ARTICLE TYPE

Epidemiological models with quadratic equation for endemic equilibria — a bifurcation atlas

Rachid Ouifki*[1] | Jacek Banasiak[1,2]

[1]Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria, South Africa
[2]Institute of Mathematics, Łódź University of Technology, Łódź, Poland

Correspondence
*Rachid Ouifki. Email: Rachid.Ouifki@up.ac.za

Summary

The existence and occurrence, especially by a backward bifurcation, of endemic equilibria is of utmost importance in determining the spread and persistence of a disease. In many epidemiological models, the equation for the endemic equilibria is quadratic, with the coefficients determined by the parameters of the model. Despite its apparent simplicity, such an equation can describe an amazing number of dynamical behaviours. In this paper, we shall provide a comprehensive survey of possible bifurcation patterns, deriving explicit conditions on the equation's parameters for the occurrence of each of them, and discuss illustrative examples.

KEYWORDS:
Epidemiological models, endemic equilibria, basic reproductive number, backward bifurcation

1 | INTRODUCTION

One of the most important parameters in the analysis of the long term behaviour of compartmental models in epidemiology is the basic reproduction number, typically denoted by R_0, that is defined as the number of infections caused in a completely susceptible population by one infective during the whole period of its infectiveness. It seems obvious that if $R_0 < 1$, then the disease should die out, while it should persist and expand if $R_0 > 1$. Typically this is the case locally; that is, for small perturbations of the disease free equilibrium (DFE)[1]. In other words, the DFE is locally stable if $R_0 < 1$ and looses its stability, when R_0 moves through $R_0 = 1$ to the region where $R_0 > 1$, where typically a new stable equilibrium, called the endemic equilibrium (EE), appears. This phenomenon is typically called the supercritical, transcritical, or forward, bifurcation[2]. In many cases, however, the system undergoes another type of bifurcation, called the subcritical, or backward, bifurcation, where a stable endemic equilibrium exists for $R_0 < 1$. It has an important implication for the control of the disease, as in such a case the classical requirement, that for eradication of the disease it is enough to bring R_0 below unity, no longer suffices (even though it is still necessary). The occurrence of such a bifurcation has been studied in a number of disease transmission models by many authors. For instance, in malaria models we can mention papers[3,4,5,6,7,8], for TB papers[9,10,11,12,13], for the bovine TB[14], for HIV[15,16], for the dengue fever[17], and for Chlamydia[18]. A common reason for the occurrence of the backward bifurcation is using an imperfect vaccination[12,19,20]. Certainly, backward bifurcations appear in other mathematical models[2]; some particular cases were studied in[21,22]. We also refer to papers[23,22] for a discussion of the conditions that make some models exhibit the backward bifurcation behaviour.

Due to the reasons mentioned above it is important to develop tools that allow for a quick detection of backward bifurcations in particular models. One of the most effective techniques, based on the centre manifold theorem, was derived in[24]. It provides explicit conditions that ensure the existence of a local branch of "small" equilibrium points emerging from the disease free equilibrium to the left of $R_0 = 1$. However, for many models, numerical evidence suggests that for a certain range of $R_0 < 1$ this subcritical branch often coexists with another branch of "larger" stable endemic equilibrium points. For an effective disease control, it is essential to know how many other (stable) branches of equilibria exist alongside the bifurcating one, and the ranges of parameters for which these branches coexist. It is thus important to emphasize here that the method of[24] provides only a local result; that is, the existence of a bifurcating branch in a small neighborhood of the disease free equilibrium for values of R_0 close to one. Hence, it is unable to determine the number of branches of endemic equilibria or inform on the ranges of parameters for which they exist. Nonetheless, its relevance lies in that it provides the set of parameters for which the (local) backward bifurcation occurs and thus sets the foundation on which further analytical and numerical investigations of the global picture of endemic equilibria can be carried out.

There are some models for which the equation for the endemic equilibria is algebraic and thus the existence of a backward bifurcation can be explicitly established by a direct analysis of the set of its roots that occur for $R_0 < 1$. If feasible, this method can provide the exact number of endemic equilibria along with their maximal ranges of the parameters ensuring their existence. However, due to the number of parameters in the original model and their interplay, the calculations can be tedious, if not impossible, even in the simplest cases. In this paper we shall focus on the case, when the endemic equilibria are positive solutions of the quadratic equation

$$\lambda^2 + B\lambda + C = 0, \tag{1}$$

where the variable λ represents the value of the endemic equilibrium, or the model's force of infection at such an equilibrium that is directly related to its value. The parameters B and C are combinations of the parameters of the model. While one could think that the analysis of a roots of a quadratic equation hardly is a topic of a research paper and, indeed, the mathematical complexity of the analysis does not go beyond an undergraduate algebra, the aim of this work is to show that even such a simple case can generate a plethora of different, often unexpected, epidemiological scenarios and to provide a comprehensive survey of them.

We begin by noting that (1) has appeared in a number of epidemiological models, see e.g.[25,26,27], that will be discussed later, and various approaches have been proposed to analyze the dependence of its solutions on the parameters B and C instead of R_0. In this paper, we provide explicit conditions that ensure that (1) has none, one or two positive roots. To emphasize the importance of R_0 in the analysis, we express B and C in terms of $\mathcal{R}_0$ (where $\mathcal{R}_0$ equals R_0 or R_0^2, as could be the case for some vector-borne disease models). As we shall see in the examples, it is convenient to rewrite B and C as $B = b\left(K - \mathcal{R}_0\right)$ and $C = c(1 - \mathcal{R}_0)$, to get

$$\lambda^2 + b(K - \mathcal{R}_0)\lambda + c\left(1 - \mathcal{R}_0\right) = 0. \tag{2}$$

For a wide variety of epidemiological models, the calculation of equilibrium points entails solving equation (2) where $b > 0$ and $c > 0$. However, for some models, which we will present later, these conditions are not always satisfied, hence the need for a systematic approach to deal with all possible cases. We note here that the conditions $c > 0$ ensures that the model has a unique endemic equilibrium point for $R_0 > 1$.

As mentioned above, the aim of this paper is to provide explicit conditions on b, c and $\mathcal{R}_0$ that ensure the existence of zero, one or two positive solutions for equation (2). We will further discuss the biological/epidemiological implications of such occurrences and present two models that present a "nonstandard" backward bifurcation behaviour:

(i) a model for the interactions between HIV and the immune system,

(ii) a malaria model with counterfeit antimalarial drugs.

The mathematics of the paper, though tedious and not always obvious, is rather elementary. We believe, however, that it is useful to theoretically explore all possible bifurcation scenarios allowed by (2), derive explicit conditions that lead to them and comprehensively and graphically summarize the findings in a way that can be directly applied to a wide variety of epidemiological models.

2 | MATHEMATICAL ANALYSIS

For convenience, we use the following notations:

1. $\Delta(\mathcal{R}_0) = b^2(K - \mathcal{R}_0)^2 - 4c(1 - \mathcal{R}_0)$. This is the discriminant of (2).
2. $\mathcal{R}_b = \frac{Kb^2 - 2c - 2\sqrt{b^2c(1-K)+c^2}}{2b^2}$.
3. $\mathcal{R}_c = \frac{Kb^2 - 2c + 2\sqrt{b^2c(1-K)+c^2}}{2b^2}$. When they exist, $\mathcal{R}_b$ and $\mathcal{R}_c$ are the real roots of $\Delta(\mathcal{R}_0) = 0$.

2.1 | Case 1: $c > 0$

We can easily see that when $\mathcal{R}_0 > 1$, equation (2) has one positive root. We focus the remainder of this section on the case $\mathcal{R}_0 \leq 1$. We will proceed by discussing the cases $b > 0$ and $b < 0$. We observe that $\Delta(0) = b^2K^2 - 4c < 0$ if and only if $-\sqrt{\frac{4c}{b^2}} < K < \sqrt{\frac{4c}{b^2}}$.

2.1.1 | Sub-case 1: $b > 0$

Proposition 1. 1. If $K \geq 1$, then equation (2) exhibits a forward bifurcation; that is, equation (2) has:

   (a) no positive roots if $\mathcal{R}_0 \in [0, 1]$,
   (b) a unique positive root if $\mathcal{R}_0 \in (1, +\infty)$.

2. If $-\sqrt{\frac{4c}{b^2}} < K < 1$, then $0 < \mathcal{R}_c < 1$ and equation (2) exhibits backward bifurcation; that is equation (2) has:

   (a) no positive roots if $\mathcal{R}_0 \in [0, \mathcal{R}_c)$,
   (b) one double positive root if $\mathcal{R}_0 = \mathcal{R}_c$,
   (c) two positive real roots if $\mathcal{R}_0 \in \left(\mathcal{R}_c, 1\right)$,
   (d) a unique positive root if $\mathcal{R}_0 \in [1, +\infty)$.

3. If $K = -\sqrt{\frac{4c}{b^2}}$, then $\mathcal{R}_c = 0$ and equation (2) exhibits an almost full backward bifurcation; that is equation (2) has:

   (a) If $\mathcal{R}_0 = 0$, then equation (2) has a double positive root,
   (b) If $\mathcal{R}_0 \in (0, 1)$, then equation (2) has two positive roots,
   (c) a unique positive root if $\mathcal{R}_0 \in [1, +\infty)$.

4. If $K < -\sqrt{\frac{4c}{b^2}}$, then $\mathcal{R}_c < 0$ and equation (2) exhibits a full backward bifurcation; that is equation (2) has:

   (a) two positive roots if $\mathcal{R}_0 \in [0, 1)$,
   (b) a unique positive root if $\mathcal{R}_0 \in [1, +\infty)$.

