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Closed form parametric solutions of nonlinear Abel-type and Riccati-type spacecraft relative motion☆

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ABSTRACT

In comparison to the conventional approach of using Cartesian coordinates to describe spacecraft relative motion, the relative orbit description using Keplerian orbital elements provides a better visualization of the relative motion due to the benefit of having only one term (anomaly) that changes with time out of the six orbital elements leading to the reduction of the number of terms to be tracked from six, as in the case of Hill coordinates, to one. In this paper, under certain assumptions and transformations, the spacecraft relative equations of motion, in terms of orbital element differences, is approximated into the nonlinear first kind Abel-type and Riccati-type differential equations. Furthermore, we present methodologies for the formulation of the close form analytical solutions of the approximated equations. As shown by the numerical simulations, the closed form solutions and the nonlinear equations are in conformity with Riccati-type equations having higher errors than the Abel-type equations. This shows that the Abel-type equation, a third order polynomial, approximated the relative motion better than the Riccati-type equation, a second order polynomial. The resulting new analytical solutions gave better insight into the relative motion dynamics and can be used for the analysis of spacecraft formation flying, proximity and rendezvous operations.

1. Introduction

Over many decades, the linearized time-invariant Hill–Clohessy–Wiltshire (HCW) [1,2] equations have been used to describe relative motion of deputy spacecraft with respect to the chief spacecraft in circular orbit. Examples of the usage of this model in spacecraft formation flying, proximity and rendezvous operations can be found in [3–11]. Similarly, the linearized Tschauner–Hempel [12–14] equations are used to describe the motion with chief in elliptical orbit. The equations were derived with the assumptions that the spacecraft are in close proximity, the Earth is spherical and by neglecting the nonlinear terms. The Hill frame coordinates have the disadvantages that their differential equations must be solved before the relative orbit geometry can be obtained and the HCW equations are initial condition dependent valid only if the relative orbit dimension is small in comparison to the chief orbit radius. Unlike in the case of Hill frame coordinates, using orbit element differences offers the advantage of better visualization of the relative orbit and slow variation because only one parameter (anomaly) has to be tracked instead of tracking six parameters of position and velocity.

Recently, due to the benefits offered by using orbital elements to describe the relative motion dynamics over the conventional approach of using Cartesian coordinates of the Hill frame, researchers have published several papers to show its effectiveness and simplicity [2,15–18]. The convenience of using orbit element differences to describe and control the desired relative orbit geometry is demonstrated in many papers. A direct linear mapping between the local Cartesian coordinates and the corresponding orbit element differences is given in [16]. The mapping was used in the construction of a hybrid continuous feedback control law. Schaub and Alfriend presented a method to establish J2 invariant relative orbits for spacecraft formation flying applications. The desired relative orbit geometry is designed using differences in mean orbit elements [19]. An algorithm for relating the orbital element changes to the relative motion variables was developed by [15]. This algorithm is used in the development of a state transition matrix that includes the effects of the chief satellite orbit eccentricity and the gravitational perturbations. The state transition matrix was developed by considering the geometry of the problem, not by solving the differential equations. In 2017, a portion of this paper was presented at

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the AAS/AIAA Astrodynamics Specialist Conference which took place in Stevenson [18].

The equation of motion of the orbital element differences is nonlinear in the variation of the true of latitude rate. The evolution nonlinear equation can be approximated into second, third and higher orders. The third order corresponds to the Abel-type equation while the second order corresponds to the Riccati type equation. Over several decades, researchers have published books and presented papers on the methods for the solution of Abel and Riccati equations under certain conditions and assumptions [20–28]. The book by [21] contains several methods which can be applied to certain Abel and Riccati equations.

In this paper, we present the construction and closed form analytical solutions of first kind nonlinear Abel-type and Riccati-type spacecraft relative equations of motion from the original nonlinear equation of the variation of the true of latitude. Also, two models of third-order (corresponding to Abel-type equation) and second-order (corresponding to the Riccati-type equation) nonlinear differential equations describing the relative motion dynamics are developed. Using well known techniques and methods, we developed analytical solutions of the Abel-type and Riccati-type spacecraft nonlinear equations of motion of the first kind. To the best knowledge of the authors, this is the first time that this approach is applied to solve spacecraft relative motion. The significant contributions of this work are that new and improved models of the relative motion, in terms of the orbital element differences, are developed and their corresponding closed form solutions are also obtained.

The paper is organized as follows. In Section 2, the review of nonlinear spacecraft relative motion in terms of orbital element differences, nonlinear Abel-type and Riccati-type first order differential equations are presented. Two models of orbital element differences approximated into cubic polynomial form (Abel-type) and quadratic polynomial form (Riccati-type) are derived in Section 3. Sections 4 and 5 gave the detail on the development of close form solutions of the Abel-type and Riccati type nonlinear spacecraft relative equations of motion. In Section 6, the nonlinear and approximated equations are compared numerically. The conclusion is given in Section 7.

2. Nonlinear spacecraft relative motion in terms of orbital-element differences

Recently, to gain a better insight into the relative motion dynamics attention is shifted to the use of orbital element differences. Using Hill coordinate frames the relative orbit is determined with the Cartesian coordinates

$$\mathbf{X} = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T \tag{1}$$

where x, y, z and $\dot{x}, \dot{y}, \dot{z}$ are the position and velocity components. All the six variables, which vary with time per the Hill–Clohessy–Wiltshire (HCW) second order differential equations that govern relative motion [2], must be determined to be able to track where the deputy spacecraft would be at a point in time. Rather than tracking all six variables continuously the dynamics is simplified by astrodynamics using Keplerian elements. This has advantage of having five constant orbital elements and one time-varying. Therefore, only one term (true anomaly) which is time varying must be tracked over time. The orbit description is simplified using orbit elements which vary slowly with the presence of perturbation forces such as third body perturbation, atmospheric and solar drag. The dynamics of the relative motion can also be described using the following six orbital element set

$$\mathbf{e} = \begin{bmatrix} a & \theta & i & q_1 & q_2 & \Omega \end{bmatrix}^T \tag{2}$$

where, a is the semi-major axis, e is the eccentricity, i is the inclination, Ω is the longitude of the ascending node, ω is the argument of periapse, and $\theta = \omega + f$ is the true latitude, $q_1 = e\cos\omega$ and $q_2 = e\sin\omega$. The relative motion between the deputy and chief can be represented using the orbit element different vector as

$$\delta\mathbf{e} = \mathbf{e}_d - \mathbf{e}_c = \begin{bmatrix} \delta a & \delta\theta & \delta i & \delta q_1 & \delta q_2 & \delta\Omega \end{bmatrix}^T \tag{3}$$

Here, $\mathbf{e}_d$ and $\mathbf{e}_c$ are the deputy and chief spacecraft orbit element vector. Taking the orbital element set in Eq. (2) as the chief spacecraft elements then the deputy spacecraft elements are $a+\delta a$, $\theta+\delta\theta$, $i+\delta i$, $q_1+\delta q_1$, $q_2+\delta q_2$, and $\Omega+\delta\Omega$. The linear mapping between the Hill frame coordinates and the orbit element differences is presented in Refs. [2,15]. Using the orbit elements, the orbit radius can be expressed as

$$r = \frac{a\left(1 - q_1^2 - q_2^2\right)}{\left(1 + q_1\cos\theta + q_2\sin\theta\right)} \tag{4}$$

with the variation

$$\delta r = \frac{r}{a}\delta a + \frac{V_r}{V_t} r\delta\theta - \frac{r}{p}\left(2aq_1 + r\cos\theta\right)\delta q_1 - \frac{r}{p}\left(2aq_2 + r\sin\theta\right)\delta q_2 \tag{5}$$

The chief radial and transverse velocity components are defined by

$$\begin{aligned} V_r &= \dot{r} = \frac{h}{p}\left(q_1\sin\theta - q_2\cos\theta\right) \\ V_t &= r\dot{\theta} = \frac{h}{p}\left(1 + q_1\cos\theta + q_2\sin\theta\right) \end{aligned} \tag{6}$$

In terms of the orbit element differences [15,16], the Cartesian coordinate relative position vector components are expressed as

$$\begin{aligned} x &= \delta r \\ y &= r\left(\delta\theta + \cos i\delta\Omega\right) \\ y &= r\left(\sin\theta\delta i - \cos\theta sini\delta\Omega\right) \end{aligned} \tag{7}$$

while the relative velocity components are expressed as

$$\begin{aligned} \dot{x} &= -\frac{V_r}{2a}\delta a + \left(\frac{1}{r} - \frac{1}{p}\right)h\delta\theta + \left(V_r a q_1 + h\sin\theta\right)\frac{\delta q_1}{p} \\ &\quad + \left(V_r a q_2 - h\cos\theta\right)\frac{\delta q_2}{p} \\ \dot{y} &= -\frac{3V_t}{2a}\delta a - V_r\delta\theta + \left(3V_t a q_1 + 2h\cos\theta\right)\frac{\delta q_1}{p} \\ &\quad + \left(3V_t a q_2 + 2h\sin\theta\right)\frac{\delta q_2}{p} + V_r\cos i\delta\Omega \\ \dot{z} &= \left(V_t\cos\theta + V_r\sin\theta\right)\delta i \\ &\quad + \left(V_t\sin\theta - V_r\cos\theta\right) sini\delta\Omega \end{aligned} \tag{8}$$

