A priori error analysis for Navier Stokes equations with Slip boundary conditions of friction type

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Abstract

The time dependent Navier Stokes equations under nonlinear slip boundary conditions are discretized by backward Euler scheme in time and finite elements in space. We derive error estimates for the semi-discrete problems. The focus on the semi discrete problem in time is to obtain convergence rate without extra regularity on the weak solution by following Nochetto, Savare and Verdi [1]. The semi discrete problem in space is analyzed with the help of the Stokes operator introduced. Finally we use the triangle inequality to derive the global a priori error estimates.

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1 Introduction

The subject of the present work is to analyze the time discretization of the two dimensional Navier Stokes equations driven by nonlinear slip boundary conditions, written as follows

$$\begin{cases} \boldsymbol{u}_t - 2\nu \operatorname{div} \varepsilon(\boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } Q_T, \\ \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } Q_T, \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0 \quad \text{on } \overline{\Omega}, \end{cases} \tag{1.1}$$

where $Q_T = \Omega \times (0, T)$, with Ω being a bounded domain in $\mathbb{R}^2$. $\boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity and pressure $p(\boldsymbol{x}, t)$. In (1.1) $\boldsymbol{f}(\boldsymbol{x}, t)$ is the external body force per unit volume, while ν is the kinematic viscosity, and the symmetry part of the velocity gradient is $2\varepsilon(\boldsymbol{u}) = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T$. $\boldsymbol{u}(\boldsymbol{x}, 0)$ is the value of $\boldsymbol{u}(\boldsymbol{x}, t)$ at $t = 0$, for which precise assumptions will be introduced below, and $\overline{\Omega}$ is the closure of Ω. To solve (1.1), boundary conditions should be introduced. We assume that the boundary of Ω, say, $\partial\Omega$ is made of two components S and Γ, such that $\overline{\partial\Omega} = \overline{S \cup \Gamma}$, with $S \cap \Gamma = \emptyset$. We assume the homogeneous Dirichlet condition on Γ, that is

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma \times (0, T). \tag{1.2}$$

On S, the velocity is decomposed following its normal and tangential part; that is

$$\boldsymbol{u} = (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} + \boldsymbol{u}_{\boldsymbol{\tau}},$$

where $\boldsymbol{n}$ is the normal outward unit vector to S, and $\boldsymbol{u}_{\boldsymbol{\tau}}$ is parallel to the tangent direction $\boldsymbol{\tau}$ which is orthogonal to $\boldsymbol{n}$. On S, we first assume the impermeability condition

$$u_N = \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad S \times (0, T) . \tag{1.3}$$

The Cauchy stress tensor is $\boldsymbol{T} = 2\nu\varepsilon(\boldsymbol{u}) - p\boldsymbol{I}$ where $\boldsymbol{I}$ is the identity tensor. Just like the velocity, the traction $\boldsymbol{Tn}$ acting on S is decomposed following its normal and tangential part; that is

$$\begin{aligned} \boldsymbol{Tn} &= (\boldsymbol{Tn} \cdot \boldsymbol{n})\boldsymbol{n} + (\boldsymbol{Tn} \cdot \boldsymbol{\tau})\boldsymbol{\tau} \\ &= (-p + 2\nu\boldsymbol{n} \cdot \varepsilon(\boldsymbol{u})\boldsymbol{n})\boldsymbol{n} + 2\nu(\boldsymbol{\tau} \cdot \varepsilon(\boldsymbol{u})\boldsymbol{n})\boldsymbol{\tau} \\ &= (\boldsymbol{Tn})_{\boldsymbol{n}} + (\boldsymbol{Tn})_{\boldsymbol{\tau}} . \end{aligned}$$

$\boldsymbol{u}_{\boldsymbol{\tau}}$ and $(\boldsymbol{Tn})_{\boldsymbol{\tau}}$ are related to each other through the Tresca's friction law [2, 3, 4]

$$\left.\begin{aligned} &\text{if} \quad |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| < g \quad \text{then} \quad \boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{0}, \\ &\text{if} \quad |(\boldsymbol{Tn})_{\boldsymbol{\tau}}| = g \quad \text{then} \quad \boldsymbol{u}_{\boldsymbol{\tau}} \neq \boldsymbol{0} \;, \; \textit{and} \quad -(\boldsymbol{Tn})_{\boldsymbol{\tau}} = g\frac{\boldsymbol{u}_{\boldsymbol{\tau}}}{|\boldsymbol{u}_{\boldsymbol{\tau}}|} \end{aligned}\right\} \text{ on } \quad S \times (0, T), \tag{1.4}$$

where $g : S \longrightarrow (0, \infty)$ is a non negative function. By assuming that the weak solution of (1.1),...,(1.4) is twice differentiable in time, Yuan Li and Rong An in [5] have derived error estimates by using a regularization procedure together with stabilized finite element approximation where the velocity and pressure are approximated by P_1 discontinuous elements. The objective of our work is to study the convergence under minimal regularity in time of the weak solution of (1.1),...,(1.4) when the implicit Euler's scheme is used. It is important to point out at this level that our result differ substantially from the one in [5], because in the former the weak solution is expected to be twice differentiable in time while in our work no such regularity is required. Instead we used the correct regularity of the weak solution and demand the forcing term to be once time differentiable in time.
We first prove the existence and uniqueness of solutions of the time discrete model associated with (1.1),...,(1.4). This is done in Section 2 (see Theorem 2.2 and Proposition 2.1). The construction of weak solutions are based on; regularization, Faedo-Galerkin, Brouwer's fixed point and passage to the limit by using appropriate compactness results. The uniqueness is derived by using classical approach which consist of assuming the existence of two solutions and exploiting the variational formulation. It should be noted that existence and uniqueness of solution can be extended to 3d without particular complications.
In Section 3, we derive some a priori estimates necessary to analyse the convergence of the

discrete in time solution.
Section 4 which can be regarded as the main contribution of this work is concerned with the actual analysis of the error between the discrete in time solution $(\boldsymbol{u}_k^n, p_k^n)$ and weak solution $(\boldsymbol{u}, p)$. Trying to obtain convergence in time with actual rate for solution of any approximate differential equation by not imposing higher differentiability in time on the solution of the continuous model is not a trivial one, as most method of proof uses the consistency arguments which is responsible of demanding higher differentiability in time of the solution. Following [1], we are able to prove the convergence of the discrete in time solution to the solution of the continuous model. Indeed we derive optimal rate of convergence in time in the absence of higher regularity in time of the solution but under moderate assumptions on the forcing term $\boldsymbol{f}$. This important result is stated in Theorem 4.1. This technique avoids the "consistency error check" by introducing dissipation quantities and exploits fine properties of sub-differentials. Similar approach has been used by Bartels [6] to analyze elasto-plastic deformation by assuming only that the weak solution is continuously differentiable in time. It is worth mentioning that obtaining error estimates for differential equation under minimal regularity in time has been the subject of interest in [7, 8]. One of the important feature in the derivation of the a priori estimate presented in [1] is the fact that a posteriori error estimate in time is obtained in the process, hence adaptive refinement strategy in time can be formulated.
This work is a continuation of our recent work [9], where the Oseen equations are treated, but the estimate on the pressure is not examined. The second objective of this article is to analyze the error committed on the pressure after time discretization. It is well known that for time dependent Navier Stokes equations, one cannot directly used the compatibility condition between the pressure and velocity to estimate the error committed on the pressure because of the presence of the time derivative on the velocity. In [5] that difficulty was lifted with the introduction of penalty parameter and pressure projection method while maintaining higher order differentiability in time. In our work, we first estimate the error caused on the time derivative of the velocity (see Theorem 4.2), and then we show that the error on the pressure is not optimal (see Theorem 4.3). In this work we show that the framework presented by R. Nochetto, G. Savare and C. Verdi in [1] is applicable to partial differential equations which are not pure gradient or sub-differential flows.
After the completion of the continuous in space discrete in time problem, we study in Section 5 the finite element formulation while the time is continuous. This semi-discrete problem has the advantage to be well posed (see Theorem 5.1), and some crucial error estimates are derived in Theorem 5.2, Theorem 5.3, and Theorem 5.4.
The third objective in this article is to use the technique presented by Alexander Mielke, Laetitia Paoli, Adrien Petrov and Ulisse Stefanelli in [10] to derive the global error generated by the fully discrete problem, a task that we achieve Theorem 5.5. This approach exploit the triangle inequality and the semi-discrete approximations treated before.
Important background information on existence of solutions of (1.1),...,(1.4) can be found in the work of Kashiwabara [11], while works on numerical analysis of stationary Stokes and Navier Stokes equations driven by nonlinear slip boundary conditions can be found in [12, 13, 14, 15, 16].

2 Preliminaries

This section is twofold. We first introduce some classical notations pertaining to the mathematical formulation of the problem (1.1),...,(1.4). Next, the weak formulation is formulated and the existence of solution is recalled. The second paragraph of the section is concerned with the time discretization of the weak introduced before. In fact we show that the discrete in time problem is well posed. The solvability together with the uniqueness results can be considered as first contribution in this work.

2.1 Variational formulation:continuous description

For the mathematical setting of our work, we next introduce standard definitions and facts in the treatment of Navier Stokes equations that are can be found in general [17], but because of (1.4), some minor modifications will be made clear when necessary. Standard notation on Lebesgue and Sobolev spaces is employed and $(\cdot,\cdot)$ denotes the L^2 scalar product, and $\|\cdot\|$ the L^2-norm. In this work, boldface characters denote vector quantities, and $\boldsymbol{H}^1(\Omega)=H^1(\Omega)^2$ and $\boldsymbol{L}^2(\Omega)=L^2(\Omega)^2$. For a mathematical formulation of the problem, we introduce the following function spaces and functionals [17].

$$\begin{array}{rcl} \mathbb{V} &=& \{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega)\ :\quad \boldsymbol{v}|_\Gamma=0\ \ ,\ \ \boldsymbol{v}\cdot\boldsymbol{n}|_S=0\},\\ \mathbb{V}_{\text{div}} &=& \{\boldsymbol{u}\in\boldsymbol{H}^1(\Omega),\ \operatorname{div}\boldsymbol{u}=0,\ \ \boldsymbol{u}|_\Gamma=\boldsymbol{0}\ ,\ \boldsymbol{u}\cdot\boldsymbol{n}|_S=0\},\\ \mathbb{H} &=& \{\boldsymbol{u}\in\boldsymbol{L}^2(\Omega),\ \operatorname{div}\boldsymbol{u}=0,\ \ \boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega}=0\}. \end{array}$$

As usual, $\phi(t)$ stands for the function $\boldsymbol{x}\in\Omega\to\phi(\boldsymbol{x},t)$. If E is a Banach space, $E^\star$ its dual, then $\langle\cdot,\cdot\rangle$ is the duality pairing between E and $E^\star$. For $x_0\in E$, the sub-differential of $\Psi:E\to\mathbb{R}$, is defined as follows:

$$z\in\partial\Psi(x_0)\text{ if and only if }\Psi(x)-\Psi(x_0)\geq\langle z,x-x_0\rangle\quad\forall x\in E. \tag{2.1}$$

With (2.1) in mind, (1.4) can also be written as follows

$$-(\boldsymbol{Tn})_{\boldsymbol{\tau}}\in\ g\partial|\boldsymbol{u}_{\boldsymbol{\tau}}|\quad\text{on }S\times(0,T), \tag{2.2}$$

where $\partial|\cdot|$ is the sub-differential of the real valued function $|\cdot|$, with $|\boldsymbol{w}|^2=\boldsymbol{w}\cdot\boldsymbol{w}$. The letter c denotes a generic positive constant which may take different values even in the same calculation, independent of the discretization parameter. We introduce the continuous bilinear form $a(\cdot,\cdot)$ given as follows

$$\begin{array}{rccl} a: & \mathbb{V}\times\mathbb{V} & \longrightarrow & \mathbb{R}\\ & (\boldsymbol{v},\boldsymbol{u}) & \longrightarrow & a(\boldsymbol{v},\boldsymbol{u})=2\nu(\varepsilon(\boldsymbol{v}),\varepsilon(\boldsymbol{u})). \end{array}$$

At this point we recall that the Korn inequality reads: there is a constant c depending only on Ω such that

$$c\int_\Omega\nabla\boldsymbol{v}:\nabla\boldsymbol{v}\ dx\leq\int_\Omega\varepsilon(\boldsymbol{v}):\varepsilon(\boldsymbol{v})\ dx\quad\text{for all}\ \ \boldsymbol{v}\in\mathbb{V}\ , \tag{2.3}$$

while the Poincaré-Friedreich inequality state that; there is constant c depending only on the domain Ω such that

$$c\int_\Omega|\boldsymbol{v}|^2\ dx\leq\int_\Omega\nabla\boldsymbol{v}:\nabla\boldsymbol{v}\ dx\quad\text{for all}\ \ \boldsymbol{v}\in\mathbb{V}\ . \tag{2.4}$$

Thus it is manifest that the norms $\|\cdot\|_1$ and $\|\nabla\cdot\|$ are equivalent on $\mathbb{V}$. Thus the coercivity of $a(\cdot,\cdot)$ is easily deduced from (2.3). Indeed

$$2\nu c\|\boldsymbol{v}\|_1^2\leq a(\boldsymbol{v},\boldsymbol{v})\quad\text{for all}\quad\boldsymbol{v}\in\mathbb{V}.$$

Next, the continuous trilinear form $b(\cdot,\cdot,\cdot)$ given as follows

$$\begin{array}{rccl} d: & \mathbb{V}\times\mathbb{V}\times\mathbb{V} & \longrightarrow & \mathbb{R}\\ & (\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) & \longrightarrow & d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}), \end{array}$$

is continuous on $H^1(\Omega)^2$, and enjoys the following properties [17]

$$|d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})|\ \leq\ c\|\boldsymbol{u}\|^{1/2}\|\nabla\boldsymbol{u}\|^{1/2}\|\nabla\boldsymbol{v}\|\|\boldsymbol{w}\|^{1/2}\|\nabla\boldsymbol{w}\|^{1/2}\ ,\quad\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{V} \tag{2.5}$$

$$d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v})\ =\ 0\ ,\ \boldsymbol{u}\in\mathbb{V}_{\text{div}},\ \text{and}\ \ \boldsymbol{v},\boldsymbol{w}\in\mathbb{V} \tag{2.6}$$

the last property implying for $\boldsymbol{u}\in\mathbb{V}_{\text{div}}$, and $\boldsymbol{v},\boldsymbol{w}\in\mathbb{V}$

$$d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=-d(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})\ \ \text{for all }\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{V}, \tag{2.7}$$

and the following bound is valid

$$|d(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v})|\ \leq\ c\|\boldsymbol{u}\|\|\nabla\boldsymbol{u}\|\|\nabla\boldsymbol{v}\|\ . \tag{2.8}$$

Finally, we introduce the functionals (continuous on $\mathbb{V}$)

$$\begin{array}{rrcl} J: & \mathbb{V} & \longrightarrow & \mathbb{R} \\ & \boldsymbol{v} & \longrightarrow & J(\boldsymbol{v}) = (g, |\boldsymbol{v}_{\boldsymbol{\tau}}|)_S, \\ \ell: & \mathbb{V} & \longrightarrow & \mathbb{R} \\ & \boldsymbol{v} & \longrightarrow & \ell(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}). \end{array} \tag{2.9}$$

The weak formulation associated to (1.1),...,(1.4), can be stated as follows:
Find $\boldsymbol{u}(t) \in \mathbb{V}_{\text{div}}$, such that

$$\begin{cases} \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega, \\ \text{and for all } \ \boldsymbol{v} \in \mathbb{V}_{\text{div}}, \quad \text{and a.e } t > 0 \\ (\boldsymbol{u}'(t), \boldsymbol{u}(t) - \boldsymbol{v}) + a(\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{v}) + d(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{v}) \\ +J(\boldsymbol{u}(t)) - J(\boldsymbol{v}) \leq \ell(\boldsymbol{u}(t) - \boldsymbol{v}). \end{cases} \tag{2.10}$$

The following result has been proved in [11] using the regularization approach together with Faedo-Galerkin and compactness arguments (following [17] (Th 3.5 page 202)).

Theorem 2.1. *Let $\boldsymbol{u}_0$ be an element of $\mathbb{V} \cap \boldsymbol{H}^2(\Omega)$, such that (1.4) is satisfied at $t = 0$. Let $g \in L^2(S)$, $\boldsymbol{f} \in W^{1,\infty}(0, T; L^2)$ with $\boldsymbol{f}(0) \in \mathbb{H}$. Then the variational problem (2.10) has a unique solution $\boldsymbol{u}$ with the regularity $\boldsymbol{u} \in L^2(0, T; \mathbb{V}) \cap L^\infty(0, T; \mathbb{H})$ and $\boldsymbol{u}' \in L^2(0, T; \mathbb{V}) \cap L^\infty(0, T; \mathbb{H})$. The pressure obtained via the incompressibility condition is such that $p \in L^\infty(0, T; L^2_0(\Omega))$.*

The following formulation equivalent to (2.10) will also be important in this work:
Find $(\boldsymbol{u}(t), p(t)) \in \mathbb{V} \times M$, such that

$$\begin{cases} \boldsymbol{u}(0) = \boldsymbol{u}_0 \text{ in } \Omega, \\ \text{and for all } \boldsymbol{v}, q \in \mathbb{V} \times M, \quad \text{and a.e } t > 0\ , \\ (\boldsymbol{u}'(t), \boldsymbol{u}(t) - \boldsymbol{v}) + a(\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{v}) + b(\boldsymbol{u}(t) - v, p(t)) + d(\boldsymbol{u}(t), \boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{v}) \\ +J(\boldsymbol{u}(t)) - J(\boldsymbol{v}) \leq \ell(\boldsymbol{u}(t) - \boldsymbol{v}), \\ b(\boldsymbol{u}(t), q) = 0\,, \end{cases} \tag{2.11}$$

with

$$b(\boldsymbol{v}, q) = -(\operatorname{div} \boldsymbol{v}, q)\,.$$

2.2 Variational formulation: time discretization

Let N be an integer, and set $k = T/N$, we shall introduce by induction a sequence of elements of $\mathbb{V}$, say $\boldsymbol{u}_0^k, \boldsymbol{u}_1^k, \cdots, \boldsymbol{u}_N^k$, where $\boldsymbol{u}_n^k$ is an approximation of $\boldsymbol{u}(t)$ we are seeking on the interval $I_{k,n} = (t_{n-1}, t_n)$ with $t_n = nk$. The interval $(0, T)$ is subdivided into N sub-interval $I_{k,n}$ of equal length, and we approximate $\boldsymbol{f}$ by $\boldsymbol{f}_n^k$ given as follows

$$\boldsymbol{f}_n^k = \frac{1}{k} \int_{t_{n-1}}^{t_n} \boldsymbol{f}(t) dt, \ \text{for } n = 1, 2, 3, \cdots, N.$$

We consider a time discretization of (2.8) using the fully implicit Euler scheme

$$\begin{cases} \text{Let } \boldsymbol{u}_0^k = \boldsymbol{u}_0, \\ \text{Find } \boldsymbol{u}_n^k \in \mathbb{V}_{\text{div}}, \text{such that for all } \ \boldsymbol{v} \in \mathbb{V}_{\text{div}}, \text{and } n = 1, 2, \cdots, N, \\ \left(\dfrac{\boldsymbol{u}_n^k - \boldsymbol{u}_{n-1}^k}{k}, \boldsymbol{u}_n^k - \boldsymbol{v} \right) + a(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k - \boldsymbol{v}) + d(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \boldsymbol{u}_n^k - \boldsymbol{v}) \\ +J(\boldsymbol{u}_n^k) - J(\boldsymbol{v}) \leq (\boldsymbol{f}_n^k, \boldsymbol{u}_n^k - \boldsymbol{v}). \end{cases} \tag{2.12}$$

Remark 2.1. *One observes that*

$$\|\boldsymbol{f}_n^k\| \leq \frac{1}{k} \int_{t_{n-1}}^{t_n} \|\boldsymbol{f}(t)\| dt \leq \frac{1}{k^{1/2}} \|\boldsymbol{f}\|_{L^2(t_{n-1}, t_n; L^2)} \leq \|\boldsymbol{f}\|_{L^\infty(t_{n-1}, t_n; L^2)}\,.$$

It worth noting that (2.12) is equivalent to

$$\begin{cases}\text{Let } \boldsymbol{u}_0^k = \boldsymbol{u}_0, \\ \text{Find } (\boldsymbol{u}_n^k, p_n^k) \in \mathbb{V}\times M, \text{such that for all } \ \boldsymbol{v}, q \in \mathbb{V}\times M, \text{and } n = 1,2,\cdots,N, \\ \left(\dfrac{\boldsymbol{u}_n^k - \boldsymbol{u}_{n-1}^k}{k}, \boldsymbol{u}_n^k - \boldsymbol{v}\right) + a(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k - \boldsymbol{v}) + b(\boldsymbol{u}_n^k - \boldsymbol{v}, p_n^k) + d(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \boldsymbol{u}_n^k - \boldsymbol{v}) \\ +J(\boldsymbol{u}_n^k) - J(\boldsymbol{v}) \leq (\boldsymbol{f}_n^k, \boldsymbol{u}_n^k - \boldsymbol{v}), \\ b(\boldsymbol{u}_n^k, q) = 0\,. \end{cases} \tag{2.13}$$

The aim is to show that the solution sequence $(\boldsymbol{u}_n^k, p_n^k)_n$ of (2.13) convergence to the solution $(\boldsymbol{u}, p)$ of (2.11) using minimal regularity in time. This task is realized with the help of the seminal contribution of Nochetto, Savare and Verdi in [1]. First, we show that the sequence $(\boldsymbol{u}_n^k)_{n\geq 1}$ given through (2.12) is well defined and that purpose, we claim that

Theorem 2.2. *Let $(\boldsymbol{f}(t), g) \in \mathbb{H}\times L^\infty(S)$ and $\boldsymbol{u}_0 \in \mathbb{H}$. Then the variational formulation (2.12) is solvable.*

