Multivariate normal estimation: the case ($n < p$)

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Abstract

Estimation in the multivariate context when the number of observations available is less than the number of variables is a classical theoretical problem. In order to ensure estimability, one has to assume certain constraints on the parameters. A method for maximum likelihood estimation under constraints is proposed to solve this problem. Even in the extreme case where only a single multivariate observation is available, this may provide a feasible solution. It simultaneously provides a simple, straightforward methodology to allow for specific structures within and between covariance matrices of several populations. This methodology yields exact maximum likelihood estimates.

Keywords

Linear growth in covariance matrices; Maximum likelihood estimation under constraints; Observations less than parameters; Proportional covariance matrices; Proportional growth in covariance matrices; Seemingly unrelated regression.

1. Introduction

This article addresses the fundamental problem of determining the exact maximum likelihood estimates (MLEs) for the mean vector and covariance matrix structures of p-dimensional multivariate normal distributions with special reference to the case where the number of observations, n, may be less than the dimension of the observations, p. In such cases, constraints on the parameters have to be introduced to ensure estimability. Such constraints are usually suggested by the underlying experimental design, as well as the nature of the data. These additional structures on parameters, mean vectors, and covariance matrices can easily be incorporated using the proposed procedure of estimation by combining the principles discussed in the two papers by Strydom and Crowther (2012, 2013). In this article, it is shown how this method even provides an estimation solution in the extreme case where a number of multivariate populations are sampled with only one observation per population. Specifi approaches for dealing with a variety of scenarios are presented.

No general maximum likelihood estimation procedures are known in the case where $n < p$. In order to ensure estimability, reparameterization is required which in essence reduces the number of parameters. Each parameterization then calls for the derivation of its own set of likelihood equations which should then be maximized. This process is cumbersome in the easiest of applications. The complexity of the problem increases in magnitude in the case of several multivariate populations. For this reason, assumptions such as equality of covariance matrices of several populations are often required. Less restrictive assumptions, such as proportionality of covariance matrices of several populations, represent a problematic estimation problem even when $n \geq p$ (Flury 1986). When $n < p$, estimation of the covariance matrices is impossible—unless restrictions are imposed. The method proposed in this article enables and illustrates straightforward maximum likelihood estimation in a variety of complicated practical applications.

Although the procedure proposed in this article is focused on relatively small samples, the techniques can be applied to model the same structures in large samples (Strydom and Crowther 2012, 2013). Many aspects of the Behrens–Fisher problem and other similar problems can be addressed in this way. However, if insuffici t sample sizes are available, but certain structures between (or within) samples may be assumed, the same procedure provides a simple approach to the estimation of the underlying parameters. Although normality is assumed in this article, the procedure may be applied similarly in a broader context to distributions in the exponential class.

In this article, the focus is on the following aspects:

- It is shown that the theory for maximum likelihood estimation under constraints provides a simple framework within which, for relatively small samples ($n < p$), the MLEs of mean vectors and covariance structures can easily be obtained.
- The scope, flexibility, and usefulness of the methodology is illustrated by estimating the covariance matrices under three different variations of growth: proportional, linear, and linear porportional.
- The accuracy of the MLEs of covariance matrices obtained when $n < p$ is presented in simulation studies.
- A practical application illustrates the potential and innovative use of the methodology in the context of a seemingly unrelated regression (SUR) model.

The theoretical background of the estimation procedure is given in Section 2. A simple example is used to show that the exact MLEs are obtained using the proposed procedure. In Section 3, the covariance matrix estimation for several multivariate populations when $n < p$ is illustrated using simulated data. The accuracy of the results is illustrated by way of simulation studies in the case of linear growth, proportional growth, as well as linear proportional growth of the covariance matrices. These underlying structures are suggested as practical and sensible assumptions or constraints that may exist within and among covariance matrices. These constraints enable estimation even in the case of relatively small sample sizes. Finally, in Section 4, a practical application to a SUR model (Greene 2012) is considered. In this case, data observed for four companies over 20 years (1935–1954) are modeled as 20 single ($n = 1$) four-dimensional multivariate observations. MLEs of both the mean vectors and covariance matrices are obtained simultaneously under diffe ent structural assumptions. The seemingly unrelated regression model is fit ed under the assumption of proportional covariance matrices allowing for different trends in growth.

