MUS-Based Partitioning for Inconsistency Measures

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Résumé

Mesurer le degré d'incohérence des bases de connaissances permet aux agents une meilleur compréhension de leur environnement. Différentes approches sémantiques et syntaxiques ont été proposées pour quantifier l'incohérence. Dans ce papier, nous proposons d'analyser les limites des approches existantes. Tout d'abord, nous explorons la propriété logique d'additivité en considérant les composantes connexes du graphe représentant les bases de connaissances. Ensuite, nous montrons comment la structure de ce graphe peut être prise en compte pour identifier d'une manière plus fine la responsabilité de chaque formule dans l'incohérence. Finalement, nous étendons notre approche pour fournir une mesure d'incohérence de la base entière en satisfaisant des propriétés définies.

Mots Clef

Mesure d'incohérence, Logique classique.

Abstract

Measuring inconsistency degrees of knowledge bases of different agents facilitates the understanding of an agent to her environment. Several semantic and syntactic based approaches have been proposed to quantify inconsistencies. By analyzing the limitations of existing approaches, we deeply explore the logical property of the Additivity condition through the connected components based on a graph representation of knowledge bases. Then, we show how the structure of this graph can be taken into account to discriminate in a fine-grained way the blame of each formula for the inconsistency. Finally, we extend our framework to provide an inconsistency measure for the whole base satisfying desired properties.

Keywords

Measuring Inconsistency, Classical Logic.

1 Introduction

Measuring inconsistency has been proved useful and attractive in diverse scenarios including software specifications [19], e-commerce protocols [2], belief merging [25], news reports [10], integrity constraints [5], requirements engineering [19], databases [20, 8], ontologies [29], semantic web [29], and network intrusion detection [21]. Indeed, we cannot expect large-sized knowledge bases (KBs, for short) inconsistency free in real intelligent systems, such as multi-agents communicating with each other to build a common KB or to perform some actions in a complex environment.

Analyzing inconsistency has gained a considerable attention and become an important issue in computer science recently [1]. Indeed, measuring inconsistency is helpful to compare different KBs and to evaluate their quality [4]. For instance, giving the opportunity for an agent to choose between different KBs, naturally he may try to choose the one that is less inconsistent.

To understand the nature of inconsistency and to quantify it in turn, a number of logic-based inconsistency measures have been studied, including the maximal η-consistency [15], measures based on variables or via multi-valued models [4, 9, 24, 10, 6, 18, 27, 17], n-consistency and n-probability [3], minimal inconsistent subsets based inconsistency measures [12, 22, 23, 28], the Shapley inconsistency value [11, 13], and the inconsistency measurement based on minimal proofs [14]. Although it is hardly possible to have a complete comparison of the proposed measures, one way to categorize the existing measures can be by the dependence of syntax or semantics : Semantic based ones aim to compute the proportion of language that is affected by the inconsistency. The inconsistency values belonging to this class are often based on some paraconsistent semantics and thus syntax independent. Whilst, syntax based ones are concerned with the minimal number of formulae that cause inconsistencies. Viewing minimal inconsistent subsets as the cornerstones of inconsistency, it is natural to derive inconsistency measures from minimal inconsistent subsets of a KB. Another possible classification of different measures originates in the measuring objective : formula oriented or knowledge base oriented.

In this paper, we propose a new approach for measuring inconsistency, both formula oriented and knowledge base oriented. It is inspired, on one hand, by the observation that existing measures fail to distinguish certain KBs which should have different inconsistency degrees. On the other hand, we explore a specific property, namely Additivity, that is rarely discussed in the literature due to the modeling difficulty [13]. This is done by analyzing connections between minimal inconsistent subsets which is shown a useful and general way to quantify more finely the inconsis-

tency responsibility of a formula or the inconsistency of a whole base. We show that measures based on this approach are different from existing inconsistency ones, and are able both to look inside the minimal inconsistent subsets, and to take into account the distribution of the contradictions among different formulae of a base. We also enhance the additivity property to be more intuitive. Our measures are shown satisfying the basic properties and the enhanced additivity. Clearly, such measures belong to syntax-dependent category.
The paper is organized as follows : Section 2 reviews syntactic approaches to measuring inconsistencies. In Section 3, we revisit the additivity property and propose the graph representation of a KB based on which we introduce the notion of MUS decomposition. This notion is then used in Section 4 to evaluate the degree of inconsistency of each formula in the KB. In Section 5, we generalize our inconsistency measure to quantify inconsistency of a whole base. Section 6 concludes by giving perspectives of this work.

