Input-to-State Stability of Non-uniform Linear Hyperbolic Systems of Balance Laws via Boundary Feedback Control

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Abstract

In this paper, a linear hyperbolic system of balance laws with boundary disturbances in one dimension is considered. An explicit candidate Input-to-State Stability (ISS)-Lyapunov function in L^2-norm is considered and discretised to investigate conditions for ISS of the discrete system as well. Finally, experimental results on test examples including the Saint-Venant equations with boundary disturbances are presented. The numerical results demonstrate the expected theoretical decay of the Lyapunov function.

Keywords

- Lyapunov function
- Hyperbolic PDE
- System of balance laws
- Feedback control

Introduction

We consider a $k \times k$ system described by the following linear hyperbolic system of balance laws with variable coefficients

$$\partial_t W(x,t) + \Lambda(x)\partial_x W(x,t) + \Pi(x)W(x,t) = 0, \quad (x,t) \in [0,l] \times [0,+\infty), \tag{1}$$

where $W := W(x,t) : [0,l] \times [0,+\infty) \to \mathbb{R}^k$ is a state vector, $\Lambda(x) = \text{ diag}\{\Lambda^+(x), -\Lambda^-(x)\}$, with $\Lambda^+(x) = \text{ diag}\{\lambda_i(x) > 0 : i = 1,\dots,m\}$ and $-\Lambda^-(x) = \text{ diag}\{\lambda_i(x) < 0 : i = m+1,\dots,k\}$, is a non-zero diagonal matrix and $\Pi(x) \in \mathbb{R}^{k\times k}$ is a non-zero matrix. Corresponding to the diagonal entries of $\Lambda(x)$, the state vector W is specified by $W = [W^+, W^-]^\top$, where $W^+ \in \mathbb{R}^m$ and $W^- \in \mathbb{R}^{k-m}$.

The system (1) is subject to an initial condition set as

$$W(x,0) = W_0(x),\ x \in (0,l), \tag{2}$$

for some function $W_0 : (0,l) \to \mathbb{R}^k$ and linear feedback boundary conditions with disturbances defined by

$$\begin{bmatrix} W^+(0,t) \\ W^-(l,t) \end{bmatrix} = K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} + Mb(t),\ t \in [0,+\infty), \tag{3}$$

where $K \in \mathbb{R}^{k\times k}$ is a constant matrix of the form $K = \begin{bmatrix} 0 & K^- \\ K^+ & 0 \end{bmatrix}$, with $K^- \in \mathbb{R}^{m\times(k-m)}$ and $K^+ \in \mathbb{R}^{(k-m)\times m}$, $M \in \mathbb{R}^{k\times k}$ is a non-zero constant diagonal matrix, and $b \in \mathbb{R}^k$ is a vector of disturbance functions. It is for such a system that the Input-to-State Stability (ISS) will be discussed in this paper.

In science and engineering, many important physical phenomena, in particular flow of fluids such as flow of shallow water, gas, traffic and electricity, have mathematical models that describe the dynamic behaviour of the flow in terms of mathematical equations. These mathematical models are mainly represented by hyperbolic systems of balance laws, e.g. Saint-Venant equations, isentropic Euler equations, or Telegrapher's equations. The solution of linear hyperbolic systems of balance laws under an initial condition, boundary conditions and initial-boundary compatibility conditions exist and are unique (see [5, 26]). Stabilisation problems with boundary controls (also called boundary feedbacks or boundary damping) of such systems have been an active research field as demonstrated by these papers, [4, 7,8,9,10,11,12, 14, 18, 19, 22]. These studies mainly focused on linear and non-linear systems in L^2-norm and H^2-norm, respectively, in the sense of exponential stability. For the most part, a strict Lyapunov function has played a central role in the investigation of conditions for stability.

Recently, the stabilisation of linear hyperbolic systems of balance laws with boundary disturbance created another dimension in the field. In [24, 27, 29], an input-to-state stability (ISS) which is an exponential stability in the presence of disturbances was introduced for hyperbolic system of conservation laws and balance laws.

Our aim is to analyse a numerical feedback boundary stability of such systems with boundary disturbance. This method has been presented in a few papers, for instance, [2, 3, 13, 15,16,17, 20]. In these studies, a discrete L^2-Lyapunov function is constructed and used to investigate conditions for exponential stability of discretised hyperbolic systems. Furthermore, the decay of the discrete L^2-Lyapunov function has been shown and numerical computations have been done to compare with analytical stability results.

In this paper, we extend our result [3] in the presence of boundary disturbances. For this reason, we discretise the ISS-Lyapunov function to investigate conditions for ISS in the sense of discrete ISS. Furthermore, the decay of ISS-Lyapunov functions is explicitly defined.

This paper is organised as follows: In Sect. 2, the problem is described. Basic definitions and theoretical results are stated and presented in Sect. 2. In Sect. 3, the numerical methods and discretisation are discussed and presented. Also the numerical results are discussed and presented in Sect. 3. The discussion in Sect. 3 is applied to computational examples in Sect. 4. Finally, conclusion and references are given at the end.

Preliminaries and Analytical Results

In this section, necessary definitions and theoretical results for the continuous problem will be presented. Firstly, reference will be made to the existence of solutions. This will be followed by a definition of a Lyapunov function and a stability proof in Theorem 1.

In this paper, the sets $\mathbb{R}^k$, $\mathbb{R}^{k\times k}$ and $\mathbb{R}_+^{k\times k}$ are the set of $k-$order real vectors, $k-$order real matrices and $k-$order positive real matrices, respectively. In addition, the sets C^0 and C^1 are the set of continuous and once continuously differentiable functions in $\mathbb{R}^k$, respectively. For a given function $f:[0,l]\to\mathbb{R}^k$, L^2-norm is defined as $\|f\|_{L^2}=\sqrt{\int_0^l |f(x)|^2 dx}$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^k$. Furthermore, $L^2(0,l)$ is called the space of all measurable functions f for which $\|f\|_{L^2}<\infty$.

In order to discuss ISS of steady-state, $W\equiv 0$, of the system (1) with initial condition (2) and boundary conditions (3), we make the following assumptions: For all $x\in[0,l]$, and $t\in[0,+\infty)$, we assume that

A1.: The real diagonal matrix Λ is of class $C^1([0,l])$.

A2.: The real matrix Π is of class $C^0([0,l])$.

A3.: The vector of boundary disturbances, b, is a class of $C^0([0,+\infty))$.

Consider Assumptions **A1**–**A3**, existence and uniqueness of a solution to the system (1) with initial condition (2) and boundary conditions (3) were discussed in detail in [21]. This was accompanied by the proof of existence and uniqueness. For brevity, such details will not be presented in the current paper.

Below, we provide a definition of ISS stability:

Definition 1

(ISS) The steady-state $W \equiv 0$ of the system (1) with the boundary conditions (3) is ISS in L^2-norm with respect to disturbance function b if there exist positive real constants $\eta > 0$, $\xi > 0$, $C_1 > 0$ and $C_2 > 0$ such that, for every initial condition $W_0(x) \in L^2((0,l);\mathbb{R}^k)$, the L^2-solution to the system (1) with initial condition (2) and boundary conditions (3) satisfies

$$\begin{aligned}\|W(\cdot,t)\|^2_{L^2((0,l);\mathbb{R}^k)} \le C_1 e^{-\eta t}\|W_0\|^2_{L^2((0,l);\mathbb{R}^k)} \\ + \frac{C_2}{\eta}\left(1+\frac{1}{\xi}\right)\sup_{s\in[0,t]}\left(|b(s)|^2\right),\ t\in[0,+\infty).\end{aligned} \tag{4}$$

Remark 1

(1) The second term on the right hand side (RHS) of the inequality (4) estimates the influence of the disturbance function $b(t)$ on the solution of the system (1) with the boundary conditions (3).

(2) A similar problem was considered in [27] for the case in which Λ and Π in Eq. (1) are constants.

(3) In [27], it was pointed out that stabilisation in the L^2-norm does not necessarily guarantee convergence of the maximum norm of $W(\cdot,t)$ over the domain $[0, l]$ in space. To guarantee such convergence, in [27], stability is considered in the H^1-norm.

(4) In this paper, analysis will be made in the L^2-norm.

Similar to Definition 1, we define an ISS-Lyapunov function as follows:

Definition 2

(L^2-ISS-Lyapunov function) For any continuously differentiable positive definite diagonal matrix $P(x) = \operatorname{diag}\{p_1(x),\ldots,p_k(x)\}$, $x\in[0,l]$, an L^2-function defined by

$$\mathcal{L}(W(\cdot,t)) = \int_0^l W^\top P(x) W dx,\ t\in[0,+\infty), \tag{5}$$

is said to be an ISS-Lyapunov function for the system (1) with the boundary conditions (3) if there exist positive real constants $\eta > 0$, $\xi > 0$ and $\nu > 0$ such that, for all functions $b(t) \in C^0([0,+\infty))$, for L^2-solutions of the system (1) satisfying the boundary conditions

(3), and for all $t \in [0, +\infty)$,

$$\frac{d\mathcal{L}(W(\cdot,t))}{dt} \leq -\eta\mathcal{L}(W(\cdot,t)) + \nu\left(1+\frac{1}{\xi}\right)\sup_{s\in[0,t]}\left(|b(s)|^2\right). \tag{6}$$

The following proposition presents preliminary results which will be used in the proof of the main result of this section in Theorem 1:

Proposition 1

Let y and z be vectors in $\mathbb{R}^k$. For any real constant $\xi > 0$, any matrix A and any positive semi-definite matrix B in $\mathbb{R}^{k\times k}$, the following holds:

(a) $$-2y^\top A(y-z) = -y^\top Ay + z^\top Az - (y-z)^\top A(y-z). \tag{7}$$

(b) $$\pm 2y^\top Bz \leq \xi y^\top By + \frac{1}{\xi}z^\top Bz. \tag{8}$$

Proof

The proof of the above statements is straightforward: a) Consider a quadratic form to obtain the Eq. (7) as follows:

$$\begin{aligned}(y-z)^\top A(y-z) &= y^\top Ay + z^\top Az - 2y^\top Az,\\ &= -y^\top Ay + z^\top Az - 2y^\top Az + 2y^\top Ay,\\ &= -y^\top Ay + z^\top Az + 2y^\top A(y-z).\end{aligned}$$

b) The following inequality implies the inequality (8):

$$\begin{aligned}&0 \leq \left(\sqrt{\xi}y \mp \frac{1}{\sqrt{\xi}}z\right)^\top B\left(\sqrt{\xi}y \mp \frac{1}{\sqrt{\xi}}z\right),\\ &= \xi y^\top By + \frac{1}{\xi}z^\top Bz \mp 2y^\top Bz.\end{aligned}$$

In Lemma 1 below, the boundedness of the Lyapunov function is established:

Lemma 1

Assume $P(x) = \operatorname{diag}\{p_1(x), \ldots, p_k(x)\}$ is a positive definite diagonal matrix for all $x \in [0, l]$. Let

$$\zeta = \min\left\{ \min_{0\le x\le l}(p_1(x)), \dots, \min_{0\le x\le l}(p_k(x)) \right\}, \text{ and}$$
$$\beta = \max\left\{ \max_{0\le x\le l}(p_1(x)), \dots, \max_{0\le x\le l}(p_k(x)) \right\},$$

where $p_i(x),\ i = 1, \dots, k$ are diagonal entries of $P(x)$. Then, the inequality

$$\zeta \int_0^l |W|^2 dx \le \mathcal{L}(W(\cdot,t)) \le \beta \int_0^l |W|^2 dx. \tag{9}$$

holds.

