Strictly positive definite functions on the real line

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Abstract

We give some necessary or sufficient conditions for a function to be strictly positive definite on $\mathbb{R}$. This problem is intimately linked with the repartition of the zeros of trigonometric polynomials.

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1. Introduction and notations

1.1. Introduction

In the theory of scattered data interpolation with linear combinations of translates of a basis function, we have to deal with strictly positive definite functions rather than positive definite ones, or more generally with conditionally strictly positive definite functions [2, 4, 16]. This ensures that the interpolation problem is always solvable. Using Bochner's characterization of continuous positive definite functions as Fourier transforms of nonnegative finite measures, it is straightforward to see that verifying strict positive definiteness reduces to checking whether the exponentials are linearly independent on a certain subset of real numbers. Some years ago, K-F. Chang published a paper on this subject [3] but unfortunately his main result is erroneous. In fact, the author asserts in [3, theorem 3.5] that any nonzero complex polynomial $t(z) = \sum_{k=1}^{K} c_k e^{i\xi_k z}$ with $\xi_k \in \mathbb{R}$ is of sine type that is there are positive constants a, b, σ and τ such that

$$ae^{\sigma|y|} \leq |t(x+iy)| \leq be^{\sigma|y|} \text{ for all } x, y \in \mathbb{R}, |y| \geq \tau.$$

Taking for instance $t(z) = e^{iz}$, we see easily that the first inequality can not be satisfied. As a consequence, the author concludes at the end of his paper that there does not exist nonzero trigonometric polynomial vanishing at the points $x_n = n + \frac{1}{8}\operatorname{sgn}(n)$ where $n \in \mathbb{Z}$, but trivially $t(x) = \sin(8\pi x)$ does.

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1.2. Notations

Let us fix some usual notations for subspaces of complex valued functions defined on the real line. We denote by C the set of continuous functions and C_0 its subspace of functions vanishing at infinity, C^k the set of functions with k continuous derivatives, $\mathcal{S}$ the Schwartz space of infinitely differentiable and rapidly decreasing functions, $\mathcal{D}$ the space of infinitely differentiable functions with compact support, L^p the set of p-power integrable functions with respect to the Lebesgue measure λ, $\mathcal{T}$ the set of trigonometric polynomials that is the functions of the form $t(x) = \sum_{k=1}^{K} c_k e^{i\xi_k x}$ where c_k are complex numbers and ξ_k real numbers, $AP(\mathbb{R})$ the set of Bohr almost periodic functions.

We will use also the notation $AP(\mathbb{Z})$ for the set of complex valued Bohr almost periodic functions on $\mathbb{Z}$. The set of real valued Bohr almost periodic functions on $\mathbb{R}$, resp. $\mathbb{Z}$, is denoted by $AP_{\mathbb{R}}(\mathbb{R})$, resp. $AP_{\mathbb{R}}(\mathbb{Z})$.

For a function $\phi \in C$, we note $\check{\phi}$, resp. $\tilde{\phi}$, the function defined by $\check{\phi}(\xi) = \phi(-\xi)$, resp. $\tilde{\phi}(\xi) = \overline{\phi(-\xi)}$.

The symbol M^+ stands for the set of nonnegative finite Borel measure on $\mathbb{R}$. The support of a measure $\mu \in M^+$ defined as the smallest closed subset of $\mathbb{R}$ whose complement has μ-measure 0 is denoted by $\operatorname{supp}\mu$. The Fourier transform of a measure $\mu \in M^+$ is defined by $\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x)$.

2. Formulation of the interpolation problem, an analogue for Bochner's theorem

Given a function $\varphi \in C$, an arbitrary set of distinct real numbers $\Xi = \{\xi_1, \dots, \xi_K\}$ and complex numbers $z_1, \dots, z_K$, the scattered data interpolation problem consists in finding a function

$$u(\xi) = \sum_{k=1}^{K} c_k \varphi(\xi - \xi_k)$$

such that $u(\xi_j) = z_j$, $j = 1 \dots K$. Although this problem is equivalent to the nonsingularity of the $K \times K$ matrix A with entries $A_{jk} = \varphi(\xi_j - \xi_k)$, one wishes to know when the interpolation matrix A is positive definite for any set Ξ. For this purpose, let us recall the following definition.

Definition 2.1. A complex valued continuous function φ is said positive definite (resp. strictly positive definite) on $\mathbb{R}$ if for every finite set of distinct real numbers $\Xi = \{\xi_1, \dots, \xi_K\}$ and every vector $(c_1, \dots, c_K) \in \mathbb{C}^K \setminus \{0\}$, the inequality

$$\sum_{j=1}^{K} \sum_{k=1}^{K} \bar{c}_j c_k \varphi(\xi_j - \xi_k) \geq 0 \text{ (resp. } > 0) \tag{1}$$

holds true.

We denote by $\mathcal{P}$ (resp. $\mathcal{P}^s$) the class of such functions. Clearly $\mathcal{P}^s \subset \mathcal{P}$.

The class of positive definite functions is fully characterized by the Bochner's theorem [1].

Theorem 2.1. *A function $\varphi \in \mathcal{P}$ if and only if $\varphi = \hat{\mu}$ where $\mu \in M^+$, φ and μ being biuniquely determined.*

Therefore we can ask for an equivalent characterization of a strictly positive definite function in terms of its Fourier transform. We state before a basic lemma.

Lemma 2.1. *Let $\mu \in M^+$ and $f \in C$ a nonnegative function. We have*

$$\int_{\mathbb{R}} f\, d\mu = 0 \Leftrightarrow f = 0 \text{ on } \operatorname{supp} \mu.$$

Proof. Let us set $X = \operatorname{supp} \mu$. Since $\mathbb{R} \setminus X$ is the largest open subset of $\mathbb{R}$ with μ-measure 0 and the set $\{f > 0\}$ is open, we have

$$\int_{\mathbb{R}} f\, d\mu = 0 \Leftrightarrow \mu(\{f > 0\}) = 0 \Leftrightarrow \{f > 0\} \subset \mathbb{R} \setminus X \Leftrightarrow X \subset \{f = 0\} \Leftrightarrow f = 0 \text{ on } X.$$

□

Proposition 2.1. *Let $\varphi = \hat{\mu} \in \mathcal{P}$. Then $\varphi \in \mathcal{P}^s$ if and only if there does not exist $t \in \mathcal{T} \setminus \{0\}$ vanishing on $\operatorname{supp} \mu$.*

Proof. Let $t \in \mathcal{T} \setminus \{0\}$. Remark first that t can be written in the form $t(x) = \sum_{k=1}^{K} c_k e^{i\xi_k x}$ where x_k are pairwise distinct real numbers and c_k are complex numbers, not all zero. We have

$$\begin{aligned}
\sum_{j=1}^{K} \sum_{k=1}^{K} \bar{c}_j c_k \varphi(\xi_j - \xi_k) &= \sum_{j=1}^{K} \sum_{k=1}^{K} \bar{c}_j c_k \int_{\mathbb{R}} e^{-i(\xi_j - \xi_k)x}\, d\mu(x) \\
&= \int_{\mathbb{R}} \sum_{j=1}^{K} \bar{c}_j e^{-i\xi_j x} \sum_{k=1}^{K} c_k e^{i\xi_k x}\, d\mu(x) \\
&= \int_{\mathbb{R}} \left| \sum_{k=1}^{K} c_k e^{i\xi_k x} \right|^2 d\mu(x) \\
&= \int_{\mathbb{R}} |t(x)|^2\, d\mu(x).
\end{aligned}$$

From lemma 2.1 the last integral is 0, i.e. $\varphi \notin \mathcal{P}^s$, if and only if t vanishes on $\operatorname{supp} \mu$. □

3. Elementary properties of strictly positive definite functions

The next result shows in particular that the class $\mathcal{P}^s$ is a convex cone, closed under multiplication.

Theorem 3.1. *Let $\varphi \in \mathcal{P}^s$ and $\psi \in \mathcal{P}$. Then*

(i) $\varphi + \psi \in \mathcal{P}^s$,
(ii) $\varphi\psi \in \mathcal{P}^s$ *provided that* $\psi \neq 0$.

Proof. Let $t \in \mathcal{T}$ and write $\varphi = \hat{\mu}$, $\psi = \hat{\nu}$ so that $\varphi + \psi = \widehat{\mu + \nu}$ and $\varphi\psi = \widehat{\mu \star \nu}$. It is clear that $\mu + \nu$ and $\mu \star \nu$ are in M^+. We have from proposition 2.1

$$\int_{\mathbb{R}} |t|^2\, d(\mu + \nu) = 0 \implies \int_{\mathbb{R}} |t|^2\, d\mu = 0 \implies t = 0.$$

For the second assertion, we remark by Fubini's theorem that

$$\int_{\mathbb{R}} |t|^2\, d(\mu \star \nu) := \int_{\mathbb{R}^2} |t(x+y)|^2\, d\mu(x)\, d\nu(y) = \int_{\mathbb{R}} d\nu(y) \int_{\mathbb{R}} |t(x+y)|^2\, d\mu(x)$$

and if we suppose $\psi \neq 0$, then $\operatorname{supp} \nu \neq \emptyset$. From lemma 2.1 and proposition 2.1, it follows that

$$\int_{\mathbb{R}} |t|^2 \, d(\mu \star \nu) = 0 \implies \int_{\mathbb{R}} |t(x+y)|^2 \, d\mu(x) = 0 \text{ for } y \in \operatorname{supp} \nu \implies t = 0.$$

□

Theorem 3.2. *Let φ be a function in $\mathcal{P}^s$. Then*

(i) *φ is hermitian i.e. $\check{\varphi} = \bar{\varphi}$,*
(ii) *$\varphi(0) > 0$ and $|\varphi(\xi)| < \varphi(0)$ for all $\xi \neq 0$,*
(iii) *$\varphi(a\cdot) \in \mathcal{P}^s$ provided that $a \neq 0$,*
(iv) *the functions $\bar{\varphi}, \Re\,\varphi$ and $|\varphi|^2$ are in $\mathcal{P}^s$,*
(v) *$h \circ \varphi \in \mathcal{P}^s$ if $h(z) = \sum_{n=0}^{\infty} a_n z^n$ is a nonconstant power series with nonnegative coefficients, converging for $z = \varphi(0)$.*

