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ASYMPTOTIC BEHAVIOR OF GAPS BETWEEN ROOTS OF WEIGHTED FACTORIALS

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Abstract. Here, we find a general method for computing the limit of differences of consecutive terms of n-th roots of weighted factorials by a sequence x_n (under some technical condition). As a consequence, we show that $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!x_{n+1}} - \sqrt[n]{n!x_n}\right) = \alpha e^{-1}$, where $\alpha \geq 1$ is the dominant root of the characteristic equation of an increasing linear sequence x_n, and e is Euler's constant.

1. Motivation

In [1], Bătineţu–Giurgiu and Stanciu ask for the limits $\lim_{n\to\infty}(a_{n+1} - a_n)$, where $a_n = \sqrt[n]{n!F_n}, a_n = \sqrt[n]{n!L_n}, a_n = \sqrt[n]{n!!F_n}$, and $a_n = \sqrt[n]{n!!L_n}$, where F_n, respectively, L_n are the Fibonacci, respectively, Lucas sequences. In this note, we introduce a general method that will find the limits of many such differences, in particular, our method is applicable to sequences of the form $a_n = \sqrt[n]{n!x_n}$, where x_n is *any* sequence under some technical assumptions (in particular, the conditions are easily satisfied by any increasing linear recurrence sequence).

2. The results

We start with the next lemma which will be used throughout.

Lemma 2.1. *We have* $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$, $\lim_{n\to\infty}\left(1 \pm \frac{1}{x_n}\right)^{x_n} = e^{\pm 1}$, *if* $0 < x_n \to \infty$ *as* $n \to \infty$.

Proof. The second limit can be found in the reader's preferred calculus book, and the second follows easily by applying Stirling's formula $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, e^{-\frac{u_n}{12n}}$ (where $0 < u_n < 1$), or Stolz-Cesàro theorem [6], which states that if $\{b_n\}_n$ is a divergent strictly monotone real sequence and $\{a_n\}_n$ is an arbitrary real sequence, such that $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = L$, then the following limit exists and $\lim_{n\to\infty}\frac{a_n}{b_n} = L$; or even as a particular case of Theorem 3.37 in [5]. $\square$

Our approach to deal with $(a_{n+1} - a_n)$ is to transform this additive problem into a multiplicative one to be in sync with the flavor of the factorial. (The problem at hand resembles the celebrated Lalescu's sequence limit: $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) = e^{-1}$.) We would like thank

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14. ABSTRACT
Here, we nd a general method for computing the limit of di erences of con- secutive terms of n-th roots of weighted factorials by a sequence xn (under some technical condition). As a consequence, we show that lim n!1   n+1 p (n + 1)!xn+1 ???? n p n!xn   = e????1 where   1 is the dominant root of the characteristic equation of an increasing linear se- quence xn, and e is Euler's constant.

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the referee pointing to the paper [3], which also contains a method for dealing with several such sequences.

Lemma 2.2. *Let* $a_n \geq 1$ *be an increasing sequence of real numbers and set* $b_n := \frac{a_{n+1}}{a_n} > 1$*. If the following conditions hold:*

$$\lim_{n\to\infty} \frac{a_n}{n} = \alpha, \lim_{n\to\infty} b_n = 1, \lim_{n\to\infty} \ln(b_n^n) = \beta,$$

for some real numbers α, β*, then* $\lim\limits_{n\to\infty}(a_{n+1} - a_n) = \alpha\beta$.