2 Preliminaries

Through this paper, we consider the propositional language $\mathcal{L}$ built from a finite set of propositional symbols $\mathcal{P}$ under connectives $\{\neg, \wedge, \vee, \rightarrow\}$. We will use $a, b, c, \ldots$ to denote propositional variables. Also, we use Greek letters $\alpha, \beta, \gamma, \ldots$ to denote propositional formulae and sets thereof are denoted $\Phi, \Psi, \Theta, \ldots$
A KB K consists of a finite set of propositional formulae. Please note that K is fixed in the sequel. We denote by $Var(K)$ the set of variables occurring in K and $|S|$ the cardinality of a set S. Moreover, a KB K is inconsistent if there is a formula α such that $K \vdash \alpha$ and $K \vdash \neg\alpha$, where $\vdash$ is the deduction in classical propositional logic.
If K is inconsistent, then one can define the notion of *Minimal Inconsistent Subset* as an unsatisfiable set of formulae $\mathcal{M}$ in K such that any of its subsets is satisfiable. Formally,

Definition 1 (MUS) *Let K be a KB and $\mathcal{M}$ be a subset of K. $\mathcal{M}$ is a minimal unsatisfiable (inconsistent) subset of K iff $\mathcal{M} \vdash \bot$ and $\forall \mathcal{M}' \subset \mathcal{M}$, $\mathcal{M}' \nvdash \bot$.*

Clearly, an inconsistency K can have multiple minimal inconsistent subsets. The set of all minimal inconsistent subsets of K is defined as $MUSes(K) = \{\mathcal{M} \subseteq K \mid \mathcal{M}\ is\ a\ MUS\}$. When a MUS is a singleton, the single formulae in it is called a *self-contradictory formula*. A formula α that is not involved in any MUS of K is called *free formula*. That is, α do not have any relationship with the inconsistency of K. We denote $SelfC(K) = \{\alpha \in K \mid \{\alpha\} \vdash \bot\}$ and use $free(K) = \{\alpha \mid \alpha \notin MUSes(K)\}$.
At the same time, we can define the *Maximal Consistent Subset* (MC) and Hitting set as follows :

Definition 2 (MC) *Let K be a KB and $\mathcal{M}$ be a subset of K. $\mathcal{M}$ is a maximal consistent subset of K iff $\mathcal{M} \nvdash \bot$ and $\forall \alpha \in K \setminus \mathcal{M}$, $\mathcal{M} \cup \{\alpha\} \vdash \bot$.*

We denote by $MCes(K)$ the set of all maximal consistent subsets of K.

2.1 Inconsistency Measures

In this section, we review some inconsistency measures. We limit our presentation to the most important and related measures to the ones proposed in this paper.
There have been a number of proposals for measuring inconsistency in KBs defined through minimal inconsistent subsets theories. In [13], Hunter and Konieczny introduce a scoring function allowing to measure the degree of inconsistency of a subset of formulae of a given KB. In other words, for a subset $K' \subseteq K$, the scoring function is defined as the diminution of the number of minimal inconsistent subsets while K' is removed (i.e. $|MUSes(K)| - |MUSes(K - K')|$). By extending the scoring function, the authors introduce an inconsistency measure of the whole base, defined as the number of minimal inconsistent subsets of K. Formally, $I_{MI}(K) = |MUSes(K)|$. In the same paper, a family of "MinInc inconsistency values" MIV based on minimal inconsistent subsets is also presented :

- $MIV_D(K, \alpha)$ is a simple measure that values 1 if α belongs to a minimal inconsistent subset and 0 otherwise.
- $MIV_{\#}$ is defined in the way of the scoring function, i.e. $MIV_{\#}(K, \alpha) = |\{\mathcal{M} \in MUSes(K) \mid \alpha \in \mathcal{M}\}|$.
- MIV_C takes into account the size of each MUS in addition to the number of $MUSes$ of K, formally $MIV_C(K, \alpha) = \sum_{\alpha \in \mathcal{M} | \mathcal{M} \in MUSes(K)} \frac{1}{|\mathcal{M}|}$.

Additionally, another inconsistency value I_M, that combines both the MUSes and the MCes, has been introduced in [7]. The I_M measure counts for a given KB, the number of its $MCes$ and its Self-contradictory formulae, i.e. $I_M(K) = |MCes(K)| + |SelfC(K)| - 1$.

