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Abstract: In this text, associated with my opening lecture at the first INDRUM Conference, I will reflect on the achievements and challenges of mathematics education research at university level, in the light of my experience both as a researcher and a university teacher. First, I will come back to the historical evolution of this field of research through the retrospective analysis of some selected episodes of my personal trajectory. I will then consider the current state of the field, presenting my vision of its strengths and weaknesses, of the challenges it faces and the resources existing to take them up.

Keywords: mathematics education, didactics, university mathematics, university teaching, advanced mathematical thinking, Calculus, anthropological theory of the didactic

INTRODUCTION

Having been invited to give the opening lecture at this first conference of the new International Network for Didactic Research in University Mathematics is an immense honour. For several reasons, this is also a big challenge. The first reason is that this field, seen as an academic field of research, has a rather long history which can be traced back up to the seventies at least. To make sense of research achievements and of the limitation of these, one needs to be aware of some elements of this history. The second reason is that the field is currently a burgeoning field of research, addressing a huge diversity of questions and mobilizing an increasing diversity of theoretical approaches for their study. Making sense of recent research achievements, through the multiplicity of existing discourses, through the multiplicity of changing contexts where they are produced, is a very challenging task. The third reason is that the vision that each one of us has of the field is necessarily today a limited vision. This is also a biased vision, shaped by each individual's personal experience. I am not an exception in this.

Reflecting on how I could take up the challenge and organize this opening lecture, I decided to structure it into two main sections. In the first one, perhaps more directed towards researchers in the field who are not necessarily aware of the details of its historical development, I will try to provide an idea of the evolution of the field, through a reflective look at some important episodes of my personal trajectory. Then, in a second part, I will discuss the current state of the field, presenting my vision of its strengths and weaknesses, of the challenges it faces and the resources existing to take these challenges up.

As already stressed, this is certainly a partial and subjective vision of the field, with which the reader is not obliged to agree. I hope, however, that it will provide some useful elements to understand how this field of research progressively constituted and developed its identity, what it was able to achieve, and to outline perspectives for its future.

ENTERING THE FIELD OF UNIVERSITY RESEARCH: A MATHEMATICS-PHYSICS ADVENTURE

I entered research in university mathematics education at the turn of the eighties, when a group of mathematicians and physicists in my university, including mathematics and physics didacticians, decided to create an experimental course for students entering the mathematics and science programme. Our main goal was to challenge the compartmentalization of the two disciplines. The design of this course was very innovative with regular lectures on topics of common interest jointly prepared and given by one mathematician and one physicist, common tests, interdisciplinary projects (Artigue 1981). The whole team met once a week and everyone attended the common lectures.

Everything worked fine with the exception of the planned common lecture on the notion of differential for which the mathematicians and physicists could not reach an agreement. This « differential clash » became a research issue that I first addressed jointly with the two didacticians of physics in the team, Laurence Viennot and Edith Saltiel, then with Marc Legrand and his colleagues in Grenoble, who were working on the teaching of the Riemann integral.

If I consider retrospectively the research we developed (Artigue, Menigaux & Viennot 1990; Alibert et al. 1988), it is rather representative of some predominant characteristics of university research in mathematics and physics education at that time. The research has indeed a clear cognitive orientation. It aims at understanding the conceptions of the differential developed by our students, the difficulties they meet with this notion and the associated processes in mathematics and in physics, and it does so through a series of questionnaires and interviews.

This research also demonstrates a strong epistemological sensitivity, perhaps more characteristic of the French didactic community. With the help of historians, we made specific efforts to understand the source of the « differential clash » observed, why mathematicians seemed so rigidly attached to their vision of the differential as a function, in fact a differential form, and why physicists seemed so rigidly attached to their pragmatic vision of differentials as small, if not infinitesimal, increments. For that purpose, we worked on primary and secondary historical sources, and also systematically studied the traces of the differential/derivative educational debate.