Proof. It is obtained in several steps.
step 1: Regularization. Note that the functional J is non differentiable at zero. Hence we introduce the parameter $\varepsilon > 0$, approaching zero and define the functional $J_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ as follows

$$J_\varepsilon(\boldsymbol{v}) = \int_S g\sqrt{|\boldsymbol{v_T}|^2 + \varepsilon^2}\ .$$

One observes that

$$\lim_{\varepsilon\to 0}(J_\varepsilon(\boldsymbol{v}) - J(\boldsymbol{v})) = 0\ .$$

The functional J_ε is convex, lower semi-continuous and twice Gateaux-differentiable with

$$\begin{aligned} \langle J_\varepsilon^{(1)}(\boldsymbol{u}), \boldsymbol{v}\rangle &= \int_S g\frac{\boldsymbol{u_T}\cdot\boldsymbol{v_T}}{\sqrt{|\boldsymbol{u_T}|^2+\varepsilon^2}}, \\ J_\varepsilon^{(2)}(\boldsymbol{u})(\boldsymbol{v}, \boldsymbol{w}) &= \int_S g\frac{(\boldsymbol{v_T}\cdot\boldsymbol{w_T})(|\boldsymbol{u_T}|^2+\varepsilon^2) - (\boldsymbol{u_T}\cdot\boldsymbol{w_T})(\boldsymbol{u_T}\cdot\boldsymbol{v_T})}{(|\boldsymbol{u_T}|^2+\varepsilon^2)^{3/2}}\ . \end{aligned} \tag{2.14}$$

The regularized problem reads:

$$\begin{cases}\text{Let } \boldsymbol{u}_0^k = \boldsymbol{u}_0, \\ \text{Find } \boldsymbol{u}_n^{k,\varepsilon} \in \mathbb{V}_{\text{div}}, \text{such that for all } \ \boldsymbol{v}\in\mathbb{V}_{\text{div}}, \text{and } n = 1,2,\cdots,N, \\ \left(\dfrac{\boldsymbol{u}_n^{k,\varepsilon} - \boldsymbol{u}_{n-1}^{k,\varepsilon}}{k}, \boldsymbol{v} - \boldsymbol{u}_n^{k,\varepsilon}\right) + a(\boldsymbol{u}_n^{k,\varepsilon}, \boldsymbol{v} - \boldsymbol{u}_n^{k,\varepsilon}) + d(\boldsymbol{u}_n^{k,\varepsilon}, \boldsymbol{u}_n^{k,\varepsilon}, \boldsymbol{v} - \boldsymbol{u}_n^{k,\varepsilon}) \\ +J_\varepsilon(\boldsymbol{v}) - J_\varepsilon(\boldsymbol{u}_n^{k,\varepsilon}) \geq (\boldsymbol{f}_n^k, \boldsymbol{v} - \boldsymbol{u}_n^{k,\varepsilon}). \end{cases} \tag{2.15}$$

Now since J_ε is differentiable and using some arguments in [18] (see page 157–158), it turn out that the problem (2.15) is equivalent to

$$\begin{cases}\text{Let } \boldsymbol{u}_0^k = \boldsymbol{u}_0, \\ \text{Find } \boldsymbol{u}_n^{k,\varepsilon} \in \mathbb{V}_{\text{div}}, \text{such that for all } \ \boldsymbol{v}\in\mathbb{V}_{\text{div}}, \text{and } n = 1,2,\cdots,N, \\ \left(\dfrac{\boldsymbol{u}_n^{k,\varepsilon} - \boldsymbol{u}_{n-1}^{k,\varepsilon}}{k}, \boldsymbol{v}\right) + a(\boldsymbol{u}_n^{k,\varepsilon}, \boldsymbol{v}) + d(\boldsymbol{u}_n^{k,\varepsilon}, \boldsymbol{u}_n^{k,\varepsilon}, \boldsymbol{v}) + \langle J_\varepsilon^{(1)}(\boldsymbol{u}_n^{k,\varepsilon}), \boldsymbol{v}\rangle = (\boldsymbol{f}_n^k, \boldsymbol{v})\,. \end{cases} \tag{2.16}$$

step 2: Galerkin approximation.
$\mathbb{V}_{\text{div}}$ is a separable Hilbert space. Hence one can find $\{\boldsymbol{\phi}_i\}_{i=1}^\infty$ an orthonormal basis of $\mathbb{V}_{\text{div}}$ such that

$$\overline{\{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, ..., \boldsymbol{\phi}_m, ...\}} = \mathbb{V}_{\text{div}}\ .$$

We let

$$\boldsymbol{W}_m = \{\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, ..., \boldsymbol{\phi}_m\}$$

the space generated by the indicated vectors. For each m, knowing $\boldsymbol{u}_{n-1}^{k,\varepsilon}$ and $\boldsymbol{f}_n^k$ one considers the Galerkin problem;

$$\begin{cases} \text{Find } \boldsymbol{u}_m^{k,\varepsilon} \in \boldsymbol{W}_m, \text{such that for all } \ \boldsymbol{v} \in \boldsymbol{W}_m, \\ (\boldsymbol{u}_m^{k,\varepsilon}, \boldsymbol{v}) + ka(\boldsymbol{u}_m^{k,\varepsilon}, \boldsymbol{v}) + kd(\boldsymbol{u}_m^{k,\varepsilon}, \boldsymbol{u}_m^{k,\varepsilon}, \boldsymbol{v}) + k\langle J_\varepsilon^{(1)}(\boldsymbol{u}_m^{k,\varepsilon}), \boldsymbol{v}\rangle \\ = (\boldsymbol{u}_{n-1}^{k,\varepsilon}, \boldsymbol{v}) + k(\boldsymbol{f}_n^k, \boldsymbol{v}) . \end{cases} \tag{2.17}$$

step 3: Brouwer fixed point (see [19], Chap IV, Corollary 1.1).

One considers the mapping $\Phi : \boldsymbol{W}_m \longrightarrow \boldsymbol{W}_m$ defined by

$$\begin{aligned} (\Phi(\boldsymbol{u})|\boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v}\rangle + ka(\boldsymbol{u}, \boldsymbol{v}) + kd(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + k\langle J_\varepsilon^{(1)}(\boldsymbol{u}), \boldsymbol{v}\rangle \\ - (\boldsymbol{u}_{n-1}^{k,\varepsilon}, \boldsymbol{v}) - k\langle \boldsymbol{f}_n^k, \boldsymbol{v}\rangle. \end{aligned} \tag{2.18}$$

Φ is continuous with the $\boldsymbol{H}^1(\Omega)$ norm. Indeed for $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{W}_m$, one has

$$\begin{aligned} (\Phi(\boldsymbol{u}_1) - \Phi(\boldsymbol{u}_2)|\boldsymbol{u}_1 - \boldsymbol{u}_2) \quad &= \quad \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2 + \nu k\|\varepsilon(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|^2 + kd(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \\ &\quad + k\langle J_\varepsilon^{(1)}(\boldsymbol{u}_1) - J_\varepsilon^{(1)}(\boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle \\ &\leq \quad \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1^2 + \nu k\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1^2 + kc\|\boldsymbol{u}_2\|_1\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1^2 \\ &\quad + k\langle J_\varepsilon^{(1)}(\boldsymbol{u}_1) - J_\varepsilon^{(1)}(\boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle. \end{aligned} \tag{2.19}$$

It is manifest that to close the inequality (2.19), we need to estimate from above the quantity $\langle J_\varepsilon^{(1)}(\boldsymbol{u}_1) - J_\varepsilon^{(1)}(\boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle$. We first note from the mean value theorem that

$$\langle J_\varepsilon^{(1)}(\boldsymbol{u}_1) - J_\varepsilon^{(1)}(\boldsymbol{u}_2), \boldsymbol{u}_1 - \boldsymbol{u}_2\rangle = \int_0^1 J_\varepsilon^{(2)}(\boldsymbol{u}_2 + \theta(\boldsymbol{u}_1 - \boldsymbol{u}_2)) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) d\theta \ . \tag{2.20}$$

One deduces from $J_\varepsilon^{(2)}$ (see (2.14)) that

$$\begin{aligned} J_\varepsilon^{(2)}(\boldsymbol{u}_2 + \theta(\boldsymbol{u}_1 - \boldsymbol{u}_2)) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) &\leq \frac{2}{\varepsilon} \int_S g|(\boldsymbol{u}_1 - \boldsymbol{u}_2)_{\boldsymbol{\tau}}|\,|(\boldsymbol{u}_1 - \boldsymbol{u}_2)_{\boldsymbol{\tau}}| \\ &\leq \frac{c}{\varepsilon}\|g\|_{L^\infty(S)}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1 \, . \end{aligned} \tag{2.21}$$

Finally returning to (2.19) with (2.20) and (2.21), we conclude that Φ is continuous.
Next, we need to find a constant r for which $(\Phi(\boldsymbol{v})|\boldsymbol{v})$ is positive outside the ball $B(0, r)$. for $\boldsymbol{v} \in \boldsymbol{W}_m$, one has

$$\begin{aligned} (\Phi(\boldsymbol{v})|\boldsymbol{v}) \quad &= \quad \|\boldsymbol{v}\|^2 + ka(\boldsymbol{v}, \boldsymbol{v}) + k\langle J_\varepsilon^{(1)}(\boldsymbol{v}), \boldsymbol{v}\rangle - (\boldsymbol{u}_{n-1}^{k,\varepsilon}, \boldsymbol{v}) - k\langle \boldsymbol{f}_n^k, \boldsymbol{v}\rangle \\ &\geq \quad \|\boldsymbol{v}\|^2 + \|\boldsymbol{v}\|_1 \left(\nu kc\|\boldsymbol{v}\|_1 - \frac{1}{c}\|\boldsymbol{u}_{n-1}^{k,\varepsilon}\| - \frac{k}{c}\|\boldsymbol{f}_n^k\| \right) \ . \end{aligned}$$

Let r be such that

$$r > \frac{\|\boldsymbol{u}_{n-1}^{k,\varepsilon}\| + k\|\boldsymbol{f}_n^k\|}{\nu kc}.$$

Then for $\boldsymbol{v} \in \boldsymbol{W}_m$ with $\|\boldsymbol{v}\|_1 = r$, $(\Phi(\boldsymbol{v})|\boldsymbol{v})$ is non-negative.
Thus Brouwer's fixed-point theorem is applicable and we deduce that, for each m, there are $\boldsymbol{u}_m^{k,\varepsilon} \in \boldsymbol{W}_m$ satisfying

$$\text{for all } \boldsymbol{v} \in \boldsymbol{W}_m, \qquad (\Phi(\boldsymbol{u}_m^{k,\varepsilon})|\boldsymbol{v}) = 0 \, . \tag{2.22}$$

We should bear in mind that (2.22) is also equivalent to (2.15) written in $\boldsymbol{W}_m$.

step 4: a priori estimates and passage to the limit.

We take $\boldsymbol{v} = \boldsymbol{u}_m^{k,\varepsilon}$ in (2.22) and obtain;

$$\|\boldsymbol{u}_m^{k,\varepsilon}\|^2 - \|\boldsymbol{u}_{n-1}^{k,\varepsilon}\|^2 + \|\boldsymbol{u}_m^{k,\varepsilon} - \boldsymbol{u}_{n-1}^{k,\varepsilon}\|^2 + 4\nu k\|\varepsilon(\boldsymbol{u}_m^{k,\varepsilon})\|^2 + 2k\langle J_\varepsilon^{(1)}(\boldsymbol{u}_m^{k,\varepsilon}), \boldsymbol{u}_m^{k,\varepsilon}\rangle = 2k(\boldsymbol{f}_n^k, \boldsymbol{u}_m^{k,\varepsilon}) \, .$$

By using the inequality (see (2.3), (2.4), Young's inequality and Remark 2.1)

$$2k(\boldsymbol{f}_n^k, \boldsymbol{u}_m^{k,\varepsilon}) \leq 2k\|\boldsymbol{f}_n^k\|\|\boldsymbol{u}_m^{k,\varepsilon}\| \leq 2ck\|\boldsymbol{f}_n^k\|\|\varepsilon(\boldsymbol{u}_m^{k,\varepsilon})\| \leq \frac{ck}{\nu}\|\boldsymbol{f}\|^2_{L^\infty(t_{n-1},t_n;L^2)} + \nu k\|\varepsilon(\boldsymbol{u}_m^{k,\varepsilon})\|^2$$

we deduce the relation

$$\begin{aligned}&\|\boldsymbol{u}_m^{k,\varepsilon}\|^2 - \|\boldsymbol{u}_{n-1}^{k,\varepsilon}\|^2 + \|\boldsymbol{u}_m^{k,\varepsilon} - \boldsymbol{u}_{n-1}^{k,\varepsilon}\|^2 + 3\nu k\|\varepsilon(\boldsymbol{u}_m^{k,\varepsilon})\|^2 + 2k\langle J_\varepsilon^{(1)}(\boldsymbol{u}_m^{k,\varepsilon}), \boldsymbol{u}_m^{k,\varepsilon}\rangle \\ &\leq \frac{ck}{\nu}\|\boldsymbol{f}\|^2_{L^\infty(t_{n-1},t_n;L^2)} \ .\end{aligned} \tag{2.23}$$

Since $J_\varepsilon^{(1)}$ is monotone, $\langle J_\varepsilon^{(1)}(\boldsymbol{u}_m^\varepsilon), \boldsymbol{u}_m^\varepsilon\rangle$ is non-negative. We then deduce that $\boldsymbol{u}_m^\varepsilon$ is bounded in $\boldsymbol{H}^1(\Omega)$ by a constant c independent of m and ε. Now owing to the compact embedding of $H^1(\Omega)$ into $L^4(\Omega)$, there exists a subsequence, still denoted by $(\boldsymbol{u}_m^\varepsilon)_m$ for convenience, which converges to $\boldsymbol{u}^\varepsilon$ weakly in $\boldsymbol{H}^1(\Omega)$ and strongly in $\boldsymbol{L}^4(\Omega)$. With equation (2.22) in mind, passing to the limit on m is obvious for linear expression, while for the nonlinear expression $(\boldsymbol{u}_m^\varepsilon \cdot \nabla)\boldsymbol{u}_m^\varepsilon$ it suffice to use the strong convergence in $\boldsymbol{L}^4(\Omega)$ (see [11, 17, 18]). It should be noted that the penalized pressure p^ε is constructed by following [19]. So the proof for the existence of solutions of (2.12) is complete. □

Proposition 2.1. *Let $\boldsymbol{f} \in L^\infty(0,T;L^2)$, $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\Omega)$ and ν such that*

$$\nu^4 - \nu^2\,\|\boldsymbol{u}_0\|^2 - \,\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2)} > 0\,.$$

There exists c independent of k such that if

$$k < \frac{\nu^4 - \nu^2\|\boldsymbol{u}_0\|^2 - \|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2)}}{c\nu\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2)}}$$

is valid, then the problem (2.12) has a unique solution.

Proof. The proof is standard and proceed as follows. Let $\boldsymbol{u}_n^k, \boldsymbol{w}_n^k$ be two solutions of problem (2.12). Then

$$\left\langle \frac{\boldsymbol{u}_n^k - \boldsymbol{u}_{n-1}^k}{k}, \boldsymbol{w}_n^k - \boldsymbol{u}_n^k\right\rangle + a(\boldsymbol{u}_n^k, \boldsymbol{w}_n^k - \boldsymbol{u}_n^k) + d(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \boldsymbol{w}_n^k - \boldsymbol{u}_n^k) + J(\boldsymbol{w}_n^k) - J(\boldsymbol{u}_n^k) \geq \langle \boldsymbol{f}_n^k, \boldsymbol{w}_n^k - \boldsymbol{u}_n^k\rangle,$$

and

$$-\left\langle \frac{\boldsymbol{w}_n^k - \boldsymbol{u}_{n-1}^k}{k}, \boldsymbol{w}_n^k - \boldsymbol{u}_n^k\right\rangle - a(\boldsymbol{w}_n^k, \boldsymbol{w}_n^k - \boldsymbol{u}_n^k) - d(\boldsymbol{w}_n^k, \boldsymbol{w}_n^k, \boldsymbol{w}_n^k - \boldsymbol{u}_n^k) + J(\boldsymbol{u}_n^k) - J(\boldsymbol{w}_n^k) \geq -\langle \boldsymbol{f}_n^k, \boldsymbol{w}_n^k - \boldsymbol{u}_n^k\rangle.$$

Adding these relations and using (2.7), (2.8), Korn's inequality and Young's inequality one obtains

$$\begin{aligned}\frac{1}{4}\|\boldsymbol{w}_n^k - \boldsymbol{u}_n^k\|^2 + \nu k\|\varepsilon(\boldsymbol{w}_n^k - \boldsymbol{u}_n^k)\|^2 \quad &\leq \quad \frac{k}{2}d(\boldsymbol{w}_n^k - \boldsymbol{u}_n^k, \boldsymbol{w}_n^k - \boldsymbol{u}_n^k, \boldsymbol{u}_n^k) \\ &\leq \quad \frac{1}{4}\|\boldsymbol{w}_n^k - \boldsymbol{u}_n^k\|^2 + c\|\varepsilon(\boldsymbol{w}_n^k - \boldsymbol{u}_n^k)\|^2\|\varepsilon(\boldsymbol{u}_n^k)\|^2 k^2\end{aligned}$$

which is re-written as follows

$$\nu\|\varepsilon(\boldsymbol{w}_n^k - \boldsymbol{u}_n^k)\|^2 \leq c\|\varepsilon(\boldsymbol{w}_n^k - \boldsymbol{u}_n^k)\|^2\|\varepsilon(\boldsymbol{u}_n^k)\|^2 k \tag{2.24}$$

where c is a constant independent of ν and k. It is then manifest that to close the inequality (2.24), we need to estimate $\|\varepsilon(\boldsymbol{u}_n^k)\|$. For that purpose, letting $\boldsymbol{v} = \boldsymbol{0}$ in (2.12), and following to the line the process of getting (2.23), we find

$$\|\boldsymbol{u}_n^k\|^2 + \|\boldsymbol{u}_n^k - \boldsymbol{u}_{n-1}^k\|^2 + k\nu\|\varepsilon(\boldsymbol{u}_n^k)\|^2 + 2kJ(\boldsymbol{u}_n^k) \leq \|\boldsymbol{u}_{n-1}^k\|^2 + c\frac{k}{\nu}\|\boldsymbol{f}_n^k\|^2. \tag{2.25}$$

Dropping some positive expressions, applying (2.3) and (2.4), (2.25) implies

$$\|\boldsymbol{u}_n^k\|^2 \leq \frac{1}{1+\nu kc}\|\boldsymbol{u}_{n-1}^k\|^2 + \frac{c\,k}{\nu(1+\nu kc)}\|\boldsymbol{f}_n^k\|^2, \tag{2.26}$$

which by induction on n leads to

$$\begin{aligned}\|\boldsymbol{u}_n^k\|^2 &\leq \frac{1}{(1+\nu kc)^n}\|\boldsymbol{u}_0\|^2 + c\frac{k}{\nu}\sum_{i=1}^{n}\frac{1}{(1+\nu kc)^i}\|\boldsymbol{f}_{n+1-i}^k\|^2 \\ &\leq \|\boldsymbol{u}_0\|^2 + \frac{c}{\nu^2}\,\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2)}. \end{aligned} \tag{2.27}$$

Now, replacing (2.27) in (2.25) and dropping some positive terms, it holds that

$$\begin{aligned}k\|\varepsilon(\boldsymbol{u}_n^k)\|^2 &\leq \frac{1}{\nu}\,\|\boldsymbol{u}_0\|^2 + \frac{c}{\nu^3}\,\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2)} + c\frac{k}{\nu^2}\|\boldsymbol{f}_n^k\|^2 \\ &\leq \frac{1}{\nu}\,\|\boldsymbol{u}_0\|^2 + \frac{c}{\nu^3}\,\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2)} + k\frac{c}{\nu^2}\,\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2)}\,. \end{aligned} \tag{2.28}$$

Replacing (2.28) in (2.24), we find

$$\Big[\nu - \frac{c}{\nu}\,\|\boldsymbol{u}_0\|^2 - \frac{c}{\nu^3}\,\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2)} - \frac{c}{\nu^2}\,k\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2)}\Big]\,\|\varepsilon(\boldsymbol{w}_n^k - \boldsymbol{u}_n^k)\|^2 \leq 0\,.$$

Clearly for unique solvability, we request that

$$\nu - \frac{c}{\nu}\,\|\boldsymbol{u}_0\|^2 - \frac{c}{\nu^3}\,\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2)} - \frac{c}{\nu^2}\,k\|\boldsymbol{f}\|^2_{L^\infty(0,T;L^2)} \geq 0\ .$$

So the proof is complete. □

3 Some A priori results

This section is a preparatory one for analysing a priori error in the next Section. We are interested here in deriving some upper bounds of the solution of the time discrete model. To fix the notation, for each fixed k, we associate in general to $\boldsymbol{\psi}_n^k$ the following approximate functions $\boldsymbol{\psi}_k$ and $\widehat{\boldsymbol{\psi}}_k$ defined as follows for $n = 1, 2, 3, ..., N$,

$$\begin{aligned} \boldsymbol{\psi}_k :\quad & [0,T] \longrightarrow \quad \mathbb{V}_{\text{div}} \\ & \quad t \longrightarrow \quad \boldsymbol{\psi}_k(t) = \boldsymbol{\psi}_n^k\ \ ,\ t \in [(n-1)k, nk], \\ \widehat{\boldsymbol{\psi}}_k :\quad & [0,T] \longrightarrow \quad \mathbb{H} \\ & \quad t \longrightarrow \quad \widehat{\boldsymbol{\psi}}_k(t) = \frac{t - t_{n-1}}{k}\boldsymbol{\psi}_n^k + \frac{t_n - t}{k}\boldsymbol{\psi}_{n-1}^k\ \ ,\ t \in [(n-1)k, nk]. \end{aligned} \tag{3.1}$$

We recall that

$$\|\boldsymbol{\psi}_k\|^2_{L^2(0,T;L^2)} = k\sum_{n=1}^{N}\|\boldsymbol{\psi}_n^k\|^2\ \ ,\qquad \|\boldsymbol{\psi}_k\|_{L^\infty(0,T;L^2)} = \sup_{1\leq n\leq N}\|\boldsymbol{\psi}_n^k\|\,. \tag{3.2}$$

We note that for

$$\delta\boldsymbol{\psi}_n^k = \frac{\boldsymbol{\psi}_n^k - \boldsymbol{\psi}_{n-1}^k}{k} \quad \text{for } n = 1, 2, 3, \cdots, N, \quad \text{and} \quad \delta\boldsymbol{\psi}_0^k = 0\,,$$

the discrete derivative in time is

$$\begin{aligned} &\widehat{\boldsymbol{\psi}}_k'(t) = \delta\boldsymbol{\psi}_n^k\ , \\ &\widehat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k = (t - t_n)\widehat{\boldsymbol{\psi}}_k' = (t - t_n)\delta\boldsymbol{\psi}_k\ , \\ &\langle\widehat{\boldsymbol{\psi}}_k', \widehat{\boldsymbol{\psi}}_k - \boldsymbol{\psi}_k\rangle = (t - t_n)\|\widehat{\boldsymbol{\psi}}_k'\|^2\,. \end{aligned} \tag{3.3}$$

In the rest of the text, we will adopt the notation $L^p(L^q)$ for $L^p(0,T;L^q)$, $L^p(H^1)$ for $L^p(0,T;H^1)$, etc.....