3 MUS partitioning in knowledge bases

There are a set of well accepted basic properties that inconsistency measures should satisfy (see Definition 3), while leaving one property Additivity debatable [12]. In this section, we propose an enhancement of the additivity property to make it more intuitive and give a way to modify the $I_M(K)$ measure which is not additive to satisfy the enhance additivity.

Definition 3 *A basic inconsistency measure I is an inconsistency measure satisfying the following properties, for all KBs K and K' and for every two formulae α and β in $\mathcal{L}$:*
(1) Consistency : $I(K) = 0$ iff K is consistent
(2) Monotony : $I(K) \leq I(K \cup K')$
(3) Free Formula Independence : if α is a free formula in $K \cup \{\alpha\}$, then $I(K \cup \{\alpha\}) = I(K)$
(4) Dominance : if $\alpha \vdash \beta$ and $\alpha \nvdash \bot$, then $I(K \cup \{\beta\}) \leq I(K \cup \{\alpha\})$

These properties result from translating the ones of Shapley's characterization [26]. The monotony property shows that the inconsistency value of a KB has to be increased while adding new formulae. And the free formula independence property states that the set of formulae not involved in any minimal inconsistent subset is not considered in the inconsistency measure. Finally the dominance property states that if we substitute a consistent formula by a logical consequence one, the inconsistency measure can not be increased.
Besides, another property called *Additivity* [1] has been proposed in [13] by translating the additivity axiom of Shapley's characterization as following [26].

Definition 4 (Additivity) *Let $K_1, \ldots, K_n$ be KBs and I an inconsistency measure. I is additive if it satisfies the following condition : If $MUSes(K_1 \cup \ldots \cup K_n) = MUSes(K_1) \oplus \ldots \oplus MUSes(K_n)$ [2], then $I(K_1 \cup \ldots \cup K_n) = I(K_1) + \ldots + I(K_n)$.*

The additivity was proposed to ensure that the inconsistency degree of a KB K can be obtained by summing up the degrees of its sub-bases K_i under the condition that $\{MUSes(K_i)\}$ is a partition of $MUSes(K)$. However, Luce and Raiffa have pointed out that the interaction of sub-bases (sub-games in [16]) is not taken into account by Additivity, which is one of the criticisms about this condition [16, 13]. Although the partitionability of $MUSes$ is used to describe a sort of interaction in Definition 4, we argue that it is not enough. Consider the following example :

Example 1 *Let $K_1 = \{a, \neg a\}, K_2 = \{\neg a, a \wedge b\}, K_3 = \{c, \neg c\}$, each of which contains only one single MUS. Consider two bases $K = K_1 \cup K_2, K' = K_1 \cup K_3$. Clearly, $MUSes(K)$= $MUSes(K_1) \oplus MUSes(K_2)$, $MUSes(K') = MUSes(K_1) \oplus MUSes(K_3)$. For any measure I, if I satisfies Additivity by Definition 4, we have $I(K) = I(K_1) + I(K_2)$ and $I(K') = I(K_1) + I(K_3)$. Taking the MIV inconsistency measure family introduced in Section 2.1, single MUS leads to the same inconsistency value. Then K and K' will have the same value, which is counterintuitive because the components of $MUSes(K')$ are less interactive thus more spreading than that of $MUSes(K)$, and in turn K' should, intuitively, contain more inconsistencies than K.*

To enhance the consideration of interaction among sub-bases, we propose the following Enhanced Additivity :

Definition 5 (Enhanced Additivity) *Let $K_1, \ldots, K_n$ be KBs and I an inconsistency measure. If $MUSes(K_1 \cup \ldots \cup K_n) = MUSes(K_1) \oplus \ldots \oplus MUSes(K_n)$ and $\{\alpha \in MUSes(K_i)\} \cap \{\beta \in MUSes(K_j)\} = \emptyset$ for all $1 \leq i \neq j \leq n$, $I(K_1 \cup \ldots \cup K_n) = I(K_1) + \ldots + I(K_n)$. I is then called an independent-additive measure.*

1. The Additivity condition is named Decomposability in [12].
2. We denote a partition $\{A, B\}$ of a set C by $C = A \oplus B$, i.e., $C = A \cup B$ and $A \cap B = \emptyset$.

Note that the enhanced additivity requires an extra precondition, which is to encode a stronger independence among sub-bases to perform additivity. Clearly, enhanced additivity implies additivity. The enhanced additivity can exclude the counterintuitive conclusions as given in Example 1 : suppose I satisfies the enhanced additivity, then we have $I(K') = I(K_1) + I(K_3)$, but not necessarily $I(K') = I(K_1) + I(K_2)$. Hence $I(K)$ is not necessarily equal to $I(K')$.
While we can see that the MIV measure family satisfies the additivity and the enhanced additivity, it is not case for the I_M measure as showed below.

