



# RINGS OVER WHICH CYCLICS ARE DIRECT SUMS OF PROJECTIVE AND CS OR NOETHERIAN

Chris Holston, Surrender Jain, André Leroy

► **To cite this version:**

Chris Holston, Surrender Jain, André Leroy. RINGS OVER WHICH CYCLICS ARE DIRECT SUMS OF PROJECTIVE AND CS OR NOETHERIAN. A Paraître Glasgow Mathematical Journal. 2009. <hal-00449861>

**HAL Id: hal-00449861**

**<https://hal.archives-ouvertes.fr/hal-00449861>**

Submitted on 23 Jan 2010

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# RINGS OVER WHICH CYCLICS ARE DIRECT SUMS OF PROJECTIVE AND CS OR NOETHERIAN

C. J. HOLSTON, S. K. JAIN, AND A. LEROY

*Dedicated to Patrick F. Smith on his 65th birthday.*

ABSTRACT.  $R$  is called a right  $WV$ -ring if each simple right  $R$ -module is injective relative to proper cyclics. If  $R$  is a right  $WV$ -ring, then  $R$  is right uniform or a right  $V$ -ring. It is shown that for a right  $WV$ -ring  $R$ ,  $R$  is right noetherian if and only if each right cyclic module is a direct sum of a projective module and a CS or noetherian module. For a finitely generated module  $M$  with projective socle over a  $V$ -ring  $R$  such that every subfactor of  $M$  is a direct sum of a projective module and a CS or noetherian module, we show  $M = X \oplus T$ , where  $X$  is semisimple and  $T$  is noetherian with zero socle. In the case that  $M = R$ , we get  $R = S \oplus T$ , where  $S$  is a semisimple artinian ring, and  $T$  is a direct sum of right noetherian simple rings with zero socle. In addition, if  $R$  is a von Neumann regular ring, then it is semisimple artinian.

## 1. INTRODUCTION AND PRELIMINARIES

The question of studying homological properties on modules that guarantee the noetherian property dates back to the 1960s, when Bass and Papp showed that a ring is right noetherian iff direct sums of injective modules are injective. Since then, there has been continuous work on finding properties on classes of modules that guarantee the ring to be right noetherian (or some other finiteness condition). For instance, if each cyclic right module is: an injective module or a projective module [5], a direct sum of an injective module and a projective module [12, 14], or a direct sum of a projective module and a module  $Q$ , where  $Q$  is either injective or noetherian [6], then the ring is right noetherian. It is also known that if every finitely generated right module is CS, then the ring is right noetherian [7]. A celebrated theorem of Osofsky-Smith states that if every cyclic module is CS then  $R$  is a  $qfd$ -ring [12]. In this paper, we will consider rings over which every cyclic right module is a direct sum of a projective module and a CS or noetherian module.

In Section 2, we first introduce a slight generalization of  $V$ -rings, which we call  $WV$ -rings (weakly  $V$ -rings). Recall that rings over which all simple modules are injective are known as  $V$ -rings [11]. A ring  $R$  is called a right  $WV$ -ring if every simple right  $R$ -module is  $R/A$ -injective for any right ideal  $A$  of  $R$  such that  $R/A \not\cong R$ . Detailed study as to how  $WV$ -rings differ from  $V$ -rings is provided in Section 2. Indeed, if  $R$  is a right  $WV$ -ring but not a right  $V$ -ring, then  $R$  must be right uniform.

In Section 3, we introduce the property (\*) for an  $R$ -module  $M$ , and say that

---

2000 *Mathematics Subject Classification.* 16D50, 16D70, 16D80.

*Key words and phrases.* Projective, CS, and Noetherian Modules.  $V$ -Rings,  $WV$ -Rings, Essential Right Ideal, Socle, Von Neumann Regular Rings, QFD-Rings, Semisimple Artinian Rings.

$M$  satisfies (\*) if we can write  $M = A \oplus B$ , where  $A$  is either a CS-module or a noetherian module, and  $B$  is a projective module. Theorem 18 (a) shows that if  $R$  is a  $V$ -ring and  $M$  is a finitely generated  $R$ -module with projective socle such that each subfactor of  $M$  satisfies (\*), then  $M = X \oplus T$ , where  $X$  is semisimple and  $T$  is noetherian with zero socle. In particular, if  $R$  is a  $V$ -ring such that each cyclic module satisfies (\*), then  $R = S \oplus T$ , where  $S$  is semisimple artinian and  $T$  is a finite direct sum of simple noetherian rings with zero socle. Theorem 18 (b) shows that for a  $WV$ -ring  $R$ ,  $R$  is noetherian iff each cyclic  $R$ -module satisfies (\*). The property (\*) has been studied in [13] for finitely generated, as well as 2-generated, modules. The proofs of the main results depend upon a series of lemmas.

