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ON SOME PROPERTIES OF THE CLASS OF STATIONARY SETS*

by Pascal Lefevre

Abstract

Some new properties of the stationary sets (defined by G. Pisier in [12]) are studied. Some arithmetical conditions are given, leading to the non-stationarity of prime numbers. It is shown that any stationary set is a set of continuity. Some examples of "big" stationary sets are given, being not sets of uniform convergence.

Key Words : Stationary sets, Sidon sets, sets of continuity, Rajchman sets, random Fourier series, Riesz products, UC sets.
AMS classifications : 42A20, 42A55, 42C10, 43A46, 43A77.

CONTENTS

*Preprint

1 Introduction, notations and definitions :

Let G be an infinite metrizable compact abelian group, equipped with its normalized Haar measure dx, and Γ its dual (discrete and countable). G will be mostly the unit circle of the complex plane and then Γ will be identified to $\mathbb{Z}$ by $p \to e_p$, where $e_p(x) = e^{2i\pi px}$.

We shall denote by $\mathcal{P}(G)$ the set of the trigonometric polynomials over G, i.e. the set of the finite sums $\sum_{\gamma\in\Gamma} a_\gamma \gamma$, where $a_\gamma \in \mathbb{C}$; this is also the vector space of functions over G spanned by Γ.

We shall denote by $C(G)$ the space of complex continuous functions over G, with the norm : $\|f\|_\infty = \sup_{x\in G} |f(x)|$. This is also the completion of $\mathcal{P}(G)$ for $\|.\|_\infty$.

$M(G)$ will denote the space of complex regular Borel measures over G, equipped with the norm of total variation. If $\mu \in M(G)$, its Fourier transform at the point γ is defined by $\hat{\mu}(\gamma) = \int_G \gamma(-x) d\mu(x)$.

$L^p(G)$ denotes the Lebesgue space $L^p(G, dx)$ with the norm :

$$\|f\|_p = \begin{cases} (\int_G |f(x)|^p dx)^{1/p} & 1 \leq p < \infty \\ \text{essup}|f(x)| & p = \infty \end{cases}$$

The map : $f \to f dx$ identifies $L^1(G)$ with a closed ideal of $M(G)$ equipped with the convolution.

If B is a normed space of functions over G, which is continuously injected in $M(G)$, and if Λ is a subset of Γ, we shall set :

$$B_\Lambda = \{f \in B / \hat{f}(\gamma) = 0 \quad \forall \gamma \notin \Lambda\}.$$

This is also the set of elements of B whose spectrum is contained in Λ.

$(\varepsilon_\gamma)_{\gamma\in\Gamma}$ will denote a Bernoulli sequence indexed by Γ, ie a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, taking values $+1$ and -1 with probability $1/2$. In the same way, $(g_\gamma)_{\gamma\in\Gamma}$ will denote a sequence of centred independent complex gaussian random variables, normalized by $\mathbb{E}|g_\gamma|^2 = 1$.

$|E|$ will denote the cardinal of a finite set E.

Let us now recall some classical definitions of lacunary sets of Γ.

Definition 1.1 *Let Λ be a subset of Γ, Λ is a Sidon set if Λ shares one of the following equivalent conditions :*

i) There is $C > 0$ such that for all $P \in \mathcal{P}_\Lambda(G)$: $\sum_{\gamma\in\Lambda} |\hat{P}(\gamma)| \leq C\|P\|_\infty$

ii) There is $C > 0$ such that for all $f \in C_\Lambda(G)$: $\sum_{\gamma\in\Lambda} |\hat{f}(\gamma)| \leq C\|f\|_\infty$

iii) *There is* $C > 0$ *such that for all* $(b_\lambda)_{\lambda\in\Lambda} \in \ell^\infty(\Lambda)$ *with* $\|b\|_\infty = 1$ *: there exists* $\mu \in M(G)$ *with* $\|\mu\| \le C$ *such that :* $\forall\lambda \in \Lambda$ *:* $\hat{\mu}(\lambda) = b_\lambda$

iv) *There is* $C > 0$ *such that for all* $(b_\lambda)_{\lambda\in\Lambda} \in c_0(\Lambda)$ *with* $\|b\|_\infty = 1$ *: there exists* $f \in L^1(G)$ *with* $\|f\|_1 \le C$ *such that :* $\forall\lambda \in \Lambda \quad \hat{f}(\lambda) = b_\lambda$.

For a deep study of Sidon sets, it may be useful to read [4],[9] or [13].

Definition 1.2 *A subset* A *of* Γ *is dissociated (resp. quasi-independent) if for every* $(n_\gamma)_{\gamma\in A} \in \{-2;\ldots;2\}^A$ *(resp.*$\forall(n_\gamma)_{\gamma\in A} \in \{-1;0;1\}^A$*) with almost all* n_γ *equal to zero :*

$$\prod_{\gamma\in A} \gamma^{n_\gamma} = 1 \Rightarrow \forall\gamma \in A : \gamma^{n_\gamma} = 1.$$

We recall that if A *is quasi-independent, then* A *is a Sidon set.*

Definition 1.3 $(F_N)_{N\ge0}$ *being an increasing sequence of finite subsets of* Γ *such that* $\bigcup_{N=0}^{\infty} F_N = \Gamma$*, then a subset* Λ *of* Γ *is a set of uniform convergence, relatively to* $(F_N)_{N\ge0}$ *(in short : UC set) if :*
$\forall f \in C_\Lambda(G)$, $(S_N f)_{N\ge0}$ *converges to* f *in* $C_\Lambda(G)$*, where :* $S_N f = \sum_{\gamma\in F_N} \hat{f}(\gamma)\gamma$.

We define, in this case, the UC *constant (denoted* $U(\Lambda)$*) as:*

$$\sup\{\|S_N(f)\|_\infty / f \in C_\Lambda(G),\ \|f\|_\infty = 1, N \ge 0\}.$$

We recall too that Λ *(included in* $\mathbb{Z}$*) is a* CUC *set if this is a* UC *set such that* $\sup_{p\in\mathbb{Z}} U(p+\Lambda)$ *is finite.*

Remark : This notion, closely linked with the choice of $(F_N)_{N\ge0}$, is particularly studied in two cases: $G = \mathbb{T}$ and G is the Cantor group. Here we shall be interested in the case $G = \mathbb{T}$, where the natural choice of $(F_N)_{N\ge0}$ is $F_N = \{-N;\ldots;N\}$. For a (non-exhaustive) set out on UC sets, one may read [8].

Definition 1.4 *Let* Λ *be included in* $\mathbb{Z}$ *:* Λ *is a set of continuity if :*
$\forall\varepsilon > 0\ \exists\delta > 0$ *s.t.* $\forall\mu \in M(\mathbb{T})$ *with* $\|\mu\| = 1$ *:*

$$\overline{\lim_{\mathbb{Z}\setminus\Lambda}} |\hat{\mu}(n)| < \delta \Rightarrow \overline{\lim_{\Lambda}} |\hat{\mu}(n)| < \varepsilon.$$

The links between the sets of continuity and some other thin sets (particularly UC ; $\Lambda(1)$; p-Sidon) where studied in [6].

Definition 1.5 *Let* $0 < p < \infty$ *and* A *a subset of* Γ. A *is a* $\Lambda(p)$ *set if :*
$\exists 0 < q < p :\ L^p_A(G) = L^q_A(G)$.
Let us mention, that, in this case, we have: $\forall r \in]0,p[$: $L^p_A(G) = L^r_A(G)$.

