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A random walk version of Robbins' problem: small horizon

Abstract In Robbins' problem of minimizing the expected rank, a finite sequence of n independent, identically distributed random variables are observed sequentially and the objective is to stop at such a time that the expected rank of the selected variable (among the sequence of all n variables) is as small as possible. In this paper we consider an analogous problem in which the observed random variables are the steps of a symmetric random walk. Assuming continuously distributed step sizes, we describe the optimal stopping rules for the cases $n = 2$ and $n = 3$ in two versions of the problem: a "full information" version in which the actual steps of the random walk are disclosed to the decision maker; and a "partial information" version in which only the relative ranks of the positions taken by the random walk are observed. When $n = 3$, the optimal rule and expected rank depend on the distribution of the step sizes. We give sharp bounds for the optimal expected rank in the partial information version, and fairly sharp bounds in the full information version.

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1. Introduction. Robbins' problem of optimal stopping can be described as follows: Let $X_1, X_2, \ldots, X_n$ be a finite sequence of independent, identically distributed (i.i.d.) random variables defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ whose common distribution is continuous (without loss of generality, uniform on $[0, 1]$), and let $\rho_k := \sum_{i=1}^{n} \mathrm{I}(X_k \geq X_i)$, $k = 1, \ldots, n$, so ρ_k denotes the rank of X_k among $X_1, \ldots, X_n$ (where we rank the observations from smallest to largest). The objective is to determine the value of

$$v_n := \inf_{\tau} \mathbb{E}(\rho_\tau),$$

where the infimum is over all stopping times adapted to the natural filtration of $X_1, \ldots, X_n$; and to describe the optimal stopping time τ_n^* which attains the infimum. Since this problem was proposed by H. Robbins at the International Conference on Sequential Search and Selection in Real Time (Amherst, 1990), many mathematicians have worked on it, but despite the real progress that has been made, several fundamental questions remain open. This is mainly due

to the fact that the optimal rule is fully history dependent (i.e. the decision to stop at time k depends on all of the values $X_1, \ldots, X_k$ observed so far); see Bruss and Ferguson [4]. In fact, the exact optimal rule is known only for $n \leq 4$. (The case $n = 2$ is trivial; the case $n = 3$ was solved by Assaf and Samuel-Cahn [1]; and the case $n = 4$ was solved recently by Dendievel and Swan [7]. The computations in the last paper are considerable.)

Most of the literature on Robbins' problem focuses on asymptotics. Bruss and Ferguson [3] proved that v_n is increasing in n, and that the limit

$$v := \lim_{n \to \infty} v_n$$

is at most 7/3. This was improved slighty by Assaf and Samuel-Cahn [1], to 2.3318 using a simple threshold rule. On the other hand, the best known lower bound for v, obtained by Bruss and Ferguson [3], is 1.908. Despite considerable efforts by these and other mathematicians, this sizable gap has not been narrowed much in the last 20 years. The pre-2005 history of the problem is beautifully laid out in the survey by Bruss [2].

In more recent times, emphasis has shifted to a continuous-time version of the problem where the arrival times and values of the observations follow a planar Poisson process (the *Poisson-embedded Robbins' problem*); see Gnedin [9]. In this context, Meier and Sőgner [11] recently constructed a hybrid strategy (part rank-based, part threshold rule) with a value of 2.32614. This is currently the best known upper bound for v, as Bruss and Swan [5] have shown that the Poisson embedded Robbins' problem has the same limiting value (as the time horizon t tends to infinity) as the discrete problem. For more details, we refer to the Ph.D. thesis of Swan [12].

Other works have studied other aspects of Robbins' problem. In particular, Gnedin and Iksanov [8] showed that $\limsup_n nE[X_{\tau_n^*}]$ is finite, proving a conjecture of Bruss and Swan [5]. In a recent paper, Chen and Goldberg [10] develop an iterative scheme for solving history-dependent optimal stopping problems and apply this to Robbins' problem, obtaining concrete error bounds. This may eventually aid numerical efforts to estimate the optimal expected rank v_n.

We note also that a simpler version of the problem, in which the decision to stop at time k may depend only on the relative rank $\tilde{\rho}_k := \sum_{i=1}^k I(X_k \geq X_i)$, was already solved completely in 1964 by Chow et al. [6]. They obtained the optimal rule explicitly, and showed that, as $n \to \infty$, the optimal expected rank converges to approximately 3.87.

In the present article we consider an analog of Robbins' problem for symmetric random walks. Our setup is as follows. Let $X_1, X_2, \ldots, X_n$ be a finite sequence of i.i.d. random variables defined on $(\Omega, \mathscr{F}, \mathbb{P})$ whose common distribution is continuous and symmetric about 0, and consider the random walk

$S_n := X_1 + X_2 + \cdots + X_n$, $n \geq 1$, with $S_0 \equiv 0$. Let

$$R_k := \sum_{i=0}^{n} \mathrm{I}(S_k \leq S_i)$$

denote the rank of S_k among $S_0, S_1, \cdots, S_n$. (Observe that here we rank the variables from largest to smallest, which is more natural when thinking of the random walk as modeling, for instance, a stock price process.) We are interested in finding a stopping time τ so as to minimize the expected rank $\mathbb{E}(R_\tau)$. We consider two versions of the problem: In the *full information version*, we assume that the values of $X_1, X_2, \ldots$ are observed completely, so we can use any stopping time τ adapted to the filtration $\{\mathscr{F}_k\}$, where $\mathscr{F}_k := \sigma(\{X_1, \ldots, X_k\}) = \sigma(\{S_0, S_1, \ldots, S_k\})$ for $k = 0, 1, \ldots, n$. By contrast, in the *relative ranks version* of the problem, we assume that only the relative ranks

$$\tilde{R}_k := \sum_{i=0}^{k} \mathrm{I}(S_k \leq S_i), \qquad k = 0, 1, \ldots, n$$

are observed, so only stopping times adapted to the filtration $\{\mathscr{G}_k\}$ may be used, where $\mathscr{G}_k := \sigma(\{\tilde{R}_0, \tilde{R}_1, \ldots, \tilde{R}_k\})$ for $k = 0, 1, \ldots, n$.

In this note, we give a complete solution to both versions of the problem when the time horizon is small ($n \leq 3$). As far as we are aware, this is the first work to give a detailed treatment of the expected rank problem for random walks.

It should be noted that in the expected rank problem for i.i.d. random variables, neither the optimal stopping rule nor the expected rank depends on the distribution of the X_i's, as long as it is continuous. As will be seen below, this is no longer the case for the random walk version of the problem. For $n \geq 3$, both the optimal rule and the optimal expected rank depend on the distribution of the steps $X_1, X_2, \ldots$ of the walk in both the full information version and the relative ranks version of the problem, though less so in the latter.

