Compact composition operators on Hardy-Orlicz and Bergman-Orlicz spaces

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Abstract. It is known, from results of B. MacCluer and J. Shapiro (1986), that every composition operator which is compact on the Hardy space H^p, $1 \leq p < \infty$, is also compact on the Bergman space $\mathfrak{B}^p = L^p_a(\mathbb{D})$. In this survey, after having described the above known results, we consider Hardy-Orlicz H^Ψ and Bergman-Orlicz $\mathfrak{B}^\Psi$ spaces, characterize the compactness of their composition operators, and show that there exist Orlicz functions for which there are composition operators which are compact on H^Ψ but not on $\mathfrak{B}^\Psi$.

Keywords. Bergman spaces, Bergman-Orlicz spaces, Blaschke product ; Carleson function, Carleson measure ; compactness ; composition operator ; Hardy spaces ; Hardy-Orlicz spaces ; Nevanlinna counting function

MSC. Primary: 47B33 ; Secondary: 30H10 ; 30H20 ; 30J10 ; 46E15

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} ;\ |z| < 1\}$ be the open unit disk of the complex plane. For $1 \leq p < \infty$, consider the Hardy space

$$H^p = \{f : \mathbb{D} \to \mathbb{C} ;\ f \text{ analytic and } \|f\|_{H^p} < +\infty\},$$

where

$$\|f\|_{H^p} = \sup_{r<1} \left[\frac{1}{2\pi} \int_0^{2\pi} |f(r\mathrm{e}^{it})|^p \, dt \right]^{1/p},$$

and the Bergman space $\mathfrak{B}^p$ (otherwise denoted by A^p or L^p_a)

$$\mathfrak{B}^p = \{f : \mathbb{D} \to \mathbb{C} ;\ f \text{ analytic and } f \in L^p(\mathbb{D}, \mathcal{A})\},$$

whose norm is defined by

$$\|f\|_{\mathfrak{B}^p} = \left[\int_{\mathbb{D}} |f(z)|^p \, d\mathcal{A}(z) \right]^{1/p},$$

*These results come from joint works with P. Lefèvre, H. Queffélec and L. Rodríguez-Piazza ([4], [5], [6]). It is an expanded version of the conference I gave at the ICM satellite conference *Functional Analysis and Operator Theory*, held in Bangalore, India, 8–11 august 2010.

where $d\mathcal{A} = \frac{dxdy}{\pi}$ is the normalized area measure on $\mathbb{D}$.

Every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ (such a function is also known as a Schur function, or function of the Schur-Agler class) defines a *composition operator* $f \mapsto C_\varphi(f) = f \circ \varphi$ which is a *bounded* linear operator

$$C_\varphi\colon H^p \to H^p \qquad \text{resp.} \quad C_\varphi\colon \mathfrak{B}^p \to \mathfrak{B}^p,$$

thanks to *Littlewood's subordination principle* (see [2], Theorem 1.7). The function φ is called the *symbol* of C_φ.

1.1 Compactness

The compactness of composition operators on Bergman spaces had been characterized in 1975 by D. M. Boyd ([1]) for $p = 2$ and by B. MacCluer and J. Shapiro in 1986 for the other p's ([9], Theorem 3.5 and Theorem 5.3):

Theorem 1.1 (Boyd (1976), MacCluer-Shapiro (1986)) *Let φ be an analytic self-map of $\mathbb{D}$. Then, for $1 \leq p < \infty$, one has:*

$$C_\varphi\colon \mathfrak{B}^p \to \mathfrak{B}^p \quad \textit{compact} \qquad \Longleftrightarrow \qquad \lim_{|z|\to 1} \frac{1 - |\varphi(z)|}{1 - |z|} = +\infty\,. \tag{1}$$

That means that $\varphi(z)$ approaches the boundary of $\mathbb{D}$ less quickly that z. That means also (but we do not need this remark in the sequel) that φ has no finite angular derivative on $\partial\mathbb{D}$ (φ has a finite angular derivative at $\omega \in \partial\mathbb{D}$ if the angular limits $\varphi^*(\omega) = \lim_{z\to\omega} \varphi(z)$ and $\varphi'(\omega) = \angle \lim_{z\to\omega} \frac{\varphi(z)-\varphi^*(\omega)}{z-\omega}$ exist, with $|\varphi^*(\omega)| = 1$; by the Julia-Carathéodory Theorem, the non-existence of such a derivative is equivalent to the right-hand side of (1): see [12], § 4.2). We actually prefer to write (1) in the following way:

$$C_\varphi\colon \mathfrak{B}^p \to \mathfrak{B}^p \quad \text{compact} \qquad \Longleftrightarrow \qquad \lim_{|z|\to 1} \frac{1 - |z|}{1 - |\varphi(z)|} = 0\,. \tag{2}$$

On the other hand, it is not difficult to see ([14], Theorem 2.1, or [12], § 3.5):

Proposition 1.2 (Shapiro-Taylor (1973)) *Let φ be an analytic self-map of $\mathbb{D}$. Then, for $1 \leq p < \infty$, one has:*

$$C_\varphi\colon H^p \to H^p \quad \textit{compact} \qquad \Longrightarrow \qquad \lim_{|z|\to 1} \frac{1 - |z|}{1 - |\varphi(z)|} = 0\,. \tag{3}$$

(this is actually an equivalence when φ is univalent ([12], § 3.2), or more generally boundedly-valent, but there are Blaschke products for which the converse of (3) is not true: see [12], § 10.2, or [7], Theorem 3.1, for a more general result, with a simpler proof).

Hence:

Corollary 1.3 *For $1 \leq p < \infty$, one has:*

$$C_\varphi \colon H^p \to H^p \quad \text{compact} \qquad \Longrightarrow \qquad C_\varphi \colon \mathfrak{B}^p \to \mathfrak{B}^p \quad \text{compact.}$$

The converse is not true.

1.2 Goal

Our goal is to replace the classical Hardy spaces H^p and Bergman spaces $\mathfrak{B}^p$ by Hardy-Orlicz spaces H^Ψ and Bergman-Orlicz spaces $\mathfrak{B}^\Psi$ and compare the compactness of composition operators $C_\varphi \colon H^\Psi \to H^\Psi$ and $C_\varphi \colon \mathfrak{B}^\Psi \to \mathfrak{B}^\Psi$. We shall detail that in Section 3.

