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Isometric and invertible composition operators on weighted Bergman spaces of Dirichlet series

Maxime Bailleul

Abstract

We show that a composition operator on weighted Bergman spaces \mathcal{A}_μ^p is invertible if and only if it is Fredholm if and only if it is an isometry.

1 Introduction

In [8], the authors defined the Hardy space \mathcal{H}^2 of Dirichlet series with square-summable coefficients. Thanks to the Cauchy-Schwarz inequality, it is easy to see that \mathcal{H}^2 is a space of analytic functions on $\mathbb{C}_{\frac{1}{2}} := \{s \in \mathbb{C}, \Re(s) > \frac{1}{2}\}$. F. Bayart introduced in [3] the more general class of Hardy spaces of Dirichlet series \mathcal{H}^p ($1 \leq p < +\infty$). In another direction, McCarthy defined in [12] some weighted Bergman Hilbert spaces of Dirichlet series and these spaces have been generalized in [2].

In order to recall how these spaces are defined, we need to recall the principle of the Bohr's point of view: let $n \geq 2$ be an integer, it can be written (in a unique way) as a product of prime numbers $n = p_1^{\alpha_1} \cdot p_k^{\alpha_k}$ where $p_1 = 2, p_2 = 3$ etc ... For $s \in \mathbb{C}$, we consider $z = (z_1, z_2, \dots) = (p_1^{-s}, p_2^{-s}, \dots)$. Then, writing

$$f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} \quad (1)$$

we get

$$f(s) = \sum_{n=1}^{+\infty} a_n (p_1^{-s})^{\alpha_1} \cdot (p_k^{-s})^{\alpha_k} = \sum_{n=1}^{+\infty} a_n z_1^{\alpha_1} \cdot z_k^{\alpha_k}.$$

So we can see a Dirichlet series as a Fourier series on the infinite-dimensional polytorus $\mathbb{T}^\infty = \{(z_1, z_2, \dots), |z_i| = 1, \forall i \geq 1\}$. We shall denote this Fourier series $D(f)$.

Let us fix now $p \geq 1$. The space $H^p(\mathbb{T}^\infty)$ is the closure of the set of analytic polynomials with respect to the norm of $L^p(\mathbb{T}^\infty, m)$ where m is the normalized Lebesgue measure on \mathbb{T}^∞ . Let f be a Dirichlet polynomial, $D(f)$ is then an analytic polynomial on \mathbb{T}^∞ by the Bohr's point of view. By definition, $\|f\|_{\mathcal{H}^p} := \|D(f)\|_{H^p(\mathbb{T}^\infty)}$ and \mathcal{H}^p is the closure of the set of Dirichlet polynomials with respect to this norm. The spaces \mathcal{H}^p and $H^p(\mathbb{T}^\infty)$ are then isometrically isomorphic.

We recall now how we can define the weighted Bergman spaces of Dirichlet series. For $\sigma > 0$, f_σ will be the translate of f by σ , *i.e.* $f_\sigma(s) := f(\sigma + s)$. We shall denote by \mathcal{P} the set of Dirichlet polynomials.

Let $p \geq 1$, $P \in \mathcal{P}$ and μ be a probability measure on $(0, +\infty)$ such that $0 \in \text{Supp}(\mu)$. Then

$$\|P\|_{\mathcal{A}_\mu^p} := \left(\int_0^{+\infty} \|P_\sigma\|_{\mathcal{H}^p}^p d\mu(\sigma) \right)^{1/p}.$$

\mathcal{A}_μ^p is the completion of \mathcal{P} with respect to this norm. When $d\mu(\sigma) = 2e^{-2\sigma} d\sigma$, these spaces are simply denoted by \mathcal{A}^p . It is shown in [2] that they are spaces of convergent Dirichlet series on $\mathbb{C}_{1/2}$.

