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Finitness of the basic intersection cohomology of a Killing foliation

Martintxo Saralegi-Aranguren[*]

Robert Wolak[†]

Université d'Artois

Uniwersytet Jagiellonski

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Abstract

We prove that the basic intersection cohomology $H^*_{\overline{p}}(M/\mathcal{F})$, where $\mathcal{F}$ is the singular foliation determined by an isometric action of a Lie group G on the compact manifold M, is finite dimensional.

This paper deals with an action $\Phi\colon G \times M \to M$ of a Lie group on a compact manifold preserving a riemannian metric on it. The orbits of this action define a singular foliation $\mathcal{F}$ on M. Putting together the orbits of the same dimension we get a stratification of M. This structure is still very regular. The foliation $\mathcal{F}$ is in fact a conical foliation and we can define the basic intersection cohomology $H^*_{\overline{p}}(M/\mathcal{F})$ (cf. [10]). This invariant becomes the basic cohomology $H^*(M/\mathcal{F})$ when the action Φ is almost free, and the intersection cohomology $H^*_{\overline{p}}(M/G)$ when the Lie group G is compact.

The aim of this work is to prove that this cohomology $H^*_{\overline{p}}(M/\mathcal{F})$ is finite dimensional. This result generalizes [3] (almost free case), [11] (abelian case) and [10] (compact case).

The paper is organized as follows. In Section 1 we present the foliation $\mathcal{F}$. The basic intersection cohomology $H^*_{\overline{p}}(M/\mathcal{F})$ associated to this foliation is studied in Section two. Twisted products are studied in Section 3. The finiteness of $H^*_{\overline{p}}(M/\mathcal{F})$ is proved in Section 4.

In the sequel M is a connected, second countable, Haussdorff, without boundary and smooth (of class C^∞) manifold of dimension m. All the maps are considered smooth unless something else is indicated.

1 Killing foliations determined by isometric actions.

We study in this work the foliations induced by isometric actions: the *Killing foliations*. These foliations are examples of the conical foliations for which the basic intersection cohomology has been defined (see [10, 11]). We present this geometrical framework in this section.

1.1 Killing foliations. A smooth action $\Phi\colon G \times M \to M$ of a Lie group G on a manifold M is a *isometric action* when there exists a riemannian metric μ on M preserved by G.

The connected components of the orbits of the action Φ determine a partition $\mathcal{F}$ on M. In fact, this partition is a singular riemannian foliation that we shall call *Killing foliation* (cf. [7]). Notice that $\mathcal{F}$ is also a conical foliation in the sense of [10, 11]. Classifying the points of M following the dimension of the leaves of $\mathcal{F}$ one gets the *stratification* $\mathsf{S}_{\mathcal{F}}$ of $\mathcal{F}$. It is determined by the equivalence relation $x \sim y \Leftrightarrow \dim G_x = \dim G_y$. The elements of $\mathsf{S}_{\mathcal{F}}$ are called *strata*.

In the particular case where the closure of G in the isometry group of (M, μ) is a compact Lie group[1] we shall say that the action Φ is a *tame action*. In fact, a smooth action $\Phi\colon G \times M \to M$ is tame if and only if it extends to a smooth action $\Phi\colon K \times M \to M$ where K is a compact Lie group containing G (cf. [6]). The group K is not unique, but we always can choose K in such a way that G is dense in K. We shall say that K is a *tamer group*. Here the strata of $\mathsf{S}_{\mathcal{F}}$ are K-invariant closed submanifolds of M.

Since the aim of this work is the study of $\mathcal{F}$ and not the action Φ itself, we can consider that the Lie group G is connected. Let us see that.

Proposition 1.1.1 *Let* $\Phi\colon G \times M \to M$ *is a tame action. Let* G_0 *be the connected component of* G *containing the unity element. The Killing folation defined by the restriction* $\Phi\colon G_0 \times M \to M$ *is also* $\mathcal{F}$.

Proof. The partition $\mathcal{F}$ is defined by this equivalence relation:

$$x \sim y \iff \exists \text{ continuous path } \alpha\colon [0,1] \to G(x) \text{ such that } \alpha(0) = x \text{ and } \alpha(1) = y.$$

Since the map $\Delta\colon G \to G(x)$, defined by $\Delta(g) = \Phi(g, x) = g \cdot x$, is a submersion (see for example [2]) then

$$x \sim y \iff \exists \text{ continuous path } \beta\colon [0,1] \to G \text{ such that } \beta(0) = e \text{ and } \beta(1) \cdot x = y,$$

and by definition of G_0

$$x \sim y \iff \exists \text{ continuous path } \beta\colon [0,1] \to G_0 \text{ such that } \beta(0) = e \text{ and } \beta(1) \cdot x = y.$$

This gives the result. ♣

When G is connected, the tamer group K has richer properties.

Proposition 1.1.2 *Let* G *be a connected Lie subgroup of a compact Lie group* K. *If* G *is dense in* K *then* $G \triangleleft K$ *and the quotient group* K/G *is commutative.*

Proof. The Lie algebra $\mathfrak{g}$ is Ad_G- invariant and hence, by density, Ad_K-invariant. Then $\mathfrak{g}$ is an ideal of $\mathfrak{k}$. The connectedness of G gives that G is a normal subgroup of K. Since Ad_G acts trivially on $\mathfrak{k}/\mathfrak{g}$, Ad_K acts trivially, too. Therefore, $\mathfrak{k}/\mathfrak{g}$ is abelian (see for example [8, pag. 628]). ♣

1.2 Particular tame actions. A *trio* is a triple (K, G, H), with K is a compact Lie group, G a normal subgroup of K and H a closed subgroup of K. We present now some tame actions associated to a trio (K, G, H). They are going to be intensively used in this work. First of all we need some definitions.

- The action $\Phi_l\colon K \times K \to K$ is defined by $\Phi_l(g, k) = g \cdot k$. For each element u of the Lie algebra $\mathfrak{k}$ of K, we shall write X^u the associated (right invariant) vector field. It is defined by $X^u(k) = T_e R_k(u)$ where $R_k\colon K \to K$ is given by $R_k(\ell) = \ell \cdot k$.

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[1]This is always the case when the manifold M is a compact.

- The action $\Phi_r \colon K \times K \to K$ is defined by $\Phi_r(g,k) = k \cdot g^{-1}$. For each element $u \in \mathfrak{k}$ of K, we shall write X_u the associated (left invariant) vector field. It is defined by $X_u(k) = -T_e L_k(u)$ where $L_k \colon K \to K$ is given by $L_k(\ell) = k \cdot \ell$.

- The action $\Psi \colon K \times K/H \to K/H$ is defined by $\Psi(g, kH) = (g \cdot k)H$. For each element $u \in \mathfrak{k}$, we shall write Y_u the associated vector field. Since the canonical projection $\pi \colon K \to K/H$ is a K-equivariant map, then we have $\pi_* X^u = Y_u$ for each $u \in \mathfrak{k}$.

- The action $\Gamma \colon H \times H \to H$ is defined by $\Gamma(g, h) = g \cdot h$. For each element u of the Lie algebra $\mathfrak{h}$ of H we write Z^u the associated (right invariant) vector field.

The associated actions we are going to use are the following.

(a) *The restriction $\Phi_l \colon G \times K \to K$, which induces the regular Killing foliation $\mathcal{K}$.*

(b) *The restriction $\Phi_r \colon G \times K \to K$, which induces the regular Killing foliation $\mathcal{K}$.*

Since $G \lhd K$, the foliation $\mathcal{K}$ is determined by the family of vector fields $\{X^u \, / \, u \in \mathfrak{g}\}$, where $\mathfrak{g}$ is the Lie algebra of G, and also by the family $\{X_u \, / \, u \in \mathfrak{g}\}$. The orbits $G(k) = Gk = kG$ have the same dimension $\dim G$.