2. Results

Let F denote the common distribution function of $X_1, X_2, \ldots$. If $n = 1$, it does not matter whether we stop at time 0 or time 1: in either case our expected rank is $3/2$ by the symmetry of X_1. For a random walk with two steps, the problem is still distribution-invariant:

THEOREM 2.1 *Let $n = 2$. Then the optimal rule in both the full information and the relative rank versions of the problem is*

$$\tau^* = \begin{cases} 1 & \text{if } S_1 > 0, \\ 2 & \text{otherwise,} \end{cases}$$

and the minimum expected rank is $\mathbb{E}(R_{\tau^*}) = 15/8$.

By contrast, for a random walk with three steps, the optimal rule and expected return depend on the distribution in both versions of the problem.

THEOREM 2.2 *Let $n = 3$. In the full information version of the problem, there is a number $x_1^* > 0$ (given implicitly as a solution of equation (11) below) such that the optimal rule is*

$$\tau^* = \begin{cases} 1 & \text{if } 0 < X_1 \leq x_1^*, \\ 2 & \text{if } X_1 > x_1^* \text{ and } X_2 \in (0, \infty) \cup (-X_1, -\tfrac{1}{2}X_1]; \text{ or} \\ & \quad X_1 \leq 0 \text{ and } X_2 \in (0, -\tfrac{1}{2}X_1] \cup (-X_1, \infty), \\ 3 & \text{in all other cases.} \end{cases} \tag{1}$$

(See Figure 1.) Moreover, $F(x_1^*) \geq \frac{1}{2} + \frac{\sqrt{2}}{4} \approx .85355$, and the optimal expected rank $\mathbb{E}(R_{\tau^*})$ satisfies the inequalities

$$2.2413 \leq \mathbb{E}(R_{\tau^*}) \leq \frac{55}{24} \approx 2.2917, \tag{2}$$

in which the upper bound is attained.
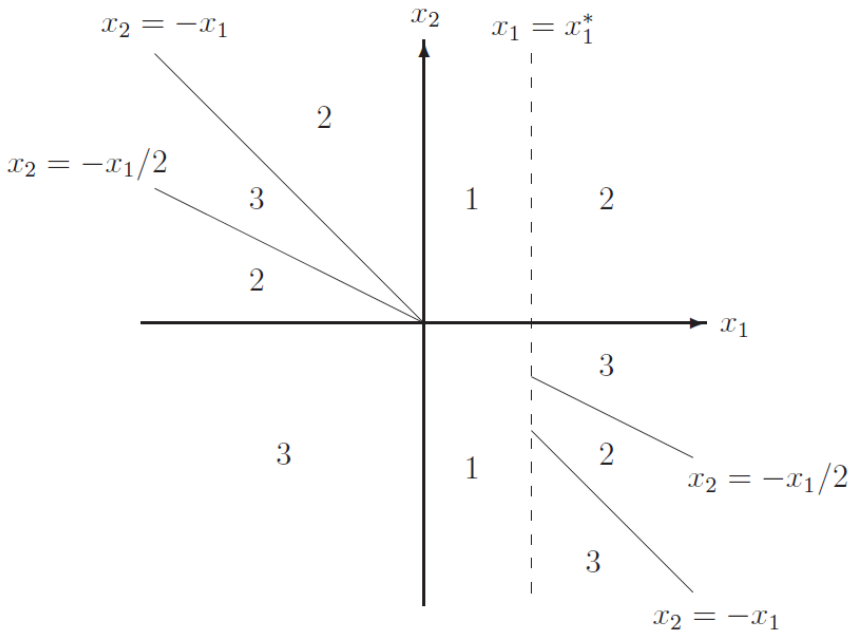

Figure 1: The stopping regions for the full-information problem with $n = 3$. Each region is labeled with the corresponding value of τ^*.

For the relative ranks version of the problem, we introduce two parameters $p = p_X$ and $q = q_X$, defined as

$$p := p_X := \mathbb{P}(0 < X_1 < X_2 < X_3 < X_1 + X_2),$$
$$q := q_X := \mathbb{P}(0 < X_1 < X_2 < X_1 + X_2 < X_3).$$

Here we use X to denote a generic random variable with the same distribution as X_1, X_2 and X_3. Observe that $p + q = \mathbb{P}(0 < X_1 < X_2 < X_3) = 1/48$ by the symmetry of the X_j's and the fact that all six permutations of (X_1, X_2, X_3) have the same joint distribution. We first give sharp bounds on p for an important class of random variables.

DEFINITION 2.3 Let $\mathcal{U}$ be the class of symmetric random variables X whose distribution function F is continuous and satisfies

$$F(x) - F(0) \geq F(x + y) - F(y) \qquad \forall x, y > 0.$$

Note that all unimodal symmetric random variables are in $\mathcal{U}$, but so is, for example, the random variable that is uniform on $(-4, -3) \cup (-2, 2) \cup (3, 4)$.

PROPOSITION 2.4 *Let* $X \in \mathcal{U}$. *Then* $0 < p_X \leq 1/96$. *These bounds are sharp, and the upper bound is attained when* X *has the uniform distribution on* $(-1, 1)$ *(or any other interval symmetric about* 0*).*

THEOREM 2.5 *Let* $n = 3$. *In the relative ranks version of the problem, the optimal rule* τ^* *is as follows:*

  a) If $p \leq q$ *(in particular, if* $X \in \mathcal{U}$*), then*

$$\tau^* = \begin{cases} 1 & \text{if } S_1 > 0, \\ 2 & \text{if } S_1 \leq 0 \text{ and } S_2 > S_1, \\ 3 & \text{otherwise.} \end{cases}$$

  b) If instead $p > q$, *then*

$$\tau^* = \begin{cases} 1 & \text{if } S_1 > 0, \\ 2 & \text{if } S_1 \leq 0 < S_2, \\ 3 & \text{otherwise.} \end{cases}$$

The optimal expected rank is

$$\mathbb{E}(R_{\tau^*}) = \min \left\{ \frac{55}{24}, \frac{109}{48} + 2p \right\}.$$

Further,

$$2.2708 \approx \frac{109}{48} < \mathbb{E}(R_{\tau^*}) \leq \frac{55}{24} \approx 2.2917,$$

and these bounds are sharp.

Note that the optimal rule and expected rank only depend on the distribution of X through the parameter p, and remain constant once p exceeds $1/96$.