Proof. 1. If $K \geq 1$, then for all $\mathcal{R}_0 \leq 1$, we have $c\left(1 - \mathcal{R}_0\right) \geq 0$ and $b(K - \mathcal{R}_0) \geq 0$ implying that (2) has no positive roots.

2. If $-\sqrt{\frac{4c}{b^2}} < K < 1$, we discuss the following cases:

   (a) If $\sqrt{\frac{4c}{b^2}} \geq 1$, then $-\sqrt{\frac{4c}{b^2}} < K < \sqrt{\frac{4c}{b^2}}$ implying that $\Delta(0) = b^2K^2 - 4c < 0$. Since $\Delta(K) = -4c(1-K) < 0$, $\Delta(-\infty) > 0$ and $\Delta(1) > 0$, then $\max(0, K) < \mathcal{R}_c < 1$ and $\mathcal{R}_b < \min(0, K)$. This implies that $\Delta(\mathcal{R}_0) < 0$ on $[0, \mathcal{R}_c)$ and $\Delta(\mathcal{R}_0) > 0$ on $(\mathcal{R}_c, 1)$. This implies that equation (2) has no real roots on $[0, \mathcal{R}_c)$, a double root at $\mathcal{R}_0 = \mathcal{R}_c$ and two real roots on $(\mathcal{R}_c, 1)$. Since $b(K - \mathcal{R}_0) < 0$ and $c(1 - \mathcal{R}_0) > 0$ for $\mathcal{R}_0 \in (\mathcal{R}_c, 1)$, then equation (2) has no positive real roots on $[0, \mathcal{R}_c)$, a double positive root at $\mathcal{R}_0 = \mathcal{R}_c$ and two positive real roots on $(\mathcal{R}_c, 1)$.

   (b) If $\sqrt{\frac{4c}{b^2}} < 1$, then we have the following cases:

      i. If $-\sqrt{\frac{4c}{b^2}} < K < \sqrt{\frac{4c}{b^2}}$ then $\Delta(0) := b^2K^2 - 4c < 0$ which leads to the same results as above.

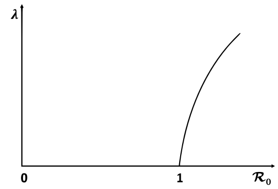
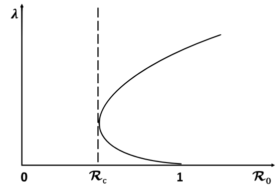
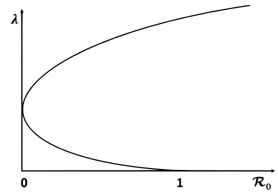
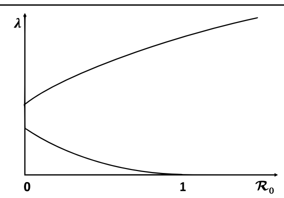
      ii. If $\sqrt{\frac{4c}{b^2}} < K < 1$ then $\Delta(0) = b^2K^2 - 4c > 0$. Since $\Delta(K) = -4c(1-K) < 0$ and $\Delta(1) > 0$, then $0 < \mathcal{R}_b < K < \mathcal{R}_c < 1$. In this case $\Delta(\mathcal{R}_0) < 0$ on $(\mathcal{R}_b, \mathcal{R}_c)$ and $\Delta(\mathcal{R}_0) > 0$ on $(0, \mathcal{R}_b) \cup (\mathcal{R}_c, 1)$. This implies that equation (2) has no real roots on $(\mathcal{R}_b, \mathcal{R}_c)$, a double root at $\mathcal{R}_0 = \mathcal{R}_b$ or $\mathcal{R}_c$ and two real roots on $(0, \mathcal{R}_b) \cup (\mathcal{R}_c, 1)$. Since for $\mathcal{R}_0 \in [0, 1)$ we have $c(1 - \mathcal{R}_0) > 0$, $b(K - \mathcal{R}_0) < 0$ for $\mathcal{R}_0 \in (\mathcal{R}_c, 1)$ and $b(K - \mathcal{R}_0) > 0$ for $\mathcal{R}_0 \in (0, \mathcal{R}_b)$, the two roots of equation (2) in $(0, \mathcal{R}_b)$ are negative while those in $(\mathcal{R}_c, 1)$ are positive.

3. If $K = -\sqrt{\frac{4c}{b^2}}$, then $\Delta(\mathcal{R}_0) = b^2\mathcal{R}_0^2 + 4\left(c - b\sqrt{c}\right)\mathcal{R}_0$. Then $\Delta(\mathcal{R}_0) = 0$ if $\mathcal{R}_0 = 0$ and $\Delta(\mathcal{R}_0) > 0$ if $\mathcal{R}_0 \in (0, 1)$. Thus, results on the roots of (2) follow in the same way as in the previous case.

4. If $K < -\sqrt{\frac{4c}{b^2}}$, then $K^2b^2 - 4c > 0$ implying that $\Delta(\mathcal{R}_0) = b^2\mathcal{R}_0^2 + \left(4c - 2Kb^2\right)\mathcal{R}_0 + \left(K^2b^2 - 4c\right) > 0$ for all $\mathcal{R}_0 \in [0, 1)$. We deduce the results on the roots of (2) in the same way as in the previous cases. We note that in this case, we have $\mathcal{R}_c < 0$.

□

TABLE 1 Number of positive roots for equation (2) when $b > 0$ and $c > 0$.

Assumptions	Number of positive roots	Bifurcation diagram
$K \geq 1$	$\begin{cases} 0, \text{ if } \mathcal{R}_0 \in [0, 1], \\ 1, \text{ if } \mathcal{R}_0 \in (1, +\infty). \end{cases}$	
$-\sqrt{\frac{4c}{b^2}} < K < 1$	$\begin{cases} 0, \text{ if } \mathcal{R}_0 \in [0, \mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_c, \\ 2, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_c, 1), \\ 1, \text{ if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$K = -\sqrt{\frac{4c}{b^2}}$	$\begin{cases} 1 \text{ double, if } \mathcal{R}_0 = 0, \\ 2, \text{ if } \mathcal{R}_0 \in (0, 1), \\ 1, \text{ if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$K < -\sqrt{\frac{4c}{b^2}}$	$\begin{cases} 2, \text{ if } \mathcal{R}_0 \in [0, 1), \\ 1, \text{ if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	

2.1.2 | Sub-case 2: $b < 0$

First we note that in this case $b(K - \mathcal{R}_0) < 0$ and $c(1 - \mathcal{R}_0) > 0$ for $\mathcal{R}_0 \in [0, 1)$. This implies that both roots of equation (2) for $\mathcal{R}_0 \in [0, 1)$ are positive whenever $\Delta(\mathcal{R}_0) > 0$. Furthermore, from $\Delta'(\mathcal{R}_0) = 2b^2\left(\mathcal{R}_0 - K\right) + 4c$, we deduce that $\Delta'(\mathcal{R}_0) = 0$ if and only if

$$\mathcal{R}_0 = \mathcal{R}_0^* := K - \frac{2c}{b^2} \quad \text{with} \quad \min \Delta(\mathcal{R}_0) = \Delta(\mathcal{R}_0^*) = 4c\left(K - 1 - \frac{c}{b^2}\right). \tag{3}$$

We have the following result:

Proposition 2. If $K > 1 + \frac{c}{b^2}$, then equation (2) exhibits a full backward bifurcation; that is, it has

1. two positive roots if $\mathcal{R}_0 \in [0, 1)$,
2. a unique positive root if $\mathcal{R}_0 \in [1, +\infty)$.

Proof. Since $K > 1 + \frac{c}{b^2}$, $\Delta\left(\mathcal{R}_0^*\right) > 0$ implying that $\Delta(\mathcal{R}_0) > 0$ for all $\mathcal{R}_0$ and hence, in particular, equation (2) has real roots on $[0, 1]$, which are both positive on $[0, 1)$. □

Next we consider the case

$$1 \le K \le 1 + \frac{c}{b^2}. \tag{4}$$

We observe an important inequality

$$1 + \frac{c}{b^2} \ge \sqrt{\frac{4c}{b^2}}, \tag{5}$$

with equality occurring only if $\frac{c}{b^2} = 1$, that follows from re-writing the above as $\left(1 - \sqrt{\frac{c}{b^2}}\right)^2 \ge 0$. Moreover, $\Delta(\mathcal{R}_0^*) \le 0$ and

$$1 - \frac{2c}{b^2} \le \mathcal{R}_0^* \le 1 - \frac{c}{b^2} < 1. \tag{6}$$

In the sequel we discuss the following cases: $\frac{4c}{b^2} < 1$, $\frac{c}{b^2} < 1 \le \frac{4c}{b^2}$ and $1 \le \frac{c}{b^2}$.

Proposition 3. Let $\frac{4c}{b^2} < 1$. Then

1. If $K = 1$, then $0 < \mathcal{R}_b < \mathcal{R}_c = 1$ and equation (2) has:
   (a) two positive roots if $\mathcal{R}_0 \in [0, \mathcal{R}_b)$,
   (b) no positive roots if $\mathcal{R}_0 \in \left(\mathcal{R}_b, 1\right]$,
   (c) a double positive root if $\mathcal{R}_0 = \mathcal{R}_b$,
   (d) a unique positive root if $\mathcal{R}_0 \in (1, +\infty)$.
2. For $1 < K < 1 + \frac{c}{b^2}$, we have $0 < \mathcal{R}_b < \mathcal{R}_c < 1$ and equation (2) has:
   (a) two positive roots if $\mathcal{R}_0 \in [0, \mathcal{R}_b) \cup \left(\mathcal{R}_c, 1\right)$,
   (b) no positive roots if $\mathcal{R}_0 \in \left(\mathcal{R}_b, \mathcal{R}_c\right)$,
   (c) a double positive root if $\mathcal{R}_0 = \mathcal{R}_b$ or $\mathcal{R}_0 = \mathcal{R}_c$,
   (d) a unique positive root if $\mathcal{R}_0 \in [1, +\infty)$.
3. If $K = 1 + \frac{c}{b^2}$, then $0 < \mathcal{R}_b = \mathcal{R}_c < 1$ and equation (2) has:
   (a) two positive roots if $\mathcal{R}_0 \ne \mathcal{R}_c$,
   (b) a double positive root if $\mathcal{R}_0 = \mathcal{R}_c$,
   (c) a unique positive root if $\mathcal{R}_0 \in [1, +\infty)$.

