ON THE LIPSCHITZ DECOMPOSITION PROBLEM IN ORDERED BANACH SPACES AND ITS CONNECTIONS TO OTHER BRANCHES OF MATHEMATICS

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Dedicated to the occasion of Ben de Pagter 65th birthday.

Abstract. Consider the following still open problem: For any Banach space X, ordered by a closed generating cone $C \subseteq X$, does there always exist Lipschitz functions $(\cdot)^{\pm} : X \to C$ satisfying $x = x^+ - x^-$ for every $x \in X$?

We discuss the connections of this problem to a large number of other branches of mathematics: set-valued analysis, selection theorems, the nonlinear geometry of Banach spaces, Ramsey theory, Lipschitz function spaces, duality theory, and tensor products of Banach spaces. We give equivalent reformulations of the problem, and, through known examples, provide circumstantial evidence that the above question could be answered in the negative.

1. Introduction

This paper is a brief survey on what we will term the *Lipschitz decomposition problem* in general ordered Banach spaces. To the author's knowledge, the following problem is unsolved and has remained open for number of years. The problem came to the author's attention in 2013, but may well be older.

Problem 1.1. Which one of the following mutually exclusive statements is true?

(1) For every Banach space X, ordered by a closed generating cone $C \subseteq X$, there exist Lipschitz functions $(\cdot)^{\pm} : X \to C$ satisfying $x = x^+ - x^-$ for every $x \in X$.
(2) There exists a Banach space X, ordered by a closed generating cone $C \subseteq X$, for which there exist no Lipschitz functions $(\cdot)^{\pm} : X \to C$ satisfying $x = x^+ - x^-$ for every $x \in X$.

We will say the Lipschitz decomposition problem is solved positively if the statement (1) in Problem 1.1 is true. It is known that certain ordered Banach spaces do admit such Lipschitz functions (cf. Section 3), hence we introduce the following terminology.

Definition 1.2 (Lipschitz decomposition property)**.** We will say that a Banach space X, ordered by closed generating cone $C \subseteq X$, has the *Lipschitz decomposition property,* if there exist Lipschitz functions $(\cdot)^{\pm} : X \to C$ satisfying $x = x^+ - x^-$ for every $x \in X$.

The Lipschitz decomposition problem has connections to a large number of other branches of mathematics: set-valued analysis, selection theorems, the Lipschitz– and uniform geometry of Banach spaces, Ramsey theory, Lipschitz function spaces, duality theory, and tensor products of Banach spaces. Discussing the relevance of each of these subjects to the problem at hand, including some very recent developments, is the main aim of this paper.

After some brief preliminaries in Section 2, we first establish the trivial cases of the Lipschitz decomposition problem in Section 3: Banach lattices and order unit spaces all have the Lipschitz decomposition property.

In Section 4, we show, similarly to modern proofs of the Bartle-Graves Theorem (cf. [3, Corollary 17.67]) as an application of Michael's Selection Theorem (Theorem 4.2), how the general problem may be translated into the language of set-valued analysis and selection theorems. A positive solution of the problem is equivalent to the existence of Lipschitz right inverses of a specific quotient map from a complete metric cone onto a Banach space (cf. Proposition 4.3).

This question, of the existence of Lipschitz right inverses of quotient maps, is discussed in Section 5 and is intimately related to the uniform– and Lipschitz geometry of Banach spaces and in particular the Lipschitz isomorphism problem: "Are Lipschitz isomorphic Banach spaces necessarily linearly isomorphic?" The existence of Lipschitz right inverses of quotient maps is a crucial part in constructing examples of Banach spaces that are Lipschitz isomorphic but not linearly isomorphic (cf. [1] and [15]). On the other hand, of particular interest are two previously known examples, one due to Lindenstrauss and Aharoni [1], and one due to Kalton [21], (in this paper: Examples 5.3 and 5.4) of quotient maps of Banach spaces that do not admit uniform– or Lipschitz right inverses. These two examples are the only such examples known to the author and provide some circumstantial evidence that the Lipschitz decomposition problem might be resolved negatively, perhaps by employing similar constructions. As such, these examples are presented in some detail. Apart from the mentioned relevance to the Lipschitz decomposition problem, the techniques employed in these examples are of independent interest. Both (arguably) employ some form of Ramsey theory.

In Section 6 we introduce Lipschitz function spaces and Banach space valued Lipschitz function spaces. Through an easy observation, we show that the Lipschitz decomposition property can be transferred to equivalent statements on such function spaces. It has long been known that scalar-valued Lipschitz function spaces are dual Banach spaces, with the Free Lipschitz space (also called the Arens-Eels space) as a predual [33]. Very recently in [17], dual-Banach-space valued Lipschitz function spaces were also shown to be dual Banach spaces, and furthermore, having a projective tensor product (with the Free Lipschitz space as tensor factor) as predual.

The observation that dual-Banach-space-valued Lipschitz function spaces have projective tensor products as preduals, connects the Lipschitz decomposition problem to tensor products and the geometric duality theory of ordered Banach spaces. In Section 7 we show that the Lipschitz decomposition property for dual Banach spaces transfers in general to an equivalent statements regarding the geometry of the projective tensor cone in a projective tensor product (having the Free Lipschitz space as tensor factor). This raises further questions about the structure of projective tensor cones in projective tensor products having a Free Lipschitz space as tensor factor, and for general projective tensor products of ordered Banach spaces, which are of relevance to the Lipschitz decomposition problem.

2. Preliminaries

We will assume that all vector spaces are over $\mathbb{R}$. Let V be a vector space. A set $C \subseteq V$ will be called a *cone* if both $C + C \subseteq C$ and $\lambda C \subseteq C$ hold for all $\lambda \geq 0$. A standard exercise establishes a bijection between cones and translation– and positive homogeneous pre-orders on V, cf. [3, Section 1.1]. We will say that C is *generating in* V, if $V = C - C$. For a subset $S \subseteq V$ we define the *conical span* of S, denoted cspan$S$, as the set of all elements of the form $\sum_{j=1}^{n} \lambda_j s_j$ for $n \in \mathbb{N}$ and for all $j \in \{1, \dots, n\}$, having $\lambda_j \geq 0$ and $s_j \in S$. For a topological vector space W with Hausdorff topology τ, we denote the topological dual of W by W^*, or $(W, \tau)^*$ if confusion may arise. The closure of a set $S \subseteq W$ will be denoted by $\overline{S}$, or $\overline{S}^{\tau}$ if confusion may arise. If W is a normed space with norm $\|\cdot\|$, we will use the symbol $\|\cdot\|$ as stand in for the norm-topology. For a cone $C \subseteq W$, we define the dual cone $C^* := \{\phi \in W^* \mid \forall c \in C,\ \phi(c) \geq 0\}$. Let X and Y be Banach spaces. Unless indicated otherwise, Banach spaces are assumed to be endowed with their norm topology. We denote the open unit ball, closed unit ball and unit sphere of X respectively by $\mathbb{B}_X$, $\mathbf{B}_X$ and $\mathbf{S}_X$. The bounded linear operators from X to Y will be denoted $B(X, Y)$ and endowed with the usual operator norm.

For (M, d) a metric space and X a Banach space, a function $f : M \to X$ is will be said to be *Lipschitz* if there exists some $K > 0$, so that for all $a, b \in M$, $\|f(a) - f(b)\| \leq K d(a, b)$. For a Lipschitz function $f : M \to X$, we define the *Lipschitz constant of* f by

$$L(f) := \inf \{K \mid \forall a, b \in M,\ \|f(a) - f(b)\| \leq K d(a, b)\}.$$

3. The trivial solutions

We point out the trivial solutions to the Lipschitz decomposition problem.

