Inverse Power-Law Escape Statistics of Open Billiards

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In a previous study of Dettmann and Georgiou, it was considered such a situation that point-wise particles repeatedly and elastically collided with the wall of a container without any particle-particle collision and eventually escaped through a small window from the container, which can be modeled by an open billiard. In the present study, point-wise particles were replaced with finite-size disks and disk-disk collisions were also considered, and inverse power laws of distributions of dwell time in the container were obtained from numerical simulations.
Key words: Open billiard, Power Law

1. Introduction

An open billiard was rather theoretically interesting but not realistic, where point-wise particles repeatedly and elastically collide with the wall of a container without particle-particle collisions, and eventually escape through a small window from the container. In this study, point-wise particles were replaced with finite-size disks and disk-disk collisions were also considered. Inverse power laws of the distribution of dwell time in the container were obtained from numerical simulations (Nakane and Miyazaki, 2020)

Whether chaotic or non-chaotic motion appears depends on the shape of the container in the closed and open billiard problem, which is an important problem in the research field of *Science on Form*. It is recommended for lay experts that they perform numerical analyses of the above-mentioned billiard problem, which is relatively easy to implement numerical procedures and confirm various dynamics including regular and chaotic temporal evolutions.

We showed that a simple escape mechanism yields not anomalous (algebraic) but normal (exponential) transport in §2. In §3, we discussed a conservative open-billiard system including disk-disk collisions and we showed numerical observations of inverse power-law escape statistics. The final section is devoted to concluding remarks.

2. Exponential Escapement Based on a Simple Escape Mechanism

We considered a semicircular container. It has a hole, which enables small particles to escape from the container as shown in Fig. 1. For the sake of simplicity, we ignored fluctuations of velocity, free path and free time of the particle, and replace instantaneous values with the average values.

N is the number of particles in the container, τ is the mean free time and $\overline{v}$ is the mean velocity of the particle. Between the time range $[\tau, \tau + \Delta t]$, the number of particles

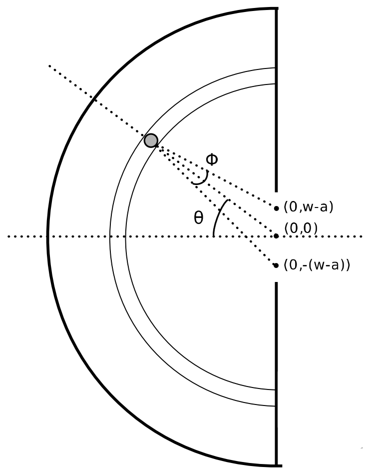

Fig. 1. Billiard model with a semicircular container.

ΔN escaping from the container is proportional to N. Its fraction was assumed to be equal to the volume ratio in the phase space satisfying conditions escaping from the container to the entire phase space.

$2w$ and a are the width of the exit and the radius of the particle, respectively. The particle can escape if the center of the particle passes the segment of $2(w - a)$ wide within the exit. We represented the position of the center of the particle using cylindrical coordinates (r, θ), whose origin is located at the center of the exit.

The center of the particle must be located within the regions bounded by two semi-circles with radii $r = \overline{v}\tau$ and $r = \overline{v}(\tau + \Delta t)$. The position of the center of the particle $(-\overline{v}\tau \cos\theta, -\overline{v}\tau \sin\theta)$ and the edges of the exit $(0, \pm(w - a))$ construct a triangle. $\Phi(\theta)$ is the opposite angle of the segment of the exit. The direction angle ϕ of the velocity vector of the particle must be located within the range of the opposite angle of the exit. The cosine theorem

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yields

$$\Phi(\theta) = \frac{(\overline{v}\tau)^2 - (w-a)^2}{(\overline{v}\tau)^2 + (w-a)^2} \frac{1}{\sqrt{1-\alpha^2 \sin^2\theta}}, \quad (1)$$

where $\alpha \equiv \dfrac{2\overline{v}\tau(w-a)}{(\overline{v}\tau)^2 + (w-a)^2}$ satisfies $0 < \alpha < 1$ due to the relationship between the arithmetic and geometric averages. Although Φ depends on θ, we replaced $\sin^2\theta$ by the average $\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin^2\theta d\theta = 1/2$, giving

$$\cos\Phi = \frac{(\overline{v}\tau)^2 - (w-a)^2}{(\overline{v}\tau)^2 + (w-a)^2} \frac{1}{\sqrt{1-\alpha^2/2}}. \quad (2)$$