Since $\delta\theta$ is the only time-varying parameter in Eq. (3) then the rate of change of the orbit element differences vector, $\delta\mathbf{e}$, is

$$\delta\dot{\mathbf{e}} = \begin{bmatrix} 0 & \delta\dot{\theta} & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{9}$$

This gives equations of relative motion of deputy with respect to the chief in terms of the orbital element differences. The true latitude rate $\dot{\theta}$, using the principle of the conservation of angular momentum h can be expressed as

$$\dot{\theta} = \frac{h}{r^2} \tag{10}$$

Using Eq. (4) and the fact that $h = \sqrt{\mu p}$ then the difference between the deputy and chief true latitude rate is

$$\delta\dot{\theta} = \left(\sqrt{\frac{\mu}{\left[(a+\delta a)\left\{1-\left(q_1+\delta q_1\right)^2-\left(q_2+\delta q_2\right)^2\right\}\right]^3}}\begin{Bmatrix} 1+\left(q_1+\delta q_1\right)\cos\left(\theta+\delta\theta\right) \\ +\left(q_2+\delta q_2\right)\sin\left(\theta+\delta\theta\right)\end{Bmatrix}^2\right) - \sqrt{\frac{\mu}{\left\{a\left(1-q_1^2-q_2^2\right)\right\}^3}}\left(1+q_1\cos\theta+q_2\sin\theta\right)^2 \tag{11}$$

Eq. (11) is the nonlinear equations of motion of the difference of latitude rate as a function of δa, $\delta\theta$, δq_1 and δq_2. The variation of Eq. (10) is

$$\delta\dot{\theta} = \frac{h}{r^2}\left(\frac{\delta p}{2p} - 2\frac{\delta r}{r}\right) \tag{12}$$

where,

$$\delta p = \frac{p}{a}\delta a - 2a\left(q_1\delta q_1 + q_2\delta q_2\right) \tag{13}$$

Using Eq. (12), Schaub and Junkins [2] approximated $\delta\dot{\theta}$ as a linear expression of $\delta\mathbf{e}$ as

$$\delta\dot{\theta} = -\frac{3h}{2ar^2}\delta a - \frac{2hV_r}{r^2V_t}\delta\theta + \left(\frac{3haq_1}{pr^2} + \frac{2h}{pr}\cos\theta\right)\delta q_1 + \left(\frac{3haq_2}{pr^2} + \frac{2h}{pr}\sin\theta\right)\delta q_2 \tag{14}$$

For a better accuracy than the linear model in Eq. (14), the nonlinear equation can be approximated into third order polynomial corresponding to Abel-type first order equation. If the expansion is done to second order only we have a Riccati-type equation.

2.1. Nonlinear Abel-type first-order differential equation

The nonlinear Abel-type first-order differential equation of first kind is one of the most known nonlinear ODEs generally used to describe majority of mathematical physics and nonlinear mechanics dynamics problems [26,27,29,30]. It has the general form

$$y'_x = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x) \tag{15}$$

Notation $d()/dx = ()'_x$ denotes total derivative. If $f_3(x) = 0$ one obtains Riccati-type ODE, if $f_3(x) = f_2(x) = 0$ one obtains a first order linear ODE, and if $f_3(x) = f_0(x) = 0$ one obtains Bernoulli equation. Intensive investigations of Eq. (15) have been carried out by researchers and these can be found in literatures and technical publications [21,26,31–33]. In terms of known (tabulated) functions, Eq. (15) does not admit general exact solutions. However, under certain restrictions and assumptions that may impose both quantitative and qualitative biases closed form analytical solutions can be obtained. Applying the admissible functional transformations

$$\begin{aligned} &\xi = \int f_3E^2dx, u = \left(y + \frac{f_2}{f_3}\right)E^{-1} \\ &E = \exp\left[\int\left(f_1 - \frac{f_2^2}{3f_3}\right)dx\right] \end{aligned} \tag{16}$$

to the general form (15) yields the canonical normal form

$$u'_\xi = u^3 + \Phi(\xi) \tag{17}$$

where,

$$\Phi(\xi) = \frac{1}{f_3E^3}\left[f_0 + \frac{1}{3}\frac{d}{dx}\left(\frac{f_2}{f_3}\right) - \frac{f_1f_2}{3f_3} + \frac{2f_2^3}{27f_3^2}\right] \tag{18}$$

Solving Equation (15) an analytical solution is (Polyanin and Zaitsev 2002 [21])

$$y(x) = E\left(C - 2\int f_3E^2dx\right)^{-1/2} - \frac{f_2}{3f_3} \tag{19}$$

A two-dimensional dynamical system associated with Abel's nonlinear equation was analyzed by [31]. Mak and Harko (2013) [34] presented new method for generating a general solution of the nonlinear first kind Abel-type differential equation from a particular one. Polyanin and Zaitsev (2002) [21], Mancas and Rosu (2013) [35] emphasized two connections between the dissipative nonlinear second order differential equations and the Abel equations which in its first kind form have only cubic and quadratic terms. They show how to obtain Abel solutions directly from the factorization of second-order nonlinear equations. Different types of Abel differential equations are presented in the mathematical handbook of exact solutions for ordinary differential [21].

2.2. Nonlinear Riccati-type first-order differential equation

In many areas of engineering and science, especially in control, optimization and systems theory, the nonlinear Riccati-type equation play a key role. The term "Riccati equation" refers to matrix equations with an analogous quadratic term in both continuous and discrete-time systems [36–39]. The general form of a first order Riccati ODE is

$$\frac{dy}{dx} = f_2(x)y^2 + f_1(x)y + f_0(x) \tag{20}$$

where, if $f_2(x) = 0$, one obtains a first order linear ODE, while if $f_0(x) = 0$, one derives a first order Bernoulli equation. Riccati equation is one of the most studied first order nonlinear differential equations that arises in different fields of mathematics and physics [21–24,40] named after the Italian mathematician Jacopo Francesco Riccati (1724). It has the form which can be considered as the lowest order nonlinear approximation to the derivative of a function in terms of the function itself. For $f_2 \equiv 0$, we obtain a linear equation and for $f_0 \equiv 0$ we have the Bernoulli equation [21].

Generally, it is well-known that only special cases can be treated because solutions to the general Riccati equation are not available. To find the general solution one needs only a particular solution. Given a particular solution $y_p = y_p(x)$ of the Riccati equation, the general solution of the Riccati equation can be written as [21–23,40]

$$y\left[f_2(x), f_1(x), y_p(x)\right] = y_p(x) + \Phi(x)\left[C - \int\Phi(x)f_2(x)dx\right]^{-1} \tag{21}$$

where C is an arbitrary constant of integration and

$$\Phi(x) = \exp\left\{\int\left[2f_2(x)y_p(x) + f_1(x)y\right]dx\right\} \tag{22}$$

The particular solution of the Riccati equation satisfies

$$\frac{dy_p}{dx} = f_2(x)y_p^2 + f_1(x)y_p + f_0(x) \tag{23}$$

The substitution $u(x) = \exp\left(-\int f_2(x)ydx\right)$ reduces the general Riccati equation to a second order linear equation

$$\frac{d^2u}{dx} - \left[\frac{1}{f_2(x)}\frac{df_2(x)}{dx} + f_1(x)\right]\frac{du}{dx} + f_0(x)f_2(x) = 0 \tag{24}$$

If a particular solution is not known and the coefficients of the Riccati equation satisfy the following specific condition

$$f_2(x) + f_1(x) + f_0(x) = 0 \tag{25}$$

the Riccati equation will have the solution

$$y = \frac{K + \int\left[f_2(x) - f_0(x)\right]E(x)dx - E(x)}{K + \int\left[f_2(x) + f_0(x)\right]E(x)dx + E(x)} \tag{26}$$

where K is an arbitrary constant of integration. If $f_2(x) \equiv 1$, and the functions $f_1(x)$ and $f_0(x)$ are polynomials satisfying the condition [21,23]

$$\Delta = f_1^2(x) - 2\frac{df_1(x)}{dx} - 4f_0(x) \equiv \text{constant} \tag{27}$$

then

$$y_\pm(x) = -\frac{1}{2}\left[f_1(x) \pm \sqrt{\Delta}\right] \tag{28}$$

are both solutions of the Riccati Eq. (15).

3. Approximation of orbital-element differences equations of motion

In this section, two approximation models of the nonlinear equation of motion of orbital element differences are presented.

3.1. Construction of Abel-type nonlinear relative motion

Here, the construction of Abel-type equations is done by approximating the nonlinear equation of motion as a third-order function of four parameters and one parameter respectively [41].