Proposition 1 *The I_M measure is neither additive nor enhanced additive.*

Proposition 2 *Let K_1 and K_2 be tow KBs such that $\{\alpha \in MUSes(K_1)\} \cap \{\beta \in MUSes(K_2)\} = \emptyset$. Then, $|MCes(K_1 \cup K_2)| = |MCes(K_1)| \times |MCes(K_2)|$.*

Because the enhanced additivity gives a more intuitive characterization of interaction among sub-sets to be added up, in the following, we are interested in restoring the enhanced additivity property of I_M measure. To simplify terminology, in the rest of the paper, we call the enhanced additivity simply Additivity by default unless other claims are made. To reach this goal, we first define the three following new concepts : *graph representation*, *MUS decomposition*, and *elementary MC*.

Definition 6 *Given a KB K with $MUSes(K) = \{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$, the graph representation of K, denoted $\mathcal{G}_{\mathcal{MUS}}(K)$, is a graph such that :*
- *each vertex is labeled by an element from $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$*
- *$\forall \mathcal{M}, \mathcal{M}' \in MUSes(K)$ such that $\mathcal{M} \cap \mathcal{M}' \neq \emptyset$, there exists an edge between $\mathcal{M}$ and $\mathcal{M}'$.*

The graph representation of a KB K allows us to obtain a structure gathering all the interconnected MUSes of K. Now, we introduce in the light of the graph representation the notion of *MUS decomposition* :

Definition 7 *Let K be a KB and $\{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$ with $\mathcal{C}_i \subseteq K$ for $1 \leq i \leq p$. $\{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$ is the MUS decomposition of K iff $\{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$ is the set of the connected components of $\mathcal{G}_{\mathcal{MUS}}(K)$.*

A MUS decomposition $\{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$ of a KB K represents a partition of the minimal inconsistent subsets of K into connected components. Clearly, it is easy (P-time) to obtain such set since each two interconnected $MUSes$ belong to the same connected component.

Proposition 3 *Let K be a KB such that $MUSes(K) \neq \emptyset$. K possesses a unique MUS decomposition.*

Definition 7 allows us to associate to a given KB K a set of sub-bases $K_1, \ldots, K_p$ such that the elements of each sub-base K_i are the formulae of the connected component $\mathcal{C}_i$.
In the following, we present an alternative to the inconsistency measure I_M so as to make it additive. To this end, we introduce the concept of *elementary MC* by using the MUS decomposition.

Definition 8 *Let K be a KB such that $K' \subseteq K$ and $\{\mathcal{C}_1, \ldots, \mathcal{C}_p\}$ be the MUS decomposition of K. K' is an elementary MC of K iff there exists a connected component $\mathcal{C}_i$ of K such that $K' \in MCes(\mathcal{C}_i)$. We denote $EMC(K)$ the set of all elementary MC of K, i.e. $EMC(K) = \bigcup_{i=1}^{p} MCes(K_i)$.*

That is, an elementary maximal consistency should be local restricted by a connected component of $MUSes(K)$.

Example 2 *Let $K = \{a \wedge d, \neg a, \neg b, b \vee \neg c, \neg c \wedge d, \neg c \vee e, c, \neg e, e \wedge d\}$. The MUS decomposition of K contains two subsets, $\mathcal{C}_1 = \{\mathcal{M}_1\}$, and $\mathcal{C}_2 = \{\mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5\}$ such that $\mathcal{M}_1 = \{\neg a, a \wedge d\}$, $\mathcal{M}_2 = \{c, \neg b, b \vee \neg c\}$, $\mathcal{M}_3 = \{c, \neg c \wedge d\}$, $\mathcal{M}_4 = \{\neg c \vee e, c, \neg e\}$, and $\mathcal{M}_5 = \{\neg e, e \wedge d\}$. Then, $EMC(K) = \{\{a \wedge d\}, \{\neg a\}, \{\neg b, b \vee \neg c, \neg c \wedge d, \neg c \vee e, e \wedge d\}, \{\neg b, b \vee \neg c, \neg c \wedge d, \neg c \vee e, \neg e\}, \{b \vee \neg c, \neg c \vee e, c, e \wedge d\}, \{\neg b, \neg c \vee e, c, e \wedge d\}, \{b \vee \neg c, c, \neg e\}, \{\neg b, c, \neg e\}\}$.*

$EMC(K)$ allows us to define an alternative of the I_M measure. This result is stated in Proposition 4.

