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Abstract

A description of right (left) quasi-duo $\mathbb{Z}$-graded rings is given. It shows, in particular, that a strongly $\mathbb{Z}$-graded ring is left quasi-duo if and only if it is right quasi-duo. This gives a partial answer to a problem posed by Dugas and Lam in [1].

A ring R with an identity is called [1] *right* (*left*) *quasi-duo* if every maximal right (left) ideal of R is two-sided. Quasi-duo rings were studied in many papers (Cf. [1], [5] and papers quoted there). The main open problem in the area asks whether the classes of left and right quasi-duo rings coincide (it is important, as it concerns the problem to what extend the notion of primitivity is left-right symmetric, Cf. [1]). This problem was also an initial motivation for our studies. Namely the results obtained in [2] on quasi-duo skew polynomial rings show that it would be interesting to examine whether it could be possible to distinct these classes within $\mathbb{Z}$-graded rings or, more generally, to describe $\mathbb{Z}$-graded right (left) quasi-duo rings. The methods of [2] are rather specific for skew-polynomial rings and one cannot apply them to $\mathbb{Z}$-graded rings. In this paper we find another approach to that problem and describe $\mathbb{Z}$-graded right (left) quasi-duo rings. This description shows, in particular, that a strongly $\mathbb{Z}$-graded ring is right quasi-duo if and only if it is left quasi-duo. Thus, for strongly $\mathbb{Z}$-graded rings, the above mentioned Dugas-Lam problem has a positive solution. As an application we also get back in another way the characterization of right (left) skew polynomial and Laurent polynomial rings obtained in [2].

The results on the Jacobson radical, the pseudoradical and maximal ideals of $\mathbb{Z}$-graded rings (see Proposition 3, Theorem 2) can be of independent interest.

All rings in this paper are associative with identity. To denote that I is an ideal (left ideal, right ideal) of a ring R we will write $I \lhd R$ ($I <_l R$, $I <_r R$). The Jacobson radical of a ring R will be denoted by $J(R)$.

It is clear that R is right (left) quasi-duo if and only if $R/J(R)$ is right (left) quasi-duo and that Jacobson semisimple right (left) quasi-duo rings are subdirect sums of division rings, so they are reduced rings. The class of right (left) quasi-duo rings is closed under homomorphic images and finite subdirect sums (Cf.[1]).

In what follows $\mathbb{Z}$ denotes the additive group of integers and R denotes a $\mathbb{Z}$-graded ring. Recall that $R = \bigoplus_{n\in\mathbb{Z}} R_n$, the direct sum of additive subgroups R_n, with $R_nR_m \subseteq R_{n+m}$ for all $n, m \in \mathbb{Z}$. If $R_nR_m = R_{n+m}$, then R is called *strongly graded*.

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Elements of $\bigcup_{n\in\mathbb{Z}} R_n$ are called *homogeneous.* Every $r \in R$ can be written as a finite sum $r = \sum_{m\le i\le n} r_i$, where $r_i \in R_i$ is called the *homogeneous component* of r of degree i. If r_m and r_n are nonzero, then the length $l(r)$ of r is defined as $n - m + 1$. Clearly a nonzero element of R is homogeneous if and only if its length is equal to 1.

An ideal I of R is called *homogeneous* if $I = \bigoplus_{n\in\mathbb{Z}}(I \cap R_n)$. The largest homogeneous ideal contained in a given ideal I of R will be denoted by $(I)_h$.

The following well known result of G. Bergman (Cf. [4]) plays a substantial role in the paper.

Theorem 1. *For every $\mathbb{Z}$-graded ring R*

(i) $J(R)$ is a homogeneous ideal;

(ii) If $r \in \bigcup_{0\neq n\in\mathbb{Z}} R_n$, then $1 + r$ is invertible if and only if r is nilpotent.

A homogeneous ideal P of R is called *graded prime* if $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for arbitrary homogeneous ideals I and J of R. It is well known and not hard to check that if P is a prime ideal of R, then $(P)_h$ is a graded prime ideal of R. It is also well known that a homogeneous ideal of a $\mathbb{Z}$-graded ring is prime if and only if it is graded prime.