Definition 1.6 *Let $1 \leq p < 2$ and Λ a subset of Γ; Λ is a p-Sidon set if there exists $C > 0$ such that for all $f \in \mathcal{P}_\Lambda(G)$: $(\sum_{\lambda\in\Lambda} |\hat{f}(\lambda)|^p)^{1/p} \leq C\|f\|_\infty$.*

The best constant C is called the p-Sidonicity constant of Λ and is denoted by $S_p(\Lambda)$. (see for example [1] or [3]). Obviously, Λ is a p-Sidon set implies Λ is a q-Sidon set for $q > p$. If Λ is a p-Sidon set and not a q-Sidon set for any $q < p$, Λ is called a true p-Sidon set.

Let us introduce too a relatively exotic norm on $\mathcal{P}(G)$, the $C^{a\cdot s}$ norm ("almost surely continuous") so defined

$$\llbracket f \rrbracket = \int_\Omega \| \sum_{\gamma\in\Gamma} \varepsilon_\gamma(\omega)\hat{f}(\gamma)\gamma\|_\infty d\mathbb{P}(\omega) \ . \tag{1}$$

Remark : Marcus and Pisier [11] showed that an equivalent norm is defined taking a gaussian sequence $(g_\gamma)_{\gamma\in\Gamma}$ instead of Bernoulli sequence $(\varepsilon_\gamma)_{\gamma\in\Gamma}$ in (1).

$C^{a\cdot s}(G)$ is, by definition, the completion of $\mathcal{P}(G)$ for the norm $\llbracket\cdot\rrbracket$. This is also the set of functions of $L^2(G)$ such that the integral in (1) is finite, or the set of functions of $L^2(G)$ such that, almost surely : $\varepsilon_\gamma(\omega)\hat{f}(\gamma) = \widehat{f^\omega}(\gamma)$ with f^ω in $C(G)$ (for the equivalence between the quantitative and the qualitative definition, we refer to [7]) and $C^{a\cdot s}(G)$ is also called the space of almost surely continuous random Fourier series.

Following the spectacular result of Drury ("the union of two Sidon sets is a Sidon set"), a lot of improvements were achieved in the 70's about such sets Λ. Rider, particularly, showed that they may be characterized by the following "a priori" inequality : $\sum_{\gamma\in\Gamma} |\hat{f}(\gamma)| \leq C \llbracket f \rrbracket$, $\forall f \in \mathcal{P}_\Lambda(G)$ and Pisier [12] then observed that they may also be characterized by the "a priori" inequality $\|f\|_\infty \leq C \llbracket f \rrbracket$, $\forall f \in \mathcal{P}_\Lambda(G)$, i.e one has the continuous inclusion $C^{a\cdot s}_\Lambda(G) \subset C_\Lambda(G)$. This led him to consider the class $\mathcal{S}$ of subsets of Γ verifying the reverse "a priori" inequality, $\llbracket f \rrbracket \leq C\|f\|_\infty$, $\forall f \in \mathcal{P}_\Lambda(G)$, which corresponds to the continuous inclusion $C_\Lambda(G) \subset C^{a\cdot s}_\Lambda(G)$. He called stationary the elements of this class $\mathcal{S}$. We have the following precise:

Definition 1.7 *A subset Λ of Γ is stationary (in short $\Lambda \in \mathcal{S}$) if:*

$$\exists C > 0,\ \forall f \in \mathcal{P}_\Lambda(G),\ \llbracket f \rrbracket \leq C\|f\|_\infty.$$

The best constant C is called the stationarity constant of Λ and is denoted $K_S(\Lambda)$.

Pisier showed that $\mathcal{S}$ contains Sidon sets and finite products of such sets. $\mathcal{S}$ is so strictly bigger than the class of Sidon sets, because of the following: if $\Lambda_1, \cdots, \Lambda_k$ are infinite Sidon sets of the groups $G_1, \cdots, G_k$: $\Lambda_1 \times \cdots \times \Lambda_k$ is a true $\frac{2k}{k+1}$-Sidon set of the group $G_1 \times \cdots \times G_k$. Bourgain ([2]) also proved that if A_1 and A_2 are infinite then :

$$A_1 \times A_2 \in \mathcal{S} \Leftrightarrow A_1 \text{ and } A_2 \in \mathcal{S} \cap \Lambda(2).$$

In spite of these results, the class $\mathcal{S}$ seems to be still badly known up to now. We plan in this work to compare it to some other class of lacunary sets of harmonic analysis, particularly UC sets and sets of continuity, which were previously defined.

We shall need in the sequel some remarkable inequalities, linked to the $[\![\cdot]\!]$ norm. The inequality of Salem-Zygmund [14] will be used in the shape of :

$$\exists C > 0\ \forall (a_n)_{n\geq 0},\ |a_n| = 1,\ \forall N \geq 1 : \left[\!\!\left[\sum_{n=0}^{N-1} a_n e_n\right]\!\!\right] \geq C\sqrt{N \log N}. \tag{2}$$

The inequality of Marcus-Pisier [11] is as follows : there exists a (numerical) constant $D > 0$ such that, for every sequence $(a_\gamma)_{\gamma\in\Gamma}$, denoting by $(a_k^*)_{k\geq 0}$ the decreasing rearrangement of $(|a_\gamma|)_{\gamma\in\Gamma}$, one has :

$$\left[\!\!\left[\sum_{\gamma\in\Gamma} a_\gamma \gamma\right]\!\!\right]_{C(G)} \geq D \left[\!\!\left[\sum_{k\geq 0} a_k^* e_k\right]\!\!\right]_{C(\mathbb{T})}. \tag{3}$$

2 Preliminary results :

In the sequel, we shall use the two previous inequalities in the following way (where c denotes a numerical constant which can vary from line to line):

Lemma 2.1 *: take $P \in \mathcal{P}(G)$; set $E_\delta = \{\gamma \in \Gamma / |\hat{P}(\gamma)| \geq \delta\}$ and $N_\delta = |E_\delta|$ $(\delta > 0)$. Then we have:*

$$[\![P]\!] \geq c\delta\sqrt{N_\delta \log N_\delta}.$$

Proof : By the contraction principle [7] we have:

$$\begin{aligned} 2\,[\![P]\!] &\geq \left[\!\!\left[\sum_{\gamma\in E_\delta} |\hat{P}(\gamma)|\gamma\right]\!\!\right] \\ &\geq \delta \left[\!\!\left[\sum_{\gamma\in E_\delta} \gamma\right]\!\!\right]. \end{aligned}$$

Using (3), we obtain:

$$\llbracket P \rrbracket \geq c\delta \left\llbracket \sum_{k=0}^{N_\delta - 1} e_k \right\rrbracket_{C(\mathbb{T})}$$

and then using (2), we have:

$$\llbracket P \rrbracket \geq c\delta \sqrt{N_\delta \log N_\delta}.$$

$\blacksquare$

Similarly to Sidon sets, there are several equivalent functional definitions for the stationary sets. Indeed, we have the following proposition :

Proposition 2.2 *The following assertions are equivalent for a stationary subset Λ of Γ:*

i) $C_\Lambda(G) \subset C_\Lambda^{a.s}(G)$

ii) There exists $K > 0$ *such that* $\forall f \in C_\Lambda(G) \quad \llbracket f \rrbracket \leq K\|f\|_\infty$.

iii) There exists $K > 0$ *such that* $\forall (\mu_\alpha) \in L^\infty(\Omega, \mathcal{A}, \mathbb{P}, M(G))$ *with* $\|\mu_\alpha\| \leq 1 \quad \mathbb{P}$ *a.s,*

there exists $\mu \in M(G)$ with $\|\mu\| \leq K$ such that $\forall \gamma \in \Lambda : \hat{\mu}(\gamma) = \int_\Omega \widehat{\mu_\alpha}(\gamma)\varepsilon_\gamma(\alpha) d\mathbb{P}(\alpha)$.