Proof. If $\frac{4c}{b^2} < 1$, then $\frac{2c}{b^2} < \frac{1}{2} < 1$, implying that $0 < \mathcal{R}_0^* < 1$.

1. If $K = 1$, then $\Delta(0) > 0$, $\Delta(1) = 0$ and $\Delta(\mathcal{R}_0^*) < 0$. Thus $0 < \mathcal{R}_b < \mathcal{R}_0^* < 1$. The statement for $\mathcal{R}_0 = 1$ follows since then (2) becomes $\lambda^2 = 0$ with no positive roots.
2. Since $K > 1$, $\Delta(0) = K^2 b^2 - 4c > 0$, $\Delta(1) > 0$ and, by $K < 1 + \frac{c}{b^2}$, $\Delta(\mathcal{R}_0^*) < 0$. Hence $0 < \mathcal{R}_b < \mathcal{R}_c < 1$. Thus
   (a) $\Delta(\mathcal{R}_0) > 0$ for $\mathcal{R}_0 \in \left(0, \mathcal{R}_b\right) \cup \left(\mathcal{R}_c, 1\right)$,
   (b) $\Delta(\mathcal{R}_0) < 0$ for $\mathcal{R}_0 \in \left(\mathcal{R}_b, \mathcal{R}_c\right)$,
   (c) $\Delta(\mathcal{R}_0) = 0$ for $\mathcal{R}_0 = \mathcal{R}_b$ or $\mathcal{R}_0 = \mathcal{R}_c$.

   and the proposition follows by the comments preceding the proposition.
3. If $K = 1 + \frac{c}{b^2}$, then $\Delta(\mathcal{R}_0^*) = 0$ and thus $0 < \mathcal{R}_b = \mathcal{R}_c = \mathcal{R}_0^* < 1$ and the statement follows as above.

Since for all $\mathcal{R}_0 \in (0, 1)$, we have $c(1 - \mathcal{R}_0) > 0$ and $b(K - \mathcal{R}_0) < 0$, then whenever a root of (2) exists it is positive. This completes the proof. □

Proposition 4. If $\frac{c}{b^2} < 1 \le \frac{4c}{b^2}$, then $\frac{2c}{b^2} < \sqrt{\frac{4c}{b^2}} < 1 + \frac{c}{b^2}$ and we have

1. If $K = 1$, then $\mathcal{R}_b \le 0 < \mathcal{R}_c = 1$ and equation (2) has:

(a) no positive roots if $\mathcal{R}_0 \in [0, 1]$,

(b) a unique positive root if $\mathcal{R}_0 \in (1, +\infty)$.

2. If $1 < K \leq \sqrt{\frac{4c}{b^2}}$, then $\mathcal{R}_c \in (0, 1)$ and equation (2) has:

   (a) no positive roots if $\mathcal{R}_0 \in [0, \mathcal{R}_c)$,

   (b) a double positive root if $\mathcal{R}_0 = \mathcal{R}_c$,

   (c) two positive roots if $\mathcal{R}_0 \in \left(\mathcal{R}_c, 1\right)$,

   (d) a unique positive root if $\mathcal{R}_0 \in [1, +\infty)$.

3. If $\sqrt{\frac{4c}{b^2}} < K < 1 + \frac{c}{b^2}$, then $0 < \mathcal{R}_b < \mathcal{R}_c < 1$ and equation (2) has:

   (a) two positive roots if $\mathcal{R}_0 \in [0, \mathcal{R}_b) \cup \left(\mathcal{R}_c, 1\right)$,

   (b) no positive roots if $\mathcal{R}_0 \in \left(\mathcal{R}_b, \mathcal{R}_c\right)$,

   (c) a double positive root if $\mathcal{R}_0 = \mathcal{R}_b$ or $\mathcal{R}_0 = \mathcal{R}_c$,

   (d) a unique positive root if $\mathcal{R}_0 \in [1, +\infty)$.

4. If $K = 1 + \frac{c}{b^2}$, then $0 < \mathcal{R}_b = \mathcal{R}_c < 1$ and equation (2) has:

   (a) two positive roots if $\mathcal{R}_0 \neq \mathcal{R}_c$,

   (b) a double positive root if $\mathcal{R}_0 = \mathcal{R}_c$,

   (c) a unique positive root if $\mathcal{R}_0 \in [1, +\infty)$.

Proof. Since $\frac{c}{b^2} < 1$, then $1 + \frac{c}{b^2} > \sqrt{\frac{4c}{b^2}}$ (due to $\frac{c}{b^2} < 1$ and (5)). Moreover, for the same reason, $\frac{2c}{b^2} < \sqrt{\frac{4c}{b^2}}$. Therefore,

1. if $K = 1 \leq \sqrt{\frac{4c}{b^2}}$, then $\Delta(0) \leq 0$ and $\Delta(1) = 0$, so $\mathcal{R}_b \leq 0$ and $\mathcal{R}_c = 1$ with $0 < \mathcal{R}_0^* < 1$ and the statements follow as in Proposition 3, item 1;

2. $1 < K \leq \sqrt{\frac{4c}{b^2}}$ implies $\Delta(0) = K^2 b^2 - 4c \leq 0$ and $\Delta(0) = 0$ for $K = \sqrt{\frac{4c}{b^2}} > \frac{2c}{b^2}$, with corresponding $0 < \mathcal{R}_0^* < 1$ (see (6)), where $\Delta(\mathcal{R}_0^*) < 0$. Also $\Delta(1) > 0$. Hence

   (a) $\Delta(\mathcal{R}_0) < 0$ for $\mathcal{R}_0 \in [0, \mathcal{R}_c)$,

   (b) $\Delta(\mathcal{R}_0) = 0$ for $\mathcal{R}_0 = \mathcal{R}_c$,

   (c) $\Delta(\mathcal{R}_0) > 0$ for $\mathcal{R}_0 \in \left(\mathcal{R}_c, 1\right)$.

3. If $\sqrt{\frac{4c}{b^2}} < K < 1 + \frac{c}{b^2}$, then $\Delta(0) = K^2 b^2 - 4c > 0$ and again $0 < \mathcal{R}_0^* < 1$ with $\Delta(\mathcal{R}_0^*) < 0$. Since $\Delta(1) > 0$, then $0 < \mathcal{R}_b < \mathcal{R}_c < 1$ and

   (a) $\Delta(\mathcal{R}_0) > 0$ for $\mathcal{R}_0 \in [0, \mathcal{R}_b) \cup \left(\mathcal{R}_c, 1\right)$,

   (b) $\Delta(\mathcal{R}_0) < 0$ for $\mathcal{R}_0 \in \left(\mathcal{R}_b, \mathcal{R}_c\right)$,

   (c) $\Delta(\mathcal{R}_0) = 0$ for $\mathcal{R}_0 = \mathcal{R}_b$ or $\mathcal{R}_0 = \mathcal{R}_c$.

4. If $K = 1 + \frac{c}{b^2}$, then $\Delta(0) = \left(1 + \frac{c}{b^2}\right)^2 b^2 - 4c > 0$ and $0 < \mathcal{R}_0^* := K - \frac{2c}{b^2} < 1$. Since $\Delta(1) > 0$ and $\Delta(\mathcal{R}_0^*) = 0$, $0 < \mathcal{R}_b = \mathcal{R}_c = \mathcal{R}_0^* < 1 < K$ and

   (a) $\Delta(\mathcal{R}_0) > 0$ for $\mathcal{R}_0 \neq \mathcal{R}_c$,

   (b) $\Delta(\mathcal{R}_0) = 0$ for $\mathcal{R}_0 = \mathcal{R}_c$.

   Since for all $\mathcal{R}_0 \in [0, 1]$, we have $b(K - \mathcal{R}_0) < 0$ (due to $K > \sqrt{\frac{4c}{b^2}} > 1$) and $c(1 - \mathcal{R}_0) > 0$, then the statements follow as in Proposition 3, item 3.

□

Proposition 5. If $1 \le \frac{c}{b^2}$, then $1 < \sqrt{\frac{4c}{b^2}} \le 1 + \frac{c}{b^2} \le \frac{2b}{c^2}$, and we have the following results:

1. If $K = 1$, then $\mathcal{R}_b \le 0 < \mathcal{R}_c = 1$ and equation (2) has:
   (a) no positive roots if $\mathcal{R}_0 \in [0, 1]$,
   (b) a unique positive root if $\mathcal{R}_0 \in (1, +\infty)$.
2. If $1 \le K < \sqrt{\frac{4c}{b^2}}$, then $\mathcal{R}_c \in (0, 1)$ and equation (2) has:
   (a) no positive roots if $\mathcal{R}_0 \in [0, \mathcal{R}_c)$,
   (b) a double positive root if $\mathcal{R}_0 = \mathcal{R}_c$,
   (c) two positive roots if $\mathcal{R}_0 \in \left(\mathcal{R}_c, 1\right)$,
   (d) a unique positive root if $\mathcal{R}_0 \in [1, +\infty)$.
3. If $\sqrt{\frac{4c}{b^2}} \le K \le 1 + \frac{c}{b^2}$, then (2) has:
   (a) two positive roots for $\mathcal{R}_0 \in [0, 1)$,
   (b) a unique positive root if $\mathcal{R}_0 \in [1, +\infty)$.