The intersection of all nonzero graded prime ideals of R will be called the *graded pseudoradical* of R. The empty intersection, by definition, is equal to R.

The following result generalizes Lemma 3.2 from [3].

Theorem 2. *Suppose that a $\mathbb{Z}$-graded ring R contains a maximal ideal M such that $(M)_h = 0$. Then the graded pseudoradical of R is nonzero.*

Proof. Let $a = \sum_{m\le i\le n} a_i$ be a nonzero element of M of minimal length, where $a_m \neq 0 \neq a_n$. Since $(M)_h = 0$, $l(a) \geq 2$.

Let C (resp. D) denote the sets of all n-th (resp. m-th) components of nonzero elements from $M \cap (\bigoplus_{m\le i\le n} R_i)$. Notice that C and D are non empty homogeneous sets depending only on M.

If R has no nonzero graded prime ideals, then the graded pseudoradical of R is equal to R, so the thesis holds.

Suppose now that we can pick a nonzero graded prime ideal Q of R. Then $M + Q = R$, so $1 = b + q$, where $b = \sum_{s\le l\le t} b_l \in M$ and $1 - b_0 \in Q$ and $b_i \in Q$, for $s \le i \le t$, $i \neq 0$. This implies that precisely one homogeneous component of $b = \sum_{s\le l\le t} b_l \in M$ is not in Q. Suppose that b is an element in M with the smallest possible length amongst the elements of M having precisely one homogeneous component not in Q. Let us write $b = \sum_{s\le l\le t} b_l \in M$ with $b_k \notin Q$.

If $k \neq t$ we claim that $C \subseteq Q$. If not then there exists $r = \sum_{m\le i\le n} r_i \in M$ such that $r_n \notin Q$. Since Q is a prime graded ideal, there is $c \in R_w$, for some $w \in \mathbb{Z}$, such that $b_k c r_n \notin Q$. Notice that $n - m + 1 = l(r) \le l(b) = t - s + 1$ and the element $u = bcr_n - b_t cr \in M$ is such that precisely one homogeneous component of u (namely u_{k+w+n}) is not in Q. Moreover, since $(bcr_n)_l = (b_t cr)_l = 0$ if $l < s + w + n$ and $u_{t+w+n} = 0$, we get $l(u) < l(b)$, which is impossible, by the choice of b. This proves the claim.

If $k = t$ we can prove in a similar way that $D \subseteq Q$.

We conclude that $CRD \subseteq Q$ for any nonzero graded prime ideal Q of R. Since M is prime and $(M)_h = 0$, the ring R is a graded prime ring and hence $CRD \neq 0$. This yields the desired result. □

In what follows we denote by $\mathcal{A}$ the set of all maximal right ideals M of R such that $R_n \not\subseteq M$, for some $0 \neq n \in \mathbb{Z}$ and by $\mathcal{B}$ the set of remaining maximal right ideals of R. Set $A(R) = \bigcap_{M\in\mathcal{A}} M$ and $B(R) = \bigcap_{M\in\mathcal{B}} M$.

It is easy to describe $B(R)$. Note that $U = \sum_{0\neq n\in\mathbb{Z}} R_{-n}R_n \lhd R_0$. It is clear that if $M \in \mathcal{B}$, then $M = M_0 + \bigoplus_{0\neq n\in\mathbb{Z}} R_n$ for a maximal right ideal M_0 of R_0 containing U. Consequently $B(R) = J + \sum_{0\neq n\in\mathbb{Z}} R_n$, where J is the ideal of R_0 containing U such that $J(R_0/U) = J/U$. In particular, $B(R)$ is a two-sided ideal of R.

If R is strongly graded, then for every $0 \neq n \in \mathbb{Z}$, $R_0 = R_nR_{-n}$. This shows that in this case $\mathcal{B} = \emptyset$, so $B(R) = R$ and $A(R) = J(R)$.