1.3 Some comments

To prove Proposition 1.2, J. H. Shapiro and P. D. Taylor used the following result ([14], Theorem 6.1):

Theorem 1.4 (Shapiro-Taylor (1973)) *If the composition operator C_φ is compact on H^{p_0} for some $1 \leq p_0 < \infty$, then it is compact on H^p for all $1 \leq p < \infty$.*

Proof. First, by Montel's Theorem, C_φ is compact on H^p if and only if $\|C_\varphi(f_n)\|_p$ converges to 0 for every sequence (f_n) in the unit ball of H^p which converges uniformly to 0 on compact subsets of $\mathbb{D}$. One then uses Riesz's factorization Theorem: if f_n is in H^p, we can write $f_n = B_n g_n$, where B_n is a Blaschke product and g_n has no zero in $\mathbb{D}$. By Montel's Theorem, we may assume that $(g_n)_n$ converge uniformly on compact sets of $\mathbb{D}$. Setting $h_n = g_n^{p/p_0}$, we get a function which is in the unit ball of H^{p_0}. Since $(g_n)_n$ converges uniformly on compact sets of $\mathbb{D}$, so does $(h_n)_n$. Its limit h belongs also to the unit ball of H^{p_0} and, by the compactness of C_φ, $\big(C_\varphi(h_n)\big)_n$ converges to $C_\varphi(h)$ in H^{p_0}.

Now the compactness of C_φ on H^{p_0} implies that $|\varphi^*| < 1$ almost everywhere (φ^* is the boundary values function of φ). In fact, let $P_n(z) = z^n$, since (P_n) is in the unit ball of H^{p_0} and converges uniformly to 0 on compact subsets of $\mathbb{D}$, one has $\|C_\varphi(P_n)\|_{p_0} \to 0$. But, if $E_\varphi = \{\xi \in \partial\mathbb{D}\,;\ |\varphi^*(\xi)| = 1\}$, one has $\|C_\varphi(P_n)\|_{p_0}^{p_0} \geq \int_{E_\varphi} |\varphi^*(\xi)|^{np_0}\, dm(\xi) \geq m(E_\varphi)$, where m is the normalized Lebesgue measure on $\partial\mathbb{D}$. Hence $m(E_\varphi) = 0$.

It follows that the sequence $\big((h_n \circ \varphi)^*\big)_n = (h_n \circ \varphi^*)_n$ converges almost everywhere to $h \circ \varphi^*$ on $\partial\mathbb{D}$. Since $\big(C_\varphi(h_n)\big)_n$ converges in the norm of $L^{p_0}(\mathbb{T})$, Vitali's convergence Theorem gives:

$$\lim_{m(E) \to 0} \sup_n \int_E |h_n \circ \varphi^*|^{p_0}\, dm = 0\,.$$

But

$$\int_E |f_n \circ \varphi^*|^p\, dm \leq \int_E |g_n \circ \varphi^*|^p\, dm = \int_E |h_n \circ \varphi^*|^{p_0}\, dm\,,$$

so Vitali's convergence Theorem again gives $\|f_n \circ \varphi\|_p \to 0$, since $f_n \circ \varphi^*$ tends to 0 *a. e.* on $\partial\mathbb{D}$. $\square$

Actually, the proof of Proposition 1.2 can be made without using Theorem 1.4, but we gave it to see that Riesz's factorization Theorem is the main tool.

Proof of Proposition 1.2. For every $z \in \mathbb{D}$, the evaluation map $e_z\colon f \in H^p \mapsto f(z)$ is a continuous linear form and $\|e_z\| \leq \frac{2^{1/p}}{(1-|z|)^{1/p}}$ (see [2], lemma in § 3.2, page 36). But actually $\|e_z\| = \frac{1}{(1-|z|^2)^{1/p}}$. Indeed, let $u_z(\zeta) = \big(\frac{1-|z|}{1-\bar{z}\zeta}\big)^{2/p}$, $|\zeta| < 1$. Then, using the Parseval formula, $\|u_z\|_p^p = \frac{1-|z|}{1+|z|}$. Therefore

$$\|e_z\| \geq \frac{|u_z(z)|}{\|u_z\|_p} \geq \frac{\frac{1}{(1+|z|)^{2/p}}}{\big(\frac{1-|z|}{1+|z|}\big)^{1/p}} = \frac{1}{(1-|z|^2)^{1/p}}\,.$$

On the other hand, it is clear, by the Cauchy-Schwarz inequality, that $|h(z)| \leq 1/(1-|z|^2)^{1/2}$ for every h in the unit ball of H^2; hence, if f is in the unit ball of H^p, and we write $f = Bg$, where B is the Blaschke product associated to the zeroes of f, we get, since $h = g^{p/2}$ is in the unit ball of H^2: $|f(z)| \leq |g(z)| = |h(z)|^{2/p} \leq 1/(1-|z|^2)^{1/p}$.

Now, $\frac{e_z}{\|e_z\|} \underset{|z|\to 1}{\overset{w^*}{\longrightarrow}} 0$ in $(H^p)^*$, because $e_z(P)(1-|z|)^{1/p} = P(z)(1-|z|)^{1/p} \underset{|z|\to 1}{\longrightarrow} 0$ for every polynomial P. Since C_φ is compact, its adjoint is also compact; hence $\|C_\varphi^*(e_z/\|e_z\|)\| \underset{|z|\to 1}{\longrightarrow} 0$, and that gives the result since $\|C_\varphi^*(e_z/\|e_z\|)\| = \frac{\|e_{\varphi(z)}\|}{\|e_z\|} = \Big(\frac{1-|z|^2}{1-|\varphi(z)|^2}\Big)^{1/p} \geq \frac{1}{2^{1/p}}\Big(\frac{1-|z|}{1-|\varphi(z)|}\Big)^{1/p}$. $\square$

For Bergman spaces, the necessary condition of compactness in Theorem 1.1 follows the same lines as in the Hardy case. B. D. MacCluer and J. H. Shapiro ([9], Theorem 5.3) proved the sufficient condition, in showing that the compactness of C_φ on one of the Bergman spaces $\mathfrak{B}^{p_0}$, $1 \leq p_0 < \infty$, implies its compactness for all the Bergman spaces $\mathfrak{B}^p$, $1 \leq p < \infty$ (see Theorem 1.5 below), and then used Boyd's result for $p = 2$ (actually, they gave a new proof of Boyd's result). To do that, since there is no Bergman version of Riesz's factorization Theorem, they had to use another tool and they used the notion of *Carleson measure*, that we shall develop in the next section. Before that, let us give a proof of Boyd's result. We follow [16], Theorem 10.3.5.