(c) *The restriction $\Psi \colon G \times K/H \to K/H$, which induces the regular Killing foliation $\mathcal{D}$.*

The foliation $\mathcal{D}$ is determined by the family of vector fields $\{Y_u \, / \, u \in \mathfrak{g}\}$. The orbits $G(kH)$ have the same dimension $\dim G - \dim(G \cap H)$.

(d) *The restriction $\Gamma \colon (G \cap H) \times H \to H$, which induces the regular Killing foliation $\mathcal{C}$.*

The foliation $\mathcal{C}$ is determined by the family of vector fields $\{Z^u \, / \, u \in \mathfrak{g} \cap \mathfrak{h}\}$. The orbits $(G \cap H)(k)$ have the same dimension $\dim(G \cap H)$.

(e) *The restriction $\Phi_r \colon GH \times K \to K$, which induces the regular Killing foliation $\mathcal{E}$.*

Notice that GH is a Lie group since G is normal in K. The foliation $\mathcal{E}$ is, in fact, determined by the vector fields $\{X_u \, / \, u \in \mathfrak{g} + \mathfrak{h}\}$. The orbits $(GH)(k)$ have the same dimension $\dim G + \dim H - \dim(G \cap H)$.

1.3 Twisted product. In order to prove the finiteness of the basic intersection cohomology we decompose the manifold in a finite number of simpler pieces. These are the twisted products we introduce now.

We fix a trio (K, G, H) and a smooth action $\Theta \colon H \times N \to N$ of H on the manifold N. The *twisted product* is the quotient $K \times_H N$ of $K \times N$ by the equivalence relation $(k, z) \sim (k \cdot h^{-1}, \Theta(h, z) = h \cdot z)$. The element of $K \times_H N$ corresponding to $(k, z) \in K \times N$ is denoted by $< k, z >$. This manifold is endowed with the tame action

$$\Phi \colon G \times (K \times_H N) \longrightarrow (K \times_H N),$$

defined by $\Phi(g, < k, z >) = < g \cdot k, z >$. We denote by $\mathcal{W}$ the induced Killing foliation.

We also use the following tame action, namely, the restriction

$$\Theta \colon (G \cap H) \times N \to N$$

whose induced Killing foliation is denoted by $\mathcal{N}$.

The canonical projection $\Pi \colon K \times N \to K \times_H N$ relates the involved foliations as follows:

(a) $\Pi_*(\mathcal{K} \times \mathcal{I}) = \mathcal{W}$, where $\mathcal{I}$ is the pointwise foliation (since the map Π is G-equivariant).

(b) $\mathsf{S}_{\mathcal{W}} = \{\Pi(K \times S) \, / \, S \in \mathsf{S}_{\mathcal{N}}\} = \Pi(\{K\} \times \mathsf{S}_{\mathcal{N}})$ (since $G_{<k,z>} = k(G \cap H)_z k^{-1}$).

2   Basic Intersection cohomology

In this section we recall the definition of the basic intersection[2] cohomology and we present the main properties we are going to use in this work. For the rest of this section, we fix a conical foliation $\mathcal{F}$ defied on a manifold M. The associated stratification is $\mathsf{S}_{\mathcal{F}}$. The regular stratum of is denoted by $R_{\mathcal{F}}$. We shall write $m = \dim M$, $r = \dim \mathcal{F}$ and $s = m - r = \operatorname{codim}_M \mathcal{F}$.

We are going to deal with differential forms on a product (manifold) $\times [0, 1[^p$, they are restrictions of differential forms defined on (manifold) $\times] - 1, 1[^p$.

2.1   Perverse forms.   Recall that a *conical chart* is a foliated diffeomorphism $\varphi \colon (\mathbb{R}^{m-n-1} \times c\mathbb{S}^n, \mathcal{H} \times c\mathcal{G}) \to (U, \mathcal{F}_U)$ where $(\mathbb{R}^{m-n-1}, \mathcal{H})$ is a simple foliation and $(\mathbb{S}^n, \mathcal{G})$ is a conical foliation without 0-dimensional leaves. We also shall denote this chart by (U, φ, S) where S is the stratum of $\mathsf{S}_{\mathcal{F}}$ verifying $\varphi(\mathbb{R}^{m-n-1} \times \{\vartheta\}) = U \cap S$.

The differential complex $\Pi_{\mathcal{F}}^*(M \times [0, 1[^p)$ of *perverse forms* of $M \times [0, 1[^p$ is introduced by induction on depth $\mathsf{S}_{\mathcal{F}}$. When this depth is 0 then

$$\Pi_{\mathcal{F}}^*(M \times [0, 1[^p) = \Omega^*(M \times [0, 1[^p).$$

Consider now the generic case. A perverse form of $M \times [0, 1[^p$ is first of all a differential form $\omega \in \Omega^*(R_{\mathcal{F}} \times [0, 1[^p)$ such that,

$$\begin{cases} \text{the pull-back} & (\varphi \times \mathbb{I}_{[0,1[^p})^* \omega \in \Omega^*\left(\mathbb{R}^{m-n-1} \times R_{\mathcal{G}} \times]0, 1[\times [0, 1[^p\right) \\[2mm] \text{extends to} & \omega_\varphi \in \Pi_{\mathcal{H} \times c\mathcal{G}}^*\left(\mathbb{R}^{m-n-1} \times \mathbb{S}^n \times [0, 1[^{p+1}\right) \end{cases}$$

for any conical chart (U, φ), where $\mathbb{I}_\bullet$ stands for the identity map. Notice that $\Omega^*(M)$ is included on $\Pi_{\mathcal{F}}^*(M)$[3].

2.2   Perverse degree.   The amount of transversality of a perverse form $\omega \in \Pi_{\mathcal{F}}^*(M)$ with respect to a singular stratum $S \in \mathsf{S}_{\mathcal{F}}$ is measured by the perverse degree $\|\omega\|_S$. We recall here the definition of local perverse degree $\|\omega\|_U \in \{-\infty\} \cup \mathbb{N}$ of ω relatively to a conical chart (U, φ, S):

1. $\|\omega\|_U = -\infty$ when $\omega_\varphi \equiv 0$ on $\mathbb{R}^{m-n-1} \times R_{\mathcal{G}} \times \{0\}$,

2. $\|\omega\|_U \le p$, with $p \in \mathbb{N}$, when $\omega_\varphi(v_0, \dots, v_p, -) \equiv 0$ where the vectors $\{v_0, \dots, v_p\}$ are tangent to the fibers of $P_\varphi \colon \mathbb{R}^{m-n-1} \times R_{\mathcal{G}} \times \{0\} \longrightarrow U \cap S$[4].

This number does not depend on the choice of the conical chart (cf. [11, Proposition 1.3.1]). Finally, we define the *perverse degree* $\|\omega\|_S$ by

$$\|\omega\|_S = \sup\left\{\|\omega\|_U \,/\, (U, \varphi, S) \text{ conical chart}\right\}.$$

The perverse degree of $\omega \in \Omega^*(M)$ verifies $\|\omega\|_S \le 0$ for any singular stratum $S \in \mathsf{S}_{\mathcal{F}}$ (cf. 2.1).

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[2] We refer the reader to [10],[11] for details.
[3] Through the restriction $\omega \mapsto \omega_{R_{\mathcal{F}}}$.
[4] The map $P_\varphi \colon \mathbb{R}^{m-n-1} \times \mathbb{S}^n \times [0, 1[\longrightarrow U$ is defined by $P_\varphi(x, y, t) = \varphi(x, [y, t])$.

