Infinitesimal Knowledges

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The notion of indivisibles and atoms arose in ancient Greece. The continuum—that is, the collection of points in a straight line segment, appeared to have paradoxical properties, arising from the 'indivisibles' that remain after a process of division has been carried out throughout the continuum. In the seventeenth century, Italian mathematicians were using new methods involving the notion of indivisibles, and the paradoxes of the continuum appeared in a new context. This cast doubt on the validity of the methods and the reliability of mathematical knowledge which had been regarded as established by the axiomatic method in geometry expounded by Aristotle's younger contemporary Euclid. The teaching of indivisibles was banned within the Society of Jesus, the Jesuits. In England, indivisibles were used by the mathematician John Wallis, and there was an acrimonious and extended feud between Wallis and the philosopher Thomas Hobbes over legitimate methods of argument in mathematics. Notions of the infinitesimal were used by Isaac Newton and Gottfried Leibniz, and were attacked by Bishop Berkeley for the vagueness of the concept and the illegitimate reasoning applied to it. This article discusses aspects of these events with reference to the book *Infinitesimal* by Amir Alexander and to other sources. Also discussed are wider issues arising from Alexander's book including: the changes in cultural sensibility associated with the growth of new mathematical and scientific knowledge in the seventeenth century, the changes in language concomitant with these changes, what constitutes valid methods of enquiry in various contexts, and the question of authoritarianism in knowledge. More general aims of this article are to widen the immediate mathematical and historical contexts in Alexander's book, to bridge a gap in conversations between mathematics and the humanities, and to relate mathematical ideas to wider human and contemporary issues.

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Key words: infinitesimals; indivisibles; paradox; axiomatic method; Euclid; Aristotle; history of mathematics; Jesuits; historical method; uncertainty in knowledge; language and knowledge; authoritarianism and knowledge

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1 Introduction

In an interview in 1972, Michel Foucault described the motivation for his investigations of the intertwining of power and knowledge: '..... if concerning a science like theoretical physics or organic chemistry, one poses the problem of its relations with the political and economic structures of society, isn't one posing an excessively complicated question? Doesn't this set the threshold of possible explanations excessively high? But ..... if one takes a form of knowledge like psychiatry, won't the question be much easier to resolve, since the epistemological profile of psychiatry is a low oneÉ..?'[1] Apart from an indication of his intellectual program and the recognition that different disciplines have differing epistemological profiles, behind Foucault's comment lies the general question of the differing epistemologies and methods for gaining knowledge in individual disciplines, and even within them, for within intellectual disciplines there need be neither a common method nor consistent criteria of validity. There is also the related comment of Aristotle: '..... it is the mark of a trained mind never to expect more precision in the treatment of a subject than the nature of that subject permits; for demanding logical demonstrations from a teacher of rhetoric is about as reasonable as accepting mere plausibility from a mathematician.'[2] Foucault's statement above seems to be an admission that because his program is too difficult to carry out in the context of physics and chemistry he will select an easier target, namely psychiatry. The question of the applicability of any conclusions he might draw in the case of psychiatry to physics and chemistry, say, is not considered. Aristotle's comment, in particular, is a caution against simply extrapolating conclusions about any claimed relationships between power and knowledge in the one context to other contexts, since the epistemological constraints in a given context are dependent upon the circumstances. This raises the further question of what constitutes reasonable grounds, in diverse contexts, for accepting the validity of statements and for the affirmation of belief.

Galileo's clash with the Roman Catholic Church has of course received very extensive scrutiny. Over the same period, mathematics was also a subject of controversy and one of its new methods, that of indivisibles, was regarded with suspicion by the Roman church, and particularly by the Jesuit order. It is a great merit of Amir Alexander's book *Infinitesimal*[3] that he has made more widely known the lesser-known controversy surrounding the use of indivisibles and infinitesimals in the new mathematical techniques of the seventeenth century. The notion and teaching of indivisibles was formally condemned by the Jesuits on numerous occasions in the first half of the seventeenth century. In England, Thomas Hobbes had made an attempt to incorporate the style of reasoning in geometry for the purposes of his political philosophy, and attacked the mathematician John Wallis for his use of indivisibles.[4] These matters receive extensive and scholarly discussion by Alexander, and implicit in them are wider issues of intellectual and cultural change, not only for the seventeenth century, but reverberating down to present times, where they are still live issues. It is a purpose here to review some of the historical background and also to consider some of the arising associated wider issues of knowledge, culture, history and language.

2 Paradox and Certainty

The term 'infinitesimal' does not seem to have been used until the later part of the seventeenth century, but 'indivisible' goes back to ancient Greece, the atomic theory of Democritus and Aristotle. Alexander in this book treats indivisibles and infinitesimals as synonymous. The gist of these notions and their accompanying paradoxes can be grasped by imagining that, using a ruler and pen, a straight line segment of finite magnitude (length) is drawn on a sheet of paper. Call this line segment a *continuum* and, although there is more than one, it suffices to consider just one. On the one hand, we can think of the continuum as a homogeneous *material* object, and if we carry out successive divisions it seems that we will reach a state where no further division is possible, as we will have obtained indivisible particles. So, in this case, the ultimate constituents of the continuum would appear to be minute, identical, non-overlapping particles – that is, indivisibles. In this case, indivisibles are conceived of as material *atoms*, all having the same exceedingly small but positive magnitude. In this case, there would be a finite number of indivisibles, for otherwise the continuum would have infinite magnitude. On the other hand, we can think of the continuum as an *ideal* object in which all parts of the continuum may be successively divided without limit. In this case the ultimate constituents of the continuum would appear to be points, all having no magnitude, there would be an infinite number (or quantity) of such indivisible points and, because they have no length, we might equally call them 'infinitesimal'. Thus, we might say that an atom is an indivisible with a positive magnitude, but a point has the property that it is an indivisible with no magnitude and is thus 'infinitesimal'. In the material context, mathematical indivisibles in seventeenth century science were associated with atomic notions of matter, but there have been many confusions and inconsistencies in use of the terms 'indivisible' and 'infinitesimal'.[5]

Now, let's follow Aristotle's thinking in a little more detail. As the continuum is divided indefinitely and throughout into continua of smaller and smaller magnitudes, a process having no end in practice but having an end in our imagination, its ultimate constituents appear to be an infinite number of points, each having no magnitude. There is an immediate paradox, for if the continuum is made up of points having no magnitude, how can the continuum itself have a positive magnitude? Noting arguments like these Aristotle rejected the notion that a line segment consists of points, saying '..... it is absurd to think that a magnitude consists of what are not magnitudes .....'.[6] After all, how can something be made out of nothings? So, if we follow the reasoning that indefinite subdivision of the continuum means that its ultimate constituents must be points, it appears that its ultimate constituents cannot be points. This contradiction is the paradox of the continuum, and it was not clarified until the late nineteenth century. Another way of looking at the paradox is to observe that that although the continuum is apparently 'composed' of points, the length of the continuum cannot be recovered from its points. This paradox underlies the controversies over the mathematical method of indivisibles in the seventeenth century, and the latter is the main subject of Alexander's book.