(a) Approximation as a Third-Order Function of Four Parameters

In the development of the first model of Abel-type relative equations of motion it is assumed that, in Eq. (11), $\delta\dot{\theta}$ is a nonlinear function of all the four parameters δa, $\delta\theta$, δq_1 and δq_2 using Taylor series expansion. Using the mean motion $n=\sqrt{\mu/a^3}$ it can be written as

$$\begin{aligned}&\delta\dot\theta = n\left(1+\frac{\delta a}{a}\right)^{-3/2} f\left(\delta q_1,\delta q_2,\delta\theta\right)\\&-n\left(1+q_1\cos\theta+q_2\sin\theta\right)^2\left(1-q_1^2-q_2^2\right)^{-3/2}\end{aligned} \tag{29}$$

where,

$$f\left(\delta q_1,\delta q_2,\delta\theta\right)=\left[\frac{\begin{pmatrix}1+\left(q_1+\delta q_1\right)\cos\left(\theta+\delta\theta\right)\\+\left(q_2+\delta q_2\right)\sin\left(\theta+\delta\theta\right)\end{pmatrix}^2}{\left(1-\left(q_1+\delta q_1\right)^2-\left(q_2+\delta q_2\right)^2\right)^{-3/2}}\right] \tag{30}$$

Using Binomial series expansion we have

$$\left(1+\frac{\delta a}{a}\right)^{-3/2}=1-\frac{3}{2a}(\delta a)+\frac{15}{8a^2}(\delta a)^2-\frac{35}{16a^3}(\delta a)^3\ldots \tag{31}$$

Applying Taylor series at the origin $(0,0,0)$ to $f\left(\delta q_1,\delta q_2,\delta\theta\right)$ and substituting the result and Eq. (31) into Eq. (29) yields a special type of Abel-type equation of first kind as

$$\delta\dot\theta = p_3(\theta)(\delta\theta)^3+p_2(\theta)(\delta\theta)^2+p_1(\theta)\delta\theta+p_0(\theta) \tag{32}$$

Eq. (32) is the first model of the third-order approximation of the rate of change of the true-anomaly. The expressions for $p_3(\theta), p_2(\theta), p_1(\theta), p_0(\theta)$ are provided in Appendix A.

(b) Approximation as a Third-Order Function of One Parameter

Here, $\delta\dot\theta$ is approximated as a third order function of only one time-varying parameter, true of latitude difference. The other three parameters, $\delta a, \delta q_1, \delta q_2$, are constants. Using series expansion technique and eliminating higher order terms above the cubic, we have the following trigonometric functions

$$\begin{aligned}\cos\left(\theta+\delta\theta\right) &\approx \frac{1}{6}\sin\theta(\delta\theta)^3-\frac{1}{2}\cos\theta(\delta\theta)^2\\&\quad-\sin\theta\delta\theta+\cos\theta\\ \sin\left(\theta+\delta\theta\right) &\approx -\frac{1}{6}\cos\theta(\delta\theta)^3-\frac{1}{2}\sin\theta(\delta\theta)^2\\&\quad+\cos\theta\delta\theta+\sin\theta\\ \cos 2\left(\theta+\delta\theta\right) &\approx \frac{4}{3}\sin 2\theta(\delta\theta)^3-2\cos 2\theta(\delta\theta)^2\\&\quad-2\sin 2\theta\,(\delta\theta)+\cos 2\theta\\ \sin 2\left(\theta+\delta\theta\right) &\approx -\frac{4}{3}\cos 2\theta(\delta\theta)^3-2\sin 2\theta(\delta\theta)^2\\&\quad+2\cos 2\theta\,(\delta\theta)+\sin 2\theta\end{aligned} \tag{33}$$

Defining M_D and M_C as

$$\begin{aligned}M_D &= \sqrt{\frac{\mu}{\left[(a+\delta a)\left\{1-\left(q_1+\delta q_1\right)^2-\left(q_2+\delta q_2\right)^2\right\}\right]^3}}\\ M_C &= \sqrt{\frac{\mu}{\left\{a\left(1-q_1^2-q_2^2\right)\right\}^3}}\end{aligned} \tag{34}$$

and using the identities in Eq. (33) simplifies Eq. (11) to the second model of special form of Abel-type equation of first kind as

$$\delta\dot\theta = k_3(\theta)(\delta\theta)^3+k_2(\theta)(\delta\theta)^2+k_1(\theta)\delta\theta+k_0(\theta) \tag{35}$$

The coefficients k_3, k_2 and k_0 are shown in Appendix B.

3.2. Construction of Riccati-type nonlinear relative motion

In a manner similar to the approach in Section 3.1, Riccati-type orbital-element differences equations of motion are developed as functions of all the four parameters δa, $\delta\theta$, δq_1 and δq_2 and as functions of only the parameter $\delta\theta$. Using Taylor series expansion with $(x,y,z)=\left(\delta q_1,\delta q_2,\delta\theta\right)$ and $\left(x_0,y_0,z_0\right)=(0,0,0)$, truncation after quadratic terms gives the first model of a Riccati-type second-order approximation of orbital element differences equations of motion as a function of all the four parameters as

$$\delta\dot\theta = s_2(\theta)(\delta\theta)^2+s_1(\theta)\delta\theta+s_0(\theta) \tag{36}$$

where,

$$\begin{aligned}&s_0(\theta)=\tfrac{1}{2}n\delta q_1\delta q_2 f_{\delta q_1\delta q_2}\Big|_{(0,0,0)}\\&+\tfrac{1}{2}n\delta q_2\delta q_1 f_{\delta q_2\delta q_1}\Big|_{(0,0,0)}\\&-\tfrac{3n}{2a}\delta a\delta q_1 f_{\delta q_1}\Big|_{(0,0,0)}-\tfrac{3n}{2a}\delta a\delta q_2 f_{\delta q_2}\Big|_{(0,0,0)}\\&+\tfrac{15n}{8a^2}(\delta a)^2 f\Big|_{(0,0,0)}+\tfrac{1}{2}n\left(\delta q_1\right)^2 f_{\delta q_1\delta q_1}\Big|_{(0,0,0)}\\&+\tfrac{1}{2}n\left(\delta q_2\right)^2 f_{\delta q_2\delta q_2}\Big|_{(0,0,0)}\\&-\tfrac{3n}{2a}f\Big|_{(0,0,0)}\delta a+nf_{\delta q_1}\Big|_{(0,0,0)}\delta q_1\\&+nf_{\delta q_2}\Big|_{(0,0,0)}\delta q_2\\&s_1(\theta)=nf_{\delta\theta}\Big|_{(0,0,0)}+\tfrac{1}{2}n\delta q_1 f_{\delta q_1\delta\theta}\Big|_{(0,0,0)}\\&+\tfrac{1}{2}n\delta q_2 f_{\delta q_2\delta\theta}\Big|_{(0,0,0)}+\tfrac{1}{2}n\delta q_1 f_{\delta\theta\delta q_1}\Big|_{(0,0,0)}\\&+\tfrac{1}{2}n\delta q_2 f_{\delta\theta\delta q_2}\Big|_{(0,0,0)}-\tfrac{3n}{2a}\delta a f_{\delta\theta}\Big|_{(0,0,0)}\\&s_2(\theta)=\tfrac{1}{2}nf_{\delta\theta\delta\theta}\Big|_{(0,0,0)}\end{aligned} \tag{37}$$

From Eq. (35), the second model of the Riccati-type second order approximation of orbital-element differences equations of motion as a function of only one parameter is obtained, taking $k_3(\theta)=0$, as

$$\delta\dot\theta = k_2(\theta)(\delta\theta)^2+k_1(\theta)\delta\theta+k_0(\theta) \tag{38}$$

This is made possible because the approximation is done with respect to only one varying parameter.

3.3. Construction of linearized relative motion

Using Taylor series expansion we have the first order approximation of Eq. (29) as

$$\begin{aligned}\delta\dot\theta &= -\frac{3n}{2a}f\Big|_{(0,0,0)}\delta a+nf_{\delta\theta}\Big|_{(0,0,0)}\delta\theta\\&\quad+nf_{\delta q_1}\Big|_{(0,0,0)}\delta q_1+nf_{\delta q_2}\Big|_{(0,0,0)}\delta q_2\end{aligned} \tag{39}$$