2.3 Basic cohomology. The basic cohomology of the foliation $\mathcal{F}$ is an important tool to study its transversal structure and plays the rôle of the cohomology of the orbit space $M/\mathcal{F}$, which can be a wild topological espace. A differential form $\omega \in \Omega^*(M)$ is *basic* if $i_X\omega = i_X d\omega = 0$, for each vector field X on M tangent to the foliation $\mathcal{F}$. For exemple, a function f is basic iff f is constant on the leaves of $\mathcal{F}$. We shall write $\Omega^*(M/\mathcal{F})$ for the complex of basic forms. Its cohomology $H^*(M/\mathcal{F})$ is the *basic cohomology* of $(M, \mathcal{F})$. We also use the *relative basic cohomology* $H^*((M, F)/\mathcal{F})$, that is, the cohomology computed from the complex of basic forms vanishing on the saturated set $F \subset M$. The basic cohomology does not use the stratification $S_{\mathcal{F}}$.

2.4 Basic intersection cohomology. A *perversity* is a map $\overline{p}\colon S_{\mathcal{F}}^{\sigma} \to \mathbb{Z} \cup \{-\infty, \infty\}$, where $S_{\mathcal{F}}^{\sigma}$ is the family of singular strata. The *constant perversity* $\overline{\iota}$ is defined by $\overline{\iota}(S) = \iota$, where $\iota \in \mathbb{Z} \cup \{-\infty, \infty\}$.

The basic intersection cohomology appears when one considers basic perverse forms whose perverse degree is controlled by a perversity. We shall put

$$\Omega^*_{\overline{p}}(M/\mathcal{F}) = \left\{ \omega \in \Pi^*_{\mathcal{F}}(M) \ / \ \omega \text{ is basic and } \max\left(\|\omega\|_s, \|d\omega\|_s\right) \le \overline{p}(S) \ \forall S \in S_{\mathcal{F}}^{\sigma} \right\}$$

the complex of basic perverse forms whose perverse degree (and that of the their derivative) is bounded by the perversity $\overline{p}$. The cohomology $H^*_{\overline{p}}(M/\mathcal{F})$ of this complex is the *basic intersection cohomology*[5] of $(M, \mathcal{F})$ relatively to the perversity $\overline{p}$.

Consider a twisted product $K\times_H N$. Perversities on $K\times_H N$ and $K \times N$ are determinate by perversities on N by the formula (cf. 1.3(b)):

$$(1) \qquad\qquad \overline{p}(K \times S) = \overline{p}(\Pi(K \times S)) = \overline{p}(S).$$

2.5 Mayer-Vietoris. This is the technique we use in order to decompose the manifold in nicer pieces. An open covering $\{U, V\}$ of M by saturated open subsets is a *basic covering*. It possesses a subordinated partition of the unity made up of basic functions defined on M (see [9]). For a such covering we have the Mayer-Vietoris short sequence

$$0 \to \Omega^*_{\overline{p}}(M/\mathcal{F}) \to \Omega^*_{\overline{p}}(U/\mathcal{F}) \oplus \Omega^*_{\overline{p}}(V/\mathcal{F}) \to \Omega^*_{\overline{p}}((U \cap V)/\mathcal{F}) \to 0,$$

where the map are defined by $\omega \mapsto (\omega, \omega)$ and $(\alpha, \beta) \mapsto \alpha - \beta$. The third map is onto since the elements of the partition of the unity are *controlled functions*, id est, elements of $\Omega^0_0(-)$ (cf. 2.2). Thus, the sequence is exact. This result is not longer true for more general coverings.

We shall use in this work the two following local calculations (see [11, Proposition 3.5.1 and Proposition 3.5.2] for the proofs).

Proposition 2.6 *Let* $(\mathbb{R}^k, \mathcal{H})$ *be a simple foliation. Consider* $\overline{p}$ *a perversity on M and define the perversity* $\overline{p}$ *on* $\mathbb{R}^k \times M$ *by* $\overline{p}(\mathbb{R}^k \times S) = \overline{p}(S)$. *The canonical projection* $\mathrm{pr} : \mathbb{R}^k \times M \to M$ *induces the isomorphism*

$$H^*_{\overline{p}}(M/\mathcal{F}) \cong H^*_{\overline{p}}\left(\mathbb{R}^k \times M/\mathcal{H} \times \mathcal{F}\right).$$

Proposition 2.7 *Let* $\mathcal{G}$ *be a conical foliation without 0-dimensional leaves on the sphere* $\mathbb{S}^n$. *A perversity* $\overline{p}$ *on* $c\mathbb{S}^n$ *gives the perversity* $\overline{p}$ *on* $\mathbb{S}^n$ *defined by* $\overline{p}(S) = \overline{p}(S\times]0, 1[)$. *The canonical projection* $\mathrm{pr} : \mathbb{S}^n\times]0, 1[\to \mathbb{S}^n$ *induces the isomorphism*

$$H^i_{\overline{p}}(c\mathbb{S}^n/c\mathcal{G}) = \begin{cases} H^i_{\overline{p}}(\mathbb{S}^n/\mathcal{G}) & \text{if } i \le \overline{p}(\{\vartheta\}) \\ 0 & \text{if } i > \overline{p}(\{\vartheta\}). \end{cases}$$

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[5]BIC for short.

In the next section we shall need the following technical Lemma.

Lemma 2.8 *Let* $\Phi\colon K \times M \to M$ *be a smooth action, where K is a compact Lie group, and let V be a fundamental vector field of this action. Consider a normal subgroup G of K and write $\mathcal{F}$ the associated conical foliation on M. Then, the interior operator $i_V\colon \Omega^*_{\overline{p}}(M/\mathcal{F}) \longrightarrow \Omega^{*-1}_{\overline{p}}(M/\mathcal{F})$ is well defined, for any perversity $\overline{p}$.*

Proof. Since the question is a local one, then it suffices to consider where M is a twisted product $K\times_H N$[6]. Notice that the blow up $\Pi\colon K \times N \to K\times_H N$ is a K-equivariant map relatively to the action $\ell \cdot (k, z) = (\ell \cdot k, z)$. This gives $\Pi_*(X^u, 0) = V$ for some $u \in \mathfrak{k}$. From Lemma 3.1 we know that it suffices to prove that the operator

$$i_{(X^u,0)}\colon \Omega^*_{\overline{p}}(K \times N/\mathcal{K} \times \mathcal{N}) \longrightarrow \Omega^{*-1}_{\overline{p}}(K \times N/\mathcal{K} \times \mathcal{N})$$

is well defined. Since $G \triangleleft K$ then the vector field X^u preserves the foliation $\mathcal{K}$. So, it suffices to prove that the operator

$$i_{(X^u,0)}\colon \Omega^*_{\overline{p}}(K \times N) \longrightarrow \Omega^{*-1}_{\overline{p}}(K \times N)$$

is well defined. This comes from the fact that X^u acts on the K-factor while the perversion conditions are measured on the N-factor (cf. (1)). ♣

3 The BIC of a twisted product

We compute now the BIC of a twisted product $K\times_H N$ (cf. 1.3) for a perversity $\overline{p}$ (cf. (1)).

Lemma 3.1 *The natural projection $\Pi\colon K \times N \to K\times_H N$ induces the differential monomorphism*

(2) $$\Pi^*\colon \Omega^*_{\overline{p}}(K\times_H N/\mathcal{W}) \longrightarrow \Omega^*_{\overline{p}}(K \times N/\mathcal{K} \times \mathcal{N}).$$

Moreover, given a differential form ω on $K\times_H R_{\mathcal{W}}$, we have:

(3) $$\Pi^*\omega \in \Omega^*_{\overline{p}}(K \times N/\mathcal{K} \times \mathcal{N}) \Longleftrightarrow \omega \in \Omega^*_{\overline{p}}(K\times_H N/\mathcal{W}).$$

Proof. Notice that the injectivity of Π^* comes from the fact that Π is a surjection. For the rest, we proceed in several steps.