In geometry, using assumed definitions and axioms, to a given proposition Euclid (fl. 300 BCE) applied a process of precise formal reasoning, a demonstration or proof, whose conclusion is the truth of the proposition. He proceeded synthetically – that is, from what is known or assumed as definitions and axioms to a conclusion. Thus, a formally presented proof emphasises exposition, demonstration, and validation, rather than discovery. Nevertheless, proof and discovery are intertwined because, strictly speaking, in mathematics there is no discovery without the validation a proof provides. As well, following a proof's argument may, in itself, be a form of discovery, and of participation in new knowledge. A proof's strength is the greater when the axioms appear more-or-less self-evident and so command general agreement. A proof is broken down into a series of small reasoning steps, and the steps may constitute a long chain. Finding, discovering or constructing a proof may require a great effort, making calls upon the imagination going beyond logic, for one cannot necessarily find the string of inferences needed for a proof by purely logical means. Ideally, each step in a proof is compelling, so that despite a possibly extended reasoning process, the conclusion also is compelling and gives a feeling of assent, truth and certainty. Consequently, assent is active and willing, and not a passive acceptance of authority – rather, it comes from the inner assent and participation of the learner. At the same time, because that assent feels so compelling, it seems to derive from an external, objective existence in harmony with our own reasoning. Consequently, mathematical knowledge conforming to the Euclidean ideal has an appearance of universality and certainty. Now, Aristotle had proposed a corresponding axiomatic method for science in the *Posterior Analytics* in which, as noted by Jonathan Barnes: 'The essential thesis is simple and striking: the sciences are properly expounded in formal axiomatised systems. What Euclid later did, haltingly, for geometry, Aristotle wanted done for every branch of human knowledge.'[7] Again, as in Euclid, knowledge conforming to this Aristotelian ideal would provide a strong feeling of universality and certainty, and Aristotle says 'demonstration is not addressed to external argument, but to argument in the mind [or the soul] – since deduction is not either.'[8]

3 The New Mathematics

The Collegio Romano was established by the Jesuit order in 1551. Mathematics in the early days of the Jesuit colleges had a low status, and at the time there was a wider debate as to the certainty of mathematical knowledge and whether Euclidean arguments conformed to the criteria for science as set by Aristotle in the *Posterior Analytics*. Christopher Clavius, born in 1538, was a German Jesuit, and by 1563 he was lecturing on mathematics at the Collegio Romano, becoming a professor of mathematics around 1565. In an essay introducing his edition of Euclid in 1574, he said:

> [If] the nobility and excellence of a science is to be judged by the certainty of the demonstrations it uses ..... without a doubt the mathematical disciplines have the first place among all others ..... The Theorems of Euclid and the rest of the mathematicians still today as for many years past, retain in the schools their true purity, their real certitude, and their strong and firm demonstrations.[9]

Similar sentiments regarding mathematics still are felt by many mathematicians today.[10] Clavius fought to get more recognition for mathematical study. When the *Ratio Studiorum*, the curriculum for the Jesuit colleges, was finalised in 1599 it specified study of Euclid's *Elements* and further mathematical topics.[11] Alexander argues that the Euclidean style of thought was lastingly influential upon the Jesuits, deriving from their vision of a true, eternal and unchallengeable order emerging from the methods and certainty of Euclidean knowledge.[12]

Following Aristotle, the paradox of the continuum remained less a matter of practical application than speculation about a familiar but strange object. However, in the seventeenth century, it reappeared in a new if somewhat hidden form within the mathematical method of indivisibles. Whereas the continuum was taken to be a length comprised of indivisible points having no magnitude, an area (or planar shape) could be taken to be comprised of 'indivisible line segments. The line segments have a magnitude (that is, length) but have no width, so the line segments are 'indivisible' or 'infinitesimal' as far as their width is concerned. As the line segments have no width, one must presume they have no area. Whereas a paradox arose by regarding the continuum as composed of points having no magnitude, in the new context paradoxes arose from regarding a planar area as composed of line segments having no area. On the one hand, the method of indivisibles was greatly effective in being able to calculate areas and volumes that were previously impossible of calculation, but on the other, contradictions analogous to the paradox of the continuum could result.

An example of a contradictory indivisibles argument discussed by Alexander is as follows.[13] Consider a rectangle that is not a square. A diagonal drawn in the rectangle splits the rectangle into two triangles having equal areas, illustrated in Figure 1. The upper triangle is 'covered' by an infinite family of horizontal line segments, while the lower is 'covered' by a corresponding infinite family of vertical line segments. Figure 1 illustrates only four horizontal segments and the four corresponding vertical segments. Each horizontal line segment is greater in length that the corresponding vertical line segment. An indivisibles argument would say that the areas of the triangles derive from the horizontal line segments covering the upper triangle, and from the vertical line segments covering the lower triangle. Because each horizontal line segment has length greater that the corresponding vertical line segment, it seemingly follows that the upper triangle has a greater area than the lower triangle. One concludes that the two triangles have areas that are equal but also are not equal, an impossible conclusion.

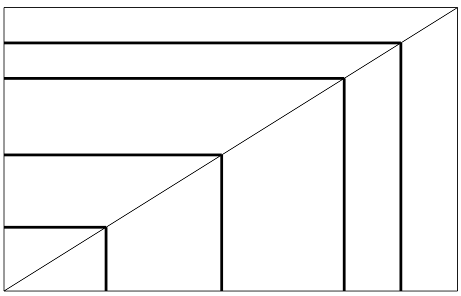

Figure 1: The rectangle paradox

The contradiction arises because the paradox of the continuum, which involves points, carries over into this planar context of areas, which involves line segments. However, the method of indivisibles could be very effective. Kepler used them in deriving his planetary laws of motion in 1609, and they were also used by Galileo.[14]

In 1635 Cavalieri published his *Geometria Indivisibilis,* which became the standard work in its time on the method of indivisibles. Cavalieri applied the new method while endeavouring to avoid a planar version of the continuum paradox, by not directly saying that a planar shape was composed of its line segments. At about the same time, Evangelista Torricelli pointed out the contradictions that could arise using the method, but used indivisibles and their paradoxes as a means for exploration and discovery rather than trying to disguise their use.[15]

The development of mathematical knowledge was running ahead of a conceptual framework that could provide the earlier geometrical standards of clarity and rigour. The method of indivisibles could not be placed on Euclidean-style foundations, and consequently it did not have the appearance of certainty so characteristic of Euclid, as well as producing paradoxes, and conclusions that were sometimes absurd. However, one should be cautious about seeing this as a type of 'political' reaction against Euclidean certainty, for although the new methods could not be made to conform to the Euclidean ideal, their success meant that their use could not be eschewed. The logic of the situation created a challenge, at least implicitly, to the perception and status of mathematical knowledge, with a corresponding challenge to Aristotle's view of science. Also, the new methods often reasoned in the opposite direction from Euclid – they started from a problem and reasoned backwards to reduce it to what was already known or accepted, rather than reasoning from the accepted and the known to deduce the unknown, as in Euclid and as envisaged by Aristotle. The new methods meant that different methods and standards in mathematics were being set – ones that respected not only formal Euclidean-style arguments, but also more informal arguments using intangible entities, and where intuition and usefulness played a greater role in the validation of knowledge. In science this change corresponded to the empiricism of Galileo and his followers, with its emphasis on observation and experiment, and to the corresponding reduced interest in proceeding from final causes as found in Aristotle. In mathematics, remarkable progress was being made, but by accepting weaker foundations, so the new knowledge came at a cost. Correspondingly, in science, the developing empirical outlook was to lead to a jettisoning of notions of certainty that science could not legitimately provide. As it was, one can understand why the new ideas more generally challenged established notions of truth and falsity, and the possibility of distinguishing between them.

4 The Ferment of Ideas

The Jesuits issued statements against indivisibles in 1606, 1608, 1613, 1615, 1632 and 1651.[16] In 1615 there was censure of the view 'the continuum is composed of a finite number of indivisibles ..... [and further] this proposition should not be allowed, even if one maintains that the indivisibles are infinite'.[17] In 1651 a prohibited opinion was, along similar lines, that an infinity in number and magnitude can be enclosed between two points, and that there may be tiny vacuums interspersed within the continuum. This was possibly directed against Galileo's ideas concerning the continuum in his *Two New sciences* (1638).[18]