We next collect useful stability results. We first claim that

Lemma 3.1. *Assume that $\boldsymbol{f} \in L^\infty(\boldsymbol{L}^2)$. Let $(\boldsymbol{u}_n^k)_n$ the sequence defined through (2.12). Then there is c independent of k such that*

$$\|\boldsymbol{u}_k\|^2_{L^\infty(L^2)} + \nu\|\nabla\boldsymbol{u}_k\|^2_{L^2(L^2)} + k\|\widehat{\boldsymbol{u}}_k'\|^2_{L^2(L^2)} \leq c\left[\|\boldsymbol{u}_0\|^2 + T\|\boldsymbol{f}\|^2_{L^\infty(L^2)}\right]\ .$$

Proof. We take successively $\boldsymbol{v} = \mathbf{0}$, and $\boldsymbol{v} = 2\boldsymbol{u}_n^k$ in (2.12). Adding the resulting equations, one obtains

$$\|\boldsymbol{u}_n^k\|^2 - \|\boldsymbol{u}_{n-1}^k\|^2 + \|\boldsymbol{u}_n^k - \boldsymbol{u}_{n-1}^k\|^2 + 4k\nu\|\varepsilon(\boldsymbol{u}_n^k)\|^2 + 2kJ(\boldsymbol{u}_n^k) = 2k(\boldsymbol{f}_n^k, \boldsymbol{u}_n^k)\ . \tag{3.4}$$

Summing (3.4) for $n = 1, \cdots, m \leq N$, we find

$$\begin{aligned}\|\boldsymbol{u}_m^k\|^2 + k^2\sum_{n=1}^{m}\|\delta\boldsymbol{u}_n^k\|^2 + 4k\nu\sum_{n=1}^{m}\|\varepsilon(\boldsymbol{u}_n^k)\|^2 + 2k\sum_{n=1}^{m}J(\boldsymbol{u}_n^k)\\ = \|\boldsymbol{u}_0\|^2 + 2k\sum_{n=1}^{m}(\boldsymbol{f}_n^k, \boldsymbol{u}_n^k),\end{aligned} \tag{3.5}$$

which implies that

$$\begin{aligned}\sup_{1\leq m\leq N}\|\boldsymbol{u}_m^k\|^2 + k^2\sum_{n=1}^{N}\|\delta\boldsymbol{u}_n^k\|^2 + 4k\nu\sum_{n=1}^{N}\|\varepsilon(\boldsymbol{u}_n^k)\|^2 + 2k\sum_{n=1}^{N}J(\boldsymbol{u}_n^k)\\ \leq \|\boldsymbol{u}_0^k\|^2 + 2k\sum_{n=1}^{N}(\boldsymbol{f}_n^k, \boldsymbol{u}_n^k)\ .\end{aligned} \tag{3.6}$$

Next from Cauchy-Schwarz, (2.4), Remark 2.1 and Young's inequality

$$\begin{aligned}2k\sum_{n=1}^{N}(\boldsymbol{f}_n^k, \boldsymbol{u}_n^k) \leq 2k\sum_{n=1}^{N}\|\boldsymbol{f}_n^k\|\|\boldsymbol{u}_n^k\| &\leq \frac{2k}{c}\sum_{n=1}^{N}\|\boldsymbol{f}_n^k\|\|\nabla\boldsymbol{u}_n^k\|\\ &\leq \frac{k}{\epsilon c}\sum_{n=1}^{N}\|\boldsymbol{f}_n^k\|^2 + k\epsilon\sum_{n=1}^{N}\|\nabla\boldsymbol{u}_n^k\|^2\\ &\leq \frac{T}{\epsilon c}\|\boldsymbol{f}\|^2_{L^\infty(L^2)} + \epsilon\|\nabla\boldsymbol{u}_k\|^2_{L^2(L^2)}\ .\end{aligned} \tag{3.7}$$

Returning to (3.6) with (3.7) and appropriate choice of ϵ, and having in mind (3.2), one obtains the result announced. □

To present the next result, we define the function $\widetilde{\boldsymbol{\psi}}_k$ as

$$\widetilde{\boldsymbol{\psi}}_k(t) = \frac{1}{k}\int_t^{t+k}\widehat{\boldsymbol{\psi}}_k(s)ds. \tag{3.8}$$

The chain's rule formula leads to

$$\widetilde{\boldsymbol{\psi}}_k'(t) = \frac{\widehat{\boldsymbol{\psi}}_k(t+k) - \widehat{\boldsymbol{\psi}}_k(t)}{k}\ ,\ \widetilde{\boldsymbol{\psi}}_k''(t) = \frac{\widehat{\boldsymbol{\psi}}_k'(t+k) - \widehat{\boldsymbol{\psi}}_k'(t)}{k}. \tag{3.9}$$

We claim that

Lemma 3.2. *Let $(\boldsymbol{u}_n^k)_n$ be the sequence defined through (2.12). If the conditions on proposition 2.1 are valid, and $\boldsymbol{f}' \in L^\infty(\boldsymbol{L}^2)$. Then there exists a constant c independent of k such that for all $n \geq 1$,*

$$\|\widehat{\boldsymbol{u}}_k'\|^2_{L^\infty(L^2)} + c\beta k\|\nabla\widehat{\boldsymbol{u}}_k'\|^2_{L^2(L^2)} + k\|\widetilde{\boldsymbol{u}}_k''\|^2_{L^2(L^2)} \leq c\frac{T}{\nu}\|\boldsymbol{f}'\|^2_{L^\infty(L^2)},$$

with

$$\beta = \left(\nu - c\left(\frac{1}{\nu k}\|\boldsymbol{u}_0\|^2 + \frac{1}{k\nu^3}\|\boldsymbol{f}\|^2_{L^\infty(L^2)} + \frac{1}{\nu^2}\|\boldsymbol{f}\|^2_{L^\infty(L^2)}\right)^{1/2}\right) > 0\ . \tag{3.10}$$

Proof. Take $\boldsymbol{v} = \boldsymbol{u}_{n+1}^k$ in (2.12) gives

$$\begin{aligned}-\left(\delta\boldsymbol{u}_n^k, \delta\boldsymbol{u}_{n+1}^k\right) - a(\boldsymbol{u}_n^k, \delta\boldsymbol{u}_{n+1}^k) - d(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \delta\boldsymbol{u}_{n+1}^k) + \frac{J(\boldsymbol{u}_n^k) - J(\boldsymbol{u}_{n+1}^k)}{k}\\ \leq -(\boldsymbol{f}_n^k, \delta\boldsymbol{u}_{n+1}^k)\,.\end{aligned} \tag{3.11}$$

Writing (2.12) at the time step t_{n+1}, and taking in the resulting equation $\boldsymbol{v} = \boldsymbol{u}_n^k$, one obtains

$$\begin{aligned}(\delta\boldsymbol{u}_{n+1}^k, \delta\boldsymbol{u}_{n+1}^k) + a(\boldsymbol{u}_{n+1}^k, \delta\boldsymbol{u}_{n+1}^k) + d(\boldsymbol{u}_{n+1}^k, \boldsymbol{u}_{n+1}^k, \delta\boldsymbol{u}_{n+1}^k) + \frac{J(\boldsymbol{u}_{n+1}^k) - J(\boldsymbol{u}_n^k)}{k} \\ \leq (\boldsymbol{f}_{n+1}^k, \delta\boldsymbol{u}_{n+1}^k)\ .\end{aligned} \tag{3.12}$$

We do 2×(3.11)+2×(3.12) and obtain

$$\begin{aligned}&2k\left(\delta\boldsymbol{u}_{n+1}^k - \delta\boldsymbol{u}_n^k, \delta\boldsymbol{u}_{n+1}^k\right) + 2k^2 a(\delta\boldsymbol{u}_{n+1}^k, \delta\boldsymbol{u}_{n+1}^k) \\ &\leq -k\left(d(\boldsymbol{u}_{n+1}^k, \boldsymbol{u}_{n+1}^k, \delta\boldsymbol{u}_{n+1}^k) - d(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \delta\boldsymbol{u}_{n+1}^k)\right) + 2k^2(\delta\boldsymbol{f}_{n+1}^k, \delta\boldsymbol{u}_{n+1}^k).\end{aligned} \tag{3.13}$$

But

$$2k\left(\delta\boldsymbol{u}_{n+1}^k - \delta\boldsymbol{u}_n^k, \delta\boldsymbol{u}_{n+1}^k\right) = k\|\delta\boldsymbol{u}_{n+1}^k\|^2 - k\|\delta\boldsymbol{u}_n^k\|^2 + k^3\|\delta^2\boldsymbol{u}_{n+1}^k\|^2, \tag{3.14}$$

and

$$\begin{aligned}d(\boldsymbol{u}_{n+1}^k, \boldsymbol{u}_{n+1}^k, \delta\boldsymbol{u}_{n+1}^k) - d(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \delta\boldsymbol{u}_{n+1}^k) &= kd(\delta\boldsymbol{u}_{n+1}^k, \boldsymbol{u}_n^k, \delta\boldsymbol{u}_{n+1}^k) \\ &\leq kc\|\varepsilon(\delta\boldsymbol{u}_{n+1}^k)\|^2\,\|\varepsilon(\boldsymbol{u}_n^k)\|\,.\end{aligned} \tag{3.15}$$

Replacing (3.14), (3.15) and (2.28) in (3.13) and utilization of Holder's inequality, Korn's inequality and Young's inequality leads to

$$\|\delta\boldsymbol{u}_{n+1}^k\|^2 - \|\delta\boldsymbol{u}_n^k\|^2 + k^2\|\delta^2\boldsymbol{u}_{n+1}^k\|^2 + c\,k\beta\|\varepsilon(\delta\boldsymbol{u}_{n+1}^k)\|^2 \leq c\frac{k}{\nu}\|\delta\boldsymbol{f}_{n+1}^k\|^2\,. \tag{3.16}$$

The quantity β is non-negative from the uniqueness condition, so adding (3.16) for $n = 0, 1, 2, \cdots, m-1$ gives

$$\|\delta\boldsymbol{u}_m^k\|^2 + k^2\sum_{n=1}^{m}\|\delta^2\boldsymbol{u}_n^k\|^2 + c\beta k\sum_{n=1}^{m}\|\varepsilon(\delta\boldsymbol{u}_n^k)\|^2 \quad\leq\quad c\frac{k}{\nu}\sum_{n=1}^{m}\|\delta\boldsymbol{f}_n^k\|^2 \leq c\frac{T}{\nu}\|\boldsymbol{f}'\|^2_{L^\infty(L^2)}\,.$$

The result announced follows after utilization of (3.1), (3.2), (3.9) and (2.3). □

4 Semi discrete problem in time: a priori error analysis

This section is inspired by work in [1, 7], and constitute a key contribution in this study. We derive a priori error estimates between the solutions of continuous and approximate problems by maintaining the same regularity of the solution of the continuous problem but requiring moderate regularity of the forcing term $\boldsymbol{f}$. We do not make use of the consistency error which is responsible of assuming higher order differentiability in time of the weak solution. The convergence results one obtains are first order with respect to the discretization parameter. Using similar techniques, S. Bartels in [6] had obtained first order convergence in time for problems in plasticity, and J.K. Djoko and J.M. Lubuma in [9] had derived first order convergence in time for Oseen equations driven by nonlinear slip boundary conditions. The work we present here is more complicated than those discussed in [6, 9]. Indeed, the convective term of the Navier Stokes equations together with the incompressibility condition and the relate unknown pressure bring significant difficulties in the analysis of the present model, moreover in [9], only the velocity field is treated but in the present work, we go one step further by analysing the pressure which is made possible by first estimating the time derivative of the velocity field.

4.1 A priori error on the velocity

The goal here is to estimate the quantity $\boldsymbol{u} - \widehat{\boldsymbol{u}}_k$. With the preliminaries results Lemma 3.1 and Lemma 3.2 in place, we claim that

Proposition 4.1. *Assume that $\boldsymbol{f} \in W^{1,\infty}(\boldsymbol{L}^2)$, and assume that the conditions on proposition 2.1 are valid. Let $\boldsymbol{u}$ be the solution of (2.10), and $(\boldsymbol{u}_k^n)_n$ be the sequence of defined through (2.12). Then there is c independent of k such that*

$$\begin{aligned}
&\sum_{n=1}^{N} \|\boldsymbol{u}(t_n) - \widehat{\boldsymbol{u}}_k(t_n)\|^2 + c\beta \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\nabla(\boldsymbol{u} - \widehat{\boldsymbol{u}}_k)\|^2 \\
\leq \quad & c\, k^2 \left\{ \frac{\nu}{\beta} + \frac{1}{\beta\nu}(\nu - \beta)^2 \right\} \frac{T}{\nu} \|\boldsymbol{f}'\|^2_{L^\infty(L^2)} + \frac{c}{\nu} k^2 \|\boldsymbol{f}'\|^2_{L^2(L^2)} + 2k^2 \sum_{n=1}^{N} \mathcal{E}_n^k,
\end{aligned}$$

with

$$-\mathcal{E}_n^k = (\widehat{\boldsymbol{u}}'_k, \delta \boldsymbol{u}_n^k) + a(\boldsymbol{u}_k, \delta \boldsymbol{u}_n^k) + d(\boldsymbol{u}_k, \boldsymbol{u}_k, \delta \boldsymbol{u}_n^k) + \delta J(\boldsymbol{u}_n^k) - (\boldsymbol{f}_k, \delta \boldsymbol{u}_n^k)\ . \tag{4.1}$$

Remark 4.1. *The analysis we embark on next bears similarities to that of Djoko and Lubuma [9] (see Theorem 4.1), the essential difference being that the non-linear convective term is introduced here. But, it is worth mentioning that the presence of the convective term makes the analysis more delicate and for completeness we have include the computations here.*

Proof. The proof is done in several steps.
Step 1: evolution equation for $\widehat{\boldsymbol{u}}'_k$.
We recall that (2.12) can be re-written as follows

$$(\widehat{\boldsymbol{u}}'_k, \boldsymbol{u}_k - \boldsymbol{v}) + a(\boldsymbol{u}_k, \boldsymbol{u}_k - \boldsymbol{v}) + d(\boldsymbol{u}_k, \boldsymbol{u}_k, \boldsymbol{u}_k - \boldsymbol{v}) + J(\boldsymbol{u}_k) - J(\boldsymbol{v}) \leq (\boldsymbol{f}_k, \boldsymbol{u}_k - \boldsymbol{v}). \tag{4.2}$$

But by linearity

$$\begin{aligned}
(\widehat{\boldsymbol{u}}'_k, \boldsymbol{u}_k - \boldsymbol{v}) =& (t_n - t)(\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}'_k) + (\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}), \\
a(\boldsymbol{u}_k, \boldsymbol{u}_k - \boldsymbol{v}) =& (t_n - t) a(\widehat{\boldsymbol{u}}'_k, \boldsymbol{u}_k - \boldsymbol{v}) + (t_n - t) a(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}'_k) + a(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}), \\
d(\boldsymbol{u}_k, \boldsymbol{u}_k, \boldsymbol{u}_k - \boldsymbol{v}) =& (t_n - t) d(\boldsymbol{u}_k, \boldsymbol{u}_k, \widehat{\boldsymbol{u}}'_k) + (t_n - t) d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) \\
& + (t_n - t) d(\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) + d(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}).
\end{aligned} \tag{4.3}$$

Returning to (4.2) with (4.3) we find

$$\begin{aligned}
&(\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) + a(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) + d(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) + J(\widehat{\boldsymbol{u}}_k) - J(\boldsymbol{v}) \\
&\qquad \leq (\boldsymbol{f}_k, \boldsymbol{u}_k - \boldsymbol{v}) + (t - t_n)(\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}'_k) + J(\widehat{\boldsymbol{u}}_k) - J(\boldsymbol{u}_k) + (t - t_n) a(\widehat{\boldsymbol{u}}'_k, \boldsymbol{u}_k - \boldsymbol{v}) \\
&\qquad\quad + (t - t_n) a(\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{u}_k) + (t - t_n) a(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}'_k) + (t - t_n) d(\boldsymbol{u}_k, \boldsymbol{u}_k, \widehat{\boldsymbol{u}}'_k) \\
&\qquad\quad + (t - t_n) d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) + (t - t_n) d(\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v})\ .
\end{aligned} \tag{4.4}$$

The fact that $J(\cdot)$ is convex implies that

$$J(\widehat{\boldsymbol{u}}_k) - J(\boldsymbol{u}_n^k) \leq \frac{t - t_{n-1}}{k} J(\boldsymbol{u}_n^k) + \frac{t_n - t}{k} J(\boldsymbol{u}_{n-1}^k) - J(\boldsymbol{u}_n^k) = (t - t_n)\delta J(\boldsymbol{u}_n^k). \tag{4.5}$$

We replace (4.5) in (4.4) and obtain

$$\begin{aligned}
&(\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) + a(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) + d(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) + J(\widehat{\boldsymbol{u}}_k) - J(\boldsymbol{v}) - (\boldsymbol{f}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) \\
\leq &(t - t_n)\left[a(\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) + d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}) + d(\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v})\right] - (t - t_n)\mathcal{E}_n^k\,.
\end{aligned} \tag{4.6}$$

Step 2: Evolution equation for $\widehat{\boldsymbol{u}}'_k - \boldsymbol{u}$.

We take $\boldsymbol{v} = \widehat{\boldsymbol{u}}_k$ in (2.10), and $\boldsymbol{v} = \boldsymbol{u}(t)$ in (4.6). One obtains

$$(\boldsymbol{u}', \boldsymbol{u} - \widehat{\boldsymbol{u}}_k) + a(\boldsymbol{u}, \boldsymbol{u} - \widehat{\boldsymbol{u}}_k) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u} - \widehat{\boldsymbol{u}}_k) + J(\boldsymbol{u}) - J(\widehat{\boldsymbol{u}}_k) \leq (\boldsymbol{f}, \boldsymbol{u} - \widehat{\boldsymbol{u}}_k), \tag{4.7}$$

and

$$\begin{aligned}
&-(\widehat{\boldsymbol{u}}'_k, \boldsymbol{u} - \widehat{\boldsymbol{u}}_k) - a(\widehat{\boldsymbol{u}}_k, \boldsymbol{u} - \widehat{\boldsymbol{u}}_k) - d(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k, \boldsymbol{u} - \widehat{\boldsymbol{u}}_k) + J(\widehat{\boldsymbol{u}}_k) - J(\boldsymbol{u}) \\
&\qquad \leq (t - t_n)\left[a(\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) + d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) + d(\widehat{\boldsymbol{u}}'_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{u})\right] \\
&\qquad\quad - (t - t_n)\mathcal{E}_n^k - (\boldsymbol{f}_k, \boldsymbol{u} - \widehat{\boldsymbol{u}}_k)\,.
\end{aligned} \tag{4.8}$$

Putting together (4.7) and (4.8), we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}-\widehat{\boldsymbol{u}}_k\|^2 + a(\boldsymbol{u}-\widehat{\boldsymbol{u}}_k, \boldsymbol{u}-\widehat{\boldsymbol{u}}_k) \qquad (4.9)\\
\leq\ & d(\widehat{\boldsymbol{u}}_k-\boldsymbol{u}, \widehat{\boldsymbol{u}}_k, \boldsymbol{u}-\widehat{\boldsymbol{u}}_k) + (t-t_n)a(\delta\boldsymbol{u}_n^k, \widehat{\boldsymbol{u}}_k-\boldsymbol{u}) + (t-t_n)d(\boldsymbol{u}_k, \delta\boldsymbol{u}_n^k, \widehat{\boldsymbol{u}}_k-\boldsymbol{u})\\
+\ & (t-t_n)d(\delta\boldsymbol{u}_n^k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k-\boldsymbol{u}) + (\boldsymbol{f}-\boldsymbol{f}_k, \boldsymbol{u}-\widehat{\boldsymbol{u}}_k) + (t_n-t)\mathcal{E}_n^k.
\end{aligned}$$

Now, it remains to bound the terms on the right hand side of (4.9). For that purpose, we first take $\boldsymbol{v}=\boldsymbol{u}_{n-1}^k$ in (2.12). That is

$$(\widehat{\boldsymbol{u}}_k', \delta\boldsymbol{u}_n^k) + a(\boldsymbol{u}_n^k, \delta\boldsymbol{u}_n^k) + d(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \delta\boldsymbol{u}_n^k) + \delta J(\boldsymbol{u}_n^k) - (\boldsymbol{f}_n^k, \delta\boldsymbol{u}_n^k) \leq 0.$$

Hence $\mathcal{E}_n^k$ defined in (4.1) is non-negative. Using the continuity of $a(\cdot,\cdot)$ and the properties of $d(\cdot,\cdot,\cdot)$ (see (2.5) and (2.8)), one obtains

$$\begin{aligned}
&\frac{d}{dt}\|\boldsymbol{u}-\widehat{\boldsymbol{u}}_k\|^2 + 2\nu c\|\nabla(\boldsymbol{u}-\widehat{\boldsymbol{u}}_k)\|^2\\
\leq\ & 2c\|\nabla\widehat{\boldsymbol{u}}_k\|\|\nabla(\boldsymbol{u}-\widehat{\boldsymbol{u}}_k)\|^2 + 2k\nu\|\nabla\widehat{\boldsymbol{u}}_k'\|\|\nabla(\widehat{\boldsymbol{u}}_k-\boldsymbol{u})\|\\
& +2kc\|\nabla\boldsymbol{u}_k\|\|\nabla\widehat{\boldsymbol{u}}_k'\|\|\nabla(\widehat{\boldsymbol{u}}_k-\boldsymbol{u})\| + 2\|\boldsymbol{f}-\boldsymbol{f}_k\|\|\boldsymbol{u}-\widehat{\boldsymbol{u}}_k\| + 2k\mathcal{E}_n^k.\\
\leq\ & 2c\|\varepsilon(\boldsymbol{u}_n^k)\|\|\nabla(\boldsymbol{u}-\widehat{\boldsymbol{u}}_k)\|^2 + \frac{3\nu k^2}{c}\|\nabla\widehat{\boldsymbol{u}}_k'\|^2 + \frac{\nu c}{3}\|\nabla(\widehat{\boldsymbol{u}}_k-\boldsymbol{u})\|^2 + \frac{k^2}{\nu}c\|\varepsilon(\boldsymbol{u}_n^k)\|^2\|\nabla\widehat{\boldsymbol{u}}_k'\|^2\\
& +\frac{\nu c}{3}\|\nabla(\widehat{\boldsymbol{u}}_k-\boldsymbol{u})\|^2 + \frac{3}{\nu c}\|\boldsymbol{f}-\boldsymbol{f}_k\|^2 + \frac{\nu c}{3}\|\nabla(\boldsymbol{u}-\widehat{\boldsymbol{u}}_k)\|^2 + 2k\mathcal{E}_n^k
\end{aligned}$$

which together with (2.28) and the definition of β (see (3.10)) gives

$$\begin{aligned}
\frac{d}{dt}\|\boldsymbol{u}-\widehat{\boldsymbol{u}}_k\|^2 &+ c\beta\|\nabla(\boldsymbol{u}-\widehat{\boldsymbol{u}}_k)\|^2\\
&\leq c\nu k^2\|\nabla\widehat{\boldsymbol{u}}_k'\|^2 + \frac{c}{\nu}k^2(\nu-\beta)^2\|\nabla\widehat{\boldsymbol{u}}_k'\|^2 + \frac{c}{\nu}\|\boldsymbol{f}-\boldsymbol{f}_k\|^2 + 2k\mathcal{E}_n^k\ . \qquad (4.10)
\end{aligned}$$

Step 3: Resolution of (4.10).