Proof : i) $\Rightarrow$ ii) : just use the closed graph theorem.
ii) $\Rightarrow$ i) : trivial.
ii) $\Rightarrow$ iii) : Take (μ_α) in $L^\infty(\Omega, \mathcal{A}, \mathbb{P}, M(G))$ with $\|\mu_\alpha\| \leq 1 \quad \mathbb{P}$ a.s.
The map $T : \mathcal{P}_\Lambda(G) \to \mathbb{C}$ defined by :

$$\forall f \in \mathcal{P}_\Lambda(G) : T(f) = \int_\Omega \mu_\alpha * f^\alpha(0) d\mathbb{P}(\alpha) \ ,$$

is a linear form on $\mathcal{P}_\Lambda(G)$, with norm bounded by K. Indeed, we have for all $f \in \mathcal{P}_\Lambda(G)$:

$$|T(f)| \leq \int_\Omega \|\mu_\alpha\|\|f^\alpha\|_\infty d\mathbb{P}(\alpha) \leq \int_\Omega \|f^\alpha\|_\infty d\mathbb{P}(\alpha) = \llbracket f \rrbracket \leq K\|f\|_\infty.$$

By the Hahn-Banach theorem, T extends to $\tilde{T}$ belonging to $C(G)^*$ with $\|\tilde{T}\| = \|T\| \leq K$. The Riesz representation theorem gives the existence of a measure μ in $M(G)$, with norm less than K such that :

$$\forall f \in C(G) : \ \tilde{T}(f) = \mu * f(0). \tag{4}$$

Testing (4) on γ belonging to Λ, we get :

$$\forall \gamma \in \Lambda : \quad \hat{\mu}(\gamma) = \tilde{T}(\gamma) = \int_\Omega \hat{\mu}_\alpha(\gamma)\varepsilon_\gamma(\alpha) d\mathbb{P}(\alpha)$$

that is we get (iii).

iii) $\Rightarrow$ ii) : Let f in $\mathcal{P}_\Lambda(G)$. By [7], $C^{a\cdot s}(G)$ embeds in $L^1(\Omega,\mathcal{A},\mathbb{P},C(G))$ and $\Big(L^1\big(\Omega,\mathcal{A},\mathbb{P},C(G)\big)\Big)^* = L^\infty\big(\Omega,\mathcal{A},\mathbb{P},M(G)\big)$, so we get :

$$[\![f]\!] = \sup\left\{ \Big|\int_\Omega \mu_\alpha * f^\alpha(0)d\mathbb{P}(\alpha)\Big| / (\mu_\alpha)\in L^\infty\big(\Omega,\mathcal{A},\mathbb{P},M(G)\big) \quad \text{with} \quad \|\mu_\alpha\|\leq 1 \ a.s.\right\}$$

Therefore : $\forall(\mu_\alpha)$ in the unit ball of $L^\infty\big(\Omega,\mathcal{A},\mathbb{P},M(G)\big)$, the condition (iii) gives : $\exists\mu\in M(G)$ with $\|\mu\|\leq K$ *s.t* $\quad\forall\gamma\in\Lambda:\hat{\mu}(\gamma)=\int_\Omega\hat{\mu}_\alpha(\gamma)\varepsilon_\gamma(\alpha)d\mathbb{P}(\alpha)$; then we have :

$$\int_\Omega \mu_\alpha * f^\alpha(0)d\mathbb{P}(\alpha) = \sum_{\gamma\in\Lambda}\hat{f}(\gamma)\Big(\int_\Omega\hat{\mu}_\alpha(\gamma)\varepsilon_\gamma(\alpha)d\mathbb{P}(\alpha)\Big)\gamma = f*\mu(0)\ ;$$

$$\Big|\int_\Omega \mu_\alpha * f^\alpha(0)d\mathbb{P}(\alpha)\Big| \leq \|\mu\|\|f\|_\infty \leq K\|f\|_\infty$$

taking the upper bound of the left estimate, on the unit ball of $L^\infty\big(\Omega,\mathcal{A},\mathbb{P},M(G)\big)$, we get : $[\![f]\!]\leq K\|f\|_\infty$, that is we get (ii). ■

One may notice that the probabilistic point of view cannot be replaced by a topological one. More precisely, one cannot exchange the notion of "almost sure convergence" and the notion of "quasi-sure convergence" in the foregoing. Λ being a subset of Γ, we consider the Cantor group $\{-1,1\}^\Lambda$ with its usual topology and denote by $r_\gamma(\alpha)$, the γ^{th} coordinate of $\alpha\in\{-1,1\}^\Lambda$, γ belonging to Λ. Let us suppose that Λ shares the following property

$$(P)\left\{\begin{array}{c} \forall f\in C_\Lambda(G),\ \exists\Omega_f,\quad \text{dense } G_\delta \quad \text{in} \quad \{-1,1\}^\Lambda\ s.t.\ \forall\alpha\in\Omega_f:\\ \exists f^\alpha\in C_\Lambda(G)\quad\text{with}\quad \widehat{f^\alpha}(\gamma)=r_\gamma(\alpha)\hat{f}(\gamma)\quad\forall\gamma\in\Lambda\end{array}\right.$$

Then Λ is necessarily a Sidon set.

This follows from the more general following lemma :

Lemma 2.3 *Let X be a Banach space. Assume that the sequence $(x_n)_{n\geq 0}$ in X has the following property :*
there exists Omega$_1$, dense G_δ in $\{-1,1\}^{\mathbb{N}}$ such that for all $\alpha\in\Omega_1$, $\displaystyle\sum_{n\geq 0} r_n(\alpha)x_n$ converges in X.

Then we have : $\displaystyle\sum_{n\geq 0}x_n$ converges unconditionaly in X.

Proof : Fix $p\geq 1$. For every $q\geq 1$, set:

$$F_q = \{\omega\in\Omega_1/\forall m', m\geq q, \|\sum_{n=m}^{m'} r_n(\omega)x_n\|\leq\frac{1}{p}\}.$$

The assumption gives: $\bigcup_{q\in\mathbb{N}^*} F_q = \Omega_1$.

Let $\omega \in \bar{F}_q \cap \Omega_1$: $\forall m' \geq m \geq q$: $\exists \alpha \in F_q$ s.t. $\forall n \leq m' : r_n(\omega) = r_n(\alpha)$.

We then have : $\| \sum_{n=m}^{m'} r_n(\omega)x_n \| = \| \sum_{n=m}^{m'} r_n(\alpha)x_n \| \leq \frac{1}{p}$ for $\alpha \in F_q$. So $\omega \in F_q$ and F_q is closed in Ω_1.

Ω_1 is a Baire space (as an intersection of dense open sets of the compact $\{-1,1\}^{\mathbb{N}}$). So we have :

$$\exists q \geq 1 \quad \overset{\circ}{F_q}^{(\Omega_1)} \neq \emptyset$$

Hence, there exists $c \in \Omega_1$ and $N \geq 1$ with the property that for all $\omega' \in \Omega_1$ such that $r_n(\omega') = r_n(c)$ for each $n \leq N$, one has, for every $m' \geq m \geq q$: $\| \sum_m^{m'} x_n r_n(\omega') \| \leq \frac{1}{p}$ (roughly speaking $B(c;N) \subset F_q$). We set $\tilde{q} = \max(N+1, q)$.