We will say a module has *fud* whenever it has finite uniform dimension. Throughout, we assume all rings are associative rings with identity, and all modules are right  $R$ -modules. Thus in our results, we shall omit the word 'right' when we want to say right noetherian, right  $WV$ -ring, etc. We shall use  $\subset_e$  to denote an essential submodule, and  $\subset_{\oplus}$  to denote a direct summand. For any undefined notation or terminology, we refer the reader to [9].

## 2. $WV$ -RINGS

A ring  $R$  is called a  $WV$ -ring if each simple  $R$ -module is  $R/A$ -injective for any right ideal  $A$  such that  $R/A \not\cong R$  (i.e.  $R/A$  is proper). Such rings need not be  $V$ -rings, as for example the ring  $\mathbb{Z}_{p^2}$  for any prime  $p$  is a  $WV$ -ring which is not a  $V$ -ring. Let us remark that Wisbauer ([16], p.190) called a module  $M$  co-semisimple if every module in the category  $\sigma(M)$  is  $M$ -injective. Following Wisbauer's definition of co-semisimple modules, a right  $WV$ -ring is a ring for which every proper cyclic right module is co-semisimple.

Let us first compare  $V$ -rings and  $WV$ -rings.

**Lemma 1.** *Let  $R$  be a  $WV$ -ring, and  $R/A$  and  $R/B$  be proper cyclic modules such that  $A \cap B = 0$ . Then  $R$  is a  $V$ -ring.*

**Proof:** Since  $R$  is a  $WV$ -ring, any simple module is  $R/A \times R/B$ -injective. Since  $R_R$  embeds in  $R/A \times R/B$ , any simple module is  $R_R$  injective, i.e.  $R$  is a  $V$ -ring.  $\square$

**Theorem 2.** *Let  $R$  be a  $WV$ -ring which is not a  $V$ -ring. Then  $R$  must be uniform.*

**Proof:** Suppose  $R$  is a  $WV$ -ring. If  $R$  is of infinite uniform dimension, then  $R$  contains a direct sum  $A \oplus B$  where both  $A$  and  $B$  are infinite direct sums of nonzero right ideals. If  $R/A \cong R$ , then  $R/A$  is projective, and hence there exists a right ideal  $C$  of  $R$  such that  $R = C \oplus A$ . But then the cyclic module  $R/C$  is isomorphic to an infinite direct sum of nonzero modules, a contradiction. Thus  $R/A$  is proper. Similarly  $R/B$  is proper, and so  $R$  is a  $V$ -ring by Lemma 1.

Assume now that  $u.dim(R) = n > 1$  is finite. Then there exist closed uniform right ideals  $U_i$  such that  $\bigoplus_{i=1}^n U_i \subset_e R$ . Now  $u.dim(R/U_1) = n-1 = u.dim(R/U_2)$ , and so  $R/U_1$  and  $R/U_2$  are proper. Hence  $R$  is a  $V$ -ring by Lemma 1.

So if  $R$  is a  $WV$ -ring but not a  $V$ -ring, we must have  $u.dim(R) = 1$ , i.e.  $R$  is uniform.  $\square$

The proofs of the following propositions 3 and 5 are straightforward and follow closely the classical ones given in Lam ([9], Lemma 3.75 p.99) or Wisbauer ([16], 23.1 p.190).

**Proposition 3.** *Let  $R$  be a ring such that  $R/I$  is proper for any nonzero right ideal  $I$ . Then the following are equivalent:*

- (a)  $R$  is a  $WV$ -ring.
- (b)  $J(R/I) = 0$  for any nonzero right ideal  $I$ .
- (c) Any nonzero right ideal  $I \neq R$  is an intersection of maximal right ideals.
- (d) If a simple  $R$ -module is contained in a cyclic module  $R/I$  where  $I \neq 0$ , then it is a direct summand of  $R/I$ .