(a) *A foliated atlas for $\pi\colon K \to K/H$.*

Since $\pi\colon K \to K/H$ is a H-principal bundle then it possesses an atlas $\mathcal{A} = \left\{\varphi\colon \pi^{-1}(U) \longrightarrow U \times H\right\}$ made up with H-equivariant charts: $\varphi(k \cdot h^{-1}) = (\pi(k), h \cdot h_0)$ if $\varphi(k) = (\pi(k), h_0)$. We study the foliation $\varphi_*\mathcal{K}$. This equivariance property gives $\varphi_*X_u = (0, Z^u)$ for each $u \in \mathfrak{g} \cap \mathfrak{h}$. Thus, the trace of the foliation $\varphi_*\mathcal{K}$ on the fibers of the canonical projection $\mathrm{pr}\colon U \times H \to U$ is C. On the other hand, since the map π is a G-equivariant map then $\pi_*\mathcal{K} = \mathcal{D}$, which gives $\mathrm{pr}_*\varphi_*\mathcal{K} = \mathcal{D}$. We conclude that $\varphi_*\mathcal{K} \subset \mathcal{D} \times C$. By dimension reasons we get $\varphi_*\mathcal{K} = \mathcal{D} \times C$. The atlas $\mathcal{A}$ is an H-equivariant foliated atlas of π.

(b) *A foliated atlas for $\Pi\colon K \times N \to K\times_H N$.*

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[6]In fact, N is an euclidean space $\mathbb{R}^a$ et Θ is an orthogonal action.

We claim that $\mathcal{A}_{\#} = \left\{ \overline{\varphi} \colon \pi^{-1}(U) \times_H N \longrightarrow U \times N \ / \ (U, \varphi) \in \mathcal{A} \right\}$ is a foliated atlas of $K \times_H N$ where the map $\overline{\varphi}$ is defined by $\overline{\varphi}(<k, z>) = (\pi(k), (\Theta((\varphi^{-1}(\pi(k), e))^{-1} \cdot k, z)))$. This map is a diffeomorphism whose inverse is $\overline{\varphi}^{-1}(u, z) = <\varphi^{-1}(u, e), z>$. It verifies

$$\overline{\varphi}_* \mathcal{W} \overset{1.3(a)}{=\!=\!=} \overline{\varphi}_* \Pi_*(\mathcal{K} \times \mathcal{I}) = \overline{\varphi}_* \Pi_*(\varphi^{-1} \times \mathbb{I}_N)_*(\mathcal{D} \times C \times \mathcal{I}).$$

A straightforward calculation shows $\overline{\varphi} \circ \Pi \circ (\varphi^{-1} \times \mathbb{I}_N) = (\mathbb{I}_U \times \Theta)$. Since C is defined by the action Γ then $\Theta_*(C \times \mathcal{I}) = \mathcal{N}$. Finally we obtain $\overline{\varphi}_* \mathcal{W} = \mathcal{D} \times \mathcal{N}$.

(c) *Last Step*.
Given $\overline{(U, \varphi)} \in \mathcal{A}_{\#}$, we have the commutative diagram

$$
\begin{array}{ccc}
U \times H \times N & \xrightarrow{\;\;\varphi^{-1} \times \mathbb{I}_N\;\;} & K \times N \\
{\scriptstyle Q} \downarrow & & \downarrow {\scriptstyle \Pi} \\
U \times N & \xrightarrow[\;\;\overline{\varphi}^{-1}\;\;]{} & K \times_H N
\end{array}
$$

where $Q(u, h, z) = (u, h^{-1} \cdot z)$, $\Pi^{-1}(\text{Im } \overline{\varphi}^{-1}) = \text{Im}\left(\varphi^{-1} \times \mathbb{I}_N\right)$ and the rows are foliated imbeddings. Now, since (2) and (3) are local questions then it suffices to prove that

- $Q^* \colon \Omega_{\overline{p}}^*(U \times N / \mathcal{D} \times \mathcal{N}) \longrightarrow \Omega_{\overline{p}}^*(U \times H \times N / \mathcal{D} \times C \times \mathcal{N})$ is well-defined, and

- $Q^* \omega \in \Omega_{\overline{p}}^*(U \times H \times N / \mathcal{D} \times C \times \mathcal{N}) \Longleftrightarrow \omega \in \Omega_{\overline{p}}^*(U \times N / \mathcal{D} \times \mathcal{N})$, for any $\omega \in \Omega^*(U \times R_N)$.

This comes from the fact that the map

$$\nabla \colon (U \times H \times N, \mathcal{D} \times C \times \mathcal{N}) \longrightarrow (U \times H \times N, \mathcal{D} \times C \times \mathcal{N}),$$

defined by $\nabla(u, h, z) = (u, h, h^{-1} \cdot z))$, is a foliated diffeomorphism[7] and $Q = \text{pr}_0 \circ \nabla$, with $\text{pr}_0 \colon U \times H \times N \to U \times N$ canonical projection (cf. Proposition 2.6).   ♣

3.2 The Lie algebra $\mathfrak{k}$.

We suppose in this paragraph that that G is also dense on K. Choose ν a bi-invariant riemannian metric on K, which exists by compactness. Consider

$$B = \left\{ u_1, \ldots u_a, u_{a+1}, \ldots, u_b, u_{b+1}, \ldots, u_c, u_{c+1}, \ldots, u_f \right\}$$

an orthonormal basis of the Lie algebra $\mathfrak{k}$ of K with $\{u_1, \ldots u_b\}$ basis of the Lie algebra $\mathfrak{g}$ of G and $\{u_{a+1}, \ldots u_c\}$ basis of the Lie algebra $\mathfrak{h}$ of H. For each indice $1 \leq i \leq f$ we shall write $X_i \equiv X_{u_i}$ and $X^i \equiv X^{u_i}$ (cf. 1.2).

Let $\gamma_i \in \Omega^1(K)$ be the dual form of X_i, that is, $\gamma_i = i_{X_i} \nu$. Notice that $\delta_{ij} = \gamma_j(X_i)$. These forms are invariant by the left action of K. Since the flow of X^j is the multiplication on the left by $\exp(t u_j)$ then $L_{X^j} \gamma_i = 0$ for each $1 \leq j \leq f$.

For the differential, we have the formula $d\gamma_l = \displaystyle\sum_{1 \leq i < j \leq f} C_{ij}^l \gamma_i \wedge \gamma_j$, where $[X_i, X_j] = \displaystyle\sum_{l=1}^{f} C_{ij}^l X_l$, and $1 \leq i, j, l \leq f$. We have several restrictions on these coefficients. Since $G \triangleleft K$ then $\mathfrak{g}$ is an ideal of $\mathfrak{k}$ and therefore we have

$$C_{ij}^l = 0 \text{ for } i \leq b < l.$$

---

[7]Since $G \cap H \triangleleft H$.