It may appear strange that the seemingly recondite matter of the structure of the continuum should be of such concern to the Jesuits. Alexander makes clear his view that the use of indivisibles represented a challenge to the certainty of knowledge promised by Euclid. He quotes the Jesuit mathematician Paul Guldin (1577-1654): 'I think that the method [of indivisibles] should be rejected for reasons that must be suppressed by never inopportune silence'. What is this silence about, that it is so opportune? This remains unclear, but Alexander considers Guldin is referring to an ideology behind Jesuit opposition: ' ..... if this flawed system were accepted, mathematics could no longer be the basis of an eternal, rational order. The Jesuit dream of a strict universal hierarchy as unchallengeable as the truths of geometry would be doomed'.[19] In any case, a reason for Jesuit concern about indivisibles was the teaching on the Eucharist. The doctrine of transubstantiation, a central dogma of the Roman church, had been reaffirmed at the Council of Trent (1545-1563). Deriving from the words of Christ, 'this is my body', the Church doctrine was that the Eucharistic sacrament changed the substance of the bread and wine into the body and blood of Christ, while at the same time the 'incidental' attributes of taste and smell of the bread and wine remained. The Jesuit statement of 1608 condemned the proposition: 'Christ exists in the Eucharist in a finitely multiplied manner, that is to say as many times as there are indivisibles of the quantity of the sacramental species, out of which indivisibles that quantity is composed'.[20] Conceiving of matter as composed of indivisibles or atoms challenged transubstantiation,[21] and the case for this being an important factor in Jesuit opposition to indivisibles has been argued by David Sherry (2018).[22] A suggestive circumstance is the anonymous letter 'G3', discovered by Pietro Redondi in the archives of the Holy Office, that was sent to the Inquisition denouncing the atomistic views of Galileo in *Il Saggiatore* (1623) as contradictory to transubstantiation.[23] Galileo's suspected support of atomism, a notion associated with indivisibles, caused concern that he was in breach of Tridentine canons concerning transubstantiation. However, the place of Euclid in the Jesuit opposition to indivisibles seems less clear.[24] Following the Jesuit statement of 1651 the use of indivisibles in seventeenth century Italy was abandoned, and signalled an end to the prominence of Italian mathematics.[25]

In considering the different attitudes to mathematical and scientific knowledge in Italy over the first half of the seventeenth century, one might be tempted to see them as being reduced to a straightforward conflict between 'conservatives' and 'progressives', corresponding more-or-less to the Aristotelian philosophers and powerful sections of the Roman Catholic Church on the one hand, and scientists and mathematicians like Galileo, Cavalieri and Torricelli on the other.[26] However, opposition to indivisibles seems to have come from the Jesuits rather than from the church as a whole (possibly suggestive is that Cavalieri himself was a priest in the Jesuat order). There is an irony in the fact that the Jesuits were influential in the new sciences and mathematics. Carla Rita Palmerino says: ' ..... if one were to systematically study the many *censurae opinionum* that preceded the Jesuit *Ordinatio* of 1651 throughout the first half of the century, one could reconstruct in quite some detail the chronological and geographical spread of the new sciences all over Europe, and their surreptitious entrance through the back door of the Jesuit educational institutions'.[27] The views of the Jesuits at the official level were not necessarily reflected in the views or behaviour of some members of the order. In the case of Galileo, there has been varying opinion on the extent to which he had support within the church, and the extent to which Galileo wished his work to support and protect the church. Stillman Drake argued that Galileo was an orthodox Catholic who sincerely saw his work as honouring and protecting the Church,[28] and Galileo argued that the church could not afford to have unchangeable attitudes to scientific statements, because the evidence in favour of the latter could become so strong that the church would then suffer damage if it had previously denied them.[29]

Alexander also considers England's role in the development of the new mathematical techniques. Thomas Hobbes is best known as the author of *Leviathan*, but Hobbes expressed an early admiration for the systematic reasoning he found in Euclid. Indeed, the force and clarity of the argument in *Leviathan* shows the clear influence of geometrical reasoning upon its author.[30] The authoritarian state described in *Leviathan* consequently has been linked to a perception that geometric-style reasoning is intrinsically authoritarian, a view to which Alexander possibly subscribes,[31] and sometimes suggested by those inclined to sociological explanations of science. An opponent of Hobbes was the mathematician John Wallis, who was appointed Savilian Professor of Geometry at Oxford in 1649. His parliamentary sympathies were almost certainly a large consideration in his appointment, and he had no established reputation in mathematics at the time, although he rapidly rectified this lack. An acrimonious controversy developed between Hobbes and Wallace over infinitesimals, algebraic techniques and appropriate methods in mathematics.

As a philosopher, Hobbes was a strict materialist, and this carried over into his mathematics. Thus, geometrical entities such as points, lines and surfaces were to be regarded either as material objects or as derived from the motion of material objects. His materialism is entwined with his view of scientific knowledge, derived from Aristotle. Thus he writes in 1666:

> 'But,' you will ask, 'what need is there for demonstrations of purely geometric theorems to appeal to motion?' I respond: First, all demonstrations are flawed, unless they are scientific, and unless they proceed from causes, they are not scientific. Second, demonstrations are flawed unless their conclusions are demonstrated by construction .....that is, by the drawing of lines. For every drawing of a line is motion: and so every demonstration is flawed, whose first principles are not contained in the definitions of motions by which figures are described.[32]

This was a return to the issue current during the time of Clavius; namely, the status of mathematics as a scientific discipline. Hobbes' solution is to make mathematics (identified with geometry) a scientific discipline in the Aristotelian sense by seeing its objects and conclusions as deriving ultimately from causes – namely, the motion of bodies. Hobbes' view was attacked by Wallis, one of the grounds being that it introduces extraneous notions from science that are irrelevant to mathematical argument which traditionally, as in Euclid, applied to ideal objects that had no material existence. Further sharp disagreements arose between Hobbes and Wallis over indivisibles, the notion of and reasoning with the infinite, the usefulness of symbols in mathematics, and Hobbes' claimed solution of the problem of constructing by ruler-and-compass a square of the same area as a given circle, shown in the nineteenth century to be an unsolvable problem.[33]

Hobbes was ill-equipped to deal with the attacks of Wallis for, although he had certain achievements in geometry, his belief that Euclidean-type methods could provide solutions to any well-posed geometrical problem was an illusion, and he had little or no awareness of the emerging power of the new algebraic and symbolic techniques in mathematics. However, in England there was no disputation of the type that had occurred in Italy, for the confrontation between Hobbes and Wallis was one of individuals, and was not seen as a wider threat to religion or society.

The discovery of the calculus by Newton and Leibniz put the use of infinitesimals and indivisibles on firmer ground than before. However, the continuing vagueness and lack of clarity in these concepts meant that questions as to the validity of the arguments used in the calculus persisted. Both Newton and Leibniz were aware of this, and in the *Principia* (1687), Newton had enunciated the notion of a limit in the new methods, but he did not use it consistently. In 1734 Bishop Berkeley wrote concerning Newton's arguments:

> It is curious to observe, what subtilty and skill this great Genius employs to struggle with an insuperable Difficulty; and through what Labyrinths he endeavours to escape the Doctrine of Infinitesimals; which as it intrudes upon him whether he will or no, so it is admitted and embraced by others without the least repugnance.[34]

Berkeley's attack on the arguments used by Newton and Leibniz was telling and described by Florian Cajori as 'so many bombs thrown into the mathematical camp'.[35] Even so, Newton may have appreciated the foundations of his calculus better than he is often given credit for,[36] and Berkeley still accepted the conclusions in the *Principia* despite its foundational problems and use of unclear concepts.[37] It was not until the nineteenth century that Newton's implied notion of limits was fully exploited to eliminate the need for infinitesimals and to place the calculus on a consistently rigorous footing.