In [7], the bounded composition operators on \mathcal{H}^2 , in other words the analytic functions $\Phi : \mathbb{C}_{\frac{1}{2}} \rightarrow \mathbb{C}_{\frac{1}{2}}$ such that for any $f \in \mathcal{H}^2$, $f \circ \Phi \in \mathcal{H}^2$, are characterized. In [3], F. Bayart generalized this result to the space \mathcal{H}^p when $p \geq 1$.

We denote by \mathcal{D} the set of functions f which admit a representation by a convergent Dirichlet series in some half-plane and for $\theta \in \mathbb{R}$, \mathbb{C}_θ will be the following half-plane $\{s \in \mathbb{C}, \Re(s) > \theta\}$. We shall denote \mathbb{C}_+ instead of \mathbb{C}_0 .

On the spaces \mathcal{A}_μ^p , the following theorems have been proved in [1]:

Theorem 1 ([1],Th1). *Let $\Phi : \mathbb{C}_{\frac{1}{2}} \rightarrow \mathbb{C}_{\frac{1}{2}}$ be an analytic function of the form $\Phi(s) = c_0s + \varphi(s)$ where $c_0 \geq 1$ and $\varphi \in \mathcal{D}$. Then C_Φ is bounded on \mathcal{A}_μ^p if and only if φ converges uniformly in \mathbb{C}_ε for every $\varepsilon > 0$ and $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$. Moreover in this case, C_Φ is a contraction.*

Theorem 2 ([1],Th2). *Let $\Phi : \mathbb{C}_{\frac{1}{2}} \rightarrow \mathbb{C}_{\frac{1}{2}}$ be in \mathcal{D} . Then*

- (i) *If C_Φ is bounded on \mathcal{A}_μ^p then Φ converges uniformly in \mathbb{C}_ε for every $\varepsilon > 0$ and $\Phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$.*
- (ii) *If Φ converges uniformly in \mathbb{C}_ε for every $\varepsilon > 0$ and $\Phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2+\eta}$ with some $\eta > 0$ then C_Φ is bounded on \mathcal{A}_μ^2 .*

In the sequel, we assume that μ is a probability measure on $(0, +\infty)$ such that $d\mu(\sigma) = h(\sigma)d\sigma$ where h is a positive continuous function.

Example. Let $\alpha > -1$, we denote μ_α the probability measure defined on $(0, +\infty)$ by

$$d\mu_\alpha(\sigma) = \frac{2^{\alpha+1}}{\Gamma(\alpha+1)} \sigma^\alpha e^{-2\sigma} d\sigma.$$

We denote the corresponding space \mathcal{A}_α^p instead of $\mathcal{A}_{\mu_\alpha}^p$.

Main Theorem. *Let $1 \leq p < +\infty$ and C_Φ be a bounded composition operator on \mathcal{A}_μ^p . The following assertions are equivalent:*

- (i) *C_Φ is invertible.*
- (ii) *C_Φ is Fredholm.*
- (iii) *C_Φ is an isometry.*
- (iv) *Φ is a vertical translation: there exists $\tau \in \mathbb{R}$ such that for every $s \in \mathbb{C}_+$, $\Phi(s) = s + i\tau$.*

We point out that the result is false on the spaces \mathcal{H}^p : F. Bayart proved that (i), (ii), (iii) are still equivalent on \mathcal{H}^p but obtained a different characterization for the isometric composition operators on \mathcal{H}^p (see [3]). For example, if Φ is defined for every $s \in \mathbb{C}_+$ by $\Phi(s) = c_0s$ with $c_0 \geq 2$, then C_Φ is an isometry on \mathcal{H}^p but not on \mathcal{A}_μ^p . The same phenomenon appears in the framework of composition operators on the unit disk (see [10]).

In order to prove the main theorem, it suffices to show that (ii) \Rightarrow (iv) and (iii) \Rightarrow (iv). Indeed, (i) \Rightarrow (ii), (iv) \Rightarrow (i) and (iv) \Rightarrow (iii) are clear.

