i}, the multiple tensor product decomposition

$$(s_1) \otimes \cdots \otimes (s_n) = (s_1 + \ldots + s_n) \oplus \ldots \tag{2.17}$$

can be used to produce new integrals from the highest weight states in the multiplets obtained.

Rather than simply multiplying the descendants on the right-hand side of (2.14), one may take their Poisson bracket instead, since they are phase-space functions. The conformal tensor product decomposition is unaffected, and the Clebsch–Gordan coefficients remains the same,

$$\sum_{l=0}^{k} (-1)^l \binom{k}{l} \frac{\Gamma(2s_1 - k + l + 1)\Gamma(2s_2 - l + 1)}{\Gamma(2s_1 - k + 1)\Gamma(2s_2 - k + 1)} \{I_{s_1,k-l}, I_{s_2,l}\}. \tag{2.18}$$

For the Calogero model, the Poisson bracket of two conformal representations generated by Liouville integrals I_{s_1} and I_{s_2}, was considered already in [33]. For the standard Liouville integrals in the simplest cases the above formula yields nothing new as we shall sketch in Section 5. Therefore, we further discuss only the pointwise products (2.14).

We remark that conformal mechanics is not required to be integrable. However, the existence of a single third-order integral together with translational and permutation invariance (i.e. identical particles) forces any classical one-dimensional system to the integrable elliptic Calogero model [34]. Yet in general, conformal invariance only generates higher-order integrals from lower-order ones, but it does not necessarily lead to integrability. The simplest example is a composition of two non-interacting systems: Calogero particles and conformal mechanics with some non-integrable potential. Evidently, the integrals of the Calogero subsystem are also integrals of the entire system, which nevertheless is not integrable. Note that requiring permutational invariance would fix the full potential to coincide with the Calogero one. In general, the constants of motion derived from tensor products are not in involution, as we will see for the Calogero model. The number of non-vanishing such integrals and their functional independence depends on the concrete conformal mechanical system under consideration.

3. $sl(2, \mathbb{R})$ structure of the integrals of motion: quantum case

In passing from the classical to the quantum model, we replace

$$\{p_i, q_j\} = \delta_{ij} \quad \longrightarrow \quad \frac{i}{\hbar}[p_i, q_j] = \delta_{ij}. \tag{3.1}$$

The expressions (1.1), (1.2) and (1.4) for the Hamiltonians and conformal group generators remain the same, except that symmetric (Weyl) ordering between momenta and coordinates must be used, which affects the dilatation

$$D = \frac{1}{2}\sum_{i=1}^{N}(p_i q_i + q_i p_i) = \sum_{i=1}^{N} p_i q_i + i\hbar N/2. \tag{3.2}$$

The hermitian generators obey the quantum commutation relations

$$[H_0, D] = -2i\hbar H_0, \qquad [K, D] = 2i\hbar K, \qquad [H_0, K] = -i\hbar D. \tag{3.3}$$

The expressions (2.4) and (2.6) for the invariant conformal generators $S_{\pm,z,x,y}$ remain unchanged, while the quantum analogue of (2.7) reads

$$[S_\alpha, S_\beta] = -i\hbar\epsilon_{\alpha\beta\gamma} S^\gamma, \tag{3.4}$$

$$[S_+, S_-] = -2i\hbar S_z, \qquad [S_z, S_\pm] = \mp i\hbar S_\pm. \tag{3.5}$$

Note that, in contrast to the well known $su(2)$ raising and lowering operators, the $sl(2, \mathbb{R})$ operators $S_\pm$ are hermitian and thus not mutually conjugate.

The Weyl ordering becomes essential in the Casimir element

$$\mathcal{I} = 4\mathbf{S}^2, \qquad \mathbf{S}^2 = -\frac{1}{2}(S_+ S_- + S_- S_+) - S_z^2 = -S_- S_+ - S_z(S_z - i\hbar). \tag{3.6}$$

Any quantum observable f defines an infinitesimal evolution map given by the operator

$$\hat{f} = \frac{i}{\hbar}[f, \cdot], \tag{3.7}$$

which is the quantum analog of the classical vector field (2.1) and reduces to it in the semiclassical limit. Again, the assignment $f \to \hat{f}$ is a Lie algebra homomorphism. In this way, we get a (not necessarily unitary) representation of the conformal algebra on the space of quantum operators. It was introduced and used for the construction of additional integrals of the quantum Calogero system [13], simplifying an earlier procedure [12].

In the adjoint action (3.7), the quantum commutation relations of the conformal group generators coincide with their classical commutators (2.5) and (2.9).

As in the classical case, any spin-s quantum integral of motion (2.3) generates a highest-weight representation (2.11) of the conformal algebra. The product of two such representations is subject to the sum rule (2.12), and the highest states (2.14) yield new integrals of motion for the quantum conformal Hamiltonian. However, since quantum physical observables are sup-

posed to be hermitian, the expressions for $I^{(s_1,s_2)}_{s_1+s_2-k}$ must be self-conjugate. In order to achieve this, it suffices to symmetrize the products of descendants in $I^{(s_1,s_2)}_s$, i.e.[1]

$$I_{s_1,k_1} I_{s_2,k_2} \quad \longrightarrow \quad \frac{1}{2}(I_{s_1,k_1} I_{s_2,k_2} + I_{s_2,k_2} I_{s_1,k_1}). \tag{3.8}$$

It appears that $I^{(s_1,s_2)}_{s_1+s_2-k}$ is simply the irreducible component of the symmetrized tensor product $(I_{s_1} \otimes I_{s_2})_+$:

$$I^{(s_1,s_2)}_{s_1+s_2-k} = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \frac{\Gamma(2s_1-k+l)\Gamma(2s_2-l+1)}{\Gamma(2s_1-k+1)\Gamma(2s_2-k+1)} \frac{1}{2}(I_{s_1,k-l} I_{s_2,l} + I_{s_2,l} I_{s_1,k-l}). \tag{3.9}$$

In the classical limit $\hbar \to 0$, it reduces to (2.14). The first nontrivial case corresponds to $k = 1$:

$$I^{(s_1,s_2)}_{s_1+s_2-1} = s_2(I_{s_1,1} I_{s_2} + I_{s_2} I_{s_1,1}) - s_1(I_{s_1} I_{s_2,1} + I_{s_2,1} I_{s_1}), \tag{3.10}$$

which is the quantum version of the classical integrals (2.15) [12,13] and produces, for $s_1 = 1$ and $I_1 = S_+$,

$$I^{(1,s)}_s = \frac{i}{2\hbar}[\mathcal{I}, I_s] \tag{3.11}$$

as the simplest quantum integrals beyond the Liouville ones.