This simplifies to

$$\delta\dot\theta = p_{11}(\theta)\delta\theta+p_{10}(\theta) \tag{40}$$

where,

$$\begin{aligned}p_{10}(\theta) &= -\frac{3n}{2a}\left\{\begin{pmatrix}1+q_1\cos\theta\\+q_2\sin\theta\end{pmatrix}^2\begin{pmatrix}1-q_1^2\\-q_2^2\end{pmatrix}^{-3/2}\right\}\delta a\\&+n\left\{\begin{matrix}2\cos\theta\begin{pmatrix}1+q_1\cos\theta\\+q_2\sin\theta\end{pmatrix}\begin{pmatrix}1-q_1^2\\-q_2^2\end{pmatrix}^{-3/2}\\+3q_1\begin{pmatrix}1-q_1^2\\-q_2^2\end{pmatrix}^{-5/2}\begin{pmatrix}1+q_1\cos\theta\\+q_2\sin\theta\end{pmatrix}^2\end{matrix}\right\}\delta q_1\\&+n\left\{\begin{matrix}2\sin\theta\begin{pmatrix}1+q_1\cos\theta\\+q_2\sin\theta\end{pmatrix}\begin{pmatrix}1-q_1^2\\-q_2^2\end{pmatrix}^{-3/2}\\+3q_2\begin{pmatrix}1-q_1^2\\-q_2^2\end{pmatrix}^{-5/2}\begin{pmatrix}1+q_1\cos\theta\\+q_2\sin\theta\end{pmatrix}^2\end{matrix}\right\}\delta q_2\end{aligned} \tag{41}$$

and

$$p_{11}(\theta)=2n\left\{\begin{pmatrix}-q_1\sin\theta\\+q_2\cos\theta\end{pmatrix}\begin{pmatrix}1\\+q_1\cos\theta\\+q_2\sin\theta\end{pmatrix}\begin{pmatrix}1\\-q_1^2\\-q_2^2\end{pmatrix}^{-3/2}\right\} \tag{42}$$

Eq. (40) is the first model, linearized spacecraft relative equation of motion and it has the same form as in the linear Equation (14). Considering the first order approximation only, Eq. (35) becomes

$$\delta\dot{\theta} = k_1(\theta)\delta\theta + k_0(\theta) \tag{43}$$

Eq. (43) is the second model of linearized equation.

4. Closed form solution of Abel-type nonlinear relative motion

The development of general solutions of Abel-Type nonlinear relative equation of motion is carried out using approach in the Refs. [21,34,42]. Substituting the transformation $\delta\theta = \delta\theta_p + u(\theta)E(\theta)$ into Eq. (32) we have the form

$$\begin{aligned} u' = E^2 p_3 \Big[& u^3 + \tfrac{1}{Ep_3}\left(3p_3\delta\theta_p + p_2\right)u^2 \\ & + \tfrac{1}{E^2p_3}\left\{3p_3\left(\delta\theta_p\right)^2 + 2p_2\delta\theta_p + p_1 - \tfrac{E'}{E}\right\}u \\ & + \tfrac{1}{E^3p_3}\left\{p_0 - \delta\theta'_p + p_3\left(\delta\theta_p\right)^3 + p_2\left(\delta\theta_p\right)^2 + p_1\left(\delta\theta_p\right)\right\}\Big] \end{aligned} \tag{44}$$

where,

$$u' = \frac{du}{d\theta}, E' = \frac{dE}{d\theta}, \delta\theta'_p = \frac{d\left(\delta\theta_p\right)}{d\theta} \tag{45}$$

It is assumed that the following is satisfied

$$\begin{aligned} & \tfrac{1}{Ep_3}\left(3p_3\delta\theta_p + p_2\right) = \beta_1(\theta) \\ & \tfrac{1}{E^2p_3}\left\{3p_3\left(\delta\theta_p\right)^2 + 2p_2\delta\theta_p + p_1 - \tfrac{E'}{E}\right\} = \beta_2(\theta) \\ & \tfrac{1}{E^3p_3}\left\{\begin{matrix} p_0 - \delta\theta'_p + p_3\left(\delta\theta_p\right)^3 \\ +p_2\left(\delta\theta_p\right)^2 + p_1\left(\delta\theta_p\right)\end{matrix}\right\} = \beta_3(\theta) \end{aligned} \tag{46}$$

Considering the case in which the system (46) satisfies $\beta_1(\theta) = \beta_2(\theta) = 0$ and $\beta_3(\theta) = \Phi(\theta)$ we have the corresponding system

$$\begin{aligned} & \tfrac{1}{Ep_3}\left(3p_3\delta\theta_p + p_2\right) = 0 \\ & \tfrac{1}{E^2p_3}\left\{3p_3\left(\delta\theta_p\right)^2 + 2p_2\delta\theta_p + p_1 - \tfrac{E'}{E}\right\} = 0 \\ & \tfrac{1}{E^3p_3}\left\{\begin{matrix} p_0 - \delta\theta'_p + p_3\left(\delta\theta_p\right)^3 \\ +p_2\left(\delta\theta_p\right)^2 + p_1\left(\delta\theta_p\right)\end{matrix}\right\} = \Phi(\theta) \end{aligned} \tag{47}$$

Solving for $\delta\theta_p$ in the first part of Eq. (47) and substituting on the second and third part of Eq. (47) the above system reduces to

$$\begin{aligned} & \delta\theta_p = -\tfrac{p_2}{3p_3} \\ & -\tfrac{p_2^2}{3p_3} + p_1 - \tfrac{E'}{E} = 0 \\ & \tfrac{1}{E^3p_3}\left\{p_0 - \tfrac{p_1p_2}{3p_3} + \tfrac{2p_2^3}{3p_3^2} + \tfrac{1}{3}\left(\tfrac{p_2}{p_3}\right)'\right\} = \Phi(\theta) \end{aligned} \tag{48}$$

From second part of Eq. (48), E is obtained as

$$E(\theta) = \exp\left[\int\left(p_1 - \frac{p_2^2}{3p_3}\right)\partial\theta\right] \tag{49}$$

The condition $\beta_1(\theta) = \beta_2(\theta) = 0$ and $\beta_3(\theta) = \Phi(\theta)$ reduces Eq. (44) to the well-known canonical form of Abel equation of first kind

$$u'_\xi = u^3(\xi) + \Phi(\xi), \xi = \int E^2 p_3 \partial\theta \tag{50}$$

Expressing the canonical form as

$$u'_\xi - u^3(\xi) - \lambda u^3(\xi) = -\lambda u^3(\xi) + \Phi(\xi) \tag{51}$$

and imposing the right-hand side to be zero we have the system of equations

$$\begin{aligned} & u'_\xi - u^3(\xi) - \lambda u^3(\xi) = 0 \\ & -\lambda u^3(\xi) + \Phi(\xi) = 0 \end{aligned} \tag{52}$$

Eq. (52) has solution

$$u(\xi) = \frac{1}{\sqrt{C - 2\int(1+\lambda)\,d\xi}}, \Phi(\xi) = \frac{\lambda}{\left[C - 2\int(1+\lambda)\,d\xi\right]^{2/3}} \tag{53}$$

Therefore, the general solution $\delta\theta = \delta\theta_p + u(\theta)E(\theta)$ can be written as

$$[\delta\theta]_{\text{Abel-Model1}} = \frac{E(\theta)}{\sqrt{C - 2\int(1+\lambda)\,d\xi}} - \frac{p_2}{3p_3} \tag{54}$$

and

$$\begin{aligned} \Phi(\xi) &= \tfrac{\lambda}{\left[C - 2\int(1+\lambda)d\xi\right]^{2/3}} \\ &= \tfrac{1}{E^3p_3}\left\{p_0 - \tfrac{p_1p_2}{3p_3} + \tfrac{2p_2^3}{3p_3^2} + \tfrac{1}{3}\tfrac{d}{d\theta}\left(\tfrac{p_2}{p_3}\right)\right\} \end{aligned} \tag{55}$$

Two cases of analytic solutions considered are shown below:

(a) Case 1: $\lambda(\xi) = 0$

Taking $\lambda(\xi) = 0$ reduces Eq. (54) to

$$[\delta\theta]_{\text{Abel-Model1}} = \frac{E(\theta)}{\sqrt{C - 2\int p_3E^2(\theta)d\theta}} - \frac{p_2}{3p_3} \tag{56}$$

(b) Case 2: $\lambda(\xi) = \xi$

Taking $\lambda(\xi) = \xi$ reduces Eq. (54) to

$$\begin{aligned} & [\delta\theta]_{\text{Abel-Model1}} \\ & = \tfrac{E(\theta)}{\sqrt{C - 2\int p_3E^2(\theta)d\theta - \left[\int p_3E^2(\theta)d\theta\right]^2}} - \tfrac{p_2}{3p_3} \end{aligned} \tag{57}$$

Using integrating factor method, the linear equation in Eq. (40) has the solution

$$[\delta\theta]_{\text{Model1(linear)}} = e^G\left(c_1 + \int e^{-G}p_{10}(\theta)d\theta\right) \tag{58}$$

where $G = \int p_{11}(\theta)d\theta$. Expanding the exponential functions in a series yields

$$\begin{aligned} & [\delta\theta(\theta)]_{\text{Model1(linear)}} = \left\{\begin{matrix} 1 + \left(\int p_{11}(\theta)d\theta\right) \\ +\frac{1}{2}\left(\int p_{11}(\theta)d\theta\right)^2 \\ +\frac{1}{6}\left(\int p_{11}(\theta)d\theta\right)^3 + \cdots \end{matrix}\right\} \\ & \left\{c_1 + \int\left(\begin{matrix} 1 - \left(\int p_{11}(\theta)d\theta\right) \\ +\frac{1}{2}\left(\int p_{11}(\theta)d\theta\right)^2 \\ -\frac{1}{6}\left(\int p_{11}(\theta)d\theta\right)^3 + \cdots \end{matrix}\right)p_{10}(\theta)d\theta\right\} \end{aligned} \tag{59}$$