Proof. By theorem 2.1 we have $\varphi = \hat{\mu}$ where $\mu \in M^+$ so that $\varphi(-\xi) = \int_{\mathbb{R}} e^{i\xi x} \, d\mu(x) = \overline{\varphi(\xi)}$.
In the definition 1, let $n = 1$ to obtain $\varphi(0) > 0$ and then let $n = 2, \xi_1 = \xi, \xi_2 = 0, c_1 = 1, c_2 = -1$ which shows that $\varphi(0)^2 - \varphi(\xi)\varphi(-\xi) = \varphi(0)^2 - |\varphi(\xi)|^2 > 0$ whenever $\xi \neq 0$.
The third assertion follows immediately from the definition 1.
Taking $a = -1$ shows that $\bar{\varphi} = \check{\varphi} \in \mathcal{P}^s$ and so are the functions $\Re\varphi = \frac{1}{2}(\varphi + \bar{\varphi})$ and $|\varphi|^2 = \varphi\bar{\varphi}$ by the preceding theorem.
Since $|a_n \varphi^n(\xi)| \leq a_n \varphi^n(0)$ for all $\xi \in \mathbb{R}$ and all $n \geq 0$ and $\sum_{n=0}^{\infty} a_n \varphi^n(0) < \infty$, the series $\sum_{n=0}^{\infty} a_n \varphi^n$ converges uniformly on $\mathbb{R}$ by the Weierstrass M-test and so its sum $h \circ \varphi$ is continuous. Since h is nonconstant, there exists a coefficient $a_{n_0} > 0$ with $n_0 \geq 1$ and hence by theorem 3.1, $a_{n_0} \varphi^{n_0} \in \mathcal{P}^s$ and each partial sum $\sum_{n \neq n_0}^{N} a_n \varphi^n$ is either 0 or in $\mathcal{P}^s$. It follows that its pointwise limit is in $\mathcal{P}$ and then $h \circ \varphi = \sum_{n \neq n_0}^{\infty} a_n \varphi^n + a_{n_0} \varphi^{n_0}$ is in $\mathcal{P}^s$. □

4. Reproducing kernel Hilbert space

Given a strictly positive definite function φ on $\mathbb{R}$, the construction of the associated reproducing kernel Hilbert space is standard. We denote by H_0 the complex linear space spanned by the functions $\varphi_\xi, \xi \in \mathbb{R}$ where $\varphi_\xi(\eta) = \varphi(\eta - \xi)$. If $u = \sum_{k=1}^{K} c_k \varphi_{\xi_k}$ and $v = \sum_{j=1}^{J} d_j \varphi_{\eta_j}$ belong to H_0, then we define a sesquilinear form (u, v) on H_0 by the formula

$$(u, v) = \sum_{j=1}^{J} \sum_{k=1}^{K} \bar{d}_j c_k \varphi(\eta_j - \xi_k). \tag{2}$$

Since $(u, v) = \sum_{k=1}^{K} c_k \overline{v(\xi_k)} = \sum_{j=1}^{J} \bar{d}_j u(\eta_j)$, the definition of (u, v) does not depend on the chosen representations of u and v. We have $(u, u) = \sum_{j=1}^{K} \sum_{k=1}^{K} \bar{c}_j c_k \varphi(\xi_j - \xi_k) \geq 0$ by assumption and $(u, v) = \overline{(v, u)}$ since φ is hermitian. An immediate consequence of (2) is the reproducing property

$$(u, \varphi_\xi) = u(\xi) \text{ for all } u \in H_0 \text{ and } \xi \in \mathbb{R} \tag{3}$$

which implies in particular $(\varphi_\eta, \varphi_\xi) = \varphi(\xi - \eta)$. From the Cauchy-Schwartz inequality, we have

$$|u(\xi)|^2 = |(u, \varphi_\xi)|^2 \leq (u, u)(\varphi_\xi, \varphi_\xi) = (u, u)\varphi(0) \tag{4}$$

so that $(u, u) = 0$ if and only if $u = 0$, hence the form $(\cdot, \cdot)$ is an inner product on H_0. In the pre-Hilbert space H_0, we see from (4) that norm convergence implies uniform convergence on $\mathbb{R}$, thus we can adjoin all limits of Cauchy sequences to obtain a Hilbert space H of continuous functions usually called the reproducing kernel Hilbert space associated with φ. As H_0 is a dense subspace of H, equations (3) and (4) remain valid in H.

Remark 4.1. Conversely, for each Hilbert space H of complex valued continuous functions on $\mathbb{R}$, norm invariant under translations and such that point evaluation functionals are continuous and linearly independent, we can associate a unique function $\varphi \in \mathcal{P}^s$ verifying $(u, \varphi(\cdot - \xi)) = u(\xi)$. Let us give the proof. Since every point evaluation $\delta_\xi : u \mapsto u(\xi)$ is continuous, by the Riesz theorem, there exists uniquely $K_\xi \in H$ such that

$$(u, K_\xi) = u(\xi) \text{ for } u \in H.$$

Denote by $e_\xi : u \mapsto u(\cdot - \xi)$ the translation operator on H. For every $u \in H$, the function $e_{-\xi}u = u(\cdot + \xi)$ belongs to H and hence we have also $(e_{-\xi}u, K_0) = u(\xi)$. Since $\|e_{-\xi}u\| = \|u\|$ for $u \in H$, we see by the formula

$$4(u, v) = (u + v, u + v) - (u - v, u - v) + i(u + iv, u + iv) - i(u - iv, u - iv)$$

that $(e_{-\xi}u, e_{-\xi}v) = (u, v)$ for $u, v \in H$. We have therefore

$$(u, e_\xi K_0) = (e_{-\xi}u, K_0) = u(\xi) = (u, K_\xi)$$

which shows that $K_\xi = e_\xi K_0$. Let us write $\varphi = K_0$ so that $K_\xi = e_\xi \varphi = \varphi(\cdot - \xi)$. We have

$$(u, \varphi(\cdot - \xi)) = (u, K_\xi) = u(\xi) \text{ for } u \in H.$$

Next the function φ is positive definite because $\varphi(\xi - \eta) = K_0(\xi - \eta) = K_\eta(\xi) = (K_\eta, K_\xi)$ which gives

$$\sum_{j=1}^{K} \sum_{k=1}^{K} \bar{c}_j c_k \varphi(\xi_j - \xi_k) = (\sum_{k=1}^{K} c_k K_{\xi_k}, \sum_{j=1}^{K} c_j K_{\xi_j}) \geq 0.$$

Suppose that φ is not strictly positive definite. Then there exist distinct points $\xi_1, \ldots, \xi_K$ and coefficients $c_1, \ldots, c_K$ not all zero such that

$$\begin{aligned}
\sum_{j=1}^{K} \sum_{k=1}^{K} \bar{c}_j c_k \varphi(\xi_j - \xi_k) = 0 &\Leftrightarrow \sum_{k=1}^{K} c_k K_{\xi_k} = 0 \text{ in } H \\
&\Leftrightarrow \sum_{k=1}^{K} \bar{c}_k (u, K_{\xi_k}) = 0 \text{ for every } u \in H \\
&\Leftrightarrow \sum_{k=1}^{K} \bar{c}_k u(\xi_k) = 0 \text{ for every } u \in H \\
&\Leftrightarrow \sum_{k=1}^{K} \bar{c}_k \delta_{\xi_k} = 0 \text{ in } H'.
\end{aligned}$$

which shows that point evaluation functionals δ_ξ are linearly dependent in H' the dual space of H.

Let $\Xi = \{\xi_1, \ldots, \xi_K\}$ be a set of distinct real numbers and $z = (z_1, \ldots, z_K) \in \mathbb{C}^K$. We define the linear map $l : H \to \mathbb{C}^K$ by $l(v) = (v(\xi_j))_{j=1,\ldots,K}$. Since the matrix A with entries $\varphi(\xi_j - \xi_k)$ is positive definite, the function $u = \sum_{k=1}^K c_k \varphi_{\xi_k}$ is uniquely determined in $l^{-1}(\{z\})$ in solving the linear system $Ac = z$ where $c = (c_1, \ldots, c_K)$. The next theorem shows that this function minimizes the norm in H under interpolation constraints.

Theorem 4.1. *The function u is the unique solution of* $\min\{\|v\|,\ l(v) = z\}$.

Proof. From the reproducing property, the linear map l is continuous since each coordinate function is continuous. We have furthermore $l^{-1}(\{z\}) = u + l^{-1}(\{0\})$, hence the set $l^{-1}(\{z\})$ is a closed affine hyperplane in H. By the projection theorem, there exists an unique solution minimizing the norm on $l^{-1}(\{z\})$. This solution is simply given by u since for all $v \in l^{-1}(\{z\})$, we have $(v - u, u) = \sum_{k=1}^K \bar{c}_k (v - u)(\xi_k) = 0$ and hence $\|v\|^2 = \|v - u + u\|^2 = \|v - u\|^2 + \|u\|^2 \geq \|u\|^2$. □

5. Some inequalities

We prove some inequalities which are consequences of definition 2.1. We recall that for $\varphi \in \mathcal{P}^s$ and $\Xi = \{\xi_1, \ldots, \xi_K\}$ a set of distinct real numbers, there exists a function $w \in H_0$ such that $w(\xi_1) = 1$ and $w(\xi_k) = 0$ for any $k \neq 1$. Equivalently, if $\Xi = \{\xi_1, \ldots, \xi_K\}$ is a set of (not necessarily distinct) real numbers and $\xi \notin \Xi$, there exists a function $w \in H_0$ vanishing on Ξ and such that $w(\xi) = 1$.

Theorem 5.1. *Let φ be a function in $\mathcal{P}^s$. Let us consider $\{\xi_1 < \ldots < \xi_K\}$ and $\{\eta_1 < \ldots < \eta_J\}$ two sets of pairwise distinct real numbers, $(c_1, \ldots, c_K)$ and $(d_1, \ldots, d_J)$ two complex vectors with nonzero entries. Then*

(i) $|\varphi(\xi) - \varphi(\eta)|^2 < 2\varphi(0)[\varphi(0) - \Re\varphi(\xi - \eta)]$ *unless* $\xi = \eta$,

(ii) $|\varphi(0)\varphi(\xi + \eta) - \varphi(\xi)\varphi(\eta)|^2 < [\varphi(0)^2 - |\varphi(\xi)|^2][\varphi(0)^2 - |\varphi(\eta)|^2]$ *unless* $\xi = 0$ *or* $\eta = 0$ *or* $\xi + \eta = 0$,

(iii) $|\sum_{k=1}^K c_k \varphi(-\xi_k)|^2 < \varphi(0) \sum_{j,k=1}^K \bar{c}_j c_k \varphi(\xi_j - \xi_k)$ *unless* $K = 1$ *and* $\xi_1 = 0$,

(iv) $|\sum_{j=1}^J \sum_{k=1}^K \bar{d}_j c_k \varphi(\eta_j - \xi_k)|^2 < \left(\sum_{j,k=1}^K \bar{c}_j c_k \varphi(\xi_j - \xi_k)\right)\left(\sum_{j,k=1}^J \bar{d}_j d_k \varphi(\eta_j - \eta_k)\right)$ *unless* $J = K, \xi_k = \eta_k$ *and* $c_k = \alpha d_k$ *for a number* $\alpha \in \mathbb{C} \setminus \{0\}$.