In particular, the above statements are equivalent when  $R$  is uniform or local.  $\square$

**Corollary 4.** *If  $R$  is a  $WV$ -ring, then  $R/J(R)$  is a  $V$ -ring.*

**Proof:** Let  $J = J(R)$ . We note that  $R$  is a  $V$ -ring iff each right ideal ( $\neq R$ ) is an intersection of maximal right ideals. If  $R$  is a  $WV$ -ring which is not uniform, then  $R$  is a  $V$ -ring (Theorem 2) and hence  $J = 0$ . So the result is clear in this case.

Thus we may assume  $R$  is uniform. By Proposition 3, every nonzero right ideal ( $\neq R$ ) is an intersection of maximal right ideals. If  $J = 0$ , then the zero ideal is also an intersection of maximal right ideals, and so  $R(= R/J)$  is a  $V$ -ring. If  $J \neq 0$ , then in  $R/J$  all right ideals ( $\neq R/J$ ) are intersections of maximals, and so  $R/J$  is a  $V$ -ring.  $\square$

**Proposition 5.** *If  $R$  is a  $WV$ -ring, then the following statements hold:*

- (a) If  $I$  is a right ideal of  $R$ , then either  $I^2 = 0$  or  $I^2 = I$ .
- (b) If  $R$  is a domain, then  $R$  is simple.
- (c) If a nonzero right ideal  $I$  of  $R$  contains a nonzero two-sided ideal, then every simple  $R$ -module is  $R/I$ -injective.
- (d) If  $R$  is a von Neumann regular ring, then  $R$  is a  $V$ -ring.
- (e) If  $R$  is a local ring and is not a  $V$ -ring, then  $R$  has exactly three right ideals.

**Proof:** (a) If  $R$  is a  $V$ -ring, it is well known that  $I^2 = I$  for every right ideal of  $R$ . Assume that  $R$  is not a  $V$ -ring. Then  $R$  is uniform (Theorem 2).

Let  $I \neq R$  be a right ideal and suppose  $I^2 \neq 0$ . By Proposition 3, both  $I$  and  $I^2$  are intersections of maximal right ideals. If  $I^2 \neq I$ , there must exist a maximal right ideal  $M$  such that  $I^2 \subseteq M$  but  $I \not\subseteq M$ . We thus have  $R = I + M$  and we can write  $1 = x + m$  for some  $x \in I, m \in M$ . This gives  $I = (x + m)I \subseteq xI + mI \subseteq I^2 + M = M$ , a contradiction. Hence  $I^2 = I$ .

(b) Let  $0 \neq a \in R$ . Since  $R$  is a domain,  $(aR)^2 \neq 0$ , so part (a) gives us  $(aR)^2 = aR$ , i.e.  $aRaR = aR$ . Since  $R$  is a domain this gives that  $RaR = R$ .

(c) Let us first remark that if  $T$  is a nonzero two-sided ideal of  $R$ , then, since  $(\frac{R}{T})T = 0$ ,  $R/T$  is a proper cyclic module.

Let  $I$  be a nonzero right ideal of  $R$  and  $T$  a nonzero two-sided ideal contained in  $I$ . Since  $R$  is a  $WV$ -ring, any simple module is  $R/T$ -injective, and hence any simple module is injective relative to  $R/I \cong \frac{R/T}{I/T}$ .

(d) Follows from Corollary 2.4, since  $J(R) = 0$ .

(e) If  $I \neq 0$  and  $I \neq R$ , then  $I$  is an intersection of maximal right ideals (Proposition 3). So  $I = J(R)$ . Thus  $R$  has at most three right ideals.  $\square$

**Corollary 6.** *If  $R$  is a  $WV$ -domain, then  $R$  is a  $V$ -domain.*  $\square$

It is known that the property of  $R$  being a  $V$ -ring is not left-right symmetric, and hence neither is the property of being a  $WV$ -ring. In fact, the property of being a

$WV$ -ring that is not a  $V$ -ring is not left-right symmetric either, as evidenced by the following example, due to Faith ([3], page 335).

**Example 7.** Let  $R = \begin{bmatrix} a & b \\ 0 & \sigma(a) \end{bmatrix}$  where  $a, b \in \mathbb{Q}(x)$  and  $\sigma$  is the  $\mathbb{Q}$ -endomorphism of  $\mathbb{Q}(x)$  such that  $\sigma(x) = x^2$ .

In this ring there are only three left ideals and hence it is a left  $WV$ -ring. It cannot be a left  $V$ -ring because  $J(R) \neq 0$ . This ring is local and is not right noetherian and thus it cannot be a right  $WV$ -ring (Proposition 5 (e)).