2 Background material

Let f be a Dirichlet series of form (1). We do not recall the definition of abscissa of simple (resp. absolute) convergence denoted by σ_c (resp. σ_a), see [13] or [14] for more details. We shall need the two other following abscissas:

$$\begin{aligned}\sigma_u(f) &= \inf\{a \mid \text{The series (1) is uniformly convergent for } \Re(s) > a\} \\ &= \text{abscissa of uniform convergence of } f.\end{aligned}$$

$$\begin{aligned}\sigma_b(f) &= \inf\{a \mid \text{the function } f \text{ has an analytic, bounded extension for } \Re(s) > a\} \\ &= \text{abscissa of boundedness of } f.\end{aligned}$$

It is easy to see that $\sigma_c(f) \leq \sigma_u(f) \leq \sigma_a(f)$. An important result is that $\sigma_u(f)$ and $\sigma_b(f)$ coincide: this is the Bohr's theorem (see [5]). This result is really useful for the study of \mathcal{H}^∞ , the algebra of bounded Dirichlet series on the right half-plane \mathbb{C}_+ (see [11]). We shall denote by $\|\cdot\|_\infty$ the norm on this space:

$$\|f\|_\infty := \sup_{\Re(s) > 0} |f(s)|.$$

We shall make a crucial use of the point evaluation in the proof of the Main Theorem: for every $p \geq 1$, the spaces \mathcal{H}^p and \mathcal{A}_μ^p are spaces of holomorphic functions on $\mathbb{C}_{1/2}$ and more precisely if δ_s is the operator of point evaluation at $s \in \mathbb{C}_{1/2}$, then by [3], Th3:

$$\|\delta_s\|_{(\mathcal{H}^p)^*} = \zeta(2\Re(s))$$

and by [2], Th1 the point evaluation is also bounded on the spaces \mathcal{A}_μ^p . Moreover $\sigma_b(f) \leq 1/2$ for any $f \in \mathcal{A}_\mu^p$. For example when $\mu = \mu_\alpha$, it is shown in [2], Cor1 that there exists a positive constant $c_{\alpha,p}$ such that for every $s \in \mathbb{C}_{1/2}$,

$$\|\delta_s\|_{(\mathcal{A}_\alpha^p)^*} \leq c_{\alpha,p} \left(\frac{\Re(s)}{2\Re(s) - 1} \right)^{\frac{2+\alpha}{p}}.$$

When $p = 2$, \mathcal{A}_μ^2 is a Hilbert space and it is easy to see that

$$\|f\|_{\mathcal{A}_\mu^2} = \left(\sum_{n=1}^{+\infty} |a_n|^2 w_h(n) \right)^{1/2}$$

where for every $n \geq 1$,

$$w_h(n) = \int_0^{+\infty} n^{-2\sigma} h(\sigma) d\sigma.$$

Thanks of the boundedness of the point evaluation at $s \in \mathbb{C}_{1/2}$, we consider the following reproducing kernels defined for every $w \in \mathbb{C}_{1/2}$ by

$$K_\mu(s, w) = \sum_{n=1}^{+\infty} \frac{n^{-\bar{s}-w}}{w_h(n)}.$$

For every $f \in \mathcal{A}_\mu^2$ and $s \in \mathbb{C}_{1/2}$, one has

$$f(s) = \langle f, K_\mu(s, \cdot) \rangle_{\mathcal{A}_\mu^2}.$$

Example. On the space \mathcal{A}_α^2 , we simply denote (w_n^α) the corresponding weight and then for every $n \geq 1$,

$$w_n^\alpha = \frac{1}{(\log(n) + 1)^{\alpha+1}}.$$

Let $\Phi : \mathbb{C}_{\frac{1}{2}} \rightarrow \mathbb{C}_{\frac{1}{2}}$ be an analytic function such that $\Phi(s) = c_0 s + \varphi(s)$ where c_0 is a nonnegative integer and $\varphi \in \mathcal{D}$. We shall say that Φ is a symbol if C_Φ is bounded on the spaces \mathcal{A}_μ^p .