Since K/G is an abelian group (cf. Proposition 1.1.2) then the induced bracket on $\mathfrak{k}/\mathfrak{g}$ is zero and therefore we have

$$C^l_{ij} = 0 \text{ for } b < i, j, l \le f.$$

These equations imply that

$$(4) \qquad\qquad d\gamma_l = 0 \text{ for each } b < l.$$

The $\mathcal{E}$-basic differential forms in $\bigwedge^*(\gamma_1, \ldots, \gamma_f)$ are exactly $\bigwedge^*(\gamma_{c+1}, \ldots, \gamma_f)$ since they are cycles and the family $\{X_1, \ldots, X_c\}$ generates the foliation $\mathcal{E}$. This gives

$$(5) \qquad\qquad H^*\left(K/\mathcal{E}\right) = \bigwedge{}^*(\gamma_{c+1}, \ldots, \gamma_f).$$

3.3 Two actions of H/H_0. The Lie group H preserves the foliation $\mathcal{N}$ since the Lie group $G \cap H$ is a normal subgroup of H. Put H_0 the connected component of H containing the unity element. Since it is a connected compact Lie group then a standard argument shows that

$$(6) \qquad\qquad \left(H^*_{\overline{p}}(N/\mathcal{N})\right)^{H_0} = H^*\left(\left(\Omega^{\cdot}_{\overline{p}}(N/\mathcal{N})\right)^{H_0}\right) = H^*_{\overline{p}}(N/\mathcal{N})$$

(cf. [5, Theorem I, Ch. IV, vol. II]). We conclude that the finite group H/H_0 acts naturally on $H^*_{\overline{p}}(N/\mathcal{N})$.

Since H_0 is a connected Lie subgroup of GH then $\left(H^*(K/\mathcal{E})\right)^{H_0} = H^*(K/\mathcal{E})$. We conclude that the finite group H/H_0 acts naturally on $H^*(K/\mathcal{E})$.

Proposition 3.4 *Let (K, G, H) be a trio with G connected and dense in K. Then*

$$H^*_{\overline{p}}(K\times_H N/\mathcal{W}) = \left(H^*(K/\mathcal{E}) \otimes H^*_{\overline{p}}(N/\mathcal{N})\right)^{H/H_0}.$$

Proof. Using the blow up $\Pi \colon K \times N \to K\times_H N$, the computation of $H^*_{\overline{p}}(K\times_H N/\mathcal{W})$ can be done by using the complex $\operatorname{Im}\left\{\Pi^*\colon \Omega^*_{\overline{p}}(K\times_H N/\mathcal{F}) \longrightarrow \Omega^*_{\overline{p}}(K \times N/\mathcal{K} \times \mathcal{N})\right\}$ (cf. Lemma 3.1). We study this complex in several steps. We fix $B = \left\{u_1, \ldots, u_f\right\}$ an orthonormal basis of $\mathfrak{k}$ as in 3.2.

$\langle i \rangle$ Description of $\Omega^*(K \times R_{\mathcal{N}})$.

A differential form $\omega \in \Omega^*(K \times R_{\mathcal{N}})$ is of the form

$$(7) \qquad\qquad \eta + \sum_{1 \le i_1 < \cdots < i_\ell \le f} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_\ell} \wedge \eta_{i_1, \ldots, i_\ell},$$

where the forms $\eta, \eta_{i_1, \ldots, i_\ell} \in \Omega^*(K \times R_{\mathcal{N}})$ verify $i_{X_j}\eta = i_{X_j}\eta_{i_1, \ldots, i_\ell} = 0$ for each $1 \le j \le f$ and each $1 \le i_1 < \cdots < i_\ell \le f$.

$\langle ii \rangle$ Description of $\Pi^*_{\mathcal{K}\times\mathcal{N}}(K \times N)$.

Since the foliation $\mathcal{K}$ is regular then we always can choose a conical chart of the form $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ where (U_1, φ_1) is a foliated chart of $(K, \mathcal{K})$ and (U_2, φ_2) is a conical chart of $(N, \mathcal{N})$. The local blow up of the chart $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ is constructed from the second factor without modifying the first one. So, the differential forms γ_i are always perverse forms and a differential form $\omega \in \Pi^*_{\mathcal{K}\times\mathcal{N}}(K \times N)$ is of the form (7) where $\eta, \eta_{i_1, \ldots, i_\ell} \in \Pi^*_{\mathcal{K}\times\mathcal{N}}(K \times N)$ verify $i_{X_j}\eta = i_{X_j}\eta_{i_1, \ldots, i_\ell} = 0$ for each $1 \le j \le f$ and each $1 \le i_1 < \cdots < i_\ell \le f$.

$\langle iii \rangle$ Description of $\Omega^*(K \times R_N / \mathcal{K} \times \mathcal{N})$.

Take $\omega \in \Omega^*(K \times R_N / \mathcal{K} \times \mathcal{N})$. Since $\mathcal{K}$ is generated by the family $\{X_j / 1 \leq j \leq b\}$ then $L_{X_j}\omega = 0$ for any $1 \leq j \leq b$, or equivalently, $R_g^*\omega = \omega$ for each $g \in G$ since G is connected. By density, $R_k^*\omega = \omega$ for each $k \in K$ and therefore $L_{X_j}\omega = 0$ for any $1 \leq j \leq f$ since K is connected. We conclude that $L_{X_j}\eta = L_{X_j}\eta_{i_1,\dots,i_\ell} = 0$ for any $1 \leq j \leq f$ and each $1 \leq i_1 < \cdots < i_\ell \leq f$. This gives $\omega \in \bigwedge^*(\gamma_1, \dots, \gamma_f) \otimes \Omega^*(R_N)$.

The $\mathcal{N}$-basic differential forms of $\Omega^*(R_N)$ are exactly $\Omega^*(R_N/\mathcal{N})$. The $\mathcal{K}$-basic differential forms of $\bigwedge^*(\gamma_1, \dots, \gamma_f)$ are exactly $\bigwedge^*(\gamma_{b+1}, \dots, \gamma_f)$ (cf. (4)). From these two facts, we get

$$\Omega^*(K \times R_N / \mathcal{K} \times \mathcal{N}) = \bigwedge{}^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega^*(R_N/\mathcal{N})$$

as differential graduate commutative algebras.

$\langle iv \rangle$ Description of $\Omega_{\overline{p}}^*(K \times N / \mathcal{K} \times \mathcal{N})$.

From $\langle ii \rangle$ and $\langle iii \rangle$ it suffices to control the perverse degree of the forms

$$\eta + \sum_{b+1 \leq i_1 < \cdots < i_\ell \leq f} \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_\ell} \wedge \eta_{i_1,\dots,i_\ell} \in \bigwedge{}^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Pi_N^*(N).$$

Consider S a stratum of S_N. From $\|\gamma_i\|_{K \times S} = 0$ and $\|\eta\|_{K \times S} = \|\eta\|_S$, we get $\|\gamma_{i_1} \wedge \dots \gamma_{i_\ell} \wedge \eta_{i_1,\dots,i_\ell}\|_{K \times S} = \|\eta_{i_1,\dots,i_\ell}\|_S$. We conclude that

$$\Omega_{\overline{p}}^*(K \times N / \mathcal{K} \times \mathcal{N}) \cong \bigwedge{}^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N})$$

(cf. 1.3(b)).

$\langle v \rangle$ Description of $\mathrm{Im}\left\{\Pi^* : \Omega_{\overline{p}}^*(K \times_H N / \mathcal{F}) \longrightarrow \Omega_{\overline{p}}^*(K \times N / \mathcal{K} \times \mathcal{N})\right\}$.