### 3. CYCLICS BEING (CS OR NOETHERIAN) $\oplus$ PROJECTIVE

Recall that an  $R$ -module  $M$  satisfies (\*) if we can write  $M = A \oplus B$ , where  $A$  is either a CS-module or a noetherian module, and  $B$  is a projective module. It was shown in [13] that a ring  $R$  is noetherian iff every 2-generated  $R$ -module satisfies (\*). We remark that it is not sufficient to assume that every cyclic satisfies (\*) in order for  $R$  to be noetherian, as may be seen from the following example [10].

**Example 8.** Let  $R$  be the ring of all formal power series

$$\left\{ \sum a_i x^i \mid a_i \in F, i \in I \right\}$$

where  $F$  is a field, and  $I$  ranges over all well-ordered sets of nonnegative real numbers.

This ring is not noetherian, but every homomorphic image is self-injective, and hence satisfies (\*).

We begin with a result that is used throughout the paper. We would like to thank a referee for drawing our attention to shock's result [15] which shortens the proof.

**Proposition 9.** Let  $C$  be a cyclic  $R$ -module such that each cyclic subfactor of  $C$  satisfies (\*), and let  $S = \text{Soc}(C)$ . Then  $C/S$  has *fud*. Furthermore, if  $R$  is a  $WV$ -ring, then  $C/S$  is noetherian.

**Proof:** Let  $E \subseteq_e C$  and  $\frac{X}{D}$  be a cyclic subfactor of  $\frac{C}{E}$ , where  $E \subseteq D \subseteq X \subseteq C$ . Then by (\*),  $\frac{X}{D} = \frac{B}{D} \oplus \frac{A}{D}$  with  $\frac{B}{D}$  projective and  $\frac{A}{D}$  CS or noetherian. Since  $D$  splits from  $B$ , essentiality shows that  $B = D$ . Theorem 1.3 of [2] then applies to give that  $\frac{C}{E}$  has *fud*. Since  $E$  was arbitrary such that  $E \subseteq_e C$ , this implies that, in particular that  $\frac{C}{E}$  has *qfd*. Then  $\frac{C}{\text{soc}(C)}$  is *fud* by Lemma 2.9 of [2]. Now assume that  $R$  is a  $WV$ -ring and let  $Z \subset Y \subseteq C/E$ , where, as above,  $E \subseteq_e C$ . If  $0 \neq x \in \text{rad}(Y/Z)$  let  $K$  be maximal in  $xR$ . Since  $\frac{C}{E}$  is singular it is proper cyclic and the simple module  $\frac{xR}{K}$  is  $\frac{C}{E}$ -injective, so it splits in  $\frac{Y/Z}{K}$ , a contradiction since  $\frac{xR}{K} \subseteq \text{rad}(\frac{Y/Z}{K})$ . We conclude that  $\text{rad}(\frac{Y}{Z}) = 0$ . Then Theorem 3.8 of [15] implies that  $\frac{C}{E}$  is noetherian. Now, Theorem 5.15 (1) in [1] shows that  $\frac{C}{\text{soc}(C)}$  is noetherian.  $\square$

For the convenience of the reader, we state below a well-known lemma (Cf. Lemma 9.1 p. 73 [1]).

**Lemma 10.** If  $M$  is a finitely generated CS-module and  $\oplus M_i$  is an infinite direct sum of nonzero submodules of  $M$ , then  $M/\oplus M_i$  cannot have finite uniform dimension.  $\square$

Under a stronger assumption on a cyclic module  $C$  than the condition (\*), namely, if every cyclic subfactor of  $C$  is projective, CS, or noetherian, we show that  $C$  is noetherian when  $R$  is a  $WV$ -ring. This will play a key role later as we work towards the general result.