This research also shows the strong desire of researchers for concretizing their research findings into educational action. The results of both the cognitive and epistemological analyses were used in order to make visible the negative effects of the current situation to mathematicians and physicists. Students declared that it was better for them not to try to understand what a differential is, and to work mechanically, both in mathematics and in physics; they were not able to distinguish between situations requiring or not the use of differential processes, and they only succeeded in solving the tasks proposed to them because they learned to detect the linguistic hints in their texts calling for the use of such processes and to mobilize the rituals within each of the two disciplines. These results were also used to develop a compromise acceptable for the two teams. Moreover, math-physics tasks in line with this compromise were designed and implemented in specific workshops in the following years (Artigue, Menigaux & Viennot 1988).

However, this research, and more globally the work carried out in this experimental course, shows that ecology and sustainability issues were not really part of our agenda. In fact, when the teams moved to other projects and teaching activities, this experimental course disappeared.

EXPERIENCING THE 'SCHIZOPHRENIA' OF UNDERGRADUATE UNIVERSITY TEACHERS: THE TEACHING OF DIFFERENTIAL EQUATIONS

The second research I would like to briefly evoke is the research I developed regarding the teaching of differential equations some years later. At that time, I was working on issues related to dynamical systems with some colleagues, with the support of a specialist of this domain, the mathematician Adrien Douady. I experienced the type of 'schizophrenia' which is rather common to those who teach undergraduate courses: the complete disconnection between what is their lived experience as mathematicians and their lived experience as university teachers. The programme of the course on ordinary differential equations for second year students I had to teach was indeed focused on the algebraic solving in finite terms of some specific forms of equations, making the students think that the goal of research in this domain was to progressively complete the book of recipes they were shown through some exemplars. Only graduate students could have access to other views.

I decided to investigate the possibility of developing a course for beginners more in line with the epistemology of this domain. Once again, the epistemological work was an essential dimension of the research, leading to the identification of three main historical strands, each one of them having its own problématique and development: the algebraic, the numerical and the geometrical-topological strand initiated much later than the two first ones by Henri Poincaré at the end of the 19th century. Only the first strand, in its most elementary forms, was part of the course (the theory of exact resolution as initiated by Condorcet and Liouville was not considered).

To address the research question I chose a methodology of didactical engineering, and the use of didactic constructs familiar to French didacticians – such as the notions of setting and tool-object dialectics due to Regine Douady (1986) – together with fundamental constructs of the Theory of Didactical Situations (TDS) (Brousseau 1997). This methodology was of course adapted to the specific context of university education. The didactical engineering was collaboratively designed with Marc Rogalski and his colleagues from the Université de Lille 1, and successfully experimented with for several years at this university (Artigue & Rogalski 1990). However, the experimentation showed that the viability of the design required a different institutional status for graphical representations to the limited heuristic status given to these in university courses; graphical representations had to be credited as a legitimate tool for reasoning and proof, of course in appropriated forms such as those developed in the engineering design (Artigue 1992). Moreover, we discovered that such a change could not be limited solely to the teaching of differential equations; for evident reasons of coherence, this change was to impact the whole approach of the Analysis course. This certainly contributes to explaining why, despite its repeated successful implementation, only the first situations of this design were more widely used. Today, conceptual tools such as the hierarchy of didactic codetermination proposed by the Anthropological Theory of the Didactic (ATD) (Chevallard 2002) help us systematically consider the different conditions and constraints governing the possible ecology of the didactical engineerings we design, beyond those situated at the level of the mathematical theme or sector directly addressed – and better anticipate their possible effects. We were less equipped to address these ecological issues thirty years ago.

These are just two examples among many others. They reflect the cognitive and epistemological focus of the research carried out at that time, and also the form that this focus was likely to take in the French didactic culture where TDS, with the its underlying systemic was the predominant theoretical approach. It also reflects the engagement of researchers in action, but without the conceptualizations that would have allowed them to seriously address dissemination and sustainability issues.