Proof of Theorem 1.1 (for $p = 2$). We only have to show that $\lim\limits_{|z|\to 1} \frac{1-|z|}{1-|\varphi(z)|} = 0$ implies the compactness of C_φ on $\mathfrak{B}^2$.

We may assume that $\varphi(0) = 0$. Indeed, if $\varphi(0) = a$, let $\psi = \varphi_a \circ \varphi$, with $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. One has $\psi(0) = 0$ and $C_\varphi = C_\psi \circ C_{\varphi_a}$, since $\varphi = \varphi_a \circ \psi$. Hence the compactness of C_ψ implies the one of C_φ. Moreover, one has $|\psi(z)| \leq |\varphi(z)|$, so that the condition $\lim_{|z|\to 1} \frac{1-|z|}{1-|\varphi(z)|} = 0$ implies that $\lim_{|z|\to 1} \frac{1-|z|}{1-|\psi(z)|} = 0$.

Let (f_n) be a sequence in the unit ball of $\mathfrak{B}^2$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then so does (f_n'). Using Taylor expansion, one has a constant $C > 0$ such that:

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mathcal{A}(z) \leq C \left[|f(0)|^2 + \int_{\mathbb{D}} (1 - |z|^2)^2 |f'(z)|^2 \, d\mathcal{A}(z) \right]$$

for every analytic function $f \colon \mathbb{D} \to \mathbb{C}$. It follows that

$$\|C_\varphi(f_n)\|_{\mathfrak{B}^2}^2 \leq C \left[|(f_n \circ \varphi)(0)|^2 + \int_{\mathbb{D}} (1 - |z|^2)^2 |(f_n \circ \varphi)'(z)|^2 \, d\mathcal{A}(z) \right]$$

for every $n \geq 1$. Since $f_n[\varphi(0)] \underset{n \to \infty}{\longrightarrow} 0$, it remains to show that the integral tends to 0.

For every $\varepsilon > 0$, we may take, by hypothesis, some $\delta > 0$, with $\delta < 1$, such that $\frac{1-|z|^2}{1-|\varphi(z)|^2} \leq \varepsilon$ for $\delta \leq |z| < 1$. This implies that:

$$\begin{aligned}
&\int_{\mathbb{D}} (1 - |z|^2)^2 |(f_n \circ \varphi)'(z)|^2 \, d\mathcal{A}(z) \\
&\qquad \leq \int_{|z|<\delta} (1 - |z|^2)^2 |(f_n' \circ \varphi)(z)|^2 |\varphi'(z)|^2 \, d\mathcal{A}(z) \\
&\qquad\qquad + \varepsilon \int_{\delta \leq |z| < 1} (1 - |\varphi(z)|^2)(1 - |z|^2) |\varphi'(z)|^2 |(f_n' \circ \varphi)(z)|^2 \, d\mathcal{A}(z) \, .
\end{aligned}$$

Denote by I_n the first integral and by J_n the second one. Since $f_n'[\varphi(z)]$ tends to 0 uniformly for $|z| \leq \delta$, I_n tends to 0. It remains to show that the sequence (J_n) is bounded. Since $1 - |z|^2 \leq 2 \log 1/|z|$, the change of variable formula (see [12], p. 179, or [16], Proposition 10.2.5) gives:

$$\begin{aligned}
J_n &\leq 2 \int_{\mathbb{D}} (1 - |\varphi(z)|^2) |\varphi'(z)|^2 |(f_n' \circ \varphi)(z)|^2 \log \frac{1}{|z|} \, d\mathcal{A}(z) \\
&= 2 \int_{\mathbb{D}} (1 - |w|^2) |f_n'(w)|^2 N_\varphi(w) \, d\mathcal{A}(w).
\end{aligned}$$

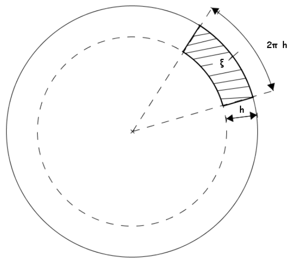
Since $\varphi(0) = 0$, Littlewood's inequality (see [12], § 10.4) reads as $N_\varphi(w) \leq \log 1/|w|$ for $w \neq 0$. We get, since $2 \log 1/|w| \approx 1 - |w|^2$ as $|w| \to 1$, a constant $C > 0$ such that:

$$J_n \leq C \int_{\mathbb{D}} (1 - |w|^2)^2 |f_n'(w)|^2 \, d\mathcal{A}(w) \, ,$$

and that ends the proof, since one easily sees (using Taylor expansion) that this last integral is less or equal than $\int_{\mathbb{D}} |f_n(w)|^2 \, d\mathcal{A}(w) \leq 1$. $\square$

1.4 Carleson measures

If $\mathbb{D}$ is the open unit disk of the complex plane and $\mathbb{T} = \partial\mathbb{D}$ is the unit circle, we denote by $\mathcal{A}$ the normalized area measure on $\mathbb{D}$ and by m the normalized Lebesgue measure on $\mathbb{T}$.



The *Carleson window* $W(\xi, h)$ centered at $\xi \in \mathbb{T}$ and of size h, $0 < h \leq 1$, is the set

$$W(\xi, h) = \{z \in \mathbb{D}\,;\ |z| > 1 - h \quad \text{and} \quad |\arg(z\bar{\xi})| < \pi h\}\,.$$

The point is that the two dimensions of the window $W(\xi, h)$ are proportional.

An α-*Carleson measure* (*Carleson measure* if $\alpha = 1$) is a measure μ on $\mathbb{D}$ such that:

$$\rho_\mu(h) := \sup_{|\xi|=1} \mu[W(\xi, h)] = O\left(h^\alpha\right).$$

We say that ρ_μ is the *Carleson function* of μ.