2. Theoretical background

Suppose that $\boldsymbol{y}_{i1}, \boldsymbol{y}_{i2}, \ldots, \boldsymbol{y}_{in_i}$ represent n_i observations of k independent random samples from $N_p(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$ distributions ($i = 1, 2, \ldots, k$). Let $\bar{\boldsymbol{y}}_i$ represent the sample mean vector

$$\bar{\boldsymbol{y}}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \boldsymbol{y}_{ij} \tag{1}$$

and $\boldsymbol{S}_i$ the matrix of mean sums of squares and cross products of the ith sample

$$\boldsymbol{S}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} \boldsymbol{y}_{ij}\boldsymbol{y}_{ij}'. \tag{2}$$

Note that n_i is not assumed to be larger than p. If any of the n_i is less than or equal to p, the MLEs in general do not exist unless additional restrictions on parameters are imposed.

Let

$$\boldsymbol{t} = \begin{pmatrix} \bar{\boldsymbol{y}}_1 \\ vec(\boldsymbol{S}_1) \\ \vdots \\ \bar{\boldsymbol{y}}_k \\ vec(\boldsymbol{S}_k) \end{pmatrix} \text{ with } E(\boldsymbol{t}) = \boldsymbol{m} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ vec(\boldsymbol{\Sigma}_1 + \boldsymbol{\mu}_1\boldsymbol{\mu}_1') \\ \vdots \\ \boldsymbol{\mu}_k \\ vec(\boldsymbol{\Sigma}_k + \boldsymbol{\mu}_k\boldsymbol{\mu}_k') \end{pmatrix} = \begin{pmatrix} \boldsymbol{m}_{11} \\ \boldsymbol{m}_{12} \\ \vdots \\ \boldsymbol{m}_{k1} \\ \boldsymbol{m}_{k2} \end{pmatrix}. \tag{3}$$

The covariance matrix of $\boldsymbol{t}$ is given by

$$\boldsymbol{V} = Cov(\boldsymbol{t}) = \begin{pmatrix} \boldsymbol{V}_{11} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{V}_{kk} \end{pmatrix} \quad \boldsymbol{V}_{ii} = \begin{pmatrix} \boldsymbol{V}_{11}^i & \boldsymbol{V}_{12}^i \\ \boldsymbol{V}_{21}^i & \boldsymbol{V}_{22}^i \end{pmatrix} \tag{4}$$

where

$$\begin{aligned}
\boldsymbol{V}_{11}^i &= \frac{1}{n_i}\boldsymbol{\Sigma}_i \\
\boldsymbol{V}_{21}^i &= \frac{1}{n_i}\left(\boldsymbol{\Sigma}_i \otimes \boldsymbol{\mu}_i + \boldsymbol{\mu}_i \otimes \boldsymbol{\Sigma}_i\right) \\
\boldsymbol{V}_{12}^i &= \boldsymbol{V}_{21}^{i\,\prime} \\
\boldsymbol{V}_{22}^i &= \frac{1}{n_i}(I_{p^2} + \boldsymbol{K})\left[\boldsymbol{\Sigma}_i \otimes \boldsymbol{\Sigma}_i + \boldsymbol{\Sigma}_i \otimes \boldsymbol{\mu}_i\boldsymbol{\mu}_i' + \boldsymbol{\mu}_i\boldsymbol{\mu}_i' \otimes \boldsymbol{\Sigma}_i\right]
\end{aligned}$$

for $i = 1, \ldots, k$. The commutation matrix $\boldsymbol{K}$ is given by $\boldsymbol{K} = \sum_{i,j=1}^{p}(\boldsymbol{H}_{ij} \otimes \boldsymbol{H}_{ij}')$ and $\boldsymbol{H}_{ij}$: $p \times p$ with $h_{ij} = 1$ and all other elements equal to zero (Muirhead 1982).