5 Cultural and Philosophical Issues

5.1 The effects on mathematical and scientific knowledge

The adoption of the new ideas meant that, at least implicitly, a different standard of mathematical argument than was to be found in Euclid became acceptable. It was a mathematics more of intuition, and of trusting an argument despite a lack of clarity and meaning in the terms. It was justified by the usefulness of its results, rather than by its own internal, self-sufficient logic. It tended to treat the validation of knowledge as deriving from successful application. There was a blurring of the distinction between usefulness of knowledge, and the level of validation and certainty of knowledge, In science, the deprecation of ultimate causes led to what Ian Hacking has called 'the evaporation of truth'.[38] Science gradually became concerned with prediction rather than truth, and with practicalities replacing metaphysical concern.[39] Contributing to this was the fact that science had come to depend upon the deduction of universal behaviour from past observation of a finite number of events. This might be taken as an axiom for science, as Newton in effect did in identifying 'the consonance of nature with itself'[40] as a necessary assumption for natural philosophy, but it lies beyond logical demonstration while not being self-evident to the extent that Euclid's axioms are. It is an irony of the seventeenth century that it produced an enormous amount of new knowledge and techniques in mathematics and science while, at the same time, and although the effects received little attention at the time, it undermined what had been seen as the foundations of knowledge. This can be viewed either negatively, as producing an unproductive scepticism, or as positive in recognising the actual limits of knowledge while at the same time clarifying the means for attaining it. At a more popular level, the mathematics and science of the seventeenth century revealed unexpected capacities in the human mind, and seemed to hold promise of resolving problems in society by analogous means. On the resulting optimism about uncovering scientific laws for society, note the comments by R. G. Collingwood:

> And if the advance of the science of human nature extends to the discovery of fundamental laws governing its manifestations, which thinkers of that age thought quite possible on the analogy of the way in which the seventeenth century scientists had discovered the fundamentals of physics, the millennium will be achieved .....The truth is that if the human mind comes to understand itself better, it thereby comes to operate in new and different ways. A race of men that has acquired the kind of self-knowledge at which the eighteenth century thinkers were aiming would act in ways not hitherto known, and these new ways of acting would give rise to new moral and social and political problems, and the millennium would be as far away as ever.[41]

Newton's achievement in the *Principia* reinforced the view of the universe as a machine, which was emerging following Galileo and Descartes. However, at one level mathematical laws can explain, but at another level they also suggest mystery. The origin of Newton's gravitational force remained mysterious and without explanation. There were objections by Leibniz, Huygens and others to the 'occult' nature of gravity and to its 'action at a distance' without apparent mechanical intervention. Hume wrote in 1757:

> While Newton seemed to draw off the veil from some of the mysteries of nature, he showed at the same time some of the imperfections of the mechanical philosophy; and thereby restored her ultimate secrets to that obscurity in which they ever did and ever will remain.[42]

Noting the criticisms of Newton's notion of gravitational force as being 'occult', one might observe that Newton's first law of motion, which says that an object not at rest will move in a straight line unless acted upon by a force, might also be regarded as equally 'occult', because the object keeps moving, apparently without an external force acting upon it, just as in the elliptical motion of the planets. One might say that Newton's law of gravitation, which gave a mathematical basis for Kepler's description of planetary motion as being around the sun in ellipses, was hinting that there was nothing 'special' about straight lines, and was hinting further at a need for a geometry of space in which a natural path of motion is not necessarily a straight line. Such geometries were found in the nineteenth century, prior to Einstein's theory of general relativity.[43]

In the later nineteenth century, Cantor and Dedekind gave formal definitions of numbers, including both rational and irrational numbers, which clarified the notion of the continuum. Newton's original notion of limits came to be systematically applied, obviating any need to refer to infinitesimals. Cantor introduced the notion of an aggregate or *set* (in German, *Menge*), by which he meant a collection of objects called *elements* of the set. These elements could be infinite in number, but by treating them collectively as a single object, a set, Cantor could apply ordinary mathematical reasoning to sets, and thus perform calculations with the infinite. This work formalised the concept of the infinite in the mathematical context, it removed the vagueness and the 'mystical' associated with the infinite, and Cantor's calculations with 'infinities' showed that there are degrees of infinity. In Cantor's theory, two infinite sets are 'equally infinite' if there is a one-to-one correspondence between the elements of the one set and the elements of the other.[44] He showed that the set of rational numbers is 'equally infinite' to the set of counting numbers $1, 2, 3, \ldots$, and that the set of rational numbers together with the irrational numbers has a strictly 'greater infinity' than the set of rational numbers by themselves. This insight also shed light on the continuum paradox for, when we think of deriving the length of the continuum from its points, we naturally think of 'counting' the points, one after the other, indefinitely. But because the continuum has a 'greater infinity' than the set of numbers $1, 2, 3, \ldots$, one can never obtain all the points in the continuum by such a counting process. Hence the failure to recover the length of the continuum from a process of 'counting' the points. The paradox remains, but rather it can be seen as deriving from the mysterious nature of the infinite rather than the failure of a counting process.

Numbers, conceived of as points in the continuum, and 'the infinite', both had come to have Euclidean-style definitions, allowing for Euclidean-style arguments with them. At this point, there was no more need for infinitesimals in calculus or mathematical analysis, although practical use of them persisted in applications of mathematics.

Mainstream mathematics now follows the axiomatic method, although generally there is a strong element of convention in its use – in practice it may be implicit rather than formal, and play a greater or lesser role in different parts of mathematics. However, although there are different attitudes to appropriate mathematical reasonings, there is a remarkable degree of consensus on what constitutes valid mathematical knowledge and proof. Formulating a strict logical argument is challenging, demanding and time-consuming even when it is possible, so that when mathematics is used and applied in many different ways by scientists, economists, statisticians and others, the emphasis is on getting practical results, rather than trying to adhere to the axiomatic method or a strict form of logical argument. This is not to be condemned in itself, rather it is a practical necessity in applications, but it does mean that such knowledge tends to retain its practical and utilitarian origins and lacks that feeling of harmony with a transcending order. That is what happens when mathematics becomes a technique for attaining immediate ends, rather than a Platonic reflection of something beyond. It should be noted also that the deduction of statements by means of strict reasoning from given assumptions is an ideal to be aimed at, rather than one that is always achieved, even in mathematics. What is accepted as a proof at one time, may not be accepted at a later, as we find in the history of mathematics. The logic of a proof may have errors, or remain incomplete. Another danger is that an argument may make use of assumptions that appear to be contained within the axioms, but in fact are not – an axiom system may be incomplete. But we should not be surprised that humanity is fallible, and in the case of mathematics the remarkable thing is not the aspect of fallibility, but rather the force and strength of proof and the wide agreement as to the validity of mathematical statements.[45]

The controversies in mathematics over infinitesimals show that, even in mathematics, an older method of enquiry that provided a higher certainty in one area of knowledge can be challenged, and even go into disuse in the face of new developments because of an inadequate framework of concepts. However, a conceptual framework incorporating new concepts or entities that may not be available at one time may become available at a later, and new concepts at a later time may validate the original method of enquiry and place its conclusions in a clearer light. This remark also applies analogously to science, where it may not be possible technically to test a theory by experiment at one time but it may be possible to test it at a later, and where future observations and knowledge of phenomena may produce a greater understanding.

5.2 History

Alexander's book is a well written, scholarly and compelling narrative that is very much worth reading. However, in historical writing where the narrative is strong there can be a tendency to value change because it is change, and to value those who are regarded as creating the future over those who are considered counterproductive to it. In relation to the latter, Alexander is suggestive of having such an underlying attitude in his book, such as when we read that John Wallis 'rescued' mathematics in the process of banishing Hobbes.[46] Here, the virtuous and the villains can be a little too clear. Despite Hobbes' stature as a political thinker, in mathematics it is very hard to imagine him exerting influence sufficient to lead to the rejection of infinitesimal arguments, whereas in Italy the Jesuits had the intellectual influence and power to crush infinitesimals. As Alexander argues, it is certainly true that the modern world would look very different had there been no development of the calculus facilitated by notions of the infinitesimal. However, the form of the modern world is the result of a whole complex of factors and historical circumstances and cannot simply be reduced to the acceptance of infinitesimals, which is the seeming tendency of some of Alexander's remarks.[47] In the longer run, the inadequacy of the initial mathematical methods was overcome and with the development of new concepts, infinitesimal arguments were no longer needed. The axiomatic method in mathematics reasserted itself, and infinitesimals have come to have as legitimate an existence as other mathematical entities even if, for most mathematicians, they occupy a niche only.