Integration of (4.10) in (t_{n-1}, t_n), and take the summation for $n=1,2,3,\cdots,N$ gives

$$\begin{aligned}
&\sum_{n=1}^N \|\boldsymbol{u}-\widehat{\boldsymbol{u}}_k\|^2 + c\beta\sum_{n=1}^N\int_{t_{n-1}}^{t_n} \|\nabla(\boldsymbol{u}-\widehat{\boldsymbol{u}}_k)\|^2 \qquad (4.11)\\
\leq\ & k^2 c\left\{\nu + \frac{1}{\nu}(\nu-\beta)^2\right\}\|\nabla\widehat{\boldsymbol{u}}_k'\|^2_{L^2(0,T;L^2)} + \frac{c}{\nu}\|\boldsymbol{f}-\boldsymbol{f}_k\|^2_{L^2(0,T;L^2)} + 2k^2\sum_{n=1}^N\mathcal{E}_n^k.
\end{aligned}$$

The error estimate announced is obtained after application of Proposition 3.2, and

$$\|\boldsymbol{f}_n^k-\boldsymbol{f}\|_{L^2(t_{n-1},t_n;L^2)} \leq c\,k\|\boldsymbol{f}'\|_{L^2(t_{n-1},t_n;L^2)}.$$

□

It is manifest that to close the a priori error in Theorem 4.1, one needs to estimate $\sum_{i=1}^N \mathcal{E}_n$. For that purpose, we claim that

Lemma 4.1. *Assume that $\boldsymbol{f}\in W^{1,\infty}(\boldsymbol{L}^2)$, and the conditions of proposition 2.1 are valid. Let $(\boldsymbol{u}_k^n)_n$ be the sequence of defined through (2.12). Then there is c, independent of k such that*

$$\sum_{n=1}^N \mathcal{E}_n^k \quad\leq\quad c\frac{T}{\nu}\|\boldsymbol{f}'\|^2_{L^\infty(L^2)}\ .$$

Proof. We recall that (4.1) can be re-written as

$$\mathcal{E}_n^k = -(\delta\boldsymbol{u}_n^k, \delta\boldsymbol{u}_n^k) - a(\boldsymbol{u}_n^k, \delta\boldsymbol{u}_n^k) - d(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \delta\boldsymbol{u}_n^k) - \delta J(\boldsymbol{u}_n^k) + (\boldsymbol{f}_n^k, \delta\boldsymbol{u}_n^k). \tag{4.12}$$

We take the equation (2.12) at the time t_{n-1} and replace $\boldsymbol{v}$ by $\boldsymbol{u}_n^k$. We find

$$-\delta J(\boldsymbol{u}_n^k) \leq (\delta\boldsymbol{u}_{n-1}^k, \delta\boldsymbol{u}_n^k) + a(\boldsymbol{u}_{n-1}^k, \delta\boldsymbol{u}_n^k) + d(\boldsymbol{u}_{n-1}^k, \boldsymbol{u}_{n-1}^k, \delta\boldsymbol{u}_n^k) - (\boldsymbol{f}_{n-1}^k, \delta\boldsymbol{u}_n^k). \tag{4.13}$$

(4.13) in (4.12) gives

$$\begin{aligned}
\mathcal{E}_n^k &\leq (\delta\boldsymbol{u}_{n-1}^k - \delta\boldsymbol{u}_n^k, \delta\boldsymbol{u}_n^k) + a(\boldsymbol{u}_{n-1}^k - \boldsymbol{u}_n^k, \delta\boldsymbol{u}_n^k) - d(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \delta\boldsymbol{u}_n^k) + d(\boldsymbol{u}_{n-1}^k, \boldsymbol{u}_{n-1}^k, \delta\boldsymbol{u}_n^k) + k(\delta\boldsymbol{f}_n^k, \delta\boldsymbol{u}_n^k)\\
&\leq -\frac{1}{2}\|\delta\boldsymbol{u}_n^k\| + \frac{1}{2}\|\delta\boldsymbol{u}_{n-1}^k\|^2 - \frac{1}{2}\|\delta\boldsymbol{u}_n^k - \delta\boldsymbol{u}_{n-1}^k\|^2 - ka(\delta\boldsymbol{u}_n^k, \delta\boldsymbol{u}_n^k) - kd(\delta\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \delta\boldsymbol{u}_n^k) + k(\delta\boldsymbol{f}_n^k, \delta\boldsymbol{u}_n^k)\\
&\leq -\frac{1}{2}\|\delta\boldsymbol{u}_n^k\| + \frac{1}{2}\|\delta\boldsymbol{u}_{n-1}^k\|^2 - \frac{1}{2}\|\delta\boldsymbol{u}_n^k - \delta\boldsymbol{u}_{n-1}^k\|^2 - ka(\delta\boldsymbol{u}_n^k, \delta\boldsymbol{u}_n^k) + ck\|\delta\boldsymbol{u}_n^k\|\|\nabla\delta\boldsymbol{u}_n^k\|\|\nabla\boldsymbol{u}_n^k\|\\
&\quad + k\|\delta\boldsymbol{f}_n^k\|\|\delta\boldsymbol{u}_n^k\|
\end{aligned}$$

which by summation for $n = 1, 2, ..., N$, and Hölder inequality, Young inequality yields

$$\begin{aligned}
&\frac{1}{2}\sum_{n=1}^{N}\|\delta\boldsymbol{u}_n^k - \delta\boldsymbol{u}_{n-1}^k\|^2 + k\nu\sum_{n=1}^{N}\|\varepsilon(\delta\boldsymbol{u}_n^k)\|^2 + \sum_{n=1}^{N}\mathcal{E}_n\\
\leq\ & -\frac{1}{2}\|\delta\boldsymbol{u}_N^k\| + ck\sum_{n=1}^{N}\|\nabla\delta\boldsymbol{u}_n^k\|^2\|\nabla\boldsymbol{u}_n^k\| + \frac{k}{c}\sum_{n=1}^{N}\|\delta\boldsymbol{f}_n^k\|\|\varepsilon(\delta\boldsymbol{u}_n^k)\|\\
\leq\ & -\frac{1}{2}\|\delta\boldsymbol{u}_N^k\| + ck\sum_{n=1}^{N}\|\nabla\delta\boldsymbol{u}_n^k\|^2\|\nabla\boldsymbol{u}_n^k\| + c\frac{k}{\nu}\sum_{n=1}^{N}\|\delta\boldsymbol{f}_n^k\|^2 + \frac{k\nu}{2}\|\varepsilon(\delta\boldsymbol{u}_n^k)\|^2.
\end{aligned}$$

So by application of (2.3), (2.28) and dropping some positive terms, we obtain

$$c\beta k\sum_{n=1}^{N}\|\nabla\delta\boldsymbol{u}_n^k\|^2 + \sum_{n=1}^{N}\mathcal{E}_n^k \quad\leq\quad c\frac{k}{\nu}\sum_{n=1}^{N}\|\delta\boldsymbol{f}_n^k\|^2 \leq c\frac{T}{\nu}\|\boldsymbol{f}'\|^2_{L^\infty(L^2)}$$

which concludes the proof. □

From Proposition 4.1 and Lemma 4.1, we deduce the following

Theorem 4.1. *Assume that $\boldsymbol{f} \in W^{1,\infty}(\boldsymbol{L}^2)$, and the conditions of proposition 2.1 hold. Let $\boldsymbol{u}$ be the solution of (2.10), and $(\boldsymbol{u}_k^n)_n$ the sequence defined through (2.12). Then there exists c independent of k such that*

$$\begin{aligned}
&\sum_{n=1}^{N}\|\boldsymbol{u}(t_n) - \widehat{\boldsymbol{u}}_k(t_n)\|^2 + \beta\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\|\nabla(\boldsymbol{u} - \widehat{\boldsymbol{u}}_k)\|^2\\
\leq\ & c\,k^2\left\{\frac{\nu}{\beta} + \frac{1}{\beta\nu}(\nu-\beta)^2\right\}\frac{T}{\nu}\|\boldsymbol{f}'\|^2_{L^\infty(L^2)} + \frac{c}{\nu}k^2\|\boldsymbol{f}'\|^2_{L^2(L^2)} + c\,k^2\frac{T}{\nu}\|\boldsymbol{f}'\|^2_{L^\infty(L^2)}\,.
\end{aligned}$$

Remark 4.2. *We only assumed that the solution $\boldsymbol{u}$ is once differentiable in time, and that $\boldsymbol{f}$ belong to $W^{1,\infty}(\boldsymbol{L}^2)$.*
The result can be extended to 3d as is suffice to use the appropriate inequalities concerning the trilinear form $d(\cdot,\cdot,\cdot)$.

4.2 A priori error on the derivative of the velocity and the pressure

Our objective is to estimate the quantities $\boldsymbol{u}' - \widehat{\boldsymbol{u}}_k'$, and $p(t_n) - p_n^k$. We first introduce some preliminaries materials needed. We first claim that

Lemma 4.2. *Let $\boldsymbol{u}$ be the solution of (2.10). Then*

$$\begin{aligned}
&(\boldsymbol{u}', \boldsymbol{u}') + a(\boldsymbol{u}, \boldsymbol{u}') + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}') + (J(\boldsymbol{u}))' = (\boldsymbol{f}, \boldsymbol{u}')\,,\\
&(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k') + a(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k') + d(\boldsymbol{u}_k, \boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k') + (J(\widehat{\boldsymbol{u}}_k))' = (\boldsymbol{f}_k, \widehat{\boldsymbol{u}}_k')\,.
\end{aligned}$$

The reader can see similar proof in [7] (see Proposition 5.4), the main difference here is the convective term.

Proof. We successively replace $\boldsymbol{v}$ by $2\boldsymbol{u}$ and $\boldsymbol{u}(t+k)$ in (2.10). Adding the resulting inequalities, we obtain

$$(\boldsymbol{u}', \boldsymbol{u}(t+k)) + a(\boldsymbol{u}, \boldsymbol{u}(t+k)) + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}(t+k) - \boldsymbol{u}) + J(\boldsymbol{u}(t+k)) \geq \ell(\boldsymbol{u}(t+k)). \quad (4.14)$$

Secondly, we replace $\boldsymbol{v}$ by $2\boldsymbol{u}$ and $\mathbf{0}$ in (2.10). Comparing the two inequalities, we find

$$(\boldsymbol{u}', \boldsymbol{u}) + a(\boldsymbol{u}, \boldsymbol{u}) + J(\boldsymbol{u}) = \ell(\boldsymbol{u}) \; . \quad (4.15)$$

Subtracting (4.15) from (4.14), we find that

$$\begin{aligned}(\boldsymbol{u}', \boldsymbol{u}(t+k) - \boldsymbol{u}) + a(\boldsymbol{u}, \boldsymbol{u}(t+k) - \boldsymbol{u}) &+ d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}(t+k) - \boldsymbol{u}) \\ &+ J(\boldsymbol{u}(t+k)) - J(\boldsymbol{u}) \geq \ell(\boldsymbol{u}(t+k) - \boldsymbol{u}) \; .\end{aligned} \quad (4.16)$$

If $k > 0$ we divide (4.16) by k and let $k \to 0$. We obtain

$$(\boldsymbol{u}', \boldsymbol{u}') + a(\boldsymbol{u}, \boldsymbol{u}') + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}') + (J(\boldsymbol{u}))' \geq \ell(\boldsymbol{u}') \; . \quad (4.17)$$

If $k < 0$, we divide (4.16) and let $k \to 0$. We find that

$$(\boldsymbol{u}', \boldsymbol{u}') + a(\boldsymbol{u}, \boldsymbol{u}') + d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}') + (J(\boldsymbol{u}))' \leq \ell(\boldsymbol{u}'). \quad (4.18)$$

Putting together (4.17) and (4.18) gives the desired result . The second result is similar and based on the variational inequality (4.2). □

Given a a non-negative real number and $\phi \in L^1(t, t+a)$. The average of ϕ over $(t, t+a)$ is the function

$$[\phi]_a(t) = \frac{1}{a}\int_t^{t+a} \phi(r)dr \, .$$

The second preliminary result reads;

Lemma 4.3. *Let $\boldsymbol{u}$ be the solution of (2.10). Then for all $\boldsymbol{v} \in \mathbb{V}_{\text{div}}$,*

$$\begin{aligned}&\left(\boldsymbol{u}', \boldsymbol{u}' - [\boldsymbol{v}]'_k + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}\right) + a\left(\boldsymbol{u}, \boldsymbol{u}' - [\boldsymbol{v}]'_k + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}\right) + d\left(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}' - [\boldsymbol{v}]'_k + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}\right) \\ &+ (J(\boldsymbol{u}))' - [J(\boldsymbol{v})]'_k + \frac{J(\boldsymbol{u}) - J(\boldsymbol{v})}{k} \leq \left(\boldsymbol{f}, \boldsymbol{u}' - [\boldsymbol{v}]'_k + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}\right) \, .\end{aligned}$$

Proof. From Lemma 4.2, and the definition of $[\boldsymbol{v}]_k$

$$\begin{aligned}\left(\boldsymbol{u}', \boldsymbol{u}' - [\boldsymbol{v}]'_k + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}\right) &= (\boldsymbol{u}', \boldsymbol{u}') + \left(-[\boldsymbol{v}]'_k + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}, \boldsymbol{u}'\right) \\ &= (\boldsymbol{f}, \boldsymbol{u}') - a(\boldsymbol{u}, \boldsymbol{u}') - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}') - (J(\boldsymbol{u}))' + \left(-[\boldsymbol{v}]'_k + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}, \boldsymbol{u}'\right) \\ &= (\boldsymbol{f}, \boldsymbol{u}') - a(\boldsymbol{u}, \boldsymbol{u}') - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}') - (J(\boldsymbol{u}))' + \left(\frac{v - v(t+k)}{k} + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}, \boldsymbol{u}'\right) \\ &= (\boldsymbol{f}, \boldsymbol{u}') - a(\boldsymbol{u}, \boldsymbol{u}') - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}') - (J(\boldsymbol{u}))' + \left(\frac{u - v(t+k)}{k}, \boldsymbol{u}'\right) \, .\end{aligned}$$

Thus

$$\begin{aligned}&\left(\boldsymbol{u}', \boldsymbol{u}' - [\boldsymbol{v}]'_k + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}\right) + a\left(\boldsymbol{u}, \boldsymbol{u}' - [\boldsymbol{v}]'_k + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}\right) + d\left(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}' - [\boldsymbol{v}]'_k + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}\right) \\ &+ (J(\boldsymbol{u}))' - [J(\boldsymbol{v})]'_k + \frac{J(\boldsymbol{u}) - J(\boldsymbol{v})}{k} \\ = \; &(\boldsymbol{f}, \boldsymbol{u}') + \frac{1}{k}\left[(\boldsymbol{u}', u - v(t+k)) + a\left(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v}(t+k)\right) + d\left(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{v}(t+k)\right) + J(\boldsymbol{u}) - J(\boldsymbol{v}(t+k))\right] \, .\end{aligned}$$

But, since $\boldsymbol{u}$ is the solution of (2.10), we have

$$(\boldsymbol{u}', \boldsymbol{u}-\boldsymbol{v}(t+k))+a(\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{v}(t+k))+d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}-\boldsymbol{v}(t+k))+J(\boldsymbol{u})-J(\boldsymbol{v}(t+k)) \leq (\boldsymbol{f}, \boldsymbol{u}-\boldsymbol{v}(t+k)).$$

Hence

$$\begin{aligned}
& \left(\boldsymbol{u}', \boldsymbol{u}' - [\boldsymbol{v}]_k' + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}\right) + a\left(\boldsymbol{u}, \boldsymbol{u}' - [\boldsymbol{v}]_k' + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}\right) + d\left(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}' - [\boldsymbol{v}]_k' + \frac{\boldsymbol{u}-\boldsymbol{v}}{k}\right) \\
& + (J(\boldsymbol{u}))' - [J(\boldsymbol{v})]_k' + \frac{J(\boldsymbol{u}) - J(\boldsymbol{v})}{k} \\
= & \left(\boldsymbol{f}, \boldsymbol{u}' + \frac{\boldsymbol{u} - \boldsymbol{v}(t+k)}{k}\right) = \left(\boldsymbol{f}, \boldsymbol{u}' + \frac{\boldsymbol{u}-\boldsymbol{v}}{k} - \frac{\boldsymbol{v}(t+k)-\boldsymbol{v}}{k}\right),
\end{aligned}$$

hence the proof is complete. □

The third preparatory result can be stated as follows

Lemma 4.4. *Let $\boldsymbol{u}_n^k$ be the solution of (2.12). Then for all $\boldsymbol{v} \in \mathbb{V}_{\text{div}}$,*

$$\begin{aligned}
& \left(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k' - [\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right) + a\left(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k' - [\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right) + d\left(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k' - [\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right) \\
& + \frac{J(\widehat{\boldsymbol{u}}_k) - J(\boldsymbol{v})}{k} - [J(\boldsymbol{v})]_k' + \delta J(\boldsymbol{u}_n^k) - \left(\boldsymbol{f}_k, \widehat{\boldsymbol{u}}_k' - [\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right) \\
\leq & \ (t-t_n)\left[a(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k') + d(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k')\right] + \frac{(t_n - t)}{k}\mathcal{E}_n^k \\
& + \frac{t-t_n}{k}\left[a(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k - \boldsymbol{v}(t+k)) + d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k - \boldsymbol{v}(t+k)) + d(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}(t+k))\right].
\end{aligned}$$

Proof. We replace $\boldsymbol{v}$ by $\boldsymbol{u}_{n-1}^k$ in (2.12). This gives

$$(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k') + a(\boldsymbol{u}_n^k, \widehat{\boldsymbol{u}}_k') + d(\boldsymbol{u}_k, \boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k') + \delta J(\boldsymbol{u}_n^k) \leq (\boldsymbol{f}_k, \widehat{\boldsymbol{u}}_k'),$$

from which we deduce that

$$\begin{aligned}
& (\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k') + a(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k') + d(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k') + \delta J(\boldsymbol{u}_n^k) - (\boldsymbol{f}_k, \widehat{\boldsymbol{u}}_k') \\
\leq & \ a(\widehat{\boldsymbol{u}}_k - \boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k') + d(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k') - d(\boldsymbol{u}_k, \boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k') \\
= & \ a(\widehat{\boldsymbol{u}}_k - \boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k') + d(\widehat{\boldsymbol{u}}_k - \boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k') + d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k') \\
= & \ (t-t_n)a(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k') + (t-t_n)d(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k') + (t-t_n)\underbrace{d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k')}_{=0}. \qquad (4.19)
\end{aligned}$$