THE INTERNATIONAL SCENE: THE AMT WORKING GROUP OF PME AND THE ICMI STUDY ON THE TEACHING AND LEARNING OF MATHEMATICS AT UNIVERSITY LEVEL

On the international scene, the state of research at that time is well reflected by the work of the Advanced Mathematic Thinking (AMT) working group of PME which I entered in the late eighties, and the synonymous book resulting from this work whose production was coordinated by David Tall (1991). This book confirms the cognitive orientation of research I already mentioned towards the study of students' learning processes, thinking modes, conceptions and difficulties, and also the strong influence of constructivist perspectives. One of the main aims of the working group was to elucidate the specific nature, if any, of what its participants called advanced mathematical thinking, and thanks to this elucidation, to better understand what differentiated learning processes at university from those experienced before by students. Even if a definitive answer is not provided in the book, some criteria are proposed in terms of relationship to abstraction, symbolism and generalization, role of definitions, formal reasoning and proof. One can also observe the important role played in research by constructions that take the form of distinctions such as that between concept definition and concept image due to Schlomo Vinner and David Tall (1981); or, emerging theories such as the construction developed by Dubinsky based on Piaget's idea of reflective abstraction that would become APOS theory (Arnon et al. 2014); or, that developed by Tall (2013) that would lead to his theory of cognitive development along three different worlds (the embodied, the symbolic and the formal). One can also note the predominance of Calculus/elementary Analysis as a mathematical theme and that many research projects were motivated by the high level of failure in the corresponding undergraduate courses, which form a gateway to any kind of scientific orientation in most universities. The concept of limit, considered as its foundational core concept, was especially addressed.

Another characteristic, well representative of educational research at that time, is the emphasis put on cognitive discontinuities and their epistemological sources in learning processes, and on the persistent difficulties generated by these discontinuities. Different theoretical constructions contributed to the conceptualization of these discontinuities – I just mention three of these: the notion of epistemological obstacle borrowed from Bachelard's epistemology (Bachelard 1938; Brousseau 1983) that shows the role played in students' resilient difficulties by forms of knowledge which have proved to be effective in other contexts (as shown by researchers such as Bernard Cornu (1991) and Anna Sierpinska (1985) for the concept of limit); the discontinuity between proceptual (Gray & Tall 1994) and formal thinking; the discontinuity between concepts that emerge as necessary ingredients of the solution of specific problems, such as the concept of derivative or integral, and concepts that respond to unifying and formalization needs, such as the concept of abstract vector space (what Aline Robert (1998), Jean-Luc Dorier (2000) and their colleagues named FUG (formalizing, unifying, generalizing) concepts).

Discontinuities were also identified between domains, for instance between algebra and analysis. I just mention below some of those which have been proved especially challenging for students:

- the change needed in the perception of equality, which, in order to understand the mechanism of analytic proofs , must be perceived as a sign expressing arbitrary level of closeness;
- the predominant role taken by inequalities over equalities, and, more than that, the transition from global perspectives regarding the solving of inequalities to a subtle combination of local and global perspectives;
- the change induced from reasoning modes based on equivalence to reasoning modes based on the use of sufficient conditions, whose effectiveness requires the ability to lose information in a controlled way, taking into account both the

respective orders of magnitude of terms in symbolic expressions and the local character of analytic reasoning.

Let me stress that what is more globally addressed here is the change and reconstructions needed in mathematical practices when moving from one domain to another – what today I would call a change in mathematical praxeologies.

As stressed earlier, research first focused on discontinuities, but progressively became more sensitive to the essential role played by connections and flexibility in teaching and learning processes. Such an evolution has been supported by the increasing attention paid to the semiotic dimension of mathematical activity in educational research, and also by the technological evolution and its specific semiotic affordances. In university research, this evolution is visible for instance in the presentation of the state of the art of research about the teaching and learning of linear algebra co-authored by Jean-Luc Dorier and Anna Sierpinska in the ICMI Study devoted to the teaching and learning of mathematics at university level (Dorier & Sierpinska 2001). This chapter makes the complexity of connections at stake in linear algebra clear: connections between different languages (geometrical, algebraic, abstract), between different registers of representations (graphical, algebraic, symbolic representations, tables), between Cartesian and parametric points of view, and synthetic-geometric, analytic-arithmetic and arithmetic-structural modes of reasoning. Moreover, analyzing teaching practices, researchers show that university teachers, most often, jump without any precaution between these different systems, underestimating the difficulties that these jumps provoke for their students. Of course, connections and flexibility are not a specificity of university mathematics, but what changes is their intensity, and the autonomy given to the students regarding their management.