Alexander's history is 'whiggish' in the sense of the term introduced by Herbert Butterfield in 1931.[48] Butterfield describes 'whig history' as 'the tendency in many historians to....praise revolutions provided they have been successful, to emphasise certain principles of progress in the past and to produce a story which is the ratification .... of the present.' However, the issue is quite nuanced, and it can be argued that the writing of history cannot totally avoid Butterfield's strictures, including some of Butterfield's own historical writing.[49] In any case, and leaving aside the question of 'whiggishness' in Alexander's history, a legitimate aim of the historian may not be simply to describe the past as it is perceived, but to try and explain how the past has influenced later times, and for that it may be necessary to include what is considered to have been to the good, but also to the bad. However, whatever has happened in the past has had more than a single effect. Historical change and actual progress may have negative effects as well as positive, they have loss as well as gain, and history may turn back on itself in the future. The dangers of 'progressive' history are that the present is glorified over the past simply because it is the present, our perception of the past becomes simplistic and may be used for questionable purposes, the comforting notion of progress is reinforced without due reason, and historical injustice may be done to those who are not perceived as in sympathy with what happened after them. Beyond that, when history is forgotten, erased, rewritten for ulterior purposes, or seen as leading inevitably to the present, the present tends to become normalised. Thus, the main dangers of all these is perhaps that they can numb our awareness of the present at it is, they hinder and even prevent us from entering into the mind of the past, and they negate the potential for our knowledge of the past to help examine and *test* the present, with its problems and its possibilities.

5.3 Language and reason

The medieval Aristotelian view of the cosmos saw a sympathy between science and religious faith. God was an absolute, and scientific knowledge was seen as based upon identifying causes. Whereas religion derived from revelation and the authority of the Church, Aristotelian science saw valid knowledge as derived logically by demonstration from causes, and there was a type of analogy between religious and scientific knowledge, with a corresponding unity of feeling and sentiment. Also, what one might call the 'smallness' of the Aristotelian cosmos encouraged such feeling. C. S. Lewis writes: 'The old language continually suggests a certain continuity between merely physical events and our most spiritual aspirations' and quotes Chaucer:

> That every natural thing that is,
> Has a natural place where it
> May best in itself nurtured fit;
> Towards which place everything,
> Through its natural inclining,
> Moves so as to come thereto.[50]

Once science moved away from the task of the identification of absolute causes, and replaced a perceived over-arching truth and unity by human observation, systematisation and prediction, a separation of religion and science inevitably grew, a separation that has steadily widened to this day. Science gradually came to be seen as a purely human activity, self-contained and free of allusive, metaphysical or divine overtones. This was noted as early as Galileo in 1638, when he indirectly refers to philosophy as a 'higher science':

> 'SIMPLICIO...... the origin of motion ..... leads one to think that there must be some very great mystery hidden in these true and wonderful results, a mystery related to the creation of the universe ..... and related also to the seat of the first cause.
>
> SALVIATI. I have no hesitation in agreeing with you. But profound considerations of this kind belong to a higher science than ours [a più alte dottrine che le nostre]. We must be satisfied to belong to that class of less worthy workmen who procure from the quarry the marble out of which, later, the gifted sculptor produces those masterpieces which lay hidden in this rough and shapeless exterior.'[51]

Here, note the metaphorical and poetic language with which Galileo expresses himself in referring to that which lies beyond scientific enquiry. This type of language, with its use of metaphor, revealing of a certain poetic sensibility, and a sense of a metaphysical transcendence, was even then destined to vanish from scientific discussion. Thus, in his *The History of the Royal Society* (1667) Thomas Sprat describes a 'language agenda' for the Society:

> But lastly, in these, and all other businesses, that have come under their care; there is one thing more, about which the [Royal] Society has been most sollicitous; and that is, the manner of their Discourse: which, unless they had been very watchful to keep in due temper, the whole spirit and vigour of their Design, had been soon eaten out, by the luxury and redundance of speech..... They have therefore been most rigorous in put|ting in execution, the only Remedy, that can be found for this extravagance: and that has been, a constant Resolution, to reject all the amplifications, digressi|ons, and swellings of style: to return back to the primitive purity, and shortness, when men deliver'd so many things, almost in an equal number of words. They have exacted from all their members, a close, naked, natural way of speaking; positive expressi|ons; clear senses; a native easiness: bringing all things as near the Mathematical plainness, as they can: and preferring the language of Artizans, Countrymen, and Merchants, before that, of Wits, or Scholars.[52]

A similar phenomenon was happening in the wider culture of the time. The dominant mode of linguistic expression can reveal more widely the changes in the sensibility of a nation and its wider culture. Thus, Matthew Arnold writes in 1880 :

> But after the Restoration the time had come when our nation felt the imperious need of a fit prose. So, too, the time had likewise come when our nation felt the imperious need of freeing itself from the absorbing preoccupation which religion in the Puritan age had exercised. It was impossible that this freedom should be brought about without some negative excess, without some neglect and impairment of the religious life of the soul ..... And as with religion so it was also with letters ..... It was impossible that a fit prose should establish itself amongst us without some touch of frost to the imaginative life of the soul ......[53]

Arnold goes on to mention Dryden and Pope as the poetic inaugurators of the new 'age of prose and reason'. In effect, Arnold here identified more broadly what T. S. Eliot in 1921 referred to in the later poetry of the seventeenth century as a 'dissociation of sensibility'.[54] Ian Robinson considered

> .... the idea of a dissociation of sensibility ..... is essential for the understanding of English history ..... The belief, the philosophical department of the new prose, that all proper discourse is propositional, was embraced with emotional fervour [at the end of the seventeenth century].[55]

Such changes in language are a contributing factor to the increasing difficulty with making any sort of metaphysical affirmation, whether that be secular or religious. This cultural shift, affecting science, literature and wider culture, can be seen in one aspect as a separating of the human and the transcendent, and as a splitting of a whole. In the absence of a counter-balancing culture this is to be expected, as language asserts its natural tendency to 'objectify' the objects it names, removing them from deeper overtones of feeling and allusion, and words and language become used predominantly for instrumental and descriptive purposes. It is precisely this instrumental, propositional use of language that has made modern science and technology possible, but at the same time its tendency to eliminate, over time, elements of the mysterious and the numinous has continually reduced the possibility of a unified spiritual and intellectual sensibility.[56] Our response to experience and our capacity for thought are determined by the words and language we can command and whose possibilities we sense. Changes in language affect our sensibilities, but are also intertwined with changing sensibilities.

5.4 Knowledge and authoritarianism

As the axiomatic and 'Euclidean' view of knowledge is often associated in the seventeenth century with institutions and philosophies regarded today as authoritarian, the question naturally arises as to whether the axiomatic descriptions of science as envisaged by Aristotle and as carried out (in part) by Euclid for geometry are intrinsically authoritarian. Alexander sees the rise of the new mathematical methods as appropriate for a 'new world' and contrasts them with 'the rigid Euclidean approach detested by the Royal society'.[57] Alexander implicitly places Euclid and geometry in the villainous camp. However, it is hardly Euclid's fault that Hobbes chose to model the case for his political philosophy upon the stringent method of argument found in Euclid. The underlying question here is the extent to which Hobbes' arguments and conclusions have validity, whatever the method of argument he used. Similarly, Euclid is not responsible for the Jesuits seeing geometry as indicating an ideal for the state of knowledge and society, if that was indeed the view of the Jesuit order. Again, the issue is the legitimacy of such adaptations or 'appropriations' of Euclidean methods in a context different from their original purpose.[58] This way of thinking is not uncommon, and it proceeds by loosely attributing 'guilt' to one thinker by association with the thought of a later. So, in connection with a perceived connection between authoritarianism in political philosophy and geometrical reasoning, we find Barnes, Bloor and Henry saying:

> The model of knowledge for Hobbes was geometrical reasoning, which could secure total and irrevocable agreement ..... Hobbes' method in natural philosophy, as in his political theory, ultimately depended upon unquestioning obedience to an absolute authority'.[59]

In fact, and in so far as geometry is concerned, 'unquestioning obedience' in Hobbes represents his conclusions, not his method. Regardless of whether one thinks geometry secures irrevocable agreement, the actual issue here is the extent to which geometrical reasoning provides an adequate and appropriate model for investigations in the very different context of political philosophy.