Next, from the definition

$$[\boldsymbol{v}]_k' = \frac{1}{k}\left(\int_t^{t+k} \boldsymbol{v}(r)dr\right)' = \frac{1}{k}(\boldsymbol{v}(t+k) - \boldsymbol{v}),$$

it appears that

$$\begin{aligned}
& \left(\widehat{\boldsymbol{u}}_k', -[\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right) + a\left(\widehat{\boldsymbol{u}}_k, -[\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right) + d\left(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k, -[\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right) \\
& + \frac{J(\widehat{\boldsymbol{u}}_k) - J(\boldsymbol{v})}{k} - [J(\boldsymbol{v})]_k' - \left(\boldsymbol{f}_n^k, -[\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right) \\
= & \ \frac{1}{k}\left(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k - \boldsymbol{v}(t+k)\right) + \frac{1}{k}a(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}(t+k)) + \frac{1}{k}d(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}(t+k)) \\
& + \frac{1}{k}\left(J(\widehat{\boldsymbol{u}}_k) - J(\boldsymbol{v}(t+k))\right) - \frac{1}{k}(\boldsymbol{f}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}(t+k)),
\end{aligned}$$

which together with (4.6) gives

$$\left(\widehat{\boldsymbol{u}}_k', -[\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right) + a\left(\widehat{\boldsymbol{u}}_k, -[\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right) + d\left(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k, -[\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right)$$
$$+\frac{J(\widehat{\boldsymbol{u}}_k) - J(\boldsymbol{v})}{k} - [J(\boldsymbol{v})]_k' - \left(\boldsymbol{f}_k, -[\boldsymbol{v}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{v}}{k}\right)$$
$$\leq \frac{(t_n - t)}{k}\mathcal{E}_n^k + \frac{t - t_n}{k} a(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k - \boldsymbol{v}(t+k)) + \frac{t - t_n}{k} d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k - \boldsymbol{v}(t+k))$$
$$+\frac{t - t_n}{k} d(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{v}(t+k))\ . \tag{4.20}$$

Finally (4.19)+(4.20) gives the desired result. □

From Lemma 4.3 and Lemma 4.4, we deduce the following

Corollary 4.1. *Let $\boldsymbol{u}$ be the solution of (2.10), and $\boldsymbol{u}_n^k$ be the solution of (2.12) such that the conditions of proposition 2.1 are valid. Then*

$$\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\boldsymbol{u}' - \widehat{\boldsymbol{u}}_k'\|^2 + \frac{2\nu}{k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\varepsilon(\widehat{\boldsymbol{u}}_k - \boldsymbol{u})\|^2$$
$$\leq \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\widehat{\boldsymbol{u}}_k'\|\,\|\boldsymbol{u}' - [\boldsymbol{u}]_k'\| + 2\nu\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\boldsymbol{u}\|_1\,\|\boldsymbol{u}' - [\boldsymbol{u}]_k'\| + 2\nu\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\widehat{\boldsymbol{u}}_k'\|_1\|\boldsymbol{u} - \widehat{\boldsymbol{u}}_k\|_1$$
$$+c\frac{2\nu}{k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\boldsymbol{u} - \widehat{\boldsymbol{u}}_k\|_1\,\|\varepsilon(\boldsymbol{u}(t+k) - \boldsymbol{u})\| + \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} 2\nu\|\widehat{\boldsymbol{u}}_k'\|_1\,\|\widehat{\boldsymbol{u}}_k - \boldsymbol{u}\|_1$$
$$+\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} 2\nu\|\widehat{\boldsymbol{u}}_k'\|_1\,\|\varepsilon(\boldsymbol{u} - \boldsymbol{u}(t+k))\| + c\|\boldsymbol{u}_k\|_1\,\|\widehat{\boldsymbol{u}}_k'\|_1\,\|\widehat{\boldsymbol{u}}_k - \boldsymbol{u}\|_1 + c\|\boldsymbol{u}_k\|_1\,\|\widehat{\boldsymbol{u}}_k'\|_1\,\|\boldsymbol{u} - \boldsymbol{u}(t+k)\|_1$$
$$+c\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\boldsymbol{u}\|\,\|\boldsymbol{u}\|_1\,\|\boldsymbol{u}' - [\boldsymbol{u}]_k'\|_1 + \frac{c}{k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\widehat{\boldsymbol{u}}_k - \boldsymbol{u}\|\,\|\boldsymbol{u} - \widehat{\boldsymbol{u}}_k\|_1\,\|\boldsymbol{u}\|_1$$
$$+\frac{c}{k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\widehat{\boldsymbol{u}}_k\|_1\,\|\widehat{\boldsymbol{u}}_k - \boldsymbol{u}\|\,\|\varepsilon(\boldsymbol{u}(t+k) - \boldsymbol{u})\| + c\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\boldsymbol{u} - \widehat{\boldsymbol{u}}_k\|_1\,\|\boldsymbol{u}\|_1\,\|\widehat{\boldsymbol{u}}_k'\|_1$$
$$+c\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\widehat{\boldsymbol{u}}_k\|_1\,\|\boldsymbol{u} - \widehat{\boldsymbol{u}}_k\|_1\,\|\widehat{\boldsymbol{u}}_k'\|_1 + \frac{k}{2}\sum_{n=1}^{N}\mathcal{E}_n^k + c\|g\|_{L^\infty(S)}\|\boldsymbol{u}_k^N - \boldsymbol{u}(t_N)\|_1$$
$$+\frac{c}{k}\|g\|_{L^\infty(S)}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\widehat{\boldsymbol{u}}_k(t+k) - \boldsymbol{u}(t+k)\|_1 + \|\boldsymbol{u} - \widehat{\boldsymbol{u}}_k\|_1$$
$$+\frac{1}{k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\boldsymbol{f}_k - \boldsymbol{f}\|\,\|\boldsymbol{u}(t+k) - \boldsymbol{u}\| + \frac{1}{k}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \|\boldsymbol{f}_k - \boldsymbol{f}\|\,\|\widehat{\boldsymbol{u}}_k - \boldsymbol{u}\|\,.$$

Proof. For $\boldsymbol{v} = \widehat{\boldsymbol{u}}_k$ in Lemma 4.3, we have

$$\left(\boldsymbol{u}', \boldsymbol{u}' - [\widehat{\boldsymbol{u}}_k]_k' + \frac{\boldsymbol{u} - \widehat{\boldsymbol{u}}_k}{k}\right)$$
$$\leq -a\left(\boldsymbol{u}, \boldsymbol{u}' - [\widehat{\boldsymbol{u}}_k]_k' + \frac{\boldsymbol{u} - \widehat{\boldsymbol{u}}_k}{k}\right) - d\left(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}' - [\widehat{\boldsymbol{u}}_k]_k' + \frac{\boldsymbol{u} - \widehat{\boldsymbol{u}}_k}{k}\right) \tag{4.21}$$
$$-(J(\boldsymbol{u}))' + [J(\widehat{\boldsymbol{u}}_k)]_k' - \frac{J(\boldsymbol{u}) - J(\widehat{\boldsymbol{u}}_k)}{k} + \left(\boldsymbol{f}, \boldsymbol{u}' - [\widehat{\boldsymbol{u}}_k]_k' + \frac{\boldsymbol{u} - \widehat{\boldsymbol{u}}_k}{k}\right)\,.$$

For $\boldsymbol{v} = \boldsymbol{u}$ in Lemma 4.4, we have

$$
\begin{aligned}
&\left(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k' - [\boldsymbol{u}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{u}}{k}\right) \\
&\le -a\left(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k' - [\boldsymbol{u}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{u}}{k}\right) - d\left(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k' - [\boldsymbol{u}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{u}}{k}\right) \\
&- \frac{J(\widehat{\boldsymbol{u}}_k) - J(\boldsymbol{u})}{k} - [J(\boldsymbol{u})]_k' + \delta J(\boldsymbol{u}_n^k) + \left(\boldsymbol{f}_k, \widehat{\boldsymbol{u}}_k' - [\boldsymbol{u}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{u}}{k}\right) \\
&+ (t - t_n) a(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k') + (t - t_n) d(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k') + \frac{(t_n - t)}{k}\mathcal{E}_n^k \\
&+ \frac{t - t_n}{k} a(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) + \frac{t - t_n}{k} a(\widehat{\boldsymbol{u}}_k', \boldsymbol{u} - \boldsymbol{u}(t+k)) + \frac{t - t_n}{k} d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) \\
&+ \frac{t - t_n}{k} d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k', \boldsymbol{u} - \boldsymbol{u}(t+k)) + \frac{t - t_n}{k} d(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) \\
&+ \frac{t - t_n}{k} d(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k, \boldsymbol{u} - \boldsymbol{u}(t+k)) \,.
\end{aligned}
\tag{4.22}
$$

But from linearity we have

$$
\begin{aligned}
\|\boldsymbol{u}' - \widehat{\boldsymbol{u}}_k'\|^2 \quad = \quad & \left(\boldsymbol{u}', \boldsymbol{u}' - [\widehat{\boldsymbol{u}}_k]_k' + \frac{\boldsymbol{u} - \widehat{\boldsymbol{u}}_k}{k}\right) - \left(\boldsymbol{u}', \widehat{\boldsymbol{u}}_k' - [\widehat{\boldsymbol{u}}_k]_k' + \frac{\boldsymbol{u} - \widehat{\boldsymbol{u}}_k}{k}\right) \\
& + \left(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k' - [\boldsymbol{u}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{u}}{k}\right) - \left(\widehat{\boldsymbol{u}}_k', \boldsymbol{u}' - [\boldsymbol{u}]_k' + \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{u}}{k}\right),
\end{aligned}
$$

which together with (4.21), (4.22) and $[\widehat{\boldsymbol{u}}_k]_k' = \widehat{\boldsymbol{u}}_k'$ gives

$$
\begin{aligned}
&\|\boldsymbol{u}' - \widehat{\boldsymbol{u}}_k'\|^2 \\
&\le -\frac{1}{k} a(\widehat{\boldsymbol{u}}_k - \boldsymbol{u}, \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) - \frac{1}{2k}\frac{d}{dt}\|\boldsymbol{u} - \widehat{\boldsymbol{u}}_k\|^2 - (\widehat{\boldsymbol{u}}_k', \boldsymbol{u}' - [\boldsymbol{u}]_k') - a(\boldsymbol{u}, \boldsymbol{u}' - [\boldsymbol{u}]_k') + a(\boldsymbol{u} - \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k') \\
&- a\left(\boldsymbol{u} - \widehat{\boldsymbol{u}}_k, \frac{\boldsymbol{u}(t+k) - \boldsymbol{u}}{k}\right) + (t - t_n) a(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k') \\
&+ \frac{t - t_n}{k}\left[a(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) + a(\widehat{\boldsymbol{u}}_k', \boldsymbol{u} - \boldsymbol{u}(t+k)) + d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) + d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k', \boldsymbol{u} - \boldsymbol{u}(t+k))\right] \\
&+ \frac{t - t_n}{k}\left[d(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) + d(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k, \boldsymbol{u} - \boldsymbol{u}(t+k))\right] - d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}' - [\boldsymbol{u}]_k') \\
&+ d\left(\widehat{\boldsymbol{u}}_k - \boldsymbol{u}, \widehat{\boldsymbol{u}}_k, \frac{\boldsymbol{u}(t+k) - \boldsymbol{u}}{k}\right) + d\left(\widehat{\boldsymbol{u}}_k - \boldsymbol{u}, \boldsymbol{u}, \frac{\boldsymbol{u} - \widehat{\boldsymbol{u}}_k}{k}\right) + d\left(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{u}, \frac{\boldsymbol{u}(t+k) - \boldsymbol{u}}{k}\right) \\
&+ d(\boldsymbol{u} - \widehat{\boldsymbol{u}}_k, \boldsymbol{u}, \widehat{\boldsymbol{u}}_k') + d(\widehat{\boldsymbol{u}}_k, \boldsymbol{u} - \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k') + \frac{(t_n - t)}{k}\mathcal{E}_n^k + \delta J(\boldsymbol{u}_n^k) - (J(\boldsymbol{u}))' + [J(\widehat{\boldsymbol{u}}_k)]_k' - [J(\boldsymbol{u})]_k' \\
&+ (\boldsymbol{f}, \boldsymbol{u}' - [\boldsymbol{u}]_k') + (\boldsymbol{f}_k - \boldsymbol{f}, \widehat{\boldsymbol{u}}_k') - \left(\boldsymbol{f}_k - \boldsymbol{f}, \frac{\boldsymbol{u}(t+k) - \boldsymbol{u}}{k}\right) + \left(\boldsymbol{f}_k - \boldsymbol{f}, \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{u}}{k}\right).
\end{aligned}
\tag{4.23}
$$

Integrating (4.23) over (t_{n-1}, t_n) and taking the sum for $n = 1, 2, 3, ..., N$, one gets

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\boldsymbol{u}' - \widehat{\boldsymbol{u}}_k'\|^2 + \frac{1}{k} \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} a(\widehat{\boldsymbol{u}}_k - \boldsymbol{u}, \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) + \frac{1}{2k} \sum_{n=1}^{N} \|\boldsymbol{u}(t_n) - \boldsymbol{u}_k^n\|^2 - \|\boldsymbol{u}(t_{n-1}) - \boldsymbol{u}_k^{n-1}\|^2$$
$$\leq - \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\widehat{\boldsymbol{u}}_k', \boldsymbol{u}' - [\boldsymbol{u}]_k') - \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} a(\boldsymbol{u}, \boldsymbol{u}' - [\boldsymbol{u}]_k') + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} a(\boldsymbol{u} - \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k')$$
$$- \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} a\left(\boldsymbol{u} - \widehat{\boldsymbol{u}}_k, \frac{\boldsymbol{u}(t+k) - \boldsymbol{u}}{k}\right)$$
$$+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \frac{t - t_n}{k} \left[a(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) + a(\widehat{\boldsymbol{u}}_k', \boldsymbol{u} - \boldsymbol{u}(t+k)) + d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) + d(\boldsymbol{u}_k, \widehat{\boldsymbol{u}}_k', \boldsymbol{u} - \boldsymbol{u}(t+k))\right]$$
$$+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \frac{t - t_n}{k} \left[d(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{u}) + d(\widehat{\boldsymbol{u}}_k', \widehat{\boldsymbol{u}}_k, \boldsymbol{u} - \boldsymbol{u}(t+k))\right] - \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}' - [\boldsymbol{u}]_k')$$
$$+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left[d\left(\widehat{\boldsymbol{u}}_k - \boldsymbol{u}, \widehat{\boldsymbol{u}}_k, \frac{\boldsymbol{u}(t+k) - \boldsymbol{u}}{k}\right) + d\left(\widehat{\boldsymbol{u}}_k - \boldsymbol{u}, \boldsymbol{u}, \frac{\boldsymbol{u} - \widehat{\boldsymbol{u}}_k}{k}\right) + d\left(\widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k - \boldsymbol{u}, \frac{\boldsymbol{u}(t+k) - \boldsymbol{u}}{k}\right)\right]$$
$$+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left[d(\boldsymbol{u} - \widehat{\boldsymbol{u}}_k, \boldsymbol{u}, \widehat{\boldsymbol{u}}_k') + d(\widehat{\boldsymbol{u}}_k, \boldsymbol{u} - \widehat{\boldsymbol{u}}_k, \widehat{\boldsymbol{u}}_k')\right] + \frac{k}{2} \sum_{n=1}^{N} \mathcal{E}_n^k$$
$$+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \delta J(\boldsymbol{u}_n^k) - (J(\boldsymbol{u}))' + [J(\widehat{\boldsymbol{u}}_k)]_k' - [J(\boldsymbol{u})]_k' + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\boldsymbol{f}, \boldsymbol{u}' - [\boldsymbol{u}]_k') + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\boldsymbol{f}_k - \boldsymbol{f}, \widehat{\boldsymbol{u}}_k')$$
$$- \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left(\boldsymbol{f}_k - \boldsymbol{f}, \frac{\boldsymbol{u}(t+k) - \boldsymbol{u}}{k}\right) + \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left(\boldsymbol{f}_k - \boldsymbol{f}, \frac{\widehat{\boldsymbol{u}}_k - \boldsymbol{u}}{k}\right) . \tag{4.24}$$

Now, from convexity of $J(\cdot)$, and definition of $[v]_k$, we obtain that

$$\begin{aligned} &\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left[\delta J(\boldsymbol{u}_n^k) - (J(\boldsymbol{u}))' + [J(\widehat{\boldsymbol{u}}_k)]_k' - [J(\boldsymbol{u})]_k'\right] dt \\ &\leq J(\boldsymbol{u}_k^N - \boldsymbol{u}(t_N)) + \frac{1}{k} \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left[J(\widehat{\boldsymbol{u}}_k(t+k) - \boldsymbol{u}(t+k)) + J(\boldsymbol{u} - \widehat{\boldsymbol{u}}_k)\right] dt \,. \end{aligned} \tag{4.25}$$

Putting back (4.25) in (4.24), and apply Cauchy-Shwarz's inequality and continuity properties of operators involved lead the inequality announced. □

We will now treat the right hand side of the inequality in Corollary 4.1. We first recall from [7] that

$$\begin{aligned} &\|\boldsymbol{u}' - [\boldsymbol{u}']_k\|_{L^2(L^2)} \leq ck \, \|\boldsymbol{u}'\|_{L^2(H^1) \cap L^\infty(L^2)} \,, \\ &\|\boldsymbol{u} - [\boldsymbol{u}]_k\|_{L^2(H^1) \cap L^\infty(L^2)} \leq ck \left(\|\boldsymbol{u}\|_{L^2(H^1) \cap L^\infty(L^2)} + \|\boldsymbol{u}'\|_{L^2(H^1) \cap L^\infty(L^2)}\right) . \end{aligned} \tag{4.26}$$

Next, we claim that

Lemma 4.5. *Let $\boldsymbol{u}$ be the solution of (2.10). Assume that $\boldsymbol{f} \in \boldsymbol{W}^{1,\infty}(\boldsymbol{L}^2)$. Then there is c independent of k such that*

$$\begin{aligned} &2\nu \int_0^T \|\varepsilon(\boldsymbol{u}(t+k) - \boldsymbol{u}(t))\|^2 dt \\ \leq \quad &\left(\|\boldsymbol{u}(k) - \boldsymbol{u}_0\|^2 + \frac{c}{\nu} k^2 T \|\boldsymbol{f}'\|^2_{L^\infty(L^2)}\right) \left\{1 + \frac{c}{\nu}\left[1 + \exp\left(\frac{c}{\nu}\|\boldsymbol{u}\|^2_{L^2(H^1)}\right)\|\boldsymbol{u}\|^2_{L^2(H^1)}\right]\|\boldsymbol{u}\|^2_{L^2(H^1)}\right\} , \end{aligned}$$

and

$$\begin{aligned}
&\|\boldsymbol{u}(k)-\boldsymbol{u}_0\|^2\\
\le\ & 2\nu k\|\boldsymbol{u}_0\|_1^2+\frac{c}{\nu}\,k\|\boldsymbol{u}_0\|_1^4+\frac{c}{\nu}\,k\|g\|^2_{L^\infty(S)}+\frac{c}{\nu}\,k\|\boldsymbol{f}\|^2_{L^\infty(L^2)}\\
&+\frac{c}{\nu}\|\boldsymbol{u}_0\|_1^2\exp\left(\frac{c}{\nu}\|\boldsymbol{u}_0\|_1^2k\right)\left\{\nu k^2\|\boldsymbol{u}_0\|_1^2+\frac{c}{\nu}\,k^2\|\boldsymbol{u}_0\|_1^4+\frac{c}{\nu}\,k^2\|g\|^2_{L^\infty(S)}+\frac{c}{\nu}\,k^2\|\boldsymbol{f}\|^2_{L^\infty(L^2)}\right\}\\
\le\ & ck\,.
\end{aligned}$$

Proof. We replace $\boldsymbol{v}$ by $\boldsymbol{u}(t+k)$ in (2.10). Next we take (2.10) at $t+k$ and replace $\boldsymbol{v}$ by $\boldsymbol{u}$. Putting these inequalities together, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t+k)-\boldsymbol{u}\|^2+2\nu\|\varepsilon(\boldsymbol{u}(t+k)-\boldsymbol{u})\|^2\\
\le\ & d(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}(t+k)-\boldsymbol{u})+d(\boldsymbol{u}(t+k),\boldsymbol{u}(t+k),\boldsymbol{u}-\boldsymbol{u}(t+k))+(\boldsymbol{f}(t+k)-\boldsymbol{f},\boldsymbol{u}(t+k)-\boldsymbol{u})\\
\le\ & d(\boldsymbol{u}-\boldsymbol{u}(t+k),\boldsymbol{u},\boldsymbol{u}(t+k)-\boldsymbol{u})+(\boldsymbol{f}(t+k)-\boldsymbol{f},\boldsymbol{u}(t+k)-\boldsymbol{u})\\
\le\ & c\|\boldsymbol{u}(t+k)-\boldsymbol{u}\|\,\|\varepsilon(\boldsymbol{u}(t+k)-\boldsymbol{u})\|\,\|\boldsymbol{u}\|_1+c\|\boldsymbol{f}(t+k)-\boldsymbol{f}\|\|\varepsilon(\boldsymbol{u}(t+k)-\boldsymbol{u})\|\,,
\end{aligned}$$

which by application of the inequality of Young gives

$$\begin{aligned}
&\frac{d}{dt}\|\boldsymbol{u}(t+k)-\boldsymbol{u}\|^2+2\nu\|\varepsilon(\boldsymbol{u}(t+k)-\boldsymbol{u})\|^2\\
&\qquad\qquad\le c\|\boldsymbol{u}(t+k)-\boldsymbol{u}\|\,\|\varepsilon(\boldsymbol{u}(t+k)-\boldsymbol{u})\|\,\|\boldsymbol{u}\|_1+\frac{c}{\nu}\|\boldsymbol{f}(t+k)-\boldsymbol{f}\|^2\,.
\end{aligned}$$

Applying Young's inequality again we obtain

$$\begin{aligned}
\frac{d}{dt}\|\boldsymbol{u}(t+k)-\boldsymbol{u}\|^2+\nu\|\varepsilon(\boldsymbol{u}(t+k)-\boldsymbol{u})\|^2\le&\frac{c}{\nu}\|\boldsymbol{u}(t+k)-\boldsymbol{u}\|^2\,\|\boldsymbol{u}\|_1^2\\
&+\frac{c}{\nu}\|\boldsymbol{f}(t+k)-\boldsymbol{f}\|^2\,.
\end{aligned}\tag{4.27}$$