• Take $m' \geq m \geq \tilde{q}$, $\omega \in \{-1,1\}^{\mathbb{N}}$ and define ω_1 by

$$\begin{cases} r_n(\omega_1) & = r_n(c) \quad \text{if } n \leq N \\ r_n(\omega_1) & = r_n(\omega) \quad \text{if } n \geq N+1. \end{cases}$$

Then the density of Ω_1 gives : $\exists \omega' \in \Omega_1$: $r_n(\omega') = r_n(\omega_1)$ for every $n \leq m'$. We then obtain for $m' \geq m \geq \tilde{q} \geq N+1$:

$$\begin{aligned} \| \sum_m^{m'} r_n(\omega)x_n \| &= \| \sum_m^{m'} r_n(\omega_1)x_n \| \\ &= \| \sum_m^{m'} r_n(\omega')x_n \| \\ &\leq \frac{1}{p} \quad \text{for} \quad \omega' \in \Omega_1 \quad \text{and} \quad \omega' \in B(c,N) \subset F_q. \end{aligned}$$

So, we get : $\sum_{n\geq 1} r_n(\omega)x_n$ converge in X for each ω of $\{-1,1\}^{\mathbb{N}}$. ∎

Corollary 2.4 *If a subset Λ of Γ has the property (P), Λ is a Sidon set.*

Proof : Let $f \in C_\Lambda(G)$ and denote Λ by $(\lambda_n)_{n\geq 0}$. We define : $x_n = \hat{f}(\lambda_n)\lambda_n$ and the sequence $(x_n)_{n\geq 0}$ verifies the assumption of lemma 2.3. So, we get : $\sum_{n\geq 0} x_n$ converges unconditionaly in $C_\Lambda(G)$. In particular, we have that $\sum_{n\geq 0} \hat{f}(\lambda_n)\lambda_n$ is unconditionaly convergent for each f in $C_\Lambda(G)$. So, $\{\lambda_n\}$ is an unconditional basis of $C_\Lambda(G)$ and Λ is a Sidon set. ∎

In [12], G. Pisier showed, using the Rudin-Shapiro polynomials, that $\mathbb{Z}$ is not a stationary set and more generally that a stationary set cannot contain arbitrarily large arithmetic progressions. It is easy to see that no infinite discrete abelian group may be a stationary set. We shall show even more in the next proposition.

Proposition 2.5 *Let $\Lambda \subset \Gamma$ a stationary set. Λ cannot contain parallelepipeds of arbitrarily large size.*

Remark We recall that a parallelepiped of size $s \geq 1$ is a set of the form

$$P = \left\{ \beta \prod_{j=1}^{s} \lambda_j^{\varepsilon_j} / \forall\ 1 \leq j \leq s : \varepsilon_j \in \{0,1\} \right\} \tag{5}$$

with $\beta, \lambda_1, \cdots, \lambda_s$ in Γ and where the λ_j are distinct.

Proof : Assuming the converse, Λ contains some parallelepiped of size s, arbitrarily big, and we may also assume that $\{\lambda_j\}$ is quasi-independent. Indeed, let P_N be a parallelepiped of size N included in Λ. With N fixed, P_N has the form (5). One can choose $\lambda_{j_1} \neq 1$ and we assume some elements $\lambda_{j_1}; \cdots; \lambda_{j_p}$ with $p \geq 1$ are chosen such that : $D_p = \{\lambda_{j_q}\}_{1 \leq q \leq p}$ is quasi-independent. We consider the set $A_p = \left\{ \prod_{q=1}^{p} \lambda_{j_q}^{\varepsilon_q} | \varepsilon_q \in \{-1; 0; 1\} \quad \text{for each} \quad 1 \leq q \leq p \right\}$, which is of cardinality less than or equal to 3^p. So the set $\{z \in \{\lambda_j\}_{1 \leq j \leq N} | z \notin A_p\}$ has a cardinality bigger than $N - |A_p|$; then bigger than $N - 3^p$. We are able to continue this construction as long as $N \geq 3^p + 1$, so we can extract $\psi(N)$ elements, forming a quasi-independent set of Γ, with $\psi(N)$ having a growth of the order of $\log N$, therefore diverging to $+\infty$.

So, in the sequel, we suppose that, the parallelepipeds of arbitrarily large size N have the form (5) with $\{\lambda_1, \cdots, \lambda_N\}$ quasi-independent.

Let us fix N and make the following construction, which generalizes that of Rudin and Shapiro : $R_0 = S_0 = \beta$; then we define by induction, for $0 \leq q \leq N-1$

$$\begin{cases} R_{q+1} &= R_q + \lambda_{q+1} S_q \\ S_{q+1} &= R_q - \lambda_{q+1} S_q \end{cases}$$

From the parallelogram law, we get : $|R_{q+1}|^2 + |S_{q+1}|^2 = 2(|R_q|^2 + |S_q|^2)$. So $|R_q|^2 + |S_q|^2 = 2^{q+1}$ and $\|R_q\|_\infty \leq 2^{\frac{q+1}{2}}$.

Now, quasi-independence gives the following properties for the polynomial R_N :

$$(6) \quad \begin{cases} R_N \in \mathcal{P}_\Lambda & (6.1) \\ |\{\gamma \in \Gamma / \hat{R}_N(\gamma) \neq 0\}| = 2^{N+1} & (6.2) \\ \|R_N\|_\infty \leq 2^{\frac{N+1}{2}} & (6.3) \\ \forall \gamma \in \Lambda \quad \hat{R}_N(\gamma) \in \{-1, 0, 1\} & (6.4) \end{cases}$$

Applying the lemma 2.1 to the polynomials R_N with $\delta = 1$, we get using (6.2) and (6.4) :

$$\exists c > 0 \quad [\![R_N]\!] \geq c 2^{\frac{N+1}{2}} \sqrt{N+1} \tag{7}$$

and the stationarity of Λ gives, using (6.1) :

$$[\![R_N]\!] \leq K_s(\Lambda) \|R_N\|_\infty \quad \text{and by (6. 3)} \quad [\![R_N]\!] \leq K_s(\Lambda) 2^{\frac{N+1}{2}} \tag{8}$$

Finally, the relations (7) and (8) lead to : $N \leq (\frac{K_S(\Lambda)}{c})^2$ which gives an upper bound for the size of the parallelepipeds able to be contained in Λ. N being arbitrarily big, the proposition follows by contradiction. ∎

Corollary 2.6 Γ *is not a stationary set.*

In the case $\Gamma = \mathbb{Z}$, we shall deduce more precise results from [10]. Let us recall that Miheev showed the following. If a set $\Lambda = \{n_j\}_{j \geq 0}$ of integers does not contain any parallelepiped of size $S \geq s$ (for some $s \geq 2$), then one has :

$$\left\{ \begin{array}{l} \text{i) } \exists m > 1, \ \exists c > 0 \text{ s.t. } n_j \geq c j^m \quad j = 1, 2, \cdots \\ \text{ii) } \displaystyle\sum_{j \geq 1} \frac{1}{n_j} \quad \text{converges.} \end{array} \right. \tag{9}$$

Corollary 2.7 *Let* $\Lambda = \{n_j\}_{j \geq 0}$ *be a stationary set of integers; then* Λ *enjoys property (9).*

From this, we easily deduce the following proposition :

Proposition 2.8 *The set of prime numbers* $(p_j)_{j \geq 1}$ *is not a stationary set.*

Proof : $\displaystyle\sum_{j \geq 1} \frac{1}{p_j} = \infty.$ ∎

Corollary 2.9 *Let* Λ *be a stationary set in* $\mathbb{Z}$*; then its upper density is zero, namely :*

$$\Delta^+(\Lambda) = \lim_N \Big(\sup_{a \in \mathbb{Z}} \frac{|\Lambda \cap \{a, \cdots, a+N\}|}{N+1} \Big) = 0.$$

3 Stationary sets and sets of continuity

In [6], the authors proved that if Λ is a UC set, included in $\mathbb{N}$, then $\mathbb{Z}^- \cup \Lambda$ is a set of continuity. We shall prove a weaker result for the stationary sets. The proof relies on the following proposition.