Proof. Let $u = \sum_{k=1}^K c_k \varphi_{\xi_k}$ and $v = \sum_{j=1}^J d_j \varphi_{\eta_j}$ be two functions in the reproducing kernel Hilbert space H associated with φ. Then we have by the Cauchy-Schwartz inequality

$$|(u, v)|^2 \leq \|u\|^2 \|v\|^2 \iff |\sum_{j=1}^J \sum_{k=1}^K \bar{d}_j c_k \varphi(\eta_j - \xi_k)|^2 \leq \Big(\sum_{j,k=1}^K \bar{c}_j c_k \varphi(\xi_j - \xi_k)\Big)\Big(\sum_{j,k=1}^J \bar{d}_j d_k \varphi(\eta_j - \eta_k)\Big)$$

and equality holds if and only u and v are linearly dependent that is, $u = \alpha v$ for a complex number $\alpha \neq 0$ since u and v are different from 0. But this is equivalent to say that

$$\sum_{k=1}^K c_k w(\xi_k) = \sum_{j=1}^J (\alpha d_j) w(\eta_j) \text{ for } w \in H_0.$$

Suppose there exists ξ_{k_0} in $\{\xi_1, \ldots, \xi_K\}$ but not in $\{\eta_1, \ldots, \eta_J\}$ and pick up a function $w \in H_0$ vanishing on $\{\xi_1, \ldots, \xi_K, \eta_1, \ldots, \eta_J\} \setminus \{\xi_{k_0}\}$ and such that $w(\xi_{k_0}) = 1$. This leads us to the contradiction $c_{k_0} = 0$. Similarly, if there exists η_{j_0} in $\{\eta_1, \ldots, \eta_J\}$ but not in $\{\xi_1, \ldots, \xi_K\}$, we would obtain $\alpha d_{j_0} = 0$ and hence $d_{j_0} = 0$. We thus have $\{\xi_1 < \ldots < \xi_K\} = \{\eta_1 < \ldots < \eta_J\}$ which shows that

$$\sum_{k=1}^{K} c_k w(\xi_k) = \sum_{k=1}^{K} \alpha d_k w(\xi_k) \text{ for } w \in H_0.$$

Choosing $w \in H_0$ vanishing on $\{\xi_1, \ldots, \xi_K\} \setminus \{\xi_k\}$ and such that $w(\xi_k) = 1$, we conclude that $c_k = \alpha d_k$ for any $k = 1, \ldots, K$.

Setting $J = 1, \eta_1 = 0, d_1 = 1$ in (*iv*), we get (*iii*). The inequality (*i*) follows from (*iii*) by setting $K = 2, \xi_1 = -\xi, \xi_2 = -\eta, c_1 = 1, c_2 = -1$.

To prove (ii) we first remark that if a matrix of the form

$$\begin{pmatrix} 1 & a & b \\ \bar{a} & 1 & c \\ \bar{b} & \bar{c} & 1 \end{pmatrix}$$

is positive definite then its determinant is positive. Computing this determinant we obtain $1 + a\bar{b}c + \bar{a}b\bar{c} > |a|^2 + |b|^2 + |c|^2$ or, equivalently $|c - \bar{a}b|^2 < (1 - |a|^2)(1 - |b|^2)$. Assume first that $\varphi(0) = 1$. Applying this last inequality to the matrix

$$(\varphi(\xi_j - \xi_k))_{j,k=1}^3 = \begin{pmatrix} 1 & \varphi(-\xi) & \varphi(\eta) \\ \varphi(\xi) & 1 & \varphi(\xi + \eta) \\ \varphi(-\eta) & \varphi(-\xi - \eta) & 1 \end{pmatrix}$$

where $\xi_1 = 0, \xi_2 = \xi$ and $\xi_3 = -\eta$ are pairwise distinct, we obtain (ii). The case $\varphi(0) \neq 1$ can be reduced to the case $\varphi(0) = 1$ by considering $\varphi/\varphi(0)$.

□

6. Sufficient conditions

6.1. Fourier transform of a nonnegative finite continuous measure

Proposition 6.1. *If $\varphi = \hat{\mu} \in \mathcal{P}$ and $\operatorname{supp} \mu$ is not a discrete set then $\varphi \in \mathcal{P}^s$.*

Proof. If $t \in \mathcal{T}$ vanishes on $\operatorname{supp} \mu$ which contains an accumulation point, it must vanish identically as an analytic function on $\mathbb{R}$. □

We recall that a measure $\mu \in M^+$ is called atomic if $\mu = \sum a_n \delta_{x_n}$ ($a_n \geq 0$ and $\sum a_n < +\infty$) and continuous if $\mu(X) = 0$ for every countable set $X \subset \mathbb{R}$. Every measure $\mu \in M^+$ can be uniquely decomposed to a sum $\mu = \mu_c + \mu_{at}$ where μ_c is continuous and μ_{at} is atomic.

Corollary 6.1. *If $\mu \in M^+$ is not atomic then $\hat{\mu} \in \mathcal{P}^s$.*

Proof. It is enough to show that $X = \operatorname{supp} \mu$ is not a discrete set. Let $\mu_c \neq 0$ be the continuous part and suppose on the contrary that X is discrete. Since X is countable we have $\mu(X) = 0$ and therefore $\mu_c(\mathbb{R}) = \mu_c(X) + \mu_c(\mathbb{R} \setminus X) = 0$ i.e. $\mu_c = 0$, a contradiction. □

Theorem 6.1. *Let $\varphi = \hat{\mu} \in \mathcal{P}$. Then $\varphi \in AP(\mathbb{R})$ if and only if μ is atomic.*

Proof. If $\mu = \sum a_n \delta_{x_n}$, then $\hat{\mu}(\xi) = \sum a_n e^{i\xi x_n}$ is almost periodic. Conversely, suppose that $\varphi \in AP(\mathbb{R})$ is almost periodic and let use the decomposition $\mu = \mu_c + \mu_{at}$. It results that $\hat{\mu}_c = \varphi - \hat{\mu}_{at}$ is also almost periodic. From the inversion formula, we have

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ix\xi} \hat{\mu}_c(\xi)\, d\xi = \mu_c(\{x\}) = 0 \text{ for } x \in \mathbb{R}$$

which shows that $\hat{\mu}_c = 0$ and hence $\mu_c = 0$. □

Corollary 6.2. *If $\varphi = \hat{\mu} \in \mathcal{P}$ is not almost periodic, in particular if* $\limsup_{|\xi|\to+\infty} |\varphi(\xi)| < \varphi(0)$, *then $\varphi \in \mathcal{P}^s$.*

6.2. Derivatives

Theorem 6.2. *Let $\varphi = \hat{\mu}$ and $\psi = \hat{\nu}$ with $\mu, \nu \in M^+$. Suppose there exists a point x_0 such that $\operatorname{supp}\mu \setminus \{x_0\} \subset \operatorname{supp}\nu$. Then $\varphi \in \mathcal{P}^s \implies \psi \in \mathcal{P}^s$.*

Proof. For $t \in \mathcal{T}$, we have also $t(\cdot)\sin(\cdot - x_0) \in \mathcal{T}$ and therefore

$$\begin{aligned} t = 0 \text{ on } \operatorname{supp}\nu &\implies t = 0 \text{ on } \operatorname{supp}\mu \setminus \{x_0\} \implies t(\cdot)\sin(\cdot - x_0) = 0 \text{ on } \operatorname{supp}\mu \\ &\implies t(\cdot)\sin(\cdot - x_0) = 0 \implies t = 0. \end{aligned}$$

□

Corollary 6.3. *Let $\varphi = \hat{\mu} \in \mathcal{P}$.*

(i) *If $\operatorname{supp}\mu \subset [0, +\infty)$ and $x\mu \in M^+$, then $\varphi \in \mathcal{P}^s \iff i\varphi' \in \mathcal{P}^s$.*
(ii) *If $x^2\mu \in M^+$, then $\varphi \in \mathcal{P}^s \iff -\varphi'' \in \mathcal{P}^s$*

Proof. By the hypothesis on the moment of the measure μ, it follows that $\varphi \in C^1$ with $i\varphi' = \widehat{x\mu}$ in the first case and $\varphi \in C^2$ with $-\varphi'' = \widehat{x^2\mu}$ in the second case. It suffices now to apply theorem 6.2 firstly with $\nu = x\mu$ and secondly with $\nu = x^2\mu$ since in both cases, we have $\operatorname{supp}\mu \setminus \{0\} \subset \operatorname{supp}\nu \subset \operatorname{supp}\mu$. □

Remark 6.1. In fact, it is well known that $x^2\mu \in M^+$ is equivalent to $\varphi \in C^2$ [11, p. 21–22].

For a function $\phi \in C^n$, $n \geq 1$, its Maclaurin series of degree $n-1$ with integral remainder is given by

$$\phi(\xi) = \sum_{k=0}^{n-1} \frac{\xi^k}{k!} \varphi^{(k)}(0) + \frac{\xi^n}{(n-1)!} \int_0^1 \varphi^{(n)}(\xi x)(1-x)^{n-1}\, dx.$$

We can therefore define the continuous function $T_n\varphi$ by the formula

$$T_n\varphi(\xi) = \begin{cases} \dfrac{\phi(\xi) - \sum_{k=0}^{n-1} \frac{\xi^k}{k!}\varphi^{(k)}(0)}{\xi^n} & \text{if } \xi \neq 0, \\ \dfrac{\varphi^{(n)}(0)}{n!} & \text{if } \xi = 0. \end{cases}$$

Corollary 6.4. *Let $\varphi = \hat{\mu}$ be a nonconstant function in $\mathcal{P}$ and n a positive integer.*

(i) *If $\operatorname{supp}\mu \subset [0, +\infty)$ and $x^{2n-1}\mu \in M^+$, then $(-1)^{n-1} iT_{2n-1}\varphi \in \mathcal{P}^s$.*

(ii) *If* $x^{2n}\mu \in M^+$*, then* $(-1)^n T_{2n}\varphi \in \mathcal{P}^s$*.*

Proof. Let us prove the first item, the same reasoning is valid for the second one. We have $\varphi \in C^{2n-1}$ and $\varphi^{(2n-1)} = (-i)^{2n-1}\widehat{x^{2n-1}\mu}$, hence the function $(-1)^{n-1} i\varphi^{(2n-1)}$ is in $\mathcal{P}$. Furthermore this function is noncontant otherwise φ should be a polynomial and hence a constant function since it is bounded. With the help of corollary 6.6, the proof is done.