**Theorem 11.** *Let  $C$  be a cyclic  $R$ -module such that each cyclic subfactor of  $C$  is either CS, noetherian, or projective.*

(a) *Then  $C$  has  $fud$ .*

(b) *If  $R$  is a  $WV$ -ring, then  $C$  is noetherian.*

**Proof:** (a) Let  $S = Soc(C)$ . By Proposition 9,  $C/S$  has  $fud$ . We show  $S$  is finitely generated.

Suppose  $S$  is infinitely generated. Write  $S = S_1 \oplus S_2$ , where  $S_1$  and  $S_2$  are both infinitely generated. Now by hypothesis,  $C/S_1$  is either CS or noetherian or projective. If projective, then  $S_1 \subset_{\oplus} C$  and hence  $S_1$  is cyclic, a contradiction as  $S_1$  is infinitely generated. If noetherian, then so is  $S/S_1 \cong S_2$ , a contradiction as  $S_2$  is infinitely generated. So  $C/S_1$  is CS. Furthermore,  $(S_1 \oplus S_2)/S_1 \cong S_2$  is infinitely generated. Since  $C/S \cong \frac{C/S_1}{(S_1 \oplus S_2)/S_1}$ , we get a contradiction by invoking Lemma 10. Hence  $S$  is finitely generated and so  $C$  has  $fud$ .

(b) Since  $C/S$  is noetherian (Proposition 9), and  $S$  has  $fud$ , it follows that  $C$  is noetherian.  $\square$

Although the proof of the next lemma is straightforward we believe that the reader will appreciate the simple technique used in the proof.

**Lemma 12.** *Let  $C$  be an  $R$ -module and  $S = Soc(C)$ . If  $C/S$  is a uniform  $R$ -module, then for any two submodules  $A$  and  $B$  of  $C$  with  $A \cap B = 0$ , either  $A$  or  $B$  is semisimple.*

**Proof:** Let  $K$  be a complement submodule of  $A$  in  $C$  containing  $B$ . Then  $A \oplus K \subset_e C$ . This yields  $Soc(A \oplus K) = S$ . Thus,  $(A \oplus K)/(Soc(A \oplus K)) \subseteq C/S$ . Since  $(A \oplus K)/(Soc(A \oplus K)) \cong A/Soc(A) \times K/Soc(K)$  and  $C/S$  is uniform as an  $R$ -module, either  $A/Soc(A)$  or  $K/Soc(K)$  is zero. So  $A = Soc(A)$  or  $K = Soc(K)$ . In other words, either  $A$  or  $K$  (and hence  $B$ ) is semisimple.  $\square$

**Lemma 13.** *If  $C$  is an  $R$ -module, and if  $C/I = A/I \oplus B/I$  is a direct sum with  $B/I$  a projective module, then  $C = A \oplus B'$ , where  $B = B' \oplus I$ .*

**Proof:** From the decomposition  $C/I = A/I \oplus B/I$ , we have  $C = A + B$ , where  $A \cap B = I$ . Since  $B/I$  is projective,  $B = B' \oplus I$  for some  $B'$ . Then  $C = A + (B' \oplus I) = A + B'$ . We claim that  $A \cap B' = 0$ .

Let  $x \in A \cap B' \subseteq A \cap B = I$ . Then  $x \in B' \cap I = 0$ . Thus  $C = A \oplus B'$ .  $\square$

**Lemma 14.** *Let  $R$  be a  $WV$ -ring. Let  $C$  be a cyclic module with a projective socle (equivalently  $S = Soc(C)$  is embeddable in  $R$ ). If  $C/S$  is a uniform  $R$ -module and each cyclic subfactor of  $C$  satisfies (\*), then  $C$  is noetherian.*

**Proof:** First assume  $R$  is a  $V$ -ring. Let  $C'/I$  be a cyclic subfactor of  $C$  and write  $C'/I = A/I \oplus B/I$  as a direct sum of a CS or noetherian module and a projective module, respectively. Then, by Lemma 13,  $C' = A \oplus B'$ , where  $B = B' \oplus I$ . Since  $\frac{C'}{Soc(C')} = \frac{C'}{C' \cap S} \cong \frac{C'+S}{S} \subseteq \frac{C}{S}$  is uniform, either  $A$  or  $B'$  is semisimple (Lemma 12).

Note both  $A$  and  $B'$  are cyclic.

Case 1:  $A$  is semisimple. Since  $A/I$  is semisimple cyclic and  $R$  is a  $V$ -ring,  $A/I$  is injective. Moreover  $A/I$  embeds in  $A \subseteq \text{Soc}(C') \subseteq \text{Soc}(C) = S$ . The hypothesis that  $S$  embeds in  $R$  yields that  $A/I$  and hence  $C'/I$  are projective.

Case 2:  $B'$  is semisimple. Since  $B/I \cong B'$  is semisimple and cyclic, it is a finite direct sum of simple injective modules. If  $A/I$  is noetherian, then clearly  $C/I$  will be also. Recall that a direct sum of a CS module and a simple module is a CS module (Cf. Lemma 7.10 [1]). Hence if  $A/I$  is CS then  $A/I \oplus B/I$  is also CS.