Now we will describe $A(R)$. Let $A_l = \{r \in R \mid R_nr \subseteq J(R)$, for every $0 \neq n \in \mathbb{Z}\}$ and $A_r = \{r \in R \mid rR_n \subseteq J(R)$, for every $0 \neq n \in \mathbb{Z}\}$.

Proposition 3. *Let R be a $\mathbb{Z}$-graded ring. Then:*

(i) $A(R) = A_l = A_r$

(ii) $A(R) \cap (\bigoplus_{0\neq n\in\mathbb{Z}} R_n) = J(R) \cap (\bigoplus_{0\neq n\in\mathbb{Z}} R_n)$.

Proof. (i). It is clear that $A_l \lhd R$. Hence $A_lR_n <_l R$, for every $0 \neq n \in \mathbb{Z}$. Since $(A_lR_n)^2 \subseteq J(R)$ and $R/J(R)$ is semiprime, $A_lR_n \subseteq J(R)$. This proves that $A_l \subseteq A_r$. Dual arguments give the opposite inclusion and show that $A_l = A_r$.

Take any $M \in \mathcal{A}$. Then $R_n \nsubseteq M$, for some $0 \neq n \in \mathbb{Z}$. Obviously $(A_r + M)R_n \subseteq M$. Thus $A_r + M \neq R$ and maximality of M implies that $A_r \subseteq M$. Consequently $A_r \subseteq A(R)$. Clearly $A(R) \cap B(R) = J(R)$, $B(R) \lhd R$ and $A(R) <_r R$, so $A(R)B(R) \subseteq J(R)$. Hence, since $\bigoplus_{0\neq n\in\mathbb{Z}} R_n \subseteq B(R)$, we get that $A(R) \subseteq A_r$.

(ii). By (i), $A(R)R_n + R_mA(R) \subseteq J(R)$, for arbitrary $n, m \in \mathbb{Z} \setminus \{0\}$. This implies that if I is the ideal of R generated by $A(R) \cap (\bigoplus_{0\neq n\in\mathbb{Z}} R_n)$, then $I^2 \subseteq J(R)$. Consequently $A(R) \cap (\bigoplus_{0\neq n\in\mathbb{Z}} R_n) \subseteq I \subseteq J(R)$. Now it is easy to complete the proof of (ii). □

Theorem 4. *If a $\mathbb{Z}$-graded ring R is right (left) quasi-duo, then R/M is a field, for every $M \in \mathcal{A}$.*

Proof. We will prove the result when R is right quasi-duo. If R is left quasi-duo, symmetric arguments can be applied. Let $M \in \mathcal{A}$. Passing to the factor ring $R/(M)_h$, we can assume without loss of generality that $(M)_h = 0$. Since R is right quasi-duo, R/M is a division ring. Making use of those two facts, one can easily check that R is a domain. Moreover, by Theorem 2, the graded pseudoradical P of R is nonzero.

Let $0 \neq n \in \mathbb{Z}$ and $a \in P_n = P \cap R_n$. Clearly a is not nilpotent, as R is a domain. Thus, by Theorem 1, $1+a$ is not invertible. Hence there exists a maximal right ideal T of R containing $1+a$. Since R is quasi-duo, $T \lhd R$. Now $(T)_h$ is a prime homogeneous ideal of R, so if $(T)_h \neq 0$, then $P \subseteq T$. This is impossible as otherwise $1 = (1+a) - a \in T$. Therefore $(T)_h = 0$. Now for every homogeneous element b of R, $ab - ba = (1+a)b - b(1+a) \in (T)_h = 0$. This shows that a belongs to the center $Z(R)$ of R and implies that $P_n \subseteq Z(R)$, for all nonzero $n \in \mathbb{Z}$. Since $M \in \mathcal{A}$, by definition, there exists $0 \neq m \in \mathbb{Z}$ such that $R_m \nsubseteq M$. In particular $R_m \neq 0$. Therefore, since P is a nonzero homogeneous ideal and R is a domain, we can pick a nonzero integer n such that $P_n \neq 0$. Then $P_0P_n \subseteq P_n \subseteq Z(R)$ and, as R is a domain, $P_0 \subseteq Z(R)$ follows. The above implies that $P \subseteq Z(R)$ and shows that the division ring $R/M = (M+P)/M$ is commutative, i.e. it is a field. □