Proof

Since the diagonal matrix $P(x) = \operatorname{diag}\{p_1(x), \dots, p_k(x)\}$ is positive definite for all $x \in [0, l]$, for every W, the following holds:

$$\zeta |W|^2 \le W^\top P(x) W \le \beta |W|^2, \ \forall\, W \in \mathbb{R}^k, \ x \in [0, l], \tag{10}$$

where

$$\zeta = \min\left\{ \min_{0\le x\le l}(p_1(x)), \dots, \min_{0\le x\le l}(p_k(x)) \right\}, \text{ and}$$
$$\beta = \max\left\{ \max_{0\le x\le l}(p_1(x)), \dots, \max_{0\le x\le l}(p_k(x)) \right\}.$$

Thus, Inequality (9) is obtained.

Further, a version of the well known Gronwall's Lemma is stated as follows:

Lemma 2

(Gronwall's Lemma) Let $y \in C^1([0, +\infty))$, $z \in \mathbb{R}$, $a \in \mathbb{R}^+$, and

$$y'(t) \le -a y(t) + z, \quad y(0) = c \ge 0, \quad t \ge 0.$$

Then

$$y(t) \le \left(c - \frac{z}{a}\right) e^{-at} + \frac{z}{a}, \quad t \ge 0.$$

Proof

The proof of a general case of Gronwall's Lemma is given in Lemma 1.1.1 in [23]. Therein the coefficients a and z are functions of t. We adopt the proof by considering constants a and z. □

We now state the stability result as follows

Theorem 1

(Stability) Assume the system (1) with the boundary conditions (3) satisfies Assumptions **A1**–**A3**. Let ξ be any positive real number and $P(x) = \text{diag}\{P^+(x), P^-(x)\}$, where $P^+(x) = \text{diag}\{p_1(x), \ldots, p_m(x)\}$ and $P^-(x) = \text{diag}\{p_{m+1}(x), \ldots, p_k(x)\}$ be a continuously differentiable positive definite diagonal matrix. Assume that the matrix

$$-\Lambda(x)P'(x) - \Lambda'(x)P(x) + \Pi^\top(x)P(x) + P(x)\Pi(x), \tag{11}$$

is positive definite for all $x \in [0, l]$ and the matrix

$$\begin{bmatrix} \Lambda^+(l)P^+(l) & 0 \\ 0 & \Lambda^-(0)P^-(0) \end{bmatrix} - (1+\xi)\, K^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} K, \tag{12}$$

is positive semi-definite. Moreover, let ν be the largest eigenvalue of the matrix

$$M^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} M.$$

Then the L^2−function defined by (5) is an ISS-Lyapunov function for the system (1) with boundary conditions (3) and parameters ξ, ν. Moreover, the steady-state $W(x,t) \equiv 0$ of the system (1) with boundary conditions (3) is ISS in L^2−norm with respect to the disturbance function b.

Remark 2

Several approaches for the construction of Lyapunov functions for hyperbolic systems have been considered (see [4, 7, 10, 12, 14]). In [6], explicit Lyapunov functions were constructed to study the exponential stability for a class of physical 2×2 hyperbolic systems with nonuniform steady states. In this paper, we consider a general quadratic Lyapunov function described in [4, 28].

At this point, we proceed with the proof of Theorem 1:

Proof

We consider the L^2−function (5) as a candidate ISS-Lyapunov function. By computing a time derivative of the candidate ISS-Lyapunov function along C^1 solutions as in [5] (see Section 5.1) and [7], we obtain

$$\frac{d\mathcal{L}(W(\cdot,t))}{dt} = -\left[W^\top \Lambda(x)P(x)W\right]_0^l - \int_0^l W^\top \left(-\Lambda(x)P'(x) - \Lambda'(x)P(x) + \Pi(x)^\top P(x) + P(x)\Pi(x)\right) W dx. \tag{13}$$

At this stage the boundary conditions (3) are inserted to obtain:

$$\begin{aligned}
&-\left[W^\top \Lambda(x)P(x)W\right]_0^l \\
&= -\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^\top \begin{bmatrix} \Lambda^+(l)P^+(l) & 0 \\ 0 & \Lambda^-(0)P^-(0) \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&\quad + \left(K\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} + Mb(t)\right)^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} \\
&\quad \times \left(K\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} + Mb(t)\right), \\
&= -\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^\top \begin{bmatrix} \Lambda^+(l)P^+(l) & 0 \\ 0 & \Lambda^-(0)P^-(0) \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&\quad + \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^\top K^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&\quad + 2\left(K\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}\right)^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} Mb(t), \\
&\quad + b(t)^\top M^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} Mb(t).
\end{aligned} \tag{14}$$

We use Proposition 1(b) on the RHS of Eq. (14) to obtain:

$$\begin{aligned}
&-\left[W^\top \Lambda(x)P(x)W\right]_0^l \\
&\leq -\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^\top \begin{bmatrix} \Lambda^+(l)P^+(l) & 0 \\ 0 & \Lambda^-(0)P^-(0) \end{bmatrix} \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&\quad + (1+\xi)\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^\top K^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} K \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&\quad + \left(1+\frac{1}{\xi}\right) b(t)^\top M^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} Mb(t), \\
&= -\begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^\top \left(\begin{bmatrix} \Lambda^+(l)P^+(l) & 0 \\ 0 & \Lambda^-(0)P^-(0) \end{bmatrix}\right. \\
&\quad \left. -(1+\xi)K^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} K\right) \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix} \\
&\quad + \left(1+\frac{1}{\xi}\right) b(t)^\top M^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} Mb(t).
\end{aligned} \tag{15}$$

Therefore, inserting Eq. (15) into Eq. (13) gives:

$$\frac{d\mathcal{L}(W(\cdot,t))}{dt} \leq -\int_0^l W^\top\left(-\Lambda(x)P'(x) - \Lambda'(x)P(x) + \Pi(x)^\top P(x) + P(x)\Pi(x)\right)$$
$$Wdx - \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}^\top \left(\begin{bmatrix} \Lambda^+(l)P^+(l) & 0 \\ 0 & \Lambda^-(0)P^-(0) \end{bmatrix}\right.$$
$$\left. -(1+\xi)K^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} K\right) \begin{bmatrix} W^+(l,t) \\ W^-(0,t) \end{bmatrix}$$
$$+\left(1+\frac{1}{\xi}\right) b(t)^\top M^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} Mb(t). \tag{16}$$

Applying the assumption that ν is the largest eigenvalue of the matrix

$$M^\top \begin{bmatrix} \Lambda^+(0)P^+(0) & 0 \\ 0 & \Lambda^-(l)P^-(l) \end{bmatrix} M,$$

using the assumption in Theorem 1 for the matrix (12), Inequality (16) is reduced to

$$\frac{d\mathcal{L}(W(\cdot,t))}{dt} \leq -\int_0^l W^\top\left(-\Lambda(x)P'(x) - \Lambda'(x)P(x) + \Pi(x)^\top P(x) + P(x)\Pi(x)\right)$$
$$Wdx + \nu\left(1+\frac{1}{\xi}\right)|b(t)|^2,$$
$$\leq -\int_0^l W^\top Q(x)Wdx + \nu\left(1+\frac{1}{\xi}\right)\sup_{s\in[0,t]}\left(|b(s)|^2\right), \tag{17}$$

where $Q(x) = -\Lambda(x)P'(x) - \Lambda'(x)P(x) + \Pi(x)^\top P(x) + P(x)\Pi(x)$. Furthermore, by the assumption in Theorem 1 for the matrix (11), i.e. positive definiteness of $Q(x)$, there exist $\eta > 0$ such that $W^\top Q(x)W \geq \eta W^\top P(x)W$. Thus, the inequality (18) below is obtained:

$$\frac{d\mathcal{L}(W(\cdot,t))}{dt} \leq -\eta\mathcal{L}(W(\cdot,t)) + \nu\left(1+\frac{1}{\xi}\right)\sup_{s\in[0,t]}\left(|b(s)|^2\right), \tag{18}$$

with $\eta = \min\left\{\min_{0\leq x\leq l} E_1(x), \ldots, \min_{0\leq x\leq l} E_k(x)\right\}$, where $E_i(x),\ i = 1,\ldots,k$, are eigenvalues of the matrix $Q(x)P^{-1}(x)$.