4. Relating angular mechanics and confined conformal mechanics

The angular part $\mathcal{I}$ of the Hamiltonian coincides with the Casimir element $\mathbf{S}^2$ of the conformal algebra (2.8). It defines a mechanical subsystem depending only on the angular coordinates and momenta u, to which we refer as angular mechanics. Its integrals of motion have been studied for general conformal mechanics [24,26] and, in particular, for the Calogero model [27].

Motion in the angular subsystem (1.5) is naturally bounded, as it is for the harmonically confined conformal system (1.1). Confined integrable systems feature quantities which oscillate in time with a fixed frequency. Examples are the angle variables, which, together with their canonically conjugated action variables, the Liouville integrals, parametrize the phase space of the system. In the case of commensurate frequencies, additional integrals exist and are expressed completely via the angles [35]. This property is known as superintegrability. Note that the existence of such integrals does not require integrability: two quantities oscillating with commensurate frequencies are sufficient. In this section we study consequences of the existence of higher-order integrals in two confined mechanical systems, namely in angular mechanics and in harmonically confined conformal mechanics. For both we will construct quantities which oscillate in time with integral multiples of a basic frequency. These frequencies are proportional to the spin projections in a finite-dimensional $sl(2, \mathbb{R})$ representation, which is realized differently in the two systems.

Let us first focus on the angular mechanics case. In order to construct the angular subsystem of a conformal mechanics model, it is suitable to express the conformal generators (2.4) and (2.2) in terms of angular coordinates and momenta, $u = (\theta_\alpha, p_{\theta_\alpha})$, and of radial ones,

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[1] Other orderings, like Weyl ordering, are also possible, but will differ only by contributions of lower-order integrals. Hence, the full set of quantum integrals does not depend on the choice of ordering.

$$r^2 = \sum_i q_i^2, \qquad r p_r = \sum_i p_i q_i, \tag{4.1}$$

via

$$S_+ = \frac{p_r^2}{2} + \frac{\mathcal{I}(u)}{2r^2}, \qquad\qquad S_- = -\frac{r^2}{2}, \qquad S_z = -\frac{p_r r}{2}, \tag{4.2}$$

$$\hat{S}_+ = p_r \frac{\partial}{\partial r} + \frac{\mathcal{I}}{r^3} \frac{\partial}{\partial p_r} + \frac{\hat{\mathcal{I}}}{2r^2}, \qquad \hat{S}_- = r \frac{\partial}{\partial p_r}, \qquad \hat{S}_z = \frac{1}{2}\left(p_r \frac{\partial}{\partial p_r} - r \frac{\partial}{\partial r} \right). \tag{4.3}$$

Using (4.2) and (4.3), it is easy to see [26] that the highest-weight condition (2.3) is equivalent to

$$\hat{\mathcal{I}} I_s = 2\big(\hat{S}_+^R - \mathcal{I} \hat{S}_-^R \big) I_s, \tag{4.4}$$

where we introduced the one-dimensional vector fields[2]

$$\hat{S}_+^R = -p_r r^2 \frac{\partial}{\partial r}, \qquad \hat{S}_-^R = \frac{1}{r} \frac{\partial}{\partial p_r}, \qquad \hat{S}_z^R = \frac{1}{2}\left(r \frac{\partial}{\partial r} + p_r \frac{\partial}{\partial p_r} \right), \tag{4.5}$$

which form another $sl(2, \mathbb{R})$ algebra. Note that $\hat{\mathcal{I}}$ acts only on the angular variables while the $\hat{S}_a$ feel just the radial dependence. Therefore, (4.4) relates the angular dependence of I_s to its radial one.

For vanishing angular part, the above new generators are *dual* to the conformal generators (4.3):

$$\hat{S}_a\big|_{\mathcal{I}=0} = R \hat{S}_a^R R, \tag{4.6}$$

where the duality map

$$R: \quad r \to 1/r, \qquad p_r \to p_r, \qquad u \to u \tag{4.7}$$

inverts the radial coordinate but leaves the angular ones unchanged. Evidently, it is not a canonical transformation, but the dual generators (4.5) obey the same algebraic relations (2.5) as the standard generators (4.3) do.

For $2s$ being integer, any spin-s integral, as defined in (2.3), can be decomposed into terms with a homogeneous radial dependence separated from the angular one,[3]

$$I_s(p_r, r, u) = \sum_{l=0}^{2s} f_{s,l}(u) \frac{p_r^{2s-l}}{r^l}. \tag{4.8}$$

The radial functions p_r^{2s-l}/r^l form a basis of the spin s-representation of the $sl(2, \mathbb{R})$ algebra (4.5), where $\hat{S}_z^R$ is diagonal. The inversion (4.6) maps them to the equivalent representation given by the $S_a|_{\mathcal{I}=0}$ acting on the polynomials $p_r^{2s-l} r^l$ of order $2s$. The conformal descendants of the integrals satisfy the following decomposition,

$$I_{s,k}(p_r, r, u) = \sum_{l=0}^{2s-k} \frac{(2s-l)!}{(2s-k-l)!} \frac{p_r^{2s-k-l}}{r^{k+l}} f_{s,l}(u) \quad \text{for } k = 0, 1, \ldots, 2s. \tag{4.9}$$

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[2] The basis defined in our previous paper [26] differs from the current one by the map $\hat{S}_\pm^R \to \hat{S}_\mp^R$ and $\hat{S}_z^R \to -\hat{S}_z^R$. In this notation it becomes equivalent to the basis of the previously defined conformal generators (4.3).

[3] In comparison to the definition of $f_{s,m}(u)$ in [26], we have multiplied a binomial factor and applied an index shift $m \to m - s$. This makes the $so(3)$ properties more apparent and simplifies further relations.

In particular, the lowest descendant ($k = 2s$) reduces to a single term independent of the radial momentum, so its radial and angular dependencies factorize:

$$I_{s,2s}(p_r, r, u) = (2s)! \, r^{-2s} f_{s,0}(u). \tag{4.10}$$

This is reminiscent of the wavefunctions of the quantum Calogero Hamiltonian [28]. The transformation (4.9) from $\{f_{s,2s-k}\}$ to $\{I_{s,k}\}$ is given by a triangular matrix with diagonal elements $(k)! \, r^{-2s}$. Therefore, it is invertible, and $f_{s,l}$ can be expressed in terms of $\{I_{s,2s}, \ldots, I_{s,2s-l}\}$.