Parameter structures can be fitted by specifying constraints $\boldsymbol{g}(\boldsymbol{m}) = \boldsymbol{0}$ on the elements of $\boldsymbol{m}$. The MLE of $\boldsymbol{m}$ under the constraints can be obtained from the expression (Strydom and Crowther 2012, 2013):

$$\widehat{\boldsymbol{m}} = \boldsymbol{t} - (\boldsymbol{G}_m\boldsymbol{V})'(\boldsymbol{G}_t\boldsymbol{V}\boldsymbol{G}_m')^*\boldsymbol{g}(\boldsymbol{t}) \tag{5}$$

where $\boldsymbol{G}_m = \dfrac{\partial \boldsymbol{g}(\boldsymbol{m})}{\partial \boldsymbol{m}}$, $\boldsymbol{G}_t = \boldsymbol{G}_m|_{\boldsymbol{m}=\boldsymbol{t}}$ and $(\boldsymbol{G}_t\boldsymbol{V}\boldsymbol{G}_m')^*$ is a generalized inverse of $\boldsymbol{G}_t\boldsymbol{V}\boldsymbol{G}_m'$. When $n_i < p$, the covariance matrices are singular and certain restrictions hold between the elements of each vector of observations $\boldsymbol{y}_i$, $i = 1, \ldots, k$. Since different sets of independent restrictions may be imposed sequentially, the result (5) also holds for singular covariance matrices $\boldsymbol{V}$ and $\boldsymbol{G}_t\boldsymbol{V}\boldsymbol{G}_m'$ (Matthews and Crowther 1995, 1998; Crowther and Shaw 1989). In general, the double-iterative procedure implies a double iteration over $\boldsymbol{t}$ and $\boldsymbol{m}$. The fi st iteration stems from the Taylor series linearization of $\boldsymbol{g}(\boldsymbol{t})$ and the second from the fact that $\boldsymbol{V}$ may be a function of $\boldsymbol{m}$.

The present issue is that the MLEs of $\boldsymbol{m}$ do not exist in general, except under constraints which are implied by assuming a particular model. In the procedure (5) for obtaining the MLEs of $\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ under constraints, the vector $\boldsymbol{t}$ of canonical statistics with the corresponding covariance matrix $\boldsymbol{V}$ is used as the point of departure for any model considered within this framework. The algorithm given by Strydom and Crowther (2012) is then used to obtain these MLEs.

The Wald test statistic may also serve as an indication of whether certain constraints exist, which may be imposed in order to ensure estimability. The examples of covariance matrix estimation in Section 3 and Section 4 implement specifica ly various types of structures (constraints) on the variances and covariances in order to obtain MLEs for the parameters. These structures are implied by the data and the theory underlying the specifi application.

2.1. Introductory example

As a simple example to demonstrate the potential of the process described in (5) and its equivalence to traditional maximum likelihood , eight observations were simulated from a multivariate normal distribution with mean zero and covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix} = \begin{pmatrix} d\boldsymbol{I}_5 + e1_5 1_5' & \boldsymbol{0} \\ \boldsymbol{0} & d\boldsymbol{I}_5 \end{pmatrix} = d\boldsymbol{I}_{10} + e\boldsymbol{c}\boldsymbol{c}' \text{ where}$$

$\boldsymbol{I}_5$ a 5×5 identity matrix, $\boldsymbol{c} = \binom{1_5}{0_5}$, 1_5 a 5×1 vector of ones, $\boldsymbol{0}_5$ a 5×1 vector of zeros, $d = 2$ and $e = 3$.