3. Proofs for the full information case

We begin by making a few simple observations:

1. If we stop at time k, then our expected overall rank in either version of the problem is

$$\mathbb{E}(R_k|\mathscr{F}_k) = \mathbb{E}(R_k|\mathscr{G}_k) = \tilde{R}_k + \frac{n-k}{2},$$

   by the symmetry of the walk.

2. When we are at a running minimum of the random walk, it is always optimal to continue. That is, if at time $0 \le k < n$ we have $\tilde{R}_k = k+1$ (or equivalently, $S_k \le S_i$ for all $i \le k$), then in view of the first observation above,

$$\mathbb{E}(R_k|\mathscr{F}_k) = k + 1 + \frac{n-k}{2}$$
$$= \frac{1}{2}\left(k + 2 + \frac{n-k-1}{2}\right) + \frac{1}{2}\left(k + 1 + \frac{n-k-1}{2}\right)$$
$$\ge \mathbb{E}(R_{k+1}|\mathscr{F}_k).$$

   Thus, continuing one more step and then stopping is at least as good as stopping immediately. This holds also when we replace $\mathscr{F}_k$ by $\mathscr{G}_k$.

3. If we are at time $n-1$ having observed $S_1, \ldots, S_{n-1}$, and if we choose to continue to the nth (and last) step, our expected rank is

$$\mathbb{E}(R_n|S_1, \ldots, S_{n-1}) = \sum_{i=0}^{n} \mathbb{P}(S_n \le S_i|S_1, \ldots, S_{n-1})$$
$$= 1 + \sum_{i=0}^{n-1} \mathbb{P}(X_n \le S_i - S_{n-1}|S_1, \ldots, S_{n-1})$$
$$= 1 + \sum_{i=0}^{n-1} F(S_i - S_{n-1})$$
$$= n + \frac{1}{2} - \sum_{i=0}^{n-2} F(X_{i+1} + \cdots + X_{n-1}),$$

   where the last step uses the symmetry of F. Note that this observation applies only to the full information version.

PROOF (PROOF OF THEOREM 2.1) Let $n = 2$, and consider first the full information case. If we take the first step and $S_1 \leq 0$, we should continue by observation 2 above, and our expected rank is $\mathbb{E}(R_2|S_1) = 2.5 - F(X_1)$ by observation 3 above. Suppose $S_1 > 0$ instead. Then $\mathbb{E}(R_2|S_1) = 2.5 - F(X_1) \geq 1.5 = \mathbb{E}(R_1|S_1)$, so we should stop, with expected rank 1.5. Thus, when we take at least the first step, the optimal expected rank is

$$\frac{1}{2}(1.5) + \int_{-\infty}^{0} \{2.5 - F(x)\}dF(x) = 2 - \int_{0}^{1/2} u \, du = \frac{15}{8}.$$

As this is less than 2 (the expected rank of S_0), we should take the first step and our optimal expected rank is $15/8$. This shows that the optimal rule is as stated in the theorem. Since τ^* uses only the relative rank of S_1 (that is, the comparison of S_1 to $0 = S_0$), it follows that this rule is optimal in the relative ranks version as well.   ■

We next consider the case $n = 3$. Here we use backward induction to determine the optimal rule. First, we define the quantities

$$V_i(X_1, \ldots, X_i) := \inf_{i \leq \tau \leq 3} \mathbb{E}(R_\tau|\mathscr{F}_i), \qquad i = 0, 1, 2, \tag{3}$$

$$W_i(X_1, \ldots, X_i) := \inf_{i < \tau \leq 3} \mathbb{E}(R_\tau|\mathscr{F}_i), \qquad i = 0, 1, 2, \tag{4}$$

where in each case, the infimum is over the set of all stopping times relative to the filtration $\{\mathscr{F}_i\}$ that take values in the specified range. In case $i = 0$, we write simply V_0 and W_0 (without any X_j's) for the quantities on the left. We also denote V_0 by V, and note that $V = \inf_\tau \mathbb{E}(R_\tau)$. Noting also that $W_i(X_1, \ldots, X_i)$ represents our expected rank given that we have decided to continue at time i, we may further observe that

$$V_i(X_1, \ldots, X_i) = \min\{\mathbb{E}(R_i|\mathscr{F}_i), W_i(X_1, \ldots, X_i)\}, \qquad i = 0, 1, 2 \tag{5}$$

by backward induction, and

$$W_i(X_1, \ldots, X_i) = \int_{-\infty}^{\infty} V_{i+1}(X_1, \ldots, X_i, x) \, dF(x), \qquad i = 0, 1 \tag{6}$$

by conditioning on the value of X_{i+1}.

Starting one step before the end of the random walk, suppose X_1 and X_2 (or equivalently, S_1 and S_2) have been observed. If $\tilde{R}_2 = 3$, then the walk is at a running minimum, so it is optimal to continue, and by observation 3,

$$V_2(X_1, X_2) = W_2(X_1, X_2) = 3.5 - F(X_2) - F(X_1 + X_2).$$

If, on the other hand, $\tilde{R}_2 = 1$, then it is optimal to stop, since $\mathbb{E}(R_2|\mathscr{F}_2) = 1.5 \leq 3.5 - F(X_2) - F(X_1 + X_2) = W_2(X_1, X_2)$. So in this case, $V_2(X_1, X_2) = 1.5$.

The interesting case is when $\tilde{R}_2 = 2$. This can happen in two different ways: (i) $X_1 > 0$ and $-X_1 < X_2 < 0$; or (ii) $X_1 < 0$ and $0 < X_2 < -X_1$. Note that in either case, $\mathbb{E}(R_2|\mathscr{F}_2) = 2.5$, which must be compared to $W_2(X_1, X_2) = 3.5 - F(X_2) - F(X_1 + X_2) = 2.5 + F(-X_2) - F(X_1 + X_2)$. Thus it is optimal to stop if $F(-X_2) \geq F(X_1 + X_2)$ and to continue otherwise; in other words, it is optimal to stop if $X_2 \leq -X_1/2$, and to continue otherwise. (Note that there could be a region of points (X_1, X_2) for which we are indifferent between stopping and continuing; this is why we do not say "it is optimal to stop if and only if $X_2 \leq -X_1/2$".)