Proposition 3.1. *Every Banach lattice has the Lipschitz decomposition property.*

Proof. An easy exercise will show that the canonical maps $(\cdot)^{\pm} : X \to X$ defined by $x^+ := x \vee 0$ and $x^- := (-x) \vee 0$ are Lipschitz [32, Theorem II.5.2]. $\square$

Proposition 3.2. *For X a Banach space and $C \subseteq X$ a closed generating cone, if there exists some $u \in C$ so that for every $x \in X$ there exists some $\lambda \geq 0$ so that $x \in (-\lambda u + C) \cap (\lambda u - C)$, then X has the Lipschitz decomposition property.*

Proof. This follows from the fact that order units are interior points of C [4, Theorem 2.8]. Since $X = \bigcup_{n \in \mathbb{N}} (-nu + C) \cap (nu - C)$, by the Baire Category Theorem, the order interval $(-u + C) \cap (u - C)$ has a non-empty interior. Let $\alpha > 0$ and $v \in X$ be such that

$$2^{-1} \alpha \mathbf{S}_X \subseteq \alpha \mathbb{B}_X \subseteq (v - u + C) \cap (v + u - C) \subseteq (v + u - C).$$

Set $w := 2\alpha^{-1}(v + u)$, and since $0 \in \mathbb{B}_X$ we have $w \in C$. For every $x \in \mathbf{S}_X$, we have $x \in w - C$ and hence define $x^+ := w \in C$ and $x^- := w - x \in C$. Clearly $x = x^+ - x^-$. Furthermore, it is easily seen that the maps $x \mapsto x^{\pm}$ are Lipschitz on $\mathbf{S}_X$. By applying the reverse triangle inequality, the positive homogeneous extensions defined as $X \ni x \mapsto \|x\| (x / \|x\|)^{\pm}$ can be seen to be Lipschitz on all of X. $\square$

In particular, the previous result shows that all finite dimensional spaces ordered by closed generating cones have the Lipschitz decomposition property.

4. Set-valued analysis and selection theorems

A naive approach to solving the Lipschitz decomposition problem positively in general would be to mimic the proof of the following theorem.

Theorem 4.1. *Let X be a Banach space and $C \subseteq X$ a closed cone. The following are equivalent:*

(1) *The cone C is generating in X.*

(2) *There exists a constant $\alpha > 0$ so that, for every $x \in X$, there exist $a, b \in C$ so that $x = a - b$ and $\|a\| + \|b\| \leq \alpha \|x\|$.*

(3) *There exists a constant $\alpha > 0$ and continuous positively homogeneous functions $(\cdot)^{\pm} : X \to C$ so that $x = x^+ - x^-$ and $\|x^+\| + \|x^-\| \leq \alpha \|x\|$.*

(4) *For every topological space T, the closed cone of all continuous functions on T taking values in C is generating in the Banach space $C(T, X)$ of all continuous X-valued functions on T with the uniform norm.*

Proof outline. It is clear that (4) $\Leftrightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). That (1) implies (4) is quite remarkable. The implication (1) $\Rightarrow$ (2) is the Klee-Andô Theorem [23, (3.2)], [5, Lemma 3], which is in essence a generalization of the usual open mapping theorem for Banach spaces, cf. [10][26]. That (2) $\Rightarrow$ (3) follows from Michael's Selection Theorem (stated below) [10, Corollary 4.2, Theorem 4.5], [26, Corollary 5.6]. $\square$

Theorem 4.2 (Michael's Selection Theorem)**.** [3, Theorem 17.66] *Let P be a paracompact space and X a Banach space. If the set-valued map $\Phi : P \to 2^X$ is closed–, convex–, and non-empty-valued and is lower hemi-continuous, then Φ has a continuous selection, i.e., there exists a continuous function $f : P \to X$ so that $f(a) \in \Phi(a)$ for all $a \in P$.*

The approach to proving (2) $\Rightarrow$ (3) in Theorem 4.1 is to consider $C \times C$ as a subset of the ℓ^1-direct sum $X \oplus X$ and the continuous additive and positively homogeneous surjection $\Sigma : C \times C \to X$ defined as $\Sigma(c, d) := c - d$ for $c, d \in C$. The set valued map $x \mapsto \Sigma^{-1}\{x\}$ can then be shown to satisfy the hypothesis of Michael's Selection Theorem. This idea is directly related to modern proofs of the Bartle-Graves Theorem, stating that continuous linear surjections (read quotient maps) between Banach spaces always have continuous (not necessarily linear) right inverses [3, Corollary 17.67].

The Lipschitz decomposition problem can then easily be rephrased as "Does the map Σ always have a Lipschitz right inverse?":

Proposition 4.3. *Let X be a Banach space, ordered by a closed generating cone $C \subseteq X$. The following are equivalent:*

(1) *The space X has the Lipschitz decomposition property.*

(2) *The continuous additive positively homogeneous map $\Sigma : C \times C \to X$, as defined above, has a Lipschitz right inverse.*

(3) *The set-valued map $X \ni x \mapsto \Sigma^{-1}\{x\}$ has a Lipschitz selection.*

We note here that the surjection $\Sigma : C \times C \to X$ is a in essence quotient map from the complete metric cone[1] $C \times C$ onto X, prompting the question "When do quotients of complete metric cones onto Banach spaces have Lipschitz right inverses?" (Not always, cf. Remark 5.5 below). Since every Banach space is a complete metric cone in its own right, this is a more general question than "When do quotient maps from Banach spaces onto Banach spaces have Lipschitz right inverses?" (Also, not always, cf. Examples 5.3 and 5.4 below).

One seemingly reasonable (but doomed) approach to solving the Lipschitz decomposition problem positively, would be to attempt to prove a Lipschitz version of Michael's Selection Theorem by replacing the word "paracompact" with "metric space" and making suitable adjustments. An observation that lends support to this idea is the appealing Lipschitz-like properties possessed by set valued maps like $x \mapsto \Sigma^{-1}\{x\}$ (cf. [25]). It is however known that no general Lipschitz version of Michael's Selection can exist [37], but it is possible to do better than mere continuity, but only negligibly [25, Corollary 4.5]. The existence of a general Lipschitz version of Michael's Selection would contradict Examples 5.3 and 5.4 presented in the next section. Hence, if this approach is to be successful in resolving the Lipschitz decomposition problem positively, then it *must* leverage specific properties of the map Σ as defined above.

5. Non-linear geometry of Banach spaces and Ramsey theory

In this section we indicate some of the connections of the Lipschitz decomposition problem to the the nonlinear geometry of Banach spaces. This is a vast and highly active research subject and we will restrict ourselves to material relevant to the Lipschitz decomposition problem. We refer the reader to Lindenstrauss and Benyamini's book [8], and Chapter 14 of the recent second edition of Albiac and Kalton's book [2] for a more thorough treatment of the subject.

The previous section reduced the Lipschitz decomposition problem to the following question "When do quotient maps from complete metric cones/Banach spaces onto Banach spaces have Lipschitz right inverses?" This question is closely related to the Lipschitz/uniform isomorphism problem for Banach spaces and its partial resolution:

Problem 5.1. Let X and Y be Banach spaces and $A : X \to Y$ a non-linear Lipschitz (uniformly continuous) bijection with Lipschitz (uniformly continuous) inverse. Are X and Y necessarily linearly isomorphic?

There exist Banach spaces (like ℓ^p-spaces [20]) for which the above question is answered affirmatively. However, examples exist of **non-separable** Banach spaces that are Lipschitz isomorphic, but not linearly isomorphic. Currently, the only known method for constructing such spaces, revolves around constructing a space X with closed non-complemented subspace E so that X/E is non-separable and the quotient map $q : X \to X/E$ has a Lipschitz right inverse. The first such example was constructed by Lindenstrauss and Aharoni in [1] and the argument is closely related to that of showing the non-complementability of c_0 in ℓ^∞ (see [35]). In a more systematic fashion, Kalton and Godefroy used this method to show that every weakly compactly generated non-separable Banach space will give rise to a

[1]For our current purpose a closed cone inside a Banach space is sufficient. See the more general definitions: [10, Definitions 2.2 and 2.3].

pair of Lipschitz isomorphic Banach spaces that are not linearly isomorphic [15, Corollary 4.4].