□

The same proof with φ' in place of φ gives the next result.

Corollary 6.5. *Let* $\varphi = \hat{\mu}$ *be a nonconstant function in* $\mathcal{P}$ *and* n *a positive integer.*

(i) *If* $x^{2n}\mu \in M^+$*, then* $(-1)^n T_{2n-1}\varphi' \in \mathcal{P}^s$*.*
(ii) *If* $\operatorname{supp}\mu \subset [0, +\infty)$ *and* $x^{2n+1}\mu \in M^+$*, then* $(-1)^{n-1} iT_{2n}\varphi' \in \mathcal{P}^s$*.*

6.3. Integration against a nonnegative measure

Theorem 6.3. *Let* $\psi \in \mathcal{P}^s$ *and* $\mu \in M^+$ *such that* $\mu(\mathbb{R} \setminus \{0\}) > 0$*. Then the function* φ *given by*

$$\varphi(\xi) = \int_{\mathbb{R}} \psi(\xi x)\, d\mu(x)$$

is in $\mathcal{P}^s$*.*

Proof. We remark that the integral is well defined since the function ψ is bounded. For an arbitrary vector $c = (c_1, \dots, c_K) \in \mathbb{C}^K$ and arbitrary real numbers $\xi_1, \dots, \xi_K$, we have

$$\sum_{j=1}^{K}\sum_{k=1}^{K} \bar{c}_j c_k \varphi(\xi_j - \xi_k) = \int_{\mathbb{R}} \sum_{j=1}^{K}\sum_{k=1}^{K} \bar{c}_j c_k \psi((\xi_j - \xi_k)x)\, d\mu(x).$$

This integral is nonnegative since ψ is positive definite. If it vanishes, the integrand equals 0 on $\operatorname{supp}\mu$ by lemma 2.1 and hence there exists an $a \neq 0$ such that $\sum_{j=1}^{K}\sum_{k=1}^{K} \bar{c}_j c_k \psi((\xi_j - \xi_k)a) = 0$ which implies $c = 0$ since the function $\psi(a\cdot)$ is in $\mathcal{P}^s$ by theorem 3.2. □

As an application we give two important classes of strictly positive definite functions. Let $\mu \in M^+$ be a measure not concentrated at 0.

(i) The function $\psi(\xi) = e^{-\xi^2}$ is in $\mathcal{P}^s$ since

$$e^{-\xi^2} = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} e^{-x^2/4} e^{-i\xi x}\, dx,$$

hence the function

$$\varphi(\xi) = \int_0^{+\infty} e^{-\xi^2 x^2}\, d\mu(x)$$

is strictly positive definite. By Bernstein–Widder theorem [17, Th. 12a, p. 160], a function admits such an integral representation if and only if $\varphi(\sqrt{\cdot})$ is not constant and completely monotone on $[0, +\infty)$ meaning that $\varphi \in C[0, \infty) \cap C^\infty(0, \infty)$ and $(-1)^k \varphi^{(k)}(\sqrt{\xi}) \geq 0$ for all $k \in \mathbb{N}_0$ and all $\xi > 0$.

(ii) We consider now the function $\psi(\xi) = (1 - |\xi|)_+$ where x_+ denotes the greater of x and 0. As before the function

$$\varphi(\xi) = \int_0^{+\infty} (1 - |\xi x|)_+ \, d\mu(x)$$

is strictly positive definite since we have

$$(1 - |\xi|)_+ = \frac{1}{2\pi} \int_{\mathbb{R}} \operatorname{sinc}^2(x/2) e^{-i\xi x} \, dx.$$

In this case, we describe the class of even continuous functions which are nonconstant, nonnegative, bounded and convex on $(0, +\infty)$ (see [11, p. 87]).

Corollary 6.6. *Let ψ be a nonconstant function in $\mathcal{P}$ and $\mu \in M^+$ such that $\hat{\mu} \in \mathcal{P}^s$. Then the function*

$$\varphi(\xi) = \int_{\mathbb{R}} \psi(\xi x) \, d\mu(x)$$

is in $\mathcal{P}^s$.

Proof. By assumption on ψ there exists $\nu \in M^+$ such that $\psi = \hat{\nu}$ and $\nu(\mathbb{R} \setminus \{0\}) > 0$. The Fubini's theorem gives next

$$\varphi(\xi) = \int_{\mathbb{R}} \hat{\nu}(\xi x) \, d\mu(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi xy} \, d\nu(y) d\mu(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi yx} \, d\mu(x) d\nu(y) = \int_{\mathbb{R}} \hat{\mu}(\xi y) \, d\nu(y),$$

hence the proof is done by theorem 6.3. □

Corollary 6.7. *Let ψ be a nonconstant function in $\mathcal{P}^s$ and $\alpha > 0$ a real number. The function*

$$\varphi(\xi) = \frac{\alpha}{\xi^\alpha} \int_0^\xi \psi(x) x^{\alpha - 1} \, dx$$

is in $\mathcal{P}^s$.

Proof. The corollary follows by putting $d\mu(x) = \alpha x^{\alpha-1} \chi_{(0,1)}(x) dx$ in corollary 6.6. □

6.4. Limit at infinity

Theorem 6.4. *Let $\varphi = \hat{\mu} \in \mathcal{P}$. If $\lim_{|\xi| \to +\infty} \varphi(\xi) = a$ in $\mathbb{C}$, then $a \geq 0$ and $\varphi - a \in \mathcal{P}$. Furthermore, we have $a < \varphi(0) \iff \varphi \in \mathcal{P}^s \iff \varphi - a \in \mathcal{P}^s$.*

Proof. From the inversion formula [11, p. 35], we have for every real x

$$\mu(\{x\}) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(\xi) e^{ix\xi} \, d\xi.$$

From the hypothesis $\lim_{|\xi| \to +\infty} \varphi(\xi) = a$, we obtain

$$\mu(\{x\}) = \begin{cases} a & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Hence we have $a \geq 0$ and $\mu = a\delta_0 + \mu_c$ where μ_c is a continuous measure in M^+. This shows that $\varphi - a = \hat{\mu}_c$ is positive definite. In particular, we have $|\varphi - a| \leq \varphi(0) - a$, hence

$$\varphi(0) = a \iff \varphi = a \iff \mu_c = 0$$

or equivalently

$$a < \varphi(0) \iff \mu_c \neq 0.$$

This is clearly equivalent to $\varphi \in \mathcal{P}^s$ or $\varphi - a \in \mathcal{P}^s$. □

6.5. Integrable positive definite functions

Theorem 6.5. *If $\varphi \in \mathcal{P} \cap L^p \setminus \{0\}$ with $0 < p < +\infty$ then $\varphi \in \mathcal{P}^s$.*

Proof. It is sufficient to show that $\varphi \in C_0$. Let $\alpha > 0$ and consider the closed set $F_\alpha = \{\xi \in \mathbb{R};\ |\varphi(\xi)| \geq 2\alpha\}$. Since φ is uniformly continuous, there exists a neighbourhood V_α of 0 such that $|\varphi(\xi)| \geq \alpha$ for every $\xi \in F_\alpha + V_\alpha$. If $\varphi \in L^p$, the Chebyshev's inequality

$$\lambda(\{\xi \in \mathbb{R};\ |\varphi(\xi)| \geq \alpha\}) \leq \alpha^{-p} \|\varphi\|_p^p$$

ensures that the set $F_\alpha + V_\alpha$ has a finite Lebesgue measure. Hence F_α is bounded otherwise there would exist a sequence $(\xi_n) \in F_\alpha$ such that $\{\xi_n + V_\alpha\} \cap \{\xi_{n+1} + V_\alpha\} \neq$ and we would have the contradiction

$$\lambda(F_\alpha + V_\alpha) \geq \sum_n \lambda(\xi_n + V_\alpha) = \sum_n \lambda(V_\alpha) = +\infty.$$

Finally we have proved that for any $\alpha > 0$, there is a compact set such that $|\varphi| < 2\alpha$ on its complement which means that $\varphi \in C_0$. □

Theorem 6.6. *Let $\varphi \in C \cap L^1$. Then $\varphi \in \mathcal{P}^s \iff \hat{\varphi} \geq 0$ and $\varphi \neq 0$.*

Proof. Suppose that $\varphi = \hat{\mu} \in \mathcal{P}^s$. We have evidently $\varphi \neq 0$ and the Riemann–Lebesgue lemma tells us that $\hat{\varphi} \in C_0$. For every test function u in the Schwartz space $\mathcal{S}$, we have

$$\begin{aligned}
\int_{\mathbb{R}} \hat{\varphi}(-x) u(x)\, dx &= \int_{\mathbb{R}} \varphi(-\xi) \hat{u}(\xi)\, d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi}\, d\mu(x) \hat{u}(\xi)\, d\xi \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix\xi} \hat{u}(\xi)\, d\xi d\mu(x) \\
&= 2\pi \int_{\mathbb{R}} u(x)\, d\mu(x).
\end{aligned}$$

Since μ is nonnegative, the last integral is nonnegative for every $u \geq 0$ in $\mathcal{S}$ which permits to conclude that $\hat{\varphi} \geq 0$. As μ is nonnegative, we conclude that $\hat{\varphi} \geq 0$. Inversely, if φ is integrable and continuous with a nonnegative Fourier transform, then $\hat{\varphi} \in L^1$ and we have the inversion formula [14, p. 15]

$$\varphi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varphi}(\xi) e^{ix\xi}\, d\xi.$$

This shows that $\varphi \in \mathcal{P}$. From the hypothesis $\varphi \in L^1$, we conclude that $\varphi \in \mathcal{P}^s$ provided that $\varphi \neq 0$. □

Remark 6.2. In the necessary part of the proof, we have also $\hat{\varphi} \in L^1$ and the equality

$$\int_{\mathbb{R}} \hat{\varphi}(-x)u(x)\,dx = 2\pi \int_{\mathbb{R}} u(x)\,d\mu(x)$$

remains valid for every function $u \in C_0$ by a density argument. It follows by the Riesz uniqueness theorem that $d\mu(x) = (2\pi)^{-1}\hat{\varphi}(-x)dx$, hence μ is absolutely continuous.