It is convenient to pass from the radial variables $1/r$ and p_r to the complex combinations

$$z = \frac{1}{\sqrt{2}}\left(p_r - \frac{i\sqrt{\mathcal{I}}}{r} \right), \qquad \bar{z} = \frac{1}{\sqrt{2}}\left(p_r + \frac{i\sqrt{\mathcal{I}}}{r} \right), \tag{4.11}$$

As any linear map, it extends to the $(2s + 1)$-dimensional space of polynomials of degree $2s$,

$$z^{2s-l}\bar{z}^l = (i\sqrt{\mathcal{I}})^s \sum_{k=0}^{2s} \frac{p_r^{2s-k}}{r^k}(\widetilde{U}^s)_{kl}, \tag{4.12}$$

where $\widetilde{U}^s$ is a $2s \times 2s$ matrix depending on $\sqrt{\mathcal{I}}$. Its elements can be derived from those of the fundamental representation, $(\widetilde{U}^{\frac{1}{2}})_{kl}$, which are determined by (4.11).

In the new basis, the dualized spin operators take the form

$$\hat{S}_z^R = \frac{1}{2}\left(\bar{z}\frac{\partial}{\partial z} + z\frac{\partial}{\partial \bar{z}} \right), \qquad \hat{S}_+^R + \mathcal{I}\hat{S}_-^R = i\sqrt{\mathcal{I}}\left(z\frac{\partial}{\partial \bar{z}} - \bar{z}\frac{\partial}{\partial z} \right),$$

$$\hat{S}_+^R - \mathcal{I}\hat{S}_-^R = i\sqrt{\mathcal{I}}\left(\bar{z}\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial z} \right). \tag{4.13}$$

The decomposition (4.8) of a conformal mechanics integral in the new monomial basis (4.12) defines shifted angular harmonics $\tilde{f}_{s,l}(u)$ via

$$I_s(z, \bar{z}, u) = \sum_{l=0}^{2s} \tilde{f}_{s,l}(u) z^{2s-l}\bar{z}^l \quad \text{with} \quad \tilde{f}_{s,l} = (i\sqrt{\mathcal{I}})^{-s} \sum_{k=0}^{2s} (\widetilde{U}^s)_{lk}^{-1} f_{s,k}. \tag{4.14}$$

We remark that our bases are not normalized. The standard, normalized, $sl(2, \mathbb{R})$ representation basis [26] is obtained by using [26]

$$\sqrt{\binom{2s}{s-m}}\, \bar{z}^{s+m} z^{s-m} \quad \text{and} \quad \sqrt{\binom{2s}{s-m}}\, \frac{p_r^{s+m}}{r^{s-m}}, \quad \text{for } -s \leq m \leq s \tag{4.15}$$

together with related normalized angular harmonics

$$f_{sm} = \binom{2s}{s-m}^{-\frac{1}{2}} f_{s,s-m} \quad \text{and} \quad \tilde{f}_{sm} = \binom{2s}{s-m}^{-\frac{1}{2}} \tilde{f}_{s,s-m}. \tag{4.16}$$

According to the last equation in (4.13), the action (4.4) of the angular Hamiltonian becomes diagonal in the new coordinates. Hence, its action on the decomposition (4.14) implies that, for $\mathcal{I} > 0$, the shifted harmonics oscillate with the frequencies equal to integral multiples of $\sqrt{\mathcal{I}}$:

$$\hat{\mathcal{I}}\tilde{f}_{sm} = -2mi\sqrt{\mathcal{I}}\tilde{f}_{sm} \quad \longrightarrow \quad \tilde{f}_{sm}(t) = e^{-2mi\sqrt{\mathcal{I}}(t-t_0)} \tilde{f}_{sm}(t_0). \tag{4.17}$$

The basic frequency $\sqrt{\mathcal{I}}$ is, of course, a constant of motion.

The map (4.12) defined by the matrix $\widetilde{U}^s$ can be expressed s-independently in terms of the conformal generators,

$$\widetilde{U} \equiv \widetilde{U}^s(\sqrt{\mathcal{I}}) = (i\sqrt{\mathcal{I}})^{-\hat{S}_z^R} e^{\frac{\pi}{2}\hat{S}_y^R}. \tag{4.18}$$

In the fundamental representation $s = 1/2$, the operators $\hat{S}_y^R$ and $\hat{S}_z^R$ are represented in terms of Pauli matrices as $\frac{i}{2}\sigma_y$ and $\frac{1}{2}\sigma_z$, respectively, but the expression naturally extends to the polynomial spin-s representation of the conformal algebra (4.13). On the basis (4.12) or (4.15), it evidently reduces to the matrix $\widetilde{U}^s$ from (4.14). From (4.18) it is clear that the latter can be expressed in terms of Wigner's small d-matrix, which is explicitly done in Appendix A. With the help of (A.7) and (4.16), the relation between the normalized original and shifted angular harmonics take the following form,

$$\tilde{f}_{sm} = \sum_{m'=-s}^{s} d_{mm'}^s(\pi/2)(i\sqrt{\mathcal{I}})^{m'-s} f_{sm'}, \tag{4.19}$$

$$f_{sm'} = \sum_{m=-s}^{s} d_{mm'}^s(\pi/2)(i\sqrt{\mathcal{I}})^{s-m'} \tilde{f}_{sm}. \tag{4.20}$$

Using the definition (2.6) and the relation $q^{-\hat{S}_z}\hat{S}_{\pm}q^{\hat{S}_z} = q^{\mp 1}\hat{S}_{\pm}$ for $q = i\sqrt{\mathcal{I}}$, which is a direct consequence of the commutation relations (2.5), the adjoint action of the operator (4.18) on the generators of the conformal algebra (4.5) or (4.13) can be calculated:

$$\hat{S}_z^R = \widetilde{U}\hat{S}_x^R\widetilde{U}^{-1}, \qquad \hat{S}_+^R + \mathcal{I}\hat{S}_-^R = 2i\sqrt{\mathcal{I}}\widetilde{U}\hat{S}_y^R\widetilde{U}^{-1}, \qquad \hat{S}_+^R - \mathcal{I}\hat{S}_-^R = -2i\sqrt{\mathcal{I}}\widetilde{U}\hat{S}_z^R\widetilde{U}^{-1}. \tag{4.21}$$

We emphasize again that the operator (4.18) is not canonical since the vector fields $\hat{S}_z^R$ and $\hat{S}_y^R$ are not Hamiltonian. The expression (4.21) is, in general, complex and multi-valued. When the potential is positive, as is the case in Calogero models, the angular part is strictly positive and the operator (4.18) is complex but single-valued. In any case, all square roots will cancel in the final expressions for the constants of motion.