Definition 9 *Let K be a KB. The inconsistency degree I'_M is defined as :*

$$I'_M(K) = \begin{cases} |EMC(K)| + |SelfC(K)| & \text{if } K \vdash \bot; \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4 *The I'_M measure is additive.*

The additivity property shows a way to take into account the connection between minimal inconsistent subsets through the connected components, and to offer then a finer grained evaluation of the inconsistencies.

4 Measuring Inconsistency using MUS-Decomposition

In this section, we use the MUS decomposition of a KB, defined in the previous section, to assign a formula oriented measure to estimate the responsibility of a formula to the inconsistency of its base.
Given two minimal inconsistent subsets and based on Definition 6, a distance, as defined below, is an assignment of a real number to each MUS pair of K.

Definition 10 *Let K be a KB. The distance between two MUSes $\mathcal{M}$ and $\mathcal{M}'$ of K, denoted $d_{MUS}(\mathcal{M}, \mathcal{M}')$, is defined as the shortest path between $\mathcal{M}$ and $\mathcal{M}'$ in the graph representation $\mathcal{G}_{MUS}(K)$.*

Next, we will extend Definition 10 to compute the distance between a formula and a MUS as follows.

Definition 11 *Let K be a KB, $\alpha \in K$ and $\mathcal{M} \in MUSes(K)$. The distance between α and $\mathcal{M}$ is defined as $d_{MUS}(\alpha, \mathcal{M}) = \min\{d_{MUS}(\mathcal{M}, \mathcal{M}') \mid \alpha \in \mathcal{M}'\}$.*

In fact, the distance between a given formula $\alpha \in K$ and a MUS $\mathcal{M}$ corresponds to the shortest path from α to $\mathcal{M}$ along a sequence of intersecting $MUSes$. Note that if α and $\mathcal{M}$ do not belong to the same connected component, this means that $\mathcal{M}$ is not reachable from α and in this case, the distance is equal to an infinite value i.e. $d_{MUS}(\alpha, \mathcal{M}) = +\infty$.

Example 3 *(Example 2 Contd.) Let $\alpha = \neg b$, we have*

$$d_{MUS}(\alpha, \mathcal{M}_2) = 0 \qquad d_{MUS}(\alpha, \mathcal{M}_3) = 1$$
$$d_{MUS}(\alpha, \mathcal{M}_4) = 1 \qquad d_{MUS}(\alpha, \mathcal{M}_5) = 2$$

We note that the distance d_{MUS} allows us to give an ordering over the minimal inconsistent subsets of K according to their distance from α.
In the following, we quantify the inconsistency value of α in the light of the distance from α with respect to reachable $MUSes$ of K. Indeed, for each formula belonging to some MUSes, there exists at least one finite distance. To compare different formulae by their inconsistency values, only finite distances are meaningful. For free formulae, all the distances will be $+\infty$. But by *Free Formula Independence* principle, they should not be contributors to inconsistency anyway. Let us note $d^{\max}_{MUS}(\alpha) = \max\{d_{MUS}(\alpha, \mathcal{M}) \mid \mathcal{M} \in MUSes(K),\ d_{MUS}(\alpha, \mathcal{M}) \neq +\infty\}$. Note that the maximum distance is not more than the cardinality of the connected component that α belongs to, that is, $d^{\max}_{MUS}(\alpha) < |\mathcal{C}|$.

Definition 12 *Let K be a KB and $\alpha \in K$. Write $\mathcal{S}_{MUS}(\alpha, k) = \{\mathcal{M} \in MUSes(K) \mid d_{MUS}(\alpha, \mathcal{M}) = k\}$. We define $\mathcal{SQ}_{MUS}(\alpha)$ as $\mathcal{SQ}_{MUS}(\alpha) = (\mathcal{S}_{MUS}(\alpha, 0), \mathcal{S}_{MUS}(\alpha, 1), \ldots, \mathcal{S}_{MUS}(\alpha, d^{\max}_{MUS}(\alpha)))$.*

Note that $\mathcal{S}_{MUS}(\alpha, k)$ represents the set of MUSes with a distance k from α, and $\mathcal{SQ}_{MUS}(\alpha)$ signifies the sequence of MUSes distributed in terms of the distance d_{MUS}.