Before stating a result about convolution of positive definite functions, we give some lemma.

Lemma 6.1. *If $\varphi \in \mathcal{P}$ and $\rho \in \mathcal{D}$ then $\varphi * \rho * \tilde{\rho} \in \mathcal{P} \cap C^\infty$.*

Proof. The regularity of the convolution product is standart. Moreover it is known that a continuous function φ is positive definite if and only if $\langle \check{\varphi}, u * \tilde{u} \rangle \geq 0$ for every $u \in \mathcal{D}$. For such a function u, we have

$$\langle (\varphi * \rho * \tilde{\rho})\check{}, u * \tilde{u} \rangle = \langle \check{\varphi}, \rho * \tilde{\rho} * u * \tilde{u} \rangle = \langle \check{\varphi}, \rho * u * (\rho * u)\tilde{} \rangle \geq 0.$$

□

Lemma 6.2. *Let $\varphi \in \mathcal{P} \cap L^p$, $1 \leq p < +\infty$. There exists a sequence (φ_n) in $\mathcal{P} \cap \mathcal{S}$ which tends to φ in L^p.*

Proof. Suppose before that $\varphi \in C^\infty$. Then the function $\varphi_n(x) = \varphi(x)e^{-x^2/n}$ is in $\mathcal{P}$ as the product of two positive definite functions and belongs to $\mathcal{S}$ since φ is bounded and infinitely differentiable. By the dominated convergence theorem, the conclusion holds.
In the general case, let us prove that the function φ is the limit in L^p of a sequence belonging to $\mathcal{P} \cap L^p \cap C^\infty$. Let u a function in $\mathcal{D}$ such that $u \geq 0$ and $\int_{\mathbb{R}} u = 1$. Then $\rho = u * \tilde{u}$ shares the same properties as u and the sequence (ρ_n) defined by $\rho_n(x) = n\rho(nx)$ is then an approximate identity. By the preceding lemma, we have $\varphi_n = \varphi * \rho_n \in \mathcal{P} \cap C^\infty$ and it is well known that $\varphi_n \to \varphi$ in L^p.
□

Theorem 6.7. *Let $\varphi \in \mathcal{P} \cap L^p$ and $\psi \in \mathcal{P} \cap L^q$ with $1 < p, q < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $\varphi \star \psi \in \mathcal{P}^s$ if $\varphi \star \psi \neq 0$.*

Proof. It is known that $\varphi \star \psi \in C_0$ and we have the Young's inequality $\|\varphi \star \psi\|_\infty \leq \|\varphi\|_p \|\psi\|_q$. From theorem 6.4 it is enough to prove that $\varphi \star \psi \in \mathcal{P}$.
Suppose first that $\varphi, \psi \in \mathcal{S}$. Then $\varphi \star \psi \in \mathcal{S}$ and it is clearly in $\mathcal{P}$ since $\widehat{\varphi \star \psi} = \hat{\varphi}\hat{\psi} \geq 0$.
Now if $\varphi \in L^p$ and $\psi \in L^q$, $1 < p, q < +\infty$, lemma 6.2 tells us that there exists some sequences (φ_n) and (ψ_n) in $\mathcal{P} \cap \mathcal{S}$ such that $\varphi_n \to \varphi$ in L^p and $\psi_n \to \psi$ in L^q. But we know from above that $\varphi_n \star \psi_n \in \mathcal{P} \cap \mathcal{S}$. The Young's inequality says that the bilinear application $(\varphi, \psi) \mapsto \varphi \star \psi$ from $L^p \times L^q \to L^\infty$ is continuous and so $\varphi_n \star \psi_n \to \varphi \star \psi$ uniformly. From definition 2.1, we see easily that the uniform limit belongs to $\mathcal{P}$. □

Recall that a function $\varphi \in L^p \cap L^q$, $1 \leq p \leq q \leq +\infty$, belongs to L^r for $p \leq r \leq q$. This elementary fact permits to give a more general result.

Corollary 6.8. *Let $\varphi \in \mathcal{P} \cap L^p$ and $\psi \in \mathcal{P} \cap L^q$ with $1 \leq p, q < +\infty$ and $\frac{1}{p} + \frac{1}{q} \geq 1$. Then $\varphi \star \psi \in \mathcal{P}^s$ if $\varphi \star \psi \neq 0$.*

Proof. We know that a positive definite function is bounded then $\varphi \in L^{p'}$ and $\psi \in L^{q'}$ with $p' \geq p$ and $q' \geq q$. It is easy to verify that we can take in particular $\frac{1}{p'} = \frac{1}{2}\Big(1 + \frac{1}{p} - \frac{1}{q}\Big)$ and $\frac{1}{q'} = \frac{1}{2}\Big(1 + \frac{1}{q} - \frac{1}{p}\Big)$ so that the two exponents are conjugate and we can therefore apply the preceding theorem. □

7. Unicity sets

From corollary 6.1 we can restrict the problem of strict positive definiteness on atomic measures $\mu_{at} \in M^+$ and we obtain by proposition 2.1 that $\hat{\mu}_{at} \in \mathcal{P}^s$ if and only if every trigonometric polynomial vanishing on the countable set $X = \operatorname{supp} \mu_{at}$ vanishes identically on $\mathbb{R}$.

In the following we call such a set X a unicity set and we denote by $\mathcal{U}$ the class of all unicity sets.

Proposition 7.1. *Let $X \in \mathcal{U}$ and Y be a countable set. We have then*

(i) $\alpha X + \beta \in \mathcal{U}$ *for* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha \neq 0$,
(ii) $X \subset Y$ *implies* $Y \in \mathcal{U}$ *(hence* $X \cup Y \in \mathcal{U}$*)*,
(iii) $X + Y \in \mathcal{U}$.

Proof.

(i) Let $t \in \mathcal{T}$ and consider the transformation $\psi(x) = \alpha x + \beta$ from $\mathbb{R}$ to itself. It is clear that $t \circ \psi \in \mathcal{T}$. So we have $t \circ \psi(X) = \{0\} \implies t \circ \psi = 0$ and hence $t = 0$ since ψ is onto.
(ii) For $t \in \mathcal{T}$ we have $t(X) \subset t(Y)$ and hence $t(Y) = \{0\} \implies t(X) = \{0\} \implies t = 0$.
(iii) We can suppose Y nonempty otherwise the result is trivial. For $y \in Y$ we have $X + y \in \mathcal{U}$ by (i) and since $X + y$ is a subset of the countable set $X + Y$ then $X + Y \in \mathcal{U}$ by (ii).

□

Proposition 7.2. *$X \cup Y \in \mathcal{U}$ implies $X \in \mathcal{U}$ or $Y \in \mathcal{U}$.*

Proof. If $X \cup Y$ is countable so are X and Y. Suppose that X and Y are not in $\mathcal{U}$. Then there exists $t_1, t_2 \in \mathcal{P} \setminus \{0\}$ such that $t_1(X) = t_2(Y) = 0$. Hence $t = t_1 t_2 \in \mathcal{P} \setminus \{0\}$ and $t(X \cup Y) = t(X) \cup t(Y) = \{0\}$ which lead to $X \cup Y \notin \mathcal{U}$. □

Since the ring of trigonometric polynomials is an integral domain, we can easily construct unicity sets from knowing one.

Proposition 7.3. *Let $X = \{x_n, n \in \mathbb{N}\} \in \mathcal{U}$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq 0$. Then the set $Y = \{y_n, n \in \mathbb{N}\}$ where $y_n \in \{x_n, \alpha x_n + \beta\}$ is also of unicity.*

Proof. Let $t \in \mathcal{T}$ such that $t(Y) = \{0\}$ i.e. $t(x_n) = 0$ or $t(\alpha x_n + \beta) = 0$. The trigonometric polynomial $t \cdot t \circ \psi$ where $\psi(x) = \alpha x + \beta$ vanishes on $X \in \mathcal{U}$ and hence on $\mathbb{R}$. We have then $t = 0$ or $t \circ \psi = 0$ which means that $t = 0$. □

We give now trivial sets which are not of unicity for $\mathcal{T}$.

Proposition 7.4.

(i) *If* $X_k = \alpha_k \mathbb{Z} + \beta_k$ *where* $\alpha_k, \beta_k \in \mathbb{R}$ *then* $\bigcup_{k=1}^{n} X_k \notin \mathcal{U}$,
(ii) *Any finite set F is not in $\mathcal{U}$.*

Proof.

(i) Since $\sin(\pi \mathbb{Z}) = 0$ then $\mathbb{Z} \notin \mathcal{U}$ and hence $\alpha_1 \mathbb{Z} + \beta_1 \notin \mathcal{U}$ by proposition 7.1. The conclusion follows inductively by proposition 7.2.

(ii) Any finite set F is a subset of $\bigcup_{\alpha\in F}\alpha\mathbb{Z}$ which is not in $\mathcal{U}$, therefore $F \notin \mathcal{U}$.

$\square$

Corollary 7.1. *$X \setminus F \in \mathcal{U}$ whenever $X \in \mathcal{U}$ and F is finite.*

Proof. Write $X = X \setminus F \cup F$ and use propositions 7.2 and 7.4. $\square$

If X is a set of reals numbers, we denote by $\{X\}$ the subset $\{\{x\}, x \in X\}$ of $[0, 1)$ where $\{x\}$ is the fractional part of x.

Corollary 7.2. *If $X \in \mathcal{U}$ then for every $\alpha \neq 0$ the set $\{\alpha X\}$ has an accumulation point in* $[0, 1]$.

Proof. It is enough to consider $\alpha = 1$ since $X \in \mathcal{U} \implies \alpha X \in \mathcal{U}$ for every $\alpha \neq 0$. If it is not the case, $(\{X\})$ is a finite set F and we have then $X \subset \bigcup_{\beta\in F}\mathbb{Z}+\beta$ which is not in $\mathcal{U}$ by proposition 7.4. $\square$

Remark 7.1. The set $X = \mathbb{N} \cup \pi\mathbb{N}$ is such that $\{\alpha X\}$ is dense in $[0, 1]$ for every $\alpha \neq 0$ but is manifestly not in $\mathcal{U}$.

8. Thick set, syndetic set, Hartman sequences

Definition 8.1. Let E be a set in $\mathbb{N}$.

