THE BLUM-HANSON PROPERTY FOR C(K) SPACES

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Abstract. We show that if K is a compact metrizable space, then the Banach space $\mathcal{C}(K)$ has the so-called Blum-Hanson property exactly when K has finitely many accumulation points. We also show that the space $\ell_\infty(\mathbb{N}) = \mathcal{C}(\beta\mathbb{N})$ does not have the Blum-Hanson property.

1. introduction

The following intriguing result is usually referred to as the *Blum-Hanson theorem* (see [3] and [6]): if T is a linear operator on a Hilbert space H with $\|T\| \leqslant 1$, and if $x \in H$ is such that $T^n x \to 0$ weakly as $n \to \infty$, then the sequence $(T^n x)$ is "strongly mixing", which means that every subsequence of $(T^n x)$ converges to 0 in the Cesáro sense; in other words,

$$\lim_{K\to\infty} \left\| \frac{1}{K} \sum_{i=1}^{K} T^{n_i} x \right\| = 0$$

for any increasing sequence of integers (n_i). (The terminology "strongly mixing" comes from [2]).

Accordingly, a Banach space X is said to have the *Blum-Hanson property* if the Blum-Hanson theorem holds true on X; that is, if T is linear operator on X such that $\|T\| \leqslant 1$, then every weakly null T-orbit is strongly mixing. For example, it was shown rather recently in [8] that $\ell_p(\mathbb{N})$ has the Blum-Hanson property for any $p \in [1, \infty)$. On the other hand, it is known since [1] that $\mathcal{C}(\mathbb{T}^2)$, the space of all continuous real-valued functions on the torus $\mathbb{T}^2$, does not have this property. Further results and references can be found in [7].

In this short note, we address the Blum-Hanson property for $\mathcal{C}(K)$ spaces. Our main result is the following:

Theorem 1.1. *Let K be a metrizable compact space. Then $\mathcal{C}(K)$ has the Blum-Hanson property if and only if K has finitely many accumulation points.*

This will be proved in the next Section. In Section 3, we obtain in much the same way one nonmetrizable result, namely that the space $\ell_\infty(\mathbb{N}) = \mathcal{C}(\beta\mathbb{N})$ fails the Blum-Hanson property. Our two results can be put together to get a single theorem on the Blum-Hanson property for spaces of bounded continuous functions, which is done in Section 4. We conclude the paper by stating explicitely the "general principle" underlying our proofs.

2. Proof of Theorem 1.1

For the "if" part of the proof, we will make use of a result from [7] which is stated as Lemma 2.1 below.

Let X be a Banach space. For any $x \in X$ and $t \in \mathbb{R}^+$, set

$$r_X(t,x) := \sup \left\{ \limsup_{n\to\infty} \|x + ty_n\| \right\},$$

where the supremum is taken over all weakly null sequences $(y_n) \subset X$ with $\|y_n\| \leqslant 1$.

Since $r_X(t,x)$ is 1-Lipschitz with respect to t, the quantity $r_X(t,x) - t$ is nonincreasing and hence it has a limit as $t \to \infty$, possibly equal to $-\infty$. Actually, this limit is nonnegative if X does not have the Schur property, i.e. there is at least one weakly null sequence in X which is not norm null.

For the needs of the present paper only, we shall say that the Banach space X has *property* (P) if, for every weakly null sequence $(x_k) \subset X$, it holds that

$$\lim_{k\to\infty} \lim_{t\to\infty} (r_X(t,x_k) - t) = 0\,. \tag{1}$$

The result we need is the following; for the proof, see the Remark just after Theorem 2.1 in [7].

Lemma 2.1. *Property* (P) *implies the Blum-Hanson property.*

An extreme example of a space with property (P) is $X := c_0(\mathbb{N})$. Indeed, if $x \in c_0$ and if (z_n) is a weakly null sequence in c_0, then

$$\limsup_{n\to\infty} \|x + z_n\|_\infty = \max(\|x\|_\infty, \limsup \|z_n\|_\infty)\,.$$

It follows that

$$r_{c_0}(t,x) = \max(\|x\|, t)\,, \tag{$*$}$$

so that $r_{c_0}(t,x) - t = 0$ whenever $t \geqslant \|x\|$, for any $x \in c_0$.

Let us also note the following useful stability property, whose proof is straightforward.

Remark 2.2. *If $X_1, \dots, X_N$ are Banach spaces with property* (P)*, then the ℓ_∞ direct sum $X_1 \oplus \cdots \oplus X_N$ also has* (P).

