INTERSECTION COHOMOLOGY OF $\mathbf{S}^1$-ACTIONS[1]

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Given a free action Φ of the circle $\mathbf{S}^1$ on a manifold M there exists a long exact sequence (the *Gysin sequence*) relating the cohomology of the manifolds M and $M/\mathbf{S}^1$:

$$(*) \qquad \cdots \to H^i(M) \xrightarrow{\oint^*} H^{i-1}(B) \xrightarrow{\wedge[e]} H^{i+1}(B) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots .$$

Here $[e] \in H^2(M/\mathbf{S}^1)$ denotes the *Euler class* of Φ and $\oint$ the integration along the fibers of the canonical projection $\pi \colon M \to M/\mathbf{S}^1$. This result has been extended to almost free actions in [9]. In this context, the orbit space is not a manifold but a Sataké manifold.

If the manifold M is compact, the Euler class vanishes if and only if there exists a locally trivial fibration $\Upsilon \colon M \to \mathbf{S}^1$ whose fibers are transverse to the orbits of Φ (see [9], [10]). Nevertheless, there are simple examples showing that the above results are not true if we allow the action Φ to have fixed points.

In this work we construct a Gysin sequence for a generic action extending $(*)$. The first important remark is that the orbit space $M/\mathbf{S}^1$ is a singular manifold (more exactly, a *stratified pseudomanifold* in the sense of [5]), possibly with boundary. Consequently, the intersection cohomology introduced by Goresky and MacPherson in [5] appears as a natural cohomology theory to study $\mathbf{S}^1$-actions. The main result of this work (Theorem 3.1.8) shows that for any perversity $\overline{r} = (0,0,0,r_5,r_6,\dots)$ there exists an exact sequence

$$\cdots \to H^i(M) \xrightarrow{\oint^*} IH^{i-1}_{\overline{r}}(M/\mathbf{S}^1, \partial(M/\mathbf{S}^1)) \xrightarrow{\wedge[e]} IH^{i+1}_{\overline{r+2}}((M-F_4)/\mathbf{S}^1) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots$$

where $\oint$ is the integration along the orbits of Φ, $\overline{r+2} = (0,1,2,r_5+2,r_6+2,\dots)$, $[e] \in IH^2_{\overline{2}}((M-F_4)/\mathbf{S}^1)$ is the Euler class of Φ, $\partial(M/\mathbf{S}^1)$ is the boundary of the orbit space and $F_4 \subset M$ is the union of the connected components of codimension 4 of the fixed point set.

The vanishing of the Euler class $[e]$ has also a geometrical interpretation. We show that $[e]=0$ is equivalent to the existence of a singular foliation, in the sense of [13], whose restriction to $M-\{$fixed points$\}$ is a locally trivial fibration over $\mathbf{S}^1$, transverse to the orbits of the action Φ (see Theorem 3.2.4). In this case the codimension of the fixed point set is at most 2.

The main tool used here is a "blow-up" of the action Φ into a free action $\widetilde{\Phi} \colon \mathbf{S}^1 \times \widetilde{M} \to \widetilde{M}$. We know that the intersection cohomology of the orbit space $M/\mathbf{S}^1$ can be calculated using a complex of differential forms of $\widetilde{M}/\mathbf{S}^1$ (see [11]). Then, we can apply the usual techniques for free actions in order to get the Gysin sequence and the Euler class.

In Section 1 we introduce the "blow-up" of the action Φ. We recall in the second section the notion of intersection differential form. Section 3 is devoted to the proof of the main results of our work: the Gysin sequence and the geometrical interpretation of the vanishing of the Euler class. In the Appendix we prove some technical Lemmas stated on previous sections.

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In a coming paper we expect to extend this study to the action of a compact Lie group and obtain a spectral sequence relating the cohomology of the manifold, the intersection cohomology of the orbit space and the Lie algebra of G.

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In this work all the manifolds are connected and smooth and "differentiable" means "of class C^∞". The cohomology $H^*(X)$ (resp. the homology $H_*(X)$) is the singular cohomology (resp. homology) of the space X with real coefficients.

1 Stratifications and unfoldings

Let $\Phi\colon \mathbf{S}^1 \times M \to M$ be an effective differentiable action of the circle $\mathbf{S}^1$ on a m-dimensional manifold M. This action induces on M a natural structure of stratified pseudomanifold, invariant by $\mathbf{S}^1$. In this section we study this structure and we construct an unfolding of M (see [11]), invariant by $\mathbf{S}^1$. Finally, we show the orbit space $M/\mathbf{S}^1$ inherits a similar structure in a natural way.

1.1 Stratification and unfolding of M

The stratification of M comes from the classification of the points of M according to their isotropy subgroups. Since the stratified pseudomanifold M is a stratified space (see [15]) it possesses an unfolding (see [1] and [12]). We recall in this paragraph these notions.

1.1.1 Definitions (see [2]). Let $\Phi\colon G \times M \to M$ an action of a closed subgroup $G \subset \mathbf{S}^1$. We will write $\Phi(g, x) = \Phi_g(x) = g \cdot x$. Throughout this paper every action will be supposed to be **effective**, that is, each Φ_g is different from the identity, for $g \neq e$. The map $\pi\colon M \to M/G$ is the canonical projection onto the **orbit space** M/G.

Consider on M the equivalence relation $\sim$ defined by $x \sim y$ iff $G_x = G_y$, where G_z denotes the **isotropy subgroup** $\{g \in G \,/\, g \cdot z = z\}$ of a point $z \in M$. The connected components of the equivalence classes of this relation are the **strata** of M, which are proper submanifolds of M. For each stratum S we will write G_S the isotropy subgroup of any point of S. There are three types of strata: **regular stratum** (if $G_S = \{\text{identity element } e\}$), **fixed stratum** (if $G_S = G$) and **exceptional stratum** (if $G_S \neq \{e\}$, G). The projection $\pi\colon S \to \pi(S)$ is a principal fibration with fiber G/G_S. The union of regular strata is an open dense subset of M (see [2, page 179]).

We will write M^G the fixed point set of M. The action is said to be a **free action** (resp. **almost free action**) if the strata of Φ are regular strata (resp. regular or exceptional strata).

Since in this section it will be necessary to deal with actions of $\mathbf{S}^1$ and with the induced actions on the links $\mathbf{S}^\ell$, we introduce the notion of good action which includes both. The action $\Phi\colon G \times M \to M$ will be said a **good action** if $G = \mathbf{S}^1$ or $M = \mathbf{S}^\ell$ and G is a finite abelian subgroup of $SO(\ell + 1)$. Notice that in this case we have the relation $\Phi(G \times S) \subset S$ for each stratum S.

Throughout this section we will suppose that Φ is a good action. In order to describe the stratification and the unfolding of M we need to recall some facts about the local structure of the action Φ.

1.1.2 Local structure of M (see [2, page 306]). Each stratum S possesses a **tubular neighborhood** $\mathcal{N}_S = (\mathcal{T}, \tau, S, D^{\ell+1})$ satisfying:

i) $\mathcal{T}$ is an open neighborhood of S,

ii) $\tau\colon \mathcal{T} \to S$ is a locally trivial fibration, with fiber the open disk $D^{\ell+1}$, whose restriction to S is the identity,

iii) there exist an orientable orthogonal action $\Psi_S\colon G_S \times \mathbf{S}^\ell \to \mathbf{S}^\ell$ and an atlas $\mathcal{A} = \{(U,\varphi)\}$ such that $\varphi\colon \tau^{-1}(U) \to U \times D^{\ell+1}$ is G_S-equivariant, that is, $\varphi g \varphi^{-1}(x,[\theta,r]) = (x,[\Psi_S(g,\theta),r])$ for each $g \in G_S$ and $(x,[\theta,r]) \in U \times c\mathbf{S}^\ell$. Here we have identified $D^{\ell+1}$ with the cone $c\mathbf{S}^\ell = \mathbf{S}^\ell \times [0,1[\ /\mathbf{S}^\ell \times \{0\}$ and written $[\theta,r]$ an element of the cone $c\mathbf{S}^\ell$, and

iv) if $g \in G$ and $\varphi_j\colon \tau^{-1}(U_j) \to U_j \times c\mathbf{S}^\ell$, $j = 1,2$, are two charts of $\mathcal{A}$ with $g \cdot U_1 \subset U_2$, then there exists a map $\gamma\colon U_1 \to O(\ell+1)$ such that $\varphi_2 g \varphi_1^{-1}(x,[\theta,r]) = (g \cdot x, [\gamma(x)\cdot\theta, r])$ for each $(x,[\theta,r]) \in U_1 \times c\mathbf{S}^\ell$.

Condition iii) implies that the structural group of $\mathcal{A}$ is the centralizer $\mathcal{Z}$ of G in $O(\ell+1)$. Condition iv) means that the group G acts on $\mathcal{T}$ by morphisms of fibration with structural group; it also implies that the map τ is equivariant. Notice that the action Ψ_S is a good action without fixed points. The charts of $\mathcal{A}$ will be said **distinguished charts** of the tubular neighborhood $\mathcal{N}_S$.

1.1.3 Stratification of M. For each integer i we put M_i the union of strata S of M with $\dim S \leq i$. This defines a filtration of M by closed subsets:

$$M = M_m \supset M_{m-1} \supset \cdots \supset M_1 \supset M_0 \supset M_{-1} = \emptyset.$$

If the subset $M_{m-1} - M_{m-2}$ is not empty then it is a submanifold, not necessarily connected, of codimension 1. The group G_S acts trivially on $S \subset M_{m-1} - M_{m-2}$ and each $g \in G_S$ acts transversally by the antipodal map. This is impossible because the action Φ is a good action. Therefore the above filtration becomes:

$$M = M_m \supset M_{m-1} = M_{m-2} = \Sigma_M \supset \cdots \supset M_1 \supset M_0 \supset M_{-1} = \emptyset.$$

For the definition of a stratified pseudomanifold we refer a reader to [6]. A stratified pseudomanifold is said to be **differentiable** if the strata are differentiable manifolds.

Proposition 1.1.4 *The above filtration endows M with a structure of differentiable stratified pseudomanifold.*

Proof. We proceed by induction on the dimension of M. For $\dim M = 0$ the Proposition is obvious. Suppose that the statement holds for each manifold with dimension strictly smaller than that of M. We first check the local structure near of a stratum S of M.

Let (U,φ) and Ψ_S be as in §1.1.2 iii). By induction hypothesis the sphere $\mathbf{S}^\ell$ is a stratified pseudomanifold with the structure induced by the action Ψ_S. We show that φ sends diffeomorphically the strata of $\tau^{-1}(U)$ to the strata of $U \times c\mathbf{S}^\ell$.

Since the isotropy subgroup of any point in $\tau^{-1}(U)$ is included in G_S, the map φ induces a diffeomorphism between $\tau^{-1}(U) \cap (M_j - M_{j-1})$ and

$$\left\{ \begin{array}{cl} \emptyset & \text{if } j \leq m-\ell-2 \\ U \times \{\text{vertex}\} & \text{if } j = m-\ell-1 \\ U \times \{(\mathbf{S}^\ell)_{j+\ell-m} - (\mathbf{S}^\ell)_{j+\ell-m-1}\} \times]0,1[& \text{if } j \geq m-\ell \end{array} \right.$$

where $\mathbf{S}^\ell = (\mathbf{S}^\ell)_\ell \supset (\mathbf{S}^\ell)_{\ell-1} = (\mathbf{S}^\ell)_{\ell-2} \supset \cdots \supset (\mathbf{S}^\ell)_0 \supset \emptyset$ is the stratification induced by Ψ_S.

If the stratum S is not regular we have $\tau^{-1}(U) \cap (M - M_{m-2}) \cong U \times \{\mathbf{S}^\ell - (\mathbf{S}^\ell)_{\ell-2}\} \times]0,1[$, which by induction hypothesis is a dense open subset of $U \times c\mathbf{S}^\ell$. Hence the open set $M - M_{m-2}$ is a dense subset of M. ♣

Remark that the trace on $\tau^{-1}(U)$ of the stratification defined by G, is the same as the stratification defined by G_S. The open $M - M_{m-2}$ is the union of regular strata.

An **isomorphism** between two differentiable stratified pseudomanifolds is a homeomorphism whose restriction to the strata is a diffeomorphism. In particular, the map φ is an isomorphism.

The **length** of M is the integer $\operatorname{len}(M)$ satisfying $M_{m-\operatorname{len}(M)} \neq M_{m-\operatorname{len}(M)-1} = \emptyset$. For example, $\operatorname{len}(M) > \operatorname{len}(\mathbf{S}^\ell)$. Notice that the action is free if and only if $\operatorname{len}(M) = 0$.

1.1.5 Equivariant unfolding. If the action Φ is free, an equivariant unfolding of M is just an equivariant trivial finite differentiable covering. In the general case, an **equivariant unfolding** of M is given by

1) a manifold $\widetilde{M}$ supporting a free action of G,

2) a continuous equivariant map $\mathcal{L}_M \colon \widetilde{M} \to M$ such that the restriction to $\widetilde{M} - \mathcal{L}_M^{-1}(\Sigma_M)$ is a finite trivial differentiable covering, and

3) for each $x_0 \in S$, S stratum non regular, and for each $\widetilde{x}_0 \in \mathcal{L}_M^{-1}(x_0)$ the following diagram commutes

$$(1) \qquad \begin{tikzcd} \widetilde{\mathcal{U}} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\mathcal{L}_M"'] & U \times \widetilde{\mathbf{S}^\ell} \times]-1,1[\arrow[d, "P"'] \\ \mathcal{U} \arrow[r, "\varphi"] & U \times c\mathbf{S}^\ell \end{tikzcd}$$

where

i) $\mathcal{U} \subset M$ and $\widetilde{\mathcal{U}} \subset \widetilde{M}$ are G_S-invariant neighborhoods of x_0 and $\widetilde{x}_0$ respectively,
ii) (U, φ) is a distinguished chart of a tubular neighborhood of S,
iii) $\widetilde{\varphi}$ is a G_S-equivariant diffeomorphism, and
iv) $P(x, \widetilde{\theta}, r) = (x, [\mathcal{L}_{\mathbf{S}^\ell}(\widetilde{\theta}), |r|])$ for a G_S-equivariant unfolding$\mathcal{L}_{\mathbf{S}^\ell} \colon \widetilde{\mathbf{S}^\ell} \to \mathbf{S}^\ell$.