We denote by $\{W_{a+1}, \dots, W_c\}$ the fundamental vector fields of the action $\Theta : H \times N \to N$ associated to the basis $\{u_{a+1}, \dots, u_c\}$. Consider now the action $\Upsilon : H \times (K \times N) \to (K \times N)$ defined by $\Upsilon(h, (k, z)) = (k \cdot h^{-1}, \Theta(h, z))$. Its fundamental vector fields associated to the basis $\{u_{a+1}, \dots, u_c\}$ are $\{(X_{a+1}, W_{a+1}), \dots, (X_c, W_c)\}$. Given $h \in H$, we take $\Upsilon_h : K \times N \to K \times N$ the map defined by $\Upsilon_h(k, z) = \Upsilon(h, (k, z))$. Then, we have

$$\mathrm{Im}\,\Pi^* = \left\{ \omega \in \bigwedge{}^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N}) \,\middle|\, \begin{array}{l} \text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } a < i \leq c\} \\ \text{(ii) } L_{X_i}\omega = -L_{W_i}\omega \text{ if } a < i \leq c\}, \\ \text{(iii) } (\Upsilon_h)^*\omega = \omega \text{ for } h \in H. \end{array} \right\}.$$

Let H_0 be the unity connected component of H. Recall that the subgroup H_0 is normal in H and that the quotient H/H_0 is a finite group. Conditions (ii) gives that ω is H_0-invariant. So, condition (iii) can be replaced by: (iv) $(\Upsilon_h)^*\omega = \omega$ for $h \in H/H_0$. Therefore

$$\mathrm{Im}\,\Pi^* = \left\{ \omega \in \bigwedge{}^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N}) \,\middle|\, \begin{array}{l} \text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } a < i \leq c\} \\ \text{(ii) } L_{X_i}\omega = -L_{W_i}\omega \text{ if } a < i \leq c\}. \end{array} \right\}^{H/H_0}.$$

Since the group H/H_0 is a finite one, we get that the cohomology $H^*(\mathrm{Im}\,\Pi^*)$ is isomorphic to $\left(H^*(A^{\cdot})\right)^{H/H_0}$, where A^* is the differential complex

$$\left\{ \omega \in \bigwedge{}^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N}) \,\middle|\, \begin{array}{l} \text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } a < i \leq c\} \\ \text{(ii) } L_{X_i}\omega = -L_{W_i}\omega \text{ if } a < i \leq c\} \end{array} \right\}.$$

So, it remains to compute $H^*(A^{\cdot})$. This computation can be simplified by using these three facts:

- $i_{W_i}\omega = L_{W_i}\omega = 0$ for each $a < i \le b$, since the foliation $\mathcal{N}$ is defined by the action of $G \cap H$.

- $i_{X_i}\gamma_j = \delta_{ij}$ for all i, j (cf. 3.2).

- $d\gamma_j = 0$ for $b < j$ (cf. (4)).

We get that A^* is the differential complex

$$\left\{ \omega \in \bigwedge{}^*(\gamma_{b+1}, \ldots, \gamma_f) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N}) \;\middle|\; \begin{array}{l} \text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } b < i \le c\} \\ \text{(ii) } 0 = L_{W_i}\omega \text{ if } b < i \le c\} \end{array} \right\} =$$

$$\bigwedge{}^*(\gamma_{c+1}, \ldots, \gamma_f) \otimes \underbrace{\left\{ \omega \in \bigwedge{}^*(\gamma_{b+1}, \ldots, \gamma_c) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N}) \;\middle|\; \begin{array}{l} \text{(i) } i_{X_i}\omega = -i_{W_i}\omega \text{ if } b < i \le c\} \\ \text{(ii) } 0 = L_{W_i}\omega \text{ if } b < i \le c\} \end{array} \right\}}_{B^*}.$$

A straightforward computation gives that the canonical writing of a form $\omega \in \bigwedge^*(\gamma_{b+1}, \ldots, \gamma_c) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N})$ verifying (i) is

(8)
$$\omega = \omega_0 + \sum_{b < i_1 < \cdots < i_\ell \le c} (-1)^\ell \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_\ell} \wedge (i_{W_{i_\ell}} \cdots i_{W_{i_1}} \omega_0)$$

for some $\omega_0 \in \Omega_{\overline{p}}^*(N/\mathcal{N})$ (cf. Lemma 2.8).

Consider now $b < i, j \le c$. Since K/G is an abelian group (cf. Proposition 1.1.2) and H is a Lie group then $[W_i, W_j] = \sum_{l=a+1}^{b} C_{ij}^l W_l$. Then, $i_{[W_i,W_j]}\omega_0 = 0$ since the foliation $\mathcal{N}$ is defined by the action of $G \cap H$. So, the canonical writing of a form $\omega \in B^*$ is (8) for some $\omega_0 \in \left\{ \eta \in \Omega_{\overline{p}}^*(N/\mathcal{N}) \;/\; L_{W_i}\eta = 0 \text{ if } b < i \le c \right\} = \left(\Omega_{\overline{p}}^*(N/\mathcal{N}) \right)^{H_0}$.

Then, the operator $\Delta \colon B^* \longrightarrow \left(\Omega_{\overline{p}}^*(N/\mathcal{N}) \right)^{H_0}$, defined by $\Delta(\omega) = \omega_0$, is a differential isomorphism. We conclude that the differential complex A^* is isomorphic to $\bigwedge^*(\gamma_{c+1}, \ldots, \gamma_f) \otimes \left(\Omega_{\overline{p}}^*(N/\mathcal{N}) \right)^{H_0}$ and therefore $H^*(A^{\cdot}) \cong H^*(K/\mathcal{E}) \otimes H_{\overline{p}}^*(N/\mathcal{N})$ (cf. (5) and (6)). Since the operator Δ is (H/H_0)-equivariant (cf. 3.3) then we get

$$H_{\overline{p}}^*(K \times_H N/\mathcal{W}) \;=\; H^*(\text{Im } \Pi^*) = \left(H^*(A^{\cdot}) \right)^{H/H_0} = \left(H^*(K/\mathcal{E}) \otimes H_{\overline{p}}^*(N/\mathcal{N}) \right)^{H/H_0}.$$

This ends the proof. ♣

3.5 Remarks.

(a) When the Lie group G is commutative then K is also commutative. Differential forms $\gamma_\bullet$ are K-invariants on the left and on the right, so $\left(H^*(K/\mathcal{E}) \right)^H = H^*(K/\mathcal{E})$ and therefore

$$H_{\overline{p}}^*(K \times_H N/\mathcal{W}) = H^*(K/\mathcal{E}) \otimes \left(H_{\overline{p}}^*(N/\mathcal{N}) \right)^{H/H_0} = H^*(K/\mathcal{E}) \otimes \left(H_{\overline{p}}^*(N/\mathcal{N}) \right)^H$$

as it has been proved in [11, Proposition 3.8.4].

(b) Since the foliation $\mathcal{E}$ is a riemannian foliation defined on a compact manifold then we know that the cohomology $H^*(K/\mathcal{E})$ is finite (cf. [4]). So, the finiteness of $H_{\overline{p}}^*(K \times_H N/\mathcal{W})$ depends on the finiteness of $H_{\overline{p}}^*(N/\mathcal{N})$.

4 Finiteness of the BIC

We prove in this section that the BIC of a Killing foliation on a compact manifold is finite dimensional. First of all, we present two geometrical tools we shall use in the proof: the isotropy type stratification and the Molino's blow up.

We fix an isometric action $\Phi\colon G \times M \to M$ on the compact manifold M. We denote by $\mathcal{F}$ the induced Killing foliation. For the study of $\mathcal{F}$ we can suppose that G is connected (see Lemma 1.1.1). We fix K a tamer group. Notice that the group G is normal in K and the quotient K/G is commutative (cf. Proposition 1.1.2).

4.1 Isotropy type stratification. The *isotropy type stratification* $\mathsf{S}_{K,M}$ of M is defined by the equivalence relation[8]:

$$x \sim y \Leftrightarrow K_x \text{ is conjugated to } K_y.$$

When depth $\mathsf{S}_{K,M} > 0$, any closed stratum $S \in \mathsf{S}_{K,M}$ is a K-invariant submanifold of M and then it possesses a K-invariant tubular neighborhood $(T, \tau, S, \mathbb{R}^m)$ whose structural group is $O(m)$. Recall that there are the following smooth maps associated with this neighborhood:

+ The *radius map* $\rho\colon T \to [0,1[$ defined fiberwise from the assignation $[x,t] \mapsto t$. Each $t \neq 0$ is a regular value of the ρ. The pre-image $\rho^{-1}(0)$ is S. This map is K-invariant, that is, $\rho(k \cdot z) = \rho(z)$.