We can now start the proof of theorem 1.1.

Proof of Theorem 1.1. Let us denote by K' the set of all accumulation points of K. We may assume that $K' \neq \varnothing$, since otherwise K is finite and hence $\mathcal{C}(K)$ is finite-dimensional.

(a) Assume first that K' is finite say $K' = \{a_1, \dots, a_N\}$, and let us show that $X := \mathcal{C}(K)$ has the Blum-Hanson property.

One may write $K = K_1 \cup \cdots \cup K_N$, where the K_i are pairwise disjoint compact sets and $K_i' = \{a_i\}$. Then $\mathcal{C}(K)$ is isometric to the ℓ_∞ direct sum $\mathcal{C}(K_1) \oplus \cdots \oplus \mathcal{C}(K_N)$, and each $\mathcal{C}(K_i)$ is isometric to the space c of all convergent sequences of real numbers.

Therefore (by Lemma 2.1 and Remark 2.2) it is enough to show that the space c has property (P).

We view c as the space $\mathcal{C}(\mathbb{N}\cup\{\infty\})$, so that c_0 is identified with the subspace of all $f\in\mathcal{C}(\mathbb{N}\cup\{\infty\})$ such that $f(\infty)=0$. We have to show that if (f_k) is a weakly null sequence in c, then $\lim_{k\to\infty}\lim_{t\to\infty}(r_c(t,f_k)-t)=0$.

Observe first that since $f_k(\infty)\to 0$ as $k\to\infty$, one can find a (weakly null) sequence $(\widetilde{f}_k)\subset c$ such that $\widetilde{f}_k\in c_0$ for all k and $\|\widetilde{f}_k-f_k\|_\infty\to 0$: just set $\widetilde{f}_k:=f_k-f_k(\infty)\mathbf{1}$.

Let (g_n) be a weakly null sequence in c with $\|g_n\|_\infty\leqslant 1$. As above, choose a (weakly null) sequence $(\widetilde{g}_n)\subset c$ such that $\|\widetilde{g}_n-g_n\|_\infty\to 0$ and $\widetilde{g}_n\in c_0$ for all n. Since $\|g_n\|_\infty\leqslant 1$, we may also asume that $\|\widetilde{g}_n\|_\infty\leqslant 1$ for all n. Then, since f_k and the $\widetilde{g}_n$ are living in c_0, we get from $(*)$ above that for any $t\in\mathbb{R}^+$ and for each $k\in\mathbb{N}$:

$$\limsup_{n\to\infty}\|\widetilde{f}_k+t\widetilde{g}_n\|_\infty\leqslant r_{c_0}(t,\widetilde{f}_k)=\max(\|\widetilde{f}_k\|_\infty,t)\,.$$

By the triangle inequality, it follows that

$$\limsup_{n\to\infty}\|f_k+tg_n\|_\infty\leqslant\|\widetilde{f}_k-f_k\|_\infty+\max(\|\widetilde{f}_k\|_\infty,t)$$

for each $k\in\mathbb{N}$ and all $t\geqslant 0$. This being true for any weakly null sequence (g_n) with $\|g_n\|_\infty\leqslant 1$, we conclude that

$$\lim_{t\to\infty}(r_c(f_k,t)-t)\leqslant\|\widetilde{f}_k-f_k\|_\infty$$

for each $k\in\mathbb{N}$, and hence that $\lim_{k\to\infty}\lim_{t\to\infty}(r_c(t,f_k)-t)=0$.

(b) Now assume that K' is infinite. Since K is metrizable, it follows that K contains a compact set S of the following form:

$$S=\bigcup_{k=1}^{\infty}\Big[\{s_{i,k};\ i\in\mathbb{N}\}\cup\{s_{\infty,k}\}\Big]\cup\{s_{\infty,\infty}\}\,,$$

where all the points involved are distinct and

- $s_{i,k}\to s_{\infty,k}$ as $i\to\infty$ for each fixed $k\geqslant 1$;
- $s_{\infty,k}\to s_{\infty,\infty}$ as $k\to\infty$;
- the sets $S_k:=\{s_{i,k};\ i\in\mathbb{N}\}\cup\{s_{\infty,k}\}$ "accumulate to $\{s_{\infty,\infty}\}$", i.e. they are eventually contained in any neighbourhood of $s_{\infty,\infty}$.

Thus, we have $S'=\{s_{\infty,k};\ k\geqslant 1\}\cup\{s_{\infty,\infty}\}$ and $S''=\{s_{\infty,\infty}\}$.