Notice that for each stratum S the restriction $\mathcal{L}_M \colon \mathcal{L}_M^{-1}(S) \to S$ is a submersion. The map $\mathcal{L}_M \colon \widetilde{\mathcal{U}} \to \mathcal{U}$ is a G_S-equivariant unfolding.

Since the construction of an equivariant unfolding is a technical point without influence for the rest of the work, the proof of the following statement can be founded in the Appendix.

Proposition 1.1.6 *For every good action $\Phi \colon G \times M \to M$ there exists an equivariant unfolding of M.*

1.2 Stratification and unfolding of B

Now, we show how the stratification and the unfolding of M induce a stratification and an unfolding in the orbit space $B = M/G$, by means of the canonical projection $\pi \colon M \to B$. To this end, we study the local structure of B.

1.2.1 Local structure of B. For each stratum S of M, the image $\pi(\mathcal{T})$ is a neighborhood of $\pi(S)$ (see §1.1.2). The map $\rho \colon \pi(\mathcal{T}) \to \pi(S)$ given by $\rho(\pi(x)) = \pi\tau(x)$ is well defined. We are going to show that $\mathcal{N}_{S/G_S} = (\pi(\mathcal{T}), \rho, \pi(S), D^{\ell+1}/G_S)$ is a tubular neighborhood of $\pi(S)$ in B.

Lemma 1.2.2 *The map $\rho \colon \pi(\tau) \to \pi(S)$ is a submersion.*

Proof. Let y_0 be a point of $\pi(S)$. We choose a distinguished chart (U, φ) of $\mathcal{N}_S$ such that:

1) $V = \pi(U)$ is a neighborhood of y_0, and

2) there exists a differentiable section σ of $\pi \colon U \to V$.

Thus, if x is a point of U there exists $g \in G$ with $g \cdot x \in \sigma(V)$. The element g is not unique, but $g' \cdot x \in \sigma(V)$ implies $g^{-1}g' \in G_S$, then $\pi(U) = \pi\sigma(V) = \sigma(V)/G_S$. Because τ is equivariant we get

$\pi\tau^{-1}(U) = \pi\tau^{-1}\sigma(V) = \tau^{-1}\sigma(V)/G_S$. Since the restriction $\varphi: \tau^{-1}\sigma(V) \to \sigma(V) \times c\mathbf{S}^\ell$ is an equivariant diffeomorphism we obtain the commutative diagram

$$\begin{tikzcd}
\tau^{-1}\sigma(V) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \sigma(V) \times c\mathbf{S}^\ell \arrow[d, "\Pi"] \\
\rho^{-1}(V) \arrow[r, "\psi"] & V \times c(\mathbf{S}^\ell/G_S)
\end{tikzcd}$$

where $p: \mathbf{S}^\ell \to \mathbf{S}^\ell/G_S$ is the canonical projection and $\Pi(y,[\theta,r]) = (\pi(y),[p(\theta),r])$. Finally, the homeomorphism ψ satisfies $pr_V\psi\pi(x) = \pi\tau(x) = \rho\pi(x)$, where $pr_V: V \times c(\mathbf{S}^\ell/G_S) \to V$ is the canonical projection. ♣

The family $\mathcal{B} = \{(V,\psi)\}$ previously constructed is an atlas of $\mathcal{N}_{S/G_S}$. Each (V,ψ) will be said a **distinguished chart** of $\mathcal{N}_{S/G_S}$. In order to simplify some calculations, we shall suppose that each V is a **cube**, that is, it is diffeomorphic to a product of intervals.

1.2.3 We have already seen that the family $\{\pi(S)\ /\ S$ stratum of $M\}$ is a partition of B in submanifolds, called **strata** of B. This leads us to the filtration

$$\cdots \supset B_j \supset B_{j-1} \supset \cdots \supset B_0 \supset B_{-1} = \emptyset,$$

where each B_j is the union of the strata of B with dimension less or equal than j. This filtration enjoys of the following three properties:

a) $B = B_n$, *where* $n = m - \dim G$,

b) $B - B_{n-1}$ *is a dense open set*, and

c) $B_{n-1} - B_{n-2} = \cup\pi(\{$*strata of codimension 2 with* $G_S = \mathbf{S}^1\})$.

In order to proof a) consider a regular stratum S. The projection $\pi: S \to \pi(S)$ is a G-principal bundle and hence $\dim \pi(S) = m - \dim G$. Let S be a stratum of M_{m-2}. Consider (U,φ) a distinguished chart of $\mathcal{N}_S$. The density of $M - M_{m-2}$ implies the existence of a m-dimensional stratum R of M satisfying $\tau^{-1}(U) \cap R \neq \emptyset$. There exists a stratum $\mathcal{R}$ of $\mathbf{S}^\ell$ (for the action Ψ_S) verifying $\varphi(\tau^{-1}(U) \cap R) = U \times \mathcal{R} \times]0,1[$. Hence, $\dim \pi(S) = \dim U \leq \dim R = m - \dim G$, and therefore $B = B_n$.

Property b) is proved in a similar way.

Finally, if $\pi(S)$ is a stratum of dimension $n-1$, we get from the previous diagram $\dim(\mathbf{S}^\ell/G_S) = 0$. Thus $G_S = \mathbf{S}^\ell$ and $\ell = 1$.

For the definition of stratified pseudomanifold with boundary we refer the reader to [5].

Proposition 1.2.4 *The filtration* $B = B_n \supset B_{n-2} = \Sigma_B \supset B_{n-3} \supset \cdots \supset B_0 \supset B_{-1} = \emptyset$, *endows* B *with a differentiable stratified pseudomanifold structure, possibly with boundary.*

Proof. Assume that the statement is true for any good action of length smaller than $\operatorname{len}(M)$. The **boundary** $\partial B = \cup\{\pi(S)$ strata of $B\ /\ G_S = \mathbf{S}^1$ and $\dim S = m-2\}$ is a manifold. According to §1.2.2, ∂B possesses a neighborhood N diffeomorphic to the product $B \times [0,1[$. It remains to show that $B - \partial B$ is a stratified pseudomanifold. We need to check the local behavior of the above filtration.

Let $\pi(S)$ be a stratum of $B - \partial B$ and $(V,\psi) \in \mathcal{B}$ a chart. According to §1.2.2, for each stratum $\pi(S_0) \neq \pi(S)$ of B meeting $\rho^{-1}(V)$ there exists a stratum σ_0 of $\mathbf{S}^\ell$ such that the diagram

$$\begin{tikzcd}
\tau^{-1}\sigma(V) \cap S_0 \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \sigma(V) \times \sigma_0 \times]0,1[\arrow[d, "\pi \times p \times \text{ identity}"] \\
\rho^{-1}(V) \cap \pi(S_0) \arrow[r, "\psi"] & V \times p(\sigma_0) \times]0,1[
\end{tikzcd}$$

commutes. By induction, the quotient $\mathbf{S}^\ell/G_S$ is a stratified pseudomanifold, with strictly positive dimension and without boundary (see §1.2.3 c)). Finally, since π and p are submersions and φ is a diffeomorphism we get that ψ is a diffeomorphism. Analogously we show that ψ sends diffeomorphically $\rho^{-1} \cap \pi(S)$ to V. Moreover ψ is an isomorphism. ♣

1.2.5 Unfolding of B. We recall the definition of unfolding of a stratified pseudomanifold given in [11]. For the case $\text{len}(M) = 0$ an unfolding of B is a finite trivial covering. Assume $\text{len}(M) > 0$. An **unfolding** of B is a continuous map $\mathcal{L}_B$ from a manifold $\widetilde{B}$ to B, such that the restriction $\mathcal{L}_B \colon \widetilde{B} - \mathcal{L}_B^{-1}(\Sigma_B) \to B - \Sigma_B$ is a diffeomorphism in each connected component and the following condition holds:

For each $y_0 \in \pi(S)$, S non regular stratum, and for each $\widetilde{y}_0 \in \mathcal{L}_B^{-1}(y_0)$ there exists a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{V}} \arrow[r, "\widetilde{\psi}"] \arrow[d, "\mathcal{L}_B"'] & V \times \widetilde{\mathbf{S}^\ell/G_S} \times]-1,1[\arrow[d, "R"'] \\
\mathcal{V} \arrow[r, "\psi"] & V \times c(\mathbf{S}^\ell/G_S)
\end{tikzcd} \tag{2}$$

where:

i) $\mathcal{V} \subset B$ and $\widetilde{\mathcal{V}} \subset \widetilde{B}$ are neighborhoods of y_0 and $\widetilde{y}_0$ respectively,
ii) $(V, \psi) \in \mathcal{B}$ is a distinguished chart of a tubular neighborhood of S/G_S,
iii) $\widetilde{\psi}$ is a diffeomorphism, and
iv) $R(x, \widetilde{\zeta}, r) = (x, [\mathcal{L}_{\mathbf{S}^\ell/G_S}(\widetilde{\zeta}), |r|])$, for an unfolding $\mathcal{L}_{\mathbf{S}^\ell/G_S} \colon \widetilde{\mathbf{S}^\ell/G_S} \to \mathbf{S}^\ell/G_S$.

Remark that for each stratum S of M the restriction $\mathcal{L}_B \colon \mathcal{L}_B^{-1}(S/G_S) \to S/G_S$ is a submersion. The existence of equivariant unfoldings for M implies the existence of unfoldings for B.

Proposition 1.2.6 *For every good action $\Phi \colon G \times M \to M$ there exists a commutative diagram*

$$\begin{tikzcd}
\widetilde{M} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\mathcal{L}_M"'] & \widetilde{B} \arrow[d, "\mathcal{L}_B"] \\
M \arrow[r, "\pi"] & B
\end{tikzcd} \tag{3}$$

where:

a) $\widetilde{\pi} \colon \widetilde{M} \to \widetilde{B}$ *is a principal fibration,*
b) $\mathcal{L}_M \colon \widetilde{M} \to M$ *is an equivariant unfolding of* M*, and*
c) $\mathcal{L}_B$ *is an unfolding of* B*.*

Proof. See Appendix. ♣

2 Differential forms

The aim of this section is to recall the notion of intersection differential forms (see [11]). We also establish a first relation between the intersection differential forms of M and those of B.

From now on we will suppose $G = \mathbf{S}^1$. We will write $\Sigma_M = M_{m-2}$ and $\Sigma_B = B_{n-2}$ the singular parts of M and B respectively. We fix two unfoldings $\mathcal{L}_M : \widetilde{M} \to M$ and $\mathcal{L}_B : \widetilde{B} \to B$ satisfying §1.2.6. By $\overline{q} = (q_2, \ldots, q_m)$ we denote a **perversity**, that is $q_2 = 0$ and $q_k \leq q_{k+1} \leq q_k + 1$ (see [5]).

2.1 Intersection differential forms

The intersection cohomology of M and B can be calculated with a complex of differential forms on $M - \Sigma_M$ and $B - \Sigma_B$ respectively. This corresponds to the complex of intersection differential forms (see [11]), which we recall now.

2.1.1 A differential form ω on $M - \Sigma_M$ (resp. $B - \Sigma_B$) is **liftable** if there exists a differential form $\widetilde{\omega}$ on $\widetilde{M}$ (resp. $\widetilde{B}$), called the **lifting** of ω, coinciding with $\mathcal{L}_M^* \omega$ on $\mathcal{L}_M^{-1}(M - \Sigma_M)$ (resp. $\mathcal{L}_B^* \omega$ on $\mathcal{L}_B^{-1}(B - \Sigma_B)$). By density this form is unique.

If the forms ω and η are liftable then the forms $\omega + \eta$, $\omega \wedge \eta$ and $d\omega$ are liftable, and we have the following relations:

$$\widetilde{\omega + \eta} = \widetilde{\omega} + \widetilde{\eta}, \ \widetilde{\omega \wedge \eta} = \widetilde{\omega} \wedge \widetilde{\eta}, \text{ and } \widetilde{d\omega} = d\widetilde{\omega}.$$

Hence, the family of liftable differential forms is a differential subcomplex of the De Rham complex of $\widetilde{M}$ (resp. $\widetilde{B}$).

2.1.2 Cartan's filtration. Let $\kappa : N \to C$ be a submersion with N and C manifolds. For each differential form $\omega \not\equiv 0$ on N we define the **perverse degree** of ω, written $||\omega||_C$, as the smallest integer k verifying:

(4) If $\xi_0, \ldots, \xi_k$ are vector fields on N tangents to the fibers of κ then

$$i_{\xi_0} \cdots i_{\xi_k} \omega \equiv 0.$$

Here i_{ξ_j} denotes the interior product by ξ_j. We will write $||0||_C = -\infty$. For each $k \geq 0$ we let $F_k\Omega_N^* = \{\omega \in \Omega^*(N) \ / \ ||\omega||_C \leq k \ \text{ and } \ ||d\omega||_C \leq k\}$. This is the **Cartan's filtration** of κ (see [3]).