At present it is not not known whether there exist separable Banach spaces that are Lipschitz isomorphic but not linearly isomorphic. By proving the following theorem, Kalton and Godefroy also eliminated the method described above in attacking in the problem in the separable case:

Theorem 5.2. [15, Corollary 3.2] *Let X be a Banach space with $E \subseteq X$ a closed subspace with X/E separable. Then the quotient map $q : X \to X/E$ has a Lipschitz right inverse if and only if E is complemented in X.*

Keeping in mind the relevance of the existence/non-existence of Lipschitz right inverses to quotient maps, as discussed in the previous section, below we present two examples of linear quotient maps that do not have admit Lipschitz right inverses (or even uniformly continuous right inverses). Although interesting in their own right, these examples provide some circumstantial evidence toward a negative solution to the Lipschitz decomposition problem. We therefore present these examples in quite some detail, but, in the interest of flow, will favor rather informal language to describe some of the most technical details.

Example 5.4 below, due to Kalton, relies on a Ramsey-type graph coloring theorem. Arguably Example 5.3 below, due to Lindenstrauss and Aharoni, also relies on Ramsey theoretic ideas in a broader sense of the term: For some fixed notion of regularity and some large collection of objects, for any choice of object made from the large collection of objects, there necessarily exists a related object (often sub-object) with the mentioned notion of regularity.

We begin with Aharoni and Lindenstrauss' example. Let $D_{\mathbb{Q}}[0,1]$ be the càdlàg space of all bounded $\mathbb{R}$-valued functions on $[0,1]$ with the uniform norm that only admit discontinuities at rational points in $[0,1]$, are right continuous everywhere, and with left limits existing everywhere on $[0,1]$. We note that for every $\varepsilon > 0$, an element in $f \in D_{\mathbb{Q}}[0,1]$ can have only finitely many discontinuities larger than ε, else, by the Bolzano-Weierstrass Theorem, there would exist a point where f is not right continuous. Therefore, with $C[0,1]$ denoting the closed subspace of all continuous $\mathbb{R}$-valued functions on $[0,1]$, the quotient $D_{\mathbb{Q}}[0,1]/C[0,1]$ is isomorphic to $c_0([0,1] \cap \mathbb{Q})$, with the coordinates of $q(f) \in c_0([0,1] \cap \mathbb{Q})$ measuring (half) the size of the discontinuities of $f \in D_{\mathbb{Q}}[0,1]$.

Example 5.3 (Aharoni-Lindenstrauss)**.** [1][8, Example 1.2] The quotient map $q : D_{\mathbb{Q}}[0,1] \to c_0([0,1] \cap \mathbb{Q})$ does not admit a Lipschitz right inverse.

Sketch of proof. Two Ramsey theoretic ideas are employed; both are straightforward:

The first Ramsey Thoeretic idea is: For every $f \in D_{\mathbb{Q}}[0,1]$, around every irrational number in $[0,1]$ there exists an open set U on which f "varies very little" and necessarily has only "small" discontinuities on U.

The second Ramsey Theoretic idea is: For distinct elements $x, y \in c_0([0,1] \cap \mathbb{Q})$ and every non-empty open set $U \subseteq [0,1]$ and $\alpha \in (0,1)$, there exists some $r \in \mathbb{Q} \cap U$ so that $\|x - y \pm \alpha \|x - y\| e_r\| = \|x - y\|$. I.e., no matter the open set U, there exists a rational number in U where a rather "large" perturbation of $x - y$ does not affect the the norm.

Assuming the existence of a Lipschitz right inverse $\tau : c_0([0,1]\cap\mathbb{Q}) \to D_{\mathbb{Q}}[0,1]$ of the quotient q, for every $\varepsilon > 0$, there exist distinct elements $x, y \in c_0([0,1]\cap\mathbb{Q})$ so that $\|\tau(x) - \tau(y)\|_\infty > (L(\tau) - \varepsilon)\,\|x - y\|$. Hence there exists a point (which, by right continuity, we may assume to be irrational) in $[0,1]$ at which $\tau(x)$ and $\tau(y)$ take on values that are "far apart" (at least $(L(\tau) - \varepsilon)\,\|x - y\|$). By the first Ramsey Theoretic idea above, there exists open set U, on which both $\tau(x)$ and $\tau(y) \in D_{\mathbb{Q}}[0,1]$ "vary very little" on U, and by construction take on values that are "far apart" on U. In particular their average $2^{-1}(\tau(x) + \tau(y))$ also "varies very little" on U.

By the second Ramsey theoretic idea, there exists a rational number $r \in U \cap \mathbb{Q}$, which allows the construction of a metric midpoint $z \in c_0([0,1]\cap\mathbb{Q})$ through a "large" perturbation at r, so that $\|x - z\| = \|y - z\| = 2^{-1}\|x - y\|$. This necessarily implies that $\tau(z) \in D_{\mathbb{Q}}[0,1]$ has a "large" discontinuity at r.

At the same time, using the Lipschitz-ness of τ, we obtain both the following inequalities:

$$\|\tau(y) - \tau(z)\|_\infty \leq 2^{-1}L(\tau)\,\|x - y\|_\infty \quad \text{and} \quad \|\tau(x) - \tau(z)\|_\infty \leq 2^{-1}L(\tau)\,\|x - y\|_\infty .$$

However, on U, the functions $\tau(y)$ and $\tau(x)$ take on values that are "far apart", (at least a distance of $(L(\tau) - \varepsilon)\,\|x - y\|$). So the only way that the above two inequalities can hold is with $\tau(z)$ taking on values that are "very close"[2] (within $2^{-1}\varepsilon\,\|x - y\|$) to the average $2^{-1}(\tau(x) + \tau(y))$ on U. But since $2^{-1}(\tau(x) + \tau(y))$ "varies very little" on U, so does $\tau(z)$, which contradicts $\tau(z)$ having a "large" discontinuity at $r \in U$. $\square$

The next example, due to Kalton, proceeds through showing that every Banach lattice X whose unit ball embeds uniform continuously into ℓ^∞ with uniform continuous inverse, must necessarily have the property that every increasing transfinite sequence (meaning: indexed by the first uncountable ordinal) in X must eventually be constant (cf. Theorem 5.7). Kalton called this property *The Monotone Transfinite Sequence Property*. This is proven through a Lipschitz-Ramsey Theorem (stated as Theorem 5.6 below), its proof proceeding through an argument in ordinal combinatorics. Once this done, by showing that ℓ^∞/c_0 does not have The Monotone Transfinite Sequence Property [21, Theorem 4.2] (again using similar methods to usual techniques employed in showing the non-complementibility of c_0 in ℓ^∞), one arrives at:

Example 5.4 (Kalton)**.** [21, Theorem 4.2] The quotient map $q : \ell^\infty \to \ell^\infty/c_0$ has no uniformly continuous right inverse.

Remark 5.5. This also provides an example of a continuous additive positively homogenous surjection from a complete metric cone onto a Banach space that has no uniform continuous right inverse. Since Σ as defined in the previous section is Lipschitz, the surjection $q \circ \Sigma : \ell^\infty_+ \times \ell^\infty_+ \to \ell^\infty/c_0$ cannot have a uniformly continuous right inverse, as that would contradict Example 5.4. We note however that the quotient ℓ^∞/c_0 is actually a Banach lattice by [32, Proposition II.5.4], and so does have the Lipschitz decomposition property.