The key point is now to construct a special continuous map $\theta:S\to S$ and to consider the associated *composition operator* C_θ acting on $\mathcal{C}(S)$. This is the same strategy as in [1], in our setting.

Fact 1. One can construct a continuous map $\theta:S\to S$ such that, denoting by θ^n the iterates of θ, the following properties hold true.

(i) $\theta^n(s)\to s_{\infty,\infty}$ pointwise on S as $n\to\infty$;

(ii) there exists an open neighbourhood V of $s_{\infty,\infty}$ in S such that

$$\sup_{s\in S} \#\{n\in\mathbb{N};\ \theta^n(s)\notin V\} = \infty\,.$$

Proof. We define the map θ as follows:

$$\begin{cases} \theta(s_{\infty,\infty}) &= s_{\infty,\infty} & \\ \theta(s_{i,k}) &= s_{i,k-1} & \text{if } k\geqslant 2\\ \theta(s_{\infty,k}) &= s_{\infty,k-1} & \text{if } k\geqslant 2\\ \theta(s_{i,1}) &= s_{i-1,i-1} & \text{if } i\geqslant 2\\ \theta(s_{\infty,1}) &= s_{\infty,\infty} & \\ \theta(s_{1,1}) &= s_{\infty,\infty} & \end{cases}$$

It is clear that θ is continuous at each point $s_{\infty,k}$, $k\geqslant 2$. Moreover, since $s_{i-1,i-1}\to s_{\infty,\infty}$ as $i\to\infty$, the map θ is also continuous at $s_{\infty,1}$ and at $s_{\infty,\infty}$. Since all other points of S are isolated, it follows that θ is continuous on S.

An examination of the orbits of θ reveals that for any $s\in S$, we have $\theta^n(s)=s_{\infty,\infty}$ for all but finitely many $n\in\mathbb{N}$. Indeed, if $s=s_{\infty,k}$ for some $k\in\mathbb{N}$, then $Orb(s,\theta)=\{s_{\infty,k},s_{\infty,k-1},\dots,s_{\infty,1},s_{\infty,\infty}\}$, whereas if $s=s_{i,k}$ for some $(i,k)\in\mathbb{N}\times\mathbb{N}$, then $Orb(s,\theta)=\{s_{i,k},s_{i,k-1},\dots,s_{i,1},s_{i-1,i-1},\dots,s_{i-1,1},s_{i-2,i-2}\dots,s_{1,2},s_{1,1},s_{\infty,\infty}\}$. So property (i) is satisfied.

Set $V:=S\backslash S_1$, where $S_1=\{s_{i,1};\ i\in\mathbb{N}\}\cup\{s_{\infty,1}\}$. This is an open (actually clopen) neighbourhood of $s_{\infty,\infty}$ in S. For any $N\in\mathbb{N}$, the orbit of $s_N:=s_{N,1}$ contains exactly N points of $S\backslash V=S_1$, namely $s_{N,1},s_{N-1,1},\dots,s_{1,1}$. So property (ii) is satisfied as well. □

From Fact 1, it is straightforward to deduce

Fact 2. The space $\mathcal{C}(S)$ does not have the Blum-Hanson property.

Proof. Let $\theta:S\to S$ be given by Fact 1, and let $C_\theta:\mathcal{C}(S)\to\mathcal{C}(S)$ be the composition operator associated with θ:

$$C_\theta u = u\circ\theta \quad \text{for all } u\in\mathcal{C}(S).$$

By property (i) above, we see that $C_\theta^n u\to u(s_{\infty,\infty})\mathbf{1}$ weakly as $n\to\infty$, for every $u\in\mathcal{C}(S)$.

Let us choose a function $f\in\mathcal{C}(S)$ such that $f(s_{\infty,\infty})=0$ and $f\equiv 1$ on $F:=S\backslash V$, where V satisfies (ii). Then $C_\theta^n f\to 0$ weakly. On the other hand, since $f\equiv 1$ on F it follows from (ii) that one can find points $s\in S$ such that $\#\{n\in\mathbb{N};\ C_\theta^n f(s)=1\}$ is arbitrarily large. So we have

$$\frac{1}{\#I}\left\|\sum_{n\in I} C_\theta^n f\right\|_\infty \geqslant 1$$

for finite sets $I\subset\mathbb{N}$ with arbitrarily large cardinality. From this, it is a simple matter to deduce that the sequence $(C_\theta^n f)$ is not strongly mixing, which concludes the proof of Fact 2. □

It is now easy to conclude the proof of Theorem 1.1, by using the following trivial observation.