I was involved in the ICMI Study just mentioned co-ordinated by Derek Holton as a member of its International Programme Committee and this was a very interesting experience. As is the case for any ICMI Study, our collective work intertwined general reflection on the themes identified in the discussion document, syntheses of research advances, and the presentation and discussion of many innovative realizations carried out in different contexts. Published ten years after the AMT book, this ICMI Study (Holton 2001) shows the diversity of issues addressed by those interested in teaching and learning at university level, not just researchers in the field: curricular and assessment issues, teaching practices, relationships between mathematics and other disciplines, affordances of technology, teacher education including that of university teachers still in an emerging state at that time. Issues related to the secondary and university transition are addressed in several chapters, but they are more widely approached than in the AMT book, considering the diversity of social and psychological moves that this transition entails for students. The research section, however, tends to show that the "socio-cultural turn", as denoted by Steve Lerman, has not yet substantially impacted research at that level.

THE SOCIO-CULTURAL TURN THROUGH THE LENS OF THE ATD

For me, in fact this socio-cultural turn was tightly linked to the incorporation of the ATD in my research perspectives. It first occurred with the supervision of Brigitte Grugeon's doctoral thesis on the transition between vocational high school and technological high school in France (Grugeon 1995), and it turned out to be so productive that I engaged Frederick Praslon, another doctoral student of mine, with adopting ATD as a macro-theoretical framework to study the secondary/university transition on the concept of derivative and its mathematical environment (Praslon 2000).

As expressed very well by Marianna Bosch and her colleagues (Bosch, Fonseca & Gascón 2004), adopting such a perspective on the secondary/university transition represents a radical move. The lens is no longer directed towards the student and her cognitive functioning or development, but towards the institutional practices that condition and constrain, both explicitly and implicitly, what she has the possibility to learn or not, and the associated systems of norms and values which remain partly tacit. In his pioneering work, Frederick Praslon in fact used the ATD to question the common at the time perspective on transition as a transition from the proceptual to the formal world, from intuitive and pragmatic reasoning modes to rigorous mathematical ones. Carefully analyzing mathematical praxeologies in scientific high school and first university year, through a diversity of institutional sources, he showed that contrary to what was often claimed by university teachers, a substantial universe around the notion of derivative was already established at the end of high school in France at least for students in the scientific stream, but that a dramatic extension of the landscape was taking place in the first six months at university, which he visualized using concept maps. He also showed that the transition between secondary and mathematics-sciences programmes at university was not a radical move from the proceptual to the formal world, from an intuitive and algorithmic Calculus to the approximation world of Analysis; it was rather an accumulation of micro-breaches, thus less visible and not appropriately addressed by the institution. The main breaches he identified are the following:

- an increasing speed in the introduction of new objects;
- a greater diversity of tasks making routinization much more difficult;
- much more autonomy given in the solving process for similar tasks;
- a new balance between the particular and the general, the *tool* and *object* dimensions of mathematical concepts;
- objects more controlled by definitions, results more systematically proved, and proofs which are no longer "the cherry on the cake" but take the status of mathematical methods.

As he evidenced, the conjunction of these breaches created a substantial gap but university teachers were not aware of it in their great majority and tended thus to under-estimate the cognitive charge induced for their students. To make university teachers and students sensitive to these changes, Praslon designed a set of tasks that could be considered in the gap between the two cultures: a priori compatible with high school knowledge but fully exotic in high schools, and at the same time not really university tasks.

I will not enter into more details. Since that time, the anthropological perspective has been used for the study of institutional transitions, with the incorporation of conceptual tools such as the hierarchy of didactic codetermination which were not available at the time of Praslon's doctoral thesis, the development of specific notions such as the notion of completeness of praxeologies (Bosch, Fonseca & Gascón 2004). These have made the identification of new characteristics of the secondary-university transition possible: incompleteness and isolation of high school praxeologies, changes in the respective importance attached to the praxis and theoretical blocks of praxeologies, and in the topogenetic distribution of roles between teacher and students. However, this is only one part of the changes potentially induced by the adoption of the ATD lens, and does not take into account more recent developments of the theory such as its design dimension based on the paradigm of "Questioning the world" and the idea of Study and Research Path (SRP) (cf., for instance, the pioneering doctorate thesis by Barquero (2009), the recent one by Cristina Oliveira (2015) and several contributions to this conference), or the refinements of the notions of technology and theory introduced by Castela and Romo Vazquez (2011) in order to better take in charge the circulation of knowledge between institutions and the reality of practices in engineering courses.