Notice that for α, $\beta \in \Omega^*(N)$ we have the following relations

$$||\alpha + \beta||_C \leq \max(||\alpha||_C \, , ||\beta||_C) \text{ and } ||\alpha \wedge \beta||_C \leq ||\alpha||_C + ||\beta||_C. \tag{5}$$

2.1.3 The allowability condition is written in terms of the Cartan's filtration of the submersions $\mathcal{L}_M : \mathcal{L}_M^{-1}(S) \to S$ and $\mathcal{L}_B : \mathcal{L}_B^{-1}(S/G_S) \to S/G_S$, where S is a stratum of M.

A liftable form ω on $M - \Sigma_M$ is a **$\overline{q}$-intersection differential form** if for each stratum S included in Σ_M the restriction of $\widetilde{\omega}$ to $\mathcal{L}_M^{-1}(S)$ belongs to $F_{q_k}\Omega^*_{\mathcal{L}_M^{-1}(S)}$, where k is the codimension of S.

Analogously, a liftable form ω on $B - \Sigma_B$ is a **$\overline{q}$-intersection differential form** if for each stratum S/G_S included in Σ_B the restriction of $\widetilde{\omega}$ to $\mathcal{L}_B^{-1}(S/G_S)$ belongs to $F_{q_k}\Omega^*_{\mathcal{L}_B^{-1}(S/G_S)}$, where k is the codimension of S/G_S.

We shall write $\mathcal{K}^*_{\overline{q}}(M)$ (resp. $\mathcal{K}^*_{\overline{q}}(B)$) the complex of $\overline{q}$-intersection differential forms. It is a differential subcomplex of the De Rham complex of $\widetilde{M}$ (resp. $\widetilde{B}$), but it is not always an algebra. It coincides with $\Omega^*(M)$ (resp. $\Omega^*(B)$) if the action Φ is free.

We show in [11] that the complex of $\overline{q}$-intersection differential forms computes the intersection cohomology. In fact we have the isomorphisms

- $H^*(\mathcal{K}_{\overline{q}}(M)) \cong IH^{\overline{p}}_*(M) \cong H_*(M) \cong H^*(M)$,

- $H^*(\mathcal{K}_{\overline{q}}(B)) \cong IH^{\overline{p}}_*(B)$,
- $H^*(\mathcal{K}_{\overline{q}}(B, \partial B)) \cong IH^{\overline{p}}_*(B, \partial B)$.

Here $\overline{p}$ denotes the complementary perversity of $\overline{q}$ (see [5]) and $\mathcal{K}_{\overline{q}}(B, \partial B)$ the complex of differential forms of $\mathcal{K}_{\overline{q}}(B)$ which vanish on ∂B. In order to make uniform the notations, we will write: $H^*(\mathcal{K}_{\overline{q}}(M)) = IH^*_{\overline{q}}(M)$, $H^*(\mathcal{K}_{\overline{q}}(B)) = IH^*_{\overline{q}}(B)$ and $H^*(\mathcal{K}_{\overline{q}}(B, \partial B)) = IH^*_{\overline{q}}(B, \partial B)$.

2.1.4 An important tool, used in Section 3 to get the Gysin sequence, is the study of the relationship between the degrees defining the Cartan's filtration on M and B. A first step in this direction is given by

$$||\widetilde{\pi}^*\eta||_S = ||\eta||_{S/G_S} \tag{6}$$

where S is a stratum of Σ_M and η is a differential form on $\mathcal{L}_B^{-1}(S/G_S)$. If the action Φ has not fixed points, then the codimensions of S and S/G_S are the same. Therefore, the equality (6) implies

$$\omega \in \mathcal{K}^*_{\overline{q}}(B) \Leftrightarrow \pi^*\omega \in \mathcal{K}^*_{\overline{q}}(M).$$

In this case the map $\pi^*\colon IH^*_{\overline{q}}(B) \to H^*(M)$ is well defined.

In order to proof (6) it suffices to remark that in the following commutative diagram

$$\begin{tikzcd}
\mathcal{L}_M^{-1}(S) \arrow[r, "\widetilde{\pi}"] \arrow[d, "\mathcal{L}_M"'] & \mathcal{L}_B^{-1}(S/G_S) \arrow[d, "\mathcal{L}_B"] \\
S \arrow[r, "\pi"] & S/G_S
\end{tikzcd}$$

the restriction of $\widetilde{\pi}$ to the fibers of $\mathcal{L}_M$ is a submersion onto the fibers of $\mathcal{L}_B$.

2.2 Invariant forms

It is well known that the De Rham cohomology of a manifold supporting an action of G is calculated by the complex of differential forms invariant by the action. The same phenomenon happens when the intersection cohomology is involved.

2.2.1 A differential form ω on $M - \Sigma_M$ is called **invariant** under the action of G if it satisfies $\Phi_g^*\omega = \omega$ for each $g \in G$. The invariant differential forms are a subalgebra of $\Omega^*(M - \Sigma_M)$, which will be denoted by $I\Omega^*(M - \Sigma_M)$. It is shown in [7] that the inclusion $I\Omega^*(M - \Sigma_M) \hookrightarrow \Omega^*(M - \Sigma_M)$ induces an isomorphism in cohomology.

The following Lemmas are devoted to prove that the operators used in [7] send the liftable differential forms to themselves. This will prove that the inclusion $I\mathcal{K}^*_{\overline{q}}(M) = I\Omega^*(M - \Sigma_M) \cap \mathcal{K}^*_{\overline{q}}(M) \hookrightarrow \mathcal{K}^*_{\overline{q}}(M)$ induces an isomorphism in cohomology.

Lemma 2.2.2 *Consider $\Phi\colon G \times M \to M$ and $\Phi'\colon G \times M' \to M'$ two actions and $f\colon M \to M'$ an equivariant differentiable map. Suppose there exists an equivariant differentiable map $\widetilde{f}\colon \widetilde{M} \to \widetilde{M'}$ with $\mathcal{L}_{M'}\widetilde{f} = f\mathcal{L}_M$. If $G_x = G_{f(x)}$ for each $x \in M$, then the map f^* sends $\mathcal{K}^*_{\overline{q}}(M')$ to $\mathcal{K}^*_{\overline{q}}(M)$.*

Proof. For each form $\omega \in \mathcal{K}^*_{\overline{p}}(M')$ the lifting of $f^*\omega$ is $\widetilde{f}^*\widetilde{\omega}$ because $\mathcal{L}^*_M f^*\omega = \widetilde{f}^*\mathcal{L}^*_{M'}\omega$ on $\widetilde{M} - \mathcal{L}_M^{-1}(\Sigma_M)$. Furthermore, for each stratum S of Σ_M there exists a stratum S' of $\Sigma_{M'}$ with $f(S) \subset S'$. This gives us the commutative diagram

$$\begin{tikzcd}
\mathcal{L}_M^{-1}(S) \arrow[r, "\widetilde{f}"] \arrow[d, "\mathcal{L}_M"'] & \mathcal{L}_{M'}^{-1}(S') \arrow[d, "\mathcal{L}_{M'}"] \\
S \arrow[r, "f"] & S'
\end{tikzcd}$$

Therefore $||\widetilde{f}^*\widetilde{\omega}||_{S'} \leq ||\widetilde{\omega}||_S$, which implies $f^*(F_{q_k}\Omega^*_{\mathcal{L}^{-1}_{M'}(S')}) \subset F_{q_k}\Omega^*_{\mathcal{L}^{-1}_{M}(S)}$ and so $f^*K^*_{\overline{q}}(M') \subset K^*_{\overline{q}}(M)$. ♣

For each manifold N, we will consider on the product $N \times M$ the action G defined by $g\cdot(x,y) = (x, g\cdot y)$ and the equivariant unfolding $\mathcal{L}_{N\times M} = \text{identity } \times \mathcal{L}_M \colon N \times \widetilde{M} \to N \times M$. We shall write $\pi_N \colon N \times M \to N$ the canonical projection.

Lemma 2.2.3 *Let Δ be a differential form on N with compact support. Then,*

$$\omega \in \mathcal{K}^*_{\overline{q}}(N \times M) \Rightarrow \oint_N \omega \wedge \pi_N^*\Delta \in \mathcal{K}^*_{\overline{q}}(M)$$

where $\oint_N$ *denotes the integration along the fibers of* π_N.

Proof. Since the fibers of $\mathcal{L}_{N\times M} \colon N \times \mathcal{L}_M^{-1}(S) \to N \times S$ are tangent to the second factor then $\pi_N^*\Delta \in \mathcal{K}^*_{\overline{0}}(N \times M)$. Hence $\omega \wedge \pi_N^*\Delta \in \mathcal{K}^*_{\overline{q}}(N \times M)$. The result follows by noticing that the N-factor is tangent to the strata. ♣

Lemma 2.2.4

$$\omega \in \mathcal{K}^*_{\overline{q}}(M) \Rightarrow \Phi^*\omega \in \mathcal{K}^*_{\overline{q}}(G \times M)$$

Proof. Apply Lemma 2.2.2 for $f = \Phi$ and $\widetilde{f} = \widetilde{\Phi}$. ♣

Lemma 2.2.5 *Let $H \colon N \times [0,1] \times M \to N \times M$ be a differentiable map defined by $H(x,t,y) = (H_0(x,t), y)$. Then*

$$\omega \in \mathcal{K}^*_{\overline{q}}(N \times M) \Rightarrow h\omega \in \mathcal{K}^*_{\overline{q}}(N \times M),$$

where $h\omega(x,y) = \int_0^1 (H^*\omega)(x,t,y)(\partial/\partial t)dt$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
N \times [0,1] \times \widetilde{M} \arrow[r, "\widetilde{H}"] \arrow[d, "\mathcal{L}_{N\times[0,1]\times\widetilde{M}}"'] & N \times \widetilde{M} \arrow[d, "\mathcal{L}_{N\times M}"] \\
N \times [0,1] \times M \arrow[r, "H"] & N \times M
\end{tikzcd}$$

where $\widetilde{H}(x,t,\widetilde{y}) = (H_0(x,t), \widetilde{y})$. Using §2.2.2 we deduce that $H^*\omega$ belongs to $\mathcal{K}^*_{\overline{q}}(N \times [0,1] \times M)$. Now, since the $[0,1]$-factor is tangent to the strata, we get that $h\omega$ belongs to $\mathcal{K}^*_{\overline{q}}(N \times M)$. ♣

The operators used in [7] to show that the inclusion $I\Omega^*(M - \Sigma_M) \hookrightarrow \Omega^*(M - \Sigma_M)$ induces an isomorphism in cohomology are composition of operators of type §2.2.3, §2.2.4 and §2.2.5. Therefore we get

Proposition 2.2.6 *The inclusion $I\mathcal{K}^*_{\overline{q}}(M) \hookrightarrow \mathcal{K}^*_{\overline{q}}(M)$ induces an isomorphism in cohomology.*

2.3 Decomposition of invariant forms

In the case of a free action, each invariant form on M is written in terms of the differential forms on the orbit space B and the fiber G. We extend this decomposition to the case of non-free actions. First we need some definitions.

2.3.1 The **fundamental vector field** X of Φ is defined by the relation $X(x) = T_e\Phi_x(1)$, where $\Phi_x(g) = g\cdot x$. This vector field is invariant by the action of G and tangent to their orbits. In particular, it vanishes on the set of fixed points. Since $\mathcal{L}_M$ is equivariant then the fundamental vector field $\widetilde{X}$ of $\widetilde{\Phi}$ and X are $(\mathcal{L}_M)_*$-related. That is, $(\mathcal{L}_M)_*\widetilde{X} = X \circ \mathcal{L}_M$.

We define the **fundamental forms** χ and $\widetilde{\chi}$ by

$$\chi = \mu(X,\ ) \text{ and } \widetilde{\chi} = \widetilde{\mu}(\widetilde{X},\ ),$$

where μ and $\widetilde{\mu}$ are two riemannian metrics on $M - \Sigma_M$ and $\widetilde{M}$ respectively. These forms depend on the choice of μ and $\widetilde{\mu}$. Improving the properties of μ and $\widetilde{\mu}$ we will have richer fundamental forms.

Lemma 2.3.2 *There exist two riemannian metrics μ and $\widetilde{\mu}$, on $M - \Sigma_M$ and $\widetilde{M}$ respectively, satisfying:*

a) μ and $\widetilde{\mu}$ are invariant,

b) $\mathcal{L}_M^*\mu = \widetilde{\mu}$ on $\widetilde{M} - \mathcal{L}_M^{-1}(\Sigma_M)$,

c) for each exceptional stratum S the differential form $\widetilde{\chi}$ is a basic form, relatively to $\mathcal{L}_M \colon \mathcal{L}_M^{-1}(S) \to S$.

d) for each fixed stratum S there exists a G_S-equivariant riemannian metric $\widetilde{\mathcal{M}}$ on $\mathbf{S}^\ell$ such that the structural group of $\mathcal{L}_M \colon \mathcal{L}_M^{-1}(S) \to S$ can be reduced to the group of isometries of $(\mathbf{S}^\ell, \widetilde{\mathcal{M}})$.

Proof. See Appendix ♣

2.3.3 A riemannian metric μ on $M - \Sigma_M$ is said to be a **good metric of** M if there exists $\widetilde{\mu}$ satisfying the previous conditions a), b), c) and d). From now on we fix a good metric μ of M. The following properties of the fundamental forms associated to μ and $\widetilde{\mu}$ arise directly from the preceding Lemma.

i) The Lie derivatives $L_X\chi$ and $L_{\widetilde{X}}\widetilde{\chi}$ are 0.
ii) $\widetilde{\chi}(\widetilde{X}) = h \neq 0$ (and we will suppose $h = 1$).
iii) $||\widetilde{\chi}||_S = 0$ if S is an exceptional stratum.
iv) $||\widetilde{\chi}||_S = 1$ if S is a fixed stratum.
v) For each fixed stratum S we have $\widetilde{\varphi}^*\widetilde{\chi_{\mathbf{S}^\ell}} = \widetilde{\chi}$ on the fibers of $\mathcal{L}_M \colon \mathcal{L}_M^{-1}(S) \to S$. Here $\widetilde{\chi_{\mathbf{S}^\ell}}$ denotes the fundamental form associated to $(\mathbf{S}^\ell, \widetilde{\mathcal{M}})$ and (φ, U) is a distinguished chart.