Proof. Since $1 \le \frac{c}{b^2}$, then $\sqrt{\frac{4c}{b^2}} \le \frac{2c}{b^2}$. Moreover,

1. if $1 = K < \sqrt{\frac{4c}{b^2}} \le \frac{2c}{b^2}$, then $\Delta(0) = b^2 - 4c < 0$, $\mathcal{R}_0^* := K - \frac{2c}{b^2} = 1 - \frac{2c}{b^2} < 0$ and $\Delta(\mathcal{R}_0^*) := 4c(K - 1 - \frac{c}{b^2}) = \frac{-4c^2}{b^2} < 0$. Thus $\mathcal{R}_b < \mathcal{R}_0^* < 0$. Furthermore, we have $\Delta(1) = 0$ which implies that $\Delta(R_0) < 0$ on $(0, 1)$. At $R_0 = 1$, equation (2), as before, has a double zero root;
2. if $1 < K < \sqrt{\frac{4c}{b^2}}$, then $\Delta(0) = K^2b^2 - 4c < 0$. Moreover, since we also have $K < \frac{2c}{b^2}$, $\mathcal{R}_0^* < 0$. Hence, by $\Delta(-\infty) > 0$ and $\Delta(1) = b^2(K - 1)^2 > 0$, we have $0 < \mathcal{R}_c < 1$ and
   (a) $\Delta(\mathcal{R}_0) < 0$ for $\mathcal{R}_0 \in [0, \mathcal{R}_c)$,
   (b) $\Delta(\mathcal{R}_0) = 0$ for $\mathcal{R}_0 = \mathcal{R}_c$,
   (c) $\Delta(\mathcal{R}_0) > 0$ for $\mathcal{R}_0 \in \left(\mathcal{R}_c, 1\right)$;
3. if $\sqrt{\frac{4c}{b^2}} \le K \le 1 + \frac{c}{b^2} \le \frac{2c}{b^2}$, then $\mathcal{R}_0^* \le 0$ and $\Delta(0) = K^2b^2 - 4c \ge 0$. Hence $\Delta(\mathcal{R}_0) \ge 0$ for $\mathcal{R}_0 \in [0, 1)$.

The statements follow from $b(K - \mathcal{R}_0) < 0$ and $c(1 - \mathcal{R}_0) > 0$ on $[0, 1]$. ☐

For $0 < K < 1$, we discuss two cases, $1 < \sqrt{\frac{4c}{b^2}}$ and $\sqrt{\frac{4c}{b^2}} \le 1$.

Proposition 6. 1. if $1 < \sqrt{\frac{4c}{b^2}}$, then for $0 < K < 1$, equation (2) has
   (a) no positive root if $\mathcal{R}_0 \in [0, 1]$,
   (b) a unique positive root if $\mathcal{R}_0 \in (1, +\infty)$;

2. If $\sqrt{\frac{4c}{b^2}} \le 1$, then
   (a) if $\sqrt{\frac{4c}{b^2}} < K < 1$, then $0 < \mathcal{R}_b < 1$ and equation (2) has:
      i. two positive roots for $\mathcal{R}_0 \in \left[0, \mathcal{R}_b\right)$,
      ii. a double positive root for $\mathcal{R}_0 = \mathcal{R}_b$,
      iii. no positive roots for $\mathcal{R}_0 \in \left(\mathcal{R}_b, 1\right]$,
      iv. a unique positive root if $\mathcal{R}_0 \in (1, +\infty)$;
   (b) if $\sqrt{\frac{4c}{b^2}} = K$, then $\mathcal{R}_b = 0$ and equation (2) has:

i. a double positive root for $\mathcal{R}_0 = 0$,
ii. no positive roots for $\mathcal{R}_0 \in (0,1]$,
iii. a unique positive root if $\mathcal{R}_0 \in (1,+\infty)$;

(c) if $0 < K < \sqrt{\frac{4c}{b^2}}$, then $\mathcal{R}_c \in (K,1)$ and equation (2) has:
i. no positive root if $\mathcal{R}_0 \in [0,1]$,
ii. a unique positive root if $\mathcal{R}_0 \in (1,+\infty)$.

Proof. 1. If $1 < \sqrt{\frac{4c}{b^2}}$ then, for $0 < K < 1$, we have $-\sqrt{\frac{4c}{b^2}} < K < \sqrt{\frac{4c}{b^2}}$ which implies that $\Delta(0) = b^2K^2 - 4c < 0$. Since $\Delta(K) = -4c(1-K) < 0$, $\Delta(-\infty) > 0$ and $\Delta(1) = b^2(K-1)^2) > 0$, $\mathcal{R}_c \in (K,1)$. Hence

(a) if $\mathcal{R}_0 \in [0,\mathcal{R}_c)$, then $\Delta(\mathcal{R}_0) < 0$ implying that equation (2) has no real roots,

(b) if $\mathcal{R}_0 = \mathcal{R}_c$, then $\Delta(\mathcal{R}_0) = 0$ implying that equation (2) has a double root which is negative because $b(K-\mathcal{R}_c) > 0$ and $c(1-\mathcal{R}_c) > 0$,

(c) if $\mathcal{R}_0 \in (\mathcal{R}_c,1]$, then $\Delta(\mathcal{R}_0) > 0$ implying that equation (2) has two roots which are negative if $\mathcal{R}_0 \in (\mathcal{R}_c,1)$ because $b(K-\mathcal{R}_0) > 0$ and $c(1-\mathcal{R}_0) > 0$, and are negative and zero if $\mathcal{R}_0 = 1$ as then $b(K-1) > 0$ and $c(1-1) = 0$.

2. If $\sqrt{\frac{4c}{b^2}} \le 1$, then we discuss the following cases:

(a) If $\sqrt{\frac{4c}{b^2}} < K$, then $\Delta(0) = b^2K^2 - 4c > 0$. Since $\Delta(K) = -4c(1-K) < 0$ and $\Delta(1) = b^2(K-1)^2) > 0$, then $0 < \mathcal{R}_b < K < \mathcal{R}_c < 1$. Hence
i. if $\mathcal{R}_0 \in \left[0,\mathcal{R}_b\right)$, then $\Delta(\mathcal{R}_0) > 0$ implying that equation (2) has two roots which are positive because $b(K-\mathcal{R}_0) < 0$ and $c(1-\mathcal{R}_0) > 0$,
ii. if $\mathcal{R}_0 = \mathcal{R}_b$, then $\Delta(\mathcal{R}_0) = 0$ implying that equation (2) has a double root which is positive because $b(K-\mathcal{R}_b) < 0$ and $c(1-\mathcal{R}_b) > 0$.
iii. if $\mathcal{R}_0 \in \left(\mathcal{R}_b,\mathcal{R}_c\right)$, then $\Delta(\mathcal{R}_0) < 0$ implying that equation (2) has no real roots,
iv. if $\mathcal{R}_0 = \mathcal{R}_c$, then $\Delta(\mathcal{R}_0) = 0$ implying that equation (2) has a double root which is negative because $b(K-\mathcal{R}_c) > 0$ and $c(1-\mathcal{R}_c) > 0$.
v. if $\mathcal{R}_0 \in \left(\mathcal{R}_c,1\right]$, then $\Delta(\mathcal{R}_0) > 0$ implying that equation (2) has two roots which are non-positive, as in item 1.c above.

(b) If $0 < K = \sqrt{\frac{4c}{b^2}}$, then $\Delta(0) = 0$, $\Delta(K) = -4c(1-K) < 0$, $\Delta(1) = b^2(K-1)^2) > 0$, and hence $\mathcal{R}_b = 0$. Then for $R_0 = 0$ equation (2) has double positive root $\lambda = \sqrt{c}$. The remaining cases are the same as in item (a).

(c) If $0 < K < \sqrt{\frac{4c}{b^2}}$, then $-\sqrt{\frac{4c}{b^2}} < K < \sqrt{\frac{4c}{b^2}}$ which implies that $\Delta(0) = b^2K^2 - 4c < 0$. Since $\Delta(K) = -4c(1-K) < 0$, $\Delta(-\infty) > 0$ and $\Delta(1) = b^2(K-1)^2) > 0$, $\mathcal{R}_b < 0$, $\mathcal{R}_c \in (K,1)$. The remaining part of the proof is similar to the first case in this proposition.

□

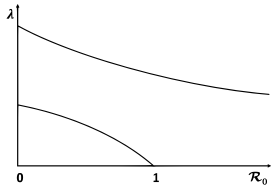
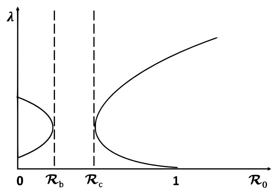
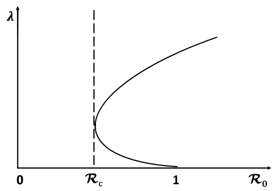
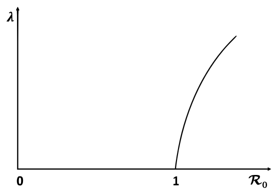
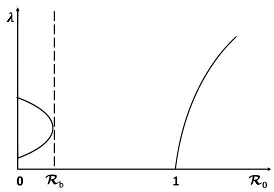
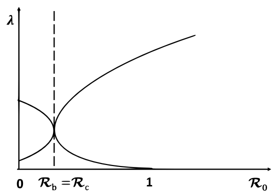
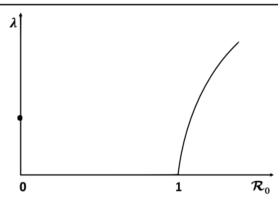
Proposition 7. If $K \le 0$, then equation (2) has:

1. no positive roots if $\mathcal{R}_0 \in [0,1]$,
2. a unique positive root if $\mathcal{R}_0 \in (1,+\infty)$.