Putting these observations together, we see that, if we had not yet stopped before, it is optimal to stop at time 2 if one of the following holds:

1. $X_1 > 0$ and $X_2 > 0$; or

2. $X_1 > 0$ and $-X_1 < X_2 \leq -X_1/2$; or

3. $X_1 < 0$ and $X_2 > -X_1$; or

4. $X_1 < 0$ and $0 < X_2 \leq -X_1/2$;

and to continue otherwise. Moreover,

$$V_2(X_1, X_2) = \min\{\tilde{R}_2 + 0.5, 1.5 + F(-X_2) + F(-X_1 - X_2)\}. \tag{7}$$

From the above facts, we can compute the optimal expected rank if we continue after the first step, that is, after observing $X_1 = x$: For $x > 0$, we have from (6) that

$$W_1(x) = 1.5\mathbb{P}(X_2 > 0) + \int_{-x/2}^{0} \{1.5 + F(-x - y) + F(-y)\}dF(y)$$

$$+ 2.5\mathbb{P}\left(-x < X_2 < -\frac{x}{2}\right) + \int_{-\infty}^{-x} \{1.5 + F(-x - y) + F(-y)\}dF(y)$$

$$= 1.5 + \int_{-x/2}^{0} \{F(-x - y) + F(-y)\}dF(y)$$

$$+ \mathbb{P}\left(-x < X_2 < -\frac{x}{2}\right) + \int_{-\infty}^{-x} \{F(-x - y) + F(-y)\}dF(y).$$

Using the symmetry of f, we can simplify this by noting that

$$\int_{-b}^{-a} F(-y)dF(y) = \int_{a}^{b} F(z)dF(z) = \frac{1}{2}[F^2(b) - F^2(a)] \tag{8}$$

and

$$\int_{-b}^{-a} F(-x - y)dF(y) = \int_{a}^{b} F(z - x)dF(z) \tag{9}$$

for $0 \leq a < b \leq \infty$; and by using also that $F(0) = 1/2$. This gives, for $x > 0$,

$$W_1(x) = \frac{15}{8} + \int_0^{x/2} F(y-x)dF(y) + \int_x^\infty F(y-x)dF(y)$$
$$+ F(x) - F\left(\frac{x}{2}\right) - \frac{1}{2}\left[F^2(x) - F^2\left(\frac{x}{2}\right)\right]. \tag{10}$$

Remembering (5), we must compare this with $\mathbb{E}(R_1|\mathscr{F}_1) = 2$. Taking limits under the integral signs in (10) (which is justified by dominated convergence), we obtain

$$\lim_{x \to \infty} W_1(x) = \frac{15}{8}$$

(which, not coincidentally, is equal to the value of the 2-step problem); and

$$\lim_{x \searrow 0} W_1(x) = \frac{15}{8} + \int_0^\infty F(y)dF(y) = \frac{15}{8} + \int_{1/2}^1 u\,du = \frac{9}{4}.$$

Since $W_1(x)$ is clearly continuous, it follows that there is a critical point $x_1^* > 0$ (not necessarily unique) such that $W_1(x_1^*) = 2$. Since $W_1(x)$ is also nonincreasing in x (an immediate consequence of the definition), we conclude that it is optimal to stop at time 1 if $0 < X_1 \leq x_1^*$, and to continue if $X_1 > x_1^*$. (Intuitively, if X_1 is very large, one should continue since the risk of falling back below the starting point of 0 is very small.) Note that x_1^* is a solution of the equation

$$\int_0^{x/2} F(y-x)dF(y) + \int_x^\infty F(y-x)dF(y)$$
$$+ F(x) - F\left(\frac{x}{2}\right) - \frac{1}{2}\left[F^2(x) - F^2\left(\frac{x}{2}\right)\right] = \frac{1}{8}. \tag{11}$$

As shown below, x_1^* is always a high quantile of the distribution of X_1.

LEMMA 3.1 *We have*

$$F(x_1^*) \geq \frac{1}{2} + \frac{\sqrt{2}}{4} \approx .85355.$$

PROOF In (10), we use $F(y-x) \geq F(-x)$ in the first integral and $F(y-x) \geq F(0) = 1/2$ in the second to obtain (using the symmetry of F),

$$W_1(x) \geq \frac{15}{8} + F(x) - \frac{1}{2}F^2(x) - F(x)F\left(\frac{x}{2}\right) + \frac{1}{2}F^2\left(\frac{x}{2}\right)$$
$$= \frac{15}{8} + F(x) - F^2(x) + \frac{1}{2}\left[F(x) - F\left(\frac{x}{2}\right)\right]^2 \tag{12}$$
$$\geq \frac{15}{8} + F(x)\{1 - F(x)\}.$$

Setting $W_1(x_1^*) = 2$ thus yields $F(x_1^*)\{1 - F(x_1^*)\} \leq 1/8$, and since we know $F(x_1^*) \geq F(0) = 1/2$, it follows that $F(x_1^*) \geq \frac{1}{2} + \frac{\sqrt{2}}{4}$. ∎

We now consider the case when $X_1 = x < 0$. Here it is always optimal to continue as we are at a running minimum, and

$$
\begin{aligned}
W_1(x) &= 1.5\mathbb{P}(X_2 > -x) + \int_{-x/2}^{-x} \{1.5 + F(-y) + F(-x-y)\}dF(y) \\
&\quad + 2.5\mathbb{P}(0 < X_2 < -x/2) + \int_{-\infty}^{0} \{1.5 + F(-y) + F(-x-y)\}dF(y) \\
&= 1.5 + \int_{-x/2}^{-x} \{F(-y) + F(-x-y)\}dF(y) \\
&\quad + \mathbb{P}(0 < X_2 < -x/2) + \int_{-\infty}^{0} \{F(-y) + F(-x-y)\}dF(y).
\end{aligned}
$$

Using (8) and (9), this leads to

$$
\begin{aligned}
W_1(x) &= \frac{19}{8} + \int_{x}^{x/2} F(y-x)dF(y) + \int_{0}^{\infty} F(y-x)dF(y) \\
&\quad - F\left(\frac{x}{2}\right) + \frac{1}{2}\left[F^2\left(\frac{x}{2}\right) - F^2(x)\right].
\end{aligned}
\tag{13}
$$

PROOF (PROOF OF THEOREM 2.2) We have already determined the optimal rule and shown, in Lemma 3.1, that $F(x_1^*) \geq \frac{1}{2} + \frac{\sqrt{2}}{4}$. It remains to prove the estimates (2) and to show that the upper bound is attained.