2.3.4 For each $\omega \in I\Omega^*(M - \Sigma_M)$ there exist two forms ω_1, $\omega_2 \in \Omega^*(B - \Sigma_B)$ such that

$$\omega = \pi^*\omega_1 + \chi \wedge \pi^*\omega_2.$$

The forms ω_1 and ω_2 are unique, in fact $\pi^*\omega_1 = i_X\omega$ and $\pi^*\omega_2 = \omega - \chi \wedge i_X\omega$. The above expression will be called the **decomposition** of ω.

Analogously, for each $\eta \in I\Omega^*(\widetilde{M})$ there exist two unique forms η_1, $\eta_2 \in \Omega^*(\widetilde{B})$ such that

$$\eta = \widetilde{\pi}^*\eta_1 + \widetilde{\chi} \wedge \widetilde{\pi}^*\eta_2.$$

This expression is called the **decomposition** of η.

If ω is liftable we get the following relation between the two decompositions:

$$\widetilde{\omega} = \widetilde{\omega_1} + \widetilde{\chi} \wedge \widetilde{\omega_2}.$$

The relation between the perverse degree of η, η_1 and η_2 is the following.

Proposition 2.3.5 *For each form* $\eta \in \Omega^*(\widetilde{M})$ *and for each stratum* S *of* M *we get*

$$||\eta||_S = \max(||\eta_1||_{S/G_S}, ||\widetilde{\mathcal{X}}||_S + ||\eta_2||_{S/G_S}).$$

Proof. By (5) and (6) it suffices to show that $||\eta||_S \geq \max(||\eta_1||_{S/G_S}, ||\widetilde{\mathcal{X}}||_S + ||\eta_2||_{S/G_S})$. We distinguish two cases.

- *S is an exceptional stratum.* Fix $k \geq 0$. The condition (4) on η is equivalent to

  $i_{\xi_0} \cdots i_{\xi_k} \widetilde{\pi}^* \eta_1 \equiv i_{\xi_0} \cdots i_{\xi_k} \widetilde{\pi}^* \eta_2 \equiv 0$ for each family $\{\xi_0, \ldots, \xi_k\}$ of vector fields tangents to the fibers of $\mathcal{L}_M \colon \mathcal{L}_M^{-1}(S) \to S$ (see §2.3.2 c)).

  From (3) this condition is equivalent to

  $i_{\xi_0} \cdots i_{\xi_k} \eta_1 = i_{\xi_0} \cdots i_{\xi_k} \eta_2 \equiv 0$ for each family $\{\xi_0, \ldots, \xi_k\}$ of vector fields tangents to the fibers of $\mathcal{L}_B \colon \mathcal{L}_B^{-1}(S/G_S) \to S/G_S$,

  which holds if and only if $k \geq \max(||\eta_1||_{S/G_S}, ||\eta_2||_{S/G_S})$. Thus $||\eta||_S \geq \max(||\eta_1||_{S/G_S}, ||\widetilde{\mathcal{X}}||_S + ||\eta_2||_{S/G_S})$ (see §2.3.3 iii)).

- *S is a fixed stratum.* Fix $k \geq 0$. Since $\widetilde{X}$ is tangent to the fibers of $\mathcal{L}_M \colon \mathcal{L}_M^{-1}(S) \to S$, condition (4) on η becomes

  $i_{\xi_0} \cdots i_{\xi_k} \widetilde{\pi}^* \eta_1 \equiv 0$ and $i_{\xi_0} \cdots i_{\xi_{k-1}} \widetilde{\pi}^* \eta_2 \equiv 0$ for each family $\{\xi_0, \ldots, \xi_k\}$ of vector fields tangents to the fibers of $\mathcal{L}_M \colon \mathcal{L}_M^{-1}(S) \to S$.

  Now we proceed as above taking into account that $||\widetilde{\mathcal{X}}||_S = 1$ (see §2.3.3 iv)). ♣

The form $i_X d\mathcal{X}$ vanishes identically. Thus, the decomposition of $d\mathcal{X}$ is reduced to $d\mathcal{X} = \pi^* e$ for a form $e \in \Omega^2(B - \Sigma_B)$, called the **Euler form** of Φ (we will also write e_μ). Remark that e is a cycle. The Euler form $\widetilde{e}$ of $\widetilde{\Phi}$ is the lifting of e.

Proposition 2.3.6 *For each stratum* S *of* M *we get*

$$||\widetilde{e}||_{S/G_S} = \begin{cases} 2 & \textit{if } S \subset M^{\mathbf{S}^1} \textit{ and } \dim S < m - 2 \\ -\infty, 0 & \textit{otherwise.} \end{cases}$$

Proof. We distinguish two cases.

- *S is an exceptional stratum.* Since $\widetilde{\mathcal{X}}$ is a basic form we get $||\widetilde{e}||_{S/G_S} = ||d\widetilde{\mathcal{X}}||_S \leq 0$ (see (6)).

  Remark that if Φ is almost free, then $e \in \mathcal{K}^2_{\bar{0}}(B)$.

- *S is a fixed stratum.* Each fiber F of $\mathcal{L}_M \colon \mathcal{L}_M^{-1}(S) \to S$ is equivariantly isometric to $(\widetilde{\mathbf{S}^\ell}, \widetilde{\mathcal{M}})$ endowed with the free action $\widetilde{\Psi_S}$. The restriction $\widetilde{\mathcal{X}}|_F$ becomes the fundamental form $\widetilde{\mathcal{Y}}$ of $\widetilde{\Psi_S}$. Then, we get the decomposition

$$\widetilde{e} = \widetilde{\varepsilon}_1 + \widetilde{\varepsilon}_2 \tag{7}$$

where the restriction $\widetilde{\varepsilon}_1|_F$ is the Euler form $\widetilde{\epsilon}$ of $\widetilde{\Psi_S}$ and $\widetilde{\varepsilon}_2|_F$ vanishes identically.

If $\ell > 1$ we claim that the Euler form $\widetilde{\epsilon} \in \Omega^2(\widetilde{\mathbf{S}^\ell/G_S})$ is not zero; in this case the restriction $\widetilde{e}|_F$ does not vanishes identically and therefore the perverse degree $||\widetilde{e}||_{S/G_S}$ is 2. In order to prove the claim it suffices to verify that $[\epsilon] \in IH^2_{\bar{0}}(\mathbf{S}^\ell/G_S)$ is non-zero. Suppose that there exists $\gamma \in \mathcal{K}^1_{\bar{0}}(\mathbf{S}^\ell/G_S)$ with $d\gamma = \epsilon$. Thus, the differential form $\mathcal{X}_{\mathbf{S}^\ell} - p^*\gamma$ is a cycle of $\mathcal{K}^1_{\bar{0}}(\mathbf{S}^\ell)$, where $\mathcal{X}_{\mathbf{S}^\ell}$ is the fundamental form of Ψ_S. Since $IH^1_{\bar{0}}(\mathbf{S}^\ell) \cong H^1(\mathbf{S}^\ell) = 0$ there exists $f \in \mathcal{K}^0_{\bar{0}}(\mathbf{S}^\ell)$ with $d\widetilde{f} = \widetilde{\mathcal{X}_{\mathbf{S}^\ell}} - \widetilde{p}^*\widetilde{\gamma}$. We have arrived to a contradiction because $\widetilde{f}\colon \widetilde{\mathbf{S}^\ell} \to \mathbf{R}$ is a differentiable map, $d\widetilde{f} \not\equiv 0$ ($d\widetilde{f}$(fundamental vector field of $\Psi_S) \equiv 1$) and $\widetilde{\mathbf{S}^\ell}$ is compact.

If $\ell = 1$ the dimension of $\widetilde{\mathbf{S}^\ell}$ is 1. Since $\widetilde{e}|_F$ is a differential 2-form, it vanishes identically. Therefore, we get $||\widetilde{e}||_{S/G_S} \leq 0$. ♣

Corollary 2.3.7 *If the action Φ has not fixed points, then for each liftable form $\omega \in \Omega^*(M - \Sigma_M)$ we have*

$$\omega \in \mathcal{K}^*_{\overline{q}}(M) \Leftrightarrow \omega_1, \omega_2 \in \mathcal{K}^*_{\overline{q}}(B). \tag{8}$$

Proof. The decomposition of $d\widetilde{\omega}$ is given by: $(d\widetilde{\omega})_1 = d\widetilde{\omega}_1 + \widetilde{e} \wedge \widetilde{\omega}_2$ and $(d\widetilde{\omega})_2 = -d\widetilde{\omega}_2$. For each stratum S of M we get $\max(||\widetilde{\omega}||_S, ||d\widetilde{\omega}||_S) = \max(||\widetilde{\omega}_1||_{S/G_S}, ||\widetilde{\omega}_2||_{S/G_S}, ||d\widetilde{\omega}_1 + \widetilde{e} \wedge \widetilde{\omega}_2||_{S/G_S}, ||d\widetilde{\omega}_2||_{S/G_S})$. Moreover, since $||\widetilde{e} \wedge \widetilde{\omega}_2||_{S/G_S} \leq ||\widetilde{e}||_{S/G_S} + ||\widetilde{\omega}_2||_{S/G_S} \leq ||\widetilde{\omega}_2||_{S/G_S}$ we obtain the relation $\max(||\widetilde{\omega}||_S, ||d\widetilde{\omega}||_S) = \max(||\widetilde{\omega}_1||_{S/G_S}, ||\widetilde{\omega}_2||_{S/G_S}, ||d\widetilde{\omega}_1||_{S/G_S}, ||d\widetilde{\omega}_2||_{S/G_S})$. Notice that the codimension of S in M is the codimension of S/G_S in B. Thus

$$\widetilde{\omega} \in F_{q_k}\Omega^*_{\mathcal{L}^{-1}_M}(S) \Leftrightarrow \widetilde{\omega}_1, \widetilde{\omega}_2 \in F_{q_k}\Omega^*_{\mathcal{L}^{-1}_B}(S/G_S),$$

from which the result holds. ♣

2.3.8 Euler class. We write F_4 the union of 4-codimensional connected components of M^G, and also its image by π. Proposition 2.3.6 shows that the restriction of the Euler form e to $B - F_4$ belongs to $\mathcal{K}^2_{\overline{2}}(B - F_4)$, where $\overline{2}$ is the perversity $(0, 1, 2, 2, \ldots)$. The class $[e] \in IH^2_{\overline{2}}(B - F_4)$ is the **Euler class** of Φ. Notice that the Euler class $[\widetilde{e}]$ of $\widetilde{\Phi}$ belongs to $H^2(\widetilde{B})$.

3 Gysin sequence

In this section we establish the Gysin sequence that relates the cohomology of M and the intersection cohomology of B. We also give a geometrical interpretation of the vanishing of the Euler class. Recall that G denotes the unitary circle $\mathbf{S}^1$.

3.1 Integration along the fibers

Differential forms on $M - \Sigma_M$ and differential forms on $B - \Sigma_B$ are related by the integration $\oint$ along the fibers of the projection π. The Gysin sequence here obtained arises from the study of this integration $\oint$.

3.1.1 For each differential form $\omega \in \Omega^*(M - \Sigma_M)$ we define $\oint \omega = \omega_2$, the **integration along the fibers** of π. The form $\oint \omega$ belongs to $\Omega^{*-1}(B - \Sigma_B)$. Notice that for each α, $\beta \in \Omega^*(B - \Sigma_B)$ we have $\oint \pi^*\alpha = 0$ and $\oint \chi \wedge \pi^*\beta = \beta$.

If the action Φ is free, the above relations show that the short sequence

$$0 \longrightarrow \Omega^*(B) \xrightarrow{\pi^*} \Omega^*(M) \xrightarrow{\oint} \Omega^{*-1}(B) \longrightarrow 0$$

is exact. The associated long exact sequence

$$\cdots \to H^i(M) \xrightarrow{\oint^*} H^{i-1}(B) \xrightarrow{\wedge[e]} H^{i+1}(B) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots \tag{9}$$

is the Gysin sequence of the free action Φ (see [7]).

If the action Φ is almost free the relation (8) shows that the integration $\oint$ defines a short exact sequence

$$0 \longrightarrow \mathcal{K}^*_{\overline{q}}(B) \xrightarrow{\pi^*} I\mathcal{K}^*_{\overline{q}}(M) \xrightarrow{\oint} \mathcal{K}^{*-1}_{\overline{q}}(B) \longrightarrow 0.$$

Since M and B are homological manifolds, the associated long exact sequence is in fact (9) (see Proposition 2.2.6 and [5, §6.4]), which has been proved already in [9].

If fixed points appear, the above relation (9) is not longer true (see §3.1.10 1)). The Gysin sequence of Φ arises from the study of the short exact sequence

$$0 \longrightarrow Ker\oint \xrightarrow{\iota} I\mathcal{K}^*_{\overline{q}}(M) \xrightarrow{\oint} Im\oint \longrightarrow 0, \tag{10}$$

where ι is the inclusion. The crucial point is to compare $Ker\oint$ and $Im\oint$ with $\mathcal{K}^*_{\overline{q}}(B)$. We will observe a shift in the perversities involved; this is due to the fact that for each fixed stratum S we have

1) codimension of S in M = (codimension of S/G_S in B) +1,

2) $||\widetilde{\chi}||_S = 1$, and

3) $||\widetilde{e}||_{S/G_S} = 2$ (except for the case $\dim S = m-2$).