Statements such as the one above by Barnes *et al* appear to suggest that an argument that secures 'irrevocable agreement' must be authoritarian – but then, are we to say that because Newton's law of gravity affects us all, it secures unquestioning 'obedience' and so must be authoritarian? The idea that an argument that succeeds by rigorous reasoning is 'authoritarian' then leaves open the question of whether *any* reasoning process is authoritarian. However, a person who accepts Hobbes' philosophy on the basis that its exposition was inspired by geometrical argument which is accepted as conclusive in the geometrical context, is surely mistaken. There is no automatic connection between the degree of intellectual certainty in one discipline and the possibility of that in another, as Aristotle pointed out long ago. The degree of certainty that is possible depends upon circumstances, as an attempt to incorporate the legitimate methods and conclusions in one discipline may result in illegitimate methods and false conclusions in the other. The strong certainties in Euclidean geometry derive from the simplicity and clarity of the fundamental entities, that have an assumed absolute existence and do not interact with the others, so that reasoning with them is tight. When the same approach is used in a non-mathematical context, the fundamental entities are not so clearly 'fundamental', they are not independent of each other, they are not as clear in the mind, and logical deduction has to bend and be replaced by a degree of judgment, greater or lesser depending on the circumstances, but in either case the tightness of the reasoning is compromised or even vitiated. This is implicit in Aristotle's dictum concerning the degrees of precision reasonably to be expected in different areas of enquiry. As well, the 'fundamental entities' underlying a geometric-style argument in a political philosophy like Hobbes' need to take account of society in its multifarious aspects in a precise way, whereas society is too diffuse and complex for that to be possible. Society is an interacting and complex organism more than it is a collection of discrete mechanical parts performing unambiguous and impersonal operations on its members.

Now, if one area of knowledge commands a greater certainty than one that has a lesser, it is not surprising that the exponents of the knowledge of lesser certainty may try and adapt the methods of the more certain to increase respect or improve their methods for their own area of knowledge. Hobbes' adoption of an approach to political philosophy influenced by geometry is a case in point. More recently, areas of study such as economics, management, marketing, and the social sciences, have made their claims to be sciences, with effect that in many contexts 'science' has a loose meaning. Another response to the epistemological status of mathematics and science is to assert that the seemingly 'exact' sciences have little or no claim to a degree of certainty anyway – and so are no more worthy of confidence or respect than any other area of knowledge. This is often associated with the idea that knowledge is 'merely' socially constructed, and also with the idea that knowledge is little more that a means of exerting power within society. Such ideas have been a thrust of the post-modern program, extending beyond the sciences and intermingled with the idea that the main function of knowledge is power, as exemplified by Foucault and others. Thus, we read:

> What, after all, is an education system, other than a ritualisation of speech a qualification and fixing of the roles for speaking subjects, the constitution of a doctrinal group, however diffuse, a distribution and an appropriation of discourse with its powers and knowledges? What is 'écriture' (the writing of the 'writers') other than a similar system of subjection, which perhaps takes slightly different forms, but forms whose main rhythms are analogous? Does not the judicial system, does not the institutional system of medicine likewise constitute, in some of their aspects at least, similar systems of subjections of and by discourse?[60]

This remarkable but loose statement raises many issues, not least being the absence of a sense of differentiation between the circumstances and natures of particular discourses or forms of education. If all education or discourse is to be seen only terms of the 'subjection' it creates, does this also apply to the discourse that produces such a statement? Discourses do not exist in isolation, but interact with each other, and such interaction, it seems, must be seen purely in terms of power, and as an attempt of the one discourse to subjugate the other. This negative perception of discourse arises when knowledge is seen solely in terms of power and its perceived social and political implications, regardless of the extent to which that knowledge has an independent status, a status which is what, after all, a search for knowledge strives to achieve.

Although postmodern ideas like these may be on the wane, the perception remains common that knowledge derived by reasoned argument is inherently authoritarian owing to its constraining nature. This attitude derives more from its perceived educational and social effects than from its reasonableness for the attainment of knowledge. There is a need to distinguish between what is inherent in knowledge, and its effects and place within society. Any use of reason can be seen, if one wills, as no more than a form of constraint and necessarily creating subjection. The accompanying notion of facts, ideas and discourses as being merely alternative creates a morass of relativism, in which relativism itself is relativised and rendered null.

Mathematics and science attain their knowledge by imposing constraints on what methods are considered likely to produce valid knowledge. It is precisely such constraints and their variants and circumstances that provide different areas of knowledge with their differing epistemological profiles.[61] There is an automatic tendency in some forms of social thought to regard the various forms of knowledge as no more than constraining discourses, or as being socially constructed and so little more than a collection of opinions. In either case, the various aspects of knowledge tend to be reduced to the same level and become a matter of convenience. This can produce a situation where one would bar logical argument from mathematics, and eliminate empirical methods in science, and so on. Some claimed knowledge may be deeply embedded in dominant discourses and attitudes in society, and may go so far as to create a form of subjection, but other knowledge may lie to a greater or lesser extent, 'outside' society and be tested by a stricter rationality, producing a strong element of necessity. The 'epistemological profile' (to use Foucault's term) of an area of knowledge is closely related to the degree of necessity that it produces.

In a mathematical context G. Hanna and H. Niels Jahnke describe the type of criticism of mathematical proof which sees proof (or demonstration) as a 'mechanism of control':

> [Those critical of proof] ..... would argue that the so-called 'Euclidean' view of mathematics is in conflict with the present values of society, which dictate that we not bow down to authority and not bow down to knowledge as infallible or irrefutable. They appear to see proof in general, and rigorous proof in particular, as a mechanism of control wielded by an authoritarian establishment to help impose upon students a body of knowledge that it regards as predetermined and infallible.[62]

The views they describe are still discernible today. However, concerning charges that proof is authoritarian they say:

> It is in the very nature of proof that the validity of the conclusion follows from the proof itself, not from any external authority. Proof conveys to students that they can reason for themselves, that they do not need to bow down to authority. The use of proof in the classroom is actually anti-authoritarian ..... it is difficult to take seriously those who challenge the use of proof in the classroom as an expression of authoritarianism and infallibility ..... nor does a significant role for proof in the classroom require mathematics educators to embrace a specific *a priori* view of mathematics.[63]

Geometry does not demand an obedience to an absolute authority, but it does require the person to follow a closely-reasoned argument, and decide whether the argument convinces, does not convince, or requires further examination. But the nature of geometry, and mathematical investigation in general, is to produce a general assent as to truth or falsity, relative to the assumptions underlying the investigation. It is in the issue of this 'general assent' that the perceived authoritarianism of geometry and mathematics lies. However, once one realises that it is a general *assent* and derives from an inner and personal acceptance and is not externally enforced, the perception of authoritarianism vanishes. What is the origin of this feeling of acceptance? Following a close logical argument provides a feeling that the argument lies beyond the person, and that the conclusion is independent of the person's will and desire. But for such a feeling to be attained, the argument must be tackled on its own terms, and the person must *experience* the argument[64] and *participate* in it, and that may require an effort, both in reasoning and imagination. The apparent certainty of Euclidean-style argument, and more informal mathematical reasoning, can be experienced as authoritarian when the person does not experience or participate in the processes of discovery or validation. Thus, Isaiah Berlin writes:

> ..... when I understand the functions of these symbols ..... mathematical truths no longer obtrude themselves as external entities forced upon me, which I must receive whether I want this or not, but as something which I now freely will in the course of the natural function of my own reasoning activity ..... '[65]

So, are we to conclude that there are two groups of people concerning mathematical knowledge? Namely, those who participate in the process of how the knowledge is attained and give a willing assent and make of it something of their own, and those who do not comprehend or understand and so regard the knowledge as authoritarian 'truths' that are forced upon them? No, for one must recognise that the capacity to participate depends upon social and cultural factors that may, and often do, act as barriers. The perception of knowledge as authoritarian need not arise from the intrinsic nature of that knowledge, but rather from the attitudes of its practitioners and the circumstances under which that knowledge and its methods are encountered.[66] It is the clarity of concepts and the tightness of the reasoning in geometry and mathematics that produce a general assent and individual acceptance, but that degree of clarity and tightness is not replicable elsewhere to the same degree. Mathematics identifies some of the logical necessities of thought while at the same time having multifarious applications in experience illustrating, as Simone Weil says, that there is no need to choose between demonstration and experience.[67]

6 Conclusion

The paradox of the continuum arises from a simple question concerning a straight line segment: is it continuous or discrete? The paradox and the associated notion of indivisibles revealed mystery within a seemingly simple and basic entity of thought. The paradox hints at an intrinsic incompleteness that enquiry and knowledge may contain. In the seventeenth century there emerged practical uses of indivisibles in mathematics and science. The resulting prohibitions and controversies are well discussed in Alexander's book, that has brought to light important and fascinating aspects of this seminal period in the history of mathematics and science. The paradoxes in the new mathematical techniques cast doubt over the certainty of the new mathematical knowledge, thus challenging the possibility of genuine knowledge in other areas. As far as the Jesuit order was concerned, it does seem that their world-view was seriously challenged, notwithstanding the technical and obscure notions of indivisibles.