The second part of this section deals with conformal mechanics in an external harmonic potential. We shall see that, again, the integrals of motion are derived from the descendants $I_{s,k}$ with the help of a Wigner rotation, similar to the angular mechanics case above. Adding a harmonic confining potential is a deformation compatible with the conformal symmetry:

$$H_\omega = H_0 + \omega^2 K = S_+ - \omega^2 S_-, \qquad \hat{H}_\omega = \hat{S}_+ - \omega^2 \hat{S}_-. \tag{4.22}$$

The last relation has the same structure as in (4.4), and it is mapped to the latter expression under the substitution

$$\hat{S}_a \to \hat{S}_a^R, \qquad \omega \to \sqrt{\mathcal{I}}. \tag{4.23}$$

However, while (4.4) is valid only for the integrals of motion I_s of H_0, (4.22) is an operator identity. Using this formal analogy between (4.4) and (4.22), one can recycle the previous subsection to express the constants of motion for H_ω in terms of the integrals for H_0. This procedure generalizes a construction previously applied to the Calogero model [16].

Using (4.21) and (4.18) and the correspondence (4.23), it is easy to see that the operator

$$U = U(\omega) = (i\omega)^{-\hat{S}_z} e^{\frac{\pi}{2}\hat{S}_y} \tag{4.24}$$

links the Hamiltonian with harmonic potential to the diagonal conformal generator,

$$U: \quad -2i\omega S_z \quad \mapsto \quad H_\omega, \qquad \hat{H}_\omega = -2i\omega U \hat{S}_z U^{-1}. \tag{4.25}$$

The operator U defines a complex-valued nonlocal map which, in contrast to its counterpart $\tilde{U}$, is canonical. Nevertheless, its action on the space spanned by the descendants (2.11) is given by a $SL(2, \mathbb{C})$ representation matrix. The transformation (4.24) mapping S_z to H_ω is determined only up to an overall z rotation from the right.[4] Our choice of (4.24) is fixed by the transformation rules

$$U: \quad S_x \quad \mapsto \quad S_z \quad \text{and} \quad 2i\, S_y \quad \mapsto \quad \omega^{-1} S_+ + \omega S_-. \tag{4.26}$$

The operator (4.24) diagonalizes also the basis (2.11) of the spin-s representation comprising the descendants of the spin-s integral of conformal mechanics. In complete analogy with (4.12), we define

$$\tilde{I}_{s,l} := (i\omega)^s \sum_{k=0}^{2s} I_{s,k} \left(U^s\right)_{kl}, \tag{4.27}$$

where the matrix elements $(U^s)_{kl}$ depend on ω. Applying the second relation in (4.25) to these shifted basis states, we get

$$\hat{H}_\omega \tilde{I}_{s,l} = -2im\omega \tilde{I}_{s,l} \quad \text{with } m = s - l. \tag{4.28}$$

Due to these eigenvalue relations, the "harmonics" (4.33) oscillate in time with integer frequencies proportional to the spin projection value,

$$\tilde{I}_{sm}(t) = e^{-2im\omega(t-t_0)} \tilde{I}_{sm}(t_0). \tag{4.29}$$

Here and in the following, we use the standard basis for $sl(2, \mathbb{R})$ representations by introducing

$$I_{sm} = \sqrt{\frac{(s+m)!}{(2s)!(s-m)!}} I_{s,s-m}, \qquad \tilde{I}_{sm} = \sqrt{\frac{(s+m)!}{(2s)!(s-m)!}} \tilde{I}_{s,s-m} \quad \text{with } -s \le m \le s, \tag{4.30}$$

which is distinguished from the previous basis by omitting the comma between indices. In this basis, the decomposition (4.9) into radial and angular parts reads

$$I_{sm} = \sum_{m'=-m}^{s} \sqrt{\binom{s+m}{m+m'}\binom{s+m'}{m+m'}} \frac{p_r^{m+m'}}{r^{2s-m-m'}} f_{sm'}, \tag{4.31}$$

where we applied the same index nomenclature to the fs given by (4.16).

As was mentioned before, the transformation U is canonical and preserves the Poisson brackets,

$$U: \quad \{I_{s_1 m_1}, I_{s_2 m_2}\} \quad \mapsto \quad \{\tilde{I}_{s_1 m_1}, \tilde{I}_{s_2 m_2}\}. \tag{4.32}$$

In particular, if some $\tilde{I}_{sm}$ are in involution, then the corresponding I_{sm} are in involution, too.

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[4] Indeed, [16] used instead the following operator for the Calogero model: $\omega^{-\hat{S}_z} e^{\frac{i\pi}{2}\hat{S}_x} = U i^{\hat{S}_z}$.

The matrix form of the action of the shift operators (4.24) and (4.18) acquires the following form,

$$\tilde{I}_{sm} = \sum_{m'=-s}^{s} d_{m'm}^{s}(\pi/2)(i\omega)^{s-m'} I_{sm'}, \tag{4.33}$$

$$I_{sm'} = \sum_{m=-s}^{s} d_{m'm}^{s}(\pi/2)(i\omega)^{m'-s} \tilde{I}_{sm}. \tag{4.34}$$

These formulae are analogous to (4.19) and (4.20). Actually, the transformation (4.33) is equivalent to the transformation (4.12), which is the inverse transpose of (4.19), according to the definition (4.14).

Finally, we would like to take advantage of the structural analogy of the two models and directly relate the corresponding shifted harmonics (4.19) and (4.33). To this end, we first substitute (4.20) into (4.31) and then insert the resulting expression for I_{sm} into (4.33). Ultimately, we arrive at

$$\tilde{I}_{sm} = \sum_{m',m_1,m_2}' c_{m_1 m_2, mm'}^{s} (i\omega)^{s-m_2} (i\sqrt{\mathcal{I}})^{s-m_1} \frac{p_r^{m_2+m_1}}{r^{2s-m_2-m_1}} \tilde{f}_{sm'}, \tag{4.35}$$

which expresses the harmonic functions of the confined conformal mechanics in terms of the harmonics of the related angular system. Here, the prime over the sum restricts the indices by the condition $m_1 + m_2 \geq 0$, so that all arguments of the binomial coefficients in

$$c_{m_1 m_2, mm'}^{s} = \sqrt{\binom{s+m_1}{m_1+m_2}\binom{s+m_2}{m_1+m_2}} \, d_{m_1 m}^{s}(\pi/2) \, d_{m' m_2}^{s}(\pi/2) \tag{4.36}$$

are positive.