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[2] Although very seldomly used in analysis, this is purely an observation about real numbers: For distinct real numbers $a, b \in \mathbb{R}$. With $K, \varepsilon > 0$ satisfying $0 < (K - \varepsilon) < |a - b| \leq K$. If $c \in \mathbb{R}$ satisfies $|a - c| \leq 2^{-1}K$ and $|b - c| \leq 2^{-1}K$, then $\left|c - 2^{-1}(a + b)\right| \leq 2^{-1}\varepsilon$.

We sketch the essentials of the proof of Kalton's Monotone Transfinite Sequence Theorem (Theorem 5.7). Let I be any set. For $n \in \mathbb{N}$, by $I^{[n]}$ we denote the set of all n-element subsets of I. Let Ω be the first uncountable ordinal. The set $\Omega^{[n]}$ is made into a graph, by defining the set of edges as all pairs of distinct elements $a, b \in \Omega^{[n]}$ which *interlace*: for which either $a_1 \leq b_1 \leq a_2 \leq b_2 \leq \dots a_n \leq b_n$ or $b_1 \leq a_1 \leq b_2 \leq \dots b_n \leq a_n$, (where we write $a = \{a_1, a_2, \dots, a_n\}$ with $a_1 < a_2 < \dots < a_n$). With the least path length metric, the graph $\Omega^{[n]}$ is a bounded metric space with diameter n.

Theorem 5.6 (Kalton's Lipschitz-Ramsey Theorem)**.** [21, Theorem 3.6] *For any $n \in \mathbb{N}$, let $c : \Omega^{[n]} \to \ell^\infty$ be Lipschitz, with Lipschitz constant L. Then there exists some $\xi \in \ell^\infty$ and uncountable set $\Theta \subseteq \Omega$, so that, for all $a \in \Theta^{[n]}$, $\|c(a) - \xi\|_\infty \leq L/2$.*

Translating into coloring language, one may interpret the previous result as follows: Coloring the vertices of the graph $\Omega^{[n]}$ by elements in ℓ^∞ through a Lipschitz map $c : \Omega^{[n]} \to \ell^\infty$ with Lipschitz constant L, in general any subset of vertices $A \subseteq \Omega^{[n]}$ with diameter n has a *color diameter,* $\operatorname{diam} c(A)$, of at most nL (read: "can have rather large variations in color: of the order of the diameter of the graph $\Omega^{[n]}$"). However, by the previous theorem, one is assured of an uncountable set of ordinals $\Theta \subseteq \Omega$, so that the color diameter, $\operatorname{diam} c(\Theta^{[n]})$, of $\Theta^{[n]} \subseteq \Omega^{[n]}$ is at most the Lipschitz constant L of the coloring c (read: "has not so large variations in color, and is independent of the diameter of $\Theta^{[n]}$"). It is important to notice here that $\Theta^{[n]}$ has diameter n, since $\Theta \subseteq \Omega$ is an uncountable set of countable ordinals.

Theorem 5.7 (Kalton's Monotone Transfinite Sequence Theorem)**.** [21, Theorem 4.1] *If the closed unit ball $\mathbf{B}_X$ of a Banach lattice X embeds uniform continuously (with uniformly continuous inverse) into ℓ^∞, then X has The Monotone Transfinite Sequence Property.*

Sketch of proof. Let $f : \mathbf{B}_X \to \ell^\infty$ be a uniformly continuous embedding with uniformly continuous inverse $g : f(\mathbf{B}_X) \to \mathbf{B}_X$. Let $(x_\mu)_{\mu \in \Omega} \subseteq X$ be an increasing transfinite sequence. Since transfinite increasing sequences are always bounded, we may assume $(x_\mu)_{\mu \in \Omega} \subseteq \mathbf{B}_X$ without loss.

Let $n \in \mathbb{N}$ be arbitrary and define the averaging map $A_n : \Omega^{[n]} \to \mathbf{B}_X$ by $A_n(\{a_1, \dots a_n\}) := \frac{1}{n}\sum_{j=1}^n x_{a_j}$. The coloring map $c_n := f \circ A_n : \Omega^{[n]} \to \ell^\infty$ can be verified to be Lipschitz (we omit the details), and we denote its Lipschitz constant by L_n. Furthermore it can be verified that the sequence (L_n) converges to zero as $n \to \infty$. By Kalton's Lipschitz-Ramsey Theorem (Theorem 5.6), there exists some uncountable set of ordinals $\Theta_n \subseteq \Omega$, so that, for all $a, b \in \Theta_n^{[n]}$ we have $\|c_n(a) - c_n(b)\|_\infty \leq L_n$.

Now, for any $n \in \mathbb{N}$, fix any $a \in \Theta_n^{[n]}$. There exists $\mu_n \in \Omega$ with $\max a < \mu_n$. For all $\nu \in \Omega$ satisfying $\max a < \mu_n < \nu$, since Θ_n is an uncountable set, there exists $b \in \Theta_n^{[n]}$ so that $\max a < \mu_n < \nu < \min b$. Using the monotonicity of the norm of the Banach lattice X, we obtain

$$\|x_\nu - x_{\mu_n}\| \leq \left\| \frac{1}{n}\sum_{j=1}^n x_{b_j} - \frac{1}{n}\sum_{j=1}^n x_{a_j} \right\| = \|g \circ c_n(b) - g \circ c_n(a)\|.$$

However $\sup_{a,b\in\Theta_n^{[n]}} \|c_n(a) - c_n(b)\|_\infty \leq L_n \to 0$ as $n \to \infty$, and, since g is uniformly continuous, we have $\sup_{a,b\in\Theta_n^{[n]}} \|g \circ c_n(b) - g \circ c_n(a)\| \to 0$ as $n \to \infty$. Taking $\mu_0 := \sup_{n\in\mathbb{N}} \mu_n \in \Omega$, we see, for all $\nu \in \Omega$ with $\mu_0 < \nu$, again by monotonicity of the norm of X, that $\|x_\nu - x_{\mu_0}\| \leq \|x_\nu - x_{\mu_n}\| \to 0$ as $n \to \infty$. Therefore $\|x_\nu - x_{\mu_0}\| = 0$ for all $\mu_0 < \nu$, and hence the increasing transfinite sequence $(x_\mu)_{\mu\in\Omega}$ is eventually constant. □

Remark 5.8. We point out that only the monotonicity of the norm of the Banach lattice was used in the above result. By a straightforward adaptation of the proof, the same is true if we replace the Banach lattice X by an ordered Banach space X for which there exists a constant $\alpha > 0$, so that for all $a, b \in X$, the inequality $0 \leq a \leq b$ implies $\|a\| \leq \alpha \|b\|$.

6. Lipschitz function spaces

We begin with some basic definitions.

Let X be a Banach space and (M, d) be a metric space. We will assume M to be pointed with some point in M fixed and labeled as $0_M \in M$. Viewing X as a metric space, we will always take $0_X := 0 \in X$. The *X-valued Lipschitz space* $\mathrm{Lip}_0(M, X)$ is defined as the set of all Lipschitz functions $f : M \to X$ satisfying $f(0_M) = 0$. The map $L(\cdot)$, as defined in Section 2, is a norm on $\mathrm{Lip}_0(M, X)$. A standard exercise will show $\mathrm{Lip}_0(M, X)$ endowed with $L(\cdot)$ is again Banach space. As usual, if $X = \mathbb{R}$, then we will write $\mathrm{Lip}_0(M)$ for $\mathrm{Lip}_0(M, \mathbb{R})$. For a closed cone $C \subseteq X$ we will define $\mathrm{Lip}_0(M, C) := \mathrm{Lip}_0(M, X) \cap C^M$. This cone is certainly non-empty as for every $c \in C$, the map $M \ni a \mapsto d(0_M, a)c$ is in $\mathrm{Lip}_0(M, C)$.