The singular observed covariance matrix of rank 8 is

$$\begin{pmatrix} 4.12 & 3.33 & 2.22 & 3.82 & 2.38 & 0.71 & -0.95 & 2.82 & 0.82 & 0.03 \\ 3.33 & 4.22 & 3.02 & 3.68 & 1.60 & 1.35 & -0.93 & 2.77 & -1.00 & -0.67 \\ 2.22 & 3.02 & 3.29 & 3.56 & 1.33 & 0.94 & -0.63 & 2.21 & -0.20 & -0.56 \\ 3.82 & 3.68 & 3.56 & 9.64 & 3.65 & -0.44 & 0.42 & 3.28 & 1.01 & 1.05 \\ 2.38 & 1.60 & 1.33 & 3.65 & 3.00 & 0.90 & 0.07 & 2.11 & 1.86 & 0.19 \\ 0.71 & 1.35 & 0.94 & -0.44 & 0.90 & 2.26 & -0.35 & 1.13 & -0.18 & -1.28 \\ -0.95 & -0.93 & -0.63 & 0.42 & 0.07 & -0.35 & 1.47 & -0.81 & -0.28 & 0.35 \\ 2.82 & 2.77 & 2.21 & 3.28 & 2.11 & 1.13 & -0.81 & 2.50 & 0.80 & -0.33 \\ 0.82 & -1.00 & -0.20 & 1.01 & 1.86 & -0.18 & -0.28 & 0.80 & 3.52 & 0.62 \\ 0.03 & -0.67 & -0.56 & 1.05 & 0.19 & -1.28 & 0.35 & -0.33 & 0.62 & 0.95 \end{pmatrix}.$$

The structure in the covariance matrix Σ implies that the restrictions $\boldsymbol{g}(\boldsymbol{m})$ required by the estimation process in (5) are given by:

$$\boldsymbol{g}(\boldsymbol{m}) = \begin{bmatrix} \boldsymbol{P}_{de} \\ \boldsymbol{P}_d \\ \boldsymbol{P}_e \\ \boldsymbol{P}_0 \\ \boldsymbol{P}_r \end{bmatrix} vec(\boldsymbol{\Sigma}) \text{ and } \boldsymbol{G}(\boldsymbol{m}) = \begin{bmatrix} \boldsymbol{P}_{de} \\ \boldsymbol{P}_d \\ \boldsymbol{P}_e \\ \boldsymbol{P}_0 \\ \boldsymbol{P}_r \end{bmatrix} \frac{\partial vec(\boldsymbol{\Sigma})}{\partial \boldsymbol{m}}$$

where the matrix

$\boldsymbol{P}_{de}$ selects and equates the diagonal elements of $vec(\boldsymbol{\Sigma}_{11})$,
$\boldsymbol{P}_d$ selects and equates the diagonal elements of $vec(\boldsymbol{\Sigma}_{22})$,
$\boldsymbol{P}_e$ selects and equates the off-diagonal elements of $vec(\boldsymbol{\Sigma}_{11})$,
$\boldsymbol{P}_0$ selects and sets the elements of $vec(\boldsymbol{\Sigma}_{12})$ equal to zero
$\boldsymbol{P}_r$ selects the elements and specifie the implied relation between the elements of $\boldsymbol{\Sigma}_{11}$ and $\boldsymbol{\Sigma}_{22}$.

The MLE of the covariance matrix $\mathbf{\Sigma}$ under these restrictions follows from (5) as a non singular matrix with $\widehat{\mathbf{\Sigma}} = \begin{pmatrix} 2.07\boldsymbol{I}_5 + 2.841_5 1_5' & \mathbf{0} \\ \mathbf{0} & 2.07\boldsymbol{I}_5 \end{pmatrix}$.