For the purpose of completing the proof, the Gronwall's Lemma 2 is applied to obtain:

$$\mathcal{L}(W(\cdot,t)) \leq e^{-\eta t}\left(\mathcal{L}(W(\cdot,0)) - \frac{\nu}{\eta}\left(1+\frac{1}{\xi}\right)\sup_{s\in[0,t]}\left(|b(s)|^2\right)\right)$$
$$+\frac{\nu}{\eta}\left(1+\frac{1}{\xi}\right)\sup_{s\in[0,t]}\left(|b(s)|^2\right),$$
$$\leq e^{-\eta t}\mathcal{L}(W(\cdot,0)) + \frac{\nu}{\eta}\left(1+\frac{1}{\xi}\right)\sup_{s\in[0,t]}\left(|b(s)|^2\right),\ t\geq 0. \tag{19}$$

Now insert the inequality in (9) into inequality (19), to obtain

$$\zeta\|W(\cdot,t)\|^2_{L^2((0,l);\mathbb{R}^k)}$$
$$\leq \beta e^{-\eta t}\|W_0\|^2_{L^2((0,l);\mathbb{R}^k)} + \frac{\nu}{\eta}\left(1+\frac{1}{\xi}\right)\sup_{s\in[0,t]}\left(|b(s)|^2\right),\ t\geq 0. \tag{20}$$

Therefore, from the inequality (20), the constant coefficients in the condition for exponential stability (4) can be assigned to $C_1 = \beta/\zeta$ and $C_2 = \nu/\zeta$, hence Theorem 1 is proved. □

Remark 3

ISS of a $k \times k$ uniform linear hyperbolic system of balance laws which can be written as

$$\partial_t W + \Lambda\partial_x W + \Pi W = 0, \tag{21}$$

where $\Lambda, M \in \mathbb{R}^{k\times k}$ are non-zero diagonal matrices, and $\Pi, K \in \mathbb{R}^{k\times k}$ are non-zero matrices with the boundary conditions (3) was analysed in [27].

Having established the stability of the continuous model, Eq. (1), we now move on to analyse the stability of the discretised form of the same equation in the next section.

Numerical Discretisation and Stability for a Balance Law with Boundary Disturbance

The discretisation of the balance law in Eq. (1) will be discussed first. This will be followed by the discrete presentation of the Lyapunov function and the stability analysis of the discrete system. In order to solve the linear hyperbolic system of balance laws (1) numerically, a first-order Finite Volume Method (FVM) is considered. Thus, the upwind scheme, is applied to discretise space together with Euler schemes for temporal discretisation. The details of the use of the approach can be found in [25]. Specifically, we fix $T > 0$ and discretise the domain with $(x,t) \in [0,l] \times [0,T]$ by taking uniform space and time step sizes as $\Delta x = l/J$ and $\Delta t = T/N$, where $J, N > 0$, respectively. The values J and N denote the number of cells in space and time, respectively. Denote grid points by

$$x_{j-\frac{1}{2}} = j\Delta x,\ j = 0,\ldots,J, \quad t^n = n\Delta t,\ n = 0,\ldots,N.$$

Further, denote left and right boundary points by $x_{-\frac{1}{2}} = 0$ and $x_{J-\frac{1}{2}} = l$, respectively. In addition, cell centres are denoted by $x_j = \left(j+\frac{1}{2}\right)\Delta x,\ j = 0,\ldots,J-1$ and the left ghost-point (outside the domain) is denoted x_{-1} and the right ghost-point is denoted x_J.

A first order numerical scheme as described in [25] is considered. The approximate cell average of the state variable, W, over the jth cell at time t^n $(n = 0,\ldots,N)$ is defined by

$$W_j^n = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} W(x,t^n)\,dx,\ j=0,\dots,J-1, \tag{22}$$

such that for a smooth solution $W(x,t)$, the integral approximation is defined as

$$\int_0^l W(x,t^n)dx = \Delta x \sum_{j=0}^{J-1} W_j^n,\ n=0,\dots,N-1. \tag{23}$$

Therefore, the solution $W(x_j,t^n)$ is approximated by W_j^n. Hence, for $n=0,\dots,N-1$, $j=0,\dots,J-1$, the non-uniform system (1) is discretised as

$$\begin{bmatrix} W{+}_j^{n+1} \\ W{-}_j^{n+1} \end{bmatrix} = \begin{bmatrix} W{+}_j^{n} \\ W{-}_j^{n} \end{bmatrix} - \frac{\Delta t}{\Delta x}\begin{bmatrix} \Lambda_{j-1}^{+} & 0 \\ 0 & -\Lambda_{j+1}^{-} \end{bmatrix}\begin{bmatrix} W{+}_j^{n} - W{+}_{j-1}^{n} \\ W{-}_{j+1}^{n} - W{-}_j^{n} \end{bmatrix} - \Delta t \Pi_j \begin{bmatrix} W{+}_j^{n} \\ W{-}_j^{n} \end{bmatrix}, \tag{24}$$

with the CFL condition is given by

$$\frac{\Delta t}{\Delta x}\max\left(\max_{0\le j\le J-1}\{|\lambda_{1j}|\},\dots,\max_{0\le j\le J-1}\{|\lambda_{kj}|\}\right) \le 1. \tag{25}$$

Consequently, the initial conditions (2) and the boundary conditions (3) are discretised as

$$W_j^0 = W_{0,j}, \quad j=0,\dots,J-1, \tag{26}$$

and

$$\begin{bmatrix} W{+}_{-1}^{n+1} \\ W{-}_{J}^{n+1} \end{bmatrix} = K \begin{bmatrix} W{+}_{J-1}^{n+1} \\ W{-}_{0}^{n+1} \end{bmatrix} + Mb^{n+1}, \quad n=0,\dots,N-1, \tag{27}$$

respectively. The boundary conditions $W{+}_{-1}^{n}$ and $W{-}_{J}^{n}$ are applied at ghost-points x_{-1} and x_J, respectively

The aim of this paper is to investigate conditions for numerical boundary feedback stabilisation in the sense of the following definitions of discrete ISS and discrete ISS-Lyapunov function.

Definition 3

(Discrete ISS) The steady-state $W_j^n \equiv 0,\ j=0,\dots,J-1,\ n=0,\dots,N-1$ of the discretised system (24) with the discretised boundary conditions (27) is discrete ISS in L^2-norm with respect to discrete disturbance function $b^n,\ n=1,\dots,N$ if there exist positive real constants $\eta>0$, $\xi>0$, $C_1>0$ and $C_2>0$ such that, for every initial condition $W_j^0 \in L^2((x_{j-\frac{1}{2}},x_{j+\frac{1}{2}});\mathbb{R}^k),\ j=0,\dots,J-1$, the L^2-solution of the discretised system (24) with initial condition (26) and boundary conditions (27) satisfies

$$\Delta x\sum_{j=0}^{J-1}|W_j^n|^2 \le C_1 e^{-\eta t^n}\Delta x\sum_{j=0}^{J-1}|W_j^0|^2 + \frac{C_2}{\eta}\left(1+\frac{1}{\xi}\right)\sup_{0\le s<n}\left(|b^s|^2\right),\ n=1,\dots,N. \tag{28}$$

Definition 4

(A discrete L^2-ISS-Lyapunov function) For any positive definite diagonal matrix $P_j = \text{ diag}\{p_{1j}, \ldots, p_{kj}\}, j = 0, \ldots, J-1$, a discrete L^2-function defined by

$$\mathcal{L}^n = \Delta x \sum_{j=0}^{J-1} W_j^{n\top} P_j W_j^n, \quad n = 0, \ldots, N, \tag{29}$$

is said to be a discrete ISS-Lyapunov function for the discretised system (24) with the discretised boundary conditions (27) if there exist positive real constants $\eta > 0$, $\xi > 0$ and $\nu > 0$ such that, for all discrete functions b^n, $n = 1, \ldots, N$, for L^2-solutions of the discretised system (24) satisfying the discretised boundary conditions (27), and for all $n = 0, \ldots, N-1$,

$$\frac{\mathcal{L}^{n+1} - \mathcal{L}^n}{\Delta t} \leq -\eta \mathcal{L}^n + \nu \left(1 + \frac{1}{\xi}\right) \sup_{0 \leq s \leq n} \left(|b^s|^2\right). \tag{30}$$

Before stating the main theorem of this section, we present two preliminary results:

Lemma 3

Assume $P_j = \text{ diag}\{p_{1j}, \ldots, p_{kj}\}$ is positive definite diagonal matrix for all $j = 0, \ldots, J-1$. Let

$$\zeta = \min\left\{\min_{0 \leq j \leq J-1} p_{1j}, \ldots, \min_{0 \leq j \leq J-1} p_{kj}\right\} \text{ and}$$
$$\beta = \max\left\{\max_{0 \leq j \leq J-1} p_{1j}, \ldots, \max_{0 \leq j \leq J-1} p_{kj}\right\},$$

where $p_{1j}, \ldots, p_{kj}$ are diagonal entries of the diagonal matrix P_j, $j = 0, \ldots, J-1$. Then, the following inequality holds:

$$\zeta \Delta x \sum_{j=0}^{J-1} |W_j^n|^2 \leq \mathcal{L}^n \leq \beta \Delta x \sum_{j=0}^{J-1} |W_j^n|^2. \tag{31}$$

Proof

Since the diagonal matrix $P_j = \text{ diag}\{p_{1j}, \ldots, p_{kj}\}$, $j = 0, \ldots, J-1$ is positive definite, for all W_j^n, $n = 0, \ldots, N-1$, we have

$$\zeta |W_j^n|^2 \leq W_j^{n\top} P_j W_j^n \leq \beta |W_j^n|^2, \; j = 0, \ldots, J-1, \tag{32}$$

Where

$$\zeta = \min\left\{\min_{0\le j\le J-1} p_{1j}, \dots, \min_{0\le j\le J-1} p_{kj}\right\} \text{ and}$$
$$\beta = \max\left\{\max_{0\le j\le J-1} p_{1j}, \dots, \max_{0\le j\le J-1} p_{kj}\right\}.$$

Then, the inequality (32) implies the inequality (31). □

Now we present an equivalent Gronwall's Lemma for the discrete case:

Lemma 4

Let $a > 0$ and $z \in \mathbb{R}$. Suppose for discrete functions $y^n,\ n = 0, \dots, N-1$,

$$\frac{y^{n+1} - y^n}{\Delta t} \le -ay^n + z, \quad y^0 = c. \tag{33}$$

Then

$$y^{n+1} \le \left(c - \frac{z}{a}\right)(1 - a\Delta t)^{n+1} + \frac{z}{a}, \quad n = 0, \dots, N-1$$

for $0 < a\Delta t < 1$. (34)