The focus in Alexander's book on the interaction and clashes between different views and uses of mathematical knowledge in the seventeenth century may seem narrow, but he is right to draw attention to the contribution of indivisibles, infinitesimals and the calculus to forming the modern world, and within and beyond the events described in the book are wider issues concerning the discovery, validation and uses of knowledge, and of their relation to society and power. The controversy over indivisibles and infinitesimals reminds us that knowledge may be actual even though it is always changing, that accepted standards for validation of knowledge may change according to circumstances, and that knowledge may challenge power and be a great deal more than a mere expression of it. Directly and indirectly, for the reader so inclined, Alexander's book provides a valuable reminder for continually evaluating for oneself what should be thought and believed.

To the extent that knowledge is concerned to try and see things as they are, rather than how it suits us to perceive them, the search for knowledge and its attainment have a moral function, although this is little recognised in a wider culture of immediate utility, of an exaltation in freedom of opinion (however uninformed), and a suspicion that knowledge has a hidden agenda posing as something more worthy. The search for knowledge should be critical, but should not proceed from a mental attitude of automatic suspicion. In the present times, personal feeling, individual desire, experience and immediate utility tend to take precedence over rationality and disinterested argument. However, the very great changes in society, technology and culture may be creating a greater awareness of the importance of how we decide what knowledge is. The proliferation of opinion made possible by social media means that knowledge and its validation are crucial issues for the longer term stability of western societies, as various levels of our culture collapse intellectual distinctions and magnify social difference. Knowledge, after all, is meant to be *shared* knowledge and the liberty of profitable public discussion is only possible under a portion of moderation, restraint and some sense of a whole. The point is well made by Burke when he says: 'Liberty, too, must be limited in order to be possessed'.[68]

All knowledge requires some sort of underlying affirmation or set of assumptions, for whatever methods there may be for attaining knowledge, they cannot fully justify themselves, and scepticism can be directed against any statement of philosophy, or even of fact. The basis for such legitimate affirmations are rarely discussed, and a need for them is little recognised now in western culture. When knowledge and language are seen almost entirely in terms of instrumental purpose, or are reduced to purely formal requirements, or are seen as little more than vehicles for the assertion of individual or group freedoms, the underlying foundations for knowledge are fragmented, obscured or ignored, resulting in a human and spiritual emptiness in society and a diminishing of intellect. There is a simultaneous need to affirm and remain positive to the human situation, but also a need for remaining at a certain distance from what we affirm, for we may need to re-evaluate it. There is a need to accept and aspire to knowledge, while at the same time accepting those elements of uncertainty and incompleteness that lie to varying extents within all affirmation and all knowledge, for it is only our acceptance of our own uncertainty and incompleteness that makes it possible to listen to the voice of others.

Notes

1. Foucault (2010, pp. 51-52). An earlier version in French is available online as *La fonction politique de l'intellectuel*, at `http://1libertaire.free.fr/MFoucault133.html`.

2. Aristotle (2004, p. 5).

3. Alexander (2014).

4. See Jesseph (1999) for the clash between Hobbes and Wallis, but also a later section in this article.

5. On the confusions over divisibility arising from a blurring of the distinction between the actual and the potential, see Waterfield's further comments in Aristotle (2008, pp. lii-liii). The distinction mentioned by Waterfield was recognised by Aristotle (1995, 316b lines 19-21, p. 517). Also, note that historically there are notions of the infinitesimal that include entities other than points. However, whereas a point has a comfortable geometric interpretation as a 'nothing' indicating (perhaps) position, an infinitesimal, as an entity that is not a point but has infinitesimal length, resists geometric intuition.

6. Barnes (1987, p. 251). See also Aristotle (2008, pp. lii-liii and 138-139).

7. Aristotle, (1975, pp. x-xi).

8. Aristotle, (1975, p. 17).

9. Alexander, (2014, p. 64).

10. Tamvakis, (2014, p. 703).

11. On Clavius see Alexander (2014, Chapter 2). Concerning Clavius and the *Ratio Studiorum* see also Feldhay (1995, Chapter 11). The status of mathematical knowledge was in wider dispute over this period. See Schöttler (2012), where he regards the issue as '...the incompatibility of Euclidean geometry with the Aristotelian understanding of science'. A major issue was the apparent lack of 'cause' in mathematics.

12. Alexander (2014, pp. 67-68 and 119-120). On the value placed on Euclid by the Jesuits see also H. Bosmans (1927, p. 77), where we read '... le grand nombre des mathématiciens de la Compagnie de Jésus resta jusqu'à la fin du XVIII siècle profondément attaché aux méthodes euclidiennes. Par un respect un peu outré, un peu suranné, pour la belle géométrie des Grecs, la plupart dÕentre eux ne surent, ou ne vouleront pas évoluer.'

13. Alexander (2014. pp. 112, 317), where Alexander mentions the originator of the paradox as Evangelista Torricelli (1608-1647).

14. See Alexander (2014, pp. 90, 124), and the *Stanford Encyclopedia of Philosophy* entry *Continuity and Infinitesimals*, available at https://plato.stanford.edu/entries/continuity.

15. A discussion of Cavalieri's method is in Jesseph (1999, pp. 40-42). On contemporaneous criticisms of Cavalieri's method and their possible motivations see Alexander (2014, pp. 152-157). On Torricelli, see Alexander (2014, pp. 111-116) and Jesseph (1999, pp. 185-187).

16. Alexander (2014, pp. 17-21, 147-149), and Palmerino (2003, Introduction and pp.187-188).

17. Palmerino (2003, p. 188.)

18. Alexander (2014, p. 148) and Galileo (1914, pp. 33-38).

19. Alexander (2014, pp. 154-157).

20. Sherry (2018, p. 368).

21. Redondi (1987, pp. 163-165), Heilbron, (2010, pp. 264-265), Hellyer (2005, chapter 5).

22. Alexander (2014) does not pursue the possible place of eucharistic doctrine in the Jesuit opposition to indivisibles, concerning which see the subsequent discussion in Sherry (2018), Alexander (2018) and Radelet-de Grave (2018).

23. Redondi (1987, pp. 157-165, 333-335).

24. Sherry (2018, p. 368), regards as hypothetical the two main views of the reasons behind Jesuit opposition to indivisibles. One is stated by Alexander as the struggle of the Jesuits to impose '..... a true, eternal, and unchallengeable order upon a seemingly chaotic reality', Alexander (2014, p. 67). The other is 'A different hypothesis ..... a conflict between the method of indivisibles and the Catholic doctrine of the Eucharist', Sherry (2018, p. 368). Whereas there is direct evidence that the Jesuits opposed indivisibles because of their perceived challenge to Eucharistic doctrine, the extent to which the challenge posed by indivisibles to Euclidean-style reasoning in mathematics was significant in this opposition is less clear. One possible reason for the importance of Euclid for the Jesuits might be the influence of Clavius, given his high regard for Euclid. But in the absence of direct evidence, one is inclined to agree with Radelet-de Grave (2018, p. 602) who says: 'Did Clavius impose Euclidean mathematics on the Jesuits and prevent them from learning the the new mathematics of indivisibles in order to maintain the hierarchy within the Company? I don't think so.' One of the points at issue in this discussion is the importance at the time of any distinction between infinitesimals and indivisibles. However, noting the subtlety of any such distinction, noting the resistance of infinitesimals to geometric intuition, and noting that the term 'infinitesimal' was not really current until later, it seems unlikely that the Jesuits were interested in the finer distinctions that might be made between indivisibles that were a threat to Eucharistic doctrine and those that might not be. However, note Sherry (2018, p. 16): '. . . it is not out of the question that a handful of Jesuit mathematicians understood that there was a middle ground between Euclid and Cavalieri, one that did not invoke the horror of indivisibles'.

