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Handbook of Mathematical Fuzzy Logic

Volume 1

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Preface

Mathematical Fuzzy Logic (MFL) is a subdiscipline of Mathematical Logic. It is a mathematical study of a certain family of formal logical systems whose algebraic semantics involve some notion of truth degree. The central rôle of truth degrees in MFL stems from three distinct historical origins of the discipline:

(1) *Philosophical motivations:* Any scientific theory is, at least initially, driven by some kind of external motivation, i.e. some independent reality one would like to understand and model by means of the theory. MFL is motivated by the need to model correct reasoning in some particular contexts where more standard systems, such as classical logic, might be considered inappropriate. Namely, these motivating contexts are those where the involved propositions suffer from a lack of precision, typically because they contain some vague predicate, i.e. a property lacking clear boundaries. Vague predicates (such as 'tall', 'intelligent', 'poor', 'young', 'beautiful', or 'simple') are omnipresent in natural language and reasoning and, thus, dealing with them is also unavoidable in linguistics. They constitute an important logical problem as clearly seen when confronting *sorites* paradoxes, where a sufficient number of applications of a legitimate deduction rule (*modus ponens*) leads from (apparently?) true premises, to a clearly false conclusion: (1) one grain of wheat does not make a heap, (2) a group of grains of wheat does not become a heap just by adding one more grain, therefore: (3) one million grains of wheat does not make a heap. One possible way to tackle this problem is the degree-based approach related to logical systems studied by MFL (for other logical approaches see e.g. [8, 13, 15]). In this proposal one assumes that truth comes in degrees which, in the case of the *sorites* series, vary from the absolute truth of 'one grain of wheat does not make a heap' to the absolute falsity of 'one million grains of wheat does not make a heap', through the intermediate decreasing truth degrees of 'n grains of wheat do not make a heap'.

(2) *Fuzzy Set Theory:* In 1965 Lotfi Zadeh proposed fuzzy sets as a new mathematical paradigm for dealing with imprecision and gradual change in engineering applications [16]. Their conceptual simplicity (a fuzzy set is nothing more than a classical set endowed with a $[0, 1]$-valued function which represents the degree to which an element belongs to the fuzzy set) provided the basis for a substantial new research area and applications such as a very popular engineering toolbox used successfully in many technological applications, in particular, in so-called *fuzzy control*. This field is referred to as *fuzzy logic*, although its mathematical machinery and the concepts investigated are largely unrelated to those typically used and studied in (Mathematical) Logic. Nevertheless, there have been some attempts to present fuzzy logic in the sense of Zadeh as a useful tool for dealing with vagueness paradoxes (see e.g. [5]). These attempts have encountered strong opposition among proponents of other theories of vagueness (see e.g. [8]).

(3) *Many-valued logics:* The 20th century witnessed a proliferation of logical systems whose intended algebraic semantics, in contrast to classical logic, have more than two truth values (for a historical account see e.g. [3]). Prominent examples are 3-valued systems like Kleene's logic of indeterminacy and Priest's logic of paradox, 4-valued systems like Dunn–Belnap's logic, n-valued systems of Łukasiewicz and Post, and even infinitely valued logics of Łukasiewicz logic [9] or Gödel–Dummett logic [2]. These systems were inspired by a variety of motivations, only occasionally related to the aforementioned vagueness problems. More recently, Algebraic Logic has developed a paradigm in which most systems of non-classical logics can be seen as many-valued logics, because they are given a semantics in terms of algebras with more than two truth values. From this point of view, many-valued logics encompass wide well-studied families of logical systems such as relevance logics, intuitionistic and superintuitionistic logics and substructural logics in general (see e.g. [4]).

Mathematical Fuzzy Logic was born at the crossroads of these three areas. At the beginning of the nineties of last century, a small group of researchers (including among others Esteva, Godo, Gottwald, Hájek, Höhle, and Novák), persuaded that fuzzy set theory could be a useful paradigm for dealing with logical problems related to vagueness, began investigations dedicated to providing solid logical foundations for such a discipline. In other words, they started developing logical systems in the tradition of Mathematical Logic that would have the $[0,1]$-valued operations used in fuzzy set theory as their intended semantics. In the course of this development, they realised that some of these logical systems were already known such as Łukasiewicz and Gödel–Dummett infinitely valued logics. Both systems turned out to be strongly related to fuzzy sets because they are $[0,1]$-valued and the truth functions interpreting their logical connectives are, in fact, of the same kind (t-norms, t-conorms, negations) as those used to compute the combination (resp. intersection, union, complement) of fuzzy sets. Several conferences and a huge funded research project (COST action 15) brought together the aforementioned scholars with researchers working on many-valued systems and fuzzy sets yielding a fertile collaborative environment. These pioneering efforts produced a number of important papers and even some monographs (especially [7], but also [6, 12]).