We denote by $\mathcal{A}_\varphi$ the pull-back measure of $\mathcal{A}$ by φ and by m_φ the pull-back measure of m by φ^*, φ^* being the boundary values function of φ. Recall that these pull-back measures are defined by $m_\varphi(E) = m[{\varphi^*}^{-1}(E)]$ and $\mathcal{A}_\varphi(E) = \mathcal{A}[\varphi^{-1}(E)]$ for every Borel set E of $\partial\mathbb{D}$ or of $\mathbb{D}$, respectively. We write ρ_φ and $\rho_{\varphi,2}$ the Carleson functions of m_φ and $\mathcal{A}_\varphi$ respectively. We call them the *Carleson function of* φ and the *Carleson function of order* 2 *of* φ.

Carleson's Theorem (see [2], Theorem 9.3) says that, for $p < \infty$, the inclusion map $I_\mu\colon f \in H^p \mapsto f \in L^p(\mu)$ is defined and bounded if and only if μ is a Carleson measure. A Bergman version has been proved by W. W. Hastings in 1975 ([3]): $J_\mu\colon f \in \mathfrak{B}^p \mapsto f \in L^p(\mu)$ is defined and bounded if and only if μ is a 2-Carleson measure. Now, composition operators $C_\varphi\colon H^p \to H^p$, resp. $C_\varphi\colon \mathfrak{B}^p \to \mathfrak{B}^p$, may be seen as inclusion maps $I_\varphi\colon H^p \to L^p(\mathbb{D}, m_\varphi)$, resp. $J_\varphi\colon \mathfrak{B}^p \to L^p(\mathbb{D}, \mathcal{A}_\varphi)$, since:

$$\|C_\varphi(f)\|_{H^p}^p = \int_{\mathbb{D}} |f|^p\, dm_\varphi \qquad \text{and} \qquad \|C_\varphi(f)\|_{\mathfrak{B}^p}^p = \int_{\mathbb{D}} |f|^p\, d\mathcal{A}_\varphi\,.$$

Hence the continuity of the composition operator C_φ, both on H^p and $\mathfrak{B}^p$, implies that m_φ is a Carleson measure and $\mathcal{A}_\varphi$ is a 2-Carleson measure. It should be stressed that in the Hardy case, m_φ is a measure on $\overline{\mathbb{D}}$, and not on $\mathbb{D}$ in general, so we have to adapt the previous notations in this case.

For compactness, one has the following result:

Theorem 1.5 (MacCluer (1985), MacCluer-Shapiro (1986))
For $1 \le p < \infty$, and every analytic self-map φ of $\mathbb{D}$, one has:

$$C_\varphi \colon H^p \to H^p \quad \text{compact} \qquad \Longleftrightarrow \qquad \rho_\varphi(h) = o\,(h), \quad \text{as } h \to 0,$$

and

$$C_\varphi \colon \mathfrak{B}^p \to \mathfrak{B}^p \quad \text{compact} \qquad \Longleftrightarrow \qquad \rho_{\varphi,2}(h) = o\,(h^2), \quad \text{as } h \to 0.$$

Recall that when the composition operator $C_\varphi \colon H^p \to H^p$ is compact, one has $|\varphi^*| < 1$ *a.e.*, and hence m_φ is supported by $\mathbb{D}$. Since these characterizations do not depend on p, one recovers Theorem 1.4 and get its Bergman counterpart.

We shall see in the next section how this theorem changes when we replace the classical Hardy and Bergman spaces by their Orlicz generalizations.

Proof of Theorem 1.5. We only prove the Bergman case; the Hardy case being analogous.

1) Assume that C_φ is compact on $\mathfrak{B}^p$. Consider, for every $a \in \mathbb{D}$, the Berezin kernel

$$H_a = \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\,\cdot$$

One has $\|H_a\|_{\mathfrak{B}^1} = 1$ and

$$\|H_a\|_\infty = \frac{(1-|a|^2)^2}{(1-|a|)^4} = \frac{(1+|a|)^2}{(1-|a|)^2} \le \frac{4}{(1-|a|)^2}\,;$$

hence, writing $a = (1-h)\xi$, $0 < h \le 1$, $|\xi| = 1$, we get $\|H_a\|_{\mathfrak{B}^p} \le (4/h^2)^{1-\frac{1}{p}}$ and the function $f_a = \big(\frac{h^2}{4}\big)^{1-\frac{1}{p}} H_a$ is in the unit ball of $\mathfrak{B}^p$. Moreover, f_a tends to 0 as $|a| \to 1$ uniformly on compact subsets of $\mathbb{D}$. Since C_φ is compact on $\mathfrak{B}^p$, one has $\|C_\varphi(f_a)\|_{\mathfrak{B}^p} \underset{|a|\to 1}{\longrightarrow} 0$. But it is easy to see that $|1-\bar{a}z| \le 5h$ when $z \in W(\xi,h)$. Hence $|f_a(z)| \ge C_p/h^{2/p}$ when $z \in W(\xi,h)$ and

$$\|C_\varphi(f_a)\|_{\mathfrak{B}^p}^p = \int_{\mathbb{D}} |f_a \circ \varphi|^p \, d\mathcal{A} \ge \int_{W(\xi,h)} |f_a|^p \, d\mathcal{A}_\varphi \ge \frac{C_p^p}{h^2}\, \mathcal{A}_\varphi[W(\xi,h)]\,.$$

Hence $\rho_{\varphi,2}(h) = o\,(h^2)$.

2) Conversely, assume that $\rho_{\varphi,2}(h) = o\,(h^2)$, and let (f_n) be a sequence in the unit ball of $\mathfrak{B}^p$ converging uniformly to 0 on compact subsets of $\mathbb{D}$, and $\varepsilon > 0$.

For every measure μ on $\mathbb{D}$, let $K_{\mu,2}(h) = \sup_{0<t\le h} \frac{\rho_\mu(t)}{t^2}$ and $K_{\varphi,2}(h) = K_{\mathcal{A}_\varphi,2}(h)$.

By hypothesis, there is a $\delta > 0$ such that $K_{\varphi,2}(\delta) \le \varepsilon$. Let μ be the measure $\mathbb{1}_{\mathbb{D}\setminus\overline{D(0,1-\delta)}}\,.\mathcal{A}_\varphi$. One has $K_{\mu,2}(1) \le 2K_{\varphi,2}(\delta)$, because, for $\delta < h \le 1$, the intersection of a window of size h with the annulus $\{z \in \mathbb{D}\,;\ 1-\delta < |z| < 1\}$ can be covered by less than $2(h/\delta)$ windows of size δ.