We refer the reader to Weaver's book [33] for a treatment of scalar-valued Lipschitz function spaces. Of note is the order structure of $\mathrm{Lip}_0(M)$ when ordered by the *standard cone* $\mathrm{Lip}_0(M)_+ := \{f \in \mathrm{Lip}_0(M) \mid \forall a \in M,\ f(a) \geq 0\}$. The space $\mathrm{Lip}_0(M)$ is then lattice ordered [33, Proposition 1.5.5]. However, depending on the structure of the metric space M, the space $\mathrm{Lip}_0(M)$ need not be a Banach lattice in general, as order intervals need not be norm-bounded. E.g., For any $\alpha > 0$, one can easily construct a (say piecewise affine) function $g \in \mathrm{Lip}_0([0, 1])$ so that, with $f(x) := x$, the function g satisfies $0 \leq g \leq f$, while $L(f) < \alpha L(g)$.

The following easy observation connects the Lipschitz decomposition property to the structure of Lipschitz function spaces:

Theorem 6.1. *Let X be a Banach space, ordered by a closed generating cone $C \subseteq X$. The following are equivalent:*

(1) *The space X has the Lipschitz decomposition property.*
(2) *The cone $\mathrm{Lip}_0(X, C)$ is generating in $\mathrm{Lip}_0(X, X)$.*
(3) *The cone $\mathrm{Lip}_0(\mathbf{S}_X \cup \{0\}, C)$ is generating in $\mathrm{Lip}_0(\mathbf{S}_X \cup \{0\}, X)$.*
(4) *For every metric space M, the cone $\mathrm{Lip}_0(M, C)$ is generating in $\mathrm{Lip}_0(M, X)$.*

Proof. It is clear that (4) implies (1), (2) and (3).

We prove (1) implies (4). Let $(\cdot)^\pm : X \to C$ be Lipschitz maps such that $x = x^+ - x^-$ for all $x \in X$. By considering the positive homogeneous extension of the restrictions $(\cdot)^\pm|_{\mathbf{S}_X}$, we may assume that $0^\pm = 0$. Hence, for any $f \in \mathrm{Lip}_0(M, X)$, by defining $f^\pm \in \mathrm{Lip}_0(M, C)$ as $f^\pm(a) := f(a)^\pm$ for all $a \in M$, we obtain $f^\pm \in \mathrm{Lip}_0(M, C)$ and $f = f^+ - f^-$. □

A classical result in the scalar valued case, due to de Leeuw, is that $\mathrm{Lip}_0(M)$ is always a dual Banach space (first proven in [11] for $M = \mathbb{R}$. See [33, Chapter 2] for a general treatment). Very recently, Weaver showed that, under mild conditions on the metric space M, preduals of $\mathrm{Lip}_0(M)$ are unique up to isometric isomorphism [34].

One construction of a predual of $\mathrm{Lip}_0(M)$ by Kalton and Godefroy [15], is through the Dixmier-Ng Theorem [28, Theorem 1] as the closed linear span of the evaluation maps $\{\delta_a : a \in M\}$ in $\mathrm{Lip}_0(M)^*$, with norm inherited from $\mathrm{Lip}_0(M)^*$. Denoted by $F(M)$, this space is called the *Free Lipschitz space* (or often *Lipschitz free-space*). We define the *standard cone* in $F(M)$ as the norm-closure of $\mathrm{cspan}\,\{\delta_a \mid a \in M\}$ and denote it by $F(M)_+$. It is straightforward to verify that $F(M)_+^* = \mathrm{Lip}_0(M)_+$.

The name Free Lipschitz space is apt. The Free Lipschitz space mimics the universal property of the free group over a set of symbols of, from any function from the set of symbols into any group G, inducing a homomorphism from the free group into the group G. Here, from any Lipschitz map from M into a Banach space X, a linear map is induced, mapping the Free Lipschitz space $F(M)$ into the Banach space X. Explicitly:

Theorem 6.2. *Let M be a metric space. For every Banach space X and for every $f \in \mathrm{Lip}_0(M, X)$, there exists a unique bounded linear operator $T_f : F(M) \to X$ with $\|T_f\| = L(f)$ making the following diagram commute:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\delta_{(\cdot)}"'] & X \\ F(M) \arrow[ur, "T_f"'] & \end{tikzcd}$$

Furthermore, the map $f \mapsto T_f$ is an isometric isomorphism from $\mathrm{Lip}_0(M, X)$ *onto* $B(F(M), X)$.

The previous result is straightforward, and has been proven independently a number of times, cf. [30, Theorem 1], [33, Theorem 2.2.4], [22, Lemma 3.2].

An elementary verification will also show, if $C \subseteq X$ is a norm-closed cone, then the isometric isomorphism from $\mathrm{Lip}_0(M, X)$ onto $B(F(M), X)$ maps the cone $\mathrm{Lip}_0(M, C)$ onto the cone $\{T \in B(F(M), X) \mid TF(M)_+ \subseteq C\}$.

Vector-valued Lipschitz spaces has seen much recent developments. Of interest here is the observation by Guerrero, Lopez-Perez and Ruedo Zoca that, as in the scalar case, with X a Banach space, the space $\mathrm{Lip}_0(M, X^*)$ is always a dual Banach space, having a vector-valued version of the Free Lipschitz space as predual [17]. Explicitly, with $\delta_a \otimes x(f) := f(a)(x)$ for all $a \in M$, $x \in X$ and $f \in \mathrm{Lip}_0(M, X^*)$, the *$X$-valued Free Lipschitz space* $F(M, X)$ is defined as the closed linear span of $\{\delta_a \otimes x \mid a \in M,\ x \in X\}$ in $\mathrm{Lip}_0(M, X^*)^*$. Moreover, viewing $B(F(M), X^*)$ as the dual of the projective tensor product $F(M)\overline{\otimes}_\pi X$ [31, Section 2.2], the isometric isomorphism between the spaces $\mathrm{Lip}_0(M, X^*)$ and $B(F(M), X^*)$ from Theorem 6.2 is wk*-to-wk* continuous, so that by [14, Exercise 3.60], the space $F(M, X)$ is isometrically isomorphic to the projective tensor product $F(M)\overline{\otimes}_\pi X$, cf. [17, Proposition 1.1].

Through the geometric duality theory of cones in Banach spaces, this last observation allows the for the Lipschitz decomposition property to be characterized by the geometry of a polar cone in projective tensor products with a Free Lipschitz space as tensor factor. This is discussed in the next section.

7. Duality of ordered Banach spaces and tensor products

We introduce the following preliminaries to describe the geometric duality theory of ordered Banach spaces. Let V and W be vector spaces and $\langle\cdot\mid\cdot\rangle : V\times W\to\mathbb{R}$ a bilinear map. If $\{\,\langle\cdot\mid w\rangle \mid w\in W\}$ and $\{\,\langle v\mid\cdot\rangle\mid v\in V\}$ separate the points of V and W respectively, we call (V,W) a *dual pair* and $\langle\cdot\mid\cdot\rangle$ a *duality*. By $\sigma(V,W)$ and $\sigma(W,V)$ we denote the smallest topologies for which all elements of $\{\,\langle\cdot\mid w\rangle\mid w\in W\}$ and $\{\,\langle v\mid\cdot\rangle\mid v\in V\}$ continuous functionals respectively. For $A\subseteq V$ and $B\subseteq W$ we define the *one-sided polars* of A *and* B *with respect to the dual pair* (V,W) by $A^{\odot} := \{w\in W\mid \forall a\in A,\ \langle a\mid w\rangle\le 1\}$ and $B^{\odot} := \{v\in V\mid \forall b\in B,\ \langle v\mid b\rangle\le 1\}$ respectively.

The following lemma forms the basis of geometric duality theory of ordered Banach spaces. Its proof is an elementary exercise in applications of the Hahn-Banach Spearation Theorem.