Fact 3. Let X be a Banach space, and let Z be a closed subspace of X. Assume that Z is 1 - *complemented* in X, i.e. there is a linear projection $\pi : X \to Z$ such that $\|\pi\| = 1$. If Z fails the Blum-Hanson property, then so does X.

Proof. If $T : Z \to Z$ and $z \in Z$ witness that Z fails the Blum-Hanson property, then $\widetilde{T} := T \circ \pi : X \to Z \subset X$ and z witness that so does X. □

It is well known that since K is metrizable, there is an isometric linear extension operator $J : \mathcal{C}(S) \to \mathcal{C}(K)$: this is a classical result due to Dugundji [4]. So the space $\mathcal{C}(S)$ is isometric to a 1-complemented subspace of $\mathcal{C}(K)$, namely $Z := J\,[\mathcal{C}(S)]$. By Fact 3, this concludes the proof of Theorem 1.1.

□

Remark 1. The above proof shows that the space $\mathcal{C}(S)$ fails the Blum-Hanson property in a very special way. Namely, there exists a composition operator C_θ on $\mathcal{C}(S)$ all whose orbits are weakly convergent and such that some weakly null orbit is not strongly mixing. As shown in [1], the same is true for the space $\mathcal{C}(\mathbb{T}^2)$. On the other hand, it is observed in [7] that this is *not* so in the space $\mathcal{C}([0,1])$, for the following reason: if $\theta : [0,1] \to [0,1]$ is a continuous map and if the iterates θ^n converge pointwise to some continuous map $\alpha : [0,1] \to [0,1]$, then the convergence is in fact uniform.

Remark 2. Our proof gives in fact the following more precise result: if K has finitely accumulation points, then $\mathcal{C}(K)$ has property (P); and otherwise, one can find an operator T on $\mathcal{C}(K)$ with $\|T\| \leqslant 1$ such that *all T - orbits are weakly convergent* and some wealy null orbit is not strongly mixing.

3. One nonmetrizable example

We have been unable to show without the metrizability assumption on K that $\mathcal{C}(K)$ fails the Blum-Hanson property if K has infinitely many accumulation points. Note that metrizability was used twice in the proof of Theorem 1.1: to ensure that if K' is infinite then K contains the special compact set S; and for the existence of an isometric (linear) extension operator $J : \mathcal{C}(S) \to \mathcal{C}(K)$.

It is known that the linear extension theorem may fail in the nonmetrizable case (see e.g. [9, Remark 2.3]). It may also happen that a compact set K has infinitely many accumulation points and yet does not contain any compact set like S. For example, this holds for $K = \beta\mathbb{N}$ (the Stone-Čech compactification of $\mathbb{N}$) because there are no nontrivial convergent sequences in $\beta\mathbb{N}$. However, in this (very) special case it is possible to adapt the proof of Theorem 1.1 to obtain the following result.

Proposition 3.1. *The space $\ell_\infty(\mathbb{N}) = \mathcal{C}(\beta\mathbb{N})$ does not have the Blum-Hanson property.*

Proof. It will be more convenient to view ℓ_∞ as $\ell_\infty(\mathbb{N} \times \mathbb{N}) = \mathcal{C}(\beta(\mathbb{N} \times \mathbb{N}))$.

Let $\theta : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be essentially the same map as in the proof of Theorem 1.1 but ignoring the limit points:

$$\begin{cases} \theta(i,k) &= (i,k-1) & \text{if } k \geqslant 2 \\ \theta(i,1) &= (i-1,i-1) & \text{if } i \geqslant 2 \\ \theta(1,1) &= (1,1) \end{cases}$$

We denote by C_θ the asociated composition operator acting on $\ell_\infty = \ell_\infty(\mathbb{N} \times \mathbb{N})$, i.e

$$C_\theta f(i,k) = f(\theta(i,k)) \quad \text{for every } (i,k) \in \mathbb{N} \times \mathbb{N}\,.$$

Set $f := \mathbf{1}_F \in \ell_\infty(\mathbb{N} \times \mathbb{N})$, where $F = \{(i,1);\ i \geqslant 1\} \backslash \{(1,1)\} = \{(i,1);\ i \geqslant 2\}$. Exactly as in the proof of Theorem 1.1, one checks that the sequence $(C_\theta^n f)$ is not strongly mixing in $\ell_\infty(\mathbb{N} \times \mathbb{N})$. So it is enough to show that, on the other hand, $C_\theta^n f \to 0$ weakly in $\ell_\infty(\mathbb{N} \times \mathbb{N})$.