Integrating (4.27) for $0\le t\le T$, and dropping the second term on the left hand side, one obtains

$$\begin{aligned}
\|\boldsymbol{u}(t+k)-\boldsymbol{u}\|^2\le&\frac{c}{\nu}\int_0^t\|\boldsymbol{u}(s+k)-\boldsymbol{u}\|^2\,\|\boldsymbol{u}(s)\|_1^2ds\\
&+\frac{c}{\nu}\int_0^t\|\boldsymbol{f}(s+k)-\boldsymbol{f}\|^2+\|\boldsymbol{u}(k)-\boldsymbol{u}_0\|^2\,.
\end{aligned}\tag{4.28}$$

Application of Gronwall's lemma gives

$$\begin{aligned}
&\|\boldsymbol{u}(t+k)-\boldsymbol{u}\|^2\\
\le\ & \|\boldsymbol{u}(k)-\boldsymbol{u}_0\|^2+\frac{c}{\nu}\int_0^t\|\boldsymbol{f}(s+k)-\boldsymbol{f}\|^2\\
&+\exp\left(\frac{c}{\nu}\int_0^t\|\boldsymbol{u}(r)\|_1^2dr\right)\int_0^t\left(\|\boldsymbol{u}(k)-\boldsymbol{u}_0\|^2+\frac{c}{\nu}\int_0^s\|\boldsymbol{f}(r+k)-\boldsymbol{f}\|^2dr\right)\|\boldsymbol{u}(s)\|_1^2ds\\
\le\ & \|\boldsymbol{u}(k)-\boldsymbol{u}_0\|^2+\frac{c}{\nu}k^2T\|\boldsymbol{f}'\|^2_{L^\infty(L^2)}\\
&+\exp\left(\frac{c}{\nu}\|\boldsymbol{u}\|^2_{L^2(H^1)}\right)\left(\|\boldsymbol{u}(k)-\boldsymbol{u}_0\|^2+\frac{c}{\nu}k^2T\|\boldsymbol{f}'\|^2_{L^\infty(L^2)}\right)\|\boldsymbol{u}\|^2_{L^2(H^1)}\,.
\end{aligned}\tag{4.29}$$

We return to (4.27) with (4.29), integrate the resulting inequality and dropping some positive terms, one has

$$\begin{aligned}
2\nu\int_0^T\|\varepsilon(\boldsymbol{u}(t+k)-\boldsymbol{u})\|^2dt\le&\left(\|\boldsymbol{u}(k)-\boldsymbol{u}_0\|^2+\frac{c}{\nu}k^2T\|\boldsymbol{f}'\|^2_{L^\infty(L^2)}\right)\\
&\left\{1+\frac{c}{\nu}\left[1+\exp\left(\frac{c}{\nu}\|\boldsymbol{u}\|^2_{L^2(H^1)}\right)\|\boldsymbol{u}\|^2_{L^2(H^1)}\right]\|\boldsymbol{u}\|^2_{L^2(H^1)}\right\}\,.
\end{aligned}\tag{4.30}$$

So the first inequality announced is proved.
From (2.10), it appears that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}-\boldsymbol{u}_0\|^2+2\nu\|\varepsilon(\boldsymbol{u}-\boldsymbol{u}_0)\|^2 \leq\ & a(\boldsymbol{u}_0,\boldsymbol{u}_0-\boldsymbol{u})+d(\boldsymbol{u},\boldsymbol{u},\boldsymbol{u}_0-\boldsymbol{u})+J(\boldsymbol{u}_0-\boldsymbol{u})+(\boldsymbol{f},\boldsymbol{u}-\boldsymbol{u}_0)\\ =\ & a(\boldsymbol{u}_0,\boldsymbol{u}_0-\boldsymbol{u})+d(\boldsymbol{u}-\boldsymbol{u}_0,\boldsymbol{u}_0,\boldsymbol{u}_0-\boldsymbol{u})+d(\boldsymbol{u}_0,\boldsymbol{u}_0,\boldsymbol{u}_0-\boldsymbol{u})\\ & +J(\boldsymbol{u}_0-\boldsymbol{u})+(\boldsymbol{f},\boldsymbol{u}-\boldsymbol{u}_0)\end{aligned}$$

which with the continuity properties and Young's inequality leads to

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{u}-\boldsymbol{u}_0\|^2+\nu\|\varepsilon(\boldsymbol{u}-\boldsymbol{u}_0)\|^2 \leq\ & \frac{c}{\nu}\|\boldsymbol{u}_0\|_1^2\|\boldsymbol{u}-\boldsymbol{u}_0\|^2+2\nu\|\boldsymbol{u}_0\|_1^2+\frac{c}{\nu}\|\boldsymbol{u}_0\|_1^4\\ & +\frac{c}{\nu}\|g\|^2_{L^\infty(S)}+\frac{c}{\nu}\|\boldsymbol{f}\|^2 .\end{aligned} \tag{4.31}$$

(4.31) is re-written as follows

$$\begin{aligned}&\|\boldsymbol{u}(t)-\boldsymbol{u}_0\|^2+\nu\int_0^t\|\varepsilon(\boldsymbol{u}(s)-\boldsymbol{u}_0)\|^2ds-\frac{c}{\nu}\|\boldsymbol{u}_0\|_1^2\int_0^t\|\boldsymbol{u}(s)-\boldsymbol{u}_0\|^2ds \\ &\leq 2\nu t\|\boldsymbol{u}_0\|_1^2+\frac{c}{\nu}\,t\|\boldsymbol{u}_0\|_1^4+\frac{c}{\nu}\,t\|g\|^2_{L^\infty(S)}+\frac{c}{\nu}\,t\|\boldsymbol{f}\|^2_{L^\infty(L^2)} .\end{aligned} \tag{4.32}$$

Dropping the second term on the left hand side of (4.31) and applying Gronwall's Lemma we find

$$\begin{aligned}& \|\boldsymbol{u}(t)-\boldsymbol{u}_0\|^2\\ \leq\ & 2\nu t\|\boldsymbol{u}_0\|_1^2+\frac{c}{\nu}\,t\|\boldsymbol{u}_0\|_1^4+\frac{c}{\nu}\,t\|g\|^2_{L^\infty(S)}+\frac{c}{\nu}\,t\|\boldsymbol{f}\|^2_{L^\infty(L^2)}\\ & +\frac{c}{\nu}\|\boldsymbol{u}_0\|_1^2\exp\left(\frac{c}{\nu}\|\boldsymbol{u}_0\|_1^2t\right)\int_0^t\left(2\nu s\|\boldsymbol{u}_0\|_1^2+\frac{c}{\nu}\,s\|\boldsymbol{u}_0\|_1^4+\frac{c}{\nu}\,s\|g\|^2_{L^\infty(S)}+\frac{c}{\nu}\,s\|\boldsymbol{f}\|^2_{L^\infty(L^2)}\right)ds\\ \leq\ & 2\nu t\|\boldsymbol{u}_0\|_1^2+\frac{c}{\nu}\,t\|\boldsymbol{u}_0\|_1^4+\frac{c}{\nu}\,t\|g\|^2_{L^\infty(S)}+\frac{c}{\nu}\,t\|\boldsymbol{f}\|^2_{L^\infty(L^2)}\\ & +\frac{c}{\nu}\|\boldsymbol{u}_0\|_1^2\exp\left(\frac{c}{\nu}\|\boldsymbol{u}_0\|_1^2t\right)\left\{\nu t^2\|\boldsymbol{u}_0\|_1^2+\frac{c}{\nu}\,t^2\|\boldsymbol{u}_0\|_1^4+\frac{c}{\nu}\,t^2\|g\|^2_{L^\infty(S)}+\frac{c}{\nu}\,t^2\|\boldsymbol{f}\|^2_{L^\infty(L^2)}\right\} .\end{aligned}$$

The proof is complete by taking $t=k$. □

From Corollary 4.1, Lemma 4.5, (4.26), and Theorem 4.1, we deduce that

Theorem 4.2. *Assume that $\boldsymbol{f}\in W^{1,\infty}(\boldsymbol{L}^2)$, and the conditions of proposition 2.1 are valid. Let $\boldsymbol{u}$ be the solution of (2.10), and $(\boldsymbol{u}_k^n)_n$ the sequence defined through (2.12). Then there is c independent of k, such that*

$$\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\|\boldsymbol{u}'-\widehat{\boldsymbol{u}}_k'\|^2+\frac{2\nu}{k}\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\|\varepsilon(\widehat{\boldsymbol{u}}_k-\boldsymbol{u})\|^2\leq c(\boldsymbol{f}',g,\boldsymbol{u}_0,\nu)\,k\ .$$

The third main contribution in the paragraph is to analyse the pressure. We claim that;

Theorem 4.3. *Let $\boldsymbol{f}\in W^{1,\infty}(\boldsymbol{L}^2)$. Assume that $(\boldsymbol{u}(t_n),p(t_n))$ is the solution of (2.11), and $(\boldsymbol{u}_n^k,p_n^k)$ is the solution of (2.13). Then*

$$\begin{aligned}\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\|p(t_n)-p_n^k\|^2 \leq\ & c(\boldsymbol{f}',g,\boldsymbol{u}_0,T,\nu)\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\left\{\|\widehat{\boldsymbol{u}}_k'-\boldsymbol{u}'(t_n)\|^2+\|\boldsymbol{u}_k-\boldsymbol{u}(t_n)\|_1^2\right\}\\ & +c\,k^2\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\|\boldsymbol{f}'(t_n)\|^2 .\end{aligned}$$

Proof. Let $\boldsymbol{w}\in\mathbb{V}\cap\{\boldsymbol{v}|_S=0\}$, for $t=t_n$ in (2.11), we take $\boldsymbol{v}$ such that $\boldsymbol{v}-\boldsymbol{u}(t_n)=\pm\boldsymbol{w}$. Then

$$(\boldsymbol{u}'(t_n),\boldsymbol{w})+a(\boldsymbol{u}(t_n),\boldsymbol{w})+b(\boldsymbol{w},p(t_n))+d(\boldsymbol{u}(t_n),\boldsymbol{u}(t_n),\boldsymbol{w})=(\boldsymbol{f}(t_n),\boldsymbol{w}) . \tag{4.33}$$

Repeating this process with (2.13) and $\boldsymbol{v} - \boldsymbol{u}_n^k = \pm \boldsymbol{w}$, one has

$$\left(\frac{\boldsymbol{u}_n^k - \boldsymbol{u}_{n-1}^k}{k}, \boldsymbol{w}\right) + a(\boldsymbol{u}_n^k, \boldsymbol{w}) + b(\boldsymbol{w}, p_n^k) + d(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \boldsymbol{w}) = (\boldsymbol{f}_n^k, \boldsymbol{w})\,. \tag{4.34}$$

Equations (4.33) and (4.34) leads to

$$\begin{aligned} b(\boldsymbol{w}, p(t_n) - p_n^k) &= \big(\widehat{\boldsymbol{u}}_k' - \boldsymbol{u}'(t_n), \boldsymbol{w}\big) + a(\boldsymbol{u}_n^k - \boldsymbol{u}(t_n), \boldsymbol{w}) + d(\boldsymbol{u}_n^k, \boldsymbol{u}_n^k, \boldsymbol{w}) \\ &\quad - d(\boldsymbol{u}(t_n), \boldsymbol{u}(t_n), \boldsymbol{w}) + (\boldsymbol{f}(t_n) - \boldsymbol{f}_n^k, \boldsymbol{w}) \\ &= \big(\widehat{\boldsymbol{u}}_k' - \boldsymbol{u}'(t_n), \boldsymbol{w}\big) + a(\boldsymbol{u}_n^k - \boldsymbol{u}(t_n), \boldsymbol{w}) + (\boldsymbol{f}(t_n) - \boldsymbol{f}_n^k, \boldsymbol{w}) \\ &\quad + d(\boldsymbol{u}_n^k - \boldsymbol{u}(t_n), \boldsymbol{u}_n^k - \boldsymbol{u}(t_n), \boldsymbol{w}) + d(\boldsymbol{u}_n^k - \boldsymbol{u}(t_n), \boldsymbol{u}(t_n), \boldsymbol{w}) \\ &\quad + d(\boldsymbol{u}(t_n), \boldsymbol{u}_n^k - \boldsymbol{u}(t_n), \boldsymbol{w})\,. \end{aligned} \tag{4.35}$$

Next, since the spaces $\mathbb{V} \cap \{\boldsymbol{v}|_S = 0\}$ and M are compatible, there is c independent of k such that

$$c\|p(t_n) - p_n^k\| \le \sup_{\boldsymbol{v} \in V \cap \{\boldsymbol{v}|_S = 0\}} \frac{b(\boldsymbol{v}, p(t_n) - p_n^k)}{\|\boldsymbol{v}\|_1},$$

which together with (4.35) implies that

$$\begin{aligned} \|p(t_n) - p_n^k\| &\le \|\widehat{\boldsymbol{u}}_k' - \boldsymbol{u}'(t_n)\| + 2\nu\|\boldsymbol{u}_k - \boldsymbol{u}(t_n)\|_1 + c\,k\|\boldsymbol{f}'(t_n)\| + c\|\boldsymbol{u}_k - \boldsymbol{u}(t_n)\|\,\|\nabla(\boldsymbol{u}_k - \boldsymbol{u}(t_n))\| \\ &\quad + c\|\nabla(\boldsymbol{u}_k - \boldsymbol{u}(t_n)\|\,\|\boldsymbol{u}(t_n)\|_1\,. \end{aligned}$$

Hence we deduce the result by taking the square on booth side, and application of Theorem 4.2 and Theorem 4.1 . □

5 Finite element approximation

In this section, we start by recalling some important notation concerning finite element approximations. Next, we formulate and study the finite element problem associated with (2.11). The semi-discrete error estimate is carried out, and finally we study the space-time discretization associated with (2.11).

5.1 Preliminaries

We assume that Ω is a polygon in two dimensions, so it can be entirely triangulated by triangles, this is say that

$$\overline{\Omega} = \bigcup_{1 \le i \le N} K_i\,.$$

We assume that the family $(K_i)_i$ is regular in the sense of Ciarlet [20]; there exists a constant τ, independent of h and K, such that

$$\text{for all } i \in \{1, ..., N\}\ , \qquad \frac{h_{K_i}}{\rho_{K_i}} = \tau_{K_i} \le \tau,$$

where h_{K_i} is the diameter of K_i and ρ_{K_i} is the diameter of the sphere inscribed in K_i. The length of an element K is $h = \max\limits_{1 \le i \le N} \{h_{K_i}\}$, and $\mathcal{T}_h$ stands for the family of conforming triangles K of $\overline{\Omega}$. We will also need $\mathcal{T}_{2h}$ a triangulation twice coarser (in practice we should construct $\mathcal{T}_{2h}$ first and then $\mathcal{T}_h$ by joining the midpoints of the edges of $\mathcal{T}_{2h}$, dividing thus each triangle of $\mathcal{T}_{2h}$ into four similar sub-triangles).
For each $K \in \mathcal{T}_h$, and for each nonnegative integer k, $P_k(K)$ is the space of restrictions to K of polynomials with 2 variables with degree less than or equal to k.
We introduce the skew-symmetric form (see R. Temam [17])

$$\begin{aligned} &\text{for all } \ \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{V} \times \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)\ , \\ &\widetilde{d}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \left\{ (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} \cdot \boldsymbol{w} + \frac{1}{2}(\operatorname{div} \boldsymbol{u})\boldsymbol{v} \cdot \boldsymbol{w} \right\}\,. \end{aligned} \tag{5.1}$$

It is noted that (2.5), (2.6), (2.7) and (2.8) are valid in the finite element spaces, and

$$\widetilde{d}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w} = d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) \text{ for any } \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbb{V}_{\rm div}\times \boldsymbol{H}^1(\Omega)\times \boldsymbol{H}^1(\Omega)\,. \tag{5.2}$$

We are going to use C^0 finite elements in our analysis, which are conformal to all these functional spaces. We consider the following finite element approximation of problem (2.11), obtained by the Galerkin method:
Find $(\boldsymbol{u}_h, p_h) \in \mathbb{V}_h \times M_h$ such that for all $(\boldsymbol{v}_h, q_h) \in \mathbb{V}_h \times M_h$,

$$\begin{cases} \boldsymbol{u}_h(0) = P\boldsymbol{u}_0\ , \\ \text{and for all } \ \boldsymbol{v}_h \in \mathbb{V}_h, \ \ \text{and a.e } t>0 \\ (\boldsymbol{u}_h'(t), \boldsymbol{u}_h(t)-\boldsymbol{v}_h) + a(\boldsymbol{u}_h(t), \boldsymbol{u}_h(t)-\boldsymbol{v}_h) + b(\boldsymbol{u}_h-\boldsymbol{v}_h, p_h) \\ +\widetilde{d}(\boldsymbol{u}_h(t), \boldsymbol{u}_h(t), \boldsymbol{u}_h(t)-\boldsymbol{v}_h) + J(\boldsymbol{u}_h(t)) - J(\boldsymbol{v}_h) \le \ell(\boldsymbol{u}_h(t)-\boldsymbol{v}_h), \\ b(\boldsymbol{u}_h, q_h) = 0, \end{cases} \tag{5.3}$$

with $P\boldsymbol{u}_0$ the L^2 projection of $\boldsymbol{u}_0$ into $\mathbb{V}_h$. The approximating spaces $\mathbb{V}_h$ and M_h are thus required to satisfy the standard Babuska-Brezzi stability condition (see [19, 21]). Hence there is $\gamma>0$ independent of h such that for all $h>0$,

$$\text{for all } \boldsymbol{q}_h \in M_h \quad , \qquad \gamma\|q_h\| \le \sup_{\boldsymbol{v}_h\in\mathbb{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_1}.$$

To be more precise, we consider the following finite dimensional spaces:

$$\begin{aligned} \mathbb{V}_h &= \left\{\boldsymbol{v}_h | \boldsymbol{v}_h \in (C^0(\overline{\Omega}))^2\cap\mathbb{V}, \ \ \boldsymbol{v}_h|_K \in P_1\times P_1, \ \ \text{for all } K\in\mathcal{T}_h\right\}\ , \\ L_h^2 &= \left\{q_h | q_h \in L^2(\Omega)\cap C^0(\overline{\Omega}), \ \ q_h|_K \in P_1, \ \ \text{for all } K\in\mathcal{T}_{2h}\right\}\ , \\ M_h &= \left\{q_h | q_h \in L_h^2, \ (q_h, 1) = 0\right\}\ . \end{aligned} \tag{5.4}$$

For the existence theory of Problem (5.3), we claim that

Theorem 5.1. *If $\boldsymbol{u}_0$ belong to $\mathbb{V}_h\cap\boldsymbol{H}^2(\Omega)$, $g\in H^1(S)\cap L^\infty(S)$, $\boldsymbol{f}$ and $\boldsymbol{f}'$ in $L^\infty(\boldsymbol{L}^2)$ with $\boldsymbol{f}(0)\in\mathbb{H}$. Then Problem (5.3) has a unique solution $(\boldsymbol{u}_h(t), p_h(t)) \in \mathbb{V}_h\times M_h$.*

The proof follow the steps used in the proof of Theorem 2.2 and will not be repeated here.