Proof

By recursively applying the inequality (33), we obtain

$$y^{n+1} \le c(1 - a\Delta t)^{n+1} + z\Delta t \sum_{r=0}^{n} (1 - a\Delta t)^r, \quad n = 0, \dots, N-1. \tag{35}$$

Then, the inequality (35) implies the inequality (34) for sufficiently small Δt, $0 < 1 - a\Delta t < 1$. □

In the sense of the definitions of discrete ISS and discrete L^2 −ISS-Lyapunov function, we state the numerical stability result as follows:

Theorem 2

(Stability) Let $T > 0$ be fixed and the CFL condition (25) hold. Let ξ be any positive real number and $P_j = \operatorname{diag}\{P_j^+, P_j^-\}$, where $P_j^+ = \operatorname{diag}\{p_{1j}, \dots, p_{mj}\}$ and $P_j^- = \operatorname{diag}\{p_{m+1j}, \dots, p_{kj}\}$, $j = 0, \dots, J-1$ be positive definite diagonal matrix. Assume that the matrix

$$
-\begin{bmatrix} \Lambda_{j-1}^{+}\left(\frac{P_{j+1}^{+}-P_j^{+}}{\Delta x}\right) & 0 \\ 0 & -\Lambda_{j+1}^{-}\left(\frac{P_j^{-}-P_{j-1}^{-}}{\Delta x}\right) \end{bmatrix}
-\begin{bmatrix} \left(\frac{\Lambda_j^{+}-\Lambda_{j-1}^{+}}{\Delta x}\right)P_{j+1}^{+} & 0 \\ 0 & -\left(\frac{\Lambda_{j+1}^{-}-\Lambda_j^{-}}{\Delta x}\right)P_{j-1}^{-} \end{bmatrix}
+\Pi_j{}^{\top}P_j+P_j\Pi_j,
$$
(36)

is positive definite for all $j=0,\dots,J-1$, and the matrix

$$
\begin{bmatrix} \Lambda_{J-1}^{+}P_J^{+} & 0 \\ 0 & \Lambda_0^{-}P_{-1}^{-} \end{bmatrix} - (1+\xi)\,K^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-} \end{bmatrix}K, \tag{37}
$$

with $P_{-1}=P(x_{-1})$ and $P_J=P(x_J)$ is positive semi-definite. Moreover, let ν be the largest eigenvalue of the matrix

$$
M^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-} \end{bmatrix}M.
$$

Then the discrete L^2-function defined by (29) is a discrete ISS-Lyapunov function for the discretised system (24) with discretised boundary conditions (27) and parameters $\xi,\ \nu$. Moreover, the steady-state $W_j^n\equiv 0,\ j=0,\dots,J-1,\ n=0,\dots,N-1$ of the discretised system (24) with discretised boundary conditions (27) is discrete ISS in L^2-norm with respect to discrete disturbance function $b^n,\ n=1,\dots,N$.

Proof

By using the discrete L^2-function (29) and the discretised system (24), the time derivative of the candidate ISS-Lyapunov function (5) is approximated as follows: for all $n=0,\dots,N-1$,

$$
\begin{aligned}
&\frac{\mathcal{L}^{n+1}-\mathcal{L}^{n}}{\Delta t} \\
&=\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left(\begin{bmatrix} W_j^{+n+1} \\ W_j^{-n+1}\end{bmatrix}^{\top}\begin{bmatrix} P_j^{+} & 0 \\ 0 & P_j^{-}\end{bmatrix}\begin{bmatrix} W_j^{+n+1} \\ W_j^{-n+1}\end{bmatrix}-\begin{bmatrix} W_j^{+n} \\ W_j^{-n}\end{bmatrix}^{\top}\begin{bmatrix} P_j^{+} & 0 \\ 0 & P_j^{-}\end{bmatrix}\begin{bmatrix} W_j^{+n} \\ W_j^{-n}\end{bmatrix}\right). \\
&=\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left(\begin{bmatrix} W_j^{+n+1} \\ W_j^{-n+1}\end{bmatrix}^{\top}\begin{bmatrix} P_j^{+} & 0 \\ 0 & P_j^{-}\end{bmatrix}\begin{bmatrix} W_j^{+n+1} \\ W_j^{-n+1}\end{bmatrix}-\begin{bmatrix} W_j^{+n} \\ W_j^{-n}\end{bmatrix}^{\top}\begin{bmatrix} P_j^{+} & 0 \\ 0 & P_j^{-}\end{bmatrix}\begin{bmatrix} W_j^{+n+1} \\ W_j^{-n+1}\end{bmatrix}\right)
\end{aligned}
$$

$$+\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\left(\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}^\top\begin{bmatrix}P_j^+&0\\0&P_j^-\end{bmatrix}\begin{bmatrix}W^{+n+1}_j\\W^{-n+1}_j\end{bmatrix}-\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}^\top\begin{bmatrix}P_j^+&0\\0&P_j^-\end{bmatrix}\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}\right).$$

$$=\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\begin{bmatrix}W^{+n+1}_j-W^{+n}_j\\W^{-n+1}_j-W^{-n}_j\end{bmatrix}^\top\begin{bmatrix}P_j^+&0\\0&P_j^-\end{bmatrix}\begin{bmatrix}W^{+n+1}_j\\W^{-n+1}_j\end{bmatrix}$$

$$+\frac{\Delta x}{\Delta t}\sum_{j=0}^{J-1}\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}^\top\begin{bmatrix}P_j^+&0\\0&P_j^-\end{bmatrix}\begin{bmatrix}W^{+n+1}_j-W^{+n}_j\\W^{-n+1}_j-W^{-n}_j\end{bmatrix}.$$

At this stage Eq. (24) will be inserted to give:

$$\frac{\mathcal{L}^{n+1}-\mathcal{L}^n}{\Delta t}$$

$$=-\Delta x\sum_{j=0}^{J-1}\left(\frac{1}{\Delta x}\begin{bmatrix}\Lambda^+_{j-1}&0\\0&-\Lambda^-_{j+1}\end{bmatrix}\begin{bmatrix}W^{+n}_j-W^{+n}_{j-1}\\W^{-n}_{j+1}-W^{-n}_j\end{bmatrix}+\Pi_j\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}\right)^\top\begin{bmatrix}P_j^+&0\\0&P_j^-\end{bmatrix}\begin{bmatrix}W^{+n+1}_j\\W^{-n+1}_j\end{bmatrix}$$

$$-\Delta x\sum_{j=0}^{J-1}\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}^\top\begin{bmatrix}P_j^+&0\\0&P_j^-\end{bmatrix}\left(\frac{1}{\Delta x}\begin{bmatrix}\Lambda^+_{j-1}&0\\0&-\Lambda^-_{j+1}\end{bmatrix}\begin{bmatrix}W^{+n}_j-W^{+n}_{j-1}\\W^{-n}_{j+1}-W^{-n}_j\end{bmatrix}+\Pi_j\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}\right),$$

$$=-\Delta x\sum_{j=0}^{J-1}\left(\frac{1}{\Delta x}\begin{bmatrix}\Lambda^+_{j-1}&0\\0&-\Lambda^-_{j+1}\end{bmatrix}\begin{bmatrix}W^{+n}_j-W^{+n}_{j-1}\\W^{-n}_{j+1}-W^{-n}_j\end{bmatrix}+\Pi_j\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}\right)^\top\begin{bmatrix}P_j^+&0\\0&P_j^-\end{bmatrix}\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}$$

$$-\Delta x\sum_{j=0}^{J-1}\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}^\top\begin{bmatrix}P_j^+&0\\0&P_j^-\end{bmatrix}\left(\frac{1}{\Delta x}\begin{bmatrix}\Lambda^+_{j-1}&0\\0&-\Lambda^-_{j+1}\end{bmatrix}\begin{bmatrix}W^{+n}_j-W^{+n}_{j-1}\\W^{-n}_{j+1}-W^{-n}_j\end{bmatrix}+\Pi_j\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}\right)$$

$$+\Delta t\Delta x\sum_{j=0}^{J-1}\left(\frac{1}{\Delta x}\begin{bmatrix}\Lambda^+_{j-1}&0\\0&-\Lambda^-_{j+1}\end{bmatrix}\begin{bmatrix}W^{+n}_j-W^{+n}_{j-1}\\W^{-n}_{j+1}-W^{-n}_j\end{bmatrix}+\Pi_j\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}\right)^\top\begin{bmatrix}P_j^+&0\\0&P_j^-\end{bmatrix}$$

$$\times\left(\frac{1}{\Delta x}\begin{bmatrix}\Lambda^+_{j-1}&0\\0&-\Lambda^-_{j+1}\end{bmatrix}\begin{bmatrix}W^{+n}_j-W^{+n}_{j-1}\\W^{-n}_{j+1}-W^{-n}_j\end{bmatrix}+\Pi_j\begin{bmatrix}W^{+n}_j\\W^{-n}_j\end{bmatrix}\right),$$