Proof. If $K \le 0$, then for all $\mathcal{R}_0 \in [0,1]$, we have $b(K-\mathcal{R}_0) > 0$ and $c(1-\mathcal{R}_0) \ge 0$. This implies that, on the interval $\mathcal{R}_0 \in [0,1]$, equation (2) either has no real roots, or two nonpositive ones. □

In this section we considered the case $c > 0$ which ensured that the model has a unique endemic equilibrium point when $R_0 > 1$. However, for some models the bifurcation equation (2) comes with $c < 0$ (see for instance[27]), therefore in the next section we will discuss the implication of this on the number of positive roots for the equation.

TABLE 2 Number of positive roots for equation (2) when $b < 0$ and $c > 0$.

Assumptions	Number of positive roots	Bifurcation diagram
$\begin{cases} K > 1 + \frac{c}{b^2} \text{ or} \\ 1 \le \frac{c}{b^2} \& \sqrt{\frac{4c}{b^2}} \le K < 1 + \frac{c}{b^2} \end{cases}$	$\begin{cases} 2, \text{if } \mathcal{R}_0 \in [0, 1), \\ 1, \text{if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$\begin{cases} \frac{4c}{b^2} < 1 \ \& \ 1 < K < 1 + \frac{c}{b^2} \text{ or} \\ \frac{c}{b^2} < 1 \le \frac{4c}{b^2} \ \& \ \sqrt{\frac{4c}{b^2}} < K < 1 + \frac{c}{b^2} \end{cases}$	$\begin{cases} 2, \text{if } \mathcal{R}_0 \in [0, \mathcal{R}_b) \cup (\mathcal{R}_c, 1), \\ 0, \text{if } \mathcal{R}_0 \in (\mathcal{R}_b, \mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_b \text{ or } \mathcal{R}_c, \\ 1, \text{if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$\begin{cases} \frac{c}{b^2} < 1 \le \frac{4c}{b^2} \ \& \ 1 < K \le \sqrt{\frac{4c}{b^2}} \text{ or} \\ 1 \le \frac{c}{b^2} \ \& \ 1 \le K < \sqrt{\frac{4c}{b^2}} \end{cases}$	$\begin{cases} 0, \text{if } \mathcal{R}_0 \in (0, \mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_c, \\ 2, \text{if } \mathcal{R}_0 \in (\mathcal{R}_c, 1), \\ 1, \text{if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$\begin{cases} K \le 0 \text{ or} \\ 1 \le \frac{4c}{b^2} \ \& \ K = 1 \text{ or} \\ 1 < \sqrt{\frac{4c}{b^2}} \ \& \ 0 < K < 1, \text{ or} \\ \sqrt{\frac{4c}{b^2}} \le 1 \ \& \ 0 < K < \sqrt{\frac{4c}{b^2}} \end{cases}$	$\begin{cases} 0, \text{if } \mathcal{R}_0 \in [0, 1], \\ 1, \text{if } \mathcal{R}_0 \in (1, +\infty). \end{cases}$	
$\begin{cases} \frac{4c}{b^2} < 1 \& K = 1 \\ \sqrt{\frac{4c}{b^2}} \le 1 \& \sqrt{\frac{4c}{b^2}} < K < 1 \text{or} \\ \sqrt{\frac{4c}{b^2}} \le 1 \& \sqrt{\frac{4c}{b^2}} < K < 1 \end{cases}$	$\begin{cases} 2, \text{if } \mathcal{R}_0 \in [0, \mathcal{R}_b), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_b, \\ 0, \text{if } \mathcal{R}_0 \in (\mathcal{R}_b, 1], \\ 1, \text{if } \mathcal{R}_0 \in (1, +\infty). \end{cases}$	
$\frac{c}{b^2} < 1 \& K = 1 + \frac{c}{b^2}$	$\begin{cases} 2, \text{if } \mathcal{R}_0 \ne \mathcal{R}_c, \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_c, \\ 1, \text{if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$\sqrt{\frac{4c}{b^2}} \le 1 \& K = \sqrt{\frac{4c}{b^2}}$	$\begin{cases} 1 \text{ double, if } \mathcal{R}_0 = 0, \\ 0, \text{if } \mathcal{R}_0 \in (0, 1], \\ 2, \text{if } \mathcal{R}_0 \in (1, +\infty). \end{cases}$	

2.2 | Case 2: $c < 0$

First, we observe that if $c < 0$, then equation (2) has one positive root when $\mathcal{R}_0 < 1$. Hence, in the sequel we focus on the case $\mathcal{R}_0 \ge 1$. Again, we will discuss the cases $b > 0$ and $b < 0$ each with the sub-cases $K > 1$, $1 + \frac{c}{b^2} < K \le 1$ and $K \le 1 + \frac{c}{b^2}$.

2.2.1 | Sub-case 1: $b > 0$

Proposition 8. 1. If $K \geq 1$, then $\mathcal{R}_c > K$ and equation (2) has:

(a) a unique positive root if $\mathcal{R}_0 \in [0, 1)$,

(b) no positive roots if $\mathcal{R}_0 \in [1, \mathcal{R}_c)$,

(c) a double positive root if $\mathcal{R}_0 = \mathcal{R}_c$,

(d) two positive roots if $\mathcal{R}_0 \in (\mathcal{R}_c, +\infty)$. In this case, as $\mathcal{R}_0$ becomes large, the lower root tends to zero while the larger root tends to infinity in some linear fashion.

2. If $1 + \frac{c}{b^2} < K < 1$, then $1 < \mathcal{R}_b < \mathcal{R}_c$ and equation (2) has:

(a) a unique positive root if $\mathcal{R}_0 \in [0, 1)$,

(b) no positive roots if $\mathcal{R}_0 \in (\mathcal{R}_b, \mathcal{R}_c)$,

(c) a double positive root if $\mathcal{R}_0 = \mathcal{R}_b$ or $\mathcal{R}_0 = \mathcal{R}_c$

(d) two positive roots if $\mathcal{R}_0 \in (1, \mathcal{R}_b) \cup (\mathcal{R}_c, +\infty)$, In this case, as $\mathcal{R}_0$ becomes large, the lower root tends to zero while the larger root tends to infinity in some linear fashion.

3. If $K \leq 1 + \frac{c}{b^2}$, then equation (2) has:

(a) a unique positive root if $\mathcal{R}_0 \in [0, 1]$,

(b) two positive roots for all $\mathcal{R}_0 \in (1, +\infty)$.

Proof. We first note that when $K < 1$, we have $b(K - \mathcal{R}_0) < 0$ for all $\mathcal{R}_0 > 1$ with $c(1 - \mathcal{R}_0) > 0$ for all $\mathcal{R}_0 > 1$ and $c(1 - \mathcal{R}_0) = 0$ if $\mathcal{R}_0 = 1$. This implies implying that whenever the roots of (2) are real, they must be positive if $\mathcal{R}_0 > 1$, and one root is positive while the other 0 if $\mathcal{R}_0 = 1$.

1. If $K \geq 1$, then we have the following cases:

(a) If $K > 1$, then $\Delta(K) = 4c\,(K - 1) < 0$, since $\Delta(1) = b^2\,(K - 1)^2 > 0$ and $\Delta(+\infty) > 0$, then $1 < \mathcal{R}_b < K < \mathcal{R}_c$ and we have $\Delta(\mathcal{R}_b) = 0$, $\Delta(\mathcal{R}_c) = 0$, $\Delta(\mathcal{R}_0) > 0$ for $1 < \mathcal{R}_0 < \mathcal{R}_b$, $\Delta(\mathcal{R}_0) < 0$ for $\mathcal{R}_b < \mathcal{R}_0 < \mathcal{R}_c$ and $\Delta(\mathcal{R}_0) > 0$ for $\mathcal{R}_c < \mathcal{R}_0$. In this case, we have the following:

i. if $1 \leq \mathcal{R}_0 < \mathcal{R}_b$ equation (2) has two roots which, since $b(K - \mathcal{R}_0) > 0$ and $c(1 - \mathcal{R}_0) \geq 0$, are either both negative, or negative and 0 (if $R_0 = 1$),

ii. if $\mathcal{R}_0 = \mathcal{R}_b$, then equation (2) a double positive root which is negative since $b(K - \mathcal{R}_b) > 0$,

iii. if $\mathcal{R}_b < \mathcal{R}_0 < \mathcal{R}_c$, then equation (2) has no roots,

iv. if $\mathcal{R}_0 = \mathcal{R}_c$, then equation (2) has a double root which is positive as $b(K - \mathcal{R}_c) < 0$,

v. if $\mathcal{R}_c < \mathcal{R}_0$, equation (2) has two real roots which are positive since $b(K - \mathcal{R}_0) < 0$ and $c(1 - \mathcal{R}_0) > 0$.

(b) The case $K = 1$ is similar to the previous one. We have again $\Delta(\mathcal{R}_0^*) < 0$ and $\Delta(0) > 0$. Furthermore, $K = 1$ implies $\mathcal{R}_0^* = 1 - \frac{2c}{b^2} > 1$. Therefore, by $\Delta(\mathcal{R}^*) < 0$, $\Delta(1) = 0$, we have $1 = \mathcal{R}_b < \mathcal{R}_0^* < \mathcal{R}_c$. Thus $\Delta(\mathcal{R}_0) > 0$ for $\mathcal{R}_0 \in [0, 1) \cup (\mathcal{R}_c, +\infty)$ and $\Delta(\mathcal{R}_0) < 0$ for $\mathcal{R}_0 \in (1, \mathcal{R}_c)$. Since $b(1 - \mathcal{R}_0) = 0$ and $c(1 - \mathcal{R}_0) = 0$ for $\mathcal{R}_0 = 1$ and $b(K - \mathcal{R}_0) < 0$ and $c(1 - \mathcal{R}_0) > 0$ for $\mathcal{R}_0 \in (1, +\infty)$, the proof of items (b)-(d) is concluded.