Mentioning these recent developments helps me make the transition to the next section of this text in which I discuss more broadly the strengths and weaknesses of this field of research, as I see them in the light of its historical evolution, and the current challenges that the field faces.

STRENGTHS, WEAKNESSES AND CHALLENGES

Strengths

There is no doubt that the field of research in university mathematics education presents evident strengths. It has developed over more than four decades, with regular efforts of syntheses. I have already evoked two of these, the AMT book and the ICMI Study volume, but in the recent years, new syntheses have been produced, such as the chapter I co-authored with Carmen Batanero and Philip Kent for the second NCTM Handbook (Artigue, Batanero & Kent 2007), the survey led by by Mike Thomas for ICME-12 (Thomas et al. 2014), or the book *Amongst Mathematicians* (Nardi 2008). All these syntheses show that a substantial amount of knowledge has been accumulated, and also that important efforts have been made to build structured, connected and coherent accounts of this knowledge. This is certainly a strength in itself.

I have stressed the importance of epistemological reflection in the emergence of this field and evoked some forms it has taken. This epistemological work is going on accompanying the development of the field, more and more benefitting from productive interactions with other communities and from the progression of their research problématiques and results. Being attached to a doctoral school structured around philosophy, history and epistemology of sciences and didactics of sciences, I have a regular experience of such productive interactions. I can also measure the fascinating evolution of epistemological perspectives since the time of the AMT working group, more and more open to the diversity of forms of life that mathematics has according to the contexts and cultures where it is practised and developed.

The emergence of the field was also characterized by the domination of cognitive and constructivist perspectives. I consider as a strength of our field the fact that we have succeeded in emancipating ourselves from these perspectives, whose limitations are evident, but also the fact evidenced by the consideration of most research publications, that this emancipation went along a reconstruction of their main outcomes, thus making possible some form of incorporation of these outcomes in the new paradigms. I personally experienced such reconstructions in the diverse doctoral theses I supervised on institutional transitions, and I see also a sign of it in the current enterprise of networking between APOS and the ATD, two theoretical constructions I tend to position at the extreme opposites of our field.

Another strength of the field is certainly its move from investigation focusing on the student to a more balanced interest in both the student and the teacher. This move is not proper to the field of research at university level as is well known, but it seems to have been more difficult to achieve at this level of schooling. Today, however, this obstacle seems finally overcome, and university teacher practices are becoming an object of study in their own. Research also investigates more and more possible strategies for the didactic acculturation of university teachers, extending the pioneering work of Barbara Jaworski and Elena Nardi at Oxford University years ago (Nardi, Jaworski & Hegedus 2005), and benefiting from the potential of new theoretical perspectives, such as those offered by the theories of community of practice (Biza, Jaworski & Hemmi 2014) and community of inquiry (Jaworski 2008).

More generally, theoretical evolution in the field is both promising and challenging. I have already evoked the increasing use in research of the ATD, the potential of which for university research has been especially analysed by Carl Winsløw in his regular lecture at ICME12 (Winsløw 2014). Furthermore the *Research in Mathematics Education* Special Issue (Nardi, Biza, González-Martín, Gueudet & Winsløw 2014) is especially insightful from this perspective, considering the affordances of a range of socio-cultural, institutional and discursive theories: ATD, TDS, instrumental and documentational approaches (Gueudet, Pepin & Trouche 2012), the theories of Communities of Practice and Communities of Inquiry, and the theory of Commognition (Sfard 2008). There is no doubt for me that these theories offer evident potential for research at university level and at the transition between secondary and university education. Without entering into more details about this potential, I would say that the most challenging perspective for me is that offered by commognition. I have personally some reservation at accepting all the implications of adopting such a radical discursive approach, but the ways it engages us to analyze communicative acts involving students and teachers (Nardi, Ryve, Stadler & Viirman 2014), to think about the teacher role and the resilience of university practices such as lecturing (Sfard 2014) is for me really insightful.