Now, the Bergman version of Carleson's Theorem (see [15], proof of Theorem 1.2, bottom of the page 117, with $\alpha = 1/2$) says that:

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu \leq C_p \, K_{\mu,2}(1) \, \|f\|_{\mathfrak{B}^p}^p$$

for every $f \in \mathfrak{B}^p$. Hence:

$$\begin{aligned} \int_{\mathbb{D}} |f_n \circ \varphi|^p \, d\mathcal{A} &= \int_{\overline{\mathbb{D}(0,1-\delta)}} |f_n(z)|^p \, d\mathcal{A}_\varphi(z) + \int_{\mathbb{D}\setminus \overline{D(0,1-\delta)}} |f_n(z)|^p \, d\mathcal{A}_\varphi(z) \\ &\leq \varepsilon + 2C_p \, \varepsilon \,, \end{aligned}$$

for n large enough, since (f_n) converges uniformly to 0 on $\overline{\mathbb{D}(0, 1-\delta)}$. It follows that $\|C_\varphi(f_n)\|_{\mathfrak{B}^p}$ converges to 0. □

2 Hardy-Orlicz and Bergman-Orlicz spaces

2.1 Orlicz spaces

An *Orlicz function* is a function $\Psi\colon [0,\infty) \to [0,\infty)$ which is positive, non-decreasing, convex and such that $\Psi(0) = 0$, $\Psi(x) > 0$ for $x > 0$ and $\Psi(x) \underset{x\to\infty}{\longrightarrow} \infty$.

Examples. $\Psi(x) = x^p$; $\Psi(x) = x^p \log(x+1)$, $1 \leq p < \infty$; $\Psi(x) = \mathrm{e}^{x^q} - 1$; $\Psi(x) = \exp[\big(\log(x+1)\big)^q] - 1$, $q \geq 1$.

If $(S, \mathcal{T}, \mu)$ is a finite measure space, the *Orlicz space* $L^\Psi(\mu)$ is the space of classes of measurable functions $f\colon S \to \mathbb{C}$ such that, for some $C > 0$:

$$\int_S \Psi(|f|/C) \, d\mu < \infty \,.$$

The norm is defined by:

$$\|f\|_\Psi = \inf\{C\,; \int_S \Psi(|f|/C) \, d\mu \leq 1\} \,.$$

For $\Psi(x) = x^p$, we get the classical Lebesgue space: $L^\Psi(\mu) = L^p(\mu)$.

2.2 Hardy-Orlicz and Bergman-Orlicz spaces

We define the Bergman-Orlicz space $\mathfrak{B}^\Psi$ by:

$$\mathfrak{B}^\Psi = \{f \in L^\Psi(\mathbb{D}, \mathcal{A})\,; \ f \text{ analytic}\} \,,$$

with the norm $\|f\|_{\mathfrak{B}^\Psi} = \|f\|_{L^\Psi(\mathbb{D},\mathcal{A})}$.

The Hardy-Orlicz space H^Ψ can be defined as in the classical case (see [4], Definition 3.2), but is is easier to define it by:

$$H^\Psi = \{f \in H^1\,; \ f^* \in L^\Psi(\mathbb{T}, m)\} \,,$$

with the norm $\|f\|_{H^\Psi} = \|f^*\|_{L^\Psi(m)}$.

As in the classical case (because $\Psi(|f|)$ is subharmonic), Littlewood's subordination principle implies that every analytic $\varphi\colon \mathbb{D} \to \mathbb{D}$ induces bounded composition operators $C_\varphi\colon \mathfrak{B}^\Psi \to \mathfrak{B}^\Psi$ and $C_\varphi\colon H^\Psi \to H^\Psi$.

2.3 Compactness

Theorem 1.5 has the following generalization ([4], Theorem 4.11) and [6], Theorem 2.5):

Theorem 2.1 *Let μ be a finite positive measure on $\mathbb{D}$, and assume that the identity maps $I_\mu\colon H^\Psi \to L^\Psi(\mu)$ and $J_\mu\colon \mathfrak{B}^\Psi \to L^\Psi(\mu)$ are defined. Then:*

1) $\displaystyle\lim_{h\to 0} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}[1/hK_\mu(h)]} = 0 \quad \Rightarrow \quad I_\mu$ *compact* $\quad \Rightarrow \quad \displaystyle\lim_{h\to 0} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}[1/\rho_\mu(h)]} = 0$*;*

2) $\displaystyle\lim_{h\to 0} \frac{\Psi^{-1}(1/h^2)}{\Psi^{-1}[1/h^2K_{\mu,2}(h)]} = 0 \quad \Rightarrow \quad J_\mu$ *compact* $\quad \Rightarrow \quad \displaystyle\lim_{h\to 0} \frac{\Psi^{-1}(1/h^2)}{\Psi^{-1}[1/\rho_\mu(h)]} = 0,$

where $\displaystyle K_\mu(h) = \sup_{0<t\leq h} \frac{\rho_\mu(t)}{t}$ *and* $\displaystyle K_{\mu,2}(h) = \sup_{0<t\leq h} \frac{\rho_\mu(t)}{t^2}$.

Actually, the sufficient conditions imply the existence of the identity maps I_μ and J_μ.

The conditions in 1), resp. 2), are equivalent if Ψ is "regular" (namely, if Ψ satisfies the condition ∇_0, whose definition is given after Proposition 4.3 below), as $\Psi(x) = x^p$, for which case both conditions in 1) read as $\displaystyle\frac{\rho_\mu(h)}{h} \underset{h\to 0}{\longrightarrow} 0$ and in 2) as $\displaystyle\frac{\rho_{\mu,2}(h)}{h^2} \underset{h\to 0}{\longrightarrow} 0$, but examples show that there is no equivalence in general (see [4], pp. 50–54 and [6], § 2).