First we estimate $W_1(x)$ by a simpler expression for $x < 0$. In (13), use $F(y-x) \geq F(0) = 1/2$ in the first integral to get

$$
\int_{x}^{x/2} F(y-x)dF(y) \geq \frac{1}{2}\left[F\left(\frac{x}{2}\right) - F(x)\right].
$$

The second integral we estimate as follows:

$$
\begin{aligned}
\int_{0}^{\infty} F(y-x)dF(y) &\geq \int_{0}^{-x} F(-x)dF(y) + \int_{-x}^{\infty} F(y)dF(y) \\
&= F(-x)\left\{F(-x) - \frac{1}{2}\right\} + \frac{1}{2}\left\{1 - F^2(-x)\right\} \\
&= \frac{1}{2}\left(1 - F(-x)\{1 - F(-x)\}\right) \\
&= \frac{1}{2}\left(1 - F(x)\{1 - F(x)\}\right).
\end{aligned}
$$

Putting these estimates back into (13) yields

$$
W_1(x) \geq \frac{23}{8} - F(x) - \frac{1}{2}F\left(\frac{x}{2}\right) + \frac{1}{2}F^2\left(\frac{x}{2}\right).
\tag{14}
$$

Note that $V_1(x) = 2$ if $0 < x \leq x_1^*$, and $V_1(x) = W_1(x)$ otherwise. At time 0 it is optimal to continue since the walk is at a minimum. Hence,

$$
\begin{aligned}
V &= \int_{-\infty}^{\infty} V_1(x) dF(x) \\
&= \int_{-\infty}^{0} W_1(x) dF(x) + 2\mathbb{P}(0 < X_1 \leq x_1^*) + \int_{x_1^*}^{\infty} W_1(x) dF(x). \quad (15)
\end{aligned}
$$

We estimate each integral separately. First, by (14),

$$
\begin{aligned}
\int_{-\infty}^{0} W_1(x) dF(x) &\geq \frac{23}{16} - \int_{-\infty}^{0} F(x) dF(x) - \frac{1}{2} \int_{-\infty}^{0} F\left(\frac{x}{2}\right) \left\{1 - F\left(\frac{x}{2}\right)\right\} dF(x) \\
&\geq \frac{23}{16} - \int_{0}^{1/2} u \, du - \frac{1}{8} \int_{-\infty}^{0} dF(x) = \frac{5}{4}.
\end{aligned}
$$

The other integral can be estimated below using (12), which gives

$$
\begin{aligned}
\int_{x_1^*}^{\infty} W_1(x) dF(x) &\geq \int_{x_1^*}^{\infty} \left[\frac{15}{8} + F(x)\{1 - F(x)\}\right] dF(x) \\
&= \frac{15}{8}\{1 - F(x_1^*)\} + \frac{1}{6} - \frac{1}{2}F^2(x_1^*) + \frac{1}{3}F^3(x_1^*).
\end{aligned}
$$

Combining this with the term $2\mathbb{P}(0 < X_1 \leq x_1^*)$ from (15) yields

$$
\begin{aligned}
\int_{0}^{\infty} V_1(x) dF(x) &= 2\left(F(x_1^*) - \frac{1}{2}\right) + \int_{x_1^*}^{\infty} W_1(x) dF(x) \\
&\geq \frac{25}{24} + \frac{1}{24}\left\{3F(x_1^*) - 12F^2(x_1^*) + 8F^3(x_1^*)\right\} \\
&\geq \frac{25}{24} - \frac{1 + \sqrt{2}}{48}.
\end{aligned}
$$

To see the last inequality, let $g(t) = 3t - 12t^2 + 8t^3$ and note that g has a local minimum value of $-\frac{1+\sqrt{2}}{2}$ at $t = \frac{1}{2} + \frac{\sqrt{2}}{4}$, so the cubic polynomial in $F(x_1^*)$ is minimized exactly when $F(x_1^*) = \frac{1}{2} + \frac{\sqrt{2}}{4}$.

Combining the estimates, we finally arrive at

$$
V \geq \frac{5}{4} + \frac{25}{24} - \frac{1 + \sqrt{2}}{48} = \frac{109 - \sqrt{2}}{48} \approx 2.24137.
$$

In the next section we will show that $55/24$ is a sharp upper bound for $\mathbb{E}(R_{\tau^*})$ in the relative ranks version of the problem. Since in the full information version we can do at least as well, it follows that $V \leq 55/24$. This is attained (in both versions of the problem) when X has the uniform distribution on $(-2, -1) \cup (1, 2)$; we leave the details to the interested reader. Thus, the proof is complete. ∎

EXAMPLES 3.2 *(a)* Let X have the uniform distribution on $(-1, 1)$. Then for $0 < x < 1$ we have

$$W_1(x) = \frac{9}{4} - \frac{x}{4} - \frac{x^2}{16},$$

so that $x_1^* = 2(\sqrt{2} - 1) \approx 0.828$; and for $-1 < x < 0$ we have

$$W_1(x) = \frac{9}{4} - \frac{3x}{4} - \frac{3x^2}{16}.$$

The optimal expected rank is

$$V = \frac{1}{2} \int_{-1}^{0} W_1(x)dx + \frac{1}{2} \cdot 2x_1^* + \frac{1}{2} \int_{x_1^*}^{1} W_1(x)dx = \frac{11}{4} - \frac{\sqrt{2}}{3} \approx 2.279.$$

(b) Let X have the standard two-sided exponential (or Laplace) distribution, with density $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$. Then for $x > 0$ we have

$$W_1(x) = \frac{15}{8} + \frac{1}{8}xe^{-x} + \frac{1}{2}e^{-x} - \frac{1}{8}e^{-2x},$$

and numerically solving $W_1(x) = 2$ gives $x_1^* \approx 1.71$. For $x < 0$ we have

$$W_1(x) = \frac{23}{8} + \frac{1}{8}xe^{x} - \frac{1}{2}e^{x} - \frac{1}{8}e^{2x}.$$

The optimal expected rank is approximately 2.271.

4. Proofs for the relative ranks case

In this section we assume that only the relative ranks $\tilde{R}_i$, $i = 0, 1, \ldots, n$ are observed. We fix $n = 3$. Before deriving the optimal rule, we prove Proposition 2.4. In what follows, $|X|_{(1)}, |X|_{(2)}$ and $|X|_{(3)}$ denote the order statistics of $|X_1|, |X_2|$ and $|X_3|$, so $(|X|_{(1)}, |X|_{(2)}, |X|_{(3)})$ is a permutation of $(|X_1|, |X_2|, |X_3|)$ with $|X|_{(1)} \le |X|_{(2)} \le |X|_{(3)}$. We let G denote the distribution function of $|X_1|$, so, by the symmetry of the X_j's, $G(x) = 2F(x) - 1$ for $x \ge 0$.