Weaknesses

This being said, there is no doubt that the field also presents some weaknesses, and here I would like to mention some of these. In the ICMI Study already mentioned, it was pointed out that research concentrated too much its efforts on the classical formation of future mathematicians despite the fact that these represented only a very small percentage of university students being taught some mathematics. This was the reason why, when I was asked to lead the authorship of the chapter on learning mathematics at post-secondary level of the second NCTM Handbook (Artigue et al 2007), I proposed as co-authors Carmen Batanero and Philip Kent, who could help us realize a more balanced perspective that paid due attention to stochastic and engineering education. However, there is no doubt that research is still biased both in terms of domains and in terms of population. I still have the impression that: fields of increasing importance in mathematics – such as the stochastic field including probability and statistics, and more generally applied and computational mathematics – are still under-investigated, and that still the practice of the mathematician researcher, and even the pure mathematician researcher, is the implicit reference in most research studies; and, that the diversity of forms of professional relationship with mathematics for which university courses may prepare graduates is still not sufficiently investigated and taken into account. Perspectives are moving as attested by the contributions at this INDRUM conference, and we are much better equipped for tackling such issues; however, interests in the field remain too unbalanced.

Another weakness in my opinion is the excessive predominance of very small-scale qualitative studies, involving a very limited number of students or teachers. Moreover, reviewing submissions or reading articles, I am also often disappointed to read that the authors have collected a huge amount of data, but that quite often the evidence they provide for supporting their claims is reduced to the micro-analysis of some very limited episodes; and, that the triangulation that is a priori possible between different levels and dimensions of analysis that would make the results more convincing is hardly present.

I have to confess also that I often have the impression that what I am reading has been already said years ago – admittedly with some variation in the discourse – but with variations that do not show evident progression of knowledge. This is for me especially the case in Calculus and Analysis, but this may be just because I have been involved much longer in that area. I do not deny the necessity of going on working on foundational concepts such as the concept of limit, incorporating new perspectives, taking into account the evolution of contexts, of populations, educational means and resources. After attending this conference where many contributions have dealt with Calculus and Analysis, in the first thematic working group as well as in the other four, I am, however, more optimistic.

The last weakness I would like to mention is the insufficient dissemination of research results towards the relevant communities or practitioners, and the very limited influence of our research on university teaching practices. Reading recent publications, for instance the three first issues of the new *International Journal of Research in Undergraduate Mathematics Education*, I find nearly the same description of standard university practices at undergraduate level as decades ago. Of course, such difficulties are not specific to the field of university mathematics education, but one could expect that, being themselves researchers, university teachers would be more open to considering research advances and what these can offer them to better understand their students and to improve their teaching practices. Unfortunately, this does not seem to be the case in general, for many reasons which range from the low institutional value attached to teaching activities, in comparison to research activities at university, to the image of the didactic discipline itself in the mathematics community, in most countries.

However, we have to acknowledge also that making sense of research results in mathematics education, converting them into something useful in practice, is not an easy task. The activities of networking between theoretical frameworks I have been involved through different projects in the last decade (Bikner-Ahsbahs & Prediger 2014), (Lagrange & Kynigos 2014) have evidenced that, even for a didactician, to make sense of other research approaches and results, just by reading the associated literature and by attending seminars or conferences, is difficult. In these projects, appropriation resulted in fact from the collaborative building of networking praxeologies on top of our own research praxeologies (Artigue & Bosch 2014), and it was very progressive. The communication between mathematicians and didacticians does not face exactly the same problems, as extensively discussed in (Fried & Dreyfus 2013), but this experience reinforced my conviction that to overcome the current limitations, we must not think in terms of dissemination of research results, but in terms of collaborative projects, building and negotiating, jointly with mathematicians and other university teachers, problématiques that make sense for all those involved, and meet their respective interests and needs. And then we must combine our respective knowledge and expertise in these projects through appropriate praxeologies.

Of course, collaborative projects have existed for decades. The two experiences I mentioned at the beginning of this text were clearly collaborative projects; the ICMI Study volume (Holton 2001) presents a range of examples; however, each of them more or less appears as a particular and isolated case. We can go further today; we have more powerful conceptual tools in order to approach ecological and institutional issues, to approach collaborative work and relationships between communities, to build, analyse and compare projects, and, last but not least, to consider the long term dynamics necessarily at stake.