In the case of $\overline{v}\tau > w - a$,

$$\cos\Phi = \frac{1-\rho^2}{1+\rho^2} \frac{1}{\sqrt{1-\frac{2\rho^2}{(1+\rho^2)^2}}} \quad (3)$$

with $0 < \rho \equiv \frac{w-a}{\overline{v}\tau} < 1$ is expanded as

$$\cos\Phi \sim 1 - \rho^2 - \frac{1}{2}\rho^4, \quad (4)$$

giving

$$\Phi \sim \sqrt{2}\rho + \frac{\sqrt{2}}{3}\rho^3. \quad (5)$$

In the case of $\overline{v}\tau < w - a$,

$$\cos\Phi = \frac{\rho^{-2}-1}{\rho^{-2}+1} \frac{1}{\sqrt{1-\frac{2\rho^{-2}}{(1+\rho^{-2})^2}}} \quad (6)$$

is expanded as

$$\cos\Phi \sim -1 + \rho^{-2} + \frac{1}{2}\rho^{-4}, \quad (7)$$

giving

$$\Phi \sim \pi - \sqrt{2}\rho^{-1} - \frac{\sqrt{2}}{3}\rho^{-3}. \quad (8)$$

The number of particles ΔN moving from the left room to the left was assumed to be equal to the number of particles staying in the left room N multiplied by the ratio of the volume of the phase space $\int_{\overline{v}\tau}^{\overline{v}(\tau+\Delta t)} dr \int_{-\pi/2}^{\pi/2} d\theta\, \Phi$ to the whole volume $\int_0^R dr \int_{-\pi/2}^{\pi/2} d\theta \int_{-\pi}^{\pi} d\varphi$, yielding $\Delta N = -\dfrac{\Phi\overline{v}\Delta t}{2\pi R}N$. Replacing $\dfrac{\Delta N}{\Delta t}$ by $\dot{N}$ gives

$$\dot{N} = \begin{cases} -\dfrac{\overline{v}}{2\pi R}(\sqrt{2}\rho + \dfrac{\sqrt{2}}{3}\rho^3) & (\overline{v}\tau > w-a), \\ -\dfrac{\overline{v}}{2\pi R}(\pi - \sqrt{2}\rho^{-1} + \dfrac{\sqrt{2}}{3}\rho^{-3}) & (\overline{v}\tau < w-a). \end{cases} \quad (9)$$

We considered the case $\overline{v}\tau > w - a$ only. The mean free path $\overline{v}\tau$ has the following relationship with the surface density of the particles in the left room $d = \dfrac{N}{\pi R^2/2}$, and with the two-dimensional scattering cross-section of the particle $2a$ as $\overline{v}\tau = \dfrac{1}{d\cdot 2a}$, giving

$$\frac{w-a}{\overline{v}\tau} = \frac{4a(w-a)}{\pi R^2}N, \quad (10)$$

where the entire container is a circle with radius R, and a separator with length $2R$ divides the container evenly into the right and the left rooms. Thus, the decay rate of the number of the particles in the container is

$$\dot{n} = -\frac{\overline{v}}{2\pi R}(\sqrt{2}\beta n + \frac{\sqrt{2}}{3}\beta^3 n^3)n, \quad (11)$$

where the number density of particles in the container is n $(0 \le n \le 1)$ and $\dfrac{4a(w-a)N}{\tau R^2}$ is β. The number density n exponentially approaches the empty state $n = 0$ for small n.

3. A Conservative Open-Billiard System with Disk-Disk Collisions

A situation in which point-wise particles repeatedly and elastically collide with the wall of a container without particle-particle collisions, and eventually escape through a small window from the container can be modeled by an open billiard (Dettman and Georgiou, 2009). In a previous study, point-wise particles were replaced with finite-size disks and disk-disk collisions were also considered. Inverse power laws of the distribution of dwell time in the container were obtained from numerical simulations. The relationship between the power laws and anomalous transport was discussed previously (Nakane and Miyazaki, 2020).

We considered here the stadium billiard, whose boundary consists of two parallel line segments, $2a$ in length, and two semicircles of radius r, as shown in Fig. 2. Positions of the left and the right edges of the opening section on one of the parallel segments was determined by h_1 and h_2, respectively (Dettman and Georgiou, 2009). The survival density of the point particles in the billiard at time t was given by

$$P(t) = \frac{(3\ln 3 + 4)((a+h_1)^2 + (a-h_2)^2)}{4(4a+2\pi r)t} + \frac{D}{t^2} + o(1/t^2), \quad (12)$$

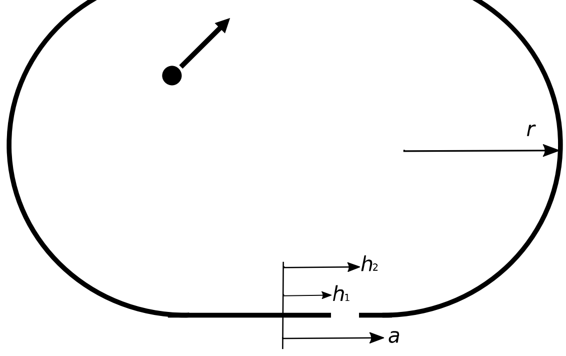

Fig. 2. Open stadium billiard.

where D was a numerical constant (Dettman and Georgiou, 2009). In the case of point particle, the survival density $P(t)$ for $a = 0.05$, $r = 0.5$, $h_1 = -0.01$, and $h_2 = 0.01$ was shown in Fig. 3, where the case of the square billiard was also drawn for comparison. In the former, an exponential decay followed by an inverse-power-law decay was observed. In the latter, an overall inverse-power-law decay was observed. The exponential decay corresponded to the exponential escapement to the empty state $n = 0$ in the second section.