(c) If $K \leq 1 + \frac{c}{b^2}$, then $\Delta(\mathcal{R}_0^*) = 4c\left(K - 1 - \frac{c}{b^2}\right) \geq 0$ implying that $\Delta(\mathcal{R}_0) \geq 0$ for all $\mathcal{R}_0 \geq 1$. In this case, (2) has two positive roots for all $\mathcal{R}_0 > 1$ which coalesce into a double positive root if $K = 1 + \frac{c}{b^2}$. For $\mathcal{R}_0 = 1$ one root is 0, while the other positive.

2. If $1 + \frac{c}{b^2} < K < 1$, then $\Delta(\mathcal{R}_0^*) < 0$, see (3). Moreover, due to $c < 0$, $\Delta(0) = b^2K^2 - 4c > 0$. Furthermore, the condition $1 + \frac{c}{b^2} < K \leq 1$ implies that $1 - \frac{c}{b^2} < \mathcal{R}_0^* = K - \frac{2c}{b^2}$. Since $c < 0$, $1 < \mathcal{R}_0^*$. Therefore, by $\Delta(\mathcal{R}^*) < 0$, $\Delta(1) = b^2\,(K - 1)^2 \geq 0$ and $\Delta(+\infty) > 0$, we have $1 \leq \mathcal{R}_b < \mathcal{R}_0^* < \mathcal{R}_c$ and $\Delta(\mathcal{R}_0) > 0$ for $\mathcal{R}_0 \in [0, \mathcal{R}_b) \cup (\mathcal{R}_c, +\infty)$ and $\Delta(\mathcal{R}_0) < 0$ for $\mathcal{R}_0 \in (\mathcal{R}_b, \mathcal{R}_c)$. This, along with $b(K - \mathcal{R}_0) < 0$ and $c(1 - \mathcal{R}_0) > 0$ on $(1, +\infty)$, completes the proof of items (b)-(d).

□

2.2.2 | Sub-case 2: $b < 0$

In this case we have the following result:

Proposition 9. 1. If $K \leq 1$, then equation (2) has:

   (a) a unique positive root if $\mathcal{R}_0 \in [0, 1)$,
   (b) no positive roots if $\in [1, +\infty)$.

2. If $K > 1$, then $\mathcal{R}_b \in (1, K)$ and equation (2) has:

   (a) a unique positive root if $\mathcal{R}_0 \in [0, 1]$,
   (b) two positive roots if $\mathcal{R}_0 \in (1, \mathcal{R}_b)$,
   (c) a double positive root if $\mathcal{R}_0 = \mathcal{R}_b$,
   (d) no positive roots if $\mathcal{R}_b < \mathcal{R}_0$.

Proof. 1. If $K \leq 1$, then, for all $\mathcal{R}_0 \geq 1$ we have $b\left(K - \mathcal{R}_0\right) > 0$ and $c(1 - \mathcal{R}_0) \geq 0$, implying that equation (2) has no positive roots.

2. Since $K > 1$, then $\Delta(K) = 4c\,(K-1) < 0$ which, by $\Delta(1) = b^2\,(K-1)^2 > 0$ and $\Delta(+\infty) > 0$, imply that $1 < \mathcal{R}_b < K < \mathcal{R}_c$ and we have $\Delta(\mathcal{R}_0) > 0$ for $1 < \mathcal{R}_0 < \mathcal{R}_b$, $\Delta(\mathcal{R}_0) < 0$ for $\mathcal{R}_b < \mathcal{R}_0 < \mathcal{R}_c$ and $\Delta(\mathcal{R}_0) > 0$ for $\mathcal{R}_c < \mathcal{R}_0$. In this case, we have:

   (a) if $\mathcal{R}_0 = 1$, then equation (2) has one positive and one 0 root,
   (b) if $\mathcal{R}_0 \in (1, \mathcal{R}_b)$, then equation (2) has two roots which are positive because $b(K - \mathcal{R}_0) < 0$ and $c(1 - \mathcal{R}_0) > 0$ on $(1, K)$,
   (c) if $\mathcal{R}_0 = \mathcal{R}_b$, then equation (2) has a double root which is positive because $b(K - \mathcal{R}_b) < 0$ and $c\left(1 - \mathcal{R}_b\right) > 0$,
   (d) if $\mathcal{R}_0 \in (\mathcal{R}_b, \mathcal{R}_c)$, then equation (2) has no real roots,
   (e) if $\mathcal{R}_0 = \mathcal{R}_c$, then equation (2) has a double root which is negative because $b(K - \mathcal{R}_c) > 0$ and $c\left(1 - \mathcal{R}_c\right) > 0$,
   (f) if $\mathcal{R}_0 \in (\mathcal{R}_c, \infty)$, then equation (2) has two roots which are negative because $b(K - \mathcal{R}_0) > 0$ and $c(1 - \mathcal{R}_0) > 0$ on $(\mathcal{R}_c, +\infty)$,

   which, combined, give (a)–(d) of item 2. of the theorem.

□

3 | EXAMPLE 1: A BASIC MODEL FOR THE INTERACTIONS BETWEEN HIV AND THE IMMUNE SYSTEM

The basic model (see e.g. Perelson[25] and Nowak[26]) considers a population of T-cells, T, which are produced at a constant rate s and die at a rate d per cell. Through interactions with the virus population, V, T-cells become infected at a constant rate β and move to the infected class I. Infected cells are assumed to lyse at constant rate δ per cell and produce new virus particles at a constant rate p which are assumed to be cleared at a constant rate σ per virus. The infection of T-cells triggers an immune response mediated by CD8 lymphocytes, Z, which are produced at a constant rate κ and reduced through either contact with infected cells at a rate ϕ or death at a rate ζ. The model describing the basic model of viral dynamics is as follows:

$$\begin{cases} \dfrac{dT}{dt} = s - \beta TV - dT \\ \dfrac{dI}{dt} = \beta TV - \alpha IZ - \delta I \\ \dfrac{dV}{dt} = pI - \sigma V \\ \dfrac{dZ}{dt} = \kappa - \phi YZ - \zeta Z \end{cases} \tag{7}$$

TABLE 3 Number of positive roots for equation (2) when $b > 0$ and $c < 0$.

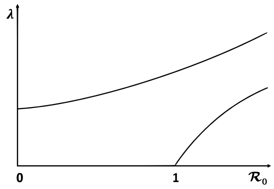
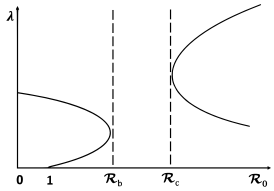
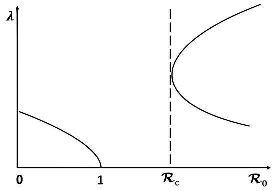
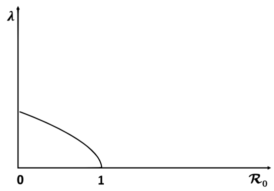
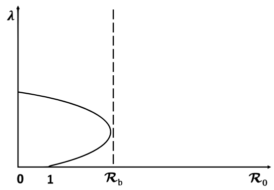
Assumptions	Number of positive roots	Bifurcation diagram
$K \leq 1 + \frac{c}{b^2}$	$\begin{cases} 1, \text{ if } \mathcal{R}_0 \in [0, 1), \\ 2, \text{ if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$1 + \frac{c}{b^2} < K < 1$	$\begin{cases} 1, \text{ if } \mathcal{R}_0 \in [0, 1), \\ 0, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_b, \mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_b \text{ or } \mathcal{R}_c, \\ 2, \text{ if } \mathcal{R}_0 \in (1, \mathcal{R}_b) \cup (\mathcal{R}_c, +\infty). \end{cases}$	
$K \geq 1$	$\begin{cases} 1, \text{ if } \mathcal{R}_0 \in [0, 1), \\ 0, \text{ if } \mathcal{R}_0 \in [1, \mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_c, \\ 2, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_c, +\infty). \end{cases}$	

TABLE 4 Number of positive roots for equation (2) when $b < 0$ and $c < 0$.

Assumptions	Number of roots	Graphical illustration
$K \leq 1$	$\begin{cases} 1, \text{ if } \mathcal{R}_0 \in [0, 1), \\ 0, \text{ for } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$K > 1$	$\begin{cases} 1, \text{ if } \mathcal{R}_0 \in [0, 1), \\ 2, \text{ if } \mathcal{R}_0 \in (1, \mathcal{R}_b), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_b, \\ 0, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_b, +\infty). \end{cases}$	

The model's virus-free equilibrium is given by $E_0 = \left(\frac{s}{d}, 0, \frac{\kappa}{\zeta}, 0\right)$ and the basic reproductive number is given by

$$R_0 = \frac{ps\beta\zeta}{d\sigma\left(\alpha\kappa + \zeta\delta\right)}.$$

Using Maple, we find that the bifurcation equation of this model is given by:

$$\lambda^2 + b\left(K - R_0\right)\lambda + c\left(1 - R_0\right) = 0, \tag{8}$$

where

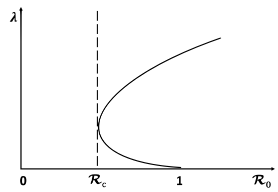
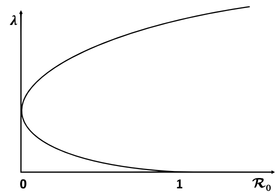
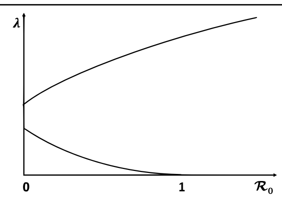
$$\begin{cases} b = \dfrac{d\psi\delta\,(\zeta\delta - s\phi)}{s\phi\,(\kappa\alpha + \psi\delta)} \\ c = \dfrac{pd\beta\psi\zeta^2\delta^2}{\sigma\phi\,(\alpha\kappa + \zeta\delta)\,(\alpha\kappa + \psi\delta)} \\ K = \dfrac{s\phi\,(\kappa\alpha + \psi\delta)}{\psi\delta\,(s\phi - \zeta\delta)} \end{cases}$$

In this example, we see that $c > 0$ while b and K can take negative values. We note that the parameter ϕ appears in the expression of b and K and not in R_0. We therefore use it for our bifurcation discussion. We have the following cases:

1. If $\phi < \frac{\zeta\delta}{s}$, then $b > 0$ and $K < 0$.
2. If $\phi \geq \frac{\zeta\delta}{s}$, then $b < 0$ and $K > 1$.