This led us to consider the following perversities:

$\overline{r} = (r_2, r_3, r_4, r_5, \ldots)$ with $r_2 = r_3 = r_4 = 0$,

$\overline{r+2} = (0, 1, 2, r_5 + 2, r_6 + 2, \ldots)$, and

$\overline{q} = (0, 1, 2, 2, r_5 + 2, r_6 + 2, \ldots)$.

We begin recalling Propositions 3.2.3 and 3.3.2 of [10].

Proposition 3.1.2 *Let A be an unfoldable pseudomanifold (possibly with boundary). Fix $I =]-\varepsilon, \varepsilon[$ an interval of* $\mathbf{R}$. *The maps $pr\colon I \times (A - \Sigma_A) \to A - \Sigma_A$ and $J\colon A - \Sigma_A \to I \times (A - \Sigma_A)$, defined respectively by $pr(t, a) = a$ and $J(a) = (t_0, a)$, for a fixed $t_0 \in I$, induce the quasi-isomorphisms:*

$$pr^*\colon \mathcal{K}^*_{\overline{q}}(A) \to \mathcal{K}^*_{\overline{q}}(I \times A) \text{ and } J^*\colon \mathcal{K}^*_{\overline{q}}(I \times A) \to \mathcal{K}^*_{\overline{q}}(A).$$

Proof (sketch). Consider $\widetilde{pr}\colon \times \widetilde{A} \to \widetilde{A}$ and $\widetilde{J}\colon \widetilde{A} \to I \times \widetilde{A}$ defined by $\widetilde{pr}(t, \widetilde{a}) = \widetilde{a}$ and $\widetilde{J}(\widetilde{a}) = (t_0, \widetilde{a})$. The two operators pr^* and J^* are well defined because, for each stratum S of A, we have $||\widetilde{pr^*\omega} = \widetilde{pr}^*\widetilde{\omega}||_{I\times S} \leq ||\widetilde{\omega}||_S$ and $||\widetilde{J^*\eta} = \widetilde{J}^*\widetilde{\eta}||_S \leq ||\widetilde{\eta}||_{I\times S}$, for any liftable form $\omega \in \Omega^*(A - \Sigma_A)$ and $\eta \in \Omega^*(I \times (A - \Sigma_A))$. In fact, these two operators are homotopic; a homotopy operator is given by $H\eta = \int_{t_0}^{-} \eta$. This comes from the following facts:

- $\widetilde{H\eta} = \int_{t_0}^{-} \widetilde{\eta}$ (on $I \times \widetilde{A}$),
- $||\widetilde{H\eta}||_{I\times S} \leq ||\widetilde{\eta}||_{I\times S}$, and
- $dH\eta - Hd\eta = (-1)^{i-1}(\eta - pr^*J^*\eta)$,

where $\eta \in \Omega^i(I \times (A - \Sigma_A))$ is a liftable form. $\clubsuit$

Proposition 3.1.3 *Let A be a n-dimensional compact unfoldable pseudomanifold. Then*

$$H^i(\mathcal{K}^*_{\overline{q}}(cA)) \cong \begin{cases} H^i(\mathcal{K}^*_{\overline{q}}(A)) & \text{if } i \leq q_{n+1} \\ 0 & \text{if } i > q_{n+1} \end{cases}$$

where the isomorphism is induced by the canonical projection $pr\colon (A-\Sigma_A)\times]0,1[\to (A-\Sigma_A)$.

Proof (sketch). The complex $\mathcal{K}^*_{\overline{q}}(cA)$ is naturally isomorphic (by restriction) to the subcomplex C^* of $\mathcal{K}^*_{\overline{q}}(A\times]-1,1[)$ made up of the forms η satisfying:

a) $\eta \equiv 0$ on $(A-\Sigma)\times\{0\}$ if (degree of η) $> q_{n+1}$,

b) $d\eta \equiv 0$ on $(A-\Sigma)\times\{0\}$ if (degree of η) $= q_{n+1}$, and

c) $\sigma^*\eta \equiv \eta$ on $(A-\Sigma_A)\times(]-1,1[-\{0\})$ where $\sigma\colon A\times]-1,1[\to A\times]-1,1[$ is defined by $\sigma(a,t)=\sigma(a,-t)$.

With the notations of the above Proposition (for $\varepsilon = 1$ and $t_0 = 0$), we get: $pr^*(\mathcal{K}^i_{\overline{q}}(A)) \subset C^i$, for $i < q_{n+1}$; $pr^*(\mathcal{K}^i_{\overline{q}}(A)\cap d^{-1}\{0\}) \subset C^i$, for $i = q_{n+1}$; $J^*C^i = \{0\}$, for $i > q_{n+1}$ and $H(C^*) \subset C^*$. The same procedure used in §3.1.2 finishes the proof. ♣

3.1.4 Kernel of $\oint$. The elements of $Ker \oint$ are the differential forms $\pi^*\omega$ verifying

i) $\omega \in \Omega^*(B-\Sigma_B)$ is a liftable form,

ii) $\widetilde{\omega} \in F_{q_k}\Omega^*_{\mathcal{L}_B^{-1}(S/G_S)}$ for each exceptional stratum S with $\dim S = n-k$ and for each fixed stratum S with $\dim S = n-k < n-4$,

iii) $\widetilde{\omega} \in F_2\Omega^*_{\mathcal{L}_B^{-1}(S/G_S)}$ for each fixed stratum S with $\dim S = n-4$, and

iv) $\widetilde{\omega} \in F_0\Omega^*_{\mathcal{L}_B^{-1}(S/G_S)}$ for each fixed stratum S with $\dim S = n-2$.

(see (6)). The two last conditions are always fulfilled. In fact, the dimension of the fibers of $\mathcal{L}_B\colon \mathcal{L}_B^{-1}(S/G_S) \to S/G_S$ are 2 and 0 respectively.

Proposition 3.1.5 *The map $\pi^*\colon IH^*_{\overline{r+2}}(B-F_4) \to H^*(Ker \oint)$ is an isomorphism.*

Proof. Consider $\mathcal{D}^*(B)$ the subcomplex of $\Omega^*(B-\Sigma_B)$ made up of the differential forms satisfying i) and ii). This complex is isomorphic to $Ker \oint$ by π^*. The relations $\overline{q} \leq \overline{r+2}$ and $q_k \leq r_{k-1}+2$, for $k \geq 6$, imply that the restriction $\mathcal{D}^*(B) \to \mathcal{K}^*_{\overline{r+2}}(B-F_4)$ is well defined. Now, it suffices to show that this restriction induces an isomorphism in cohomology. First of all notice that for each stratum S the space $\mathbf{S}^\ell/G_S$ is a homological manifold. We have several possibilities:

1) *$B = V\times c(\mathbf{S}^\ell\ /G_S)$ and $G_S \neq \mathbf{S}^1$.* We have $F_4 = \emptyset$ and $\mathcal{D}^*(B) = \mathcal{K}^*_{\overline{q}}(B)$. The result comes from the fact that B is a homological manifold (see [5, §6.4]).

2) *$B = V\times c(\mathbf{S}^\ell/G_S)$, $G_S = \mathbf{S}^1$ and $\ell > 3$.* We have $F_4 = \emptyset$, the local calculations of the intersection cohomology give $IH^j_{\overline{r+2}}(B) \cong IH^j_{\overline{r+2}}(\mathbf{S}^\ell/G_S)$ if $j \leq r_\ell + 2$, and $IH^j_{\overline{r+2}}(B) \cong 0$ otherwise.

On the other hand, the operators used in §3.1.2 and §3.1.3 preserve the Cartan's filtration. Following the same procedure used there, we get:

$$H^*(\mathcal{D}(B)) \cong H^*(\mathcal{D}(c(\mathbf{S}^\ell/G_S)) \cong H^*\{\omega \in \mathcal{K}^j_{\overline{q}}((\mathbf{S}^\ell/G_S)\times]-1,1[) \text{ such that}$$

$$\begin{array}{lll} \text{a)} & \omega \equiv 0 & \text{on } (\mathbf{S}^\ell/G_S - \Sigma_{\mathbf{S}^\ell/G_S})\times\{0\} \text{ if } j > q_{\ell+1} = r_\ell + 2, \\ \text{b)} & d\omega \equiv 0 & \text{on } (\mathbf{S}^\ell/G_S - \Sigma_{\mathbf{S}^\ell/G_S})\times\{0\} \text{ if } j = q_{\ell+1} = r_\ell + 2, \text{ and} \\ \text{c)} & \sigma^*\omega \equiv \omega & \text{on } (\mathbf{S}^\ell/G_S - \Sigma_{\mathbf{S}^\ell/G_S})\times\{0\} \quad \}), \end{array}$$

$$\cong H^*(\{\omega \in \mathcal{K}^j_{\overline{q}}(\mathbf{S}^\ell/G_S)\ /\ \omega \equiv 0 \text{ if } j > r_\ell + 2, \text{ and } d\omega \equiv 0 \text{ if } j = r_\ell + 2\}),$$

which is isomorphic to $IH^*_{\overline{r+2}}(B)$.

3) $B = V \times c(\mathbf{S}^\ell/G_S)$, $G_S = \mathbf{S}^1$ *and* $\ell = 3$. We have $\mathcal{K}^*_{\overline{r+2}}(B - F_4) = \mathcal{K}^*_{\overline{r+2}}(V \times (\mathbf{S}^3/G_S)\times]0,1[)$. The local calculations of the intersection cohomology show $IH^*_{\overline{r+2}}(B - F_4) \cong H^*(\mathbf{S}^3/G_S)$.

Using the same procedure as before, we get:

$$H^*(\mathcal{D}(B)) \cong H^*(\mathcal{D}(c(\mathbf{S}^3/G_S))) \cong H^*(\{\omega \in \mathcal{K}^*_{\overline{q}}((\mathbf{S}^3/G_S)\times]-1,1[) \;/\; \sigma^*\omega = \omega\})$$

$\cong H^*(\mathbf{S}^3/G_S)$.

4) $B = V \times c(\mathbf{S}^\ell/G_S)$, $G_S = \mathbf{S}^1$ *and* $\ell = 1$. We have $\Sigma_B = \emptyset$ and therefore $\mathcal{D}^*(B) = \mathcal{K}^*_{\overline{r+2}}(B) = \{$liftable forms of $\Omega^*(B)\}$.

5) *General case*. The space B possesses a cover by open sets $\mathcal{W} = \{W\}$ and every W satisfies one of the previous conditions. We finish the proof if we construct a subordinated partition of unity $\{f\}$ such that

$$\omega \in \mathcal{K}^*_{\overline{q}}(B - F_4) \text{ (resp. } \mathcal{D}^*(B)) \Rightarrow f\omega \in \mathcal{K}^*_{\overline{q}}(B - F_4) \text{ (resp. } \mathcal{D}^*(B)). \tag{11}$$

To this end, take $\{f\}$ a partition of unity made up of controlled functions (see [15]). It is easy to check that each function f is a liftable one (see [10, §4.1.5]). Since the lifting $\widetilde{f}$ is constant on the fibers of each $\mathcal{L}_B\colon \mathcal{L}_B^{-1}(S/G_S) \to S/G_S$ we get $||\widetilde{f}||_{S/G_S} = ||d\widetilde{f}||_{S/G_S} \leq 0$. Therefore (11) holds. ♣

3.1.6 Image of $\oint$. Recall that for a liftable differential form $\omega = \pi^*\alpha + \chi \wedge \pi^*\beta$ on $I\Omega^*(M - \Sigma_M)$ the perverse degrees $||\widetilde{\omega}||_S$ and $||d\widetilde{\omega}||_S$, where S is a stratum of Σ_M, are calculated by:

$$\begin{aligned}
||\widetilde{\omega}||_S &= \max(||\widetilde{\alpha}||_{S/G_S}, ||\widetilde{\chi}||_S + ||\widetilde{\beta}||_{S/G_S}) \quad \text{and}\\
||d\widetilde{\omega}||_S &= \max(||d\widetilde{\alpha} + \widetilde{e} \wedge \widetilde{\beta}||_{S/G_S}, ||\widetilde{\chi}||_S + ||d\widetilde{\beta}||_{S/G_S}).
\end{aligned}$$

Therefore, a differential form $\pi^*\beta$ belongs to the image of $\oint$ if and only if there exists a differential form α satisfying:

$$\begin{array}{rll}
\text{i)} & \alpha, \beta \in \Omega^*(B - \Sigma_B) & \text{are liftable forms,}\\
\text{ii)} & \widetilde{\alpha}, \widetilde{\beta} \in F_{q_k}\Omega^*_{\mathcal{L}_B^{-1}(S/G_S)} & \text{for each exceptional stratum } S \text{ with } \dim S = n - k,\\
\text{iii)} & \widetilde{\beta} \in F_{q_k - 1}\Omega^*_{\mathcal{L}_B^{-1}(S/G_S)}, & \text{for each fixed stratum } S \text{ with } \dim S = n - k \leq n - 4\\
 & ||\widetilde{\alpha}||_{S/G_S} \leq q_k \;\text{ and} & \\
 & ||d\widetilde{\alpha} + \widetilde{e} \wedge \widetilde{\beta}||_{S/G_S} \leq q_k & \\
\text{iv)} & \widetilde{\beta}|_{S/G_S} \equiv 0 & \text{for each fixed stratum } S \text{ with } \dim S = n - 2.
\end{array}$$

The relations $\overline{r} \leq \overline{q}$ and $r_{k-1} \leq q_k - 1$, for $k \geq 4$, imply that $\mathcal{K}^*_{\overline{r}}(B, \partial B)$ is a subcomplex of $Im \oint$ (taking $\alpha = 0$). Moreover we have

Proposition 3.1.7 *The inclusion $\mathcal{K}^*_{\overline{r}}(B, \partial B) \hookrightarrow Im \oint$ induces an isomorphism in cohomology.*

Proof. We consider several cases.