+ The *contraction* $H\colon T \times [0,1] \to T$ defined fiberwisely from $([x,t],r) \mapsto [x,rt]$. The restriction $H_t\colon T \to T$ is an embedding for each $t \neq 0$ and $H_0 \equiv \tau$. We shall write $H(z,t) = t \cdot z$. This map is K-invariant, that is, $t \cdot (k \cdot z) = k \cdot (t \cdot z)$.

The hyper-surface $D = \rho^{-1}(1/2)$ is the *tube* of the tubular neighborhood. It is a K-invariant submanifold of T. Notice that the map

$$\nabla\colon D \times [0,1[\longrightarrow T,$$

defined by $\nabla(z,t) = (2t) \cdot z$ is a K-equivariant smooth map, where K acts trivially on the $[0,1[$-factor. Its restriction $\nabla\colon D \times]0,1[\longrightarrow T \backslash S$ is a K-equivariant diffeomorphism.

Denote S_{min} the union of closed (minimal) strata and choose T_{min} a disjoint family of K-invariant tubular neighborhoods of the closed strata. The union of associated tubes is denoted by D_{min}. Notice that the induced map $\nabla_{min}\colon D_{min} \times]0,1[\longrightarrow T_{min} \backslash S_{min}$ is a K-equivariant diffeomorphism.

4.2 Molino's blow up. The Molino' blow up [7] of the foliation $\mathcal{F}$ produces a new foliation $\widehat{\mathcal{F}}$ of the same kind but of smaller depth. We suppose depth $\mathsf{S}_{K,M} > 0$. The *blow up* of M is the compact manifold

$$\widehat{M} = \left\{ \left(D_{min} \times]-1,1[\right) \bigsqcup \left((M \backslash S_{min}) \times \{-1,1\} \right) \right\} \Big/ \sim,$$

where $(z,t) \sim (\nabla_{min}(z,|t|), t/|t|)$, and the map $\mathcal{L}\colon \widehat{M} \longrightarrow M$ defined by

$$\mathcal{L}(v) = \begin{cases} \nabla_{min}(z,|t|) & \text{if } v = (z,t) \in D_{min} \times]-1,1[\\ z & \text{if } v = (z,j) \in (M \backslash S_{min}) \times \{-1,1\}. \end{cases}$$

Notice that $\mathcal{L}$ is a continuous map whose restriction $\mathcal{L}\colon \widehat{M} \backslash \mathcal{L}^{-1}(S_{min}) \to M \backslash S_{min}$ is a K-equivariant smooth trivial 2-covering.

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[8]For notions related with compact Lie group actions, we refer the reader to [1].

Since the map ∇_{min} is K-equivariant then Φ induces the action $\widehat{\Phi}: K \times \widehat{M} \longrightarrow \widehat{M}$ by saying that the blow-up $\mathcal{L}$ is K-equivariant. The open submanifolds $\mathcal{L}^{-1}(T_{min})$ and $\mathcal{L}^{-1}(T_{min} \setminus S_{min})$ are clearly K-diffeomorphic to $D_{min} \times]-1, 1[$ and $D_{min} \times (]-1, 0[\cup]0, 1[)$ respectively.

The restriction $\widehat{\Phi}: G \times \widehat{M} \longrightarrow \widehat{M}$ is an isometric action with K as a tamer group. The induced Killing foliation is $\widehat{\mathcal{F}}$. Foliations $\mathcal{F}$ and $\widehat{\mathcal{F}}$ are related by $\mathcal{L}$ which is a foliated map. Moreover, if S is a not minimal stratum of $\mathsf{S}_{K,M}$ then there exists an unique stratum $S' \in \mathsf{S}_{K,\widehat{M}}$ such that $\mathcal{L}^{-1}(S) \subset S'$. The family $\{S' \mid S \in \mathsf{S}_{K,M}\}$ covers $\widehat{M}$ and verifies the relationship: $S_1 \prec S_2 \Leftrightarrow S'_1 \prec S'_2$. We conclude the important property

(9) $$\text{depth } \mathsf{S}_{K,\widehat{M}} < \text{depth } \mathsf{S}_{K,M}.$$

4.3 Finiteness of a tubular neighborhood.

We suppose depth $\mathsf{S}_{K,M} > 0$. Consider a closed stratum $S \in \mathsf{S}_{K,M}$. Take $(T, \tau, S, \mathbb{R}^m)$ a K-invariant tubular neighborhood. We fix a base point $x \in S$. The isotropy subgroup K_x acts orthogonally on the fiber $\mathbb{R}^m = \tau^{-1}(x)$. So, the induced action $\Lambda_x: G_x \times \mathbb{R}^m \to \mathbb{R}^m$ is an isometric action, it gives the Killing foliation $\mathcal{N}$ on $\mathbb{R}^m$.

Proposition 4.3.1 *If the BIC of $(\mathbb{R}^m, \mathcal{N})$ is finite dimensional then the BIC of $(T, \mathcal{F})$ is also finite dimensional.*

Proof. We proceed in two steps.

(a) <u>$K_y = K_x$ for each $y \in S$.</u>

The canonical projection $\pi: S \to S/K$ is an homogeneous bundle with fiber K/K_x. For any open subset $V \subset S/K$ the pull back $\tau^{-1}\pi^{-1}(V)$ is a K-invariant subset of T, then we can apply the Mayer-Vietoris technics to this kind of subsets (cf. 2.5).

Since the manifold S/K is a compact one then we can find a finite good covering $\{U_i \mid i \in I\}$ of it (cf. [2]). An inductive argument on the cardinality of I reduces the proof of the Lemma to the case where $T = \tau^{-1}\pi^{-1}(V)$, where V is a contractible open subset of S/K.

Here, the manifold T is K-equivalently diffeomorphic to $V \times \left(K \times_{K_x} \mathbb{R}^m\right)$, where K does not act on the first factor. So, the natural retraction of V to a point gives a K-equivariant retraction of T to the twisted product $K \times_{K_x} \mathbb{R}^m$. Now the result comes directly from 3.5(b) since (K, G, K_x) is a trio.

(b) <u>*General case*.</u>

The stratum S is K-equivariantly diffeomorphic to the twisted product $K \times_{N(K_x)} F$ where $N(K_x)$ is the normalizer of K_x on K and $F = S^{K_x}$. So, the tubular neighborhood T is K-equivariantly diffeomorphic to the twisted product $K \times_{N(H)} N$ where N is the manifold $\tau^{-1}(F)$. The previous case gives that the BIC of $(N, \mathcal{F}_N)$ is finite dimensional. Now the result comes directly from 3.5(b) since $(K, G, N(K_x))$ is a trio. ♣

The main result of this work is the following

Theorem 4.4 *The BIC of the foliation determined by an isometric action on a compact manifold is finite dimensional.*

Proof. Let $\mathcal{F}$ be a Killing foliation defined on a compact manifold M induced by an isometric action $\Phi: G \times M \to M$ where G is a Lie group. Without loss of generality we can suppose that the Lie group G is a connected one (cf. Lemma 1.1.1). We fix a tamer group K. We know that G is normal in K and the quotient group K/G is commutative (cf. Proposition 1.1.2).