An illustration of the various bifurcation behaviours exhibited by this model is presented in tables 5 and 6 .

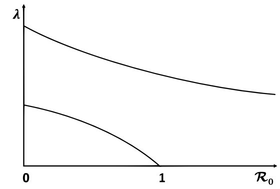
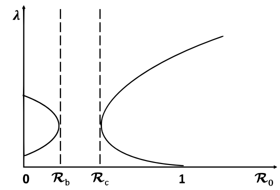
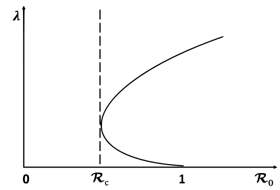
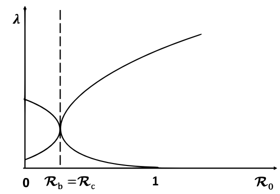
TABLE 5 Number of positive roots for equation (8) when $\phi < \frac{\zeta\delta}{s}$, (i.e. $b > 0$ and $K < 0$).

Assumptions	Number of positive roots	Bifurcation diagram
$-\sqrt{\frac{4c}{b^2}} < K < 0$	$\begin{cases} 0, \text{ if } \mathcal{R}_0 \in [0, \mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_c, \\ 2, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_c, 1), \\ 1, \text{ if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$K = -\sqrt{\frac{4c}{b^2}}$	$\begin{cases} 1 \text{ double, if } \mathcal{R}_0 = 0, \\ 2, \text{ if } \mathcal{R}_0 \in (0, 1), \\ 1, \text{ if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$K < -\sqrt{\frac{4c}{b^2}}$	$\begin{cases} 2, \text{ if } \mathcal{R}_0 \in [0, 1), \\ 1, \text{ if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	

4 | EXAMPLE 2: A MALARIA MODEL FOR HUMANS WITH EFFECTIVE AND INEFFECTIVE TREATMENT

This model was recently developed in[27] to study the effects of the use of ineffective drugs on malaria control in Ghana. In this model, the total population of humans is divided into five classes: susceptible humans (S_h), exposed humans (E_h), infectious humans (I_h), partially recovered humans (R_h) and fully recovered humans (T_h). The population of mosquitoes is divided into

TABLE 6 Number of positive roots for equation (8) when $\phi \geq \frac{\zeta\delta}{s}$, (i.e. $b < 0$ and $K > 1$).

Assumptions	Number of positive roots	Bifurcation diagram
$\begin{cases} K > 1 + \frac{c}{b^2} \text{ or} \\ 1 \leq \frac{c}{b^2} \& \sqrt{\frac{4c}{b^2}} \leq K < 1 + \frac{c}{b^2} \end{cases}$	$\begin{cases} 2, \text{ if } \mathcal{R}_0 \in [0,1), \\ 1, \text{ if } \mathcal{R}_0 \in [1,+\infty). \end{cases}$	
$\begin{cases} \frac{4c}{b^2} < 1 \;\&\; 1 < K < 1 + \frac{c}{b^2} \text{ or} \\ \frac{c}{b^2} < 1 \leq \frac{4c}{b^2} \;\&\; \sqrt{\frac{4c}{b^2}} < K < 1 + \frac{c}{b^2} \end{cases}$	$\begin{cases} 2, \text{ if } \mathcal{R}_0 \in [0,\mathcal{R}_b) \cup (\mathcal{R}_c, 1), \\ 0, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_b, \mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_b \text{ or } \mathcal{R}_c, \\ 1, \text{ if } \mathcal{R}_0 \in [1,+\infty). \end{cases}$	
$\begin{cases} \frac{c}{b^2} < 1 \leq \frac{4c}{b^2} \;\&\; 1 < K \leq \sqrt{\frac{4c}{b^2}} \text{ or} \\ 1 \leq \frac{c}{b^2} \;\&\; 1 \leq K < \sqrt{\frac{4c}{b^2}} \end{cases}$	$\begin{cases} 0, \text{ if } \mathcal{R}_0 \in (0,\mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_c, \\ 2, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_c, 1), \\ 1, \text{ if } \mathcal{R}_0 \in [1,+\infty). \end{cases}$	
$\frac{c}{b^2} < 1 \& K = 1 + \frac{c}{b^2}$	$\begin{cases} 2, \text{ if } \mathcal{R}_0 \neq \mathcal{R}_c, \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_c, \\ 1, \text{ if } \mathcal{R}_0 \in [1,+\infty). \end{cases}$	

three classes: susceptible mosquitoes $\left(S_m\right)$, exposed mosquitoes $\left(E_m\right)$ and infectious mosquitoes $\left(I_m\right)$. The model proposed in[27] to describe the dynamics of disease transmission and recovery reads as follows:

$$\begin{cases} \frac{dS_h}{dt} = \alpha_h + \phi T_h + \theta R_h - \left(\lambda_h + \mu_h\right) S_h \\ \frac{dE_h}{dt} = \lambda_h S_h - \left(\nu_h + \mu_h\right) E_h \\ \frac{dI_h}{dt} = \nu_h E_h + \xi R_h - \left(\eta + \gamma + \delta_h + \mu_h\right) I_h \\ \frac{dR_h}{dt} = \eta I_h - \left(\xi + \theta + \mu_h\right) R_h \\ \frac{dT_h}{dt} = \gamma I_h - \left(\phi + \mu_h\right) T_h \\ \frac{dS_m}{dt} = \lambda_m S_m - \left(\nu_m + \mu_{m_1} + \mu_{m_2} N_m\right) S_m \\ \frac{dE_m}{dt} = \nu_m S_m - \left(\mu_{m_1} + \mu_{m_2} N_m\right) E_m \\ \frac{dI_m}{dt} = \nu_m E_m - \left(\mu_{m_1} + \mu_{m_2} N_m\right) I_m \end{cases}$$

where $\lambda_h = \frac{a\beta_h I_m}{N_h}$ and $\lambda_m = \frac{a\beta_m\left(I_h + \rho R_h\right)}{N_h}$. The model's basic reproductive number as calculated by the authors is given by

$$R_0 = \sqrt{\frac{a^2 \beta_h \beta_m \nu_h \nu_m \left(\nu_{33} + \rho\eta\right) r_m}{\alpha_m \chi \nu_{11} \nu_{44} K_h \mu_{m_2}}},$$

where $\nu_{11} = \nu_h + \mu_h$, $\nu_{33} = \xi + \theta + \mu_h$, $\nu_{44} = \nu_m + \alpha_m$ and $\chi = \left(\gamma + \eta + \delta_h\right)\left(\theta + \mu_h\right) + \xi\left(\gamma + \delta_h\right)$.

In their investigation of the model's endemic equilibrium points, the authors obtained a bifurcation equation which we rewrite as follows:

$$I_h^{*2} + b\left(K - R_0^2\right) I_h^* + c\left(1 - R_0^2\right) = 0 \tag{9}$$

where

$$\begin{cases} b = \dfrac{\alpha_m^2 \mu_h \nu_{11} \nu_{33} \nu_{44} K_h \chi \left(\phi + \mu_h\right) B_2}{\alpha_m^2 \delta_h \nu_{11} \nu_{33} \nu_{44} \chi \left(\phi + \mu_h\right) \left(\delta^* - \delta_h\right)}, \\ c = \dfrac{\alpha_m^2 \mu_h \alpha_h \nu_{11} \nu_{33} \nu_{44} K_h \chi \left(\phi + \mu_h\right)}{\alpha_m^2 \delta_h \nu_{11} \nu_{33} \nu_{44} \chi \left(\phi + \mu_h\right) \left(\delta_h - \delta^*\right)}, \\ K = \dfrac{2\delta_h - \delta^*}{B_2}, \\ \delta^* = \dfrac{a \beta_m \mu_h \left(\nu_{33} + \rho\eta\right)}{\alpha_m \nu_{33}}, \\ B_2 = \dfrac{\nu_{11}\chi}{\nu_h \nu_{33}} - \dfrac{\phi\gamma}{\phi + \mu_h} > 0. \end{cases}$$

This is an interesting example whereby b and c can be positive or negative and K can take any positive or negative value (at least theoretically). We first note that the results of the approach described above depend on the sign of $\delta^* - \delta_h$. Second, the investigation of all possible cases of b, c, K and R_0 discussed in the next sections is possible because there are parameters that appear in b, c and K and do not appear in R_0 (such as ϕ) and there are also parameters that appear in R_0 and do not appear in b, c and K (such as r_m or μ_{m_2}). In fact, we have the following properties:

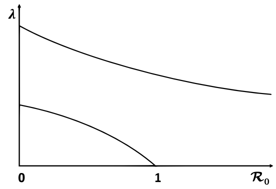
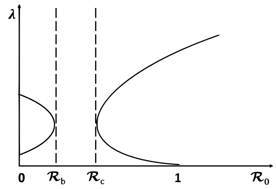
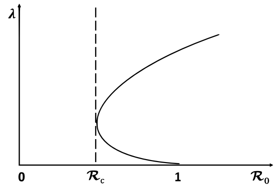
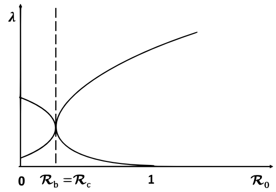
1. $b > 0$ if and only if $\delta_h < \delta^*$ in this case we also have $c < 0$
2. $K > 0$ if and only if $\delta^* < 2\delta_h$
3. $K > 1$ if and only if $\delta^* < 2\delta_h - B_2$
4. $2\delta_h - B_2 < \delta_h$ if and only if $\delta_h < B_2$,

This leads us to discuss the cases $\delta_h < B_2$ and $\delta_h > B_2$.