5. The rational Calogero model with harmonic potential

In this section we specialize to the Calogero model and employ the well known matrix-model description. Firstly, for the unconfined Calogero Hamiltonian H_0 (1.2), we work out the explicit form of the integrals of motion (2.14) composed of the conformal descendants of two standard Liouville integrals. Secondly, for the confined Calogero Hamiltonian H_ω (1.1), we present a simple expression for the oscillating observable $\tilde{I}_{sm}$ of Section 4. Thirdly, we act with the Hamiltonian vector field $\hat{\mathcal{I}}$ related to the angular Hamiltonian on the N standard Liouville integrals of H_ω and obtain the additional $N-1$ integrals for this model. The functional independence of all $2N-1$ integrals is proven explicitly, demonstrating that they comprise a complete system. A similar property is already known for H_0 [24,26].

The Hamiltonians (1.1) and (1.2) can be obtained by $SU(N)$ reduction, respectively, from the hermitian matrix models [15,36,37][5]

$$H_\omega = \frac{1}{2}(P^2) + \frac{\omega^2}{2}(Q^2) \quad \text{and} \quad H_0 = \frac{1}{2}(P^2). \tag{5.1}$$

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[5] We apply the same notation H_0 and H_ω for the Calogero systems and the related matrix models. All other notations like I_{sm} or $\tilde{I}_{sm}$ are preserved also.

Here, P and Q are hermitian matrices containing the canonical momenta and coordinates, subject to

$$\{P_{ij}, Q_{i'j'}\} = \delta_{ij'}\delta_{ji'}.$$ (5.2)

The round brackets denote the SU(N) trace,

$$(X) := \text{tr}\, X.$$ (5.3)

The matrix Hamiltonians (5.1) describe a homogeneous N^2-dimensional oscillator and a free particle in $\mathbb{R}^{N^2}$, respectively. In the matrix-model reduction, the Calogero coupling g is recovered by the gauge-fixing relation

$$[P, Q] = -ig(1 - e \otimes e) \quad \text{with } e = (1, 1, \ldots, 1).$$ (5.4)

Up to SU(N) transformations, the coordinates and Lax matrix of the Calogero model (1.2) provide the solution of above equation [15,37]:

$$Q_{ij} = \delta_{ij}q_i, \qquad P_{ij} = \delta_{ij}p_i + (1 - \delta_{ij})\frac{ig}{q_i - q_j}.$$ (5.5)

As a result, the matrix Hamiltonians (5.1) are reduced to the corresponding Calogero models (1.1) and (1.2).

The generators (1.2) and (1.4) of the conformal algebra (1.3) acquire the following form in the matrix-model representation:

$$S_z = -\frac{1}{2}(PQ), \qquad S_- = -\frac{1}{2}(Q^2), \qquad S_+ = \frac{1}{2}(P^2).$$ (5.6)

Furthermore, the standard Liouville integrals and their descendants (2.11) of the Calogero Hamiltonian H_0 are obtained from the reduction of [33]:

$$I_s = (P^{2s}) \quad \text{and} \quad I_{s,l} = \frac{(2s)!}{(2s - l)!}(P^{2s-l}Q^l)_{\text{sym}},$$ (5.7)

respectively. Here, $s \le N/2$, and the index 'sym' means symmetrization over all orderings of P, Q matrices inside the trace, e.g.

$$(P^n Q)_{\text{sym}} = (P^n Q), \qquad (P^2 Q^2)_{\text{sym}} = \frac{2}{3}(P^2 Q^2) + \frac{1}{3}(PQPQ).$$ (5.8)

The symmetrized traces can be computed by means of the generating function

$$((P + vQ)^{2s}) = \sum_{l=0}^{2s} \binom{2s}{l}(P^{2s-l}Q^l)_{\text{sym}} v^l = \sum_{l=0}^{2s} \frac{v^l}{l!}I_{s,l},$$ (5.9)

which can be considered as an extension of Newton's binomial formula.

Using (5.7), we can write down the integral (2.14) in terms of symmetrized traces,

$$I_{s_1+s_2-k}^{(s_1,s_2)} = \frac{(2s_1)!(2s_2)!}{(2s_1 - k)!(2s_2 - k)!}\sum_{l=0}^{k}(-1)^l\binom{k}{l}(P^{2s_1-k+l}Q^{k-l})_{\text{sym}}(P^{2s_2-l}Q^l)_{\text{sym}}.$$

(5.10)

The coefficient in front of the sum is not essential and can be ignored. Substituting $s_1 = 1$, $s_2 = n/2$ and $k = 1$ in above equation, we arrive at the integral proportional to $\{\mathcal{I}, I_{\frac{n}{2}}\}$, as was shown in (2.16),

$$I_{\frac{n}{2}}^{(1,\frac{n}{2})} \sim \hat{\mathcal{I}} I_{\frac{n}{2}} \sim (PQ)(P^n) - (P^2)(P^{n-1}Q). \tag{5.11}$$

For $n = 2$ it vanishes, but the remaining $N - 1$ integrals together with the Liouville ones (P^n), $1 \leq n \leq N$, constitute a complete system of constants of motion of H_0. The functional independence of this set can be seen from the free-particle limit $g = 0$ where, according to (5.5), the matrices P, Q are diagonal. This property reveals the role of the angular part: it generates the full system of integrals for H_0 by acting on the Liouville ones. Below we will show that this role of $\mathcal{I}$ extends to the confined system H_ω as well. For the particular case of $s_1 = s_2 = n/2$ and $k = 2$ in (5.10), we find the integral

$$I_{n-2}^{(\frac{n}{2},\frac{n}{2})} \sim (P^{n-2}Q^2)_{\text{sym}}(P^n) - (P^{n-1}Q)^2. \tag{5.12}$$

At $n = 2$ we recover the Casimir element of (5.6) describing the angular mechanics.