PROOF (PROOF OF PROPOSITION 2.4) It is easy to see, for *any* continuous symmetric distribution, that $p > 0$. For instance, choose $x_0 > 0$ so that $F(2x_0) > F(x_0)$; such a point certainly exists. Then, by symmetry (since X_1, X_2, X_3 are i.i.d.),

$$48p = \mathbb{P}\big(|X|_{(3)} < |X|_{(1)} + |X|_{(2)}\big) \ge \mathbb{P}(x_0 < X_i \le 2x_0 \text{ for } i = 1, 2, 3)$$
$$= \big(F(2x_0) - F(x_0)\big)^3 > 0.$$

Now assume $X \in \mathscr{U}$; this implies $G(x+y) \leq G(x) + G(y)$ for all $x, y \geq 0$. We calculate

$$
\begin{aligned}
16q &= \frac{1}{3}\mathbb{P}\big(|X|_{(3)} > |X|_{(1)} + |X|_{(2)}\big) = \mathbb{P}\big(|X_3| > |X_1| + |X_2|\big) \\
&= \int_0^\infty \int_0^\infty \{1 - G(x+y)\} dG(x) dG(y) \\
&\geq \int_0^\infty \int_0^\infty \{1 - G(x) - G(y)\}^+ dG(x) dG(y) \\
&= \int_0^1 \int_0^1 (1 - u - v)^+ du\, dv = \frac{1}{6}.
\end{aligned}
$$

The only inequality in this calculation becomes an equality when X is uniformly distributed on $(-1, 1)$. Thus, $p \leq 1/96$, and this bound is attained for the uniform distribution on $(-1, 1)$.

It remains to show that the lower bound $p > 0$ is sharp. To this end, let $G(x) = x^\delta$ for $0 \leq x \leq 1$ and $\delta > 0$. (This corresponds with a density $f(x) = \frac{\delta}{2}|x|^{\delta-1}$ for $x \in (-1, 1)\backslash\{0\}$.) For $0 \leq u \leq v \leq 1$, we have

$$
G\big(G^{-1}(u) + G^{-1}(v)\big) \leq G\big(2G^{-1}(v)\big) = G\big(2v^{1/\delta}\big) \leq 2^\delta v.
$$

Hence, for this G,

$$
\begin{aligned}
16q &= \int_0^\infty \int_0^\infty \{1 - G(x+y)\} dG(x) dG(y) \\
&= \int_0^1 \int_0^1 \big\{1 - G\big(G^{-1}(u) + G^{-1}(v)\big)\big\}\, du\, dv \\
&\geq \int_0^1 \int_0^1 \Big(1 - 2^\delta \max\{u, v\}\Big) du\, dv \\
&\to \int_0^1 \int_0^1 (1 - \max\{u, v\}) du\, dv = \frac{1}{3} \qquad \text{as } \delta \to 0.
\end{aligned}
$$

Thus q gets arbitrarily close to $1/48$, and $p = (1/48) - q$ gets arbitrarily close to 0, for sufficiently small $\delta > 0$. ∎

We will use the parameters p and q to express the probabilities of all 24 possible rank orderings of S_0, S_1, S_2 and S_3. Let

$$
\Delta := \{|X|_{(3)} < |X|_{(1)} + |X|_{(2)}\}
$$

be the event that the absolute step sizes satisfy the triangle inequality, and note that $\mathbb{P}(\Delta) = 48p$. Define the σ-algebras

$$
\begin{aligned}
\mathscr{D} &:= \{\emptyset, \Delta, \Delta^c, \Omega\}, \qquad \mathscr{S} := \sigma(\{\mathrm{sgn}(X_i) : i = 1, 2, 3\}), \\
\mathscr{C} &:= \sigma(\{\mathrm{sgn}(|X_i| - |X_j|) : i, j = 1, 2, 3, i \neq j\}).
\end{aligned}
$$

Permutation	Reflection	Probability
$0 > S_1 > S_2 > S_3$	$0 < S_1 < S_2 < S_3$	$1/8$
$0 > S_1 > S_3 > S_2$	$0 < S_1 < S_3 < S_2$	$1/16$
$0 > S_2 > S_1 > S_3$	$0 < S_2 < S_1 < S_3$	$1/24$
$0 > S_2 > S_3 > S_1$	$0 < S_2 < S_3 < S_1$	$1/48$
$0 > S_3 > S_1 > S_2$	$0 < S_3 < S_1 < S_2$	$(1/48) + 2p$
$0 > S_3 > S_2 > S_1$	$0 < S_3 < S_2 < S_1$	$2q$
$S_2 > 0 > S_1 > S_3$	$S_2 < 0 < S_1 < S_3$	$1/48$
$S_2 > 0 > S_3 > S_1$	$S_2 < 0 < S_3 < S_1$	$2p$
$S_2 > S_3 > 0 > S_1$	$S_2 < S_3 < 0 < S_1$	$2q$
$S_3 > 0 > S_1 > S_2$	$S_3 < 0 < S_1 < S_2$	$2q$
$S_3 > 0 > S_2 > S_1$	$S_3 < 0 < S_2 < S_1$	$(1/48) + 2p$
$S_3 > S_2 > 0 > S_1$	$S_3 < S_2 < 0 < S_1$	$1/16$

Table 1: The probabilities of the 24 possible rank orderings of $0 = S_0, S_1, S_2$ and S_3. The first column lists permutations with $S_1 < 0$; the second lists the ones with $S_1 > 0$. By symmetry, the two permutations in each row have the same probability.

Here we set $\mathrm{sgn}(x) := 1$ if $x \geq 0$ and $= -1$ if $x < 0$. Thus, $\mathscr{D}$ is the σ-algebra generated by Δ, $\mathscr{S}$ is the σ-algebra generated by the signs of X_1, X_2 and X_3, and $\mathscr{C}$ is the σ-algebra generated by the mutual comparisons of $|X_1|, |X_2|$ and $|X_3|$. Observe that the σ-algebras $\mathscr{C}, \mathscr{D}$ and $\mathscr{S}$ are independent by the symmetry of X. This makes it easy to calculate the probabilities of the 24 permutations of the random walk. For example,

$$\mathbb{P}(0 > S_3 > S_2 > S_1)$$
$$= \mathbb{P}\big(\{X_1 < 0, X_2 > 0, X_3 > 0\} \cap \{|X_1| > |X_2|, |X_1| > |X_3|\} \cap \Delta^c\big)$$
$$= \mathbb{P}(X_1 < 0, X_2 > 0, X_3 > 0)\, \mathbb{P}(|X|_{(3)} = |X_1|)\, \mathbb{P}(\Delta^c)$$
$$= \frac{1}{8} \cdot \frac{1}{3} \cdot 48q = 2q.$$

The other probabilities can be derived similarly; we list them in Table 1.