Thus, any cyclic subfactor of the cyclic module  $C$  is either CS or noetherian or projective. Therefore,  $C$  is noetherian by Theorem 11.

Now if  $R$  is not a  $V$ -ring, then it is uniform (Theorem 2). Since  $S$  embeds in  $R$ ,  $S$  is trivially noetherian. So  $C$  is noetherian because  $C/S$  is noetherian (Proposition 9).  $\square$

**Proposition 15.** *Let  $R$  be a  $V$ -ring. Let  $C$  be a cyclic  $R$ -module with essential and projective socle. Suppose each cyclic subfactor of  $C$  satisfies (\*). Then  $C$  is semisimple.*

**Proof:** Let  $S = \text{Soc}(C)$ . We know by Proposition 9 that  $C/S$  has *fud*. Suppose  $C/S \neq 0$ . Then  $C/S$  contains a nonzero cyclic uniform submodule. Thus we can find  $u \in C$  with  $U = (uR + S)/S \cong uR/\text{Soc}(uR)$  uniform. Since  $\text{Soc}(uR) \subseteq_{\oplus} S$  we know that  $\text{Soc}(uR)$  is projective. Moreover, every cyclic subfactor of  $uR$  also satisfies (\*). Thus Lemma 14 implies that  $uR$  is noetherian.  $\text{Soc}(uR)$  is then a finite direct sum of simple modules, and hence it is injective. Since  $\text{Soc}(uR) \subseteq_e uR$ ,  $uR = \text{Soc}(uR)$ . This yields  $U = 0$ , a contradiction. Thus  $C/S = 0$ , that is,  $C = S$ , completing the proof.  $\square$

**Remark 16.** *We note that the above proposition does not apply to a  $WV$ -ring  $R$  which is not a  $V$ -ring. In this case  $R_R$  is uniform and the only projective submodule of a cyclic  $R$ -module is the zero submodule.*

**Theorem 17.** *Let  $R$  be a von Neumann regular  $WV$ -ring such that each cyclic  $R$ -module satisfies (\*). Then  $R$  is semisimple artinian.*

**Proof:** By Proposition 5 (d),  $R$  is a  $V$ -ring. Let  $S = \text{Soc}(R)$ . Then  $R/S$  has *fud* and hence it is semisimple artinian [8]. Let  $T$  be a complement of  $S$ . Then  $T$  embeds essentially in  $R/S$ . Thus  $T = 0$ . Hence  $S \subseteq_e R$ . So by Proposition 15,  $R$  is semisimple artinian.  $\square$

Finally, we prove the following general result.

**Theorem 18.** (a) *Let  $R$  be a  $V$ -ring. Let  $M$  be a finitely generated  $R$ -module with projective socle. Suppose each cyclic subfactor of  $M$  satisfies (\*). Then  $M$  is noetherian, and  $M = X \oplus T$ , where  $X$  is semisimple and  $T$  is noetherian with zero socle.*

*In particular, if  $R$  is a  $V$ -ring such that each cyclic module satisfies (\*), then  $R = S \oplus T$ , where  $S$  is semisimple artinian and  $T$  is a finite direct sum of simple noetherian rings with zero socle.*

(b) *For a  $WV$ -ring  $R$ ,  $R$  is noetherian iff each cyclic  $R$ -module satisfies (\*).*

**Proof:** (a) First, assume  $M$  is cyclic. Let  $S_0 = \text{Soc}(M)$  and let  $T_0$  be a complement of  $S_0$  in  $M$ . Consider the cyclic module  $X_0 = M/T_0$ . Then  $S_0$  is essentially embeddable in  $X_0$ . Since  $\text{Soc}(X_0) \cong S_0$ ,  $X_0$  is semisimple by Proposition 15. So  $X_0$ , and hence  $S_0$ , is a finite direct sum of simples. In particular  $S_0$  is injective and we have  $M = S_0 \oplus T_0$ . Since  $M/S_0$  is noetherian (Proposition 9),  $T_0$  is noetherian and obviously it has zero socle.

In general,  $M = \sum_{i=1}^n x_i R$ . By above, each  $x_i R$  is noetherian, and hence  $M$  is noetherian.  $X = \text{Soc}(M)$  is finitely generated and injective by hypothesis. Therefore  $M = X \oplus T$ , where  $X$  is semisimple and  $T$  is noetherian with zero socle.