For $\sigma > 0$, we denote Φ_σ the translate of Φ by σ : $\Phi_\sigma(s) := \Phi(\sigma + s)$.

When $c_0 \geq 1$, thanks to the Theorem 1 we know that Φ is a symbol if and only φ converges uniformly on \mathbb{C}_ε for every $\varepsilon > 0$ and $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$. In this case, it is easy to see that for every $\sigma > 0$, $\Phi_\sigma - \sigma$ is also a symbol: indeed let $\sigma > 0$ and $s \in \mathbb{C}_+$, then

$$\begin{aligned} \Re(\Phi_\sigma(s) - \sigma) &= \Re(c_0(\sigma + s)) + \Re(\varphi(\sigma + s)) - \sigma \\ &> \sigma(c_0 - 1) + \Re(s) > 0 \end{aligned}$$

because $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$ and $s \in \mathbb{C}_+$. Point out that this result can be seen as the Schwarz's lemma in this framework.

3 Proof of $(ii) \Rightarrow (iv)$

With help of Proposition 4.2 from [7], F. Bayart proved the following useful lemma.

Lemma 1 ([3],Lem11). *Let Φ be a symbol. If Φ is not a vertical translation then there exists ε and $\eta > 0$ such that*

$$\Phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}.$$

Proof of $(ii) \Rightarrow (iv)$. We follow ideas from [3],Th14. Assume that Φ is not a vertical translation. By the previous lemma, there exists ε and $\eta > 0$ such that

$$\Phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}.$$

We remark that each element of $\text{Im}(C_\Phi)$ is defined and bounded on $\mathbb{C}_{1/2-\varepsilon}$: indeed $\Phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}$ and if $f \in \mathcal{A}_\mu^p$, f is bounded on $\mathbb{C}_{1/2+\eta}$ (because $\sigma_b(f) \leq 1/2$).

Now by lemma [3],Lem9 we know that there exists $f \in \mathcal{H}^p$ such that the line $\Re(s) = 1/2$ is both abscissa of convergence and natural boundary for f . Because of the inclusion $\mathcal{H}^p \subset \mathcal{A}_\mu^p$, f belongs to \mathcal{A}_μ^p . We consider the following infinite dimensional subspace of \mathcal{A}_μ^p :

$$F = \text{span}\{n^{-s}f, n \geq 1\} = f\mathcal{P}.$$

We shall show that $F \cap \text{Im}(C_\Phi) = \{0\}$ and consequently $\text{Codim}(\text{Im}(C_\Phi)) = +\infty$ which is a contradiction with (ii) .

Let $h \in F \cap \text{Im}(C_\Phi)$, there exists $P \in \mathcal{P}$ such that $h = Pf$. If $h \neq 0$, there exists s_0 such that $\Re(s_0) = 1/2$ and $P(s_0) \neq 0$. But in this case, f extends beyond $\mathbb{C}_{1/2}$ and then we obtain a contradiction because the line $\Re(s) = 1/2$ is a natural boundary for f . Finally $F \cap \text{Im}(C_\Phi) = \{0\}$. \square

4 Proof of (iii) \Rightarrow (iv)

First we shall show that if C_Φ is an isometry then $c_0 \geq 1$. We need the following result.

Lemma 2. $\lim_{\Re(s) \rightarrow +\infty} \|\delta_s\|_{(\mathcal{A}_\mu^1)^*} = 1$.