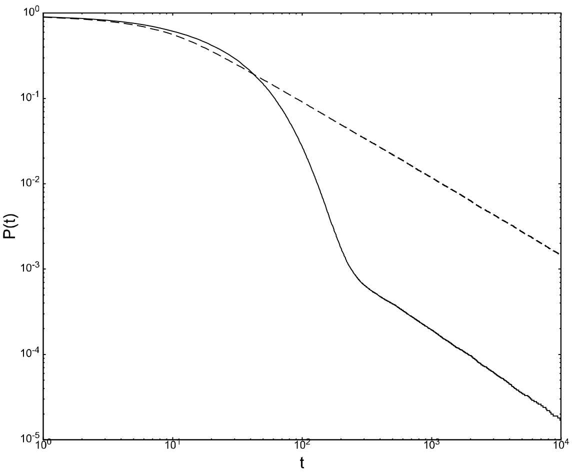

Fig. 3. Double logarithmic plot of the survival densities $P(t)$ against time t for the point particles in the square (dashed line) and the stadium (solid line) billiards.

Next, using an event-driven numerical method (Isobe, 1999), we plotted the scaled survival density $TP(t)$ for $a = 0.05$, $r = 0.5$, $h_1 = -0.01$, and $h_2 = 0.01$ against scaled time t/T for stadium and square billiards, as shown in Figs. 4 and 5, respectively, where the four cases of radii a of the disks 1.0×10^{-7}, 1.0×10^{-6}, 1.0×10^{-5}, 1.0×10^{-4} were chosen and the corresponding mean free times (mean disk-disk collision interval) T were shown in the captions. Since disk-disk interactions were considered, the results were different from the case of point particles.

For the stadium billiard, it was implied that the survival densities obeyed a scaling form $P(t) = T^{-1}\phi(T^{-1}t)$, where ϕ was a scaling function. This was not the case with the square billiard.

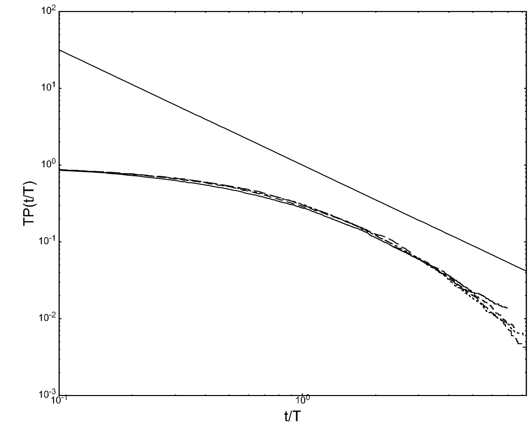

Fig. 4. Double logarithmic plot of the scaled survival densities $TP(t)$ against scaled time t/T for the stadium billiard. The radius $a = 1.0 \times 10^{-7}$ with the mean free time $T = 995.6$ (solid line), $a = 1.0 \times 10^{-6}$ with $T = 138.4$ (dashed line), $a = 1.0 \times 10^{-5}$ with $T = 13.92$ (dotted line), and $a = 1.0 \times 10^{-4}$ with $T = 1.420$ (dash-dot-dash line). The power law $P(t) \propto t^{-3/2}$ are also shown (upper solid line).

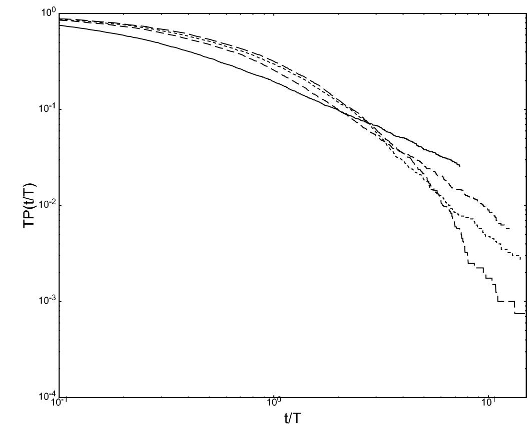

Fig. 5. Double logarithmic plot of the scaled survival densities $TP(t)$ against scaled time t/T for the square billiard. The radius $a = 1.0 \times 10^{-7}$ with the mean free time $T = 1018$ (solid line), $a = 1.0 \times 10^{-6}$ with $T = 126.1$ (dashed line), $a = 1.0 \times 10^{-5}$ with $T = 12.51$ (dotted line), and $a = 1.0 \times 10^{-4}$ with $T = 1.246$ (dash-dot-dash line).

4. Concluding Remarks

We observed the algebraic escapement of the open billiard following inverse power laws of the probability density function of the dwell time in the container by including disk-disk interactions.

The scaling form $P(t) = T^{-1}\phi(T^{-1}t)$ held not for the square billiard but for the stadium one. Since both the disk-disk and the disk-boundary collisions have orbital instability, we conjecture that the volume ratio of the chaotic or irregular region of the high-dimensional phase space for the stadium billiard is much larger than that for the square billiard, where the disk-boundary collisions do not have any orbital instability. We leave this conjecture to future studies.

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