Proof. We write

$$\lim_{n\to\infty}(a_{n+1} - a_n) = \lim_{n\to\infty} a_n\,(b_n - 1) = \lim_{n\to\infty} \frac{a_n}{n} \cdot \frac{b_n - 1}{\ln(b_n)} \cdot \ln(b_n^n).$$

Then,

$$\begin{aligned}\lim_{n\to\infty} \frac{b_n - 1}{\ln(b_n)} &= \lim_{n\to\infty} \frac{1}{\ln(b_n)^{\frac{1}{b_n-1}}} = \frac{1}{\lim_{n\to\infty}\ln(b_n)^{\frac{1}{b_n-1}}} \\ &= \frac{1}{\lim_{n\to\infty}\ln(1 + (b_n - 1))^{\frac{1}{b_n-1}}} \\ &= \frac{1}{\ln\left(\lim_{n\to\infty}(1 + (b_n - 1))^{\frac{1}{b_n-1}}\right)} = \frac{1}{\ln e} = 1.\end{aligned}$$

The claim is shown. □

Theorem 2.3. *Let* x_n *be an increasing second-order recurrent sequence of real numbers satisfying* $x_{n+1} = ax_n + bx_{n-1}$*,* $a \geq 0$*, under some initial conditions* $x_0 \geq 0, x_1 > 0$*,* $\Delta = a^2 + 4b \geq 0$*. Assume that* $\alpha = \frac{a+\sqrt{a^2+4b}}{2} \geq 1$ *is the dominant root of the associated characteristic equation for* x_n*. We have the following limits:*

(*i*) *If* $a_n = \sqrt[n]{n!x_n}$*, then* $\lim\limits_{n\to\infty}(a_{n+1} - a_n) = \dfrac{\alpha}{e}$.

(*ii*) *If* $a_n = \sqrt[n]{(2n)!!x_n}$*, or* $a_n = \sqrt[n]{(2n-1)!!x_n}$*, then* $\lim\limits_{n\to\infty}(a_{n+1} - a_n) = \dfrac{2\alpha}{e}$.

Proof. We show (*i*) first. We first assume that the sequence is nondegenerate, that is, $\Delta = a^2 + 4b \neq 0$. Let $\alpha = \frac{a+\sqrt{a^2+4b}}{2}, \bar{\alpha} = \frac{a-\sqrt{a^2+4b}}{2}$ be the roots of the associated characteristic equation $x^2 - ax - b = 0$, and so

$$x_n = A\alpha^n + B\bar{\alpha}^n, \text{ where } A = \frac{x_1 - x_0\bar{\alpha}}{\Delta} > 0, B = \frac{x_0\alpha - x_1}{\Delta} < 0, \Delta = \sqrt{a^2 + 4b}.$$

Given our assumptions, we see that $A \geq |B| = -B$ and $\alpha > |\bar{\alpha}|$.

We will check the conditions of Lemma 2.2. We will use the inequalities (for $n \geq 1$)

$$\min\left\{x_2, \frac{A}{\alpha^2}\right\}\alpha^{n-2} \leq x_n \leq (A - B)\alpha^n. \tag{2.1}$$

The upper bound follows easily since $\alpha > |\bar{\alpha}|$ and so $x_n = A\alpha^n + B\bar{\alpha}^n \leq A\alpha^n + |B||\bar{\alpha}|^n \leq (A + |B|)\alpha^n$. We now show the lower bound. If n is odd, then $x_n = A\alpha^n + B\bar{\alpha}^n > A\alpha^n$

(since $B < 0, \bar{\alpha} < 0$). We next assume that n is even. The lower bound will be shown in this case if we can prove that $x_n = A\alpha^n + B\bar{\alpha}^n = \alpha^n \left(A - |B| \left(\frac{\bar{\alpha}}{\alpha}\right)^n\right) \geq \alpha^n \frac{x_2}{\alpha^2}$. Since the sequence $A - |B| \left(\frac{\bar{\alpha}}{\alpha}\right)^n$ is increasing with respect to even n, then $A - |B| \left(\frac{\bar{\alpha}}{\alpha}\right)^n \geq A - |B| \left(\frac{\bar{\alpha}}{\alpha}\right)^2 = \frac{x_2}{\alpha^2}$.