Proof. Let $s \in \mathbb{C}_1$. By the reproducing kernel property on \mathcal{A}_μ^2 (or just by a simple computation), for any Dirichlet polynomial we have

$$P(s) = \int_0^{+\infty} \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^T P(\sigma + it) \overline{K_\mu(s, \sigma + it)} dt d\mu(\sigma).$$

Now by definition of the norm of Dirichlet polynomials in \mathcal{H}^1 (see definition 1 from [3]), we have

$$\|P\|_{\mathcal{A}_\mu^1} = \left(\int_0^{+\infty} \lim_{T \rightarrow +\infty} \frac{1}{2T} \int_{-T}^T |P(\sigma + it)| dt d\mu(\sigma) \right).$$

Consequently

$$|P(s)| \leq \|P\|_{\mathcal{A}_\mu^1} \times \|K_\mu(s, \cdot)\|_\infty.$$

Now

$$\|K_\mu(s, \cdot)\|_\infty = \sup_{w \in \mathbb{C}_+} \left| \sum_{n=1}^{+\infty} \frac{n^{-\bar{s}+w}}{w_h(n)} \right| \leq \sum_{n=1}^{+\infty} \frac{n^{-\Re(s)}}{w_h(n)}$$

and we point out that $w_h(1) = 1$ so

$$\limsup_{\Re(s) \rightarrow +\infty} \|\delta_s\|_{(\mathcal{A}_\mu^1)^*} \leq \lim_{\Re(s) \rightarrow +\infty} \sum_{n=1}^{+\infty} \frac{n^{-\Re(s)}}{w_h(n)} = 1.$$

On the other hand, it is clear that $\|\delta_s\|_{(\mathcal{A}_\mu^1)^*} \geq 1$ and then we obtain the result. \square

Proposition 1. *Let Φ be a symbol. If C_Φ is a contraction then $c_0 \geq 1$.*

Proof. Let $s \in \mathbb{C}_{1/2}$. For every $f \in \mathcal{A}_\mu^p$ we have

$$|f \circ \Phi(s)| \leq \|\delta_s\|_{(\mathcal{A}_\mu^p)^*} \|f \circ \Phi\| \leq \|\delta_s\|_{(\mathcal{A}_\mu^p)^*} \|C_\Phi\| \|f\|$$

and then

$$\frac{\|\delta_{\Phi(s)}\|_{(\mathcal{A}_\mu^p)^*}}{\|\delta_s\|_{(\mathcal{A}_\mu^p)^*}} \leq \|C_\Phi\|.$$

By inclusion of the spaces \mathcal{A}_μ^p and the fact that $\mathcal{H}^p \subset \mathcal{A}_\mu^p$ with $\|\cdot\|_{\mathcal{A}_\mu^p} \leq \|\cdot\|_{\mathcal{H}^p}$ we obtain:

$$\frac{\|\delta_{\Phi(s)}\|_{(\mathcal{H}^p)^*}}{\|\delta_s\|_{(\mathcal{A}_\mu^1)^*}} \leq \|C_\Phi\|.$$

By Theorem 3 from [3], we obtain

$$\zeta(2\Re(\Phi(s)))^{1/p} \times \|\delta_s\|_{(\mathcal{A}_\mu^1)^*}^{-1} \leq \|C_\Phi\|.$$

Now assume $c_0 = 0$, then $\Phi(s) = \varphi(s) = \sum_{n=1}^{+\infty} c_n n^{-s}$ and $\Re(c_1) > 1/2$ (see proof of Lemma 3.3 from [7]). Finally thanks to the Lemma 2, when $\Re(s)$ goes to infinity we get

$$\|C_\Phi\| \geq \zeta(2\Re(c_1))^{1/p} > 1$$

and consequently C_Φ is not a contraction. \square

Remark. In the previous Lemma we actually used that for every $s \in \mathbb{C}_{1/2}$, $\delta_s \circ C_\Phi = \delta_{\Phi(s)}$.