5.2 Continuous in time finite element approximation in space: A priori error analysis

The goal of this paragraph is to estimate difference between the solutions of problem (5.3) and problem (2.11).
We recall that from the properties of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, one can define a Stokes projector (see [19, 22]) that is,

$$(I_h, J_h) : \mathbb{V}\times M \to \mathbb{V}_h\times M_h$$

defined as follows:

$$\begin{cases} \text{for all } \ (\boldsymbol{v}_h, q_h) \in \mathbb{V}_h\times M_h, \\ a(I_h\boldsymbol{v}-\boldsymbol{v}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, J_h q - q) = 0, \\ b(I_h\boldsymbol{v}, q_h) = 0, \end{cases} \tag{5.5}$$

which satisfy the property

$$\frac{1}{h}\|\boldsymbol{v}-I_h\boldsymbol{v}\| + \|\nabla(\boldsymbol{v}-I_h\boldsymbol{v})\| \le ch\|\boldsymbol{v}\|_2 \ \ , \ \ \|J_hq-q\| \le ch\|q\|_1\,. \tag{5.6}$$

Moreover if $\boldsymbol{v}$ is such that $\boldsymbol{v}|_S \in \boldsymbol{H}^2(S)$, then

$$\|\boldsymbol{v}-I_h\boldsymbol{v}\|_S \le c\|\boldsymbol{v}\|_{H^2(S)}\,. \tag{5.7}$$

One of the principal result in this paragraph is the following

Theorem 5.2. *Let $(\boldsymbol{u}, p)$ be the solution of Problem (2.11), and $(\boldsymbol{u}_h, p_h)$ the finite element solution defined through (5.3).*
Assume that; $\boldsymbol{u} \in L^\infty(\boldsymbol{H}^2(\Omega))$, $p \in L^2(H^1(\Omega))$, $\boldsymbol{u}_t \in L^2(\boldsymbol{H}^2(\Omega))$.
Then there is a generic constant c independent of h such that

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^\infty(L^2)} &\leq \|\boldsymbol{u} - I_h\boldsymbol{u}\|_{L^\infty(L^2)} + \left[\|I_h\boldsymbol{u}(0) - \boldsymbol{u}_h(0)\| + F^{1/2}\right] \exp\left(c\|\boldsymbol{u}\|_{L^2(H^1)}\right), \\
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(H^1)} &\leq \|\boldsymbol{u} - I_h\boldsymbol{u}\|_{L^2(H^1)} + \|I_h\boldsymbol{u}(0) - \boldsymbol{u}_h(0)\| + F^{1/2} \\
&\quad + c\|\boldsymbol{u}\|_{L^2(H^1)}\left(\|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^\infty(L^2)} + \|I_h\boldsymbol{u} - \boldsymbol{u}\|_{L^\infty(L^2)}\right),
\end{aligned}$$

with

$$\begin{aligned}
F &= c\|I_h\boldsymbol{u}' - \boldsymbol{u}'\|^2_{L^2(L^2)} + \|\boldsymbol{u}'\|_{L^2(L^2)}\|I_h\boldsymbol{u} - \boldsymbol{u}\|_{L^2(L^2)} + c\|\boldsymbol{u}\|_{L^2(H^1)}\|I_h\boldsymbol{u} - \boldsymbol{u}\|_{L^2(H^1)} \\
&\quad + \nu c\|I_h\boldsymbol{u} - \boldsymbol{u}\|^2_{L^2(H^1)} + c\|p - J_hp\|^2_{L^2(L^2)} + c\|\boldsymbol{u}\|_{L^\infty(H^1)}\|I_hu - \boldsymbol{u}\|^2_{L^2(H^1)} \\
&\quad + c\|\boldsymbol{u}\|^2_{L^\infty(H^1)}\|I_h\boldsymbol{u} - \boldsymbol{u}\|_{L^1(H^1)} + c\|p\|_{L^2(L^2)}\|I_h\boldsymbol{u} - \boldsymbol{u}\|_{L^2(H^1)} \\
&\quad + \|g\|_{L^\infty(S)}\|I_h\boldsymbol{u} - \boldsymbol{u}\|_{L^1(H^1)} + \|\boldsymbol{f}\|_{L^2(V^*)}\|I_h\boldsymbol{u} - \boldsymbol{u}\|_{L^2(H^1)}.
\end{aligned}$$

Proof. We split the error as follows

$$\begin{aligned}
\boldsymbol{u} - \boldsymbol{u}_h &= (\boldsymbol{u} - I_h\boldsymbol{u}) + (I_h\boldsymbol{u} - \boldsymbol{u}_h), \\
p - p_h &= (p - J_hp) + (J_hp - p_h).
\end{aligned}$$

The first parts $\boldsymbol{u} - I_h\boldsymbol{u}$ and $p - J_hp$ can be estimate with (5.6), while $I_h\boldsymbol{u} - \boldsymbol{u}_h$ and $J_hp - p_h$ will be estimated using the variational formulations (5.3) and (2.11).
First, we let $\boldsymbol{v} = \boldsymbol{u}_h$ in $(2.11)_2$, $\boldsymbol{v}_h = I_h\boldsymbol{u}$ in $(5.3)_2$. One finds

$$(\boldsymbol{u}', \boldsymbol{u} - \boldsymbol{u}_h) \leq a(\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) + b(\boldsymbol{u}_h - \boldsymbol{u}, p) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) + J(\boldsymbol{u}_h) - J(\boldsymbol{u}) + \ell(\boldsymbol{u} - \boldsymbol{u}_h),$$

and

$$\begin{aligned}
(\boldsymbol{u}_h', \boldsymbol{u}_h - I_h\boldsymbol{u}) \leq{}& a(\boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + b(I_h\boldsymbol{u} - \boldsymbol{u}_h, p_h) + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + J(I_h\boldsymbol{u}) - J(\boldsymbol{u}_h) \\
&+ \ell(\boldsymbol{u}_h - I_h\boldsymbol{u}).
\end{aligned}$$

Putting them together gives

$$\begin{aligned}
&(\boldsymbol{u}', \boldsymbol{u} - \boldsymbol{u}_h) + (\boldsymbol{u}_h', \boldsymbol{u}_h - I_h\boldsymbol{u}) \\
\leq{}& a(\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) + a(\boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + b(\boldsymbol{u}_h - \boldsymbol{u}, p) + b(I_h\boldsymbol{u} - \boldsymbol{u}_h, p_h) \\
&+ \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) + J(I_h\boldsymbol{u}) - J(\boldsymbol{u}) + \ell(\boldsymbol{u} - I_h\boldsymbol{u}) \\
\leq{}& a(\boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}) - a(I_hu - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + a(I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
&+ b(\boldsymbol{u}_h - I_h\boldsymbol{u}, p - J_hp) + b(\boldsymbol{u}_h - I_h\boldsymbol{u}, J_hp) + b(I_h\boldsymbol{u} - \boldsymbol{u}, p) + b(I_h\boldsymbol{u} - \boldsymbol{u}_h, p_h) \\
&+ \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) + J(I_h\boldsymbol{u} - \boldsymbol{u}) + \ell(\boldsymbol{u} - I_h\boldsymbol{u}).
\end{aligned} \tag{5.8}$$

Secondly since we are dealing with conforming spaces, $(2.11)_3$, and $(5.3)_3$ imply that

$$\text{for all } q_h \in M_h, \quad b(\boldsymbol{u} - \boldsymbol{u}_h, q_h) = 0,$$

which by linearity gives

$$\text{for all } q_h \in M_h\ ,\ b(I_h\boldsymbol{u} - \boldsymbol{u}_h, q_h) = b(I_h\boldsymbol{u} - \boldsymbol{u}, q_h). \tag{5.9}$$

For $(\boldsymbol{v}_h, q_h) = (I_h\boldsymbol{u} - \boldsymbol{u}_h, J_hp)$, the Stokes operator (5.5) reads

$$\begin{aligned}
&a(I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + b(I_h\boldsymbol{u} - \boldsymbol{u}_h, J_hp - p) = 0, \\
&b(I_h\boldsymbol{u} - \boldsymbol{u}, J_hp) = 0,
\end{aligned}$$

which together with (5.9) in (5.8) gives

$$\begin{aligned}
&(\boldsymbol{u}', \boldsymbol{u} - \boldsymbol{u}_h) + (\boldsymbol{u}_h', \boldsymbol{u}_h - I_h\boldsymbol{u}) \\
&\leq a(\boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}) - a(I_hu - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + a(I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
&\quad + b(\boldsymbol{u}_h - I_h\boldsymbol{u}, p - J_hp) + b(I_h\boldsymbol{u} - \boldsymbol{u}, p) + b(I_h\boldsymbol{u} - \boldsymbol{u}_h, p_h) \\
&\quad + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) + J(I_h\boldsymbol{u} - \boldsymbol{u}) + \ell(\boldsymbol{u} - I_h\boldsymbol{u}).
\end{aligned} \tag{5.10}$$

For $q_h = p_h$ in (5.9) and making use of $(5.5)_2$, one has

$$b(I_h\boldsymbol{u} - \boldsymbol{u}_h, p_h) = b(I_h\boldsymbol{u} - \boldsymbol{u}, p_h) = 0\,.$$

Then (5.10) is reduced to

$$\begin{aligned}
&(\boldsymbol{u}', \boldsymbol{u} - \boldsymbol{u}_h) + (\boldsymbol{u}_h', \boldsymbol{u}_h - I_h\boldsymbol{u}) \\
\le & a(\boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}) - a(I_h u - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + a(I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
&+ b(\boldsymbol{u}_h - I_h\boldsymbol{u}, p - J_h p) + b(I_h\boldsymbol{u} - \boldsymbol{u}, p - J_h p) \\
&+ \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) + J(I_h\boldsymbol{u} - \boldsymbol{u}) + \ell(\boldsymbol{u} - I_h\boldsymbol{u})\,.
\end{aligned} \tag{5.11}$$

The expression $\widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u})$ is treated by using the same properties satisfy by $d(\cdot,\cdot,\cdot)$ (see the definition (5.1)).

$$\begin{aligned}
\widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) &= \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h - I_h\boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
&= \widetilde{d}(\boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h)\,, \\
\widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) &= \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - I_h\boldsymbol{u}) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u})\,.
\end{aligned}$$

Hence

$$\begin{aligned}
& \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u}) \\
= & \widetilde{d}(\boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) - \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}) \\
= & \widetilde{d}(\boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}) \\
= & \widetilde{d}(\boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}_h - I_h\boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(I_h\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
& + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u})\,.
\end{aligned} \tag{5.12}$$

(5.12) in (5.11) implies that

$$\begin{aligned}
&(\boldsymbol{u}', \boldsymbol{u} - \boldsymbol{u}_h) + (\boldsymbol{u}_h', \boldsymbol{u}_h - I_h\boldsymbol{u}) + a(I_h\boldsymbol{u} - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
\le & a(\boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}) + a(I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + b(\boldsymbol{u}_h - I_h\boldsymbol{u}, p - J_h p) + b(I_h\boldsymbol{u} - \boldsymbol{u}, p - J_h p) + J(I_h\boldsymbol{u} - \boldsymbol{u}) \\
& + \ell(\boldsymbol{u} - I_h\boldsymbol{u}) + \widetilde{d}(\boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}_h - I_h\boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(I_h\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
& + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u})\,.
\end{aligned} \tag{5.13}$$

Thirdly,

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|I_h\boldsymbol{u} - \boldsymbol{u}_h\|^2 + a(I_h\boldsymbol{u} - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
= & ((I_h\boldsymbol{u})' - \boldsymbol{u}_h', I_h\boldsymbol{u} - \boldsymbol{u}_h) + a(I_h\boldsymbol{u} - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
= & ((I_h\boldsymbol{u})', I_h\boldsymbol{u} - \boldsymbol{u}_h) + (\boldsymbol{u}_h', \boldsymbol{u}_h - I_h\boldsymbol{u}) + a(I_h\boldsymbol{u} - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
= & ((I_h\boldsymbol{u})' - \boldsymbol{u}', I_h\boldsymbol{u} - \boldsymbol{u}_h) + (\boldsymbol{u}', I_h\boldsymbol{u} - \boldsymbol{u}) \\
& + (\boldsymbol{u}', \boldsymbol{u} - \boldsymbol{u}_h) + (\boldsymbol{u}_h', \boldsymbol{u}_h - I_h\boldsymbol{u}) + a(I_h\boldsymbol{u} - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h).
\end{aligned} \tag{5.14}$$

Next replacing (5.13) in (5.14), and using standard inequality we find

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|I_h\boldsymbol{u} - \boldsymbol{u}_h\|^2 + a(I_h\boldsymbol{u} - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
\le & ((I_h\boldsymbol{u})' - \boldsymbol{u}', I_h\boldsymbol{u} - \boldsymbol{u}_h) + (\boldsymbol{u}', I_h\boldsymbol{u} - \boldsymbol{u}) + a(\boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}) + a(I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
& + b(\boldsymbol{u}_h - I_h\boldsymbol{u}, p - J_h p) + b(I_h\boldsymbol{u} - \boldsymbol{u}, p - J_h p) + J(I_h\boldsymbol{u} - \boldsymbol{u}) + \ell(\boldsymbol{u} - I_h\boldsymbol{u}) + \widetilde{d}(\boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
& + \widetilde{d}(\boldsymbol{u}_h - I_h\boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(I_h\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}) \\
\le & \|(I_h\boldsymbol{u})' - \boldsymbol{u}'\|\,\|I_h\boldsymbol{u} - \boldsymbol{u}_h\| + \|\boldsymbol{u}'\|\,\|I_h\boldsymbol{u} - \boldsymbol{u}\| + \nu\|\boldsymbol{u}\|_1\,\|I_h\boldsymbol{u} - \boldsymbol{u}\|_1 \\
& + \nu\|I_h\boldsymbol{u} - \boldsymbol{u}\|_1\,\|I_h\boldsymbol{u} - \boldsymbol{u}_h\|_1 + c\|\boldsymbol{u}_h - I_h\boldsymbol{u}\|_1\,\|p - J_h p\| \\
& + c\|I_h\boldsymbol{u} - \boldsymbol{u}\|_1\|p - J_h p\| + c\|g\|_{L^\infty(S)}\|I_h\boldsymbol{u} - \boldsymbol{u}\|_S \\
& + \|\boldsymbol{f}\|_{\mathbb{V}^*}\|I_h\boldsymbol{u} - \boldsymbol{u}\|_1 + c\|\nabla\boldsymbol{u}_h\|\|\nabla(I_h\boldsymbol{u} - \boldsymbol{u})\|\|\nabla(I_h\boldsymbol{u} - \boldsymbol{u}_h)\| \\
& + c\|\boldsymbol{u}_h - I_h\boldsymbol{u}\|\|\boldsymbol{u}_h - I_h\boldsymbol{u}\|_1\|\nabla\boldsymbol{u}\| + c\|\nabla\boldsymbol{u}\|\|I_h u - \boldsymbol{u}\|_1\|I_h\boldsymbol{u} - \boldsymbol{u}_h\|_1 \\
& + c\|\nabla\boldsymbol{u}\|\,\|\boldsymbol{u}\|_1\|I_h\boldsymbol{u} - \boldsymbol{u}\|_1\,.
\end{aligned}$$

Noting that the norm induced by $a(\cdot,\cdot)$ is equivalent to the H^1-norm on $\mathbb{V}$, and the utilization of Young's inequality, (5.15) leads to

$$\begin{aligned}
&\frac{d}{dt}\|I_h\boldsymbol{u}-\boldsymbol{u}_h\|^2 + ca(I_h\boldsymbol{u}-\boldsymbol{u}_h, I_h\boldsymbol{u}-\boldsymbol{u}_h)\\
\leq\ & c\|(I_h\boldsymbol{u})'-\boldsymbol{u}'\|^2 + \|\boldsymbol{u}'\|\,\|I_h\boldsymbol{u}-\boldsymbol{u}\| + c\|\boldsymbol{u}\|_1\,\|I_h\boldsymbol{u}-\boldsymbol{u}\|_1 + \nu c\|I_h\boldsymbol{u}-\boldsymbol{u}\|_1^2 + c\|p-J_hp\|^2\\
&+c\left(\|\boldsymbol{u}\|_1^2+\|\boldsymbol{u}_h\|_1^2+\|g\|_{L^\infty(S)}+\|\boldsymbol{f}\|_{\mathbb{V}'}\right)\|I_h\boldsymbol{u}-\boldsymbol{u}\|_1 + c\|\boldsymbol{u}\|_1\|I_hu-\boldsymbol{u}\|_1^2 \qquad (5.15)\\
&+c\|\boldsymbol{u}\|_1^2\|\boldsymbol{u}_h-I_h\boldsymbol{u}\|^2\,.
\end{aligned}$$

Solving the differential inequality (5.15), we find

$$\|I_h\boldsymbol{u}-\boldsymbol{u}_h\|^2 \quad\leq\quad \left[\|I_h\boldsymbol{u}(0)-\boldsymbol{u}_h(0)\|^2+F\right]\exp\left(c\int_0^T\|\nabla\boldsymbol{u}(s)\|^2ds\right), \qquad (5.16)$$

with

$$\begin{aligned}
F \quad=\quad & c\|I_h\boldsymbol{u}'-\boldsymbol{u}'\|_{L^2(L^2)}^2 + \|\boldsymbol{u}'\|_{L^2(L^2)}\|I_h\boldsymbol{u}-\boldsymbol{u}\|_{L^2(L^2)} + c\|\boldsymbol{u}\|_{L^2(H^1)}\|I_h\boldsymbol{u}-\boldsymbol{u}\|_{L^2(H^1)}\\
&+\nu c\|I_h\boldsymbol{u}-\boldsymbol{u}\|_{L^2(H^1)}^2 + c\|p-J_hp\|_{L^2(L^2)}^2 + c\|\boldsymbol{u}\|_{L^\infty(H^1)}\|I_hu-\boldsymbol{u}\|_{L^2(H^1)}^2\\
&+c\|\boldsymbol{u}\|_{L^\infty(H^1)}^2\|I_h\boldsymbol{u}-\boldsymbol{u}\|_{L^1(H^1)} + c\|p-J_hp\|_{L^2(L^2)}\|I_h\boldsymbol{u}-\boldsymbol{u}\|_{L^2(H^1)}\\
&+\|g\|_{L^\infty(S)}\|I_h\boldsymbol{u}-\boldsymbol{u}\|_{L^1(H^1)} + \|\boldsymbol{f}\|_{L^2(V^*)}\|I_h\boldsymbol{u}-\boldsymbol{u}\|_{L^2(H^1)}\,.
\end{aligned}$$

So the first part of the theorem is obtained after application of the triangle inequality. Next, by integrating the differential equation (5.15) over $(0,T)$ and using (2.3), we obtain

$$\|I_h\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(H^1)}^2 \leq \|I_h\boldsymbol{u}(0)-\boldsymbol{u}_h(0)\|^2 + F + c\|\boldsymbol{u}\|_{L^2(H^1)}^2\|\boldsymbol{u}_h-I_h\boldsymbol{u}\|_{L^\infty(L^2)}^2\,. \qquad (5.17)$$

Application of the triangle inequality in (5.17) will complete of the theorem. □

It is noted that for $(\boldsymbol{v},q)\in\boldsymbol{H}^2\times\boldsymbol{H}^1$, $((I_hv)',(J_hq)')=(I_hv',J_hq')$. Hence the estimates (5.6) is applicable for handling $\|I_h\boldsymbol{u}'-\boldsymbol{u}'\|$.

Remark 5.1. *It is clear with the properties (5.6) that*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^\infty(L^2)} = 0(h^{1/2}) \ \textit{and}\ \ \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(H^1)} \quad=\quad 0(h^{1/2})\,.$$

The rate of convergence obtained is not surprising for the variational inequality of second kind (see [11, 14, 20, 23]). The regularity $\boldsymbol{u}'(t)\in\boldsymbol{H}^2$ is needed for the estimation of $\|I_h\boldsymbol{u}'-\boldsymbol{u}'\|$ via (5.6).

In the analysis below, the following inequality will be crucial: there exists a constant $c>0$ such that for all $\boldsymbol{v}_h\in\mathbb{V}_h$ and $K\in\mathcal{T}$, we have

$$\|\nabla\boldsymbol{v}_h\|_K \leq ch_K^{-1}\|\boldsymbol{v}_h\|_K\,. \qquad (5.18)$$

The next, result is crucial for estimating the pressure. We claim that

Theorem 5.3. *Under the assumptions of Theorem 5.2, the following estimates hold true:*

$$\begin{aligned}
&\|\boldsymbol{u}'-\boldsymbol{u}_h'\|_{L^2(L^2)}^2 + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^\infty(H^1)}^2\\
\leq\ & c\,h^4\|\partial_t\boldsymbol{u}\|_{L^2(H^2)}^2 + c\,h^2\|\boldsymbol{u}\|_{L^\infty(H^2)}^2 + a(I_h\boldsymbol{u}_0-\boldsymbol{u}_h(0), I_h\boldsymbol{u}_0-\boldsymbol{u}_h(0))\\
&+c\,h^4\|\partial_t\boldsymbol{u}\|_{L^2(H^2)}^2 + c\,h^2\|\boldsymbol{u}\|_{L^2(H^2)}^2 + c\|\boldsymbol{u}\|_{L^\infty(H^2)}^2\|I_h\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(H^1)}^2\\
&+c\,h^3\|\boldsymbol{u}\|_{L^4(H^2)}^4 + c\|\boldsymbol{u}_h\|_{L^2(H^2)}^2\|I_h\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(H^2)}^2
\end{aligned}$$

where c is a positive constant independent of h.