At this stage Eq. (24) is substituted again and Proposition 1 is applied to obtain the following:

$$
\begin{aligned}
&\frac{\mathcal{L}^{n+1}-\mathcal{L}^{n}}{\Delta t} \\
&\quad = -2\sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} - W^{+n}_{j-1} \\ W^{-n}_{j+1} - W^{-n}_{j}\end{bmatrix}^{\top}\begin{bmatrix}\Lambda^{+}_{j-1}P^{+}_{j} & 0 \\ 0 & -\Lambda^{-}_{j+1}P^{-}_{j}\end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix} \\
&\qquad - \Delta x\sum_{j=0}^{J-1} W_j^{n\top}\left(\Pi_j^{\top}P_j + P_j\Pi_j\right)W_j^{n} \\
&\qquad + \Delta t\,\Delta x\sum_{j=0}^{J-1}\left(\frac{1}{\Delta t}\begin{bmatrix} W^{+n+1}_{j} - W^{+n}_{j} \\ W^{-n+1}_{j} - W^{-n}_{j}\end{bmatrix}\right)^{\top}\begin{bmatrix}P^{+}_{j} & 0 \\ 0 & P^{-}_{j}\end{bmatrix}\left(\frac{1}{\Delta t}\begin{bmatrix} W^{+n+1}_{j} - W^{+n}_{j} \\ W^{-n+1}_{j} - W^{-n}_{j}\end{bmatrix}\right). \\
&\quad \le -\sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}^{\top}\begin{bmatrix}\Lambda^{+}_{j-1}P^{+}_{j} & 0 \\ 0 & \Lambda^{-}_{j+1}P^{-}_{j}\end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix} \\
&\qquad + \sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1}\end{bmatrix}^{\top}\begin{bmatrix}\Lambda^{+}_{j-1}P^{+}_{j} & 0 \\ 0 & \Lambda^{-}_{j+1}P^{-}_{j}\end{bmatrix}\begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1}\end{bmatrix} \\
&\qquad - \Delta x\sum_{j=0}^{J-1} W_j^{n\top}\left(\Pi_j^{\top}P_j + P_j\Pi_j\right)W_j^{n} \\
&\qquad + \Delta t\,\Delta x\sum_{j=0}^{J-1}\left(\frac{1}{\Delta t}\begin{bmatrix} W^{+n+1}_{j} - W^{+n}_{j} \\ W^{-n+1}_{j} - W^{-n}_{j}\end{bmatrix}\right)^{\top}\begin{bmatrix}P^{+}_{j} & 0 \\ 0 & P^{-}_{j}\end{bmatrix}\left(\frac{1}{\Delta t}\begin{bmatrix} W^{+n+1}_{j} - W^{+n}_{j} \\ W^{-n+1}_{j} - W^{-n}_{j}\end{bmatrix}\right).
\end{aligned}
\tag{38}
$$

For all $n = 0, \ldots, N-1,\ j = 0, \ldots, J-1$, we have

$$
\begin{aligned}
&\sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1}\end{bmatrix}^{\top}\begin{bmatrix}\Lambda^{+}_{j-1}P^{+}_{j} & 0 \\ 0 & \Lambda^{-}_{j+1}P^{-}_{j}\end{bmatrix}\begin{bmatrix} W^{+n}_{j-1} \\ W^{-n}_{j+1}\end{bmatrix} \\
&\quad = \sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}^{\top}\begin{bmatrix}\Lambda^{+}_{j}P^{+}_{j+1} & 0 \\ 0 & \Lambda^{-}_{j}P^{-}_{j-1}\end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix} \\
&\qquad - \begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0}\end{bmatrix}^{\top}\begin{bmatrix}\Lambda^{+}_{J-1}P^{+}_{J} & 0 \\ 0 & \Lambda^{-}_{0}P^{-}_{-1}\end{bmatrix}\begin{bmatrix} W^{+n}_{J-1} \\ W^{-n}_{0}\end{bmatrix} \\
&\qquad + \begin{bmatrix} W^{+n}_{-1} \\ W^{-n}_{J}\end{bmatrix}^{\top}\begin{bmatrix}\Lambda^{+}_{-1}P^{+}_{0} & 0 \\ 0 & \Lambda^{-}_{J}P^{-}_{J-1}\end{bmatrix}\begin{bmatrix} W^{+n}_{-1} \\ W^{-n}_{J}\end{bmatrix}.
\end{aligned}
\tag{39}
$$

We substitute (39) into (38) to obtain:

$$
\begin{aligned}
\frac{\mathcal{L}^{n+1}-\mathcal{L}^{n}}{\Delta t} \le &-\sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}^{\top}\begin{bmatrix}\Lambda^{+}_{j-1}P^{+}_{j} & 0 \\ 0 & \Lambda^{-}_{j+1}P^{-}_{j}\end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix} \\
&+ \sum_{j=0}^{J-1}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix}^{\top}\begin{bmatrix}\Lambda^{+}_{j}P^{+}_{j+1} & 0 \\ 0 & \Lambda^{-}_{j}P^{-}_{j-1}\end{bmatrix}\begin{bmatrix} W^{+n}_{j} \\ W^{-n}_{j}\end{bmatrix} \\
&- \Delta x\sum_{j=0}^{J-1} W_j^{n\top}\left(\Pi_j^{\top}P_j + P_j\Pi_j\right)W_j^{n}
\end{aligned}
$$

$$
\begin{aligned}
&+\Delta t\,\Delta x\sum_{j=0}^{J-1}\left(\frac{1}{\Delta t}\begin{bmatrix} W^{+}{}_{j}^{n+1}-W^{+}{}_{j}^{n} \\ W^{-}{}_{j}^{n+1}-W^{-}{}_{j}^{n}\end{bmatrix}\right)^{\top}\begin{bmatrix} P_j^{+} & 0 \\ 0 & P_j^{-}\end{bmatrix}\\
&\left(\frac{1}{\Delta t}\begin{bmatrix} W^{+}{}_{j}^{n+1}-W^{+}{}_{j}^{n} \\ W^{-}{}_{j}^{n+1}-W^{-}{}_{j}^{n}\end{bmatrix}\right)\\
&-\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}^{\top}\begin{bmatrix} \Lambda_{J-1}^{+}P_J^{+} & 0 \\ 0 & \Lambda_0^{-}P_{-1}^{-}\end{bmatrix}\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}\\
&+\begin{bmatrix} W^{+}{}_{-1}^{n} \\ W^{-}{}_{J}^{n}\end{bmatrix}^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}\begin{bmatrix} W^{+}{}_{-1}^{n} \\ W^{-}{}_{J}^{n}\end{bmatrix},\ n=0,\dots,N-1.
\end{aligned}
\tag{40}
$$

The boundary conditions (27), the inequality (8) in Proposition 1 and the assumption in Theorem 2 are used to simplify the boundary term in (40) as follows:

$$
\begin{aligned}
&-\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}^{\top}\begin{bmatrix} \Lambda_{J-1}^{+}P_J^{+} & 0 \\ 0 & \Lambda_0^{-}P_{-1}^{-}\end{bmatrix}\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}\\
&\quad+\begin{bmatrix} W^{+}{}_{-1}^{n} \\ W^{-}{}_{J}^{n}\end{bmatrix}^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}\begin{bmatrix} W^{+}{}_{-1}^{n} \\ W^{-}{}_{J}^{n}\end{bmatrix}\\
&=-\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}^{\top}\begin{bmatrix} \Lambda_{J-1}^{+}P_J^{+} & 0 \\ 0 & \Lambda_0^{-}P_{-1}^{-}\end{bmatrix}\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}\\
&\quad+\left(K\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}+Mb^{n}\right)^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}\left(K\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}+Mb^{n}\right)\\
&=-\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}^{\top}\begin{bmatrix} \Lambda_{J-1}^{+}P_J^{+} & 0 \\ 0 & \Lambda_0^{-}P_{-1}^{-}\end{bmatrix}\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}\\
&\quad+\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}^{\top}K^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}K\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}\\
&\quad+2\left(K\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}\right)^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}(Mb^{n})+b^{n\top}M^{\top}\\
&\quad\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}Mb^{n},\\
&\le-\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}^{\top}\begin{bmatrix} \Lambda_{J-1}^{+}P_J^{+} & 0 \\ 0 & \Lambda_0^{-}P_{-1}^{-}\end{bmatrix}\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}\\
&\quad+\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}^{\top}K^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}K\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}\\
&\quad+\xi\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}^{\top}K^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}K\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}\\
&\quad+\frac{1}{\xi}b^{n\top}M^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}Mb^{n}+b^{n\top}M^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}Mb^{n},\\
&=-\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}^{\top}\begin{bmatrix} \Lambda_{J-1}^{+}P_J^{+} & 0 \\ 0 & \Lambda_0^{-}P_{-1}^{-}\end{bmatrix}\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}\\
&\quad+(1+\xi)\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}^{\top}K^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}K\begin{bmatrix} W^{+}{}_{J-1}^{n} \\ W^{-}{}_{0}^{n}\end{bmatrix}\\
&\quad+\left(1+\frac{1}{\xi}\right)b^{n\top}M^{\top}\begin{bmatrix} \Lambda_{-1}^{+}P_0^{+} & 0 \\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}Mb^{n},\\
&\le\nu\left(1+\frac{1}{\xi}\right)|b^{n}|^{2},\ n=0,\dots,N-1,
\end{aligned}
\tag{41}
$$

where ν is the largest eigenvalue of the matrix

$$M^{\top}\begin{bmatrix}\Lambda_{-1}^{+}P_0^{+} & 0\\ 0 & \Lambda_J^{-}P_{J-1}^{-}\end{bmatrix}M. \tag{42}$$

Inserting (41) into (40), for all $n = 0, \ldots, N-1$, we have

$$\begin{aligned}\frac{\mathcal{L}^{n+1}-\mathcal{L}^{n}}{\Delta t} &\leq -\sum_{j=0}^{J-1}\begin{bmatrix}W_j^{+n}\\ W_j^{-n}\end{bmatrix}^{\top}\begin{bmatrix}\Lambda_{j-1}^{+}P_j^{+}-\Lambda_j^{+}P_{j+1}^{+} & 0\\ 0 & \Lambda_{j+1}^{-}P_j^{-}-\Lambda_j^{-}P_{j-1}^{-}\end{bmatrix}\begin{bmatrix}W_j^{+n}\\ W_j^{-n}\end{bmatrix}\\ &\quad -\Delta x\sum_{j=0}^{J-1}W_j^{n\top}\left(\Pi_j^{\top}P_j+P_j\Pi_j\right)W_j^{n}+\nu\left(1+\frac{1}{\xi}\right)|b^n|^2\\ &\quad +\Delta t\Delta x\sum_{j=0}^{J-1}\left(\frac{1}{\Delta t}\begin{bmatrix}W_j^{+n+1}-W_j^{+n}\\ W_j^{-n+1}-W_j^{-n}\end{bmatrix}\right)^{\top}\\ &\quad \begin{bmatrix}P_j^{+} & 0\\ 0 & P_j^{-}\end{bmatrix}\left(\frac{1}{\Delta t}\begin{bmatrix}W_j^{+n+1}-W_j^{+n}\\ W_j^{-n+1}-W_j^{-n}\end{bmatrix}\right)\\ &= -\Delta x\sum_{j=0}^{J-1}W_j^{n\top}\Theta_jW_j^{n}+\nu\left(1+\frac{1}{\xi}\right)|b^n|^2+\mathcal{O}(\Delta t),\end{aligned} \tag{43}$$

where for $j = 0, \ldots, J-1$,

$$\begin{aligned}\Theta_j = &-\begin{bmatrix}\Lambda_{j-1}^{+}\left(\frac{P_{j+1}^{+}-P_j^{+}}{\Delta x}\right) & 0\\ 0 & -\Lambda_{j+1}^{-}\left(\frac{P_j^{-}-P_{j-1}^{-}}{\Delta x}\right)\end{bmatrix}\\ &-\begin{bmatrix}\left(\frac{\Lambda_j^{+}-\Lambda_{j-1}^{+}}{\Delta x}\right)P_{j+1}^{+} & 0\\ 0 & -\left(\frac{\Lambda_{j+1}^{-}-\Lambda_j^{-}}{\Delta x}\right)P_{j-1}^{-}\end{bmatrix}\\ &+P_j\Pi_j+\Pi_j^{\top}P_j.\end{aligned} \tag{44}$$