2.4 Compactness for composition operators

Nevertheless, for composition operators C_φ, the following theorem (see [4], Theorem 4.19 and [6], Theorem 3.1), which is one of the main result of this survey, says that:

$$K_\varphi(h) \approx \rho_\varphi(h)/h \quad \text{and} \quad K_{\varphi,2}(h) \approx \rho_{\varphi,2}(h)/h^2.$$

Theorem 2.2 *There is a universal constant $C > 0$ such that, for every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$, one has, for every $0 < \varepsilon < 1$ and $h > 0$ small enough, for every $\xi \in \mathbb{T}$:*

1) $\quad m[\varphi^* \in W(\xi, \varepsilon h)] \leq C\,\varepsilon\, m[\varphi^* \in W(\xi, h)]$*;*

2) $\quad \mathcal{A}[\varphi \in W(\xi, \varepsilon h)] \leq C\,\varepsilon^2\, \mathcal{A}[\varphi \in W(\xi, h)]$.

For fixed h, this expresses that the measure m_φ is a Carleson measure and $\mathcal{A}_\varphi$ is a 2-Carleson measure. The theorem says that this is true at "all scales".

It follows that:

Theorem 2.3 *For every analytic self-map $\varphi \colon \mathbb{D} \to \mathbb{D}$, one has:*

$$1) \quad C_\varphi \colon H^\Psi \to H^\Psi \text{ compact} \iff \lim_{h\to 0} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}[1/\rho_\varphi(h)]} = 0;$$

$$2) \quad C_\varphi \colon \mathfrak{B}^\Psi \to \mathfrak{B}^\Psi \text{ compact} \iff \lim_{h\to 0} \frac{\Psi^{-1}(1/h^2)}{\Psi^{-1}[1/\rho_{\varphi,2}(h)]} = 0.$$

Let us give a very vague idea of the proof of Theorem 2.2, in the Bergman case (the proof in the Hardy case, though following the same ideas, is different, and actually more difficult). By setting $f = h/(1-\varphi)$, it suffices to show that

$$\mathcal{A}(\{|f| > \lambda\}) \leq \frac{K}{\lambda^2}\, \mathcal{A}(\{|f| > 1\}) \qquad \text{for } \lambda \geq \lambda_0 > 0\,, \tag{4}$$

for every analytic function $f \colon \mathbb{D} \to \Pi^+ = \{z \in \mathbb{C}\,;\ \Re\mathrm{e}\, z > 0\}$ such that $|f(0)| \leq \alpha_0$, for some $\alpha_0 > 0$. But the fact that $\mathcal{A}_\varphi$ is a 2-Carleson measure writes:

$$\mathcal{A}(\{|f| > \lambda\}) \leq \frac{C}{\lambda^2}\, |f(0)|\,, \qquad \lambda > 0\,, \tag{5}$$

where $f = h/(1-\varphi)$. One has hence to replace $|f(0)|$ in the majorization by $\mathcal{A}(\{|f| > 1\})$. To that effect, one splits the disk $\mathbb{D}$ into pieces which are "uniformly conform" to $\mathbb{D}$ and on which we can use (5). However, it is far from being so easy, and we refer to [6] (and [4] for the Hardy case) for the details.

3 Compactness on H^Ψ versus compactness on $\mathfrak{B}^\Psi$

Thanks to Theorem 2.3, in order to compare the compactness of the composition operator C_φ on H^Ψ and on $\mathfrak{B}^\Psi$, we have to compare $\rho_\varphi(h)$ and $\rho_{\varphi,2}(h)$. But if one reads their definitions:

$$\rho_\varphi(h) = \sup_{|\xi|=1} m[\varphi^* \in W(\xi, h)]$$

and

$$\rho_{\varphi,2}(h) = \sup_{|\xi|=1} \mathcal{A}[\varphi \in W(\xi, h)]\,,$$

that does not seem straightforward. We shall compare them in an indirect way, by using the Nevanlinna counting function.

3.1 Nevanlinna counting function

The *Nevanlinna counting function* counts how many pre-images each element has, with a weight which decreases when this pre-image approaches $\partial\mathbb{D}$. Namely:

$$N_\varphi(w) = \sum_{\varphi(z)=w} \log \frac{1}{|z|}$$

for $w \in \varphi(\mathbb{D})$ and $w \neq \varphi(0)$. One sets $N_\varphi(w) = 0$ for the other $w \in \mathbb{D}$.

Our second main theorem asserts that the Nevanlinna counting function of φ is equivalent to its Carleson function (see [5]).

Theorem 3.1 *There exists a universal constant $C > 1$ such that, for every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ and for $h > 0$ small enough, we have:*

$$\frac{1}{C} \sup_{w \in W(\xi, h/C)} N_\varphi(w) \leq m_\varphi[W(\xi, h)] \leq \frac{C}{\mathcal{A}[W(\xi, Ch)]} \int_{W(\xi, Ch)} N_\varphi(z)\, d\mathcal{A}(z)\,.$$

Now, if one defines the Nevanlinna function of order 2 by:

$$N_{\varphi,2}(w) = \sum_{\varphi(z)=w} \left[\log \frac{1}{|z|}\right]^2$$

for $w \in \varphi(\mathbb{D})$ and $w \neq \varphi(0)$ and $N_{\varphi,2}(w) = 0$ for the other $w \in \mathbb{D}$, one has easily (see [13], Proposition 6.6):

$$N_{\varphi,2}(w) = 2 \int_0^1 N_\varphi(r, w)\, \frac{dr}{r}\,,$$

where $N_\varphi(r, w) = \displaystyle\sum_{\varphi(z)=w, |z|<r} \log \frac{r}{|z|}$ is the restricted Nevanlinna function. But, since $N_\varphi(r, w) = N_{\varphi_r}(w)$ with $\varphi_r(z) = \varphi(rz)$, one gets, by integrating in polar coordinates:

Corollary 3.2 *There is some universal constant $C > 1$ such that, for every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$ and for $h > 0$ small enough:*

$$\frac{1}{C}\, \rho_{\varphi,2}(h/C) \leq \sup_{|w| \geq 1-h} N_{\varphi,2}(w) \leq C\, \rho_{\varphi,2}(Ch)$$

But now, it is easy to compare N_φ and $N_{\varphi,2}$: one has:

$$N_{\varphi,2}(w) \leq [N_\varphi(w)]^2$$

(simply because the ℓ_2-norm is smaller than the ℓ_1-norm). Therefore:

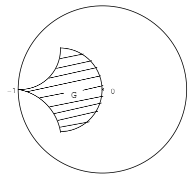
Theorem 3.3 *There is a universal constant $C > 1$ such that, for every analytic self-map $\varphi\colon \mathbb{D} \to \mathbb{D}$, one has for $h > 0$ small enough and every $\xi \in \mathbb{T}$:*

$$\mathcal{A}_\varphi[W(\xi, h)] \leq C \left(m_\varphi[W(\xi, Ch)]\right)^2$$

We can now compare the compactness on H^Ψ and on $\mathfrak{B}^\Psi$.