(i) E is called thick if it contains arbitrarily long intervals i.e. for every $k \in \mathbb{N}$, there exists $n \in E$ such that $[n, n + k] \subset E$,
(ii) E is called syndetic if it has bounded gaps i.e. there exists $k \in \mathbb{N}$ such that $E \cap [n, n+k] \neq \emptyset$ for every $n \in \mathbb{N}$

Let us now define the notion of Hartman sequences.

Definition 8.2. A sequence (x_n) in $\mathbb{R}$ is Hartman-uniformly distributed (H-u.d.) in $\mathbb{R}$ if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i t x_n} = 0 \text{ for every } t \neq 0.$$

A sequence (x_n) in $\mathbb{Z}$ is Hartman-uniformly distributed (H-u.d.) in $\mathbb{Z}$ if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i t x_n} = 0 \text{ for every } t \in \mathbb{R} \setminus \mathbb{Z}.$$

Similarly, we can define the notion of Hartman-well distributed (H-w.d.) sequence by requiring that the limit in the summation $\sum_{n=1+M}^{N+M}$ is uniform in $M \in \mathbb{N}$.

Remark 8.1. By using the Weyl criterion [8, Theorem 2.1, p. 7], we observe that a sequence (x_n) is H-u.d. in $\mathbb{R}$ if and only if the sequence (tx_n) is uniformly distributed modulo 1 for every real $t \neq 0$.

Let G denote the group $\mathbb{R}$ or $\mathbb{Z}$. If f is an almost periodic (a.p.) function on G, it admits a mean value denoted by $M(f)$. For $G = \mathbb{R}$, we have

$$M(f) = \lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} f(x)\,dx \quad \text{uniformly in } a \in \mathbb{R},$$

whereas for $G = \mathbb{Z}$, the following identity holds

$$M(f) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=n+1}^{n+N} f(k) \quad \text{uniformly in } n \in \mathbb{Z}.$$

The following result gives rise to an interesting relation between H-u.d. sequences and almost periodic functions on G [8, p. 298].

Theorem 8.1. *A sequence (x_n) is H-u.d. in G if and only if for every a.p. function f on G, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = M(f).$$

Similarly, a sequence (x_n) is H-w.d. in G if and only if for every a.p. function f on G, we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1+M}^{N+M} f(x_n) = M(f) \text{ uniformly in } M \in \mathbb{N}.$$

Corollary 8.1. *Let f be an a.p. function on G and E a thick set in $\mathbb{N}$ which contains intervals I_N with $\operatorname{card} I_N \to +\infty$. Then f vanishes identically on G whenever one of the two following conditions is satisfied*

(i) *(x_n) is H-u.d. in G and* $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |f(x_n)| = 0$,

(ii) *(x_n) is H-w.d. in G and* $\lim_{N\to\infty} \frac{1}{\operatorname{card} I_N}\sum_{n\in I_N} |f(x_n)| = 0$.

Proof. It is enough to use the fact that $M(|f|) = 0$ implies $f = 0$. □

Example 8.1. The following sequences are Hartman-u.d. in $\mathbb{R}$ [8] :

(i) $(n^\alpha \log^\beta n)_{n\geq 2}$ with $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$ and $\beta \in \mathbb{R}$; (ii) $(n^k \log^\beta n)_{n\geq 2}$ with $k \in \mathbb{N}$ and $\beta \neq 0$; (iii) $(\log^\beta n)$ with $\beta > 1$; (iv) $(n^k \log\log n)_{n\geq 2}$ with $k \geq 1$.

The following theorem provides many examples of H-u.d. sequences in $\mathbb{Z}$ [12, Theorem 1].

Theorem 8.2. *If (x_n) is H-u.d. in $\mathbb{R}$ then the sequence $([x_n])$ of integral parts is H-u.d. in $\mathbb{Z}$.*

Example 8.2. Let $p(x) = a_0 + a_1 x + \cdots + a_k x^k$ be a polynomial over $\mathbb{R}$ of degree at least 2 and suppose that $a_1, \cdots a_k$ do not lie in a singly generated additive subgroup of the reals. Then the sequence $(p(n))$ is H-w.d. in $\mathbb{R}$ [8, Ch. 4, Example 5.4] and the sequence $([p(n)])$ is H-w.d. in $\mathbb{Z}$ [15]

Let us notice that the sequence $(p(n))$ is proved to be H-u.d. in [8] but it is even H-w.d. as a consequence of a result of Lawton [9].

9. Some unicity sets

We will use a primarily result in the sequel.

Lemma 9.1. *Let α be a real number, $f \in AP(\mathbb{R})$ and $\psi \in AP_{\mathbb{R}}(\mathbb{R})$. Then the function g defined by $g(y) = f(\alpha y + \psi(y))$ belongs to $AP(\mathbb{R})$.*

Proof. The function $e^{i\psi}$ is a.p. since e^{iz} is uniformly continuous on every compact subset of $\mathbb{C}$ and ψ is bounded [5, Th. 1.7]. Hence the function $e^{i(\alpha y+\psi(y))}$ is a.p. as a product of two a.p. functions. If $t(y) = \sum_{k=1}^{K} c_k e^{ix_k y} \in \mathcal{T}$ then $t(\alpha y + \psi(y)) = \sum_{k=1}^{K} c_k e^{ix_k(\alpha y+\psi(y))}$ is a.p. as a finite sum of a.p. functions. But f being a.p. is the uniform limit of trigonometric polynomials (t_n). Hence g is also a.p. as the uniform limit of the sequences of a.p. functions $(t_n(\alpha y + \psi(y))$. $\square$

We give first some results for H-u.d. sequences in $\mathbb{R}$.

Theorem 9.1. *Let $\alpha \in \mathbb{R} \setminus \{0\}$, $\psi \in AP_{\mathbb{R}}(\mathbb{R})$, (y_n) a H-u.d. sequence in $\mathbb{R}$ and (ε_n) a real sequence such that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\varepsilon_n| = 0$. Then $X = \{x_n = \alpha y_n + \psi(y_n) + \varepsilon_n, n \in \mathbb{N}\}$ is in $\mathcal{U}$.*

Proof. We assume $\alpha > 0$ without loss of generality. Let $t \in \mathcal{T}$ such that $t(X) = 0$ and set $z_n = \alpha y_n + \psi(y_n)$. We have

$$0 = t(x_n) = t(z_n + \varepsilon_n) = t(z_n) + t'(\theta_n)\varepsilon_n \text{ with } \theta_n \in \mathbb{R}.$$

Since t' is bounded, we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |t(z_n)| = 0,$$

$$i.e. \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |g(y_n)| = 0 \text{ where } g(y) = t(\alpha y + \psi(y)).$$

Since the function $g(y) = t(\alpha y + \psi(y))$ is a.p. we deduce from corollary 8.1 that g vanishes identically on $\mathbb{R}$. As the function $h(y) = \alpha y + \psi(y)$ is continuous on $\mathbb{R}$, $h(-\infty) = -\infty$ and $h(+\infty) = +\infty$, we have $h(\mathbb{R}) = \mathbb{R}$ and hence $t = 0$ $\square$

Remark 9.1. We can take n in a thick set E provided that (y_n) is H-w.d. and $\lim_{n\to\infty, n\in E} \varepsilon_n = 0$.

We now give results for H-u.d. sequences in $G = \mathbb{Z}$ or $\mathbb{R}$.

Theorem 9.2. *Let $\alpha \in \mathbb{R}$, $\psi \in AP_{\mathbb{R}}(G)$, (y_n) a Hartman sequence in G and (ε_n) a real sequence such that $\varepsilon_n \neq 0$ and $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\varepsilon_n| = 0$. Then $X = \{x_n = \alpha y_n + \psi(y_n) + \varepsilon_n, n \in \mathbb{N}\}$ is in $\mathcal{U}$.*

Proof. Let $t \in \mathcal{T}$ such that $t(X) = 0$ and set $z_n = \alpha y_n + \psi(y_n)$. We have

$$0 = t(x_n) = t(z_n + \varepsilon_n) = t(z_n) + t'(\theta_n)\varepsilon_n \text{ with } \theta_n \in \mathbb{R}.$$

Since t' is bounded, we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |t(z_n)| = 0,$$

$$i.e. \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |g(y_n)| = 0 \text{ where } g(y) = t(\alpha y + \psi(y)), y \in G.$$

Since the function g is a.p. on G, we deduce from corollary 8.1 that $t(\alpha y + \psi(y)) = 0$ for any $y \in G$. In particular, we have $t(z_n) = 0$ for any $n \in \mathbb{N}$. Let us now suppose that $t^{(l)}(\alpha y + \psi(y)) = 0$ for $y \in G$ and $0 \leq l \leq L$. The Taylor formula gives

$$0 = t(x_n) = \frac{t^{(L+1)}(z_n)}{(L+1)!}\varepsilon_n^{L+1} + \frac{t^{(L+2)}(\theta_n)}{(L+2)!}\varepsilon_n^{L+2}$$

with $\theta_n \in \mathbb{R}$ and dividing by $\varepsilon_n^{L+1} \neq 0$, we get

$$0 = \frac{t^{(L+1)}(z_n)}{(L+1)!} + \frac{t^{(L+2)}(\theta_n)}{(L+2)!}\varepsilon_n.$$

Since the function $t^{(L+2)}$ is bounded, it follows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |t^{(L+1)}(z_n)| = 0,$$

$$i.e. \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |g(y_n)| = 0 \text{ where } g(y) = t^{(L+1)}(\alpha y + \psi(y)), y \in G.$$

The function g being a.p. on G, we obtain from corollary 8.1 that $t^{(L+1)}(\alpha y + \psi(y)) = 0$ for any $y \in G$. We conclude that $t^{(l)}(\alpha y + \psi(y)) = 0$ for any $y \in G$ and $l \in \mathbb{N}_0$, hence $t = 0$ as t is analytic. □

Remark 9.2. We have the same remark as before.

It is easy to show that the sequence (n^2) is not H-u.d. in $\mathbb{Z}$. In the following, we will give a result for polynomials.

The following result is classic in diophantine approximation [6, p. 31].

Lemma 9.2. *Let $p_1, \ldots, p_K$ be real polynomials. For every $\varepsilon > 0$, the set of positive integers n such that*

$$\max_{1\leq k\leq K} |e^{ip_k(n)} - e^{ip_k(0)}| < \varepsilon$$

is syndetic.