For the second Abel-Type equation model (35), using the same approach as above, substituting the transformation $\delta\theta = \delta\theta_p + z(\theta)V(\theta)$ we have the form

$$\begin{aligned} z' = V^2 k_3 \Big[& z^3 + \tfrac{1}{Vk_3}\left(3k_3\delta\theta_p + k_2\right)z^2 \\ & + \tfrac{1}{V^2k_3}\left\{3k_3\left(\delta\theta_p\right)^2 + 2k_2\delta\theta_p + k_1 - \tfrac{V'}{V}\right\}z \\ & + \tfrac{1}{V^3k_3}\left\{k_0 - \delta\theta'_p + k_3\left(\delta\theta_p\right)^3 + k_2\left(\delta\theta_p\right)^2 + k_1\left(\delta\theta_p\right)\right\}\Big] \end{aligned} \tag{60}$$

with the canonical form

$$z'_\zeta = z^3(\zeta) + \Psi(\zeta) \tag{61}$$

where,

$$\zeta = \int V^2k_3d\theta, V(\theta) = \exp\left[\int\left(k_1 - \frac{k_2^2}{3k_3}\right)d\theta\right] \tag{62}$$

The canonical form can be written as

$$z'_\zeta - z^3(\zeta) - \eta z^3(\zeta) = -\eta z^3(\zeta) + \Psi(\zeta) \tag{63}$$

and imposing the right-hand side to be zero we have the system of equations

$$z'_\zeta - z^3(\zeta) - \eta z^3(\zeta) = 0, -\eta z^3(\zeta) + \Psi(\zeta) = 0 \tag{64}$$

These equations have the solutions

$$z(\zeta) = \frac{1}{\sqrt{C - 2\int(1+\eta)\,d\zeta}}, \Psi(\zeta) = \frac{\eta}{\left[C - 2\int(1+\eta)\,d\zeta\right]^{2/3}} \tag{65}$$

Table 1
Chief and deputy spacecraft orbital elements for Cases A, B, C and D.

	Case A		Case B		Case C		Case D	
	Chief	Deputy	Chief	Deputy	Chief	Deputy	Chief	Deputy
a (km)	7500	7500	7500	7510	9500	9500	10 500	10 500
e	0	0.1	0	0.12	0.01	0.0309556	0.010105	0.10215
i (deg)	30	30.97	0	1.97	45	45.155	15	15.055
Ω (deg)	45	45.5097	75	75.98	30	30.5	10.002	10.005
ω (deg)	60	60.5	30	30.5	275	275.55	5.05	5.105
f (deg)	0	0	0	0	320	320.5	0	0.005

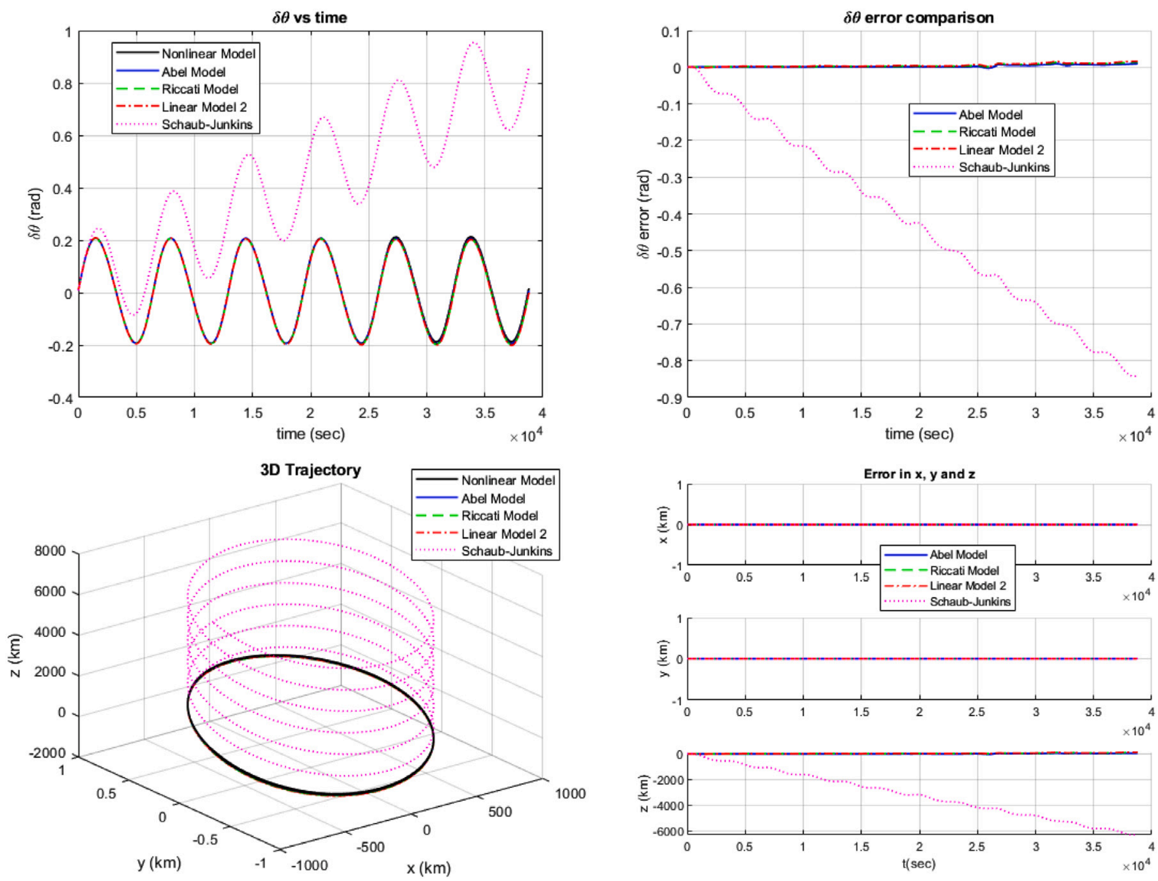

Fig. 1. Case A trajectories.

Therefore, the general solution is

$$[\delta\theta]_{\text{Abel-Model2}} = \frac{V(\theta)}{\sqrt{C - 2\int (1+\eta)\,\partial\zeta}} - \frac{k_2}{3k_3} \tag{66}$$

and

$$\Psi(\zeta) = \frac{\eta}{\left[C - 2\int (1+\eta)\partial\zeta\right]^{2/3}} = \frac{1}{V^3 k_3}\left\{k_0 - \frac{k_1 k_2}{3k_3} + \frac{2k_2^3}{3k_3^2} + \frac{1}{3}\left(\frac{k_2}{k_3}\right)'\right\} \tag{67}$$

Two cases of analytic solutions considered are shown below:

(a) Case 1: $\eta(\zeta) = 0$

Taking $\eta(\zeta) = 0$ reduces Eq. (66) to

$$[\delta\theta]_{\text{Abel-Model2}} = \frac{V(\theta)}{\sqrt{C - 2\int k_3 V^2(\theta)d\theta}} - \frac{k_2}{3k_3} \tag{68}$$

(b) Case 2: $\eta(\zeta) = \zeta$

Taking $\eta(\zeta) = \zeta$ reduces Eq. (66) to

$$[\delta\theta]_{\text{Abel-Model2}} = \frac{V(\theta)}{\sqrt{C - 2\int k_3 V^2(\theta)d\theta - \left[\int k_3 V^2(\theta)d\theta\right]^2}} - \frac{k_2}{3k_3} \tag{69}$$

Similarly, using integrating factor method, the linear equation has the solution

$$[\delta\theta]_{\text{Model2(linear)}} = e^E\left(c_2 + \int e^{-E}k_0(\theta)d\theta\right) \quad E = \int k_1(\theta)d\theta \tag{70}$$

where, c_2 is an arbitrary constant of integration. Expanding the exponential functions in a series gives

$$[\delta\theta]_{\text{Model2(linear)}} = \left\{\begin{matrix} 1 + \left(\int k_1(\theta)d\theta\right) + \frac{1}{2}\left(\int k_1(\theta)d\theta\right)^2 \\ +\frac{1}{6}\left(\int k_1(\theta)d\theta\right)^3 + \cdots \end{matrix}\right\} \left\{c_2 + \int \begin{pmatrix} 1 - \left(\int k_1(\theta)d\theta\right) \\ +\frac{1}{2}\left(\int k_1(\theta)d\theta\right)^2 \\ -\frac{1}{6}\left(\int k_1(\theta)d\theta\right)^3 + \cdots \end{pmatrix} k_0(\theta)d\theta\right\} \tag{71}$$

5. Closed form solution of Riccati-type nonlinear spacecraft relative equation of motion

The approach in Polyanin and Zaitsev [21], and Haaheim and Stein [40] is followed for the formulation of the general solution of the