Finally, let  $S = \text{Soc}(R)$  which is clearly projective in a  $V$ -ring and let  $T$  be its complement. Then, as shown above,  $R$  is a right noetherian  $V$ -ring. Therefore,  $R$  is a direct sum of simple noetherian rings ([4], page 70). So,  $R = S \oplus T$ , where  $S$  is semisimple artinian and  $T$  is a finite direct sum of simple noetherian rings with zero socle.

(b) Note that if  $R$  is a  $WV$ -ring and not a  $V$ -ring, then  $R$  is uniform (Theorem 2). In this case,  $\text{Soc}(R)$  is either zero or a minimal right ideal. Since  $R/\text{Soc}(R)$  is noetherian (Proposition 9), we conclude that  $R$  is noetherian. The converse is obvious.  $\square$

**Remark 19.** (a) *Although Theorem 17 is a consequence of Theorem 18, the short proof given for Theorem 17 is of independent interest. More generally, if  $R$  is a  $WV$ -ring in which each non-nil right ideal contains a nonzero idempotent and every cyclic  $R$ -module satisfies (\*), then  $R$  is semisimple artinian [8].*

(b) *Readers familiar with the Wisbauer Category  $\sigma[M]$  may observe that the results in this paper can be more generally stated in  $\sigma[M]$ , where  $M$  is a finitely generated module.*

**Acknowledgment** We would like to thank the referees for giving us valuable and helpful comments.

Portions of this paper were presented at the International Conference on Ring and Module Theory, Hacettepe University, Ankara, Turkey, August 2008, and also at the conference on Rings and Modules, Universidade de Lisboa, Lisbon, Portugal, September 2008.

#### REFERENCES

- [1] N. V. Dung, D. V. Huynh, P.F. Smith and R. Wisbauer, *Extending modules*, Longman Scientific and technical, Harlow, 1994
- [2] N. V. Dung, D. V. Huynh, and R. Wisbauer, *On modules with finite uniform and Krull dimension*, Arch. Math. 57, 122-13 (1991).
- [3] C. Faith, *Algebra: rings, modules and categories*, Springer-Verlag, New York-Berlin, 1973.
- [4] C. Faith, *Rings and things and a fine array of twentieth century associative algebra*, Mathematical Surveys and Monographs 65, AMS, Providence, RI, 1999.
- [5] S. C. Goel, S. K. Jain and S. Singh, *Rings whose cyclic modules are injective or projective*, Proc. Amer. Math. Soc. 53, 16-18 (1975).
- [6] D. V. Huynh and S. T. Rizvi, *An affirmative answer to a question on noetherian rings*, J. Algebra Appl. 7, 47-59 (2008).
- [7] D. V. Huynh, S. T. Rizvi and M. F. Yousif, *Rings whose finitely generated modules are extending*, J. Pure Appl. Algebra 111, 325-328 (1996).
- [8] I. Kaplansky, *Topological representation of algebras II*, Trans. Amer. Math. Soc. 68, 62-75 (1950).
- [9] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics 189, Springer-Verlag New York, Inc., 1999.



- [10] L. S. Levy, *Commutative rings whose homomorphic images are self-injective*, Pacific J. Math. 18, 149-153 (1966).
- [11] G. O. Michler, O. E. Villamayor, *On rings whose simple modules are injective*, J. Algebra 25, 185-201 (1973).
- [12] B. Osofsky and P. F. Smith, *Cyclic modules whose quotients have all complement submodules direct summands*, J. Algebra 139, 342-354 (1991).
- [13] S. Plubtieng and H. Tansee, *Conditions for a ring to be noetherian or artinian*, Comm. Algebra 30(2), 783-786 (2002).
- [14] P. F. Smith, *Rings characterized by their cyclic modules*, Canad. J. Math. 24, 93-111 (1979).
- [15] R.C. Shock, *Dual generalizations of the artinian and noetherian conditions*, Pacific J. Math. 54, 227-235 (1974).
- [16] R. Wisbauer, *Foundations of module and ring theory*, Gordon and Breach, Reading (19991).

DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO-45701, USA  
*E-mail address:* `holston@math.ohiou.edu`

DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO-45701  
*E-mail address:* `jain@ohio.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARTOIS, FRANCE  
*E-mail address:* `andre.leroy@univ-artois.fr`