As a result of this work, fuzzy logics have become a respectable family in the broad landscape of non-classical logics studied by Mathematical Logic. It has been clearly shown that fuzzy logics can be seen as a particular kind of many-valued system (or substructural logic) whose intended semantics is typically based on algebras of linearly ordered truth values. In order to distinguish it from the works on fuzzy set theory misleadingly labeled as *fuzzy logic*, the study of these systems has been called *Mathematical Fuzzy Logic*. Moreover, being a subdiscipline of Mathematical Logic it has acquired the typical core agenda of this field and is studied by many mathematically-minded researchers regardless of its original motivations. Therefore, in the last years we have seen the blossoming of MFL with a plethora of works on propositional, modal, predicate (first and higher order) logics, their semantics (algebraic, relational, game-theoretic), proof theory, model theory, complexity and (un)decidability issues, etc. There has also been an intense discussion regarding the rôle of MFL in the study of vagueness and, in general, in the study of reasoning with imprecise information. It is now clear that MFL cannot be *the theory* or even *the logic* of vagueness. However, many of the philosophical arguments against fuzzy logic as a logic of vagueness in fact do not apply to MFL but to fuzzy logic in the sense of Zadeh. In fact, there has been significant philosophical work on vagueness using elements of MFL, e.g. [14].

The handbook More than one decade after the monographs [6, 7, 12] were published, this handbook aims to be a new up-to-date systematic presentation of the best developed areas of MFL. Since they already constitute a very thick mathematical corpus, we have purposefully decided to leave all motivations and applications out of the book and concentrate only on the presentation of the theory. Therefore, this is a book on pure Mathematical Logic, focusing on the study of a particular family of many-valued non-classical logics. One will find here neither a presentation of fuzzy set theory and its applications nor any discussion of vagueness or any other philosophical or linguistic issue whatsoever. Nevertheless, the intended audience of the book can still be reasonably wide, comprising at least the following groups of readers: (1) students of Logic that should find here a systematic presentation of MFL where they can study the discipline from scratch, (2) experts on MFL that may use it as a reference book for consultation, (3) readers interested in fuzzy set theory and its applications looking for the logical foundations of (some parts) of the area, and (4) readers interested in philosophical and linguistical issues related to the vagueness phenomenon looking for a mathematical apparatus amenable for dealing with some aspects of those issues.

It must be also emphasized that this is not a book written by a single team of authors, but a collection of chapters prepared by distinguished experts on each area. However, the editors have encouraged a reasonable level of homogeneity between the chapters, as regards their structure and notation. It has been conceived as a two volume set with consecutive page enumeration and a global index at the end of each volume. The first volume starts with a gentle introduction to MFL assuming only some basic knowledge on classical logic. The second chapter presents and develops a general and uniform framework for MFL based on the notions of methods of Abstract Algebraic Logic. The third chapter is a presentation of the deeply developed proof theory of fuzzy logics (an extensive treatment can also be found in the monograph [10]). The fourth chapter presents the standard algebraic framework for fuzzy logics based on classes of semilinear residuated lattices. The fifth chapter closes the first volume with the study of Hájek's BL logic and its algebraic counterpart. The second volume of this handbook starts with the sixth chapter devoted to another widely studied fuzzy logic, Łukasiewicz logic Ł, and MV-algebras as its algebraic semantics (one may also consult the monographs [1, 11]). The seventh chapter deals with a third distinguished fuzzy logic, Gödel–Dummett logic, and its variants. The eighth chapter studies fuzzy logics in expanded languages providing greater expressive power. The ninth chapter collects the known results on functional representation of fuzzy logics and their free algebras. The last chapters are dedicated to complexity issues: Chapter X studies the computational complexity of propositional fuzzy logics, while Chapter XI is devoted to the arithmetical hierarchy of first-order predicate fuzzy logics.

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Editors

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