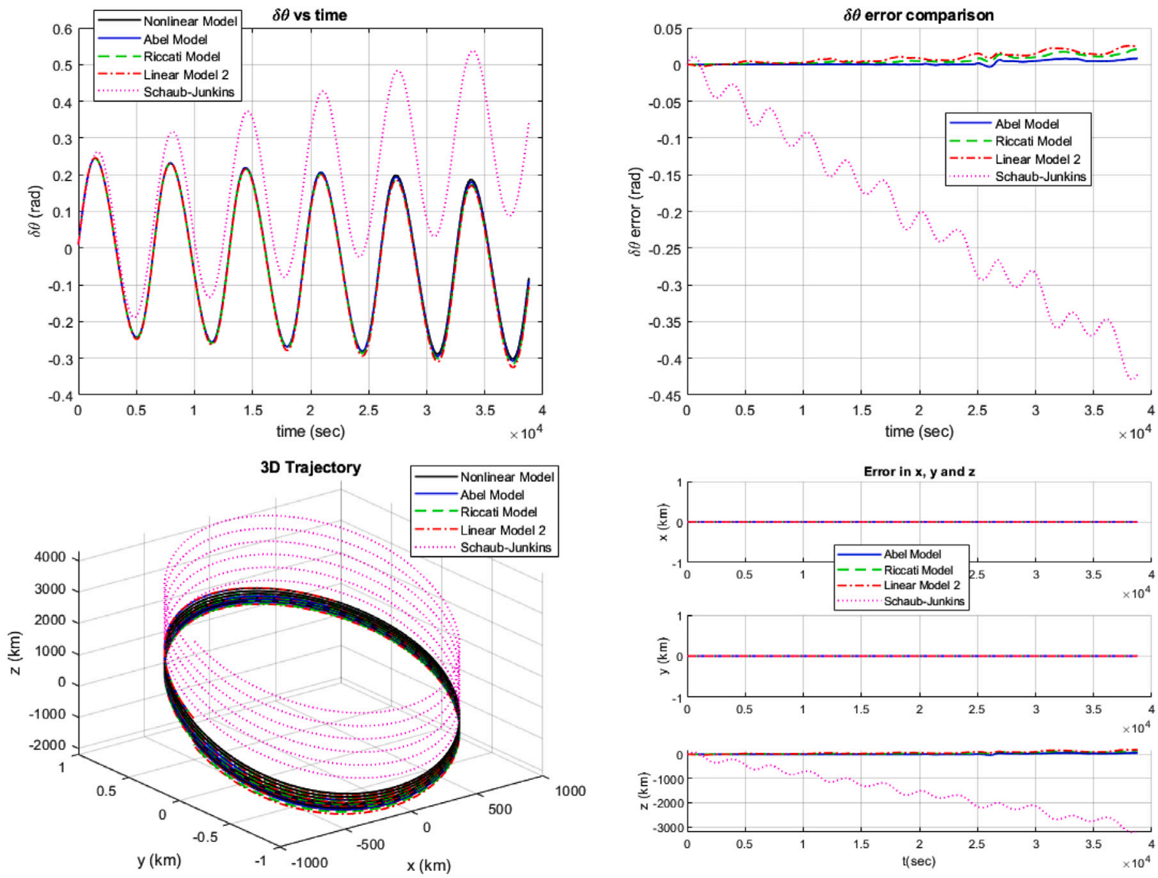

Fig. 2. Case B trajectories.

Riccati Equation. It is a well-known fact that once a particular solution $\delta\theta_p = \delta\theta_p(\theta)$ of the Riccati equation (36) is known then the general solution of the equation can be written as

$$\delta\theta = \delta\theta_p(\theta) + \frac{1}{z(\theta)} \tag{72}$$

Upon substitution of Eq. (72) into Eq. (36) we have linear differential equation

$$\frac{dz}{d\theta} + \left\{s_1(\theta) + 2s_2(\theta)\delta\theta_p\right\} z + s_2(\theta) = 0 \tag{73}$$

with the solution

$$z(\theta) = z_0 e^{-\Psi(\theta)} - e^{-\Psi(\theta)} \int_{\delta\theta_0}^{\delta\theta} s_2(\theta) e^{\Psi(\theta)} d\theta \tag{74}$$

where

$$\Psi(\theta) = \int_{\delta\theta_0}^{\delta\theta} \left[s_1(\theta) + 2s_2(\theta)\delta\theta_p\right] d\theta, z_0 = \frac{1}{\delta\theta_0 - \delta\theta_{s_0}} \tag{75}$$

Therefore, the general solution can be expressed as

$$\begin{aligned} &\delta\theta\left(s_2(\theta), s_1(\theta), \delta\theta_p\right)_{\text{Riccati−Model1}} \\ &= \delta\theta_p + e^{\Psi(\theta)}\left[\frac{1}{\delta\theta_0 - \delta\theta_{s_0}} - \int_{\delta\theta_0}^{\delta\theta} e^{\Psi(\theta)} s_2(\theta) d\theta\right]^{-1} \end{aligned} \tag{76}$$

Expanding the general solutions in a series gives

$$\begin{aligned} &\delta\theta\left(s_2(\theta), s_1(\theta), \delta\theta_p\right)_{\text{Riccati−Model1}} = \delta\theta_p \\ &+ e^{\Psi(\theta)}\left(\delta\theta_0 - \delta\theta_{p_0}\right)\left[1 + \left(\delta\theta_0 - \delta\theta_{p_0}\right)\int_{\delta\theta_0}^{\delta\theta} e^{\Psi(\theta)} s_2(\theta)\partial\theta \right. \\ &+ \left(\delta\theta_0 - \delta\theta_{p_0}\right)^2\left(\int_{\delta\theta_0}^{\delta\theta} e^{\Psi(\theta)} s_2(\theta)\partial\theta\right)^2 \\ &\left. + \left(\delta\theta_0 - \delta\theta_{p_0}\right)^3\left(\int_{\delta\theta_0}^{\delta\theta} e^{\Psi(\theta)} s_2(\theta)\partial\theta\right)^3 + \cdots\right] \end{aligned} \tag{77}$$

where

$$e^{\Psi(\theta)} = 1 + \Psi(\theta) + \frac{\Psi(\theta)^2}{2} + \frac{\Psi(\theta)^3}{6} + \cdots \tag{78}$$

The general solution of the model 2 of Riccati equation in Eq. (38) is found in a similar manner to the approach in Model 1. Let the general solution be represented as

$$\delta\theta = \delta\theta_p(\theta) + \frac{1}{w(\theta)} \tag{79}$$

Substituting Eq. (79) into Eq. (38) gives differential equation

$$\frac{dw}{d\theta} + \left\{k_1(\theta) + 2k_2(\theta)\delta\theta_p\right\} w + k_2(\theta) = 0 \tag{80}$$

Eq. (80) has the solution

$$w(\theta) = w_0 e^{-\beta(\theta)} - e^{-\beta(\theta)} \int_{\delta\theta_0}^{\delta\theta} k_2(\theta) e^{\beta(\theta)} d\theta \tag{81}$$

where

$$\beta(\theta) = \int_{\delta\theta_0}^{\delta\theta} \left[k_1(\theta) + 2k_2(\theta)\delta\theta_p\right] d\theta, w_0 = \frac{1}{\delta\theta_0 - \delta\theta_{p_0}} \tag{82}$$

Therefore, the general solution can be expressed as

$$\begin{aligned} &\delta\theta\left(k_2(\theta), k_1(\theta), \delta\theta_p\right)_{\text{Model2(Riccati)}} \\ &= \delta\theta_p + e^{\beta(\theta)}\left[\frac{1}{\delta\theta_0 - \delta\theta_{p_0}} - \int_{\delta\theta_0}^{\delta\theta} e^{\beta(\theta)} k_2(\theta) d\theta\right]^{-1} \end{aligned} \tag{83}$$