1) $B = V \times c(\mathbf{S}^\ell/G_S)$ *and* $G_S \neq \mathbf{S}^1$. We have $\mathcal{K}^*_{\overline{r}}(B, \partial B) = \mathcal{K}^*_{\overline{r}}(B)$ and $Im \oint = \mathcal{K}^*_{\overline{q}}(B)$. The result comes from the fact that B is a homological manifold.

2) *$B = V \times c(\mathbf{S}^\ell/G_S)$, $G_S = \mathbf{S}^1$ and $\ell > 1$.* We have $\mathcal{K}^*_{\overline{r}}(B, \partial B) = \mathcal{K}^*_{\overline{r}}(B)$ and therefore

$$H^j(\mathcal{K}^*_{\overline{r}}(B, \partial B)) \cong \begin{cases} H^j(\mathbf{S}^\ell/G_S) & \text{if } j \le r_\ell \\ 0 & \text{if } j > r_\ell \end{cases}$$

(see §3.1.2 and §3.1.3).

On the other hand, remark that we can change in iii) the form $\widetilde{e}$ by the (pullback of the) Euler form $\widetilde{\epsilon}$ of $\widetilde{\Psi_S}$ (see (7)). Since the operators used in §3.1.2 and §3.1.3 preserve the form $\widetilde{\epsilon}$ we get, following the same procedure used there, the isomorphisms:

$$H^*(Im \oint) \cong H^*(Im \oint : I\mathcal{K}^*_{\overline{q}}(c\mathbf{S}^\ell) \to \Omega^{*-1}(c(\mathbf{S}^\ell - \Sigma_{\mathbf{S}^\ell})/G_S))$$

$$\cong H^*(\{\beta \in \mathcal{K}^j_{\overline{q}}((\mathbf{S}^\ell/G_S)\times]-1,1[) \ / \ \exists \alpha \in \mathcal{K}^{j+1}_{\overline{q}}((\mathbf{S}^\ell/G_S)\times]-1,1[) \text{ satisfying}$$

$$\begin{array}{lll} \text{a)} & \alpha \equiv \beta \equiv 0 & \text{on } (\mathbf{S}^\ell/G_S - \Sigma_{\mathbf{S}^\ell/G_S}) \times \{0\} \text{ if } j \ge r_\ell + 2, \\ \text{b)} & d\alpha + \epsilon \wedge \beta \equiv d\beta \equiv 0 & \text{on } (\mathbf{S}^\ell/G_S - \Sigma_{\mathbf{S}^\ell/G_S}) \times \{0\} \text{ if } j = r_\ell + 1, \text{ and} \\ \text{c)} & \sigma^*\alpha \equiv \alpha \text{ and } \sigma^*\beta \equiv \beta & \text{on } (\mathbf{S}^\ell/G_S - \Sigma_{\mathbf{S}^\ell/G_S}) \times \{0\} \quad \}) \end{array}$$

$$\cong H^*(\{\beta \in \mathcal{K}^j_{\overline{q}}(\mathbf{S}^\ell/G_S) \ / \ \exists \alpha \in \mathcal{K}^{j+1}_{\overline{q}}(\mathbf{S}^\ell/G_S) \text{ satisfying}$$

$$\begin{array}{lll} \text{a)} & \alpha \equiv \beta \equiv 0 & \text{if } j \ge r_\ell + 2, \text{ and} \\ \text{b)} & d\alpha + \epsilon \wedge \beta \equiv d\beta \equiv 0 & \text{if } j = r_\ell + 1 \quad \}). \end{array}$$

These calculations imply directly

$$H^j(Im \oint) \cong \begin{cases} H^j(\mathbf{S}^\ell/G_S) & \text{if } j \le r_\ell \\ 0 & \text{if } j \ge r_\ell + 2. \end{cases}$$

Consider now a cycle β in $\mathcal{K}^{r_\ell+1}_{\overline{q}}(\mathbf{S}^\ell/G_S)$ with $d\alpha + \epsilon \wedge \beta \equiv 0$, for some $\alpha \in \mathcal{K}^{r_\ell+2}_{\overline{q}}(\mathbf{S}^\ell/G_S)$. Since the action Ψ_S has not fixed points, the map $\wedge[\epsilon]\colon H^{r_\ell+1}(\mathcal{K}^*_{\overline{q}}(\mathbf{S}^\ell/G_S)) \to H^{r_\ell+3}(\mathcal{K}^*_{\overline{q}}(\mathbf{S}^\ell/G_S))$ is a monomorphism (see §3.1.1). Thus, there exists $\gamma \in \mathcal{K}^{r_\ell}_{\overline{q}}(\mathbf{S}^\ell/G_S)$ with $d\gamma = \beta$. This implies the vanishing of $H^{r_\ell+1}(Im \oint)$ and therefore the isomorphism $H^*(Im \oint) \cong H^*(\mathcal{K}_{\overline{r}}(B, \partial B))$.

3) *$B = V \times c(\mathbf{S}^\ell/G_S)$, $G_S = \mathbf{S}^1$ and $\ell = 1$.* We have $B = V \times [0, 1[$ and therefore $Im \oint = \mathcal{K}^*_{\overline{r}}(B, \partial B) = \{ \text{ liftable forms } \} \cap \Omega^*(B, \partial B)$.

4) *General case.* Same procedure followed in §3.1.5 5). ♣

We arrive to the main result of this work.

Theorem 3.1.8 *Let $\Phi\colon \mathbf{S}^1 \times M \to M$ be an action of $\mathbf{S}^1$ on a manifold M. For each perversity $\overline{r} = (0, 0, 0, r_5, r_6, \ldots)$ there exists a long exact sequence*

$$\cdots \to H^i(M) \xrightarrow{\oint^*} IH^{i-1}_{\overline{r}}(M/\mathbf{S}^1, \partial(M/\mathbf{S}^1)) \xrightarrow{\wedge[e]} IH^{i+1}_{\overline{r+2}}((M - F_4)/\mathbf{S}^1) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots \tag{12}$$

where

a) $\oint$ is the integration along the fibers of the projection $\pi\colon M \to M/\mathbf{S}^1$,

b) $\overline{r+2} = (0, 1, 2, r_5 + 2, r_6 + 2, \ldots)$,

c) F_4 is the union of 4-codimensional connected components of the fixed point set of Φ, and

d) $[e] \in IH^2_{\overline{2}}((M - F_4)/\mathbf{S}^1)$ is the Euler class of Φ.

Proof . Consider the perversity $\overline{q} = (0,1,2,2,r_5+2,r_6+2,\ldots)$. The short exact sequence

$$0 \longrightarrow Ker\oint \xrightarrow{\iota} I\mathcal{K}^*_{\overline{q}}(M) \xrightarrow{\oint} Im\oint \longrightarrow 0$$

produces the exact long sequence

$$\cdots \to H^i(M) \xrightarrow{\oint^*} H^{i-1}(Im\oint) \xrightarrow{\delta} H^{i+1}(Ker\oint) \xrightarrow{\iota^*} H^{i+1}(M) \to \cdots$$

(see (10) and Proposition 2.2.6). The connecting operator of this sequence is defined by $\delta[\beta] = [\pi^*(e\wedge\beta)]$. The result comes now from Propositions 3.1.5 and 3.1.7. ♣

Corollary 3.1.9 *Let $\Phi\colon \mathbf{S}^1 \times M \to M$ be an action of $\mathbf{S}^1$ on a manifold M. If the codimension of the fixed point set is at least 5, we get the following exact sequence*

$$\cdots \to H^i(M) \xrightarrow{\oint^*} H^{i-1}(M/\mathbf{S}^1) \xrightarrow{\wedge[e]} IH^{i+1}_{\frac{-}{2}}(M/\mathbf{S}^1) \xrightarrow{\pi^*} H^{i+1}(M) \to \cdots.$$

Proof. By hypothesis we have $F_4 = \emptyset$ and $\partial M/\mathbf{S}^1 = \emptyset$. Applying Theorem 3.1.8, for $\overline{r} = \overline{0}$, and [5, page 153] the result follows. ♣

3.1.10 Remarks.

1) The sequence (12) does not degenerate necessarily in (9). In fact, consider $\mathbf{S}^{2\ell+1}$ the unit sphere of $\mathbf{C}^{\ell+1}$, where the product induces the action $\Psi\colon \mathbf{S}^1 \times \mathbf{S}^{2\ell+1} \to \mathbf{S}^{2\ell+1}$. Identify $\mathbf{S}^{2\ell+2}$ with the suspension $\Sigma\mathbf{S}^{2\ell+1} = \mathbf{S}^{2\ell+1}\times[-1,1]/\{\mathbf{S}^{2\ell+1}\times\{1\}\,,\,\mathbf{S}^{2\ell+1}\times\{-1\}\}$. Consider the action $\Phi\colon \mathbf{S}^1\times\mathbf{S}^{2\ell+2} \to \mathbf{S}^{2\ell+2}$ defined by $\Phi(\theta,[x,t]) = [\Psi(\theta,x),t]$. If $\ell \geq 2$ then $\partial(\mathbf{S}^{2\ell+2}/\mathbf{S}^1) = F_4 = \emptyset$ and the sequence (12) becomes

$$\cdots \to H^i(\mathbf{S}^{2\ell+2}) \to H^{i-1}(\Sigma\mathbf{CP}^\ell) \to IH^{i+1}_{\frac{-}{2}}(\Sigma\mathbf{CP}^\ell) \to H^{i+1}(\mathbf{S}^{2\ell+2}) \to \cdots.$$

On the other hand, the sequence (9)

$$\cdots \to H^i(\mathbf{S}^{2\ell+2}) \to H^{i-1}(\Sigma\mathbf{CP}^\ell) \to H^{i+1}(\Sigma\mathbf{CP}^\ell) \to H^{i+1}(\mathbf{S}^{2\ell+2}) \to \cdots$$

cannot be exact, therefore it is different from (12).

For $\ell = 1$ we get

$$\cdots \to H^i(\mathbf{S}^4) \to H^{i-1}(\mathbf{S}^3) \to H^{i+1}(\mathbf{S}^2) \to H^{i+1}(\mathbf{S}^4) \to \cdots,$$

and for $\ell = 0$ we obtain

$$\cdots \to H^i(\mathbf{S}^2) \to H^{i-1}([0,1],\{0,1\}) \to H^{i+1}([0,1]) \to H^{i+1}(\mathbf{S}^2) \to \cdots.$$

2) Up to a non-zero factor, the Euler class of Φ does not depend on the choice of the good metric. Indeed, let μ_1 and μ_2 be two good metrics of M. Suppose first that $\partial(M/\mathbf{S}^1) = \emptyset$. For $\overline{r} = \overline{0}$ we obtain from the two Gysin sequences:

$$H^0(M/\mathbf{S}^1) \xrightarrow{\wedge[e_{\mu_j}]} IH^2_{\frac{-}{2}}((M-F_4)/\mathbf{S}^1) \xrightarrow{\pi^*} H^2(M) \quad j = 1,2.$$

The space $H^0(M/\mathbf{S}^1)$ is of dimension one, then, by exactness, $\dim Ker\pi^* \leq 1$ and $Im(\wedge[e_{\mu_1}]) = Ker\pi^* = Im(\wedge[e_{\mu_2}])$. Now, there exists $\lambda \in \mathbf{R} - \{0\}$ such that $[e_{\mu_1}] = \lambda[e_{\mu_2}]$.

If $\partial(M/\mathbf{S}^1) \neq \emptyset$ we get the above result for $M/\mathbf{S}^1 - \partial(M/\mathbf{S}^1)$. Now it suffices to apply the isomorphism $IH^*_{\frac{-}{2}}(M/\mathbf{S}^1) \cong IH^*_{\frac{-}{2}}(M/\mathbf{S}^1 - \partial(M/\mathbf{S}^1))$, induced by restriction, to get the result.

In particular, the fact that the Euler class of Φ respect to the metric μ vanishes does not depend of the choice of the good metric μ.

3) If the action Φ has not fixed points, we obtain two exact sequences:

$$\cdots \xrightarrow{\pi^*} H^i(M) \xrightarrow{\oint} H^{i-1}(M/\mathbf{S}^1) \xrightarrow{\wedge[e]} H^{i+1}(M/\mathbf{S}^1) \xrightarrow{\pi^*} \cdots$$

$$\cdots \xrightarrow{\pi^*} H^i(M) \xrightarrow{\oint} H^{i-1}(M/\mathbf{S}^1) \xrightarrow{\wedge[E]} H^{i+1}(M/\mathbf{S}^1) \xrightarrow{\pi^*} \cdots.$$

The first is (12) and the second one is given by [10]. Here E denotes the Euler form associated to a global invariant riemannian metric on M. The same argument used in 2) shows that $[e]$ and $[E]$ are that there exists $\lambda \in \mathbf{R} - \{0\}$ such that $[e] = \lambda[E]$.

3.2 Vanishing of the Euler class

Consider Φ an almost free action on a compact manifold M. The Euler class $[e] \in H^2(M/\mathbf{S}^1)$ vanishes if and only if there exists a locally trivial fibration $\Upsilon \colon M \to \mathbf{S}^1$, whose fibers are transverse to the orbits of Φ (see [9], [10]).