Lemma 9.3. *Let $\varepsilon > 0$, $f \in AP(\mathbb{R})$ and p a real polynomial. For every $n_0 \in \mathbb{N}$, the set of positive integers n such that*

$$|f(p(n)) - f(p(n_0))| < \varepsilon$$

is syndetic.

Proof. Let $n_0 \in \mathbb{N}$ and $\varepsilon > 0$.

(i) By lemma 9.2 the set

$$E = \{n \in \mathbb{N} : \max_{1\leq k\leq K} |e^{ip_k(n+n_0)} - e^{ip_k(n_0)}| < \varepsilon\}$$

is syndetic and so is the set

$$E + n_0 = \{n \in \mathbb{N} : \max_{1\leq k\leq K} |e^{ip_k(n)} - e^{ip_k(n_0)}| < \varepsilon\}.$$

(ii) Let $t(x) = \sum_{k=1}^{K} c_k e^{i\lambda_k x} \in \mathcal{T}$. Setting $C = \sum_{k=1}^{K} |c_k|$, we have

$$|t(p(n)) - t(p(n_0))| = |\sum_{k=1}^{K} c_k e^{i\lambda_k p(n)} - \sum_{k=1}^{K} c_k e^{i\lambda_k p(n_0)}| \leq C \max_{1\leq k\leq K} |e^{i\lambda_k p(n)} - e^{i\lambda_k p(n_0)}|.$$

Hence the set $\{n \in \mathbb{N} : |t(p(n)) - t(p(n_0))| < \varepsilon\}$ contains the syndetic set

$$\{n \in \mathbb{N} : C \max_{1\leq k\leq K} |e^{i\lambda_k p(n)} - e^{i\lambda_k p(n_0)}| < \varepsilon\}.$$

(iii) Let $t \in \mathcal{T}$ such that $|f(x) - t(x)| < \frac{\varepsilon}{3}$ for every $x \in \mathbb{R}$. We have

$$\begin{aligned}|f(p(n)) - f(p(n_0))| &\leq |f(p(n)) - t(p(n))| + |t(p(n)) - t(p(n_0))| + |t(p(n_0)) - f(p(n_0))| \\ &< \frac{\varepsilon}{3} + |t(p(n)) - t(p(n_0))| + \frac{\varepsilon}{3}.\end{aligned}$$

Hence the set $\{n \in \mathbb{N} : |f(p(n)) - f(p(n_0))| < \varepsilon\}$ contains the set $\{n \in \mathbb{N} : |t(p(n)) - t(p(n_0))| < \frac{\varepsilon}{3}\}$ which is syndetic.

□

Lemma 9.4. *Let $f \in AP(\mathbb{R})$ and p a real polynomial such that $\lim_{n\to\infty, n\in E} f(p(n)) = 0$ where E is a thick set of $\mathbb{N}$. Then $f \circ p$ vanishes on $\mathbb{N}$.*

Proof. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that

$$|f(p(n))| < \frac{\varepsilon}{2} \text{ for every } n \in E' = \{n \in E : n > N\}$$

and a syndetic set E'' such that

$$|f(p(n)) - f(p(n_0))| < \frac{\varepsilon}{2} \text{ for every } n \in E''.$$

Since E' is thick, the set $E' \cap E''$ is nonempty i.e. there exists an integer n which verifies $|f(p(n_0))| \leq |f(p(n))| + |f(p(n)) - f(p(n_0))| < \varepsilon$. Hence $f(p(n_0)) = 0$ since ε is arbitrary. □

Theorem 9.3. *Let $\alpha \in \mathbb{R}$, $\psi \in AP_{\mathbb{R}}(\mathbb{R})$, p a real polynomial and (ε_n) a real sequence such that $\varepsilon_n \neq 0, n \in E$ and $\lim_{n\to\infty, n\in E} \varepsilon_n = 0$ where E is a thick set of $\mathbb{N}$. Then $X = \{x_n = \alpha p(n) + \psi(p(n)) + \varepsilon_n, n \in E\}$ is in $\mathcal{U}$.*

Proof. Let $t \in \mathcal{T}$ such that $t(X) = 0$ and set $z_n = \alpha p(n) + \psi(p(n)), n \in E$. We have for $n \in E$,

$$0 = t(x_n) = t(z_n + \varepsilon_n) = t(z_n) + t'(\theta_n)\varepsilon_n \text{ with } \theta_n \in \mathbb{R}.$$

Since t' is bounded, we get

$$\lim_{n\to\infty, n\in E} t(z_n) = 0,$$

$$i.e. \lim_{n\to\infty, n\in E} g(p(n)) = 0 \text{ where } g(y) = t(\alpha y + \psi(y)), y \in \mathbb{R}.$$

Since the function g is a.p. on $\mathbb{R}$, we have from lemma 9.4 that $g \circ p = 0$ on $\mathbb{N}$ i.e. $t(\alpha p(n) + \psi(p(n)) = 0$ for any $n \in \mathbb{N}$ and in particular $t(z_n) = 0$ for any $n \in E$.

Let us now suppose that $t^{(l)}(z_n) = 0$ for $n \in \mathbb{N}$ and $0 \leq l \leq L$. The Taylor formula gives

$$0 = t(x_n) = \frac{t^{(L+1)}(z_n)}{(L+1)!}\varepsilon_n^{L+1} + \frac{t^{(L+2)}(\theta_n)}{(L+2)!}\varepsilon_n^{L+2}$$

with $\theta_n \in \mathbb{R}$ and dividing by $\varepsilon_n^{L+1} \neq 0$, we get

$$0 = \frac{t^{(L+1)}(z_n)}{(L+1)!} + \frac{t^{(L+2)}(\theta_n)}{(L+2)!}\varepsilon_n.$$

Since the function $t^{(L+2)}$ is bounded, it follows that

$$\lim_{n\to\infty, n\in E} |t^{(L+1)}(z_n)| = 0,$$

$$i.e. \lim_{n\to\infty, n\in E} g(p(n)) = 0 \text{ where } g(y) = t^{(L+1)}(\alpha y + \psi(y)), y \in \mathbb{R}.$$

The function g being a.p. on $\mathbb{R}$, we have from lemma 9.4 that $g \circ p = 0$ on $\mathbb{N}$ i.e. $t^{(L+1)}(\alpha p(n) + \psi(p(n)) = 0$ for any $n \in \mathbb{N}$ and in particular $t^{(L+1)}(z_n) = 0$ for any $n \in E$. We conclude that $t^{(l)}(z_n) = 0$ for any $n \in E$ and $l \in \mathbb{N}_0$, hence $t = 0$ as t is analytic. □

Krein and Levin [10] gave a nice result about the repartition of the real part of the zeros of a trigonometric polynomial.

Theorem 9.4. *Consider a nonzero trigonometric polynomial with at least one zero in $\mathbb{C}$. Then the real part of its complex zeros forms a nondecreasing sequence $(a_n)_{-\infty}^{\infty}$ given by*

$$a_n = \lambda n + \psi(n)$$

where $\lambda > 0$ and $\psi \in AP_{\mathbb{R}}(\mathbb{Z})$.

From this theorem we obtain the following corollary.

Corollary 9.1. *If $t \in \mathcal{P} \setminus \{0\}$ then its number of zeros $N(x)$ occurring in the interval $[x-1, x+1]$ is bounded by some constant not depending on x.*

Proof. Let $(a_n)_{-\infty}^{\infty}$ denote the sequence of the real part of its complex zeros as given in theorem 9.4. Since ψ is a almost periodic, there exists some constant M such that $|\psi| < M - 1$ and we have therefore

$$\begin{aligned} N(x) &\leq \text{card}\,\{n \in \mathbb{Z} \ : x - 1 \leq a_n \leq x + 1\} \\ &\leq \text{card}\,\{n \in \mathbb{Z} \ : x - M < \frac{\pi}{\Delta} n < x + M\} \\ &\leq \frac{2\Delta M}{\pi} \end{aligned}$$

□

Corollary 9.2. *Let X be a countable subset of reals such that $\sup_{k\in\mathbb{Z}}$ card $(X \cap [k, k+1]) = +\infty$. Then $X \in \mathcal{U}$.*

In the following, we will prove that the gap between two consecutive distinct zeros of a nonzero trigonometric polynomial can not goes to 0 along certain subsets of $\mathbb{R}$.

Proposition 9.1. *Let L be a natural number and (m_k), (n_k) two sequences of integers verifying*

$$m_k + 1 \leq n_k \leq m_k + L \text{ for any } k \in \mathbb{N}.$$

Suppose that for any $\psi \in AP(\mathbb{Z})$ such that $\lim_{k\to\infty}\psi(m_k) = 0$, we have $\psi(m_k) = 0$ for every $k \in \mathbb{N}$. Then there exists $k \in \mathbb{N}$ such that $\varphi(n_k) = 0$ whenever $\varphi \in AP(\mathbb{Z})$ and $\lim_{k\to\infty}\varphi(n_k) = 0$.

Proof. Suppose on the contrary that $\varphi(n_k) \neq 0$ for any $k \in \mathbb{N}$. Without loss of generality we can also suppose that $|\varphi| \leq 1$. Let us set $A_k = [m_k + 1, m_k + L]$, $k \in \mathbb{N}$. The function

$$\psi(n) = \prod_{j=1}^{L} \varphi(n + j)$$

is in $AP(\mathbb{Z})$ as the product of almost periodic functions. We have furthermore

$$|\psi(m_k)| \leq |\varphi(n_k)|$$

and so $\lim_{k\to\infty} \psi(m_k) = 0$ which implies that $\psi(m_k) = 0$ for any $k \in \mathbb{N}$. Hence there exists at least one point $n_{k,1} \in A_k$ such that $\varphi(n_{k,1}) = 0$. Suppose now that we have found l distinct points $n_{k,1}, \dots, n_{k,l} \in A_k$ where φ vanishes $(0 < l < L - 1)$. We form all the products

$$\psi_{i_1,\dots,i_l}(n) = \prod_{j=1, j\notin\{i_1,\dots,i_l\}}^{L} \varphi(n + j) \text{ where } 1 \leq i_1 < \ldots < i_l \leq L$$

which are again functions in $AP(\mathbb{Z})$. We have

$$|\psi_{i_1,\dots,i_l}(m_k)| \leq |\varphi(n_k)|$$

since the set $\cup_{j=1, j\notin\{i_1,\dots,i_l\}}^{L}\{m_k + j\}$ contains $L - l$ points i.e. at least one point among the $l + 1$ distinct points $n_k, n_{k,1}, \dots, n_{k,l}$. We deduce that $\lim_{k\to\infty} \psi_{i_1,\dots,i_l}(m_k) = 0$ and so $\psi_{i_1,\dots,i_l}(m_k) = 0$ for any $k \in \mathbb{N}$. In particular, we have

$$\prod_{j=1, m_k+j\notin\{n_{k,1},\dots,n_{k,l}\}}^{L} \varphi(m_k + j) = 0$$

which implies that there exists $n_{k,l+1} \in A_k \setminus \{n_k, n_{k,1}, \dots, n_{k,l}\}$ such that $\varphi(n_{k,l+1}) = 0$. We conclude that $\varphi(n) = 0$ for any $n \in A_k \setminus \{n_k\}$. Let us consider now the almost periodic function

$$\psi(n) = \sum_{j=1}^{L} \varphi(n + j).$$

We have $\psi(m_k) = \sum_{j=1}^{L} \varphi(m_k + j) = \varphi(n_k) \to 0$ as $k \to \infty$. Hence for any $k \in \mathbb{N}$, $\psi(m_k) = 0$ i.e. $\varphi(n_k) = 0$ which contradicts the hypothesis. $\square$

The next result is proved in [7, Prop. 14'].