From (2.1), we see that $\lim_{n\to\infty} \sqrt[n]{x_n} = \alpha$. We infer,

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{\sqrt[n]{n! x_n}}{n} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \cdot \lim_{n\to\infty} \sqrt[n]{x_n} = \frac{\alpha}{e}, \tag{2.2}$$

from Lemma 2.1 and the previous analysis. Next, for $b_n = \frac{a_{n+1}}{a_n}$, we have

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)! x_{n+1}}}{\sqrt[n]{n! x_n}} \\
&= \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} \cdot \frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}} \\
&= \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}/(n+1)}{\sqrt[n]{n!}/n} \cdot \frac{n}{n+1} \cdot \frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}} \\
&= 1.
\end{aligned}$$

Further, $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{A\alpha^{n+1} + B\bar{\alpha}^{n+1}}{A\alpha^n + B\bar{\alpha}^n} = \lim_{n\to\infty} \frac{\alpha^{n+1}\left(A + B\frac{\bar{\alpha}^{n+1}}{\alpha^{n+1}}\right)}{\alpha^n\left(A + B\frac{\bar{\alpha}^n}{\alpha^n}\right)} = \alpha$, and so,

$$\begin{aligned}
\ln \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{(n+1)! x_{n+1}}}{\sqrt[n]{n! x_n}}\right)^n &= \ln \lim_{n\to\infty} \frac{((n+1)!)^{n/(n+1)} x_{n+1}^{n/(n+1)}}{n! x_n} \\
&= \ln \lim_{n\to\infty} \frac{(n+1)!((n+1)!)^{-1/(n+1)} x_{n+1} x_{n+1}^{-1/(n+1)}}{n! x_n} \\
&= \ln \lim_{n\to\infty} \frac{n+1}{\sqrt[n+1]{(n+1)!}} \cdot \lim_{n\to\infty} \frac{x_{n+1}}{x_n} \cdot \lim_{n\to\infty} x_{n+1}^{-1/(n+1)} \qquad (2.3) \\
&= \ln(e \cdot \alpha \cdot \alpha^{-1}) = 1.
\end{aligned}$$

Thus, by Lemma 2.2, $\lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)! x_{n+1}} - \sqrt[n]{n! x_n}\right) = \frac{\alpha}{e}$.

We next assume that the sequence x_n is degenerate, and so, $\Delta = 0$. Therefore, $x_n = (A + Bn)\alpha^n$, where $\alpha = \frac{a}{2}, A = x_0, B = \frac{x_1}{\alpha} - x_0$ (it is obvious that if $\Delta = 0$, then $a\alpha \neq 0$). As before, for $b_n = \frac{a_{n+1}}{a_n}$,

$$\lim_{n\to\infty} \frac{a_n}{n} = \frac{\alpha}{e}, \quad \lim_{n\to\infty} b_n = 1, \quad \lim_{n\to\infty} \ln(b_n^n) = 1,$$

and consequently, $\lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)! x_{n+1}} - \sqrt[n]{n! x_n}\right) = \frac{\alpha}{e}$.

We now show (ii). Recall that

$$\begin{aligned}
(2n-1)!! &= \frac{(2n)!}{2^n\, n!}, \\
(2n)!! &= 2^n\, n!
\end{aligned}$$

Thus, if $a_n = \sqrt[n]{(2n)!!x_n}$, then

$$\lim_{n\to\infty}(a_{n+1}-a_n) = 2\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!x_{n+1}} - \sqrt[n]{n!x_n}\right) = \frac{2\alpha}{e},$$

by the previous work. We now assume that $a_n = \sqrt[n]{(2n-1)!!x_n} = \frac{1}{2}\sqrt[n]{\frac{(2n)!}{n!}x_n}$. As before, we will check the conditions of Lemma 2.2.