PROOF (PROOF OF THEOREM 2.5) Recall the filtration $\{\mathscr{G}_i\}$ defined by $\mathscr{G}_i = \sigma(\tilde{R}_1, \ldots, \tilde{R}_i)$, $i = 0, 1, 2, 3$. Analogously to (3) and (4) we define

$$\tilde{V}_i(\tilde{R}_1, \ldots, \tilde{R}_i) := \inf_{i \leq \tau \leq 3} \mathbb{E}(R_\tau | \mathscr{G}_i), \qquad i = 0, 1, 2,$$
$$\tilde{W}_i(\tilde{R}_1, \ldots, \tilde{R}_i) := \inf_{i < \tau \leq 3} \mathbb{E}(R_\tau | \mathscr{G}_i), \qquad i = 0, 1, 2.$$

In case $i = 0$ we write simply $\tilde{V}_0$ and $\tilde{W}_0$ for the left hand sides, and we denote $\tilde{V}_0$ also by $\tilde{V}$. As in the full information case (cf. (5)),

$$\tilde{V}_i(\tilde{R}_1, \ldots, \tilde{R}_i) = \min\{\mathbb{E}(R_i|\mathscr{G}_i), \tilde{W}_i(\tilde{R}_1, \ldots, \tilde{R}_i)\}.$$

a) Assume first that $p \leq q$; recall that this is the case for all unimodal distributions. Suppose $\tilde{R}_1$ and $\tilde{R}_2$ have been observed. Equivalently, the mutual comparisons between $0 = S_0$, S_1 and S_2 are known. We consider the six possible permutations one by one:

Case 1. Suppose $0 < S_1 < S_2$. The probability of this event is $1/4$. Here $\mathbb{E}(R_2|\mathscr{G}_2) = 1.5 \leq \mathbb{E}(R_3|\mathscr{G}_2)$, since R_3 takes the value 1 with (conditional) probability $1/2$, and otherwise takes at least the value 2. Thus, it is optimal to stop, and $\tilde{V}_2(\tilde{R}_1, \tilde{R}_2) = 1.5$.

Case 2. Suppose $0 < S_2 < S_1$. This happens with probability $1/8$. Here $\mathbb{E}(R_2|\mathscr{G}_2) = 2.5$, whereas

$$\mathbb{E}(R_3|\mathscr{G}_2) = 8\left[1 \cdot \frac{1}{24} + 2 \cdot \frac{1}{48} + 3 \cdot 2q + 4 \cdot \left(2p + \frac{1}{48}\right)\right] = \frac{7}{3} + 16p,$$

where we used the third, fourth, sixth and eleventh rows of Table 1. Since $p \leq 1/96$, it follows that $\mathbb{E}(R_3|\mathscr{G}_2) \leq 2.5$. Hence, it is optimal to continue, and $\tilde{V}_2(\tilde{R}_1, \tilde{R}_2) = \frac{7}{3} + 16p$.

Case 3. Suppose $S_2 < 0 < S_1$. This happens with probability $1/8$. Here the walk is at a minimum, so it is optimal to continue, and

$$\tilde{V}_2(\tilde{R}_1, \tilde{R}_2) = \mathbb{E}(R_3|\mathscr{G}_2) = 8\left[1 \cdot \frac{1}{48} + 2 \cdot 2p + 3 \cdot 2q + 4 \cdot \frac{1}{16}\right] = \frac{34}{12} + 16q.$$

Case 4. Suppose $S_1 < 0 < S_2$. This happens with probability $1/8$. Here the walk is at a maximum, so as in Case 1 it is optimal to stop, and $\tilde{V}_2(\tilde{R}_1, \tilde{R}_2) = 1.5$.

Case 5. Suppose $S_1 < S_2 < 0$. This happens with probability $1/8$. Here $\mathbb{E}(R_2|\mathscr{G}_2) = 2.5$, whereas

$$\mathbb{E}(R_3|\mathscr{G}_2) = 8\left[1 \cdot \left(2p + \frac{1}{48}\right) + 2 \cdot 2q + 3 \cdot \frac{1}{48} + 4 \cdot \frac{1}{24}\right] = \frac{7}{3} + 16q \geq 2.5,$$

using that $q \geq 1/96$. Thus, it is optimal to stop, and $\tilde{V}_2(\tilde{R}_1, \tilde{R}_2) = 2.5$.

Case 6. Suppose $S_2 < S_1 < 0$. This occurs with probability $1/4$. Since we are at a minimum, it is optimal to continue and

$$\tilde{V}_2(\tilde{R}_1, \tilde{R}_2) = \mathbb{E}(R_3|\mathscr{G}_2) = 4\left[1 \cdot 2q + 2 \cdot \left(2p + \frac{1}{48}\right) + 3 \cdot \frac{1}{16} + 4 \cdot \frac{1}{8}\right] = \frac{37}{12} + 8p.$$

This completes the analysis of the situation after two steps.

We assume next that $\tilde{R}_1$ has been observed; that is, we know whether $S_1 > 0$ or $S_1 < 0$. If $S_1 > 0$, then $\mathbb{E}(R_1|\mathcal{G}_1) = 2$, whereas

$$
\begin{aligned}
\tilde{W}_1(\tilde{R}_1) &= \sum_{k=1}^{3} \mathbb{P}(\tilde{R}_2 = k|\mathcal{G}_1)\tilde{V}_2(\tilde{R}_1, k) \\
&= \mathbb{P}(S_2 > S_1 > 0|S_1 > 0) \cdot (1.5) + \mathbb{P}(S_1 > S_2 > 0|S_1 > 0) \cdot \left(\frac{7}{3} + 16p\right) \\
&\qquad + \mathbb{P}(S_1 > 0 > S_2|S_1 > 0) \cdot \left(\frac{34}{12} + 16q\right) \\
&= \frac{1}{2} \cdot (1.5) + \frac{1}{4}\left(\frac{7}{3} + 16p\right) + \frac{1}{4}\left(\frac{34}{12} + 16q\right) = \frac{102}{48} > 2,
\end{aligned}
$$

where we used the results from Cases 1-3 above. Thus, it is optimal to stop, and $\tilde{V}_1(\tilde{R}_1) = 2$.