Proof. Let $(\boldsymbol{w},\boldsymbol{w}_h)\in\mathbb{V}\times\mathbb{V}_h$ such that $\boldsymbol{v}-\boldsymbol{u}=\pm\boldsymbol{w}$ with $\boldsymbol{w}|_S=0$ and $\boldsymbol{v}_h-\boldsymbol{u}_h=\pm\boldsymbol{w}_h$ with $\boldsymbol{w}_h|_S=0$. Then the velocity equations in (2.11) and (5.3) lead to

$$\begin{aligned}
(\boldsymbol{u}',\boldsymbol{w}) + a(\boldsymbol{u},\boldsymbol{w}) + b(\boldsymbol{w},p) + \widetilde{d}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{w}) &\quad=\quad \langle\boldsymbol{f},\boldsymbol{w}\rangle,\\
(\boldsymbol{u}_h',\boldsymbol{w}_h) + a(\boldsymbol{u}_h,\boldsymbol{w}_h) + b(\boldsymbol{w}_h,p_h) + \widetilde{d}(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{w}_h) &\quad=\quad \langle\boldsymbol{f},\boldsymbol{w}_h\rangle\,.
\end{aligned}$$

Since $\mathbb{V}_h$ is a subset of $\mathbb{V}$ one can replace $\boldsymbol{w}$ by $\boldsymbol{w}_h$, and subtract the resulting equations. This gives

$$(\boldsymbol{u}' - \boldsymbol{u}_h', \boldsymbol{w}_h) = b(\boldsymbol{w}_h, p_h - p) - a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) - \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}_h)\,,$$

which is re-written thanks to (5.5) as follows

$$\begin{aligned} & (I_h\boldsymbol{u}' - \boldsymbol{u}_h', \boldsymbol{w}_h) + a(I_h\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, J_h p - p_h) \\ = \;& (I_h\boldsymbol{u}' - \boldsymbol{u}', \boldsymbol{w}_h) + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) - \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}_h)\,. \end{aligned} \tag{5.19}$$

From the incompressibility equations of (2.11) and (5.3), one deduces that

$$\text{for all } q_h \in M_h\ ,\quad b(\boldsymbol{u} - \boldsymbol{u}_h, q_h) = 0\,,$$

which with the help of (5.5) implies that

$$\text{for all } q_h \in M_h\ ,\quad b(I_h\boldsymbol{u} - \boldsymbol{u}_h, q_h) = 0\,. \tag{5.20}$$

From (5.20) we deduce that $b(I_h\boldsymbol{u}' - \boldsymbol{u}_h', q_h) = 0$. So taking $\boldsymbol{w}_h = I_h\boldsymbol{u}' - \boldsymbol{u}_h'$ in (5.19), and using (5.6), we find

$$\begin{aligned} & \|I_h\boldsymbol{u}' - \boldsymbol{u}_h'\|^2 + \frac{1}{2}\frac{d}{dt} a(I_h\boldsymbol{u} - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\ = \;& (I_h\boldsymbol{u}' - \boldsymbol{u}', I_h\boldsymbol{u}' - \boldsymbol{u}_h') + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') - \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h' - I_h\boldsymbol{u}') \\ \leq \;& \|I_h\boldsymbol{u}' - \boldsymbol{u}'\|\|I_h\boldsymbol{u}' - \boldsymbol{u}_h'\| + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') - \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h' - I_h\boldsymbol{u}') \\ \leq \;& c\,h^2\|\partial_t\boldsymbol{u}\|_2\|I_h\boldsymbol{u}' - \boldsymbol{u}_h'\| + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') - \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h' - I_h\boldsymbol{u}') \end{aligned} \tag{5.21}$$

From the linearity of $\widetilde{d}(\cdot,\cdot,\cdot)$, we have

$$\begin{aligned} & \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') - \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h' - I_h\boldsymbol{u}') \\ = \;& \widetilde{d}(\boldsymbol{u}, \boldsymbol{u} - I_h\boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') + \widetilde{d}(\boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h' - I_h\boldsymbol{u}') \\ & + \widetilde{d}(\boldsymbol{u} - I_h\boldsymbol{u}, \boldsymbol{u}_h - I_h\boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') + \widetilde{d}(\boldsymbol{u} - I_h\boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') \\ & + \widetilde{d}(\boldsymbol{u} - I_h\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') + \widetilde{d}(I_h\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h' - I_h\boldsymbol{u}') \\ = \;& I_1 + I_2 + I_3\,. \end{aligned} \tag{5.22}$$

We bound I_1 as follows

$$\begin{aligned} I_1 \;=\;& \widetilde{d}(\boldsymbol{u}, \boldsymbol{u} - I_h\boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') + \widetilde{d}(\boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h' - I_h\boldsymbol{u}') \\ \leq\;& \|\boldsymbol{u}\|_{L^\infty}\|\nabla(\boldsymbol{u} - I_h\boldsymbol{u})\|\|\boldsymbol{u}' - I_h\boldsymbol{u}'\| + \|\boldsymbol{u}\|_{L^\infty}\|\nabla(I_h\boldsymbol{u} - \boldsymbol{u}_h)\|\|I_h\boldsymbol{u}' - \boldsymbol{u}_h'\| \\ \leq\;& c\,h\|\boldsymbol{u}\|_2\|\boldsymbol{u}' - I_h\boldsymbol{u}'\| + c\|\boldsymbol{u}\|_2\|\nabla(I_h\boldsymbol{u} - \boldsymbol{u}_h)\|\|I_h\boldsymbol{u}' - \boldsymbol{u}_h'\|\,, \end{aligned} \tag{5.23}$$

where (5.6) has been used, and c a constant depending on Ω and independent of h. The second term I_2 is treated using the continuity properties of $\widetilde{d}(\cdot,\cdot,\cdot)$ together with (5.18) and (5.6)

$$\begin{aligned} I_2 \;=\;& \widetilde{d}(\boldsymbol{u} - I_h\boldsymbol{u}, \boldsymbol{u}_h - I_h\boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') + \widetilde{d}(\boldsymbol{u} - I_h\boldsymbol{u}, I_h\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') \\ \leq\;& \boldsymbol{c}\|\nabla(\boldsymbol{u} - I_h\boldsymbol{u})\|\,\|\nabla(\boldsymbol{u}_h - I_h\boldsymbol{u})\|\,\|\nabla(\boldsymbol{u}_h' - I_h\boldsymbol{u}')\| + \boldsymbol{c}\|\boldsymbol{u} - I_h\boldsymbol{u}\|\,\|\nabla(I_h\boldsymbol{u} - \boldsymbol{u})\|\|\nabla(\boldsymbol{u}_h' - I_h\boldsymbol{u}')\| \\ \leq\;& \frac{\boldsymbol{c}}{h}\|\nabla(\boldsymbol{u} - I_h\boldsymbol{u})\|\,\|\nabla(\boldsymbol{u}_h - I_h\boldsymbol{u})\|\,\|\boldsymbol{u}_h' - I_h\boldsymbol{u}'\| + \frac{\boldsymbol{c}}{h}\|\boldsymbol{u} - I_h\boldsymbol{u}\|\,\|\nabla(I_h\boldsymbol{u} - \boldsymbol{u})\|\,\|\boldsymbol{u}_h' - I_h\boldsymbol{u}'\| \\ \leq\;& c\|\boldsymbol{u}\|_2\|\nabla(\boldsymbol{u}_h - I_h\boldsymbol{u})\|\,\|\boldsymbol{u}_h' - I_h\boldsymbol{u}'\| + c\,h^2\|\boldsymbol{u}\|_2^2\|\boldsymbol{u}_h' - I_h\boldsymbol{u}'\|\,, \end{aligned} \tag{5.24}$$

with c a constant depending on Ω and independent of h. Finally,

$$\begin{aligned} I_3 \;=\;& \widetilde{d}(\boldsymbol{u} - I_h\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}_h' - I_h\boldsymbol{u}') + \widetilde{d}(I_h\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h' - I_h\boldsymbol{u}') \\ \leq\;& \|\boldsymbol{u} - I_h\boldsymbol{u}\|_{L^4}\|\nabla\boldsymbol{u}\|_{L^4}\|\boldsymbol{u}_h' - I_h\boldsymbol{u}'\| + \|I_h\boldsymbol{u} - \boldsymbol{u}_h\|_{L^4}\|\nabla\boldsymbol{u}_h\|_{L^4}\|\boldsymbol{u}_h' - I_h\boldsymbol{u}'\| \\ \leq\;& c\|\boldsymbol{u}\|_2\|\boldsymbol{u} - I_h\boldsymbol{u}\|^{1/2}\|\nabla(\boldsymbol{u} - I_h\boldsymbol{u})\|^{1/2}\|\boldsymbol{u}_h' - I_h\boldsymbol{u}'\| + c\|\boldsymbol{u}_h\|_2\|\nabla(I_h\boldsymbol{u} - \boldsymbol{u}_h)\|\|\boldsymbol{u}_h' - I_h\boldsymbol{u}'\| \\ \leq\;& ch^{3/2}\|\boldsymbol{u}\|_2^2\|\boldsymbol{u}_h' - I_h\boldsymbol{u}'\| + c\|\boldsymbol{u}_h\|_2\|\nabla(I_h\boldsymbol{u} - \boldsymbol{u}_h)\|\|\boldsymbol{u}_h' - I_h\boldsymbol{u}'\|\,, \end{aligned} \tag{5.25}$$

where (5.6) has been used and c is a positive constant depending on Ω and independent of h. Now collecting (5.25), (5.24) and (5.23), and using (5.6), Young's inequality, the inequality (5.21) gives

$$\begin{aligned}
&\|I_h\boldsymbol{u}' - \boldsymbol{u}_h'\|^2 + \frac{d}{dt}a(I_h\boldsymbol{u} - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
\leq & c\,h^4\|\partial_t\boldsymbol{u}\|_2^2 + c\|\boldsymbol{u}\|_2^2h^2 + c\|\boldsymbol{u}\|_2^2\|\nabla(I_h\boldsymbol{u} - \boldsymbol{u}_h)\|^2 \\
&+c\|\boldsymbol{u}\|_2^4h^4 + c\|\boldsymbol{u}\|_2^4h^3 + c\|\boldsymbol{u}_h\|_2^2\|\nabla(I_h\boldsymbol{u} - \boldsymbol{u}_h)\|^2 .
\end{aligned} \tag{5.26}$$

We integrate (5.26) from 0 to t, we find

$$\begin{aligned}
& \|I_h\boldsymbol{u}' - \boldsymbol{u}_h'\|^2_{L^2(0,T;L^2)} + a(I_h\boldsymbol{u} - \boldsymbol{u}_h, I_h\boldsymbol{u} - \boldsymbol{u}_h) \\
\leq\ & a(I_h\boldsymbol{u}_0 - \boldsymbol{u}_h(0), I_h\boldsymbol{u}_0 - \boldsymbol{u}_h(0)) + c\,h^4\|\partial_t\boldsymbol{u}\|^2_{L^2(H^2)} + c\,h^2\|\boldsymbol{u}\|^2_{L^2(H^2)} \\
& +c\|\boldsymbol{u}\|^2_{L^\infty(H^2)}\|I_h\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^1)} + c\,h^3\|\boldsymbol{u}\|^4_{L^4(H^2)} \\
& +c\|\boldsymbol{u}_h\|^2_{L^2(H^2)}\|I_h\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^2)} .
\end{aligned}$$

So the proof is complete after application of the triangle inequality. □

The error for the pressure estimation is stated as follows

Theorem 5.4. *Let $(\boldsymbol{u}, p)$ be the solution of Problem (2.11), $(\boldsymbol{u}_h, p_h)$ the finite element solution defined through (5.3). Under the assumptions on Theorem 5.2, there is a generic constant c independent of h such that*

$$\begin{aligned}
\|p - p_h\|^2_{L^2(L^2)} \leq\ & c\|J_hp - p\|^2_{L^2(L^2)} + c\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^1)} + c\|\boldsymbol{u}\|^2_{L^\infty(H^1)}\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^1)} \\
& +c\|\boldsymbol{u}_h - I_h\boldsymbol{u}\|^2_{L^\infty(H^1)}\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^1)} + c\|I_h\boldsymbol{u} - \boldsymbol{u}\|^2_{L^\infty(H^1)}\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^1)} \\
& +c\|\boldsymbol{u}\|^2_{L^\infty(H^1)}\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^1)} + \|\boldsymbol{u}' - \boldsymbol{u}_h'\|^2_{L^2(L^2)}
\end{aligned}$$

Proof. Let $(\boldsymbol{w}, \boldsymbol{w}_h) \in \mathbb{V}\times\mathbb{V}_h$ such that $\boldsymbol{v} - \boldsymbol{u} = \pm\boldsymbol{w}$ with $\boldsymbol{w}|_S = 0$ and $\boldsymbol{v}_h - \boldsymbol{u}_h = \pm\boldsymbol{w}_h$ with $\boldsymbol{w}_h|_S = 0$. Then (2.11) and (5.3) lead to

$$\begin{aligned}
(\boldsymbol{u}', \boldsymbol{w}) + a(\boldsymbol{u}, \boldsymbol{w}) + b(\boldsymbol{w}, p) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) &= \langle \boldsymbol{f}, \boldsymbol{w}\rangle, \\
(\boldsymbol{u}_h', \boldsymbol{w}_h) + a(\boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p_h) + \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) &= \langle \boldsymbol{f}, \boldsymbol{w}_h\rangle .
\end{aligned}$$

Since $\mathbb{V}_h$ is a subset of $\mathbb{V}$ one can replace $\boldsymbol{w}$ by $\boldsymbol{w}_h$, and subtract the resulting equations. This gives

$$b(\boldsymbol{w}_h, p_h - p) = a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}_h) - \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) + (\boldsymbol{u}' - \boldsymbol{u}_h', \boldsymbol{w}_h) ,$$

which by linearity is

$$\begin{aligned}
b(\boldsymbol{w}_h, p_h - J_hp) &= -b(\boldsymbol{w}_h, J_hp - p) + a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}_h) - \widetilde{d}(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) + (\boldsymbol{u}' - \boldsymbol{u}_h', \boldsymbol{w}_h) \\
&= -b(\boldsymbol{w}_h, J_hp - p) + a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + \widetilde{d}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{w}_h) + \widetilde{d}(\boldsymbol{u}_h - I_h\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) \\
&\quad +\widetilde{d}(I_h\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + \widetilde{d}(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w}_h) + (\boldsymbol{u}' - \boldsymbol{u}_h', \boldsymbol{w}_h).
\end{aligned} \tag{5.27}$$

We recall that since $(\mathbb{V}_h, M_h)$ is inf-sup stable, then $(\mathbb{V}_{0h}, M_h)$ is also inf-sup stable with $\mathbb{V}_{0h}$ a set of $\boldsymbol{v}_h \in \mathbb{V}_h$ with $\boldsymbol{v}_h|_S = 0$. Thus (5.27) together with the inf-sup condition leads to existence of c independent of h such that

$$\begin{aligned}
c\|p_h - J_hp\| \leq\ & \sup_{\boldsymbol{v}_h\in\mathbb{V}_{0h}} \frac{b(\boldsymbol{v}_h, p_h - J_hp)}{\|\boldsymbol{v}_h\|_1} \\
\leq\ & \|J_hp - p\| + \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + c\|\boldsymbol{u}\|_1\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + c\|\boldsymbol{u}_h - I_h\boldsymbol{u}\|_1\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \\
& +c\|I_h\boldsymbol{u} - \boldsymbol{u}\|_1\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + c\|\boldsymbol{u}\|_1\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|\boldsymbol{u}' - \boldsymbol{u}_h'\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|p_h - J_hp\|^2 \leq\ & c\|J_hp - p\|^2 + c\|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 + c\|\boldsymbol{u}\|_1^2\|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 + c\|\boldsymbol{u}_h - I_h\boldsymbol{u}\|_1^2\|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 \\
& +c\|I_h\boldsymbol{u} - \boldsymbol{u}\|_1^2\|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 + c\|\boldsymbol{u}\|_1^2\|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 + \|\boldsymbol{u}' - \boldsymbol{u}_h'\|^2 .
\end{aligned} \tag{5.28}$$

Integrating (5.28) from 0 to T, we find

$$\|p_h - J_h p\|^2_{L^2(L^2)} \leq c\|J_h p - p\|^2_{L^2(L^2)} + c\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^1)} + c\|\boldsymbol{u}\|^2_{L^\infty(H^1)}\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^1)}$$
$$+c\|\boldsymbol{u}_h - I_h\boldsymbol{u}\|^2_{L^\infty(H^1)}\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^1)} + c\|I_h\boldsymbol{u} - \boldsymbol{u}\|^2_{L^\infty(H^1)}\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^1)}$$
$$+c\|\boldsymbol{u}\|^2_{L^\infty(H^1)}\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(H^1)} + \|\boldsymbol{u}' - \boldsymbol{u}'_h\|^2_{L^2(L^2)}\,. \tag{5.29}$$

Finally application of the inequality of triangle conclude the proof of the theorem. □

It clear from (5.6), Theorem 5.3, Theorem 5.2 that

$$\|p - p_h\|^2_{L^2(L^2)} = 0(h^{1/2}).$$

5.3 Fully discrete analysis

In this paragraph, we formulate the space time discretization of the problem and study its convergence following the approach in [10]. Indeed, A. Mielke, L. Paoli and U. Stefanelli in [10] regard the global error as decomposition for the semi-discrete errors. In that approach the fully discrete approximation is viewed as a time approximation of the semi-discrete scheme in space.
Denoting by $\boldsymbol{u}^n_{k,h}, p^n_{k,h}$ as space-time approximation of $\boldsymbol{u}(\boldsymbol{x},t)$ and $p(\boldsymbol{x},t)$ respectively, we consider the fully discretization of problem (2.11):
Find $(\boldsymbol{u}^n_{k,h}, p^n_{k,h}) \in \mathbb{V}_h \times M_h$ such that for all $(\boldsymbol{v}_h, q_h) \in \mathbb{V}_h \times M_h$,

$$\begin{cases} \boldsymbol{u}^0_{k,h}(0) = P\boldsymbol{u}_0\ , \\ \text{and for all}\ \ \boldsymbol{v}_h \in \mathbb{V}_h, \quad \text{and a.e}\ t > 0 \\ \left(\dfrac{\boldsymbol{u}^n_{k,h} - \boldsymbol{u}^{n-1}_{k,h}}{k}, \boldsymbol{u}^n_{k,h} - \boldsymbol{v}_h\right) + a(\boldsymbol{u}^n_{k,h}, \boldsymbol{u}^n_{k,h} - \boldsymbol{v}_h) + b(\boldsymbol{u}^n_{k,h} - \boldsymbol{v}_h, p^n_{k,h}) \\ +\widetilde{d}(\boldsymbol{u}^n_{k,h}, \boldsymbol{u}^n_{k,h}, \boldsymbol{u}^n_{k,h} - \boldsymbol{v}_h) + J(\boldsymbol{u}^n_{k,h}) - J(\boldsymbol{v}_h) \leq (\boldsymbol{f}^n_k, \boldsymbol{u}^n_{k,h} - \boldsymbol{v}_h), \\ b(\boldsymbol{u}^n_{k,h}, q_h) = 0, \end{cases} \tag{5.30}$$

with $P\boldsymbol{u}_0$ the L^2 projection of $\boldsymbol{u}_0$ into $\mathbb{V}_h$. The existence of solutions of (5.30) resemble the existence of solutions of (2.13) treated in Theorem 2.2 and Proposition 2.1.

Lemma 5.1. *The solution $(\boldsymbol{u}^n_{k,h})_n$ of (5.30) remains bounded if the following sense*

$$\|\boldsymbol{u}^n_k\|^2 \leq \|\boldsymbol{u}_0\|^2 + cT\|\boldsymbol{f}\|^2_{L^\infty(L^2)}, \quad n = 1, 2, 3, ...N,$$
$$\nu\, k \sum_{n=1}^{N} \|\varepsilon(\boldsymbol{u}^n_k)\|^2 \leq \|\boldsymbol{u}_0\|^2 + cT\|\boldsymbol{f}\|^2_{L^\infty(L^2)}\,.$$

proof. Choose successively $\boldsymbol{v}_h = \boldsymbol{0}$, $\boldsymbol{v}_h = 2\boldsymbol{u}^n_{k,h}$ and $q_h = p^n_{k,h}$ in (5.30). One finds

$$2(\boldsymbol{u}^n_{k,h} - \boldsymbol{u}^{n-1}_{k,h}, \boldsymbol{u}^n_{k,h}) + 2ka(\boldsymbol{u}^n_{k,h}, \boldsymbol{u}^n_{k,h}) + 2kJ(\boldsymbol{u}^n_{k,h}) = 2k(\boldsymbol{f}^n_k, \boldsymbol{u}^n_{k,h}),$$

this is

$$\|\boldsymbol{u}^n_{k,h}\|^2 - \|\boldsymbol{u}^{n-1}_{k,h}\|^2 + \|\boldsymbol{u}^n_{k,h} - \boldsymbol{u}^{n-1}_{k,h}\|^2 + 2ka(\boldsymbol{u}^n_{k,h}, \boldsymbol{u}^n_{k,h}) + 2kJ(\boldsymbol{u}^n_{k,h}) = 2k(\boldsymbol{f}^n_k, \boldsymbol{u}^n_{k,h})\,.$$

Taking the summation over n, then one obtains

$$\begin{aligned}
& \|\boldsymbol{u}_{k,h}^n\|^2 + \sum_{j=1}^n \|\boldsymbol{u}_{k,h}^j - \boldsymbol{u}_{k,h}^{j-1}\|^2 + 2\nu k \sum_{j=1}^n \|\varepsilon(\boldsymbol{u}_{k,h}^j)\|^2 + 2k \sum_{j=1}^n J(\boldsymbol{u}_{k,h}^j) \\
= \quad & \|\boldsymbol{u}_{k,h}^0\|^2 + 2k \sum_{j=1}^n (\boldsymbol{f}_k^j, \boldsymbol{u}_{k,h}^j) \\
\leq \quad & \|P\boldsymbol{u}_0\|^2 + 2k \sum_{j=1}^n \|\boldsymbol{f}_k^j\| \|\boldsymbol{u}_{k,h}^j\| \\
\leq \quad & \|P\boldsymbol{u}_0\|^2 + 2ck \sum_{j=1}^n \|\boldsymbol{f}_k^j\| \|\varepsilon(\boldsymbol{u}_{k,h}^j)\| \\
\leq \quad & \|P\boldsymbol{u}_0\|^2 + c\frac{k}{\nu} \sum_{j=1}^n \|\boldsymbol{f}_k^j\|^2 + \nu k \sum_{j=1}^n \|\varepsilon(\boldsymbol{u}_{k,h}^j)\|^2 \,,
\end{aligned}$$

that is

$$\|\boldsymbol{u}_{k,h}^n\|^2 + \sum_{j=1}^n \|\boldsymbol{u}_{k,h}^j - \boldsymbol{u}_{k,h}^{j-1}\|^2 + \nu k \sum_{j=1}^n \|\varepsilon(\boldsymbol{u}_{k,h}^j)\|^2 + 2k \sum_{j=1}^n J(\boldsymbol{u}_{k,h}^j) \quad \leq \quad \|P\boldsymbol{u}_0\|^2 + c\frac{k}{\nu} \sum_{j=1}^n \|\boldsymbol{f}_k^j\|^2 \,.$$

The result is obtained by using a discrete version of Gronwall's lemma, the fact that $\|P\boldsymbol{u}_0\| \leq \|\boldsymbol{u}_0\|$ and $k \sum_{j=1}^n \|\boldsymbol{f}_k^j\|^2 \leq \|\boldsymbol{f}\|_{L^2(L^2)}^2 = T\|\boldsymbol{f}\|_{L^\infty(L^2)}^2$. □

Theorem 5.5. *Under the assumptions of Theorem 5.2, Theorem 5.4, Theorem 4.1, Theorem 4.3, then*

$$\sup_{0\leq t\leq T} \|\boldsymbol{u} - \widehat{\boldsymbol{u}}_{k,h}\|_1 = 0(h^{1/2} + k) \quad \textit{and} \quad \|p(t_n) - p_{k,h}^n\| = 0(h^{1/2} + k) \,.$$

proof. From triangle inequality

$$\begin{aligned}
\|\boldsymbol{u} - \widehat{\boldsymbol{u}}_{k,h}\|_1 \quad & \leq \quad \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_{k,h}\|_1 \,, \\
\|p - p_{k,h}^n\| \quad & \leq \quad \|p - p_h\| + \|p_h - p_{k,h}^n\| \,.
\end{aligned}$$

Application of results derived in Theorem 5.2, Theorem 5.4, Theorem 4.1, Theorem 4.3 conclude the proof. □

6 Conclusion

In this work we have analyzed a priori error analysis for the space-time discretization of the Navier Stokes equations under nonlinear slip boundary conditions. The semi discrete problem in time is formulated using backward Euler approach and detailed a priori error analysis is presented with the help of the framework discussed in [1]. We obtained convergence rate for the velocity, pressure without demanding extra regularity on the weak solution. For the semi discrete problem in space, we use finite element approximation and derive a priori error estimates after the construction of a Stokes operator adapted to this situation. Finally, we combine the two semi-discrete problems and make use of the triangle inequality to derive the global a priori error estimates.

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