25. See Alexander (2014, pp.178-180) and Sherry (2018, p. 367).

26. For the differing interpretations of the Galileo affair see Feldhay (1995), especially Chapter 1.

27. Palmerino (2003), Introduction. Also, on Jesuits and science see Feingold (2003), and Feldhay (1995) especially Chapters 11 and 12.

28. Drake (2001), Introduction.

29. Galileo (1957, pp. 206-210).

30. Gaskin (1996, p. xliii) , Grant (1966, pp. 110-111) and Macpherson (1985, pp. 17-30).

31. Alexander (2014, pp. 218, 255-256, 277-278).

32. Quoted in Jesseph (1999, p. 135).

33. On Wallis see Alexander (2014, pp. 231 - 242). See Jesseph (1999), especially Chapter 4), for the overall dispute between Hobbes and Wallis. On Hobbes' criticism of Wallis and indivisibles, see Jesseph (1999, pp. 179-182, pp. 187-188). See also Jesseph (2018).

34. Berkeley, (1734, p. 8).

35. Cajori (1919, p. 57). See also the discussions in Boyer (1949, Chapter 6), and Jesseph (1993, Chapter 6).

36. See Pourciau (2001) on Newton's implied but definite notion of limits, and see also Grattan-Guiness (1997, pp. 242-298). Concerning Leibniz's arguments, see Katz and Sherry (2012, pp. 1550-1558).

37. Jesseph (1999, pp. 199-200 and 226-230).

38. Hacking (2004).

39. Galileo (1914, p. 194) reveals an awareness by Galileo of the limitations of the new science.

40. Newton (1846, p. 384) (from Rule III in the 'Rules of Reasoning in Philosophy'). Newton's principle of the 'consonance of Nature with itself' is fundamental to scientific knowledge, as it underlies the derivation of general behaviour from only particular observations of that behaviour.

41. Collingwood (1992, pp. 81-85).

42. Hume (1854, chapter LXXI, p. 476), also available at urlhttp://oll.libertyfund.org/titles/793 p. 542. See also *Hume's Newtonianism and Anti-Newtonianism* in *The Stanford Encyclopaedia of Philosophy*, available at `https://plato.stanford.edu/entries/hume-newton`, section 4.2: 'Hume ..... does not view Newton's achievement as a decisive advance in knowledge of nature but, instead, as decisive evidence for the claim that nature will remain unknowable in principle'. See also Chomsky, (2002, pp. 52-53) where, in referring obliquely to the body-mind dualism of René Descartes he says: 'Newton exorcised the machine, he left the ghost intact'. Chomsky comments on the criticism of Newton on the grounds that gravitation was 'occult', and notes that Newton's defence involved the acceptance of a 'weaker model of intelligibility'.

43. Nikolai Lobachevsky and János Bolyai independently discovered non-Euclidean geometries in the 1820s. They are important in Einstein's theory of general relativity, according to which space is 'curved' in the presence of matter, so enabling elliptical motion to be perceived as 'normal'.

44. Cantor (1955).

45. There is a discussion of changing notions of mathematical proof and certainty in mathematics in Vavilov (2019). Euclid provides an example where there is an incomplete axiom system – in 1882 M. Pasch identified an assumption used in some of Euclid's proofs that

could not be derived from the axioms. On this, see Davis and Hersch (1981, pp. 159-161).

46. Alexander (2014, p. 276).

47. Alexander (2014, pp. 289-294).

48. Butterfield (1931).

49. Cronin (2012).

50. Lewis, (2004, pp. 92 - 94). Lewis quotes the Middle English from Chaucer's *Hous of Fame*, II, lines 730 - 736. The translation here is by A. S. Kline. Lewis also says ' ..... this is the medieval synthesis itself, the whole organisation of their theology, science, and history into a single, complex, harmonious mental Model of the Universe ..... They are bookish. They are indeed very incredulous of books. They find it hard to believe that anything an old *auctour* has said is simply untrue.' Lewis' comments are pertinent for considering the response of the Jesuits and the Aristotelian philosophers to the new ideas that could not be accommodated by the old synthesis. Also pertinent is Foucault (1981 p. 58): 'It was indispensable, in the Middle Ages, that a text should be attributed to an author, since this was an index of truthfulness. A proposition was considered as drawing even its scientific value from its author. Since the seventeenth century this function has steadily been eroded in scientific discourse.'

51. Galileo (1914, p. 194.)

52. Sprat, (1667, pp. 111-113). It is perhaps worth noting that Sprat's own prose remains intermediate and not infrequently rhetorical, and bears marks of the more florid style he is rejecting as inimical to knowledge.

53. Arnold (1964, pp. 235-260.)

54. Eliot (1975, pp. 64-65).

55. Robinson (1982, pp. 260-272).

56. On the problem of non-objectifying language see Ott (1967). Although set in a Christian and religious context, it has secular relevance concerning language and its objects and also indicates possibilities for thinking about metaphysical (non-objective) entities. On language and its effect upon our conceptual range see also Robinson (2018), especially in the chapter on *The Divine*.

57. Alexander (2014, p. 277). On Hobbes and the Royal Society see Skinner (1969).

58. Note the view ' ....never ....expect more precision in a subject than the nature of that subject permits', Aristotle (2004, p. 5). Note also the recognition of the varying epistemological profiles for different areas of intellectual enquiry in Foucault (2010, pp. 51-52).

59. Barnes, Bloor and Henry (1996, p.193).

60. Foucault (1981). This is from an inaugural lecture given at the Collège de France in 1970. The lecture contain sharp historical and analytic insights, but in this brief section the analytic and descriptive tone changes abruptly and becomes more strident and rhetorical. Whereas the word *constrainte* had been used to refer to the constraints dis-

courses impose, the word now used is the stronger *assujettisements*, translated in English as 'subjection' or sometimes 'subjugation'. It's as though Foucault has suddenly suddenly decided to reveal his 'true' agenda, or is it simply a lapse that has that effect? In any case, it gives a strong indication of the place from which Foucault speaks. The lecture in French is at `https://litterature924853235.files.wordpress.com/2018/06/ebook-michel-foucault-l-ordre-du-discours.pdf`.

61. Varying epistemological profiles are recognised in Foucault (2010, p. 51).

62. Hanna and Jahnke (1996, p. 890).

63. Hanna and Jahnke (1996, pp. 890-892). The point about proof not demanding any particular view of mathematics is important, for proof is compatible with an 'absolutist' or Platonic concept of mathematics but also with one that regards mathematical knowledge as socially constructed. In any case, 'socially constructed' knowledge can have a high degree of objectivity, depending on the type of knowledge and the constructive process.

64. This point is also made in Hacking (1999, p. 89): '..... As real as anything we know. People who have never *experienced* a mathematical proof (the feeling of, as Wittgenstein puts it, "the hardess of the logical must") seldom grasp what Platonistic mathematicians are on about.'

65. Berlin (1958, p. 26). It is not clear that this is Berlin's own view, as he is describing a certain philosophical attitude towards liberty. But that it was his view seems likely from the context.

66. The issue is examined in Plato (1956), in which Socrates argues, in effect, that mathematical knowledge is latent in every individual.

67. Weil (2002, p. 69).

68. Burke, (1999, pp. 288-289).

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