Let us consider the following statement

$$\mathfrak{A}(U,\mathcal{F}) = \text{"The BIC } \boldsymbol{H}^*_{\overline{p}}(U/\mathcal{F}) \text{ is finite dimensional for each perversity } \overline{p}\text{,"}$$

where $U \subset M$ is a K-invariant submanifold. We prove $\mathfrak{A}(M,\mathcal{F})$ by induction on $\dim M$. The result is clear when $\dim M = 0$. We suppose $\mathfrak{A}(W,\mathcal{F})$ for any K-invariant compact submanifold W of M with $\dim W < \dim M$ and we prove $\mathfrak{A}(M,\mathcal{F})$. We proceed in several steps.

First step: 0-depth. Let us suppose depth $\mathsf{S}_{K,M} = 0$. Since $G \triangleleft K$ and K_x is conjugated to K_y then G_x is conjugated to G_y, $\forall x, y \in M$. We get that the foliation $\mathcal{F}$ is a (regular) riemannian foliation (cf. [7]). Its BIC is just the basic cohomology (cf. 2.3). Then $\mathfrak{A}(M,\mathcal{F})$ comes from [4].

Second step: Inside M. Let us suppose depth $\mathsf{S}_{K,M} > 0$. The family $\{M \backslash S_{min}, T_{min}\}$ is a basic covering of M and the we get the exact sequence (cf. 2.5)

$$0 \to \Omega^*_{\overline{p}}(M/\mathcal{F}) \to \Omega^*_{\overline{p}}((M \backslash S_{min})/\mathcal{F}) \oplus \Omega^*_{\overline{p}}(T_{min}/\mathcal{F}) \to \Omega^*_{\overline{p}}((T_{min} \backslash S_{min})/\mathcal{F}) \to 0.$$

The Five Lemma gives

$$\mathfrak{A}(T_{min} \backslash S_{min}, \mathcal{F}), \;\; \mathfrak{A}(T_{min}, \mathcal{F}) \;\; \text{and} \;\; \mathfrak{A}(M \backslash S_{min}, \mathcal{F}) \Longrightarrow \mathfrak{A}(M, \mathcal{F}).$$

Since $T_{min} \backslash S_{min}$ is K-diffeomorphic to $D_{min} \times]0, 1[$ (cf. (cf. 4.1)) then $\mathfrak{A}(D_{min}, \mathcal{F}) \Longrightarrow \mathfrak{A}(T_{min} \backslash S_{min}, \mathcal{F})$. The inequality $\dim D_{min} < \dim M$ gives

$$\mathfrak{A}(T_{min}, \mathcal{F}) \;\; \text{and} \;\; \mathfrak{A}(M \backslash S_{min}, \mathcal{F}) \Longrightarrow \mathfrak{A}(M, \mathcal{F}).$$

In order to prove $\mathfrak{A}(T_{min}, \mathcal{F})$ it suffices to prove $\mathfrak{A}(T, \mathcal{F})$ where $(T, \tau, S, \mathbb{R}^m)$ a K-invariant tubular neighborhood of closed stratum S of $\mathsf{S}_{K,M}$. Following Proposition 4.3.1 we have

$$\mathfrak{A}(\mathbb{R}^m, \mathcal{N}) \Longrightarrow \mathfrak{A}(T, \mathcal{F}) \Longrightarrow \mathfrak{A}(T_{min}, \mathcal{F}).$$

Consider the orthogonal decomposition $\mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, where $\mathbb{R}^{m_1} = (\mathbb{R}^m)^{G_x}$. The only fixed point of the restriction $\Lambda_x \colon G_x \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_2}$ is the origin. So, there exists a Killing foliation[9] $\mathcal{G}$ on the sphere $\mathbb{S}^{m_2-1}$ with $(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}, \mathcal{F}) = (\mathbb{R}^{m_1} \times c\mathbb{S}^{m_2-1}, \mathcal{I} \times c\mathcal{G})$. Propositions 2.6 and 2.7 give:

$$\mathfrak{A}(\mathbb{S}^{m_2-1}, \mathcal{G}) \Longrightarrow \mathfrak{A}(\mathbb{R}^{m_1} \times c\mathbb{S}^{m_2-1}, \mathcal{I} \times c\mathcal{G}) \Longrightarrow \mathfrak{A}(\mathbb{R}^m, \mathcal{N}).$$

Finally, since $\dim \mathbb{S}^{m_2-1} < m \leq \dim T \leq \dim M$ we have

(10) $$\mathfrak{A}(M \backslash S_{min}, \mathcal{F}) \Longrightarrow \mathfrak{A}(M, \mathcal{F}).$$

Third step: Blow-up. Let us suppose depth $\mathsf{S}_{K,M} > 0$. The family $\left\{\mathcal{L}^{-1}(M \backslash S_{min}), \mathcal{L}^{-1}(T_{min})\right\}$ is a basic covering of $\widehat{M}$ and the we get the exact sequence (cf. 2.5)

$$0 \to \Omega^*_{\overline{p}}\left(\widehat{M}/\widehat{\mathcal{F}}\right) \to \Omega^*_{\overline{p}}\left(\mathcal{L}^{-1}(M \backslash S_{min})/\widehat{\mathcal{F}}\right) \oplus \Omega^*_{\overline{p}}\left(\mathcal{L}^{-1}(T_{min})/\widehat{\mathcal{F}}\right) \to \Omega^*_{\overline{p}}\left(\mathcal{L}^{-1}(T_{min} \backslash S_{min})/\widehat{\mathcal{F}}\right) \to 0.$$

Following 4.2 we have that

- $\mathcal{L}^{-1}(M \backslash S_{min})$ is K-diffeomorphic to two copies of $M \backslash S_{min}$,

- $\mathcal{L}^{-1}(T_{min})$ is K-diffeomorphic to $D_{min} \times] - 1, 1[$,

- $\mathcal{L}^{-1}(T_{min} \backslash S_{min})$ is K-diffeomorphic to $D_{min} \times (] - 1, 0[\cup]0, 1[)$.

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[9] It is given by the orthogonal action $\Lambda_x \colon G_x \times \mathbb{S}^{m_2-1} \to \mathbb{S}^{m_2-1}$.

Now, the Five Lemma gives

$$\mathfrak{A}(D_{min}, \widehat{\mathcal{F}}) \text{ and } \mathfrak{A}\left(\widehat{M}, \widehat{\mathcal{F}}\right) \Longrightarrow \mathfrak{A}(M \backslash S_{min}, \mathcal{F}).$$

But, the inequality $\dim D_{min} < \dim M$ gives

(11) $$\mathfrak{A}\left(\widehat{M}, \widehat{\mathcal{F}}\right) \Longrightarrow \mathfrak{A}(M \backslash S_{min}, \mathcal{F}).$$

Forth step: Final blow-up. When depth $S_{K,M} = 0$ we get $\mathfrak{A}(M, \mathcal{F})$ from the First step. Let us suppose depth $S_{K,M} > 0$. From (10) and (11) we get

$$\mathfrak{A}\left(\widehat{M}, \widehat{\mathcal{F}}\right) \Longrightarrow \mathfrak{A}(M, \mathcal{F}).$$

with depth $S_{K,\widehat{M}} <$ depth $S_{K,M}$ (cf. (9)). By iterating this procedure we get

$$\mathfrak{A}\left(\widetilde{M}, \widetilde{\mathcal{F}}\right) = \mathfrak{A}\left(\widehat{\widetilde{\widetilde{\mathcal{M}}}}, \widehat{\widetilde{\widetilde{\mathcal{F}}}}\right) \Longrightarrow \cdots \Longrightarrow \mathfrak{A}\left(\widehat{M}, \widehat{\mathcal{F}}\right) \Longrightarrow \mathfrak{A}(M, \mathcal{F}),$$

with depth $S_{K,\widetilde{M}} = 0$. We finish the proof by applying again the First Step.   ♣

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