Proposition 3.1 *Let Λ be a stationary set in Γ and $\delta > 0$.*
Then the following holds

$$(10) \qquad \forall \mu \in M_\Lambda(G): \quad |\{\gamma \in \Lambda / |\hat{\mu}(\gamma)| \geq \delta\}| \leq \exp\left(\frac{c\|\mu\|^2}{\delta^2}\right)$$

where c is an absolute constant depending only on Λ.
That is, for each μ belonging to M_Λ :

$$(11) \qquad \{\hat{\mu}(\gamma)\}_{\gamma \in \Lambda} \in \ell^{\psi,\infty}$$

where $\psi(t) = e^{t^2} - 1$ and $\ell^{\psi,\infty}$ denotes the space $\{(a_n)/ \sup_{n\geq 1} \psi^{-1}(n) a_n^* < \infty\}$. (a_n^*) being the decreasing rearrangement of $\{|a_n|\}_{n\geq 1}$.

The proof of proposition 3.1 uses the following lemma :

Lemma 3.2 *Let Λ be a stationary set in Γ ; then one has :*

$$(12) \qquad \exists c > 0 \quad \forall \mu \in M_\Lambda(G) \quad \forall h \in L^2(G) : [\![\mu * h]\!] \leq c\|\mu\|_M \|h\|_2.$$

Proof of the lemma : Fix μ in $M_\Lambda(G)$.
Let us first observe that the operator $T_\mu : C(G) \to C^{a.s}(G)$, defined by $T_\mu(h) = \mu * h$, is bounded.
Indeed, $\forall f \in C(G) \quad \mu * f \in C_\Lambda(G)$ hence :

$$[\![T_\mu(f)]\!] = [\![f * \mu]\!] \leq K_S(\Lambda)\|f * \mu\|_\infty \leq K_S(\Lambda)\|\mu\|\|f\|_\infty .$$

Let us recall ([12]) that $C^{a.s}(G)^*$ can be identified to $M_{2,\psi}$, the space of multipliers from $L^2(G)$ to $L^\psi(G)$, hence for each m in $M_{2,\psi}$ and for each ω in Ω, one has : $m^\omega \in M_{2,\psi}$ and $\|m^\omega\|_{M_{2,\psi}} = \|m\|_{M_{2,\psi}}$ ($M_{2,\psi}$ is a space admitting the characters as unconditional basis) where $m^\omega(n) := \varepsilon_n(\omega) m_n$.
So, we have by duality : $\forall \omega \in \Omega : m \to T_\mu^*(m^\omega)$ is bounded from $M_{2,\psi}$ to $M(G)$ and $\|T_\mu^*(m^\omega)\|_{M(G)} \leq \|T_\mu\|\|m\|_{M_{2,\psi}}$.
Therefore: $\forall \omega \in \Omega \quad T_\mu^*(m^\omega) = (\mu * m)^\omega \in M(G)$ and so ([7]) $\mu * m \in L^2(G)$.
Consequently, we have the diagram

$$\begin{tikzcd} M_{2,\psi} \arrow[rr, "T_\mu^*"] \arrow[dr, "U"'] & & M(G) \\ & L^2(G) \arrow[ur, "\text{injection}"'] & \end{tikzcd}$$

and by duality again, we have the following factorization :

$$\begin{tikzcd} C(G) \arrow[rr, "T_\mu"] \arrow[dr, "\text{injection}"'] & & C^{a.s}(G) \\ & L^2(G) \arrow[ur, "U^*"'] & \end{tikzcd}$$

that is : $\exists c>0 : \forall h \in C(G) : [\![T_\mu(h)]\!] = [\![U^*(h)]\!] \leq c\|\mu\|\|h\|_2$ and the density of $C(G)$ in $L^2(G)$ leads to :

$$\exists c>0 \quad \forall h \in L^2(G) \quad [\![T_\mu(h)]\!] \leq C\|\mu\|.\|h\|_2$$

■

Remark : It may be noticed that it is easy to prove the same result using the Kahane-Katznelson-De Leeuw theorem :

$$\exists c>0\, \forall h \in L^2(G),\, \exists f \in C(G)\ s.t : \|f\|_\infty \leq c\|h\|_2 \quad \text{and} \quad \forall \gamma \in \Gamma : |\hat{f}(\gamma)| \geq |\hat{h}(\gamma)|.$$

Another proof, similar to the one given here, can be made through the Pietsch factorization theorem, noticing that T_μ is 2-summing.

Proof of proposition 3.1 : Let μ belonging to $M_\Lambda(G)$ and $\delta>0$. Let $\Lambda_\delta = \{\gamma \in \Lambda / |\hat{\mu}(\gamma)| \geq \delta\}$; denote by Λ'_δ any finite subset of Λ_δ.

$$f := \frac{1}{|\Lambda'_\delta|^{1/2}} \sum_{\gamma\, \in \Lambda'_\delta} \gamma \quad \in L^2(G) \quad \text{and} \quad \|f\|_2 = 1\ .$$

Using lemma 3.2, (12) leads to :

$$\exists c>0 \quad \text{such that} \quad [\![f*\mu]\!] \leq c\|\mu\| \tag{13}$$

Observing that : $\forall \gamma \in \Lambda'_\delta : \widehat{f*\mu}(\gamma) = \frac{1}{|\Lambda'_\delta|^{1/2}}\hat{\mu}(\gamma)$, lemma 2.1 leads to the inequality :

$$\exists c'>0 \quad [\![f*\mu]\!] \geq c' \frac{\delta}{|\Lambda'_\delta|^{1/2}} \Big(|\Lambda'_\delta| \log|\Lambda'_\delta|\Big)^{1/2} = c'\delta\big(\log|\Lambda'_\delta|\big)^{1/2}.$$

Consequently, we obtain via (13):

$$\exists c_1>0, \quad c_1\|\mu\| \geq \delta(\log|\Lambda'_\delta|)^{1/2}\ .$$

Taking the upper bound on all finite subsets Λ'_δ of Λ_δ, we obtain that Λ_δ itself is finite and that : $\exists c_1>0 \quad c_1\|\mu\| \geq \delta\big(\log|\Lambda_\delta|\big)^{1/2}$; equivalently

$$\exists c_1>0 \quad \forall \delta>0 : |\Lambda_\delta| \leq \exp\Big(\frac{c_1^2\|\mu\|^2}{\delta^2}\Big)$$

where c_1 doesn't depend on μ; this proves (10).
This also can be written : $\exists D>0 \quad \forall \delta>0\, \forall \mu \in M_\Lambda, \quad |\Lambda_\delta| \leq \psi(\frac{D\|\mu\|}{\delta})$.
Let $(b_j)_{j\geq 1}$ the decreasing rearrangement of $\{|\hat{\mu}(\gamma)|\}_{\gamma\in\Lambda}$. Given $n \in \mathbb{N}^*$ and $\ell \in \mathbb{N}^*$ such that : $b_\ell \geq \frac{D\|\mu\|}{\psi^{-1}(n)}$, we apply the previous result with $\delta = \big(\psi^{-1}(n)\big)^{-1} D\|\mu\|$ to get:

$$n \geq |\{\gamma \in \Lambda / |\hat{\mu}(\gamma)| \geq \delta\}| = |\{p \in \mathbb{N}^* / b_p \geq \delta\}| \geq \ell$$

so, in particular, $b_n \leq \delta$ and $\sup_n b_n\psi^{-1}(n) \leq D\|\mu\|$; this proves (11). ■

An immediate corollary is :

Corollary 3.3 *Each stationary set* Λ *of* Γ *is a Rajchman set. That is :*

$$\forall \mu \in M_\Lambda(G) \quad \lim_{\gamma \to \infty} \hat{\mu}(\gamma) = 0 \ .$$

We may also deduce the following stronger result.