Theorem 3.4 *Under some condition on Ψ, one has:*

$$C_\varphi\colon H^\Psi \to H^\Psi \quad \textit{compact} \quad \Longrightarrow \quad C_\varphi\colon \mathfrak{B}^\Psi \to \mathfrak{B}^\Psi \quad \textit{compact.}$$



This condition has not a very nice statement:

$$\forall A > 0,\ \exists x_A > 0,\ \exists B \geq A: \qquad \Psi[A\Psi^{-1}(x^2)] \leq \big(\Psi[B\Psi^{-1}(x)]\big)^2, \qquad x \geq x_A$$

(though, setting $\chi_A(x) = \Psi[A\Psi^{-1}(x)]$, it writes better $\chi_A(x^2) \leq [\chi_B(x)]^2$), but it is satisfied in many cases:

- if Ψ grows moderately; namely, satisfies the condition Δ_2, *i.e.* $\Psi(2x) \leq C\,\Psi(x)$ for x large enough; this is the case for $\Psi(x) = x^p$, and we recover the classical case of Corollary 1.3;
- if Ψ grows quickly; namely, satisfies the condition Δ^2, *i.e.* for some $\alpha > 1$, $[\Psi(x)]^2 \leq \Psi(\alpha x)$ for x large enough; for instance if $\Psi(x) = \mathrm{e}^{x^q} - 1$, $q \geq 1$;
- but also for $\Psi(x) = \exp[\big(\log(x+1)\big)^2] - 1$.

Nevertheless ([6], Theorem 4.2):

Theorem 3.5 *There exists a symbol φ and an Orlicz function Ψ such that the composition operator $C_\varphi\colon H^\Psi \to H^\Psi$ is compact, and moreover in all Schatten classes $S_p(H^2)$, $p > 0$, whereas $C_\varphi\colon \mathfrak{B}^\Psi \to \mathfrak{B}^\Psi$ is* not *compact.*

The symbol φ is a conformal map from $\mathbb{D}$ onto the domain G, represented on the picture, delimited by three circular arcs of radii $1/2$.

The Carleson function of φ is "small" whereas its Carleson function of order 2 is "small":

$$\rho_\varphi(h) \leq C\,\mathrm{e}^{-\pi/4h}$$
$$\rho_{\varphi,2}(h) \geq (1/C)\,\mathrm{e}^{-\pi/h}$$

Now, it remains to construct a concave and piecewise linear function F (and Ψ will be F^{-1}) in such way that:

$$\lim_{x\to\infty} \frac{F(x)}{F(\mathrm{e}^{\pi x/4})} = 0 \quad \text{and} \quad \limsup_{x\to\infty} \frac{F(x^2)}{F(\mathrm{e}^{\pi x})} > 0\,,$$

in order that $\displaystyle\lim_{h\to 0} \frac{\Psi^{-1}(1/h)}{\Psi^{-1}[1/\rho_\varphi(h)]} = 0$, but $\displaystyle\limsup_{h\to 0} \frac{\Psi^{-1}(1/h^2)}{\Psi^{-1}[1/\rho_{\varphi,2}(h)]} > 0$. □

Remark. Let us stress that the compactness of $C_\varphi \colon H^\Psi \to H^\Psi$ always implies the compactness of $C_\varphi \colon \mathfrak{B}^{\Psi^2} \to \mathfrak{B}^{\Psi^2}$. Indeed, if $\tilde\Psi(x) = [\Psi(x)]^2$, then $\tilde\Psi^{-1}(t^2) = \Psi^{-1}(t)$, so we get $\tilde\Psi^{-1}(1/h^2)/\tilde\Psi^{-1}\big(1/\nu_{\varphi,2}(h)\big) \leq \Psi^{-1}(1/h)/\Psi^{-1}\big(1/\nu_\varphi(h)\big)$, since one has $\nu_{\varphi,2}(h) \leq [\nu_\varphi(h)]^2$, where $\nu_\varphi(h) = \sup_{|w|\geq 1-h} N_\varphi(w)$.

4 Final remarks

4.1 Modulus of the symbol and compactness on Bergman-Orlicz spaces

What about the compactness of C_φ on the Bergman-Orlicz spaces and the behaviour of the modulus of its symbol φ?

We prove in [4], Theorem 5.7, that the compactness of $C_\varphi \colon \mathfrak{B}^\Psi \to \mathfrak{B}^\Psi$ implies that:

$$\lim_{|a|\to 1} \frac{\Psi^{-1}\left[\dfrac{1}{(1-|\varphi(a)|)^2}\right]}{\Psi^{-1}\left[\dfrac{1}{(1-|a|)^2}\right]} = 0\,. \tag{6}$$

The proof is essentially the same as the proof of Proposition 1.2 given in Section 1.3. But it follows also from Theorem 2.3 and Corollary 3.2. Indeed, these results have the following consequence:

Theorem 4.1 *For every analytic self-map $\varphi \colon \mathbb{D} \to \mathbb{D}$ and for every Orlicz function Ψ, the composition operator $C_\varphi \colon \mathfrak{B}^\Psi \to \mathfrak{B}^\Psi$ is compact if and only if:*

$$\lim_{h\to 0} \frac{\Psi^{-1}(1/h^2)}{\Psi^{-1}\big(1/\nu_{\varphi,2}(h)\big)} = 0\,, \tag{7}$$

where $\nu_{\varphi,2}(h) = \sup_{|w|\geq 1-h} N_{\varphi,2}(w)$.