Example 4 *(Example 2 Contd.) For $\alpha = \neg b$, we have $\mathcal{S}_{MUS}(\alpha, 0) = \{\mathcal{M}_2\}$, $\mathcal{S}_{MUS}(\alpha, 1) = \{\mathcal{M}_3, \mathcal{M}_4\}$, $\mathcal{S}_{MUS}(\alpha, 2) = \{\mathcal{M}_5\}$. Then, $\mathcal{SQ}_{MUS}(\alpha) = \{\{\mathcal{M}_2\}, \{\mathcal{M}_3, \mathcal{M}_4\}, \{\mathcal{M}_5\}\}$.*
Figure 1 depicts a graphical representation of the connected components of K. Also, Figure 2 represents the distribution of MUSes according to their distances to α.
It is easy to check that $MIV_{\#}$ measure is just a characterization of the set $\mathcal{S}_{MUS}$ when applying uniquely the minimum distance from the formulae. More formally,

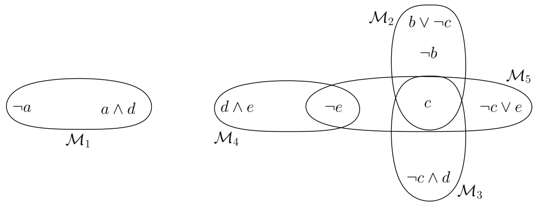

FIGURE 1 – Connected components of K

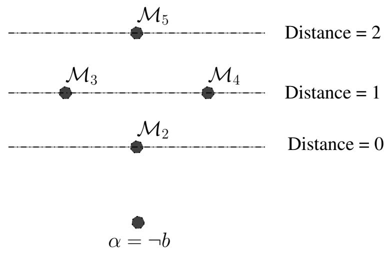

FIGURE 2 – Distribution of MUSes according to α

Proposition 5 *Let K be a KB and $\alpha \in K$. Then, $MIV_{\#}(K, \alpha) = |\mathcal{S}_{MUS}(\alpha, 0)|$.*

Proposition 5 shows that the $MIV_{\#}$ value considers only the nearest neighbors of α. However, this measure is not sufficiently discriminating for our purposes, since it takes into account only the first level of MUSes. Indeed, let us consider the formulae $d \wedge e$ and $\neg c \vee e$ of Example 2. According to $MIV_{\#}$ or MIV_C measures, these two formulae have the same inconsistency value. However, according to Figure 1 these two formulae do not have the same structural properties. Indeed, the MUS containing $\neg c \vee e$ is more connected than the one of $d \wedge e$. Hence, one has to go beyond the nearest neighbors to obtain a finer-grained measure. A first inconsistency measure can be defined as follows :

Definition 13 *Let K be a KB. We define DIM_C as :*

$$DIM_C(\alpha, K) = \frac{|\mathcal{S}_{MUS}(\alpha, 0)|}{d^{\max}_{MUS}(\alpha) + 1}.$$

Unlike $MIV_{\#}$, the DIM_C value takes into account the structure of the connected components by considering the nearest and the farthest $MUSes$. More precisely, the DIM_C measure aims to assign a better value to the formulae having numerous nearest neighbors and remaining $MUSes$ concentrated around. Put differently, while two formulae have the same number of neighbors of distance 0, the distance from the farthest MUS allows us to find out the most inconsistent one. Note that if α is a self-contradictory formula, then DIM_C measure takes value one, i.e. $DIM_C(\alpha, K) = 1$.
Let us now illustrate the behavior of the DIM_C measure in the next example.

Example 5 *(Example 2 Contd.) It is not hard to see :*

$DIM_C(a \wedge d, K) = 1$	$DIM_C(\neg a, K) = 1$
$DIM_C(\neg b, K) = \frac{1}{3}$	$DIM_C(\neg c \wedge d, K) = \frac{1}{3}$
$DIM_C(\neg e, K) = 1$	$DIM_C(b \vee \neg c, K) = \frac{1}{3}$
$DIM_C(d \wedge e, K) = \frac{1}{3}$	$DIM_C(c, K) = \frac{3}{2}$
$DIM_C(\neg c \vee e, K) = \frac{1}{2}$	

Notice that now we can make a distinction between $d \wedge e$, and $\neg c \vee e$ since $DIM_C(d \wedge e, K) < DIM_C(\neg c \vee e, K)$.

However, the problem remains between the formulae $d \wedge e$ and $\neg b$. In order to make the measure more accurate, we propose to extend this measure by not only considering the farthest MUS, but the whole structure of the connected components, which leads to the second inconsistency measure defined as follows :

Definition 14 *Let K be a KB. We define DIM_H as :*

$$DIM_H(\alpha, K) = \sum_{\substack{\mathcal{M} \in MUSes(K) \\ d_{MUS(\alpha, \mathcal{M}) \neq +\infty}}} \frac{1}{d_{MUS}(\alpha, \mathcal{M}) + 1}.$$

We can see more clearly by the following example that DIM_H can give a more precise view of the conflict brought by each formula.