First, since $\lim_{n\to\infty}\frac{\sqrt[n]{(2n)!}}{(2n)^2} = \frac{1}{e^2}$ (by a simple application of Lemma 2.1), then (regardless of whether x_n is degenerate or not)

$$\lim_{n\to\infty}\frac{a_n}{n} = \frac{1}{2}\lim_{n\to\infty}\frac{\sqrt[n]{\frac{(2n)!}{n!}x_n}}{n} = 2\lim_{n\to\infty}\frac{\sqrt[n]{(2n)!}}{(2n)^2}\cdot\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\cdot\lim_{n\to\infty}\sqrt[n]{x_n} = 2\cdot\frac{1}{e^2}\cdot e\cdot\alpha = \frac{2\alpha}{e}$$

Similarly,

$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\frac{\sqrt[n+1]{\frac{(2n+2)!}{(n+1)!}x_{n+1}}}{\sqrt[n]{\frac{(2n)!}{n!}x_n}}\\ &= \lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+2)!}\,\sqrt[n]{n!}}{\sqrt[n]{(2n)!}\,\sqrt[n+1]{(n+1)!}}\cdot\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}}\\ &= \lim_{n\to\infty}\frac{\frac{\sqrt[n+1]{(2n+2)!}}{(2n+2)^2}\cdot\frac{\sqrt[n]{n!}}{n}}{\frac{\sqrt[n]{(2n)!}}{(2n)^2}\cdot\frac{\sqrt[n+1]{(n+1)!}}{n+1}}\cdot\frac{n(2n+2)^2}{(n+1)(2n)^2}\cdot\frac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n]{x_n}}\\ &= 1.\end{aligned}$$

Lastly, observe that

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+2)!}}{\sqrt[n]{(2n)!}} = \lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+2)!}/(2n+2)^2}{\sqrt[n]{(2n)!}/(2n)^2}\cdot\frac{(2n+2)^2}{(2n)^2} = 1,$$

which implies that $\lim_{n\to\infty}\ln(b_n^n) = 1$, and consequently, $\lim_{n\to\infty}(a_{n+1}-a_n) = \frac{2\alpha}{e}$. □

The next corollary solves immediately the posed problem B-1151, along with B-1160:(2) and (4).

Corollary 2.4. *Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio, and e be Euler's constant. Then:*

(*i*) $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!F_{n+1}} - \sqrt[n]{n!F_n}\right) = \frac{\phi}{e}$,

(*ii*) $\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!L_{n+1}} - \sqrt[n]{n!L_n}\right) = \frac{\phi}{e}$,

(*iii*) $\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!F_{n+1}} - \sqrt[n]{(2n-1)!!F_n}\right) = \frac{2\phi}{e}$,

(*iv*) $\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!L_{n+1}} - \sqrt[n]{(2n-1)!!L_n}\right) = \frac{2\phi}{e}$,

(*v*) $\lim_{n\to\infty}\left(e_{n+1}\cdot\sqrt[n+1]{(n+1)!F_{n+1}} - e_n\sqrt[n]{n!F_n}\right) = \phi$,

(*vi*) $\lim_{n\to\infty}\left(e_{n+1}\cdot\sqrt[n+1]{(n+1)!L_{n+1}} - e_n\sqrt[n]{n!L_n}\right) = \phi$.

One would wonder if the method is extendable to other sequences x_n. The same proof we have used for the second-order linear sequence will work for any sequence $\{x_n\}$, under some technical conditions (see the theorem below).

Consequently, the following generalization of Theorem 2.3 will hold.

Theorem 2.5. *Let x_n be any increasing sequence of positive real numbers with exponential growth, precisely, $\lim_{n\to\infty} \sqrt[n]{x_n} = \alpha$ (or, equivalently, $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \alpha$). We have*

$$\lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)!x_{n+1}} - \sqrt[n]{n!x_n} \right) = \frac{\alpha}{e},$$
$$\lim_{n\to\infty} \left(\sqrt[n+1]{(2n+1)!!x_{n+1}} - \sqrt[n]{(2n-1)!!x_n} \right) = \frac{2\alpha}{e},$$
$$\lim_{n\to\infty} \left(\sqrt[n+1]{(2n+2)!!x_{n+1}} - \sqrt[n]{(2n)!!x_n} \right) = \frac{2\alpha}{e}.$$