We show now that if the action Φ has fixed points, the vanishing of the Euler class $[e] \in IH^2_{\frac{2}{2}}((M - F_4)/\mathbf{S}^1)$ has also a geometrical interpretation, for that we need some preliminary results.

Lemma 3.2.1 *If the Euler class of Φ vanishes then the codimension of $M^{\mathbf{S}^1}$ is at most two.*

Proof. Let S be a fixed stratum on M. Since the Euler class of Φ vanishes then the Euler class of Ψ_S also vanishes. From §3.1.10 3) and [10, §4.3] we deduce $H^1(\mathbf{S}^\ell) \neq 0$ and therefore $\ell = 1$. ♣

Lemma 3.2.2 *Suppose that M is compact and that the codimension of $M^{\mathbf{S}^1}$ is two. There exist a compact manifold $\widehat{M}$, an almost free action $\widehat{\Phi} \colon \mathbf{S}^1 \times \widehat{M} \to \widehat{M}$ and a commutative diagram*

$$\begin{tikzcd}
& \widetilde{M} \arrow[ld, "\mathcal{L}_{\widehat{M}}"'] \arrow[rd, "\mathcal{L}_M"] & \\
\widehat{M} \arrow[rr, "\widehat{\mathcal{L}}_M"] & & M
\end{tikzcd}$$

where

i) $\widehat{\mathcal{L}}_M$ is an equivariant differentiable map,
ii) the restriction of $\widehat{\mathcal{L}}_M$ to each connected component C of $\widehat{M} - \widehat{\mathcal{L}}_M^{-1}(M^{\mathbf{S}^1})$ is a diffeomorphism,and
iii) the adherence $\overline{C}$ is manifold with boundary $\widehat{\mathcal{L}}_M^{-1}(M^{\mathbf{S}^1})$.

There also exist two good metrics μ and $\hat{\mu}$, of M and $\widehat{M}$ respectively, such that $\widehat{\mathcal{L}}_M^*\mu = \hat{\mu}$, on $\widehat{M} - \widehat{\mathcal{L}}_M^{-1}(\Sigma_M)$.

Proof. For the first part we proceed as in §4.1.1, taking $M^{\mathbf{S}^1}$ instead of $M_{m-\ell-1}$.

For the second one we remark that the set of fixed points $M^{\mathbf{S}^1}$ has a neighborhood on M which is diffeomorphic to $M^{\mathbf{S}^1} \times D^2$. The restriction of the above diagram to this neighborhood becomes

$$\begin{tikzcd}
& M^{\mathbf{S}^1}\times\widetilde{\mathbf{S}^1}\times]-1,1[\arrow[dl, "\mathcal{L}_{\widehat{M}}"'] \arrow[dr, "\mathcal{L}_M"] & \\
M^{\mathbf{S}^1}\times\mathbf{S}^1\times]-1,1[\arrow[rr, "\widehat{\mathcal{L}}_M"] & & M^{\mathbf{S}^1}\times D^2
\end{tikzcd}$$

where $\widehat{\mathcal{L}}_M(x,\theta,r)=(x,[\theta,|r|])$, $\mathcal{L}_{\widehat{M}}(x,\widetilde{\theta},r)=(x,\mathcal{L}_{\mathbf{S}^1}(\widetilde{\theta}),r)$, $\mathcal{L}_M(x,\widetilde{\theta},r)=(x,[\mathcal{L}_{\mathbf{S}^1}(\widetilde{\theta}),|r|])$ and $\mathcal{L}_{\mathbf{S}^1}\colon\widetilde{\mathbf{S}^1}\to\mathbf{S}^1$ is a trivial covering.

Out of that neighborhood we take μ the restriction of a good metric of M and $\hat{\mu}=\widehat{\mathcal{L}}_M^*\mu$. Inside we consider: $\widetilde{\mu}=\nu+\mathcal{L}_{\mathbf{S}^1}^*d\theta+dr^2$, $\hat{\mu}=\nu+d\theta+dr^2$ and $\mu=\nu+d\theta+dr^2$ where ν is a riemannian metric on $M^{\mathbf{S}^1}$, $d\theta$ is an invariant metric on $\mathbf{S}^1$ and dr^2 is the canonical metric on $]-1,1[$. It is easy to see that they satisfy the given conditions. ♣

Lemma 3.2.3 *Suppose that the codimension of $M^{\mathbf{S}^1}$ is two. The Euler class of Φ and the Euler class of $\Phi'\colon\mathbf{S}^1\times(M-M^{\mathbf{S}^1})\to(M-M^{\mathbf{S}^1})$ vanish simultaneously.*

Proof. The orbit space $M/\mathbf{S}^1$ is an homological manifold with boundary $M^{\mathbf{S}^1}/\mathbf{S}^1$. Thus, the inclusion $(M-M^{\mathbf{S}^1})/\mathbf{S}^1\hookrightarrow M/\mathbf{S}^1$ induces an isomorphism $H^*((M-M^{\mathbf{S}^1})/\mathbf{S}^1)\cong H^*(M/\mathbf{S}^1)$. We have finish, because the Euler class of Φ' is the restriction of the Euler class of Φ, for a good metric. ♣

A singular foliation $\mathcal{F}$ (see [13]) on M is said to be **transverse** to Φ if for each point $x\in M-M^{\mathbf{S}^1}$ the leaf of $\mathcal{F}$ and the orbit of Φ passing trough x, are transverse.

Theorem 3.2.4 *Let $\Phi\colon\mathbf{S}^1\times M\to M$ be an action of $\mathbf{S}^1$ on a compact manifold M. The following statements are equivalent:*

a) the Euler class $[e]\in IH^2_{\overline{2}}((M-F_4)/\mathbf{S}^1)$ vanishes, and

b) there exists a singular foliation transverse to Φ, whose restriction to $M-M^{\mathbf{S}^1}$ is a locally trivial fibration over $\mathbf{S}^1$.

Proof. If there are not fixed points, the result was already proved in [10, §4.1], by means of §3.1.10 3). Then, we can suppose $M^{\mathbf{S}^1}\neq\emptyset$.

a) ⇒ b) Take μ and $\hat{\mu}$ the metrics given by Lemma 3.2.2. Since $\mathcal{L}_M^*\mu=\hat{\mu}$ we get $\mathcal{L}_M^*[e]=[\hat{e}]$ and therefore $[\hat{e}]=0$. By §3.1.10 3) and [10, §4.1], there exists a locally trivial fibration $\Upsilon\colon\widehat{M}\to\mathbf{S}^1$ transverse to the fibers of $\widehat{\Phi}$.

Let C be a connected component of $\widehat{M}-\widehat{\mathcal{L}}_M^{-1}(M^{\mathbf{S}^1})$. It is easily checked that the distribution $(\widehat{\mathcal{L}}_M)_*(Ker\,\Upsilon_*\cap T\overline{C})$ is locally of finite type, therefore it defines a singular foliation $\mathcal{F}$ (see [10, pages 185-186]). By §3.2.2 ii), the foliation $\mathcal{F}$ is transverse to Φ. So, it remains to verify that the restriction of $\mathcal{F}$ to $M-M^{\mathbf{S}^1}$ is defined by a locally trivial fibration over $\mathbf{S}^1$.

Since the restriction $\widehat{\mathcal{L}}_M\colon C\to(M-M^{\mathbf{S}^1})$ is a diffeomorphism it suffices to show that $\Upsilon\colon C\to\mathbf{S}^1$ is a locally trivial fibration. Take a fiber N of Υ we get $\widehat{M}\cong N\times[0,1]/\sim$, where $(x,0)\sim(f(x),1)$, for a diffeomorphism $f\colon N\to N$. The fibration Υ becomes $\Upsilon([x,t])=e^{2\pi it}$ and the action is tangent to the $[0,1]$-factor. Since C is invariant, there exists a submanifold $N_0\subset N$, invariant by f, such that $C\cong N_0\times[0,1]/\sim$. This finishes the proof.

b) ⇒ a) We show first that the codimension of $M^{\mathbf{S}^1}$ is two. Let S be a fixed stratum of Φ. The locally trivial fibration given by b) is defined by a closed differential form. Since $\mathbf{S}^\ell$ is an invariant submanifold of $M-M^{\mathbf{S}^1}$ then the restriction of the above form defines a locally trivial fibration on $\mathbf{S}^\ell$ transverse to Ψ_S (see [14]). From [10, §4.1, and §4.3] we deduce that the Euler class of Ψ_S vanishes, and therefore $\ell=1$.

Consider on $M - M^{\mathbf{S}^1}$ an equivariant riemannian metric ν such that: i) the leaves of $\mathcal{F}$ and the orbits of Φ are orthogonal, and ii) $\nu(X,X) = 1$. Thus, the associated characteristic form $\mathcal{X}$ is a cycle. That is, the Euler class $[E]$ (in the sense of [10]) of $\Phi' \colon \mathbf{S}^1 \times (M - M^{\mathbf{S}^1}) \to (M - M^{\mathbf{S}^1})$ vanishes. By §3.1.10 3), the Euler class $[e']$ of Φ' also vanishes. Now we apply Lemma 3.2.3 ♣

As in [10, §4.3 and §4.4], we obtain

Corollary 3.2.5 *Under the conditions of the previous Theorem, if B has not boundary and $H^1(M) = 0$ then the Euler class of Φ is non-zero.*

Proof. If the Euler class of Φ is 0 then the action Φ is almost free (consider §3.2.1 and $\partial B = \emptyset$) and we can apply [10, §4.3]. ♣

The example §3.1.10 1), with $\ell = 0$, show that the hypothesis $\partial B = \emptyset$ is necessary.

Corollary 3.2.6 *Under the conditions of the previous Theorem, if the Euler class of Φ vanishes, then any equivariant unfolding $\widetilde{M}$ of M possesses a finite covering of the form $N \times \mathbf{S}^1$.*

Proof. Let μ be a good metric of M. The relation $\mathcal{L}_M^*[e] = [\widetilde{e}]$ imply the Euler class of $\widetilde{\Phi}$ vanishes. Therefore, $\widetilde{M}$ has a finite covering of the form $N \times \mathbf{S}^1$ (see [10]). ♣

4 Appendix

The Appendix is devoted to the proofs of Proposition 1.1.6, Proposition 1.2.6 and Lemma 2.3.2.

4.1 Proof of Proposition 1.1.6

The construction of the equivariant unfolding that we exhibit now is the equivariant version of [1]. We need the two following Lemmas.

Lemma 4.1.1 *Suppose len$(M) = \ell + 1 > 0$. Then there exists a manifold $\widehat{M}$ supporting an action of G and a continuous equivariant map $\widehat{\mathcal{L}}_M \colon \widehat{M} \to M$ such that:*

a) len$(\widehat{M}) <$ len(M),

b) $\widehat{\mathcal{L}}_M \colon (\widehat{M} - \widehat{\mathcal{L}}_M^{-1}(M_{m-\ell-1})) \longrightarrow (M - M_{m-\ell-1})$ is a finite trivial differentiable covering, and

c) for each stratum S of dimension $m - \ell - 1$, for each $x_0 \in S$ and for each $\hat{x}_0 \in \widehat{\mathcal{L}}_M^{-1}(x_0)$ there exists a commutative diagram

(13)
$$\begin{tikzcd}
\widehat{\mathcal{U}} \arrow[r, "\widehat{\varphi}"] \arrow[d, "\widehat{\mathcal{L}}_M"'] & U \times \mathbf{S}^\ell \times]-1,1[\arrow[d, "Q"] \\
\mathcal{U} \arrow[r, "\varphi"] & U \times c\mathbf{S}^\ell
\end{tikzcd}$$

where

i) *$\mathcal{U} \subset M$ and $\widehat{\mathcal{U}} \subset \widehat{M}$ are neighborhoods of x_0 and $\hat{x}_0$ respectively,*
ii) *(U, φ) is a distinguished chart of a tubular neighborhood of S,*
iii) *$\hat{\varphi}$ is a G_S-equivariant diffeomorphism, and*
iv) *$Q(x, \theta, r) = (x, [\theta, |r|])$.*

Proof. We follow the process of “removing a tubular neighborhood” of [8] (see also [4]). In fact, we shall take the double of the original construction in order to avoid corners.

Let $\mathcal{S}$ be the family of strata of M with dimension $m-\ell-1$. We choose for each $S\in\mathcal{S}$ a tubular neighborhood $\mathcal{N}_S=(\mathcal{T}_S,\tau_S,S,D^{\ell+1})$ as in §1.1.2. Notice that the map $\bigcup_{S\in\mathcal{S}}\tau_S\colon\bigcup_{S\in\mathcal{S}}\mathcal{T}_S\to\bigcup_{S\in\mathcal{S}}S$ is equivariant. For each $S\in\mathcal{S}$ consider

$$D_S=\{x\in\mathcal{T}_S\ /\ \varphi(x)=(\tau_S(x),[\theta,\frac{1}{2}]),\ (U,\varphi)\in\mathcal{A}\}.$$

It follows from §1.1.2 ii) that $\bigcup_{S\in\mathcal{S}}D_S$ is a submanifold of M of codimension 1. The map

$$F\colon(\bigcup_{S\in\mathcal{S}}D_S)\times(]-1,1[-\{0\})\longrightarrow\bigcup_{S\in\mathcal{S}}(\mathcal{T}_S-S)$$

defined by $F(z,r)=\varphi^{-1}(\tau_S(z),[\theta,|r|])$, where $\varphi(z)=(\tau_S(z),[\theta,\frac{1}{2}])$, is a two-fold equivariant differentiable trivial covering.

We define now $\widehat{\mathcal{L}}_M\colon\widehat{M}\to M$.