Example 6 *(Example 2 Contd.) We have :*

$DIM_H(a \wedge d, K) = 1$	$DIM_H(\neg a, K) = 1$
$DIM_H(\neg b, K) = \frac{7}{3}$	$DIM_H(\neg c \wedge d, K) = \frac{7}{3}$
$DIM_H(\neg e, K) = 3$	$DIM_H(b \vee \neg c, K) = \frac{7}{3}$
$DIM_H(d \wedge e, K) = \frac{13}{6}$	$DIM_H(c, K) = \frac{7}{2}$
$DIM_H(\neg c \vee e, K) = \frac{7}{3}$	

Using DIM_H measure, the formula $d \wedge e$ has now an inconsistency value $\frac{13}{6}$ less than $\frac{7}{3}$ the one of $\neg b$.

The DIM_H measure could be refined by using the following notion of a weighting function that assigns a weight to each MUS in the connected component. The idea is that a weight represents the significance of each MUS with respect to their distance from the given formula, and we get then a better assignment of inconsistency responsibility to formulae. These weights can take into consideration other criteria like the degree of each MUS in the graph representation. A general definition is stated as follows.

Definition 15 *Let K be a KB. We define DIM_W as :*

$$DIM_W(\alpha, K) = \sum_{\substack{\mathcal{M} \in MUSes(K) \\ d_{MUS(\alpha, \mathcal{M}) \neq +\infty}}} \frac{w(\mathcal{M})}{d_{MUS}(\alpha, \mathcal{M}) + 1}.$$

Where $w(\mathcal{M}) \in \mathbb{R}$ represents the weight associated to $\mathcal{M}$.

The following result shows that the DIM_W measure can be expressed by using the sequence $\mathcal{SQ}_{MUS}(\alpha)$. The idea is that $w(\mathcal{M})$ only depends on the distance between $\mathcal{M}$ and α.

Proposition 6 *Let K be a KB and $\alpha \in K$. We have*

$$DIM_W(\alpha, K) = \sum_{\mathcal{S}_{MUS} \in \mathcal{SQ}_{MUS}} w(i) \times \frac{|\mathcal{S}_{MUS}(\alpha, i)|}{(i+1)}.$$

5 Measuring inconsistencies of a whole base

This section is devoted to the definition of an inconsistency measurement for a whole KB.
To address this question, let us firstly give a general characterization of our measure with respect to the additivity property. Then, we discuss different measures obtained by different restrictions of the general case.

Definition 16 *Let K be a KB. Let $\mathcal{CC} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ be the connected components of K. The inconsistency measure of K, denoted $I_{\mathcal{CC}}(K)$, is defined as $I_{\mathcal{CC}}(K) = \sum_{i=1}^{i=n} \delta(K_i)$ where δ is a function taking its values in $\mathbb{R}$ and K_i represents the sub-base $K_{\mathcal{C}_i}$.*

The above general definition allows for a range of possible measures to be defined in various ways. Next we introduce some extensions of $I_{\mathcal{CC}}$ to instantiate our framework. Let us review some resulting instances of $I_{\mathcal{CC}}$ according to some δ functions. The simplest one is obtained when $\delta(K_i) = 1$. In this case, we get a measure that assigns to K the number of its connected components i.e. $I_{\mathcal{CC}}(K) = |\mathcal{CC}|$. Note that this measure in not monotonic. Indeed, adding new formulae to the knowledge can decrease the number of connected components. For instance, if we consider the KB $K = \{a, \neg a, b, \neg b\}$ that contains two connected components $\mathcal{C}_1 = \{a, \neg a\}$ and $\mathcal{C}_2 = \{b, \neg b\}$; now adding the formula $a \vee b$ to K leads to a new KB containing the unique connected component $\mathcal{C} = \{a, \neg a, b, \neg b, a \vee b\}$. Note that this simple measure considers each connected component as an inseparable entity.
Moreover, when we consider $\delta(K_i) = |\mathcal{C}_i|$, the $I_{\mathcal{CC}}$ measure leads to an existing measure. More precisely, as $|\mathcal{C}_i|$ corresponds exactly to the number of $MUSes$ of K_i involved in the connected component $\mathcal{C}_i$, it is obvious to see that $I_{\mathcal{CC}}(K)$ is equal to I_{MI} measure i.e. $I_{\mathcal{CC}}(K) = |MUSes(K)|$. This second value is of little interest, since it states exactly the fact that the inconsistency value only takes into account only the number of the minimal inconsistent subsets of the base.
In the following, we propose to deeply explore the properties of additivity and monotony to define a new inconsistency measure.