Proposition 9.2. *Let p be a real polynomial. There exists some real $\theta(p) \in (0,1)$ such that for any $\varepsilon > 0$, $n_0 \in \mathbb{N}$ and $f \in AP(\mathbb{Z})$, the set*

$$\{n \in \mathbb{N} \ : |f([p(n) - \theta]) - f([p(n_0) - \theta])| < \varepsilon$$

is syndetic.

As in lemma 9.4, we deduce the next result.

Corollary 9.3. *Let p be a real polynomial, $f \in AP(\mathbb{Z})$ and E a thick set of $\mathbb{N}$. With $\theta(p) \in (0,1)$ chosen as in proposition 9.2, we have $f([p(n) - \theta]) = 0$ for any $n \in \mathbb{N}$ provided that* $\lim_{n\to\infty, n\in E} f([p(n) - \theta]) = 0$.

Theorem 9.5. *Let (x_n) and (ε_n) be two real sequences such that (x_n) is increasing and $\varepsilon_n > 0$ for any $n \in \mathbb{N}$, p a real polynomial and E a thick set in $\mathbb{N}$. We suppose that*

(i) $\lim_{n\to+\infty, n\in E} \varepsilon_n = 0$,
(ii) *there exists $T > 0$ such that $p(n) \le x_n \le p(n) + T$ for any $n \in E$.*

Then the set $X = \{x_n, x_n + \varepsilon_n \ : n \in E\}$ is in $\mathcal{U}$.

Proof. Let us assume there exists $t \in \mathcal{T} \setminus \{0\}$ such that $t(X) = 0$. By theorem 9.4, $(x_n)_{n\in E}$ is a subsequence of a sequence (a_n) of the form $a_n = \lambda n + \psi(n)$ where $\lambda > 0$ and $\psi \in AP_{\mathbb{R}}(\mathbb{Z})$ i.e. $x_k = a_{n'_k}$ for any $k \in E$. We can clearly take $n'_k = \max\{n | a_n = x_k\}$. We remark now that the gap $s(n) = a_{n+1} - a_n = \lambda + \psi(n+1) - \psi(n)$ is an almost periodic function on $\mathbb{Z}$ which verifies

$$0 < s(n'_k) = a_{n'_k+1} - a_{n'_k} \le x_k + \varepsilon_k - x_k \text{ for any } k \in E.$$

From assumption (i), we have then

$$\lim_{k\to+\infty, k\in E} s(n'_k) = 0$$

and from (ii), we have for any $k \in E$

$$p(k) \le x_k = \lambda n'_k + \psi(n'_k) \le p(k) + T.$$

If $M = \sup_{x\in\mathbb{R}} |\psi(x)|$, we get

$$q(k) + 1 \le n'_k \le q(k) + L - 2$$

where $q(k) = \dfrac{p(k) - M}{\lambda} - 1$ and $L = \left[\dfrac{2M + T}{\lambda}\right] + 4$.
Let $\theta \in (0,1)$ and set $m'_k(\theta) = [q(k) - \theta]$. We have

$$q(k) - \theta + 1 \le n'_k \le q(k) - \theta + 1 + L - 2,$$

and hence

$$m'_k(\theta) + 1 \le n'_k \le m'_k(\theta) + L.$$

From corollary 9.3, we can choose θ such that for any $f \in AP(\mathbb{Z})$

$$\lim_{k\to\infty, k\in E} f(m'_k(\theta)) = 0 \implies f(m'_k(\theta)) = 0 \text{ for any } k \in E.$$

With such a θ fixed, we write for simplicity m'_k instead of $m'_k(\theta)$. Since E is countably infinite, there exits an increasing bijection $\chi : \mathbb{N} \to E$. We define now the sequences (m_k) and (n_k) by

$$m_k = m'_{\chi(k)} \text{ and } n_k = n'_{\chi(k)} \text{ for any } k \in \mathbb{N}.$$

These two sequences verify the assumption of proposition 9.1 and since

$$\lim_{k\to\infty} s(n_k) = \lim_{k\to+\infty, k\in E} s(n'_k) = 0,$$

we conclude that $s(n_k) = 0$ for some $k \in \mathbb{N}$ i.e. $s(n'_k) = 0$ for some $k \in E$.

□

10. Generalization to conditionally positive definite functions

In this section, we show that the problem of determining whether a conditionally positive definite function is in fact strictly conditionally positive definite reduces to the problem of knowing whether the support of a measure is in the zero set of a nonzero trigonometric polynomial. Therefore we can apply the preceding results.

Definition 10.1. A continuous function φ is said conditionally positive definite (resp. strictly positive definite) of order m on $\mathbb{R}$ if for every set of distinct real numbers $\Xi = \{\xi_1, \ldots, \xi_K\}$ and every vector $(c_1, \ldots, c_K) \in \mathbb{C}^K \setminus \{0\}$ satisfying

$$\sum_{j=1}^{K} c_j x_j^l = 0 \text{ for all integer } l \leq m-1, \tag{5}$$

we have

$$\sum_{j=1}^{K} \sum_{k=1}^{K} \bar{c}_j c_k \varphi(\xi_j - \xi_k) \geq 0 \text{ (resp. } > 0). \tag{6}$$

We denote by $\mathcal{P}_m$ (resp. $\mathcal{P}_m^s$) the class of such functions.

In [13], Sun gives a complete characterization of all conditionally positive definite functions of a given order m. We state here his result.

Theorem 10.1. *Let $\varphi \in C$. In order for φ to be conditionally positive definite of order m it is necessary and sufficient that φ has the following integral representation*

$$\varphi(\xi) = \int_{\mathbb{R}\setminus\{0\}} \Big(e^{-i\xi x} - \kappa(x) \sum_{l=0}^{2m-1} \frac{(-i\xi)^l}{l!}\Big) d\mu(x) + \sum_{l=0}^{2m} a_l \frac{(-i\xi)^l}{l!},$$

where μ is a nonnegative Borel measure on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int_{0<|x|\leq 1} x^{2m}\, d\mu(x) < \infty \text{ and } \int_{|x|\geq 1} d\mu(x) < \infty.$$

The function κ is an analytic function in $\mathcal{S}$ such that $\kappa - 1$ has a zero of order $2m+1$ at the origin and the complex numbers a_l are such that a_{2m} is nonnegative.

Lemma 10.1. *Let $l \le 2m-1$. For every set of real numbers $\Xi = \{\xi_1, \ldots, \xi_K\}$ and every vector $(c_1, \ldots, c_K) \in \mathbb{C}^K$ satisfying*

$$\sum_{j=1}^{K} c_j x_j^l = 0 \text{ for all integer } l \le m-1,$$

we have

$$\sum_{j=1}^{K}\sum_{k=1}^{K} \bar{c}_j c_k (\xi_j - \xi_k)^l = 0. \tag{7}$$

Proof. From Leibniz formula, we have

$$\sum_{j=1}^{K}\sum_{k=1}^{K} \bar{c}_j c_k (\xi_j - \xi_k)^l = \sum_{n=0}^{l} (-1)^n \binom{l}{n} \sum_{j=1}^{K} \bar{c}_j \xi_j^{l-n} \sum_{k=1}^{K} c_k \xi_k^n.$$

Since $l \le 2m-1$, we have either $l-n \le m-1$ or $n \le m-1$ for every integer $n \le l$, hence the sum vanishes. □

Proposition 10.1. *Let φ a function with the integral representation as in theorem 10.1. Then $\varphi \in \mathcal{P}_m^s$ if and only if there does not exist $t \in \mathcal{T} \setminus \{0\}$ vanishing on supp μ.*

Proof. Let $\xi_1, \ldots, \xi_K$ be distinct points in $\mathbb{R}$ and $c_1, \ldots, c_K$ be complex numbers not all zero, satisfying the condition 5. Then we have by lemma 10.1

$$\begin{aligned}\sum_{j=1}^{K}\sum_{k=1}^{K} \bar{c}_j c_k \varphi(\xi_j - \xi_k) &= \sum_{j=1}^{K}\sum_{k=1}^{K} \bar{c}_j c_k \int_{\mathbb{R}} e^{-i(\xi_j-\xi_k)x}\, d\mu(x) + (-1)^m \frac{a_{2m}}{(2m)!} \sum_{j=1}^{K}\sum_{k=1}^{K} \bar{c}_j c_k (\xi_j - \xi_k)^{2m} \\ &= \int_{\mathbb{R}} \left| \sum_{k=1}^{K} c_k e^{i\xi_k x} \right|^2 d\mu(x) + \frac{a_{2m}}{(m!)^2} \left| \sum_{k=1}^{K} c_k \xi_k^m \right|^2 \\ &= \int_{\mathbb{R}} |t(x)|^2\, d\mu(x) + \frac{a_{2m}}{(m!)^2} |t^{(m)}(0)|^2,\end{aligned}$$

where we set $t(x) = \sum_{k=1}^{K} c_k e^{i\xi_k x}$.
From lemma 2.1, we have $\varphi \notin \mathcal{P}_m^s$ if and only if there exists a nonzero trigonometric polynomial t vanishing on supp μ and such that $t^{(m)}(0) = 0$. But the last condition is vacuous. It suffices to see that if $t \in \mathcal{T}$ vanishes on supp μ then the trigonometric polynomial $q(x) = \sin^{m+1}(x)t(x)$ equals 0 on supp μ and we have furthermore $q^{(m)}(0) = 0$. □

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