1. If $\delta_h < B_2$, then $2\delta_h - B_2 < \delta_h < 2\delta_h$. Thus, we have the following cases:
   (a) If $\delta^* < 2\delta_h - B_2$, then $K > 1$. Moreover, we have $\delta^* < \delta_h$ implying that $b < 0$ and $c > 0$.
   (b) If $2\delta_h - B_2 < \delta^* < \delta_h$, then $K < 1$ and $\delta^* < 2\delta_h$ implying that $K > 0$. Moreover, the condition $\delta^* < \delta_h$ implies that $b < 0$ and $c > 0$.
   (c) If $\delta_h < \delta^*$, then $K < 1$, $b > 0$ and $c < 0$.
2. If $\delta_h > B_2$, then $\delta_h < 2\delta_h - B_2 < 2\delta_h$. We thus discuss the following cases:
   (a) If $\delta^* < \delta_h$, then $b < 0$ and $c > 0$. Moreover, we have $\delta^* < 2\delta_h$ implying that $K > 0$ and $\delta^* + B_2 < 2\delta_h$ implying that $K < 1$
   (b) If $\delta_h < \delta^* < 2\delta_h - B_2$, then $b > 0$, $c < 0$ and $K > 1$.
   (c) If $2\delta_h - B_2 < \delta^*$, then $b > 0$, $c < 0$ and $K \leq 1$.

Various bifurcation behaviours exhibited by this model is presented in tables 7 – 10 .

TABLE 7 Number of positive roots for equation (9) when $\delta_h < B_2$ and $\delta^* \leq 2\delta_h - B_2$, (i.e. $b < 0, c > 0$ and $K > 1$).

Assumptions	Number of positive roots	Bifurcation diagram
$\begin{cases} K > 1 + \frac{c}{b^2} \text{ or} \\ 1 \leq \frac{c}{b^2} \& \sqrt{\frac{4c}{b^2}} \leq K < 1 + \frac{c}{b^2} \end{cases}$	$\begin{cases} 2, \text{ if } \mathcal{R}_0 \in [0, 1), \\ 1, \text{ if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$\begin{cases} \frac{4c}{b^2} < 1 \ \& \ 1 < K < 1 + \frac{c}{b^2} \text{ or} \\ \frac{c}{b^2} < 1 \leq \frac{4c}{b^2} \ \& \ \sqrt{\frac{4c}{b^2}} < K < 1 + \frac{c}{b^2} \end{cases}$	$\begin{cases} 2, \text{ if } \mathcal{R}_0 \in [0, \mathcal{R}_b) \cup (\mathcal{R}_c, 1), \\ 0, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_b, \mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_b \text{ or } \mathcal{R}_c, \\ 1, \text{ if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$\begin{cases} \frac{c}{b^2} < 1 \leq \frac{4c}{b^2} \ \& \ 1 < K \leq \sqrt{\frac{4c}{b^2}} \text{ or} \\ 1 \leq \frac{c}{b^2} \ \& \ 1 \leq K < \sqrt{\frac{4c}{b^2}} \end{cases}$	$\begin{cases} 0, \text{ if } \mathcal{R}_0 \in (0, \mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_c, \\ 2, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_c, 1), \\ 1, \text{ if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	
$\frac{c}{b^2} < 1 \& K = 1 + \frac{c}{b^2}$	$\begin{cases} 2, \text{ if } \mathcal{R}_0 \neq \mathcal{R}_c, \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_c, \\ 1, \text{ if } \mathcal{R}_0 \in [1, +\infty). \end{cases}$	

5 | CONCLUSIONS

In this paper we have provided a comprehensive survey of all the possible bifurcation patterns that can occur in epidemiological models in which the endemic equilibria satisfy a quadratic equation. Of practical importance are Tables 1-4, where we summarized all the bifurcation cases that may emerge from such models. We note that some of these bifurcation results do not seem to have obvious epidemiological interpretation, making their occurrence in practice debatable. Nonetheless, in the two presented examples we managed to show possible interplays of the models' parameters that lead to such unexpected bifurcation patterns. Of course, even then one needs to check if the parameter values that drive such a behaviour are realistic.

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Conflict of interest

The authors declare no potential conflict of interests.

TABLE 8 Number of positive roots for equation (9) when $\{\delta_h < B_2$ and $2\delta_h - B_2 < \delta^* < \delta_h\}$ or $\{\delta_h > B_2$ and $\delta^* < \delta_h\}$, (i.e. $b < 0, c > 0$ and $0 < K < 1$).

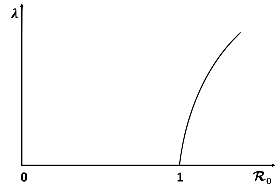
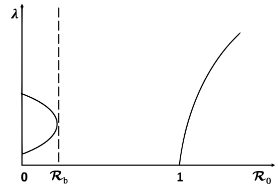
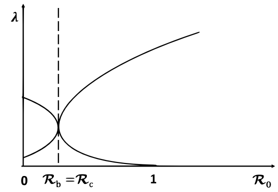
Assumptions	Number of positive roots	Bifurcation diagram
$\begin{cases} K \le 0 \text{ or} \\ 1 \le \frac{4c}{b^2} \ \& \ K = 1 \text{ or} \\ 1 < \sqrt{\frac{4c}{b^2}} \ \& \ 0 < K < 1, \text{ or} \\ \sqrt{\frac{4c}{b^2}} \le 1 \ \& \ 0 < K < \sqrt{\frac{4c}{b^2}} \end{cases}$	$\begin{cases} 0, \text{ if } \mathcal{R}_0 \in [0,1], \\ 1, \text{ if } \mathcal{R}_0 \in (1,+\infty). \end{cases}$	
$\begin{cases} \frac{4c}{b^2} < 1 \& K = 1 \\ \sqrt{\frac{4c}{b^2}} \le 1 \& \sqrt{\frac{4c}{b^2}} < K < 1 \text{or} \\ \sqrt{\frac{4c}{b^2}} \le 1 \& \sqrt{\frac{4c}{b^2}} < K < 1 \end{cases}$	$\begin{cases} 2, \text{ if } \mathcal{R}_0 \in [0,\mathcal{R}_b), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_b, \\ 0, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_b,1], \\ 1, \text{ if } \mathcal{R}_0 \in (1,+\infty). \end{cases}$	
$\frac{c}{b^2} < 1 \& K = 1 + \frac{c}{b^2}$	$\begin{cases} 2, \text{ if } \mathcal{R}_0 \neq \mathcal{R}_c, \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_c, \\ 1, \text{ if } \mathcal{R}_0 \in [1,+\infty). \end{cases}$	

TABLE 9 Number of positive roots for equation (9) when $\{\delta_h < B_2$ and $\delta_h < \delta^*\}$ or $\{\delta_h > B_2$ and $2\delta_h - B_2 < \delta^*\}$, (i.e. $b > 0, c < 0$ and $K < 1$).

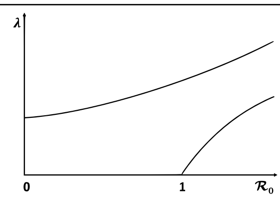
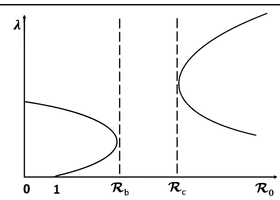
Assumptions	Number of positive roots	Bifurcation diagram
$K \le 1 + \frac{c}{b^2}$	$\begin{cases} 1, \text{ if } \mathcal{R}_0 \in [0,1), \\ 2, \text{ if } \mathcal{R}_0 \in [1,+\infty). \end{cases}$	
$1 + \frac{c}{b^2} < K < 1$	$\begin{cases} 1, \text{ if } \mathcal{R}_0 \in [0,1), \\ 0, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_b,\mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_b \text{ or } \mathcal{R}_c, \\ 2, \text{ if } \mathcal{R}_0 \in (1,\mathcal{R}_b) \cup (\mathcal{R}_c,+\infty). \end{cases}$	

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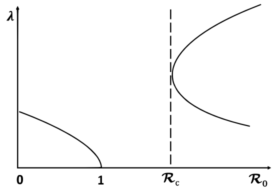
TABLE 10 Number of positive roots for equation (9) when $\{\delta_h > B_2$ and $\delta_h < \delta^*\}$, (i.e. $b > 0, c < 0$ and $K > 1$).

<table>
<tr><th>Assumptions</th><th>Number of positive roots</th><th>Bifurcation diagram</th></tr>
<tr><td>$K > 1$</td><td>$\begin{cases} 1, \text{ if } \mathcal{R}_0 \in [0, 1), \\ 0, \text{ if } \mathcal{R}_0 \in [1, \mathcal{R}_c), \\ 1 \text{ double, if } \mathcal{R}_0 = \mathcal{R}_c, \\ 2, \text{ if } \mathcal{R}_0 \in (\mathcal{R}_c, +\infty). \end{cases}$</td><td>
</td></tr>
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