Definition 17 *Let K be a KB. A conditional independent MUS partition of K, is defined as a set $\{K_1, \ldots, K_n\}$ such that :*
(1) $\forall i$, $K_i \subseteq K$ and $K_i \vdash \perp$
(2) $MUSes(K_1 \cup \ldots \cup K_n) = MUSes(K_1) \oplus \ldots \oplus MUSes(K_n)$
(3) $\forall i \neq j$, $K_i \cap K_j = \emptyset$.

According to Definition 17, K can be written using the set $\{K_1, \ldots, K_n\}$ as $K = K_1 \cup \ldots \cup K_n \cup R$ where R is a subset of K and $\{K_1, \ldots, K_n\}$ is a conditional independent MUS partition of K. Otherwise, when removing the set of formulae R from K the remaining base can be partitioned into sub-bases $K_1, \ldots, K_n$ that are inconsistent, disjoints and having disjoint MUSes.
It is clear that generally for a given KB K, there exist different subsets $R \subseteq K$ such that Definition 17 holds. Moreover, if $K = K_1 \cup \ldots \cup K_n \cup R$, then there exists $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\} \subseteq MUSes(K)$ a conditional independent MUS partition of K. In other words, $K = \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_n \cup R'$ where $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset, \forall i \neq j$. Indeed, it is sufficient to pick a MUS $\mathcal{M}_i$ from each K_i since $K_i \vdash \perp$ and consider $R' = R \cup \{K_1 \setminus \mathcal{M}_1\} \cup \ldots \cup \{K_n \setminus \mathcal{M}_n\}$.
Let us now characterize an inconsistency measure I in the light of both additivity and monotony properties. Using Definition 17, $I(K) = I(K_1 \cup \ldots \cup K_n \cup R)$. Using monotony property, we have $I(K) \geq I(K_1 \cup \ldots \cup K_n)$. Finally by additivity, we conclude that $I(K) \geq I(K_1) + \ldots + I(K_n)$.
Let us denote by $\mu_{\max}(K)$ the maximal cardinality of sets $\{K_1, \ldots, K_n\}$ satisfying the conditions (1), (2) and (3) of the Definition 17. $\mu_{\max}(K)$ corresponds to the number of maximal connected components that can be obtained while removing some formulae from K i.e., the maximal value taken by n. By considering the maximal conditional independent partition value $\mu_{\max}$, one can deduce that if the measure I is additive then, $I(K) \geq I(K_1) + \ldots + I(K_{\mu_{\max}(K)})$. Now, by using this bound one can define a new inconsistency measure as stated in Definition 18.

Definition 18 *Let K be a KB. We define the inconsistency measure of K as $I'_{\mathcal{CC}}(K) = \mu_{\max}(K)$.*

Proposition 7 *The inconsistency measure $I'_{\mathcal{CC}}(K)$ is additive and monotonic.*

Example 7 *Let us consider $K = \{a, \neg a, a \vee b, \neg b, b, c, \neg c \wedge d, \neg d \wedge e \wedge f, \neg e, \neg f\}$. K has two connected components $\mathcal{C}_1$ and $\mathcal{C}_1$ such that $K_1 = \{a, \neg a, a \vee b, \neg b, b\}$ and $K_2 = \{c, \neg c \wedge d, \neg d \wedge e \wedge f, \neg e, \neg f\}$. The conditional independent MUS partition of K_1 of maximum size is equal to 2 and can be obtained from K_1 by removing $a \vee b$. For K_2, for all removed subset of K_2, the number of resulting connected components cannot exceed 1. Then, we have $I'_{\mathcal{CC}}(K) = I'_{\mathcal{CC}}(K_1) + I'_{\mathcal{CC}}(K_2) = 3$.*

6 Conclusion

We proposed in this paper a new framework for defining inconsistency values that allow to associate each formula with its degree of responsibility for the inconsistency of a whole KB. We showed that such a framework can be extended to measure the inconsistency of the whole base. We also proposed an enhanced additivity property to better capture its intuition according to the debate existing in the literature.
In the future, we plan to study the computational complexity of our inconsistency measures , and then develop algorithms and implementations of computing inconsistency degrees.

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