Before passing to the confined model H_ω, we briefly consider the issue of the H_0 integrals (2.18) obtained by taking Poisson brackets. Let us take Liouville integrals (5.7) for I_{s_1} and I_{s_2}. In the $k = 0$ special case, (2.18) reduces to $\{I_{s_1}, I_{s_2}\} = 0$, since I_{s_1} and I_{s_2} are in involution. For $k = 1$ one finds the Liouville integral $I_{s_1+s_2-1}$. The $k = 2$ case vanishes again, as can be calculated using (5.2) and (5.7). In general, the relations (5.5) imply that the Poisson brackets in (2.18) evaluate to

$$\{I_{s_1,k_1}, I_{s_2,k_2}\} \sim (s_1 k_2 - s_2 k_1) I_{s_1+s_2-1,k_1+k_2-1} + \dots, \tag{5.13}$$

where the remaining terms are of order $O(g)$ and thus vanish in the free-particle limit. Their structure is more complicated: for higher spins they may contain, besides traces, also mean values $\langle e | \dots | e \rangle$ of products of P, Q matrices. In total, one obtains the whole infinite algebra of observables of the Calogero model [38].

For studying the Calogero system in an external harmonic potential, it is most suitable to employ the creation and annihilation combinations

$$A^\pm = \frac{1}{\sqrt{2\omega}} P \pm i\sqrt{\frac{\omega}{2}} Q. \tag{5.14}$$

In terms of these, the Hamiltonian H_ω reads [36]

$$H_\omega = \omega(A^+ A^-) \quad \text{with} \quad \{A_{ij}^-, A_{i'j'}^+\} = i\delta_{ij'}\delta_{ji'}. \tag{5.15}$$

The matrix variables $A^\pm$ oscillate in time with frequency ω:

$$\dot{A}^\pm = \{H_\omega, A^\pm\} = \pm i\omega A^\pm \quad \longrightarrow \quad A^\pm(t) = e^{\pm i\omega(t-t_0)} A^\pm(t_0). \tag{5.16}$$

Using the canonical brackets in (5.15) and the expressions (5.6) for the $SL(2,\mathbb{R})$ generators, one can calculate the action of the transformation (4.24) on the phase-space variables of the matrix model:

$$UP = e^{-i\frac{\pi}{4}} A^-, \qquad UQ = e^{i\frac{\pi}{4}} A^+. \tag{5.17}$$

Recall that this transformation maps the diagonal generator S_z to the Hamiltonian H_ω according to (4.25). It turns into the analogous transformation $\widetilde{U}$ given by (4.11) upon substituting

$$(P, Q) \to (p_r, r^{-1}), \qquad (\sqrt{\omega} A^-, \sqrt{\omega} A^+) \to (z, \bar{z}), \qquad \omega \to \sqrt{\mathcal{I}}. \tag{5.18}$$

In this context, the analogy between the confined Calogero model and the related angular system becomes more transparent in the matrix model description. It expands also to the homogeneous polynomials, which form spin-s representations of conformal algebras. According to (4.30) and (5.7), the matrix form of original and shifter normalized states are:

$$I_{sm} = \sqrt{\binom{2s}{s-m}} \left(P^{s+m} Q^{s-m}\right)_{\text{sym}}, \tag{5.19}$$

$$\tilde{I}_{sm} = \sqrt{\binom{2s}{s-m}} \omega^s \left((A^-)^{s+m} (A^+)^{s-m}\right)_{\text{sym}}. \tag{5.20}$$

According to (4.29), $\tilde{I}_{sm}$ oscillates with the frequency $-2m\omega$, which also can be seen from (5.16).

In fact, the trace of any product of $A^{\pm}$ matrices,

$$\left(A^{\sigma_1} \ldots A^{\sigma_n}\right) \quad \text{with } \sigma_i \in \{+, -\}$$

oscillates with integer frequency equal to $\omega \sum_i \sigma_i$. Any product of such observables with a vanishing total sum of the σ_i will be an integral of motion of the Calogero system (1.1). The number of such integrals is significantly higher than $2N - 1$, but they are not independent.

The Liouville integrals of the Hamiltonian H_ω can be extracted from this general set using either the Lax-pair method or the symmetries of the original matrix model (5.1) or (5.15). We recall the second way described in the review [36]. In terms of the hermitian left-multiplication generator or Lax matrix A^+A^-, one has

$$\tilde{I}_n = \left((A^+A^-)^n\right). \tag{5.21}$$

The related flow is not symplectic, because the left (or right) multiplication does not preserve the Poisson brackets in (5.15), as the adjoint action does. The matrix elements of A^+A^- obey the $U(N)$ commutation relations, and the Liouville integrals (5.21) can be identified with the Casimir elements of that group [36].

The Liouville integrals $\tilde{I}_n$ are unsymmetrized analogs of the integrals $\tilde{I}_{s=n\,m=0}$ (5.20) with integer spin $s = n \in \mathbb{N}$. They form another set of Liouville integrals [16]. Up to a numerical factor, the integrals $\tilde{I}_{n0}$ and $\tilde{I}_n$ coincide in their term of highest power in the momenta, corresponding to the free-particle limit $g = 0$. Note that, according to (4.32), the descendants I_{n0} also are in involution. The first integral is proportional to the Hamiltonian and corresponds to the central $U(1)$ part,

$$\tilde{I}_1 = \left(A^+A^-\right) = \omega^{-1} H_\omega = \omega^{-1} H_0 + \omega K. \tag{5.22}$$

The remaining two bilinear traces of (5.14) are also related to the conformal generators. Using their matrix form (5.6) and (2.4), we obtain:

$$\left(A^+A^+\right) = \omega^{-1} H_0 - \omega K - i D, \qquad \left(A^-A^-\right) = \omega^{-1} H_0 - \omega K + i D. \tag{5.23}$$

From these equations it is easy to express the angular Hamiltonian, described by the Casimir element (1.5) of the conformal algebra, in terms of $A^{\pm}$ matrices. It has a rather simple form:

$$\mathcal{I} = \left(A^+A^-\right)^2 - \left(A^+A^+\right)\left(A^-A^-\right). \tag{5.24}$$

Now recall that although the conformal generators (1.3) are not symmetries of the Hamiltonian H_ω, their Casimir element is conserved: $\dot{\mathcal{I}} = \{H_\omega, \mathcal{I}\} = 0$. The invariant $\mathcal{I}$ is not a Liouville

integral since does not commute with the whole set (5.21). Its Poisson brackets with the Liouville integrals give rise to additional integrals like for H_0:

$$J_n = \{\mathcal{I}, \tilde{I}_n\} = 2in\big[\big(A^-A^-\big)\big(A^+A^+\big(A^+A^-\big)^{n-1}\big) - \big(A^+A^+\big)\big(A^-A^-\big(A^+A^-\big)^{n-1}\big)\big].$$

(5.25)

Since $J_1 = 0$, we find $2N - 1$ integrals $(\tilde{I}_1, \ldots, \tilde{I}_N, J_2, \ldots J_N)$, which appear to form a complete set of integrals for H_ω. Their functional independence is shown in Appendix B. This confirms the superintegrability of the Calogero model in the harmonic potential. The construction of the additional integrals J_n is similar to the one for the free Calogero Hamiltonian described by Eq. (5.11) above.