Theorem 3.4 *Any stationary subset of* $\mathbb{Z}$ *is a set of continuity.*

Proof : Let Λ be a stationary subset of $\mathbb{Z}$. Arguing by contradiction, assume that : $\exists \varepsilon > 0 \quad \forall \delta > 0 \quad \exists \mu \in M(\mathbb{T})$ with $\|\mu\| = 1$: $\overline{\lim}_{n \notin \Lambda} |\hat{\mu}(n)| \le \delta$ and $\overline{\lim}_{n \in \Lambda} |\hat{\mu}(n)| > \varepsilon$; we then have :

$$\exists m = m(\delta) \quad s.t. \quad \forall n \notin \Lambda \quad \text{with} \quad |n| \ge m(\delta) : |\hat{\mu}(n)| \le \delta$$

Let us choose a sequence $(h_j)_{j \ge 0}$ in Λ such that :

$$(14) \qquad \begin{cases} |\hat{\mu}(h_j)| > \varepsilon \quad \text{for all} \quad j \ge 0 \\ \forall p \ge 1 \quad |h_p| \ge \displaystyle\sum_{j=0}^{p-1} |h_j| + m \quad \text{and} \quad |h_0| \ge m \\ \{h_j\}_{j \ge 0} \quad \text{is a dissociated set} \end{cases}$$

Let $N \ge 1$ and $\nu = \mu * R_N - \displaystyle\sum_{n \notin \Lambda} \widehat{\mu * R_N}(n) e_n$,

R_N being the Riesz product : $\displaystyle\prod_{j=1}^{N} [1 + Re(e_{h_j})]$.

ν belongs to M_Λ, so applying proposition 3.1 to ν, we obtain :
there exists $C > 0$ for all $\varepsilon_1 > 0$:

$$(15) \qquad \varepsilon_1^2 \log |\Lambda_{\varepsilon_1}| \le C \|\nu\|^2 \le C \big[\|\mu * R_N\| + \| \sum_{n \notin \Lambda} \widehat{\mu * R_N}(n) e_n \| \big]^2$$

(where Λ_{ε_1} denotes the set $\{n \in \mathbb{Z} / |\hat{\nu}(n)| \ge \varepsilon_1\}$).

But

$$(16) \qquad \|\mu * R_N\| \le \|\mu\| \|R_N\|_1 \le 1$$

and

$$(17) \qquad \| \sum_{n \notin \Lambda} \widehat{\mu * R_N}(n) e_n \| \le \| \sum_{n \notin \Lambda} \widehat{\mu * R_N}(n) e_n \|_2 .$$

One notices that

$$(18) \qquad \|R_N\|_2^2 = \sum_{\substack{s = \sum_{k=1}^{N} \varepsilon_k h_k \\ \varepsilon_k = -1, 0, 1}} |\widehat{R_N}(s)|^2 = \sum_{t=0}^{N} C_N^t \frac{1}{4^t} = \big(\frac{5}{4}\big)^N .$$

In fact, if $s = \sum_{k=1}^{N} \varepsilon_k h_k$ with $\varepsilon_k = -1, 0, 1$ and $\sum_{k=1}^{N} |\varepsilon_k| = t$, $\widehat{R_N}(s) = \frac{1}{2^t}$. On the other hand, $\widehat{R_N}(s) \neq 0$ only for $s = \sum_{k=1}^{N} \varepsilon_k h_k$ with $\varepsilon_k \in \{-1, 0, 1\}$ (and in that case, $|s| \geq m$). So, in this case, for $s \notin \Lambda$:

$$|\hat{\mu}(s)| \leq \delta. \tag{19}$$

Therefore, (17), (18) and (19) lead to :

$$\| \sum_{n \notin \Lambda} \widehat{\mu * R_N}(n) e_n \|_M \leq \delta \big(\frac{5}{4}\big)^{N/2}. \tag{20}$$

$\forall 1 \leq p \leq N : |\hat{\mu}(h_p)| |\widehat{R_N}(h_p)| \geq \frac{\varepsilon}{2}$ hence : $h_p \in \Lambda_{\varepsilon/2}$ so $\{h_1, \cdots, h_N\} \subset \Lambda_{\varepsilon/2}$ and $|\Lambda_{\varepsilon/2}| \geq N$, therefore we get from (15), (16) and (20) the inequality

$$\big(\frac{\varepsilon}{2}\big)^2 \log N \leq C[1 + \delta (\frac{5}{4})^{N/2}]^2. \tag{21}$$

Now, let us adjust N such that : $(\frac{\varepsilon}{2})^2 \log N > 4C$ and δ such that : $\delta < (\frac{5}{4})^{-N/2}$. (21) leads to a contradiction. Therefore, we have proved theorem 3.4 by contradiction. ■

4 Stationary sets and UC sets

Let us recall that G. Pisier proved the existence of some stationary sets that are not Sidon (conversely, any Sidon set is trivially stationary). We shall generalize this result by exhibiting a class of stationary sets that are not UC sets. Thus, it is possible to construct stationary subsets of $\mathbb{Z}$ rather big in the following sense : $\forall k \geq 1 :\ \exists \Lambda_k$ stationary, $\exists \delta_k > 0$

$$\forall N \geq 1 : \quad |\Lambda \cap [-N, N]| \geq \delta_k (\log N)^k.$$

Theorem 4.1 *Let E be a dissociated set in Γ, $E = \{\lambda_j\}_{j \geq 1}$. Let k be an integer bigger than* 1. *Then*

$$\Lambda_k := \Big\{ \prod_{p=1}^{k} \lambda_{j_p}^{\varepsilon_p} /\ 1 \leq p \leq k; \varepsilon_p \in \{-1, 1\}; j_p \geq 1 \text{ distinct} \Big\}$$

is a stationary subset of Γ.

Proof : We first follow the method of Blei in [1]. In fact, we have:

$$\Lambda_k = \Big\{ \prod_{p=1}^{k} \lambda_{j_p} | j_p \quad \text{distinct} \Big\} \quad \cup \bigcup_{\ell=0}^{k-1} \Big\{ \prod_{p=1}^{\ell} \lambda_{j_p} \prod_{p=\ell+1}^{k} \bar{\lambda}_{j_p} | j_p \quad \text{distinct} \Big\}$$

so that any f in $\mathcal{P}_{\Lambda_k}(G)$ can be written (in the following: $\sum_{(j_p)}{}'$ will mean $j_1 < \cdots < j_\ell$ and $j_{\ell+1} < \cdots < j_k$ for $0 \leq \ell \leq k-1$, and $j_1 < \cdots < j_k$ for $\ell = k$):

$$f = \sum_{\ell=0}^{k-1} \Big(\sum_{(j_p)}{}' \quad \hat{f}(\lambda_{j_1} \cdots \lambda_{j_\ell} \bar{\lambda}_{j_\ell+1} \cdots \bar{\lambda}_{j_k}) \lambda_{j_1} \cdots \lambda_{j_\ell} \bar{\lambda}_{j_{\ell+1}} \cdots \bar{\lambda}_{j_k} \Big)$$
$$+ \sum_{(j_p)}{}' \hat{f}(\prod_{p=1}^{k} \lambda_{j_p}) \prod_{p=1}^{k} \lambda_{j_p}$$