Theorem 5. *A $\mathbb{Z}$-graded ring R is right (left) quasi-duo if and only if R_0 is right (left) quasi-duo and $R/A(R)$ is a commutative ring.*

Proof. Suppose that R is right quasi-duo. Let M be a maximal right ideal of R_0. Clearly MR is a proper right ideal of R. Consequently MR is contained in a maximal right ideal T of R. Since R is right quasi-duo, $T \lhd R$. It is clear that $M = T \cap R_0$, so $M \lhd R_0$. Thus R_0 is a right quasi-duo ring.

When $\mathcal{A} \neq \emptyset$, Theorem 4 implies that $R/A(R)$ is a subdirect sum of fields, so it is a commutative ring. If $\mathcal{A} = \emptyset$, then $A(R) = R$ and the ring $R/A(R)$ is also commutative.

Suppose now that R_0 is right quasi-duo and $R/A(R)$ is commutative. Let I be the ideal of R generated by $\bigcup_{0 \neq n \in \mathbb{Z}} R_n$. Then, by Proposition 3(i), $IA(R) \subseteq J(R)$. Hence $(I \cap A(R))^2 \subseteq J(R)$ and semiprimeness of $J(R)$ implies that $I \cap A(R) \subseteq J(R)$. This shows that $R/J(R)$ is a homomorphic image of a subdirect sum of rings R/I and $R/A(R)$. Clearly R/I is a homomorphic image of R_0. Consequently both R/I and $R/A(R)$ are right quasi-duo, so, further, $R/J(R)$ and R are right quasi-duo.

When R is left quasi-duo, symmetric arguments apply. □

Theorem 5 immediately gives the following

Corollary 6. *Suppose a $\mathbb{Z}$-graded ring R is right quasi-duo. Then:*
1. R_0 is right quasi-duo;
2. R is left quasi-duo iff R_0 is left quasi-duo.

We know, by the remark made just before Proposition 3, that $A(R) = J(R)$, provided R is strongly $\mathbb{Z}$-graded. Thus, by Theorem 5, we get:

Corollary 7. *Suppose that R is strongly $\mathbb{Z}$-graded. Then R is right quasi-duo iff R is left quasi-duo iff $R/J(R)$ is commutative.*

Now, as an application of Theorem 5, we will get characterizations of right (left) quasi-duo skew polynomial rings and skew Laurent polynomial rings obtained in [2].

Let σ be an endomorphism of a ring S and $S[x;\sigma]$ be the associated skew polynomial ring with coefficients from S written on the left. Denote by $N(S)$ the set $\{s \in S \mid s\sigma(s)\cdots\sigma^n(s) = 0$, for some positive integer $n\}$. Clearly $N(S) = \{s \in S \subseteq S[x;\sigma] \mid (sx)^n = 0$, for some positive integer $n\}$. Let $N(S)[x;\sigma]$ be the set of all polynomials from $S[x;\sigma]$ which have all their coefficients in $N(S)$. Notice also that $\sigma(N(S)) \subseteq N(S)$. Thus, if $N(S) \lhd S$ then $N(S)[x;\sigma] \lhd S[x;\sigma]$, σ induces an endomorphism, also denoted by σ, on $S/N(S)$ and $(S/N(S))[x;\sigma] \simeq S[x;\sigma]/N(S)[x;\sigma]$.