These estimated values may also be verifi d directly by making use of the fact that if the covariance matrix $\mathbf{\Sigma}$ has the structure $\mathbf{\Sigma} = d\boldsymbol{I} + e\boldsymbol{c}\boldsymbol{c}'$, then $\mathbf{\Sigma}^{-1}$ has the same structure (Graybill 1983), namely

$$\mathbf{\Sigma}^{-1} = \delta \boldsymbol{I} + \epsilon \boldsymbol{c}\boldsymbol{c}'$$

for some δ and ϵ. The MLEs of δ, ϵ, and $\mathbf{\Sigma}^{-1}$ are obtained from the likelihood function:

$$\begin{aligned} L(\boldsymbol{y}) &\propto k\ exp\left(-\frac{1}{2}\boldsymbol{y}'\mathbf{\Sigma}^{-1}\boldsymbol{y} \right) \\ &= k\ exp\left[-\frac{1}{2}\boldsymbol{y}'(\delta \boldsymbol{I} + \epsilon \boldsymbol{c}\boldsymbol{c}')\boldsymbol{y} \right] \\ &= k\ exp\left\{ -\frac{1}{2}\delta \boldsymbol{y}'\boldsymbol{y} + \epsilon (\boldsymbol{c}'\boldsymbol{y})^2 \right\}. \end{aligned}$$

The MLEs follow directly as:

$$\widehat{\delta} = \frac{p(p-1)}{p\boldsymbol{y}'\boldsymbol{y} - 2(\boldsymbol{c}'\boldsymbol{y})^2} \quad \text{and} \quad \widehat{\epsilon} = \frac{1}{(1'\boldsymbol{y})^2} - \delta \frac{2}{p}.$$

Since the transformation of $\mathbf{\Sigma}$ to $\mathbf{\Sigma}^{-1}$ is one to one, the MLE of $\mathbf{\Sigma}$ is the inverse of the MLE of $\mathbf{\Sigma}^{-1}$. This yields exactly the same estimated values for d and e as given above.

Traditional maximum likelihood estimation may increase drastically in difficulty with more complicated structures. With the maximum likelihood procedure outlined in (5), the problem may usually be resolved in a straightforward way.

3. Covariance matrix estimation

The case of relatively few observations can only be addressed if the existence of underlying structures between and/or within covariance matrices can be assumed. The constraints in essence reduce the number of independent parameters to be estimated. Under this scenario, the procedure for maximum likelihood estimation under constraints described in the previous section provides an elegant and direct solution to this problem.

In the framework of procedure (5), when the mean vectors are assumed to be zero, the canonical statistics with expected values simplify to

$$\boldsymbol{t} = \begin{pmatrix} vec(\boldsymbol{S}_1) \\ \vdots \\ vec(\boldsymbol{S}_k) \end{pmatrix}, \quad E(\boldsymbol{t}) = vec(\mathbf{\Sigma}) = \begin{pmatrix} vec(\mathbf{\Sigma}_1) \\ \vdots \\ vec(\mathbf{\Sigma}_k) \end{pmatrix} = \begin{pmatrix} \boldsymbol{m}_1 \\ \vdots \\ \boldsymbol{m}_k \end{pmatrix} = \boldsymbol{m} \tag{6}$$

and $\boldsymbol{V} = Cov(\boldsymbol{t}) = block(\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3, \boldsymbol{V}_4, \boldsymbol{V}_5)$ a block-diagonal matrix with typical submatrix

$$\boldsymbol{V}_{\mathbf{i}} = (I_{p^2} + \boldsymbol{K})\,[\mathbf{\Sigma}_i \otimes \mathbf{\Sigma}_i]$$

for $i = 1, \ldots, 5$ and $\boldsymbol{K}$ the commutation matrix (cf., (4)).

In the next three subsections, simulations are given where the number of multivariate observations in each case presented is less than or equal to the dimension of the observations and constraints have to be imposed on the parameters in order to obtain MLEs. Traditionally, constraints such as equal covariance matrices are assumed. Assumptions like, for example, proportionality of covariance matrices could in many instances provide a much more realistic and desirable solution. Proportionality of covariance matrices implies equal correlation matrices, but allows for different covariance structures. Structures between and within the covariance matrices may be imposed which would ensure estimability.