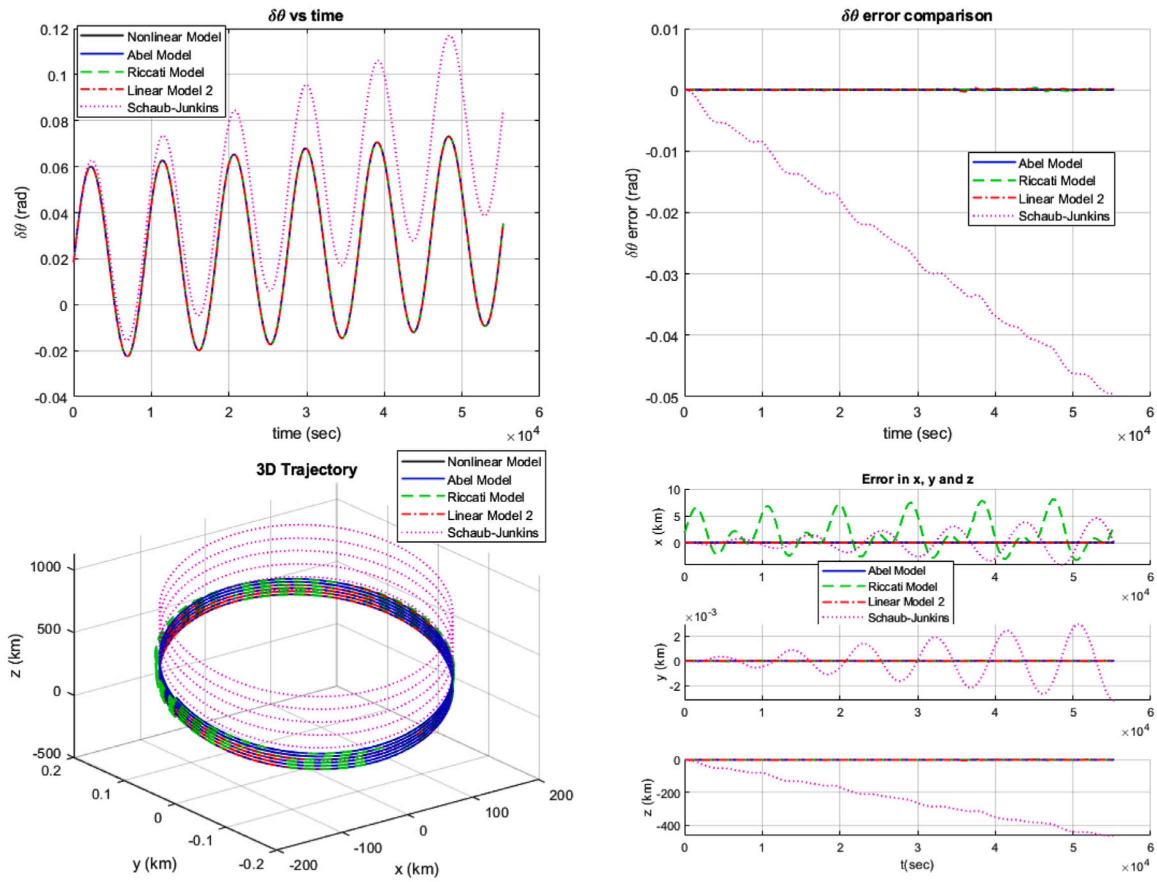

Fig. 3. Case C trajectories.

In series form, Eq. (83) becomes

$$\begin{aligned}\delta\theta\left(k_2(\theta),k_1(\theta),\delta\theta_p\right)_{\text{Model2(Riccati)}} &= \delta\theta_p \\ &+e^{\beta(\theta)}\left(\delta\theta_0-\delta\theta_{p_0}\right)\Big[1+\left(\delta\theta_0-\delta\theta_{p_0}\right)\int_{\delta\theta_0}^{\delta\theta}e^{\beta(\theta)}k_2(\theta)d\theta \\ &+\left(\delta\theta_0-\delta\theta_{p_0}\right)^2\left(\int_{\delta\theta_0}^{\delta\theta}e^{\beta(\theta)}k_2(\theta)d\theta\right)^2 \\ &+\left(\delta\theta_0-\delta\theta_{p_0}\right)^3\left(\int_{\delta\theta_0}^{\delta\theta}e^{\beta(\theta)}k_2(\theta)d\theta\right)^3+\cdots\Big]\end{aligned} \tag{84}$$

where,

$$e^{\beta(\theta)} = 1+\beta(\theta)+\frac{\beta(\theta)^2}{2}+\frac{\beta(\theta)^3}{6}+\cdots \tag{85}$$

6. Numerical simulations

The nonlinear, Abel-type, Riccati-type and linear models of spacecraft relative motion are compared numerically using four different cases A, B, C and D shown in Table 1. The orbital elements semi-major axis, a, eccentricity, e, inclination, i, right ascension of ascending node, Ω, argument of perigee, ω and true anomaly, f of the Chief and Deputy spacecraft are shown in the table. For the (x,y,z) plots the linear mapping between Hill frame coordinates and orbit element differences given by Schaub and Junkins [2]

$$\mathbf{X} = [\mathbf{A}\,(\mathbf{e})]\,\delta\mathbf{e} \tag{86}$$

was used.

Cases A and B show special scenario where chief spacecraft is positioned in the circular orbit with zero eccentricity. Cases C and D show a scenario where chief is positioned in elliptical orbit. In the plots, the linear model 1 which corresponds to the Schaub and Junkins [2] derivation is referred to as Schaub–Junkins model. As shown in all the Cases, the Abel-type model gave the best representation of the relative motion based on the fact that it has more terms than both the Riccati and linear model. The Schaub–Junkins model has the largest error than all the other models. This can be attributed to the presence of more constant terms which led to the drifting of the motion (see Figs. 1–4).

7. Conclusion

This paper has shown two approximated models of Abel-type (third-order polynomial) and Riccati-type (second-order polynomial) spacecraft equations of relative motion. Also, using standard transformation techniques, we developed closed form solutions of the models using well-known approaches of finding the solution of Abel and Riccati equations. Using both Models the relative motion of the deputy spacecraft with respect to the chief spacecraft are described for both elliptical and circular chief orbits. The models, which have only true-of latitude as time-varying, captured the dynamics better than using position and velocity in which all the six parameters vary with time. These Models can be used for spacecraft control, analysis and maneuver planning.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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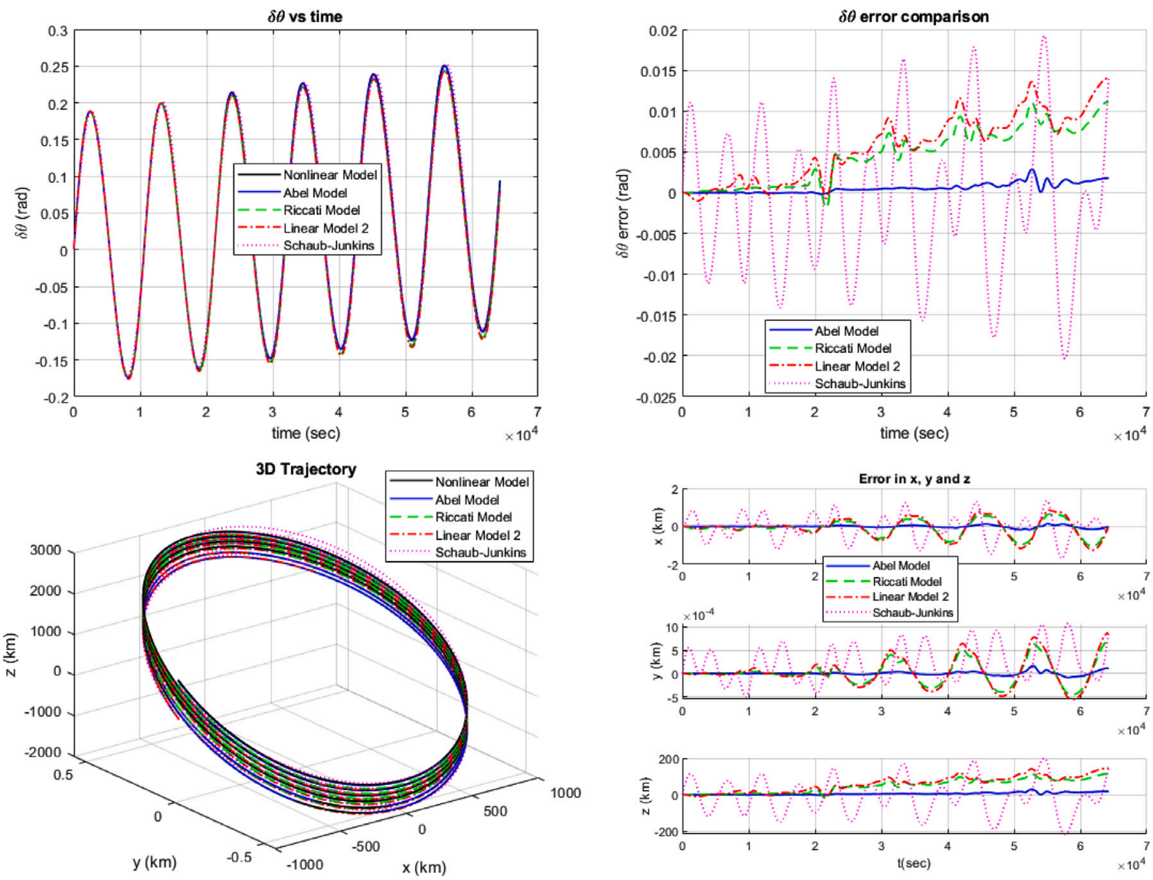

Fig. 4. Case D trajectories.