On the other hand, if $S_1 < 0$, then the walk is at a minimum, and so

$$
\begin{aligned}
\tilde{V}_1(\tilde{R}_1) = \tilde{W}_1(\tilde{R}_1) &= \sum_{k=1}^{3} \mathbb{P}(\tilde{R}_2 = k|\mathcal{G}_1)\tilde{V}_2(\tilde{R}_1, k) \\
&= \mathbb{P}(S_2 > 0 > S_1|S_1 < 0) \cdot (1.5) + \mathbb{P}(0 > S_2 > S_1|S_1 < 0) \cdot (2.5) \\
&\qquad + \mathbb{P}(0 > S_1 > S_2|S_1 < 0) \cdot \left(\frac{37}{12} + 8p\right) \\
&= \frac{1}{4} \cdot (1.5) + \frac{1}{4} \cdot (2.5) + \frac{1}{2} \cdot \left(\frac{37}{12} + 8p\right) = \frac{61}{24} + 4p,
\end{aligned}
$$

using the results from Cases 4-6 above. We thus obtain

$$
\begin{aligned}
\tilde{V} = \tilde{W}_0 &= \mathbb{P}(S_1 > 0)\tilde{V}_1(1) + \mathbb{P}(S_1 < 0)\tilde{V}_1(2) \\
&= \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \left(\frac{61}{24} + 4p\right) = \frac{109}{48} + 2p.
\end{aligned}
$$

Since $0 < p \leq 1/96$ we have $109/48 < \tilde{V} \leq 55/24$, and both bounds are sharp.

b) Now suppose $p \geq q$. The analysis being very similar, we only indicate at which points it differs from the preceding case. We focus first on the situation after two steps. Note that in Case 2 it is now optimal to stop, with expected rank 2.5. On the other hand, in Case 5 it now becomes optimal to continue, with expected rank $\frac{7}{3} + 16q$. The calculation of $\tilde{V}_1(\tilde{R}_1)$ changes as follows: If $S_1 > 0$, we now obtain

$$
\tilde{W}_1(\tilde{R}_1) = \frac{101}{48} + 4q.
$$

Version	Description	Expected Rank
Full Information	Lower bound	2.2413
Full Information	Laplace Distribution	≈ 2.271
Full Information	Uniform Distribution	≈ 2.279
Full Information	Maximum	$\frac{55}{24} = 2.291\overline{6}$
Relative Ranks	Greatest Lower Bound	$\frac{109}{48} \approx 2.2708$
Relative Ranks	Laplace Distribution	$\frac{73}{32} = 2.28125$
Relative Ranks	Uniform Distribution	$\frac{55}{24} = 2.291\overline{6}$
Relative Ranks	Maximum	$\frac{55}{24} = 2.291\overline{6}$
Both Versions	Stopping Immediately	2.5

Table 2: Summary of results for $n = 3$

This is still greater than $2 = \mathbb{E}(R_1|\mathscr{G}_1)$, so it remains optimal to stop, and $\tilde{V}_1(\tilde{R}_1) = 2$. On the other hand, if $S_1 < 0$ then we get $\tilde{V}_1(\tilde{R}_1) = \tilde{W}_1(\tilde{R}_1) = 31/12$. The optimal expected rank is thus $\tilde{V} = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{31}{12} = \frac{55}{24}$.

Finally, a close examination of the above analysis reveals that the optimal rule is as stated in the theorem. ∎

EXAMPLE 4.1 For the two-sided exponential (or Laplace) distribution, we have $p = 1/192$ and $\tilde{V} = 73/32 = 2.28125$. For the Uniform$(-1, 1)$ distribution, $p = 1/96$ and $\tilde{V} = 55/24 \approx 2.2917$. We summarize the numerical results of this paper in Table 2.

5. Concluding remarks

We have derived the optimal stopping rules for $n = 2$ and $n = 3$, in both the full information version and the relative rank version of the problem. When $n = 3$, both the optimal rule and the optimal expected rank depend on the distribution of the step sizes, though less so in the relative rank version.

For the full information version of the problem it seems unlikely, in light of the complexity of the optimal rule already for $n = 3$, that the problem can be solved exactly for even moderately large values of n. In the relative ranks version, the exact solution can probably be found for a few larger values of n, though we have not attempted to do so. One difficulty is that, while for $n = 3$ precisely half of the probabilities of the 24 permutations did not depend on the distribution of X, this proportion seems to decrease rapidly as n grows larger. Moreover, the probabilities that depend on the distribution can do so in more complicated ways. Even when $n = 4$, for instance, we might

need to consider probabilities such as $\mathbb{P}(|X|_{(4)} < |X|_{(1)} + |X|_{(2)} + |X|_{(3)})$, $\mathbb{P}(|X|_{(4)} < |X|_{(1)} + |X|_{(2)})$, $\mathbb{P}(|X|_{(4)} + |X|_{(1)} < |X|_{(3)} + |X|_{(2)})$, etc. That said, since there are essentially only finitely many different stopping rules to consider, a "brute force" computer algorithm could in principle come up with the optimal rule as long as n is not too large.

A more interesting approach, however, would be to develop relatively simple stopping rules which perform well asymptotically for large n, and to aim for reasonably sharp upper and lower bounds on the ratio $V^{(n)}/n$, where $V^{(n)}$ denotes the optimal expected rank for an n-step problem.

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Problem Robbinsa dla błądzenia losowego: mały horyzont
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Streszczenie W problemie Robbinsa celem jest zatrzymanie sekwencyjnych obserwacji skończonego ciągu niezależnych zmiennych losowych o tym samym rozkładzie tak, aby zminimalizować oczekiwaną rangę zatrzymanej zmiennej. Niniejsza praca poświęcona jest analogonowi problemu Robbinsa, w którym obserwowane zmienne losowe są wartościami symetrycznego błądzenia losowego. Zakładamy, że długości kroków są symetrycznymi zmiennymi losowymi o rozkładzie typu ciągłego. Opisujemy optymalne reguły zatrzymania dla przypadków $n = 2$ i $n = 3$ w dwóch wersjach problemu: wersja z pełną informacją, w której rzeczywiste długości kroków losowych są jawne i znane podejmującemu decyzje statystykowi, oraz wersja z częściową informacją, w której obserwowane są tylko względne ciągi pozycji zajmowanych przez ciągły, symetryczny, spacer losowy. Dla $n = 3$ optymalna strategia i oczekiwana ranga zależą od rozkładu długości kroków. Otrzymano ostre oszacowania dla wartości oczekiwanej otrzymanej rangi dla wersji problemu z częściowa informacją oraz lepsze oszacowania dla problemu z pełną informacja.

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Słowa kluczowe: proces Ornsteina-Uhlenbecka, FLOC, estymacja, rozkład stabilny.

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