By the assumption in Theorem 2 for $\Theta_j,\ j = 0, \ldots, J-1$, there exist a positive real number $\eta > 0$ such that for every $W_j^n,\ n = 0, \ldots, N-1,\ j = 0, \ldots, J-1$, we have $W_j^{n\top}\Theta_jW_j^n \geq \eta W_j^{n\top}P_jW_j^n$. Thus, (43) is approximated as

$$\frac{\mathcal{L}^{n+1}-\mathcal{L}^{n}}{\Delta t} \leq -\eta\mathcal{L}^{n}+\nu\left(1+\frac{1}{\xi}\right)\sup_{0\leq s\leq n}\left(|b^s|^2\right)+\mathcal{O}(\Delta t),\ n = 0, \ldots, N-1, \tag{45}$$

with

$$\eta = \min\{\min_{0\leq j\leq J-1}E_{1j}, \ldots, \min_{0\leq j\leq J-1}E_{kj}\},$$

where $E_{1j}, \ldots, E_{kj}$ are eigenvalues of $\Theta_jP_j^{-1}$, $j = 0, \ldots, J-1$.

From Lemma 4 and by using $(1-\eta\Delta t)^{n+1} \leq e^{-\eta(n+1)\Delta t} \leq e^{-\eta t^{n+1}}$, we have

$$
\begin{aligned}
\mathcal{L}^{n+1} \le & \left(\mathcal{L}^0 - \frac{\nu}{\eta}\left(1+\frac{1}{\xi}\right) \sup_{0\le s\le n} \left(|b^s|^2\right)\right)(1-\eta\Delta t)^{n+1} \\
& + \frac{\nu}{\eta}\left(1+\frac{1}{\xi}\right) \sup_{0\le s\le n} \left(|b^s|^2\right), \\
& \le e^{-\eta t^{n+1}} \mathcal{L}^0 + \frac{\nu}{\eta}\left(1+\frac{1}{\xi}\right) \sup_{0\le s\le n} \left(|b^s|^2\right),\ n=0,\dots,N-1.
\end{aligned} \tag{46}
$$

Thus, for all $j=0,\dots,J-1$, and $n=0,\dots,N-1$, by using (31) and (46), we have

$$
\zeta\Delta x \sum_{j=0}^{J-1} |W_j^{n+1}|^2 \le \beta e^{-\eta t^{n+1}} \Delta x \sum_{j=0}^{J-1} |W_j^0|^2 + \frac{\nu}{\eta}\left(1+\frac{1}{\xi}\right) \sup_{0\le s\le n} \left(|b^s|^2\right). \tag{47}
$$

Therefore, to show that (47) implies the condition for the discrete ISS (28), we let $C_1 = \beta/\zeta$ and $C_2 = \nu/\zeta$. Hence, the proof of Theorem 2 is completed. □

Remark 4

Similar to the discrete system (24), for uniform steady-state case, the system (21) is discretised as follows

$$
\begin{bmatrix} W^{+}{}_j^{n+1} \\ W^{-}{}_j^{n+1} \end{bmatrix} = \begin{bmatrix} W^{+}{}_j^{n} \\ W^{-}{}_j^{n} \end{bmatrix} - \frac{\Delta t}{\Delta x} \begin{bmatrix} \Lambda^+ & 0 \\ 0 & -\Lambda^- \end{bmatrix} \begin{bmatrix} W^{+}{}_j^{n} - W^{+}{}_{j-1}^{n} \\ W^{-}{}_{j+1}^{n} - W^{-}{}_j^{n} \end{bmatrix} - \Delta t \Pi \begin{bmatrix} W^{+}{}_j^{n} \\ W^{-}{}_j^{n} \end{bmatrix} \tag{48}
$$

The proof of Theorem 2 applies to the discretised system (48) as a special case. Here we have provided a numerical stability result for a more general case and, as a side-effect, for the particular case in Eq. (21).

In this section an analysis of the discrete Lyapunov function which results from a numerical discretisation of such an analytical Lyapunov function has been discussed. An Euler scheme was applied for temporal discretisation of a system. An upwind scheme was also applied for the spatial discretisation. The ISS-stability for such discretised systems was proved. In the next section, the results established here are applied to a linear example, the Saint-Venant model and the isothermal Euler model. This section endeavors to also demonstrate how values of the parameters in the Lyapunov function are delimited.

Computational Applications and Results

The results of the previous section will now be tested computationally on specific examples. We will start by presenting an example of a linear hyperbolic system of balance laws with constant coefficients in Sect. 4.1. The second example will be a Saint-Venant system of equations which will be discussed in Sect. 4.2. In Sect. 4.3, the isothermal Euler equations will be discussed as a third example. The derivation of the equilibrium and the choice of requisite parameters for such models will be discussed in detail.

Linear Hyperbolic 2×2 Systems of Balance Laws with Uniform Coefficients and Boundary Disturbances

To show the numerical analysis working, we analyse a 2×2 uniform linear system. For this reason, we consider the system (1) with uniform matrix coefficients of the form

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.3 \end{bmatrix},$$

an initial condition of the form

$$\begin{bmatrix} w_1(x,0) \\ w_2(x,0) \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}, \ x \in (0,1), \tag{49}$$

boundary conditions (3) with $m_1 = m_2 = 0.5$ and the rate of the boundary disturbance functions taken as $b_1(t) = -b_2(t) = 0.01\sin(\pi t), \ t > 0$. Then, the discretisation of the system with initial condition and boundary conditions are given by (24) - (27).

Let the CFL condition, $\lambda \frac{\Delta t}{\Delta x} \leq 1$, where $\lambda = \max\{\lambda_1, |\lambda_2|\} = 1$ holds for a fixed $T > 0$. Define a positive definite discrete diagonal matrix by $P_j := \text{diag}\{p_1\exp(-\mu x_j), p_2\exp(\mu x_j)\}, \ p_1 > 0, p_2 > 0, \mu > 0, \ j = 0, \ldots, J-1$. Based on the assumptions in Theorem 2, the parameters η, ν, κ_{12} and κ_{21} can be chosen by

$$\kappa_{12}^2 \leq \frac{|\lambda_{20}|p_{2_{-1}}}{(1+\xi)\lambda_{1_{-1}}p_{1_0}}, \quad \text{and} \quad \kappa_{21}^2 \leq \frac{\lambda_{1_{J-1}}p_{1_J}}{(1+\xi)|\lambda_{2_J}|p_{2_{J-1}}},$$

for all $\xi > 0$,

$$\eta := \min\left\{\min_{0\leq j\leq J-1}\eta_{1_j}, \min_{0\leq j\leq J-1}\eta_{2_j}\right\},$$

where η_{1_j} and η_{2_j} are eigenvalues of $\Theta_j P_j^{-1}, \ j = 0, \ldots, J-1$ (Θ_j is given in (44)), and

Hence, we compute a comparison of the discrete ISS-Lyapunov function and its upper bound for CFL $= 0.75$ and CFL $= 0.99$ in Tables 1 and 2, respectively.

Table 1 The comparison of the upper bound of Lyapunov function with discrete Lyapunov function

J	$\|\mathcal{L}^n_{\mathrm{up}} - \mathcal{L}^n\|_{L^\infty}$	$\|\mathcal{L}^n_{\mathrm{up}} - \mathcal{L}^n\|_{L^2}$	μ	η	ν
100	0.13921	0.31661	0.575	0.33709	0.11075
200	0.13187	0.30186	0.575	0.33729	0.11091
400	0.12686	0.29161	0.575	0.3374	0.11099
800	0.12338	0.28447	0.575	0.33745	0.11103
1600	0.12095	0.27948	0.575	0.33747	0.11105

Under CFL $= 0.75$, $\Delta x = \frac{1}{J}$, $\Delta t = \Delta x$CFL, $\xi = 1$, $T = 12$, $\kappa_{12} = \kappa_{21} = 0.75$, $m_1 = 1 - \kappa_{12}$ and $m_2 = 1 - \kappa_{21}$

Table 2 The comparison of the upper bound of Lyapunov function with discrete Lyapunov function

J	$\|\mathcal{L}^n_{\mathrm{up}} - \mathcal{L}^n\|_{L^\infty}$	$\|\mathcal{L}^n_{\mathrm{up}} - \mathcal{L}^n\|_{L^2}$	μ	η	ν
100	0.11984	0.24123	0.575	0.33709	0.11075
200	0.11849	0.23885	0.575	0.33729	0.11091
400	0.11751	0.23712	0.575	0.3374	0.11099
800	0.11683	0.23588	0.575	0.33745	0.11103
1600	0.11634	0.23501	0.575	0.33747	0.11105

Under CFL $= 0.99$, $\Delta x = \frac{1}{J}$, $\Delta t = \Delta x$CFL, $\xi = 1$, $T = 12$, $\kappa_{12} = \kappa_{21} = 0.75$, $m_1 = 1 - \kappa_{12}$ and $m_2 = 1 - \kappa_{21}$

From Tables 1 and 2 above, it can be observed that both L^∞ and L^2 norms demonstrate general decay.