Appendix A. Coefficients of first model of Abel-type equation

The coefficients of first model of Abel-type equation are

$$\begin{aligned}
p_0(\theta) = & -\tfrac{3n}{2a}\delta a\delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} - \tfrac{3n}{2a}\delta a\delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} \\
& + \tfrac{n}{2}\delta q_1\delta q_2 f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} + \tfrac{n}{2}\delta q_2\delta q_1 f_{\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
& + \tfrac{15n}{8a^2}(\delta a)^2 f\Big|_{(0,0,0)} + \tfrac{n}{2}\left(\delta q_1\right)^2 f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
& + \tfrac{n}{2}\left(\delta q_2\right)^2 f_{\delta q_2\delta q_2}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\left(\delta q_1\right)^2 f_{\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
& - \tfrac{3n}{4a}\delta a\delta q_1\delta q_2 f_{\delta q_1\delta q_2}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\delta q_2\delta q_1 f_{\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
& - \tfrac{3n}{4a}\delta a\left(\delta q_2\right)^2 f_{\delta q_2\delta q_2}\Big|_{(0,0,0)} + \tfrac{15n}{8a^2}(\delta a)^2\delta q_1 f_{\delta q_1}\Big|_{(0,0,0)} \\
& + \tfrac{15n}{8a^2}(\delta a)^2\delta q_2 f_{\delta q_2}\Big|_{(0,0,0)} - \tfrac{35n}{16a^3}(\delta a)^3 f\Big|_{(0,0,0)} \\
& + \tfrac{n}{6}\Big\{\left(\delta q_1\right)^3 f_{\delta q_1\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
& + \left(\delta q_1\right)^2\left(\delta q_2\right) f_{\delta q_1\delta q_1\delta q_2}\Big|_{(0,0,0)} \\
& + \left(\delta q_1\right)^2\left(\delta q_2\right) f_{\delta q_1\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
& + \left(\delta q_1\right)\left(\delta q_2\right)^2 f_{\delta q_1\delta q_2\delta q_2}\Big|_{(0,0,0)} \\
& + \left(\delta q_2\right)\left(\delta q_1\right)^2 f_{\delta q_2\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
& + \left(\delta q_2\right)^2\left(\delta q_1\right) f_{\delta q_2\delta q_1\delta q_2}\Big|_{(0,0,0)} \\
& + \left(\delta q_2\right)^2\left(\delta q_1\right) f_{\delta q_2\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
& + \left(\delta q_2\right)^3 f_{\delta q_2\delta q_2\delta q_2}\Big|_{(0,0,0)}\Big\} \\
& - \tfrac{3n}{2a} f\Big|_{(0,0,0)}\delta a + n f_{\delta q_1}\Big|_{(0,0,0)}\delta q_1 + n f_{\delta q_2}\Big|_{(0,0,0)}\delta q_2
\end{aligned} \tag{87}$$

$$\begin{aligned}
p_1(\theta) = & \; n f_{\delta\theta}\Big|_{(0,0,0)} - \tfrac{3n}{2a}\delta a f_{\delta\theta}\Big|_{(0,0,0)} + \tfrac{n}{2}\delta q_1 f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} \\
& + \tfrac{n}{2}\delta q_1 f_{\delta\theta\delta q_1}\Big|_{(0,0,0)} + \tfrac{n}{2}\delta q_2 f_{\delta q_2\delta\theta}\Big|_{(0,0,0)} \\
& + \tfrac{n}{2}\delta q_2 f_{\delta\theta\delta q_2}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\delta q_1 f_{\delta q_1\delta\theta}\Big|_{(0,0,0)} \\
& - \tfrac{3n}{4a}\delta a\delta q_2 f_{\delta q_2\delta\theta}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a\delta q_1 f_{\delta\theta\delta q_1}\Big|_{(0,0,0)} \\
& - \tfrac{3n}{4a}\delta a\delta q_2 f_{\delta\theta\delta q_2}\Big|_{(0,0,0)} + \tfrac{15n}{8a^2}(\delta a)^2 f_{\delta\theta}\Big|_{(0,0,0)} \\
& + \left(\delta q_1\right)^2 f_{\delta q_1\delta q_1\delta\theta}\Big|_{(0,0,0)} + \left(\delta q_1\right)\left(\delta q_2\right) f_{\delta q_1\delta q_2\delta\theta}\Big|_{(0,0,0)} \\
& + \left(\delta q_1\right)^2 f_{\delta q_1\delta\theta\delta q_1}\Big|_{(0,0,0)} + \left(\delta q_1\right)\left(\delta q_2\right) f_{\delta q_1\delta\theta\delta q_2}\Big|_{(0,0,0)} \\
& + \left(\delta q_2\right)^2 f_{\delta q_2\delta q_2\delta\theta}\Big|_{(0,0,0)} + \left(\delta q_2\right)\left(\delta q_1\right) f_{\delta q_2\delta\theta\delta q_1}\Big|_{(0,0,0)} \\
& + \left(\delta q_2\right)^2 f_{\delta q_2\delta\theta\delta q_2}\Big|_{(0,0,0)} + \left(\delta q_1\right)^2 f_{\delta\theta\delta q_1\delta q_1}\Big|_{(0,0,0)} \\
& + \left(\delta q_1\right)\left(\delta q_2\right) f_{\delta\theta\delta q_1\delta q_2}\Big|_{(0,0,0)} + \left(\delta q_2\right)\left(\delta q_1\right) f_{\delta\theta\delta q_2\delta q_1}\Big|_{(0,0,0)} \\
& + \left(\delta q_2\right)^2 f_{\delta\theta\delta q_2\delta q_2}\Big|_{(0,0,0)} + \left(\delta q_2\right)\left(\delta q_1\right) f_{\delta q_2\delta q_1\delta\theta}\Big|_{(0,0,0)}
\end{aligned} \tag{88}$$

$$\begin{aligned}
p_2(\theta) = & \; \tfrac{n}{2} f_{\delta\theta\delta\theta}\Big|_{(0,0,0)} - \tfrac{3n}{4a}\delta a f_{\delta\theta\delta\theta}\Big|_{(0,0,0)} \\
& + \tfrac{n}{6}\Big\{\left(\delta q_1\right) f_{\delta q_1\delta\theta\delta\theta}\Big|_{(0,0,0)} + \left(\delta q_2\right) f_{\delta q_2\delta\theta\delta\theta}\Big|_{(0,0,0)} \\
& + \left(\delta q_2\right) f_{\delta\theta\delta q_2\delta\theta}\Big|_{(0,0,0)} + \left(\delta q_1\right) f_{\delta\theta\delta q_1\delta\theta}\Big|_{(0,0,0)} \\
& + \left(\delta q_2\right) f_{\delta\theta\delta\theta\delta q_2}\Big|_{(0,0,0)} + \left(\delta q_1\right) f_{\delta\theta\delta\theta\delta q_1}\Big|_{(0,0,0)}\Big\}
\end{aligned} \tag{89}$$

$$p_3(\theta) = f_{\delta\theta\delta\theta\delta\theta}\Big|_{(0,0,0)}$$

Appendix B. Coefficients of second model of Abel-type equation

$$k_0(\theta) = \begin{bmatrix} M_D\left(1+\frac{1}{2}\left(q_1+\delta q_1\right)^2+\frac{1}{2}\left(q_2+\delta q_2\right)^2\right) \\ -M_C\left(1+\frac{1}{2}q_1^2+\frac{1}{2}q_2^2\right) \\ +\left\{\begin{matrix} \frac{1}{2}M_D\left(\left(q_1+\delta q_1\right)^2-\left(q_2+\delta q_2\right)^2\right) \\ -M_C\left(\frac{1}{2}q_1^2-\frac{1}{2}q_2^2\right)\end{matrix}\right\}\cos 2\theta \\ +\left\{M_D\left(q_1+\delta q_1\right)\left(q_2+\delta q_2\right)-M_C q_1 q_2\right\}\sin 2\theta \\ +\left\{2M_D\left(q_1+\delta q_1\right)-2M_C q_1\right\}\cos\theta \\ +\left\{2M_D\left(q_2+\delta q_2\right)-2M_C q_2\right\}\sin\theta \end{bmatrix} \tag{90}$$

and

$$k_1(\theta) = M_D\left\{\begin{matrix} 2\left(q_1+\delta q_1\right)\left(q_2+\delta q_2\right)\cos 2\theta \\ +\left(-\left(q_1+\delta q_1\right)^2+\left(q_2+\delta q_2\right)^2\right)\sin 2\theta \\ +2\left(q_2+\delta q_2\right)\cos\theta-2\left(q_1+\delta q_1\right)\sin\theta \end{matrix}\right\}$$

$$k_2(\theta) = M_D\left\{\begin{matrix} \left(-\left(q_1+\delta q_1\right)^2+\left(q_2+\delta q_2\right)^2\right)\cos 2\theta \\ -2\left(q_1+\delta q_1\right)\left(q_2+\delta q_2\right)\sin 2\theta \\ -\left(q_1+\delta q_1\right)\cos\theta-\left(q_2+\delta q_2\right)\sin\theta \end{matrix}\right\} \tag{91}$$

$$k_3(\theta) = \frac{1}{3}M_D\left\{\begin{matrix} -4\left(q_1+\delta q_1\right)\left(q_2+\delta q_2\right)\cos 2\theta \\ +2\left(\left(q_1+\delta q_1\right)^2-\left(q_2+\delta q_2\right)^2\right)\sin 2\theta \\ -\left(q_2+\delta q_2\right)\cos\theta+\left(q_1+\delta q_1\right)\sin\theta \end{matrix}\right\}$$

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