Now, since $N_{\varphi,2}\big(\varphi(z)\big) \geq \big(\log(1/|z|)\big)^2 \geq (1-|z|)^2$, we get that the compactness of $C_\varphi \colon \mathfrak{B}^\Psi \to \mathfrak{B}^\Psi$ implies (6). But Theorem 7 gives a partial converse:

Theorem 4.2 *For every* univalent *(or more generally, boundedly-valent) analytic self-map $\varphi \colon \mathbb{D} \to \mathbb{D}$ and for every Orlicz function Ψ, the composition operator $C_\varphi \colon \mathfrak{B}^\Psi \to \mathfrak{B}^\Psi$ is compact if and only if one has* (6)*.*

Let us recall that φ bounded-valent means that there is an integer $L \geq 1$ such that the equation $\varphi(z) = w$ has at most L solution(s) in $\mathbb{D}$, for every $w \in \mathbb{D}$; we then have $N_{\varphi,2}(w) \leq L(1-|z|)$, where $\varphi(z) = w$, with $|z| > 0$ minimal, and Theorem 4.2 follows. □

Compactness of C_φ also is equivalent to (6), for every symbol φ, if one adds a condition on the Orlicz function Ψ.

Proposition 4.3 *Assume that the Orlicz function Ψ satisfies the condition ∇_0. Then the composition operator $C_\varphi \colon \mathfrak{B}^\Psi \to \mathfrak{B}^\Psi$ is compact if and only if* (6) *holds.*

Condition ∇_0 is the "regularity" condition mentioned after Theorem 2.1, which gives the equivalence between the necessary and sufficient conditions. Ψ satisfies ∇_0 if (see [4], Definition 4.5 and Proposition 4.6): there is $x_0 > 0$ such that, for every $B > 1$, there exists $c_B > 1$ such that:

$$\frac{\Psi(Bx)}{\Psi(x)} \leq \frac{\Psi(c_B B y)}{\Psi(y)} \qquad \text{for } x_0 \leq x \leq y. \tag{8}$$

Let us point out that it is satisfied, in particular, when $\log \Psi(\mathrm{e}^x)$ is convex, or if Ψ satisfies the condition Δ^2 ([4], Proposition 4.7).

Proof. We may assume that $\varphi(0) = 0$. Then Littlewood's inequality writes $N_\varphi(w) \leq \log 1/|w|$, $w \neq 0$, and gives, for some constant $C > 1$:

$$N_{\varphi,2}(w) \leq C \sup_{\varphi(z)=w} (1-|z|) \sum_{\varphi(z)=w} \log \frac{1}{|z|} \leq C^2 (1-|w|) \sup_{\varphi(z)=w} (1-|z|)\,.$$

By hypothesis, for every $A > 0$, we have, with $w = \varphi(z)$:

$$\Psi^{-1}\left[\frac{1}{(1-|z|)^2}\right] \geq A\, \Psi^{-1}\left[\frac{1}{(1-|\varphi(z)|)^2}\right]$$

for $|z|$ close enough to 1. With $h = 1 - |w|$, this writes:

$$1 - |z| \leq 1 \Big/ \sqrt{\Psi\big[A\Psi^{-1}(1/h^2)\big]}\,.$$

We get hence:

$$\nu_{\varphi,2}(h) \leq \frac{C^2 h}{\sqrt{\Psi\big[A\Psi^{-1}(1/h^2)\big]}}$$

if $h > 0$ is small enough. It follows that we shall have:

$$\lim_{h\to 0} \frac{\Psi^{-1}(1/h^2)}{\Psi^{-1}\big(1/\nu_{\varphi,2}(h)\big)} = 0$$

if for every $B > 1$, we can find $A > 0$ such that:

$$\Psi^{-1}\left[\frac{\sqrt{\Psi[A\Psi^{-1}(1/h^2)]}}{C^2 h}\right] \geq B\Psi^{-1}(1/h^2)\,.$$

Setting $x = \Psi^{-1}(1/h^2)$, it suffices to have:

$$\big[\Psi(Bx)\big]^2 \leq \Psi(x)\, \Psi\big((A/C^4)x\big)\,, \tag{9}$$

for $x > 0$ big enough, since $\Psi\big((A/C^4)x\big) \leq \Psi(Ax)/C^4$ by convexity.

When $\Psi \in \nabla_0$, (8) gives (9) with $y = Bx$ and $A = C^4 c_B B^2$. $\square$

4.2 Blaschke products

We may ask about the converse implication: does the compactness of C_φ on $\mathfrak{B}^\Psi$ imply its compactness on H^Ψ? Even in the Hilbertian case $\mathfrak{B}^2 - H^2$, this is not the case (see [12], pp. 183–185): there is a Blaschke product B (whose associated composition operator is an isometry from H^2 into itself) with no angular derivative, so C_B is compact on $\mathfrak{B}^2$. Another example (a Blaschke product also) is given in [4], when $\Psi(x) = \mathrm{e}^{x^2} - 1$. More generally:

Theorem 4.4 *For every Orlicz function Ψ satisfying the condition ∇_0, there is a Blaschke product B, whose associated composition operator C_B is an isometry from H^Ψ into itself, but such that C_B is compact on $\mathfrak{B}^\Psi$.*

Proof. Indeed ([7], Theorem 3.1), for every function $\delta\colon (0,1) \to (0,1/2]$ such that $\delta(t) \underset{t\to 0}{\longrightarrow} 0$, there is a Blaschke product B such that:

$$1 - |B(z)| \geq \delta(1 - |z|), \qquad \text{for all } z \in \mathbb{D}\,. \tag{10}$$

By replacing $B(z)$ by $zB(z)$, we may assume that $B(0) = 0$ (note that $1 - |zB(z)| \geq 1 - |B(z)|$). Then C_B is an isometry from H^Ψ into itself. Indeed, one can see ([10], Theorem 1) that if φ is an inner function, then the pull-back measure m_φ is equal to $P_a.m$, when P_a is the Poisson kernel at $a = \varphi(0)$. When $\varphi(0) = 0$, one has $m_\varphi = m$ and C_φ is an isometry.

Hence, taking, for $t > 0$ small enough:

$$\delta(t) = \frac{1}{\sqrt{\Psi\big[\sqrt{\Psi^{-1}(1/t^2)}\big]}}\,,$$

we get

$$\frac{\Psi^{-1}\Big[\dfrac{1}{(1 - |B(z)|)^2}\Big]}{\Psi^{-1}\Big[\dfrac{1}{(1 - |z|)^2}\Big]} \leq \frac{1}{\sqrt{\Psi^{-1}\Big[\dfrac{1}{(1 - |z|)^2}\Big]}}\,,$$

and we get (6). Hence, assuming that Ψ satisfies the condition ∇_0, C_B is compact on $\mathfrak{B}^\Psi$, by Proposition 4.3. □

Remark. Another, but different, survey on this topic can be found in [11]

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