We note that in the quantum case the Calogero model becomes algebraically integrable for certain discrete values of the coupling constant (including zero), meaning that there exists an $(N + 1)$th conserved charge in involution with the N Liouville integrals (for a recent review see [40] and references therein). It can be considered as a supercharge, since it squares to a polynomial in the Liouville integrals [41]. Such nonlinear supersymmetry algebras have been identified and applied in various single-particle quantum mechanical systems [42]. The additional integral is constructed as a product of certain intertwiners, which relate Calogero models at different 'algebraically integrable' values of the coupling and thus link them to the free-particle system. A similar algebra, via a nonlinear extension of $osp(2 \,|\, 2)$, arises as a dynamical quantum symmetry in a supersymmetric single-particle system [43]. Finally, for arbitrary values of the Calogero coupling, the conformal algebra helps to construct a nonlocal map to the free-particle system [21,22]. It would be interesting to relate these different instances.

6. Conclusions

In this article we have studied the relations between integrals of motion and related oscillating quantities of general conformal mechanics, the corresponding angular subsystem and the conformal mechanics in an external harmonic potential, which stems from the properties of $sl(2, \mathbb{R})$ and its representations. Below we describe briefly the main results.

First, we have considered a conformal mechanical system with at least one higher-order integral of motion, which generates a representation of the conformal algebra [20]. Decomposing the tensor product of two (or more) such representations, we have constructed new conserved charges. Applying this approach to the Liouville integrals of the rational Calogero model, we have reproduced all additional integrals responsible for its superintegrability [11,13].

Second, starting with one or more integrals of motion of conformal mechanics, a system of oscillating observables with commensurate frequencies has been constructed for two related systems: for the angular sub-mechanics and for the original system in a confining harmonic potential. Their products with vanishing total frequency yield integrals of motion. In both cases we have employed appropriate conformal group rotation operators. As a result, we have found the precise relation between the oscillating quantities of confined conformal mechanics and of the related angular subsystem.

Third, we have studied the Hamiltonian vector field action of the angular Calogero system on the standard Liouville integrals of the Calogero system with an external harmonic potential. We have proved that this action results in additional integrals which, together with the Liouville ones, form a complete and independent system. This generalizes a similar property for the unconfined Calogero Hamiltonian.

The results of the current article may be extended to mechanical systems with superconformal symmetry. Similar work in this direction has been done in [44]. In the $\mathcal{N} = 2$ case, the dynamical symmetry enhances to the orthosymplectic Lie superalgebra $osp(2 \,|\, 2)$ [45]. Then new integrals of motion can be constructed from known ones using the tensor product of two orthosymplectic representations, repeating the steps performed in Sections 2 and 3 for the $sl(2, \mathbb{R})$ case.

Acknowledgements

Appendix A

In this appendix we calculate the transformation matrix (4.12), then express it in terms of Wigner's small d-matrix.

The explicit expression for the inverse $\widetilde{U}^s$-matrix from (4.14) can be obtained by substitution of the inverse transformations (4.11) into the decomposition (4.8), with subsequent comparison of the obtained angular coefficients with $f_{s,k}$ in (4.8). As a result, we get:

$$\left(\widetilde{U}^s\right)^{-1}_{lk} = (i\sqrt{\mathcal{I}})^{s-k} b^s_{lk}, \tag{A.1}$$

where the above b-matrix satisfies the following relation:

$$2^{-s}(z+\bar{z})^{2s-k}(\bar{z}-z)^k = \sum_{l=0}^{2s} b^s_{lk} z^{2s-l}\bar{z}^l. \tag{A.2}$$

Its explicit form can be calculated using Newton's binomial formula:

$$\begin{aligned}
b^s_{lk} &= \sum_{t=\max(0,l+k-2s)}^{\min(k,l)} \frac{(-1)^{k+t}}{2^s} \binom{k}{t}\binom{2s-k}{l-t} \\
&= \sum_{t=\max(0,l+k-2s)}^{\min(k,l)} i\frac{(-1)^{k+t}}{2^s} \frac{k!(2s-k)!}{t!(k-t)!(l-t)!(2s-l-k+t)!}.
\end{aligned} \tag{A.3}$$

Note that the sum over t in (A.3) is taken over all non-negative values of the four factorials in the denominator.

Then it is easy to see that the map $z \to (z+\bar{z})/\sqrt{2}$, $\bar{z} \to (\bar{z}-z)/\sqrt{2}$ is orthogonal. Therefore, the matrix (A.3) is orthogonal too:

$$\sum_i b^s_{li} b^s_{ki} = \delta_{lk}. \tag{A.4}$$

Hence, from (A.1) we obtain:

$$\left(\widetilde{U}^s\right)_{kl} = (i\sqrt{\mathcal{I}})^{k-s} b^s_{kl}. \tag{A.5}$$

Next, following [26], we express the matrix b_{kl}^s in terms of Wigner's small d-matrix, which describes an SU(2) rotation around the y axis: $d_{m'm}^s(\beta) = \langle sm'| \exp(-\beta \hat{S}_y^R)|sm\rangle$. Comparing the last expression in (A.3) with the formula (2) in §4.3 of [39] we get:

$$b_{lk}^s = \sqrt{\frac{k!(2s-k)!}{l!(2s-l)!}} d_{s-l\,s-k}^s(\pi/2). \tag{A.6}$$

Finally, using (A.5) and (A.1), we obtain the explicit forms of the $\widetilde{U}^s$-matrix and its inverse:

$$\left(\widetilde{U}^s\right)_{s-m'\,s-m} = \sqrt{\frac{(s-m)!(s+m)!}{(s-m')!(s+m')!}} d_{m'm}^s(\pi/2)(i\sqrt{\mathcal{I}})^{s-m'}, \tag{A.7}$$

$$\left(\widetilde{U}^s\right)_{s-m\,s-m'}^{-1} = \sqrt{\frac{(s-m')!(s+m')!}{(s-m)!(s+m)!}} d_{mm'}^s(\pi/2)(i\sqrt{\mathcal{I}})^{m'-s}. \tag{A.8}$$