Proof. The proof is indeed similar, by using Lemma 2.2 and equations (2.2) and (2.3), however we need to motivate our claim that $\lim_{n\to\infty} \sqrt[n]{x_n} = \alpha$ is equivalent to $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \alpha$. That follows easily from the inequalities (true for any sequence of real numbers $x_n > 0$; see [5, Theorem 3.37])

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \le \liminf_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

The proof is done. $\square$

In particular, the theorem above will be true for any increasing r-order linear recurrence sequence x_n (of initial conditions x_i, $0 \le i \le r-1$) [4], under some natural conditions. Assuming the characteristic equation of x_n has real roots α_i, $1 \le i \le s$, of multiplicity m_i, then

$$x_n = p_1(n)\alpha_1^n + p_2(n)\alpha_2^n + \cdots + p_s(n)\alpha_s^n,$$

where p_i's are polynomials of degree $m_i - 1$. Next, we assume that $\alpha := \alpha_1 \ge 1$ is the dominant root and so, there exist two nonzero polynomials G, H such that

$$G(n)\alpha^n \le x_n \le H(n)\alpha^n,$$

which is needed to infer that $\lim_{n\to\infty} \sqrt[n]{x_n} = \alpha$.

Having achieved this level of generalization, we inquire whether we can weigh the involved sequences differently. We are able to prove the following theorem (which has as a consequence a solution to [2]).

Theorem 2.6. *Let $\{u_n\}_n, \{v_n\}_n$ be two sequences such that $\lim_{n\to\infty} u_n = \beta$ and $\lim_{n\to\infty} n\,(u_n - v_n) = \gamma$ (consequently, $\lim_{n\to\infty} (u_n - v_n) = 0$ and so, $\lim_{n\to\infty} v_n = \beta$). Further, let $\{x_n\}$ be a sequence as in the previous theorem with $\sqrt[n]{x_n} = \alpha$, and $a_n = \sqrt[n]{n!x_n}$. Then,*

$$\lim_{n\to\infty} (u_n a_{n+1} - v_n a_n) = \frac{\alpha(\beta+\gamma)}{e}.$$

Proof. We first write

$$\begin{aligned} u_n a_{n+1} - v_n a_n &= u_n a_{n+1} - u_n a_n + u_n a_n - v_n a_n \\ &= u_n(a_{n+1} - a_n) + (u_n - v_n) a_n \\ &= u_n(a_{n+1} - a_n) + n(u_n - v_n)\frac{a_n}{n}. \end{aligned}$$

By our assumptions, Theorem 2.5 along with (2.2) (for the general sequence x_n), we infer that

$$\begin{aligned} &\lim_{n\to\infty} u_n(a_{n+1} - a_n) = \frac{\beta\alpha}{e}, \\ &\lim_{n\to\infty} \frac{a_n}{n} = \frac{\alpha}{e}, \\ &\lim_{n\to\infty} n(u_n - v_n) = \gamma, \end{aligned}$$

from which the claim follows. □

We omit the (easy) details, but as an application, if we let $e_n = \left(1 + \frac{1}{n}\right)^n$, and apply our theorem with $u_n := e, v_n := e_n$, or $u_n := e_{n+1}, v_n = e_n$ (along with $x_n = F_n$, respectively, $x_n = L_n$), we get the remaining Problem B-1160:(1) and (3) (we use the fact that $\lim_{n\to\infty} n(e - e_n) = \frac{e}{2}$, an easy consequence of the convergence error of e_n to e)

$$\begin{aligned} &\lim_{n\to\infty} \left(e \sqrt[n+1]{(n+1)!F_{n+1}} - e_n \sqrt[n]{n!F_n} \right) = \frac{\phi(e + e/2)}{e} = \frac{3\phi}{2}, \\ &\lim_{n\to\infty} \left(e \sqrt[n+1]{(n+1)!L_{n+1}} - e_n \sqrt[n]{n!L_n} \right) = \frac{3\phi}{2}. \end{aligned}$$

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