Let us define F in $\mathcal{P}\,\underbrace{(G \times \cdots \times G)}_{k \text{ times}}$ by $F = \sum_{\ell=0}^{k} F_\ell$ where:

$$(22) \quad \left\{ \begin{array}{l} F_k = \displaystyle\sum_{\substack{(j_p) \\ \text{distinct}}} \hat{f}(\prod_{p=1}^{k} \lambda_{j_p}) \lambda_{j_1} \otimes \cdots \otimes \lambda_{j_k} \ ; \ F_k \in \mathcal{P}_{E \times \cdots \times E}(G^k) \\ \\ \text{and} \\ \\ F_\ell = \displaystyle\sum_{\substack{(j_p) \\ \text{distinct}}} \quad \sum_{\substack{\varepsilon_i = \pm 1 \\ \varepsilon_1 + \cdots + \varepsilon_k = 2l-k}} \hat{f}(\lambda_{j_1} \cdots \lambda_{j_\ell} \bar{\lambda}_{j_{\ell+1}} \cdots \bar{\lambda}_{j_k}) \lambda_{j_1}^{\varepsilon_1} \otimes \cdots \otimes \lambda_{j_k}^{\varepsilon_k} \end{array} \right.$$

In the sequel, the case $\ell = 0$ and $\ell = k$ are treated in the same way. Fixing $0 \leq \ell \leq k-1$, F_ℓ is symmetrized by writting :

$$(23) \quad \left\{ \begin{array}{ll} F_\ell = & \sum_{m=1}^{k} (-1)^{m+k} \sum \widehat{F_\ell}\Big(\lambda_{j_1}; \cdots; \lambda_{j_\ell}; \bar{\lambda}_{j_{\ell+1}}; \cdots; \bar{\lambda}_{j_k}\Big) \times \\ & \psi_S(\lambda_{j_1}) \cdots \psi_S(\lambda_{j_\ell}) \overline{\psi_S(\lambda_{j_{\ell+1}})} \cdots \overline{\psi_S(\lambda_{j_k})} \end{array} \right.$$

where the second sum runs over the subsets S of $\{1, \cdots, k\}$ with cardinal m and over the indices (j_p) distinct ($1 \leq p \leq k$) and where $\psi_S(\gamma)(g_1, \cdots, g_k)$ is equal to $\sum_{r \in S} \gamma(g_r)$ with $(g_1, \cdots, g_k) \in G^k$.

Fixing (again) m in $\{1, \cdots, k\}$ and S included in $\{1, \cdots, k\}$ with $|S| = m$, we note $\tilde{F}$ for :

$$\sum_{\substack{(j_p) \\ \text{distinct}}} \widehat{F_\ell}\Big(\lambda_{j_1}, \cdots, \lambda_{j_\ell}, \bar{\lambda}_{j_{\ell+1}}, \cdots, \bar{\lambda}_{j_k}\Big) \psi_S(\lambda_{j_1}) \cdots \psi_S(\lambda_{j_\ell}) \overline{\psi_S(\lambda_{j_{\ell+1}})} \cdots \overline{\psi_S(\lambda_{j_k})}$$

(noticing that $\psi_S(\bar{\gamma}) = \overline{\psi_S(\gamma)}$).

One has : $\tilde{F} \in \mathcal{P}_{E \times \cdots \times E \times \bar{E} \times \cdots \times \bar{E}}$.

Fix $g_1, \cdots, g_k$ in G and set

(24) $$V := \tilde{F}(g_1, \cdots, g_k) \ .$$

Introducing the measure ν defined by the Riesz product : $\prod_{\gamma \in E} [1 + Re(e^i \gamma)]$, we have

$$\hat{\nu}(\lambda_{j_1} \cdots \lambda_{j_\ell} \bar{\lambda}_{j_{\ell+1}} \cdots \bar{\lambda}_{j_k}) = \frac{e^{i\ell} e^{-i(k-\ell)}}{2^k} := a_\ell \ .$$

There is a polynomial P_ℓ (depending only on k and ℓ) such that :

$$P_\ell(a_\ell) = 1 \quad \text{and} \quad P_\ell(a_t) = 0 \quad \text{whenever} \quad t \neq \ell \ .$$

We now set $\mu_\ell = P_\ell(\nu)$ (where the product on $M(G)$ is convolution) and observe that :

(25) $$\begin{cases} \hat{\mu}_\ell(\lambda_{j_1} \cdots \lambda_{j_t} \bar{\lambda}_{j_{t+1}} \cdots \bar{\lambda}_{j_k}) = \delta_{t,\ell} \quad \text{(Kronecker's symbol)} \\ \text{for any } (j_1, \cdots, j_k) \text{ distinct} \\ \\ \text{and} \\ \\ \|\mu_\ell\| \leq C_k \ ; C_k \text{ depending only on } k. \end{cases}$$

At last, we consider the Riesz product :

$$\mathcal{R} = \prod_{j \geq 0} [1 + Re(\eta_j \lambda_j)]$$

where : $\eta_j = \frac{\psi_S(\lambda_j)}{2^m}(g_1, \cdots, g_k)$ (notice that : $|\eta_j| \leq 1$).
One easily checks that (remember (24)) : $V = 2^{mk} 2^k \mu_\ell * \mathcal{R} * f(0)$ and concludes, using (25), that :

$$|V| \leq 2^{mk} 2^k \|\mu_\ell\|_M \|\mathcal{R}\|_M \|f\|_\infty \leq 2^{mk} 2^k C_k \|f\|_\infty \ ,$$

and then, taking the upper bound on G^k, we obtain :

$$\|\tilde{F}\|_\infty \leq 2^{mk} 2^k C_k \|f\|_\infty \ .$$

Now, considering (23) and the previous majorization, we have :

$$\|F\|_\infty \leq \sum_{\ell=0}^{k} \|F_\ell\|_\infty \leq \sum_{\ell=0}^{k} \sum_{m=1}^{k} \sum_{|S|=m} 2^{mk} 2^k C_k \|f\|_\infty$$

and

(26) $$\|F\|_\infty \leq A_k \|f\|_\infty$$

(with $A_k = k 2^k (2^k + 1)^k C_k$).

E being a dissociated set is a Sidon set; so is $E \cup \bar{E}$, hence using the Pisier theorem ([12]) : $\underbrace{E_1 \times \cdots \times E_1}_{k \text{ times}}$ is a stationary set in Γ^k, E_1 denoting $E \cup \bar{E}$. So, there is $B_k > 0$, such that $\forall F \in \mathcal{P}_{E_1 \times \cdots \times E_1}(G^k)$:

$$\int_\Omega \| \sum_{\beta \in E_1^k} \mathcal{E}_\beta(\omega) \hat{F}(\beta) \beta \|_\infty d\mathbb{P}(\omega) \leq B_k \|F\|_\infty. \tag{27}$$