Lemma 7.1. [24, Lemmas 2.1 and 2.4]. *Let (V,W) be a dual pair. Let $A,B\subseteq V$ and $C\subseteq V$ a cone and $\{A_i\}_{i\in I}$ a collection of nonempty subsets of V. Then*

(1) *The set $A^{\odot}$ contains zero, is convex and is $\sigma(W,V)$-closed in W.*
(2) *If $A\subseteq B$, then $B^{\odot}\subseteq A^{\odot}$.*
(3) *For $\lambda>0$, we have $(\lambda A)^{\odot}=\lambda^{-1}(A^{\odot})$.*
(4) *The set $(\bigcup_{i\in I}A_i)^{\odot}$ equals $\bigcap_{i\in I}A_i^{\odot}$.*
(5) *The set $A^{\odot\odot}$ equals the $\sigma(V,W)$-closed convex hull of $\{0\}\cup A$.*
(6) *If, for every $i\in I$, the set A_i is $\sigma(V,W)$-closed, convex and contains zero, then $\left(\bigcap_{i\in I}A_i\right)^{\odot}$ equals the $\sigma(W,V)$-closed convex hull of $\bigcup_{i\in I}A_i^{\odot}$.*
(7) *The set $C^{\odot}\subseteq W$ is a $\sigma(W,V)$-closed cone and $C^{\odot}=-C^*$.*
(8) *If A is convex and contains zero, then $(A\cap C)^{\odot}\subseteq\overline{(A^{\odot}+C^{\odot})}^{\sigma(W,V)}$.*
(9) *If A is $\sigma(V,W)$-closed, convex and contains zero and C is $\sigma(V,W)$-closed, then $(A\cap C)^{\odot}=\overline{(A^{\odot}+C^{\odot})}^{\sigma(W,V)}$.*
(10) *If A is convex and contains zero, then $(A+C)^{\odot}=A^{\odot}\cap C^{\odot}$.*
(11) *If A is $\sigma(V,W)$-closed, convex and contains zero, then $A=\bigcap_{\lambda>1}\lambda A$.*

Typical of the geometric duality theory of ordered Banach spaces are theorems like Theorem 7.2, below, which relate the geometry of a cone with that of its dual cone. This result, and results that are closely related to it, are well-known and have long been studied by many authors. See for example the following list of references, which is not claimed to be exhaustive: [16], [13], [9], [27], [19], [6], [7], [29].

The oldest references that the author is aware of are that of Grosberg and Krein from 1939 [16], and of Ellis from 1964 [13, Theorem 8] which assert the following:

Theorem 7.2. *Let $\alpha>0$, X be a Banach space and $C\subseteq X$ a norm closed cone.*

(1) *The following are equivalent:*
   (a) *For every $v,w\in X$, if $x\in(v+C)\cap(w-C)$, then $\|x\|\le\alpha\max\{\|v\|,\|w\|\}$.*
   (b) *For every $\eta\in X^*$, there exist $\psi,\phi\in C^*$ so that $\eta=\psi-\phi$ and $\|\phi\|+\|\psi\|\le\alpha\|\eta\|$.*
(2) *The following are equivalent:*

(a) *For every* $\varepsilon > 0$ *and* $x \in X$, *there exist* $a, b \in C$ *so that* $x = a - b$ *and* $\|a\| + \|b\| \leq (\alpha + \varepsilon)\|x\|$.

(b) *For every* $\phi, \psi \in X^*$, *if* $\eta \in (\phi + C^*) \cap (\psi - C^*)$, *then* $\|\eta\| \leq \alpha \max\{\|\phi\|, \|\psi\|\}$.

As indicated by the list of references above, Theorem 7.2 and similar results have been rediscovered quite a few times, and the terminology employed by different authors tend to be quite fragmented. Also proofs of results like Theorem 7.2, analogues to the implication (2)(b) to (2)(a) tend to become somewhat involved, relying on technical results (see for example the proof of [13, Theorem 8]; proof of [27, Proposition 6]; Lemmas 1.1.3 and 1.1.5 and Theorem 1.1.4 from [7], and Proposition 1.3.1, Lemma 1.3.2 and Theorem 2.1.5 from [6]). These results do require some investment to verify and understand. We will prove Theorem 7.2, as a special case of a result from [24]. Our reason for providing this sketch is that we hope, in our presentation, to abstract out the main features to the more casual reader and make clear that the reason for these mentioned technicalities are, when boiled down, the "wanting to erase a weak-closure" in a certain set inclusion (see the proof of Theorem 7.2 below). Crucial in performing this "closure erasure" successfully are Lemmas 7.3 and 7.4 below.

Before stating these lemmas, we introduce some further terminology from Jameson [19, Appendix]: For a topological vector space (V, τ) we say a set $A \subseteq V$ is τ-*cs-compact* if, for any sequences $(x_n) \subseteq A$ and $(\lambda_n) \subseteq [0, 1]$ satisfying $\sum_{n=1}^{\infty} \lambda_n = 1$, the series $\sum_{j=1}^{\infty} \lambda_n x_n$ converges with respect to the τ-topology and its limit is an element of A. We say a set $A \subseteq V$ is τ-*cs-closed* if for any sequences $(x_n) \subseteq A$ and $(\lambda_n) \subseteq [0, 1]$ satisfying $\sum_{n=1}^{\infty} \lambda_n = 1$, **if** the series $\sum_{j=1}^{\infty} \lambda_n x_n$ converges with respect to the τ-topology, **then** its limit is an element of A.

The first lemma is a characterization of Banach spaces as having exactly norm-cs-compact closed unit balls and proof is an easy exercise in using the absolute convergence characterization of Banach spaces, cf. [14, Lemma 1.22].

Lemma 7.3. *A normed space is a Banach space if and only if its closed unit ball is norm-cs-compact.*

The second lemma is the main tool in "closure erasure". We note explicitly that the reader should notice in Lemma 7.4 the erasure of the weak-closure by paying an arbitrarily small price in scaling up the set A.

Lemma 7.4. *Let* X *be a Banach space,* $D \subseteq X$ *and* $A \subseteq X$ *a* $\sigma(X, X^*)$-*cs-closed set. Let* $G \subseteq X$ *be a weak-cs-compact set. If* $G \subseteq D \subseteq \overline{A}^{\sigma(X,X^*)}$ *and, for all* $r > 0$ *and* $d \in D$, *we have* $(d + rG) \cap A \neq \emptyset$, *then* $D \subseteq \lambda A$ *for all* $\lambda > 1$.

To the author's knowledge, this lemma is originally due to Batty and Robinson [7, Lemma 1.1.3], and proven in a slightly more general form in [24, Lemma 2.3].

We now turn to a proof Theorem 7.2, here adapted as a special case from [24, Theorem 3.4]. The main idea is to reformulate the statements into equivalent statements involving specific set inclusions. The proof then becomes an exercise in the one-sided polar calculus (cf. Lemma 7.1) and "closure erasure" by applying Lemma 7.4.