For each simulation, relatively small random samples $\boldsymbol{Y}_i : p \times n_i$, $p > n_i$, for $i = 1, 2, 3, 4, 5$ generated from $N(0, \boldsymbol{\Sigma}_i)$ distributions, are considered. These small samples result in the unbiased singular sample covariance matrix estimates $\frac{1}{n_i}\boldsymbol{Y}_i\boldsymbol{Y}_i'$, $i = 1, 2, 3, 4, 5$ with the ith estimate having rank n_i. Structures between and within covariance matrices are imposed to ensure estimability and are illustrated with the simulations in the following subsections.

Due to the complexity of estimation and the problem of relatively small sample sizes, equality of covariance matrices is often assumed in the simultaneous modeling of mean vectors and covariance structures. In the following subsections, the accuracy of MLEs under less restrictive but practically sensible assumptions, is illustrated. These assumptions are proportional growth, linear growth, and linear proportional growth of covariance matrices with a compound symmetry structure. For each scenario, the random samples generated are described, the theory (constraints) formulated, followed by a short summary of results accompanied by a table with specific detail.

3.1. Simulation of proportional growth in covariances

The columns of $\boldsymbol{Y}_i : 6 \times 2$ for $i = 1, 2, 3, 4, 5$ represent five independent samples of size two generated from $N(0, \boldsymbol{\Sigma}_i)$ distributions where

$$\boldsymbol{\Sigma}_i = \rho^{i-1}\begin{pmatrix} \alpha & \beta & \beta & \beta & \beta & \beta \\ \beta & \alpha & \beta & \beta & \beta & \beta \\ \beta & \beta & \alpha & \beta & \beta & \beta \\ \beta & \beta & \beta & \alpha & \beta & \beta \\ \beta & \beta & \beta & \beta & \alpha & \beta \\ \beta & \beta & \beta & \beta & \beta & \alpha \end{pmatrix}, \tag{7}$$

i.e., where consecutive population covariance matrices are proportional. These small samples result in the unbiased singular sample covariance matrix estimates $\frac{1}{2}\boldsymbol{Y}_i\boldsymbol{Y}_i'$, $i = 1, 2, 3, 4, 5$ and each estimate has rank 2.

3.1.1. Constraints

The proposed methodology only requires specification of the constraints implied by the underlying structural relationships. The structures within and between the $\boldsymbol{\Sigma}_i$'s are represented by the constraints $\boldsymbol{g}(\boldsymbol{m}) = \binom{g_1(\boldsymbol{m})}{g_2(\boldsymbol{m})}$ where $\boldsymbol{g}_1(\boldsymbol{m})$ implies the compound symmetry structure within covariance matrices and $\boldsymbol{g}_2(\boldsymbol{m})$ implies the proportionality between the covariance matrices.

The compound symmetry structure on each of the five covariance matrices is specified by

$$\boldsymbol{g}_1(\boldsymbol{m}) = \begin{pmatrix} \boldsymbol{I}_5 \otimes (\mathbf{Q}_\alpha \boldsymbol{C}_\alpha) \\ \boldsymbol{I}_5 \otimes (\mathbf{Q}_\beta \boldsymbol{C}_\beta) \end{pmatrix} vec(\boldsymbol{\Sigma}) \quad \text{with derivative} \quad \boldsymbol{G}_1(\boldsymbol{m}) = \begin{pmatrix} \boldsymbol{I}_5 \otimes (\mathbf{Q}_\alpha \boldsymbol{C}_\alpha) \\ \boldsymbol{I}_5 \otimes (\mathbf{Q}_\beta \boldsymbol{C}_\beta) \end{pmatrix} \frac{\partial vec(\boldsymbol{\Sigma})}{\partial \boldsymbol{m}} \tag{8}$$

where

$\boldsymbol{C}_\alpha$ selects the diagonal elements of each $\boldsymbol{\Sigma}_i$

$\mathbf{Q}_\alpha$ equates all the diagonal elements of each $\boldsymbol{\Sigma}_i$

$\boldsymbol{C}_\beta$ selects the off-di gonal elements of each $\boldsymbol{\Sigma}_i$

$\boldsymbol{Q}_\beta$ equates all the off-di gonal elements of each $\boldsymbol{\Sigma}_i$

$\dfrac{\partial vec(\boldsymbol{\Sigma})}{\partial \boldsymbol{m}} = \boldsymbol{I}_{5p^2}$.