Appendix B

In this appendix we prove that the $2N-1$ integrals $(I_1, \ldots, I_N, J_2, \ldots J_N)$ defined by (5.21) and (5.25), are functionally independent. Here, we omit the tilde on $\tilde{I}_i$, since here we deal only with H_ω and its integrals. It suffices to prove their independence for the free-particle limit $g \to 0$, since this projects to the highest-order term in momenta for the polynomials I_n and J_n. In this limit, the reduced P and Q matrices given by (5.5) are diagonal:

$$A = \text{diag}(a_1, \ldots, a_N) + O(g), \qquad a_i = \frac{p_i}{\sqrt{2\omega}} - i\sqrt{\frac{\omega}{2}} q_i =: \sqrt{\rho_i} e^{-i\varphi/2}. \tag{B.1}$$

For the integrals we thus have

$$I_n = \sum_{i=1}^N \rho_i^n \quad \text{and} \quad J_n = 4n \sum_{i,j=1}^N \rho_i \rho_j^n \sin(\varphi_i - \varphi_j). \tag{B.2}$$

The functional independence of this set of integrals is equivalent to the nondegeneracy of the Jacobian matrix

$$\frac{\partial(I_1, \ldots, I_N, J_2, \ldots J_N)}{\partial(\rho_1, \ldots, \rho_N, \varphi_2, \ldots, \varphi_N)} = \begin{pmatrix} \dfrac{\partial(I_1, \ldots, I_N)}{\partial(\rho_1, \ldots, \rho_N)} & \dfrac{\partial(I_1, \ldots, I_N)}{\partial(\varphi_2, \ldots, \varphi_N)} \\[2mm] \dfrac{\partial(J_2, \ldots, J_N)}{\partial(\rho_1, \ldots, \rho_N)} & \dfrac{\partial(J_2, \ldots, J_N)}{\partial(\varphi_2, \ldots, \varphi_N)} \end{pmatrix}. \tag{B.3}$$

Due to the obvious relations

$$\frac{\partial I_n}{\partial \varphi_k} = 0, \qquad \frac{\partial I_n}{\partial \rho_k} = n\rho_k^{n-1}, \qquad \frac{\partial J_n}{\partial \varphi_k} = \sum_i (\rho_k \rho_i^n - \rho_i \rho_k^n) \cos(\varphi_k - \varphi_i), \tag{B.4}$$

the Jacobi matrix has block-triangular form, so its determinant is given by the product

$$\left|\frac{\partial(I_1, \ldots, I_N)}{\partial(\rho_1, \ldots, \rho_N)}\right| \left|\frac{\partial(J_2, \ldots, J_N)}{\partial(\varphi_2, \ldots, \varphi_N)}\right|. \tag{B.5}$$

The first term is proportional to the Vandermonde determinant

$$\left|\frac{\partial(I_1, \ldots, I_N)}{\partial(\rho_1, \ldots, \rho_N)}\right| = N! \prod_{1 \le i < j \le N} (\rho_j - \rho_i), \tag{B.6}$$

while the second one is more complicated. It can be presented as a product of two rectangular matrices with dimensions $(N - 1) \times N$ and $N \times (N - 1)$. Indeed, according to (B.4),

$$\frac{\partial J_n}{\partial \varphi_k} = \sum_{i=1}^{N} \rho_i^{n-1} B_{ik} \quad \text{with } B_{ik} = b_{ik} - \delta_{ik} \sum_{l=1}^{N} b_{lk} \quad \text{and} \quad b_{ik} = \rho_i \rho_k \cos(\varphi_i - \varphi_k). \tag{B.7}$$

It is possible to express the matrix equation (B.7) in terms of square $N \times N$ matrices with an additional row and column via

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ \rho_1 & \partial J_2/\partial \varphi_2 & \dots & \partial J_2/\partial \varphi_N \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{N-1} & \partial J_N/\partial \varphi_2 & \dots & \partial J_N/\partial \varphi_N \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \rho_1 & \rho_2 & \dots & \rho_N \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{N-1} & \rho_2^{N-1} & \dots & \rho_N^{N-1} \end{pmatrix}$$
$$\cdot \begin{pmatrix} 1 & B_{12} & \dots & B_{1N} \\ 0 & B_{22} & \dots & B_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & B_{N2} & \dots & B_{NN} \end{pmatrix}. \tag{B.8}$$

Indeed, its restriction to the $\partial J/\partial \varphi$ block coincides with (B.7). The first column of this matrix product is obvious, while the zeros in the first row appear due to the property $\sum_{i=1}^{N} B_{ij} = 0$, which follows from the definition (B.7).

This relation factorizes out the Vandermonde determinant from the Jacobian $\frac{\partial(J_2 \dots J_N)}{\partial(\varphi_2 \dots \varphi_N)}$. The remaining matrix can be further reduced by extracting the diagonal matrix $B_{ik} = \tilde{B}_{ik}\rho_k$, where $\tilde{B}$ is obtained by the substitution

$$b_{ik} \to \tilde{b}_{ik} = \rho_i \cos(\varphi_i - \varphi_k) \tag{B.9}$$

in the definition of B in (B.7). Together with (B.8) this implies

$$\left| \frac{\partial(J_2, \dots, J_N)}{\partial(\varphi_2, \dots, \varphi_N)} \right| = \prod_{i=2}^{N} \rho_i \prod_{1 \leq i < j \leq N} (\rho_j - \rho_i) M_{11}, \tag{B.10}$$

where M_{ij} denotes a minor of the matrix $\tilde{B}$.

In the simplest case of equal phases $\varphi_i = \varphi$, this determinant is given by characteristic polynomial of the rank-one matrix $\tilde{b}_{ij} = \rho_j$ with the value $\rho = \sum_{i=1}^{N} \rho_i$ for the eigenvalue variable, which can be easy calculated:

$$M_{11} = \det(\tilde{B}_{ij} - \rho \delta_{ij}) = \rho^{N-1} \rho_1. \tag{B.11}$$

Therefore, the Jacobian vanishes at special points only. The matrix $\tilde{B}$ generically depends analytically on the arguments φ_i, so its determinant can vanish only at a set of measure zero in the space of φ_i.

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