Proof of Theorem 7.2. With "$\oplus_p$" denoting the usual ℓ^p-direct sum for $1 \leq p \leq \infty$, we consider the canonical duality of the spaces $X \oplus_\infty X$ and $X^* \oplus_1 X^*$. We define the following sets

$$\begin{aligned}\Xi_\infty &:= \{(a,b)\in X\oplus_\infty X \mid a=b\}\,,\\ \Xi_1 &:= \Xi_\infty \cap \mathbf{B}_{X\oplus_\infty X}\,,\\ C\oplus_\infty(-C) &:= \{(a,b)\in X\oplus_\infty X \mid -b,a\in C\}\,,\end{aligned}$$

and

$$\begin{aligned}\Sigma_1 &:= \{(\phi,\psi)\in X^*\oplus_1 X^* \mid \|\phi+\psi\|\le 1\}\,,\\ \Sigma_0 &:= \{(\phi,\psi)\in X^*\oplus_1 X^* \mid \phi+\psi=0\}\,,\\ (-C^*)\oplus_1 C^* &:= \{(\phi,\psi)\in X\oplus_1 X \mid -\phi,\psi\in C^*\}\,.\end{aligned}$$

We prove (1). Assume (1)(a). This is equivalent to

$$(\mathbf{B}_{X\oplus_\infty X} + C\oplus_\infty(-C))\cap \Xi_\infty \subseteq \alpha\Xi_1.$$

With $\mathrm{wk}:=\sigma(X\oplus_\infty X, X^*\oplus_1 X^*)$, the above inclusion implies[3]

$$\overline{(\mathbf{B}_{X\oplus_\infty X} + C\oplus_\infty(-C))}^{\mathrm{wk}}\cap \Xi_\infty \subseteq \alpha\Xi_1.$$

Now, by taking one-sided polars (cf. Lemma 7.1), with $\mathrm{wk}^*:=\sigma(X^*\oplus_1 X^*, X\oplus_\infty X)$, we obtain

$$\begin{aligned}\Sigma_1 &\subseteq \overline{\alpha\left(\overline{\mathbf{B}_{X\oplus_\infty X} + C\oplus_\infty(-C)}^{\mathrm{wk}}\right)^{\odot} + \Sigma_0}^{\mathrm{wk}^*}\\ &\subseteq \overline{\alpha(\mathbf{B}_{X\oplus_\infty X} + C\oplus_\infty(-C))^{\odot} + \Sigma_0}^{\mathrm{wk}^*}\\ &= \overline{(\alpha\mathbf{B}_{X^*\oplus_1 X^*}\cap(-C^*)\oplus_1 C^*) + \Sigma_0}^{\mathrm{wk}^*}.\end{aligned}$$

Since $\mathbf{B}_{X^*\oplus_1 X^*}$ is wk*-compact, the set $(\alpha\mathbf{B}_{X^*\oplus_1 X^*}\cap(-C^*)\oplus_1 C^*)+\Sigma_0$ can be seen to already be wk*-closed, so that taking the closure is redundant and we erase it. Hence $\Sigma_1\subseteq(\alpha\mathbf{B}_{X^*\oplus_1 X^*}\cap(-C^*)\oplus_1 C^*)+\Sigma_0$, which is equivalent to (1)(b).

Assume (1)(b), which is equivalent to $\Sigma_1\subseteq(\alpha\mathbf{B}_{X^*\oplus_1 X^*}\cap(-C^*)\oplus_1 C^*)+\Sigma_0$. Taking one-sided polars (cf. Lemma 7.1) with $\mathrm{wk}:=\sigma(X\oplus_\infty X, X^*\oplus_1 X^*)$, we obtain

$$(\mathbf{B}_{X\oplus_\infty X} + C\oplus_\infty(-C))\cap\Xi_\infty \subseteq \overline{\mathbf{B}_{X\oplus_\infty X} + C\oplus_\infty(-C)}^{\mathrm{wk}}\cap\Xi_\infty\subseteq\alpha\Xi_1,$$

which is equivalent to (1)(a).

We adjust the definitions of $\Sigma_{(\cdot)}$ and $\Xi_{(\cdot)}$ appropriately, and prove (2).

Assume (2)(a). This is equivalent to, for every $\varepsilon>0$, that $\Sigma_1\subseteq(\alpha+\varepsilon)\mathbf{B}_{X\oplus_1 X}\cap C\oplus_1(-C)+\Sigma_0$. Taking one-sided polars (cf. Lemma 7.1) with $\mathrm{wk}^*:=\sigma(X^*\oplus_\infty X^*, X\oplus_1 X)$, yields

$$(\mathbf{B}_{X^*\oplus_\infty X^*} + (-C^*)\oplus_\infty C^*)\cap\Xi_\infty\subseteq\overline{(\mathbf{B}_{X^*\oplus_\infty X^*} + (-C^*)\oplus_\infty C^*)}^{\mathrm{wk}^*}\cap\Xi_\infty$$

[3]Let $(x,x)\in\overline{(\mathbf{B}_{X\oplus_\infty X}+C\oplus_\infty(-C))}^{\|\cdot\|_\infty}\cap\Xi_\infty$. Then there exists sequences $((a_n,b_n))\subseteq\mathbf{B}_{X\oplus_\infty X}$ and $((c_n,-d_n))\subseteq C\oplus_\infty(-C)$ so that $(a_n,b_n)+(c_n,-d_n)\to(x,x)$ as $n\to\infty$. For every $n\in\mathbb{N}$, let $p_n:=(a_n+c_n)-(b_n-d_n)$ and consider the sequence $S:=((a_n,b_n+p_n)+(c_n,-d_n))=((a_n+c_n,a_n+c_n))\subseteq\Xi_\infty$. This sequence S converges to (x,x) and, since $p_n\to 0$ as $n\to\infty$, for every $\varepsilon>0$ the tail of S eventually lies in $((1+\varepsilon/\alpha)\mathbf{B}_{X\oplus_\infty X}+C\oplus_\infty(-C))\cap\Xi_\infty\subseteq(1+\varepsilon/\alpha)\alpha\Xi_1$, and hence $\|(x,x)\|_\infty\le(\alpha+\varepsilon)$. But this holds for every $\varepsilon>0$, so $\|(x,x)\|_\infty\le\alpha$ and therefore $(x,x)\in\alpha\Xi_1$. Because the wk-closure and $\|\cdot\|_\infty$-closure of convex sets coincide, we obtain $\overline{(\mathbf{B}_{X\oplus_\infty X}+C\oplus_\infty(-C))}^{\mathrm{wk}}\cap\Xi_\infty\subseteq\alpha\Xi_1$.

$$\subseteq \bigcap_{\varepsilon>0}(\alpha+\varepsilon)\Xi_1 = \alpha\Xi_1.$$

Conversely, assume (2)(b), which is equivalent to $(\mathbf{B}_{X^*\oplus_\infty X^*} + (-C^*)\oplus_\infty C^*)\cap \Xi_\infty \subseteq \alpha\Xi_1$. With $\mathrm{wk}^* := \sigma(X^*\oplus_\infty X^*, X\oplus_1 X)$, the set $\mathbf{B}_{X^*\oplus_\infty X^*}$ is wk*-compact, so that $(\mathbf{B}_{X^*\oplus_\infty X^*} + (-C^*)\oplus_\infty C^*)$ is wk*-closed. Taking one-sided polars (cf. Lemma 7.1) with $\mathrm{wk} := \sigma(X\oplus_1 X, X^*\oplus_\infty X^*)$, yields

$$\Sigma_1 \subseteq \overline{(\alpha\mathbf{B}_{X\oplus_1 X}\cap(C\oplus_1(-C)) + \Sigma_0}^{\mathrm{wk}}.$$

We note that since $\mathbf{B}_{X\oplus_1 X}$ is not necessarily wk-compact, the argument employed in the previous paragraph to erase the closure is unavailable here. By Lemma 7.3, $\mathbf{B}_{X\oplus_1 X}$ is norm-cs-compact, hence also wk-cs-compact. We note that the wk-closure and norm-closure of convex sets coincide, and that the sum $(\alpha\mathbf{B}_{X\oplus_1 X}\cap C\oplus_1(-C)) + \Sigma_0$ is wk-cs-closed, being the sum of a wk-cs-compact set and a wk-cs-compact set. Applying Lemma 7.4 to the inclusion below,

$$\mathbf{B}_{X\oplus_1 X} \subseteq \Sigma_1 \subseteq \overline{(\alpha\mathbf{B}_{X\oplus_1 X}\cap C\oplus_1(-C)) + \Sigma_0}^{\mathrm{wk}},$$

we obtain, for every $\varepsilon>0$ that $\Sigma_1 \subseteq (C\oplus_1(-C)\cap(\alpha+\varepsilon)\mathbf{B}_{X\oplus_1 X}) + \Sigma_0$. This is equivalent to (2)(a). □

Theorem 7.2 and the duality $(F(M)\overline{\otimes}_\pi X)^* = B(F(M), X^*) \simeq \mathrm{Lip}_0(M, X^*)$ observed in Section 6, suggests that it is possible to characterize the Lipschitz decomposition property in terms of the geometry of cones in projective tensor products with Free Lipschitz spaces. The remainder of the current section will be devoted to this.