Saint-Venant Equations

We consider flow of water in the presence of flow rate measurements error at the boundaries. One of the causes of disturbances of a flow of water along an open channel can be a measurement error at the ends of the channel. Thus, we study a flow of water along a prismatic channel with a rectangular cross-section, a length of l units and constant bottom slope. We consider boundary measurements in this flow. The model of the flow is described by Saint-Venant equations (see [1, 7]) of the form

$$\begin{aligned} &\partial_t H + \partial_x (HV) = 0, \\ &\partial_t V + \partial_x \left(\frac{1}{2}V^2 + gH\right) + \left(C_f \frac{V^2}{H} - gS_b\right) = 0,\ x \in [0, l],\ t \in [0, +\infty), \end{aligned} \tag{50}$$

where H and V denote the depth and velocity of the water, respectively. Other constants, g, C_f, and S_b represent the gravitational constant, a friction parameter and the constant bottom slope of the channel, respectively.

We set an initial condition

$$H(x,0) = H_0(x), \quad V(x,0) = V_0(x), \quad x \in (0,l), \tag{51}$$

and boundary conditions with disturbances

$$V(0,t) = k_0 H(0,t) + b_1(t), \quad V(l,t) = k_l H(l,t) + b_2(t), \quad t \in [0,\infty), \tag{52}$$

where k_0, k_l are boundary control parameters, and b_1, b_2 are disturbance functions.

We assume $H(x,t) > 0,\ V(x,t) > 0$ for all $x \in [0,l]$ and $t \in [0,+\infty)$. A non-uniform steady-state solution, $H^*(x),\ V^*(x)$ to the system (50) satisfies

$$\begin{aligned} &H^*(x)V^*(x) = Q^*, \\ &\frac{d}{dx}\left(\frac{V^{*2}(x)}{2} + gH^*(x)\right) + \left(C_f \frac{V^{*2}(x)}{H^*(x)} - gS_b\right) = 0, \quad x \in [0,l]. \end{aligned} \tag{53}$$

Clearly $Q^* > 0$ and the system of differential Eq. (53) can be written in the following form

$$\begin{aligned} \frac{dH^*}{dx} &= \frac{C_f Q^{*2} - gS_b H^{*3}(x)}{Q^{*2} - gH^{*3}(x)}, \quad x \in [0,l], \\ \frac{dV^*}{dx} &= \frac{V^{*2}(x)\left(\frac{C_f V^{*3}(x)}{Q^*} - gS_b\right)}{gQ^* - V^{*3}(x)}, \quad x \in [0,l]. \end{aligned} \tag{54}$$

For sub-critical flow (i.e. $V^2 < gH$), the system (50) is linearized as follows

$$\partial_t Z(x,t) + A(x)\partial_x Z(x,t) + \left(\frac{dA}{dx} + B(x)\right) Z(x,t) = 0, \tag{55}$$

where $(x,t) \in [0,l] \times [0,+\infty)$, $Z(x,t) = [H - H^*, V - V^*]^\top$,

$$A(x) = \begin{bmatrix} V^* & H^* \\ g & V^* \end{bmatrix}, \quad \text{and} \quad B(x) = \begin{bmatrix} 0 & 0 \\ -\frac{C_f V^{*2}(x)}{H^{*2}(x)} & \frac{2C_f V^*(x)}{H^*(x)} \end{bmatrix}.$$

Define the Riemann coordinates by

$$W = [w_1, w_2]^\top = L(x)Z \quad \text{with} \quad L(x) = \begin{bmatrix} \sqrt{\frac{g}{H^*(x)}} & 1 \\ -\sqrt{\frac{g}{H^*(x)}} & 1 \end{bmatrix}, \tag{56}$$

where L is a matrix of left eigenvectors of A i.e.

$$L(x)A(x)L^{-1}(x) = \Lambda(x),$$

with

$$\Lambda(x) = \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix} = \begin{bmatrix} V^*(x) + \sqrt{gH^*(x)} & 0 \\ 0 & V^*(x) - \sqrt{gH^*(x)} \end{bmatrix}$$

and

$$L^{-1}(x) = \frac{1}{2}\begin{bmatrix} \sqrt{\frac{H^*(x)}{g}} & -\sqrt{\frac{H^*(x)}{g}} \\ 1 & 1 \end{bmatrix}.$$

The system (55), the initial condition (51) and the boundary conditions (52) are expressed as (1), (2) and (3), respectively, with

$$\Pi(x) = \left[L(x)\left(\frac{dA}{dx} + B(x)\right) - \Lambda(x)\frac{dL}{dx}\right] L^{-1}(x), \quad (57)$$

$f(x) := w_1(x,0)$, $g(x) := w_2(x,0)$, $\kappa_{12} := \frac{k_0\sqrt{\frac{H^*(0)}{g}}-1}{1+k_0\sqrt{\frac{H^*(0)}{g}}} \neq 1$, $\kappa_{21} := \frac{k_l\sqrt{\frac{H^*(l)}{g}}-1}{1+k_l\sqrt{\frac{H^*(l)}{g}}} \neq 1$,

$m_1 := 1 - \kappa_{12}$ and $m_2 := 1 - \kappa_{21}$.

For a numerical analysis and computations, we take an example from [28]. Thus, a non-uniform steady-state solution $(H^*(x), V^*(x))$ can be obtained from the differential Eq. (54) with $H^*(0) = 2m$, $V^*(0) = 0.5m/s$, $C_f = 0.002$, $g = 9.81m/s^2$ and $S_b = \frac{2C_f V^{*2}(0)}{gH^*(0)}$. We set an initial condition $(H(x,0), V(x,0)) = (2.5, 4\sin(\pi x))$ for $x \in [0,1]$. The rate of the boundary disturbance functions taken as $b_1(t) = -b_2(t) = 0.01\sin(\pi t),\ t > 0$.

In Fig. 1, it can be observed that as μ increases the rate of decay of the ISS-Lyapunov function in the presence of boundary disturbance increases. Hence, in the sense of the definition of discrete ISS, the steady-state $W_j^n \equiv 0,\ j = 0,\ldots,J-1,\ n = 0,\ldots,N-1$ of the discretised system with the discretised boundary conditions is discrete ISS in L^2-norm with respect to discrete disturbance function $b^n,\ n = 1,\ldots,N$.

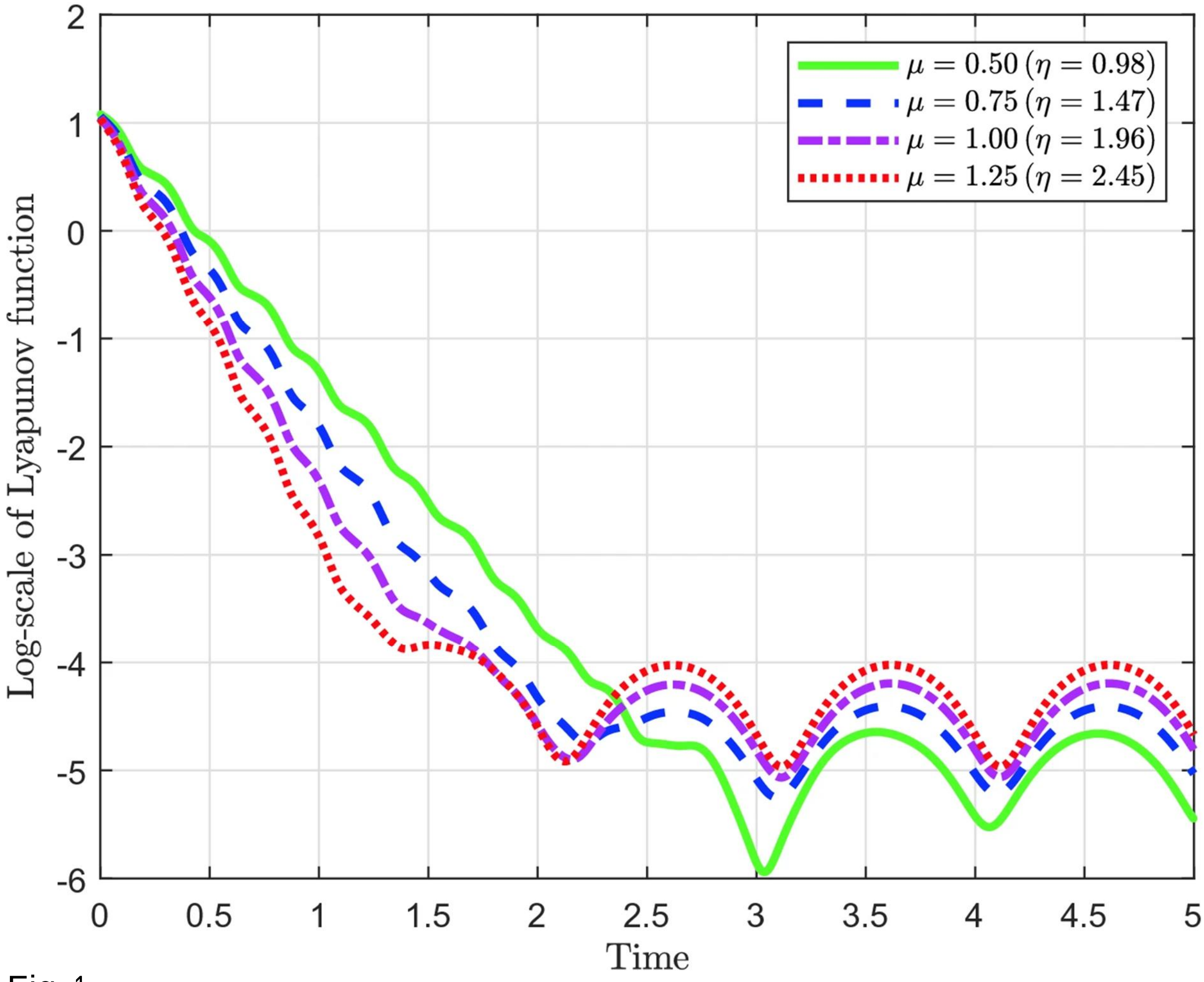

Fig. 1

Log-scale of the decay of Lyapunov function for Saint-Venant equations for different values of μ. The choice of parameters are $p_1 = 1, p_2 = 1, \xi = 1, \kappa_{12} = 0.6, \kappa_{21} = 0.8\exp(-\mu), m_1 = 1 - \kappa_{12}$, $m_2 = 1 - \kappa_{21}$ with $l = 1$, $J = 100$ and $T = 5$ under CFL $= 0.75$

Isothermal Euler Equations

Similar to the flow of water, a measurement error is a cause of disturbances of a flow of gas through a pipeline. Thus, we study a flow of ideal gas in pipelines with a measurement error. We denote δ as a diameter of the pipe, f_g as a friction factor and a as the speed of sound. The model of the flow is described by isothermal Euler equations (see [15]) of the form

$$\begin{aligned} &\partial_t \rho + \partial_x q = 0, \\ &\partial_t q + \partial_x \left(\frac{q^2}{\rho} + a^2 \rho \right) + \frac{f_g q|q|}{2\delta\rho} = 0, \; x \in [0, l], \; t \in [0, +\infty), \end{aligned} \tag{58}$$

where $\rho = \rho(x,t)$ is the density of the gas and $q = q(x,t)$ is the mass flux in the pipe.