In this case, the matrix

$$\boldsymbol{C}_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\boldsymbol{Q}_\alpha = \boldsymbol{I}_6 - \frac{1}{6}\boldsymbol{1}\boldsymbol{1}'$. Similarly for $\boldsymbol{C}_\beta$ and $\boldsymbol{Q}_\beta$.

When proportional covariance matrices are assumed, a typical subvector of the constraints $\boldsymbol{g}_2(\boldsymbol{m})$ is given by:

$$\boldsymbol{g}_{2i}(\boldsymbol{m}) = vec(\boldsymbol{\Sigma}_i) - \rho.vec(\boldsymbol{\Sigma}_{i-1}) = 0, \quad i = 2, 3, 4, 5 \tag{9}$$

where ρ is a measure of growth and

$$\boldsymbol{G}_{2i}(\boldsymbol{m}) = \frac{\partial vec(\boldsymbol{\Sigma}_i)}{\partial \boldsymbol{m}} - \rho \frac{\partial vec(\boldsymbol{\Sigma}_{i-1})}{\partial \boldsymbol{m}} - vec(\boldsymbol{\Sigma}_{i-1}) \frac{\partial \rho}{\partial \boldsymbol{m}}.$$

It is sufficient to use only the firs two elements of each $\boldsymbol{\Sigma}_i$ in the restrictions, since equality of diagonal and off-diagonal elements are specified as well. The growth factor, ρ, may be determined in various ways, e.g., $\rho = \boldsymbol{\Sigma}_2[1,1]/\boldsymbol{\Sigma}_1[1,1]$ or $\rho = (\boldsymbol{\Sigma}_5[1,1]/\boldsymbol{\Sigma}_1[1,1])^{\frac{1}{4}}$ where $\boldsymbol{\Sigma}_i[1,1]$ is the variance of the fi st variable measured in the ith sample, $i = 1, \ldots, 5$. Alternatively, ρ may be determined by

$$\rho = (\text{ sum of elements of } \boldsymbol{\Sigma}_4) \,/\, (\text{ sum of elements of } \boldsymbol{\Sigma}_3) = \boldsymbol{a}'\boldsymbol{m}/\boldsymbol{b}'\boldsymbol{m} = S_4/S_3 \tag{10}$$

with $\boldsymbol{a}' = (\boldsymbol{0}'_{36}\ \boldsymbol{0}'_{36}\ \boldsymbol{0}'_{36}\ \boldsymbol{1}'_{36}\ \boldsymbol{0}'_{36})$, $\boldsymbol{b}' = (\boldsymbol{0}'_{36}\ \boldsymbol{0}'_{36}\ \boldsymbol{1}'_{36}\ \boldsymbol{0}'_{36}\ \boldsymbol{0}'_{36})$ and derivative

$$\frac{\partial \rho}{\partial \boldsymbol{m}} = (S_3\boldsymbol{a}' - S_4\boldsymbol{b}')/S_3^2.$$

3.1.2. Results

For 1000 simulated samples with $\boldsymbol{\Sigma}_i$ given in (7), $\alpha = 5$, $\beta = 3$, $n_i = 2$, $i = 1, \ldots, 5$ and pro-portionality constant or growth factor $\rho = 1.3$, the MLEs of ρ and $\boldsymbol{\Sigma}_i$ were calculated. Conver-gence of the double-iterative process is not guaranteed, but in general it converges in relatively few iterations.

On the left-hand side in Table 1, the population values for the covariance matrices and growth factor are given. The average of the MLEs of the covariance matrices and growth factor calculated for 1000 simulations are given on the right-hand side in Table 1.