- $\widehat{M}$ is the quotient of $\left\{(M-\bigcup_{S\in\mathcal{S}}S)\times\{-1,1\}\right\}\cup\left\{(\bigcup_{S\in\mathcal{S}}D_S)\times]-1,1[\right\}$ by the equivalence relation generated by:

  $$(x,j)\sim(z,r)\text{ iff }|r|=jr\text{ and }x=F(z,|r|).$$

  In the terminology of [8], the manifold $\widehat{M}$ is the double of $M\odot\bigcup_{S\in\mathcal{S}}S$.

- $$\widehat{\mathcal{L}}_M(y)=\begin{cases}x & \text{if } y=(x,j)\in(M-\bigcup_{S\in\mathcal{S}}S)\times\{-1,1\}\\ F(z,|r|) & \text{if } y=(z,r)\in(\bigcup_{S\in\mathcal{S}}D_S)\times]-1,1[\ .\end{cases}$$

The set $\widehat{M}$ is a manifold supporting an action of G (taking the trivial action on $\{-1,1\}$ and $]-1,1[$). The map $\widehat{\mathcal{L}}_M$ is an equivariant continuous function. By construction $\operatorname{len}(\widehat{M})=\operatorname{len}(M)-1$ and the restriction of $\widehat{\mathcal{L}}_M$ to $\widehat{M}-\widehat{\mathcal{L}}_M^{-1}(M_{m-\ell-1})$ is a finite trivial covering. This gives a) and b).

In order to check c) we first notice that near $S\in\mathcal{S}$ the map $\widehat{\mathcal{L}}_M$ becomes

$$\widehat{\mathcal{L}}_M\colon D_S\times]-1,1[\to\mathcal{T}_S\text{ defined by }\widehat{\mathcal{L}}_M(z,r)=F(z,|r|).$$

Consider (U,φ) a distinguished chart of $\mathcal{N}_S$ with $x_0\in U$ and take $\mathcal{U}=\tau^{-1}(U)$ and $\widehat{\mathcal{U}}=\widehat{\mathcal{L}}_M^{-1}(\mathcal{U})$ which is $(\tau^{-1}(U)\cap D_S)\times]-1,1[$. They satisfy i) and ii).

Define the map $\hat{\varphi}$ by $\hat{\varphi}(z,r)=(\tau_S(z),\theta,r)$, where $\varphi(z)=(\tau_S(z),[\theta,\frac{1}{2}])$; it is a G_S-diffeomorphism satisfying iv).

Since the isotropy subgroup of any point of $\widehat{\mathcal{U}}$ is included in G_S we conclude that the trace on $\widehat{\mathcal{U}}$ of the stratification defined by G is the stratification defined by G_S. Therefore $\hat{\varphi}$ is an isomorphism, which gives iii). ♣

It is shown in [8] that the manifold $\widehat{M}$ is unique. So, for any equivariant diffeomorphism $f\colon M\to N$ there exists an equivariant diffeomorphism $\widehat{f}\colon\widehat{M}\to\widehat{N}$ with $\mathcal{L}_N\widehat{f}=f\mathcal{L}_M$.

4.1.3 Proof of Proposition 1.1.6

Assume inductively that the statement is true for any good action of length smaller than $\operatorname{len}(M)$. Let $\widehat{\mathcal{L}}_M\colon\widehat{M}\to M$ be the equivariant map given by §4.1.1. Recall that $\operatorname{len}(\widehat{M})<\operatorname{len}(M)$, therefore by induction there exists an equivariant unfolding $\mathcal{L}_{\widehat{M}} \colon \widetilde{M} \to \widehat{M}$. We consider the composition $\mathcal{L}_M = \mathcal{L}_{\widehat{M}} \widehat{\mathcal{L}}_M$, which verifies §1.1.5 1) and 2). It remains to verify 3).

Let x_0 be a point of a non regular stratum S and let $\widetilde{x}_0$ be a point of $\mathcal{L}_M^{-1}(x_0)$. If $x_0 \notin M_{m-\ell-1}$ we consider $\hat{x}_0 \in \widehat{M}$ with $\widehat{\mathcal{L}}_M(\hat{x}_0) = x_0$ and we apply the induction hypothesis to $\widehat{\mathcal{L}}_M$. If $x_0 \in M_{m-\ell-1}$ we apply §4.1.1 and we obtain the commutative diagram (13).

Defining $\widetilde{\mathcal{U}} = \mathcal{L}_{\widehat{M}}^{-1}(\widehat{\mathcal{U}})$, we get i) and ii). Since $\hat{\varphi}$ is a G_S-equivariant isomorphism the composition $\hat{\varphi}\mathcal{L}_{\widehat{M}}$ is a G_S-equivariant unfolding. By uniqueness of $\hat{}$ we have $U \times \mathbf{S}^\ell \times]-1,1[\cong U \times \widetilde{\mathbf{S}^\ell} \times]-1,1[$. This gives iii) and iv). ♣

4.2 Proof of Proposition 1.2.6

Let $\mathcal{L}_M \colon \widetilde{M} \to M$ be an equivariant unfolding of M (see Proposition 1.1.6). Since $\mathcal{L}_M$ is equivariant it induces the continuous map $\mathcal{L}_B \colon \widetilde{M}/G = \widetilde{B} \to B$ defined by $\mathcal{L}_B(\widetilde{\pi}(\widetilde{x})) = \pi\mathcal{L}_M(\widetilde{x})$. Then a) and b) hold. In order to prove c) assume inductively that the statement is true for any good action of length smaller than len(M). In particular, for any non regular stratum S we have a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathbf{S}^\ell} \arrow[r, "\widetilde{p}"] \arrow[d, "\mathcal{L}_{\mathbf{S}^\ell}"'] & \widetilde{\mathbf{S}^\ell}/G_S \arrow[d, "\mathcal{L}_{\mathbf{S}^\ell/G_S}"] \\
\mathbf{S}^\ell \arrow[r, "p"] & \mathbf{S}^\ell/G_S ,
\end{tikzcd} \tag{16}$$

satisfying a), b) and c).

Take $y_0 \in \pi(S)$, $\widetilde{y}_0 \in \mathcal{L}_B^{-1}(y_0)$, $x_0 = \pi(y_0)$ and $\widetilde{x}_0 = \widetilde{\pi}(\widetilde{y}_0)$. Consider the diagram (1) given by Proposition 1.1.6. We can choose the open set U small enough to have:

1) $V = \pi(U)$ is a neighborhood of y_0, and

2) a differentiable section σ of $\pi \colon U \to V$.

Define $\mathcal{V} = \rho^{-1}(V)$ and $\widetilde{\mathcal{V}} = \mathcal{L}_B^{-1}(\mathcal{V})$. We get i) and ii). Following the same method used in the proof of Proposition 1.2.2 and using the equivariance of $\mathcal{L}_M$, we can write:

$$\mathcal{V} = \mathcal{L}_B^{-1}\rho^{-1}(V) = \widetilde{\pi}\mathcal{L}_M^{-1}\pi^{-1}\pi\tau^{-1}\sigma(V) = \mathcal{L}_M^{-1}\tau^{-1}\sigma(V)/G_S.$$

Since the restriction $\widetilde{\varphi} \colon \mathcal{L}_M^{-1}\tau^{-1}\sigma(V) \to \sigma(V) \times \widetilde{\mathbf{S}^\ell} \times]-1,1[$ is a G_S-equivariant diffeomorphism (see §1.1.5), it induces the homeomorphism $\widetilde{\psi} \colon \widetilde{\mathcal{V}} \to V \times \widetilde{\mathbf{S}^\ell}/G_S \times]-1,1[$. The map $\widetilde{\psi}$ satisfies iii). Finally, for each $\widetilde{\pi}(\widetilde{x}) \in \widetilde{\mathcal{V}}$ we can write:

$$\begin{aligned}
R\widetilde{\psi}\widetilde{\pi}(\widetilde{x}) &= R\{\pi \times \widetilde{p} \times \text{identity}\}\widetilde{\varphi}(\widetilde{x}) && \text{(definition of } \widetilde{\psi}) \\
&= \Pi P\widetilde{\varphi}(\widetilde{x}) && \text{(see (16))} \\
&= \Pi\varphi\mathcal{L}_M(\widetilde{x}) && \text{(see (1))} \\
&= \psi\pi\mathcal{L}_M(\widetilde{x}) && \text{(see §1.2.2)} \\
&= \psi\mathcal{L}_B\widetilde{\pi}(\widetilde{x}) && \text{(see (3))}
\end{aligned}$$

from which iv) is satisfied. ♣

4.3 Proof of Lemma 2.3.2

We prove this result for any good action, where we suppose $X \equiv \widetilde{X} \equiv 0$ if $G \neq \mathbf{S}^1$.

Assume inductively that the statement is true for any good action such that the length of the induced stratification is smaller than $\operatorname{len}(M)$. It suffices to construct two riemannian metrics ν and $\widetilde{\nu}$, on $M-\Sigma_M$ and $\widetilde{M}$ respectively, satisfying b), c) and d); in this case the metrics:

$$\mu=\int_G \Phi_g^*\nu \quad \text{and} \quad \widetilde{\mu}=\int_G \widetilde{\Phi}_g^*\widetilde{\nu} \tag{13}$$

(see [2, page 304]), verify a), b), c) and d).

In order to get ν and $\widetilde{\nu}$ we proceed in two steps: (I) construction of two riemannian metrics $\nu_{\mathcal{U}}$ and $\widetilde{\nu}_{\mathcal{U}}$ on open sets of the type $\mathcal{U}-\Sigma_M$ and $\widetilde{\mathcal{U}}$ respectively (see (1)), satisfying b), c), d) and (II) pasting them by a partition of unity.

(I) Fix an open set $\mathcal{U}$ as in (1). Consider $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ two riemannian metrics, on $\mathbf{S}^\ell-\Sigma_{\mathbf{S}^\ell}$ and $\widetilde{\mathbf{S}^\ell}$ respectively, satisfying b), c) and d), which exist by induction. We can suppose $\widetilde{\mathcal{M}}$ invariant by the structural group of $\mathcal{L}_M\colon\mathcal{L}_M^{-1}(S)\to S$ (averaging as in (13)). By means of (U,φ) we identify $(\mathcal{U},\widetilde{\mathcal{U}},\mathcal{L}_M)$ with $(U\times c\mathbf{S}^\ell, U\times\widetilde{\mathbf{S}^\ell}\times]-1,1[,P)$.

There exists a decomposition $T\widetilde{\mathcal{U}}=T(\widetilde{\mathbf{S}^\ell}\times]-1,1[)\oplus\widetilde{E}$, where $\widetilde{X}$ is tangent to $\widetilde{E}$ if S is an exceptional stratum. The map P induces a decomposition $T(\mathcal{U}-\Sigma_M)=T\{(\mathbf{S}^\ell-\Sigma_{\mathbf{S}^\ell})\times]0,1[\}\oplus TU$. We define $\nu_{\mathcal{U}}=\mu_U+\mathcal{M}+dr^2$ and $\widetilde{\nu}_{\mathcal{U}}=P^*\mu_U+\widetilde{\mathcal{M}}+dr^2$ where μ_U is any riemannian metric on U. We need to check properties b), c) and d).

b) $P^*\nu_{\mathcal{U}}=P^*\mu_U+\mathcal{L}_{\mathbf{S}^\ell}^*\mathcal{M}+dr^2=P^*\mu_U+\widetilde{\mathcal{M}}+dr^2=\widetilde{\nu}_{\mathcal{U}}$.

c) For any exceptional stratum R the natural projection $pr\colon U\times\widetilde{\mathbf{S}^\ell}\times]-1,1[\to U\times\widetilde{\mathbf{S}^\ell}$ induces a map $pr\colon\mathcal{L}_M^{-1}(R)\to\mathcal{L}_M^{-1}(S)$ with $\mathcal{L}_M=\mathcal{L}_M\circ pr$. So, it suffices to check c) on $\mathcal{L}_M^{-1}(S)$. Here, we have: $\widetilde{\nu}_{\mathcal{U}}(\widetilde{X})=P^*\mu_U(\widetilde{X})=\mu_U(X)$.

d) For each stratum R meeting $\mathcal{U}$ the fibers of $\mathcal{L}_M\colon\mathcal{L}_M^{-1}(R)\to R$ are included on the fibers of P. Each of the fibers of P is G_S-equivariantly isometric to $(\widetilde{\mathbf{S}^\ell},\widetilde{\mathcal{M}})$. We apply now the induction hypothesis.

(II) Let $\Xi=\{\mathcal{U}\}$ and $\widetilde{\Xi}=\{\widetilde{\mathcal{U}}\}$ be coverings of M and $\widetilde{M}$ respectively made up of open sets as in (I). Consider $\{\nu_{\mathcal{U}},\widetilde{\nu}_{\mathcal{U}}\}_{\mathcal{U}\in\Xi}$ a family of riemannian metrics satisfying b), c) and d). Fix a partition of unity $\{f_{\mathcal{U}}\colon\mathcal{U}\to[0,1]\}$ subordinated to Ξ. Notice that the family $\{\widetilde{f}_{\mathcal{U}}=f_{\mathcal{U}}\mathcal{L}_M\colon\widetilde{\mathcal{U}}\to[0,1]\}$ is a partition of unity subordinated to $\widetilde{\Xi}$. Define the riemannian metrics $\nu=\sum\limits_{\Xi}f_{\mathcal{U}}\nu_{\mathcal{U}}$ on $M-\Sigma_M$ and $\widetilde{\nu}=\sum\limits_{\widetilde{\Xi}}\widetilde{f}_{\mathcal{U}}\widetilde{\nu}_{\mathcal{U}}$ on $\widetilde{M}$. It is easily checked that ν and $\widetilde{\nu}$ satisfy b) and c) . For d) we use §1.1.2 iv). ♣

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