On the other hand, fixing f in $\mathcal{P}_{\Lambda_k}(G)$, one observes that:

$$\int_\Omega \Big\| \sum_{\ell=0}^{k} \sum_{(j_p)}{}' \mathcal{E}_{\ell, j_1, \cdots, j_k}(\omega) \hat{f}(\lambda_{j_1} \cdots \lambda_{j_\ell} \bar{\lambda}_{j_{\ell+1}} \cdots \bar{\lambda}_{j_k}) \lambda_{j_1} \cdots \lambda_{j_\ell} \bar{\lambda}_{j_{\ell+1}} \cdots \bar{\lambda}_{j_k} \Big\|_{C(G)} d\mathbb{P}(\omega)$$

is less than :

$$\int_\Omega \Big\| \sum_{\ell=0}^{k} \sum_{\substack{(j_p) \\ \text{distinct}}} \mathcal{E}_{\ell, j_1, \cdots, j_k}(\omega) \hat{f}(\lambda_{j_1} \cdots \lambda_{j_\ell} \bar{\lambda}_{j_{\ell+1}} \cdots \bar{\lambda}_{j_k}) \lambda_{j_1} \otimes \cdots \otimes \lambda_{j_\ell} \otimes \bar{\lambda}_{j_{\ell+1}} \otimes \cdots \otimes \bar{\lambda}_{j_k} \Big\|_{C(G^k)} d\mathbb{P}(\omega)$$

then (using the contraction principle), this gives:

$$(28) \begin{cases} [\![f]\!] \leq [\![F]\!] \\ \text{where } F \text{ as the form given by (22).} \end{cases}$$

Combining (26), (27) and (28), we obtain : $[\![f]\!] \leq A_k B_k \|f\|_\infty$ and Λ_k is a stationary set of Γ. ∎

Corollary 4.2 *There are some stationary subsets of $\mathbb{Z}$, that are not UC sets.*

Proof : For example, if H a Hadamard sequence, $H - H + H$ is not a UC set ([5] or [8]) but is a stationary set by theorem 4.1. More generally, let us recall that if E, F are infinite sets included in $\mathbb{N}$, $E - F$ is not a CUC set. ∎

Let us recall some known facts of [6].

Definition 4.3 *A pair (Q, R) of subsets of $\mathbb{Z}$ is an alternating pair of size N if $|Q| = |R| = N$ and, writting $Q = (q_n)_{1 \leq n \leq N}$ and $R = (r_n)_{1 \leq n \leq N}$, one has: $q_2 - q_1 \leq r_2 - r_1 < q_3 - q_1 \leq r_3 - r_1 < \cdots$.*

In [6], the authors show that, UC sets included in $\mathbb{N}$ cannot contain such differences of alternating pairs for sizes too big (the boundedness of the size depending on the UC constant).

On the other hand, let us construct a particular dissociated set E. We consider the set $\Lambda_1 = \bigcup_{n \text{odd}} (4^{n!} + E_n)$ and the set $\Lambda_2 = \bigcup_{n \text{even}} (4^{n!} - E_n)$ where

$E_n = \{4^{n!-k}\}_{\substack{k \text{ odd} \\ 1\le k<n}}$ if n odd and $E_n = \{4^{(n-1)!-k}\}_{\substack{k \text{ even} \\ 2\le k<n}}$ if n even. The set E is $\Lambda_1 \cup \Lambda_2$.

Corollary 4.4 *The same conclusion as 4.2 holds, noticing that it is possible to construct stationary set included in* $\mathbb{N}$ *containing translates of difference* $Q-R$ *arising from an alternating pair* (Q,R) *of size arbitrarily large.*

Proof : With the previous notations, it suffices to consider the set $\Lambda = E+E$. Applying theorem 4.1, one has that Λ is stationary. On the other hand Λ is not a UC set.

Indeed, for odd n, Λ contains translates of the difference $E_n - E_{n+1}$. As Λ is included in $\mathbb{N}$, it should be a CUC set too (e.g.[8]) and then $\Lambda - 4^{n!}$ should be a UC set with bounded UC-constant. On the other hand, $\Lambda - 4^{n!}$ contains difference arising from the alternating pair (E_n, E_{n+1}). There is a contradiction for n large enough ([6]). ■

Corollary 4.5 *For all* $k \ge 1$*, there are some stationary sets* Λ_k *verifying :*

$$\exists \delta_k > 0 \quad \forall N \ge 1 \quad |\Lambda_k \cap [-N,N]| \ge \delta_k (\log N)^k.$$

Proof : It suffices to consider the set $\Lambda_k = \{3^{m_1} + \cdots + 3^{m_k} | m_j \quad \text{distinct}\}$ and one easily checks that: $\forall N \ge 2k$:

$$|\Lambda_k \cap \{0, \cdots, k3^N\}| \ge (N+2)\cdots(N+2-k) \ge (\frac{N+2}{2})^k.$$

■

Remark Theorem 4.1 is optimal in this sense that there is E Sidon such that $\mathbb{N} \subset E+E$ and then $E+E$ not stationary in $\mathbb{Z}$. Indeed, it suffices to consider $E = \{10^n + n\}_{n\in\mathbb{N}} \cup \{-10^n\}_{n\in\mathbb{N}}$.

General remark Essentially we used the fact that $[\![\cdot]\!]$ is an unconditional norm for characters satisfying : $\exists \psi : \mathbb{R}^+ \to \mathbb{R}^+$ s.t $\lim_{x\to+\infty} \psi(x) = +\infty$ and $\forall A \subset \Gamma$, finite $\left[\!\!\left[\sum_{\gamma\in A} \gamma\right]\!\!\right] \ge \psi(|A|) \left\|\sum_{\gamma\in A}\gamma\right\|_2$. So, it may be noticed that the previous results hold for some other lacunary sets of harmonic analysis, for example p-Sidon sets. Using the same previous methods, this is easy to refind all of them essentially replacing *stationary* by *p-Sidon*.

More precisely, the proposition 3.1, for example, can be strengthened, when Λ is a p-Sidon set ($1 \le p < 2$): we recover, by other methods, an inequality due to Edwards (see e.g.[6]). Denoting $\frac{2p}{2-p}$ by r, there exists a constant $C > 0$ such that:

$$\forall \mu \in M_\Lambda(G), \quad \|\hat{\mu}\|_{\ell_r} \le C\|\mu\|.$$

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References

[1] R. Blei, Sidon partitions and p-Sidon sets, Pacific J. Math 65 (1976), 307-313.

[2] J. Bourgain, Une remarque sur les ensembles stationnaires, Publication Math d'Orsay, exp. 2 (1981-82).

[3] M. Bożejko and T. Pytlik, Some types of lacunary series, Colloq. Math. 25 (1972) 117-124.

[4] M. Dechamps-Gondim, Sur les compacts associés aux ensembles lacunaires, les ensembles de Sidon et quelques problèmes ouverts, Publ. Math. Orsay 84-01 (1984).

[5] J.J.F Fournier, Two UC-sets whose union is not a UC set, Proc. AMS, 84 (1982) 69-72,

[6] Fournier-Pigno, Analytic and arithmetic properties of thin sets, Pacific J. Math. 105 (1983) 115-141.

[7] J.P. Kahane, Some random series of functions, Cambridge 5 (1985).

[8] P. Lefevre, Sur les ensembles de convergence uniforme, Publ. Math. Orsay, 94-24 (1994).

[9] J.M. Lopez and K.A. Ross, Sidon sets, Lecture notes in Pures and Appli. Math. 13, Marcel Dekker NY (1975).

[10] IM. Miheev, Trigonometric series with gaps, Analysis Math. 9 (1983) 43-55.

[11] Marcus-Pisier, Random Fourier series with application to harmonic analysis, Annals of Math Studies. Princeton University Press.

[12] G. Pisier, Sur l'espace de Banach des séries de Fourier aléatoires presque surement continues, Sem. Géometrie des Espaces de Banach. Ecole Polytechnique 1977-78.

[13] G. Pisier, De nouvelles caractérisations des ensembles de Sidon, Math. analysis and Appl. - Academic Press (7B).

[14] Salem and Zygmund, Some properties of trigonometric series whose terms have random signs, Acta. Math. 91, 1954, 245-301.

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