Viewing $\ell_\infty(\mathbb{N} \times \mathbb{N})$ as $\mathcal{C}(\beta(\mathbb{N} \times \mathbb{N}))$, we have to show that $C_\theta^n f(\mathcal{U}) \to 0$ for every ultrafilter $\mathcal{U}$ on $\mathbb{N} \times \mathbb{N}$. Let us fix such an ultrafilter $\mathcal{U}$.

Since $C_\theta^n f = C_\theta^n \mathbf{1}_F = \mathbf{1}_{\theta^{-n}(F)}$ when considered as an element of $\ell_\infty(\mathbb{N} \times \mathbb{N})$, we have for any $n \in \mathbb{N}$:

$$C_\theta^n f(\mathcal{U}) = \begin{cases} 1 & \text{if } \theta^{-n}(F) \in \mathcal{U} \\ 0 & \text{if } \theta^{-n}(F) \notin \mathcal{U} \end{cases}$$

So we need to prove that if n is large enough, then $\theta^{-n}(F) \notin \mathcal{U}$.

Observe first that if we set $S_1 := \mathbb{N} \times \{1\}$, then $\theta^{-n}(S_1) \cap S_1$ is finite for every $n \in \mathbb{N}$. This is readily checked from the definition of θ. Indeed, for each $s = (i,1) \in S_1$, the first $n \in \mathbb{N}$ such that $\theta^n(s) \in S_1$ is at least equal (in fact, exactly equal) to i; so for each fixed n there are at most n points $s \in S_1$ such that $\theta^n(s) \in S_1$.

Since $F \subset S_1$ and θ is finite-to-one, it follows that $\theta^{-n}(F) \cap \theta^{-n'}(F)$ is finite whenever $n \neq n'$.

Now, assume without loss of generality that $\theta^{-n}(F) \in \mathcal{U}$ for more than one $n \in \mathbb{N}$. Then, by what we have just observed, $\mathcal{U}$ contain a finite set. Hence, $\mathcal{U}$ is a principal ultrafilter, defined by some point $s_0 \in \mathbb{N} \times \mathbb{N}$. On the other hand, we know from the definition of the map θ that $\theta^n(s_0) = (1,1)$ for all but finitely many $n \in \mathbb{N}$. Since $(1,1) \notin F$, it follows that $\theta^{-n}(F) \notin \mathcal{U}$ for all but finitely many n.

□

From Proposition 3.1, we immediately deduce

Corollary 3.2. *The space $L_\infty = L_\infty(0,1)$ does not have the Blum-Hanson property. Likewise, if H is an infinite-dimensional Hilbert space, then the space $\mathcal{B}(H)$ of all bounded operators on H does not have the Blum-Hanson property.*

Proof. This is clear since these two spaces contain a 1 - complemented isometric copy of ℓ_∞. □

4. Further remarks

For any topological space E, let us denote by $\mathcal{C}_b(E)$ the space of all real-valued, bounded continuous functions on E. Putting together Theorem 1.1 and Proposition 3.1, we obtain the following result.

Theorem 4.1. *If T is a metrizable topological space, then $\mathcal{C}_b(T)$ has the Blum-Hanson property exactly when T is compact and has finitely many accumulation points.*

Proof. By Theorem 1.1, it is enough to show that if $\mathcal{C}_b(T)$ has the Blum-Hanson property, then T is compact. Now, if T is not compact, it contains a countably infinite closed discrete set S (thanks to the metrizability assumption). By Dugundji's extension theorem, $\mathcal{C}_b(T)$ then contains a 1-complemented isometric copy of $\mathcal{C}_b(S)$. Since $\mathcal{C}_b(S)$ is isometric to $\ell_\infty(\mathbb{N})$, it follows from Proposition 3.1 that $\mathcal{C}_b(T)$ does not have the Blum-Hanson property. □

To conclude this paper, and since this may be useful elsewhere, we isolate the following kind of criterion for detecting the failure of the Blum-Hanson property in $\mathcal{C}_b(T)$ for a not necessarily metrizable topological space T.