Following Ryan's [31] and Wittstock's [36], we introduce the following terminology and notation for projective tensor products and projective tensor cones. For Banach spaces X and Y, we denote the projective tensor norm by π (cf. [31, Chapter 2]) and denote the projective tensor product of X and Y by $X\overline{\otimes}_\pi Y$. Let $C\subseteq X$ and $D\subseteq Y$ be norm-closed cones. We define the *projective tensor cone* $C\overline{\otimes}_\pi D$ as the norm closure of $\mathrm{cspan}\{c\otimes d \mid (c,d)\in C\times D\}\subseteq X\overline{\otimes}_\pi Y$. We will view $B(X,Y^*)$ as the dual of $X\overline{\otimes}_\pi Y$ through the duality $\langle \sum_{i=1}^\infty x_i\otimes\phi_i \mid T\rangle := \sum_{i=1}^\infty \phi_i(Tx_i)$ for $T\in B(X,Y^*)$ and $\sum_{i=1}^\infty x_i\otimes\phi_i \in X\overline{\otimes}_\pi Y$ (cf. [31, Section 2.2]). Using that $d\in D$ if and only if $\phi(d)\geq 0$ for all $\phi\in D^*$ [4, Theorem 2.13], it can be seen that $(C\overline{\otimes}_\pi D)^* = \{T\in B(X,Y^*) \mid TC\subseteq D^*\}$.

The following is an immediate consequence of Theorem 7.2 and the observation $(X\overline{\otimes}_\pi Y)^* = B(X,Y^*)$.

Corollary 7.5. *Let $\alpha>0$ and X and Y be Banach spaces and $C\subseteq X$ and $D\subseteq Y$ norm-closed cones. The following are equivalent:*

(a) *For $v,w\in X\overline{\otimes}_\pi Y$, if $u\in(v+C\overline{\otimes}_\pi D)\cap(w-C\overline{\otimes}_\pi D))$, then $\pi(u)\leq \alpha\max\{\pi(v),\pi(w)\}$.*

(b) *For every $T\in B(X,Y^*)$, there exist operators $R,S\in\{T\in B(X,Y^*)\mid TC\subseteq D^*\}$ so that $T=R-S$ and $\|R\|+\|S\|\leq\alpha\|T\|$.*

Proof. Since $(X\overline{\otimes}_\pi Y)^* = B(X,Y^*)$ (cf. [31, Section 2.2]) and since $(C\overline{\otimes}_\pi D)^* = \{T\in B(X,Y^*)\mid TC\subseteq D^*\}$, the equivalence follows immediately from Theorem 7.2(1). □

Let M be any metric space and X a Banach space. In Section 6 we have observed that $\mathrm{Lip}_0(M, X^*)$ is isometrically isomorphic to $B(F(M), X^*)$. Now applying Corollary 7.5, the Lipschitz decomposition property can be transferred to statements on the geometry of the projective tensor cone $F(M)_+\overline{\otimes}_\pi C$ in the projective tensor product $F(M)\overline{\otimes}_\pi X$.

Corollary 7.6. *Let X be a Banach space with $C \subseteq X$ a closed cone. The following are equivalent:*

(a) *The dual space X^* (ordered by the dual cone C^*) has the Lipschitz decomposition property.*

(b) *There exists some $\alpha > 0$, so that for every $w, v \in F(X)\overline{\otimes}_\pi X$, if $u \in (w + F(X)_+\overline{\otimes}_\pi C) \cap (v - F(X)_+\overline{\otimes}_\pi C)$, then $\pi(u) \leq \alpha \max\{\pi(v), \pi(w)\}$.*

(c) *For every metric space M, there exists some $\alpha > 0$, so that for every $w, v \in F(M)\overline{\otimes}_\pi X$, if $u \in (w + F(M)_+\overline{\otimes}_\pi C) \cap (v - F(M)_+\overline{\otimes}_\pi C)$, then $\pi(u) \leq \alpha \max\{\pi(v), \pi(w)\}$.*

Proof. This follows from Theorems 6.1, 4.1, 6.2 and 7.5. ☐

We end this section by raising some final questions that are relevant to the current discussion.

Let X be a Banach space ordered by a closed cone $C \subseteq X$. Since $\mathrm{Lip}_0(X)_+$ is generating in $\mathrm{Lip}_0(X)$, by Theorems 4.1 and 7.2, there exists a constant $\alpha > 0$ so that, for $v, w \in F(X)$, if $u \in (v + F(X)_+) \cap (w - F(X)_+)$, then $\|u\| \leq \alpha \max\{\|v\|, \|w\|\}$. This property is usually called *normality* in the literature. Similarly, by Theorems 4.1 and 7.2, if C^* is generating X^*, then X also has such a normality property. Therefore, using Corollary 7.6, the Lipschitz decomposition problem can be resolved for dual Banach spaces if one can answer the question: "If X is normal, is the projective tensor product $F(X)\overline{\otimes}_\pi X$ necessarily also normal, when ordered by the projective tensor cone $F(X)_+\overline{\otimes}_\pi C$?"; or more generally: "Do projective tensor products ordered by projective tensor cones always preserve normality of its tensor factors?"

By Theorem 6.2, we have $\mathrm{Lip}_0(X, X)$ is isometrically isomorphic to $B(F(X), X)$. If X is not a dual Banach space, Corollary 7.6 is not available. Certainly one may apply the general duality result, Theorem 7.2, to $B(F(X), X)$ and its dual $B(F(X), X)^*$, but this observation is not of use without knowledge of the structure of $B(F(X), X)^*$ that could be exploited to show that $B(F(X), X)^*$ is normal when ordered by the dual cone $\{T \in B(F(X), X) \mid TF(X)_+ \subseteq C\}^*$. Since $B(F(X), X) \subseteq B(F(X), X^{**}) = (F(X)\overline{\otimes}_\pi X^*)^*$, one is tempted to consider the pair $(B(F(X), X), F(X)\overline{\otimes}_\pi X^*)$ with the duality defined by restriction. However, the space $B(F(X), X)$ need not separate the points of $F(X)\overline{\otimes}_\pi X^*$ in general (cf. [18]).

Assume further that either X^* or $F(X)$ has the approximation property[4]. Then the space $B(F(X), X)$ does separate the points of $F(X)\overline{\otimes}_\pi X^*$ (cf. [12, Corollary 3 p. 65]), and $B(F(X), X)$ lies wk*-dense (read $\sigma(B(F(X), X^{**}), (F(X)\overline{\otimes}_\pi X^*))$-dense) in $B(F(X), X^{**})$ (cf. [18, Proposition 2.2]). If C is generating in X, then so is C^{**} in X^{**} (cf. Theorems 4.1 and 7.2). Keeping Theorem 6.2 in mind, brings us

[4]Of some relevance here is Kalton and Godefroy's result [15, Theorem 5.3] showing that bounded approximation properties transfer between X and $F(X)$.

into the situation described in the previous paragraph with X^{**} being a dual Banach space, while introducing the further question: "When, if ever, does the cone $\{T \in B(F(X), X^{**}) \mid TF(X)_+ \subseteq C^{**}\}$ being generating in $B(F(X), X^{**})$, imply that the cone $\{T \in B(F(X), X) \mid TF(X)_+ \subseteq X\}$ is generating in the wk*-dense subspace $B(F(X), X)$?"

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