Lemma 8. *Suppose that the skew polynomial ring $S[x;\sigma]$ is right (left) quasi-duo. Then $J(S[x;\sigma]) \subseteq N(S)[x;\sigma] \subseteq A(S[x;\sigma])$.*

Proof. Since $S[x;\sigma]$ is right (left) quasi-duo, the ring $S[x;\sigma]/J(S[x;\sigma])$ is reduced, so every nilpotent element of $S[x;\sigma]$ belongs to $J(S[x;\sigma])$. Thus, in particular, $xN(S) \subseteq J(S[x;\sigma])$ and consequently $Sx^nN(S) \subseteq J(S[x;\sigma])$, for all $n > 0$. The ring $S[x;\sigma]$ is $\mathbb{Z}$-graded in the canonical way and the last inclusion together with Proposition 3(i) yield $N(S) \subseteq A(S[x;\sigma])$. This shows that $N(S)[x;\sigma] \subseteq A(S[x;\sigma])$.

Let $ax^n \in J(S[x;\sigma])$, for some $n > 0$. Then, by Theorem 1, ax^n and x^na are also nilpotent elements of $S[x;\sigma]$ and so $x^na \in J(S[x;\sigma])$. Hence $Sx^mx^{n-1}a \subseteq J(S[x;\sigma])$, for all $m > 0$ and Proposition 3(i) shows that $x^{n-1}a \in J(S[x;\sigma])$. Repeating this procedure we obtain $xa \in J(S[x;\sigma])$ and Theorem 1 implies that $a \in N(S)$. Since $J(S[x;\sigma])$ is a homogenous ideal, we obtain $J(S[x;\sigma]) \subseteq N(S)[x;\sigma]$. □

Corollary 9. ([2]) *$S[x;\sigma]$ is right (left) quasi-duo if and only if S is right (left) quasi-duo, $N(S) \lhd S$, $J(S[x;\sigma]) = J(S) \cap N(S) + N(S)[x;\sigma]x$ and $(S/N(S))[x;\sigma]$ is a commutative ring.*

Proof. Suppose that the ring $S[x;\sigma]$ is right (left) quasi-duo. Then, by Proposition 3(i), $A(S[x;\sigma]x) \subseteq J(S[x;\sigma])$. Thus, by Lemma 8, we get $A(S[x;\sigma]) = N(S)[x;\sigma]$. This implies that $N(S)$ is an ideal of S. Now, by Theorem 5, the ring $(S/N(S))[x;\sigma] \simeq S[x;\sigma]/N(S)[x;\sigma]$ is commutative.

Since $B(S[x;\sigma]) = J(S) + S[x;\sigma]x$ and $J(T) = A(T) \cap B(T)$, we also obtain $J(S[x;\sigma]) = J(S) \cap N(S) + N(S)[x;\sigma]x$.

Conversely, by making use of Proposition 3(i), it is evident that when $J(S[x;\sigma]) = J(S) \cap N(S) + N(S)[x;\sigma]$, then $A(S[x;\sigma]) = N(S)[x;\sigma]$. Now if the ring $(S/N(S))[x;\sigma]$ is commutative and S is right (left) quasi-duo, then $S[x;\sigma]$ is right (left) quasi-duo, by Theorem 5. $\square$

Corollary 10. ([2]) *Let σ be an automorphism of a ring S. Then the skew Laurent polynomial ring $S[x,x^{-1};\sigma]$ is right (left) quasi-duo if and only if $N(S) \lhd S$, $J(S[x,x^{-1};\sigma]) = N(S)[x,x^{-1};\sigma]$ and $(S/N(S))[x,x^{-1};\sigma]$ is a commutative ring.*

Proof. Since $S[x,x^{-1};\sigma]$ is a strongly graded, $A(S[x,x^{-1};\sigma]) = J(S[x,x^{-1};\sigma])$.

Suppose now that $S[x,x^{-1};\sigma]$ is right (left) quasi-duo. Then, as $N(S)x$ consists of nilpotent elements, $N(S)[x,x^{-1};\sigma] \subseteq J(S[x,x^{-1};\sigma])$. The opposite inclusion follows immediately from Theorem 1. Obviously $N(S) \lhd S$ and by Theorem 5, $(S/N(S))[x,x^{-1};\sigma]$ is a commutative ring. This proves the only if" part. The "if" part is a direct consequence of Theorem 5. $\square$

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