Proof of (iii) \Rightarrow (iv). Assume that C_Φ is an isometry. By the last lemma, $c_0 \geq 1$ and then we know that $\Phi : \mathbb{C}_+ \rightarrow \mathbb{C}_+$ thanks to the Theorem 1. One has

$$\|2^{-s}\|_{\mathcal{A}_\mu^p} = \|2^{-\Phi}\|_{\mathcal{A}_\mu^p}.$$

Now by [2], Th6,

$$\int_0^{+\infty} \|2^{-\sigma-\bullet}\|_{\mathcal{H}^p}^p - \|2^{-\Phi(\sigma+\bullet)}\|_{\mathcal{H}^p}^p d\mu(\sigma) = 0.$$

But

$$\|2^{-\Phi(\sigma+\bullet)}\|_{\mathcal{H}^p} = \|2^{-\sigma-(\Phi(\sigma+\bullet)-\sigma)}\|_{\mathcal{H}^p} = \|C_{\Phi_\sigma-\sigma}(2^{-\sigma-\bullet})\|_{\mathcal{H}^p}.$$

Thanks to the Schwarz's lemma in this framework (recall that $c_0 \geq 1$) we know that $\Phi_\sigma - \sigma : \mathbb{C}_+ \rightarrow \mathbb{C}_+$. So by the Theorem 1, $C_{\Phi_\sigma-\sigma}$ is a bounded composition operator on \mathcal{H}^p and $\|C_{\Phi_\sigma-\sigma}\| \leq 1$. Then

$$\|2^{-\Phi(\sigma+\bullet)}\|_{\mathcal{H}^p} \leq \|2^{-\sigma-\bullet}\|_{\mathcal{H}^p}.$$

Consequently $2^{-\sigma} = \|2^{-\sigma-\bullet}\|_{\mathcal{H}^p} = \|2^{-\Phi(\sigma+\bullet)}\|_{\mathcal{H}^p}$ for every $\sigma > 0$ (recall that h is a positive continuous function). Now by Lemma 1, if Φ is not a vertical translation, there exists ε and $\eta > 0$ such that $\Phi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}$ and then for every $\sigma > 1/2 - \varepsilon$,

$$2^{-\sigma} = \|2^{-\Phi(\sigma+\bullet)}\|_{\mathcal{H}^p} \leq \|2^{-\Phi(\sigma+\bullet)}\|_{\mathcal{H}^\infty} \leq 2^{-1/2-\eta}$$

and this is obviously false. \square

Remark. Let μ be a probability measure on $(0, +\infty)$ such that $0 \in \text{Supp}(\mu)$ and $d\mu = h d\sigma$ where h is a nonnegative function. If there exists an open interval I such that h is positive on I then the theorem still holds. It is a consequence of the following lemma and some easy adaptations of the previous proof.

Lemma 3. *Let Φ be a symbol with $c_0 \geq 1$. If Φ is not a vertical translation then for every $\varepsilon > 0$, there exists $\eta = \eta_\varepsilon > 0$ such that $\Phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_{\varepsilon+\eta}$.*

Proof. First we assume that φ is non constant then $\varphi : \mathbb{C}_+ \rightarrow \mathbb{C}_+$ and by Proposition 4.2 from [7], there exists $\vartheta > 0$ such that $\varphi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_\vartheta$ and consequently $\Phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_{c_0\varepsilon+\vartheta}$. In this case, it suffices to choose $\eta = (c_0 - 1)\varepsilon + \vartheta$ which is positive because $c_0 \geq 1$.

If φ is constant equals to $i\tau$ ($\tau \in \mathbb{R}$) and $c_0 > 1$ then $\Phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_{c_0\varepsilon}$ and it suffices to choose $\eta = (c_0 - 1)\varepsilon$.

If φ is constant and equals to $c_1 \in \mathbb{C}_+$ and $c_0 \geq 1$, $\Phi(\mathbb{C}_\varepsilon) \subset \mathbb{C}_{c_0\varepsilon+\Re(c_1)}$ and it suffices to choose $\eta = (c_0 - 1)\varepsilon + \Re(c_1)$. \square

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