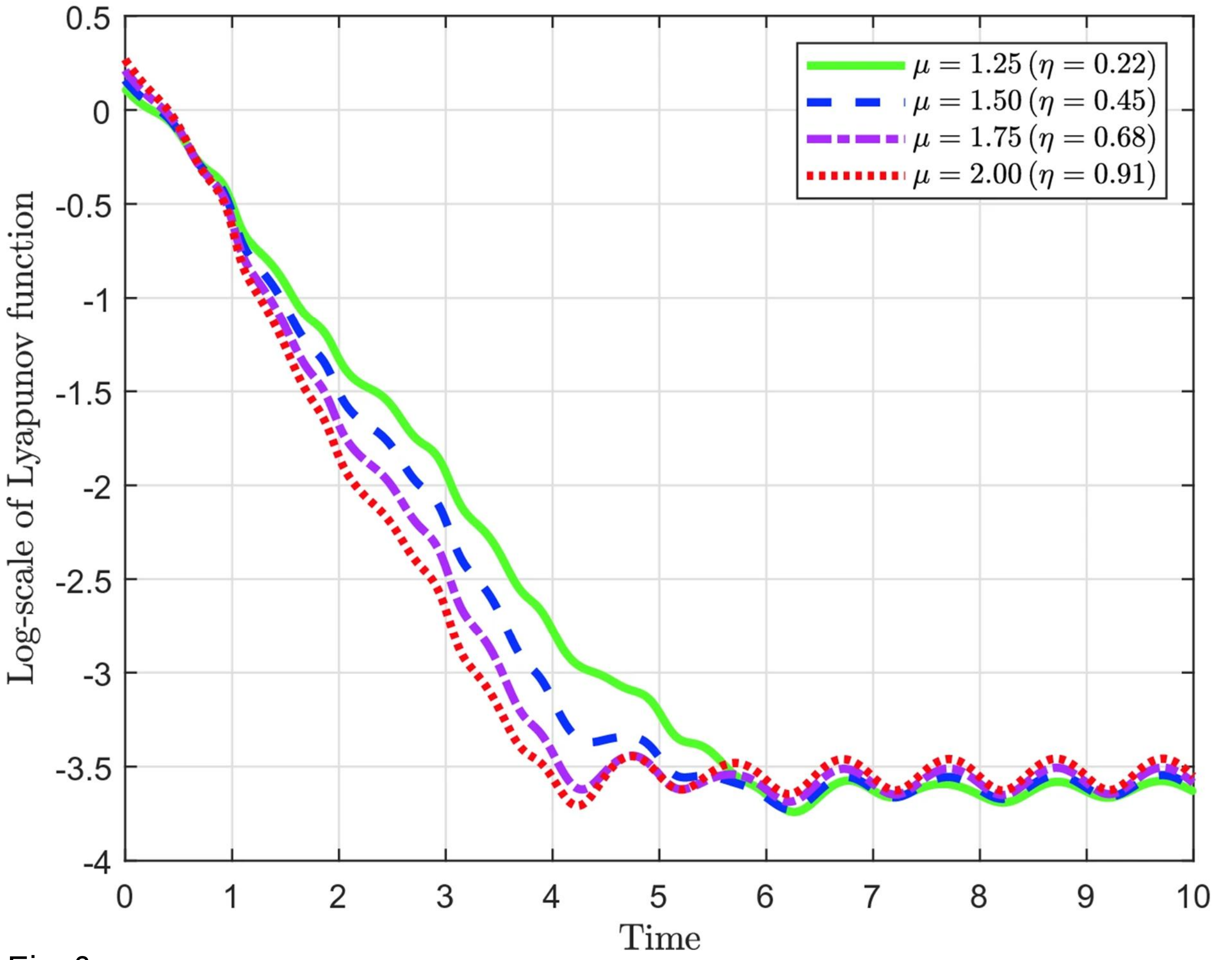

Fig. 2

Log-scale of the decay of Lyapunov function for isothermal Euler equations for different values of μ. The choice of parameters are $p_1 = 1, p_2 = 1, \xi = 1, \kappa_{12} = 0.6, \kappa_{21} = 0.75\exp(-\mu), m_1 = 1 - \kappa_{12}$, and $m_2 = 1 - \kappa_{21}$ with $l = 1$, $J = 100$ and $T = 10$ under CFL $= 0.75$

We set an initial condition

$$\rho(x,0) = \rho_0(x), \quad q(x,0) = q_0(x), \quad x \in (0,l), \tag{59}$$

and boundary conditions with disturbances

$$q(0,t) = k_0\rho(0,t) + b_1(t), \quad q(l,t) = k_l\rho(l,t) + b_2(t), \quad t \in [0,\infty), \tag{60}$$

where k_0, k_l are parameters and b_1, b_2 are disturbance functions.

We assume the flow of the gas is from $x = 0$ to $x = l$, i.e., $q > 0$. A non-uniform steady-state solution to the system (58) satisfies

$$q^* = \text{const.}, \quad \text{and} \quad \frac{d\rho^*}{dx} = -\frac{\frac{f_g q^{*2}}{2\delta\rho^*(x)}}{\left(a^2 - \frac{q^{*2}}{\rho^{*2}(x)}\right)}, \; x \in [0, l]. \tag{61}$$

For subsonic flow (i.e. $\frac{q}{\rho} < a$), the system (58) is linearised as (55) with $Z(x,t) = [\rho - \rho^*, q - q^*]^\top$,

$$A(x) = \begin{bmatrix} 0 & 1* \\ a^2 - \frac{q^{*2}}{\rho^{*2}(x)} & \frac{2q^*}{\rho^*(x)} \end{bmatrix}, \quad \text{and} \quad B(x) = \begin{bmatrix} 0 & 0 \\ -\frac{f_g q^{*2}}{2\delta\rho^{*2}(x)} & \frac{f_g q^*}{\delta\rho^*(x)} \end{bmatrix}.$$

Define the Riemann coordinates for the linearised isothermal Euler equations by (56) with

$$L(x) = \begin{bmatrix} \frac{1}{\lambda_1(x)} & 1 \\ \frac{1}{\lambda_2(x)} & 1 \end{bmatrix}, \quad \Lambda(x) = \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix} = \begin{bmatrix} \frac{q^*}{\rho^*(x)} + a & 0 \\ 0 & \frac{q^*}{\rho^*(x)} - a \end{bmatrix}$$

and

$$L^{-1}(x) = \frac{\lambda_1(x)\lambda_2(x)}{\lambda_1(x) + \lambda_2(x)} \begin{bmatrix} 1 & -1 \\ \frac{-1}{\lambda_2(x)} & \frac{1}{\lambda_1(x)} \end{bmatrix}.$$

Then, the linearised isothermal Euler equations, initial condition (59) and the boundary conditions (60) can be written as (1), (2) and (3), respectively with (57), $f(x) := w_1(x,0)$, $g(x) := w_2(x,0)$, $\kappa_{12} := \frac{k_0 + 1/\lambda_1(0)}{k_0 + 1/\lambda_2(0)} \neq 1$, $\kappa_{21} := \frac{k_l + 1/\lambda_2(l)}{k_l + 1/\lambda_1(l)} \neq 1$, $m_1 := 1 - \kappa_{12}$ and $m_2 := 1 - \kappa_{21}$.

We take an example $\rho^*(0) = 3, q^*(x) = 0.2, \; x \in [0,1]$ with the parameters given by $a = 1, \frac{f_g}{\delta} = 1$. Thus

$$\rho^*(x) = \frac{3}{\exp\left(\frac{\text{LambertW}(-1, -225\exp(x-225))}{2} - \frac{x}{2} + \frac{225}{2}\right)}.$$

Besides, we set an initial condition by $w_1(x,0) = w_2(x,0) = \cos(2\pi x), \; x \in (0,1)$ and the rate of the boundary disturbance functions taken as $b_1(t) = -b_2(t) = 0.01\sin(\pi t), \; t > 0$.

Again in Fig. 2, it can be observed that as μ increases the rate of decay of the ISS-Lyapunov function in the presence of boundary disturbance increases. Hence, in the sense of the definition of discrete ISS, the steady-state $W_j^n \equiv 0, \; j = 0, \ldots, J-1, \; n = 0, \ldots, N-1$ of the discretised system with the discretised boundary conditions is discrete ISS in L^2-norm with respect to discrete disturbance function $b^n,\ n = 1, \ldots, N$.

Conclusion

In this paper, we presented the discretisation of a linear hyperbolic system of balance laws with boundary disturbance. For numerical discretisation, we used a finite volume method. Specifically, we used upwind scheme. We also discretised an L^2-ISS-Lyapunov function to investigate conditions for ISS of the discretised system. Finally, the result was applied to a linear problem and a relevant physical problem: Saint-Venant equations and numerical simulations are computed in order to test the results and compare with analytical results. We also showed numerical simulation for the isothermal Euler equations. The properties that have been proved analytically can also be established computationally.

This work leaves more questions open. There is need to analyse Lyapunov functions for nonlinear differential equations. Analysis of numerical artefacts such as numerical viscosity need to be carefully examined. Such numerical artefacts may have an influence on the rate of convergence of the discrete results.

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