Challenges and resources

There are strengths and weaknesses, but there are also many challenges, old and new challenges. I will focus here on some of those raised by the fact that we live in a fast moving world.

How can we maintain some connection between the living field of mathematics, so dynamic and diverse, and undergraduate mathematics education, both in terms of content and practice?

There is no doubt that, contrary to graduate mathematics education, undergraduate mathematics education is poorly connected to the mathematics of today in most universities. It is generally argued that the limited mathematical background of undergraduate students makes the connection with the sophisticated world of current mathematics impossible. This may be the case if we consider that this connection must necessarily be expressed in terms of operational knowledge. However, if we consider mathematics as a part of human culture, we must admit that, as any cultural form, our mathematical culture is not reducible to its operational part. The distinction between different forms of relationships with mathematical objects and practices opens the landscape towards alternative didactic strategies and practices. Those developed and used by the very active community engaged in the popularization of mathematics could be a source of inspiration. However, as was evidenced by ICMI Study 16 which addressed this topic (Barbeau & Taylor 2009), up to recently at least, didactic research has paid limited attention to popularization practices, and more generally informal mathematics education. Moreover, as far as I know, the didactic community has still limited contact with the community of research in science communication with which the collaboration could certainly be helpful. There are thus resources that could be more systematically explored for addressing this challenge.

How can we make our students really experience the subtle and original combination mathematics currently offers of experimental and deductive games, thanks to the evolution of technology?

Technological evolution has substantially impacted professional mathematical practices, in particular by providing much more powerful tools for supporting an experimental dimension of mathematical practices that has always existed, and by making this experimental work more visible and sharable (cf. for instance the journal *Experimental Mathematics*). However, in many places, undergraduate mathematics education seems still blind to this evolution, even when those in charge make extensive use of technology in their professional activity. Making visible the experimental dimension of mathematics tends to be perceived as an obstacle to the entrance into the deductive game of mathematics aimed at, leading in some first year programmes to the banishment of any technological tool in algebra or analysis courses. Overcoming such a limited epistemological view and its negative consequences is a challenge that researchers in mathematics education have faced for decades, but there is no doubt that the situation remains critical today at university level in many places, contributing to the rupture with mathematical practices in secondary education.

How can we address the dramatic changes that the technological evolution more generally induces in the ways we and our students access information and resources, learn, communicate, interact, work and produce with others?

In fact, the changes induced by the technological evolution do not limit to those just evoked. The digital era in which we have entered induces dramatic changes in the way we access information and build knowledge, in the way we communicate, interact and work. New pedagogical strategies develop, such as reversed pedagogy, MOOCs and diverse forms of hybrid pedagogy, which need to be studied. The number of on line resources increases exponentially as well as the diversity of learning sources, and modalities of use. Once again, didactic research offers promising tools to take up this challenge, the documentational approach initiated by Ghislaine Gueudet and Luc Trouche (Gueudet et al. 2014; Gueudet, Pepin & Trouche 2012) being one of the most recent ones.

And, finally, how can we make our students consider mathematics as a resource for thinking about this fast moving world, questioning it, and trying to make it a bit better?

CONCLUSION

Opening the first INDRUM Conference, I have tried to share with the participants the experience of a researcher who has been active in the field of university mathematics education for more than three decades. I organized my reflection around the historical evolution of this field of research convinced that this could help understand its current state and better appreciate its achievements, identify its strengths and weaknesses, as well as perspectives for future research. I have tried to make clear that substantial advances have been made, both from a theoretical and empirical point of view, that knowledge has progressively accumulated, and that, even if weaknesses still exist, we are today reasonably equipped to take up the many challenges that we have to face. As pointed out in the introduction, the vision I have given of the field is certainly both partial and subjective, and shaped by my own experience and by the research and educational cultures in which it has mainly developed. I hope, however, that it has found resonance with the perspectives and experiences of many participants, and that it has been a stimulus for the discussions and work carried out during the three days of the INDRUM conference.

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