Lemma 4.2. *Let T be a Hausdorff topological space. Assume that there exists a subset S of T which is normal as a topological space, such that the following properties hold true.*

1. *One can find a continuous map $\theta : S \to S$ and a point $a \in S$ such that*
   - (i) *$\theta^n(s) \to a$ pointwise on S as $n \to \infty$;*
   - (ii) *there exists an open neighbourhood V of a such that*
   $$\sup_{s\in S} \#\{n \in \mathbb{N};\ \theta^n(s) \notin V\} = \infty\,;$$
   - (iii) *there exists a further open neighbourhood W of a with $\overline{W} \subset V$ such that, for any infinite set $\mathbf{N} \subset \mathbb{N}$, one can find $n_1, \dots, n_p \in \mathbf{N}$ such that the set $\theta^{-n_1}(S\backslash W) \cap \cdots \cap \theta^{-n_p}(S\backslash W)$ is finite.*
2. *There is a linear isometric extension operator $J : \mathcal{C}_b(S) \to \mathcal{C}_b(T)$.*

Then, one can conclude that the space $\mathcal{C}_b(T)$ fails the Blum-Hanson property.

Proof. By (2), it is enough to show that $\mathcal{C}_b(S)$ does not have the Blum-Hanson property. This will of course be done by considering the composition operator $C_\theta : \mathcal{C}_b(S) \to \mathcal{C}_b(S)$.

Since $\overline{W} \subset V$ by (iii) and S is normal, one can choose a function $f \in \mathcal{C}_b(S)$ such that $f \equiv 0$ on $\overline{W}$ and $f \equiv 1$ on $F := S\backslash V$. By condition (ii) in (1), the sequence $(C_\theta^n f)$ is not strongly mixing; so we just need to check that $C_\theta^n f \to 0$ weakly in $\mathcal{C}_b(S)$.

Being Hausdorff and normal, the space S is completely regular; so the space $\mathcal{C}_b(S)$ is canonically isometric with $\mathcal{C}(\beta S)$, where βS is the Stone-Čech compactification of S. The latter can be described as the space of all *z-ultrafilters* on S, i.e maximal filters of zero sets for functions in $\mathcal{C}_b(S)$, or, equivalently (since S is normal) maximal filters of closed subsets of S; see [5]. Therefore, what we have to do is to show that

$$\lim_{n\to\infty} \left[\lim_{\mathcal{U}} f(\theta^n(s)) \right] = 0 \quad \text{for any } z\text{-ultrafilter } \mathcal{U} \text{ on } S\,.$$

If $\mathcal{U}$ is a "principal" z-ultrafilter defined by some $s_0 \in S$, i.e. $\mathcal{U}$ is convergent with limit s_0, then $\lim_{\mathcal{U}} f(\theta^n(s)) = f(\theta^n(s_0))$ for all n, so the result is clear since $f(\theta^n(s_0)) \to f(a) = 0$ as $n \to \infty$ by (i).

Now, let us assume that $\mathcal{U}$ is not principal. Then $\mathcal{U}$ does not contain any finite set. Indeed, if a maximal filter of closed sets contains a finite union of closed sets $F_1 \cup \cdots \cup F_N$, then it has to contain one of the F_i by maximality; so, if $\mathcal{U}$ were to contain a finite set, then it would contain a singleton and hence would be principal in a trivial way. By (iii), it follows that $\theta^{-n}(S\backslash W) \notin \mathcal{U}$ for all but finitely many $n \in \mathbb{N}$; and since $\mathcal{U}$ is a maximal filter of closed sets, this implies that $\theta^{-n}(\overline{W}) \in \mathcal{U}$ for all but finitely many n. Since $f \equiv 0$ on $\overline{W}$, it follows that $\lim_{\mathcal{U}} f(\theta^n(s)) = 0$ for all but finitely many n, which concludes the proof.

□

Remark 1. This lemma would be much neater if condition (iii) above could be dispensed with; but we don't know how to prove the lemma without it. The proof of Theorem 1.1 shows that when S is compact, (i) and (ii) alone are enough for $\mathcal{C}(S)$ to fail the Blum-Hanson property. At the other extreme, the proof of Proposition 3.1 shows that when S is discrete (and infinite), one can find a map $\theta : S \to S$ satisfying (i), (ii) and a property stronger than (iii).

Remark 2. When S is compact, condition (iii) actually follows from (i). Indeed, let W be any open neighbourhood of a, and assume that (iii) fails for W and some infinite set $\mathbf{N} \subset \mathbb{N}$. Then, by compactness we have $\bigcap_{n\in\mathbf{N}} \theta^{-n}(S\backslash W) \neq \varnothing$. But if $s \in \bigcap_{n\in\mathbf{N}} \theta^{-n}(S\backslash W)$ then $\theta^n(s)$ does not tend to a as $n \to \infty$, which contradicts (i).

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