3.2. Simulation of linear growth

In this subsection, the accuracy of the MLEs of covariance matrices with compound symmetry structure growing linearly, is illustrated. Random samples $\boldsymbol{Y}_i : 6 \times 2$ for $i = 1, 2, 3, 4, 5$ were

Table 1. Simulation of proportional growth. $n_1 = 2, n_2 = 2, n_3 = 2, n_4 = 2, n_5 = 2$.

Population covariance matrices Growth factor $\rho = 1.3$						Estimated covariance matrices Estimated growth factor $\rho = 1.3047$					
1			$\boldsymbol{\Sigma}_1$			1			$\boldsymbol{\Sigma}_1$		
5.000	3.000	3.000	3.000	3.000	3.000	5.137	3.071	3.071	3.071	3.071	3.071
3.000	5.000	3.000	3.000	3.000	3.000	3.071	5.137	3.071	3.071	3.071	3.071
3.000	3.000	5.000	3.000	3.000	3.000	3.071	3.071	5.137	3.071	3.071	3.071
3.000	3.000	3.000	5.000	3.000	3.000	3.071	3.071	3.071	5.137	3.071	3.071
3.000	3.000	3.000	3.000	5.000	3.000	3.071	3.071	3.071	3.071	5.137	3.071
3.000	3.000	3.000	3.000	3.000	5.000	3.071	3.071	3.071	3.071	3.071	5.137
2			$\boldsymbol{\Sigma}_2 = 1.3 \times \boldsymbol{\Sigma}_1$			2			$\boldsymbol{\Sigma}_2 = 1.304725 \times \boldsymbol{\Sigma}_1$		
6.500	3.900	3.900	3.900	3.900	3.900	6.702	4.007	4.007	4.007	4.007	4.007
3.900	6.500	3.900	3.900	3.900	3.900	4.007	6.702	4.007	4.007	4.007	4.007
3.900	3.900	6.500	3.900	3.900	3.900	4.007	4.007	6.702	4.007	4.007	4.007
3.900	3.900	3.900	6.500	3.900	3.900	4.007	4.007	4.007	6.702	4.007	4.007
3.900	3.900	3.900	3.900	6.500	3.900	4.007	4.007	4.007	4.007	6.702	4.007
3.900	3.900	3.900	3.900	3.900	6.500	4.007	4.007	4.007	4.007	4.007	6.702
3			$\boldsymbol{\Sigma}_3 = 1.3 \times \boldsymbol{\Sigma}_2$			3			$\boldsymbol{\Sigma}_3 = 1.304725 \times \boldsymbol{\Sigma}_2$		
8.450	5.070	5.070	5.070	5.070	5.070	8.744	5.229	5.229	5.229	5.229	5.229
5.070	8.450	5.070	5.070	5.070	5.070	5.229	8.744	5.229	5.229	5.229	5.229
5.070	5.070	8.450	5.070	5.070	5.070	5.229	5.229	8.744	5.229	5.229	5.229
5.070	5.070	5.070	8.450	5.070	5.070	5.229	5.229	5.229	8.744	5.229	5.229
5.070	5.070	5.070	5.070	8.450	5.070	5.229	5.229	5.229	5.229	8.744	5.229
5.070	5.070	5.070	5.070	5.070	8.450	5.229	5.229	5.229	5.229	5.229	8.744
4			$\boldsymbol{\Sigma}_4 = 1.3 \times \boldsymbol{\Sigma}_3$			4			$\boldsymbol{\Sigma}_4 = 1.304725 \times \boldsymbol{\Sigma}_3$		
10.985	6.591	6.591	6.591	6.591	6.591	11.409	6.822	6.822	6.822	6.822	6.822
6.591	10.985	6.591	6.591	6.591	6.591	6.822	11.409	6.822	6.822	6.822	6.822
6.591	6.591	10.985	6.591	6.591	6.591	6.822	6.822	11.409	6.822	6.822	6.822
6.591	6.591	6.591	10.985	6.591	6.591	6.822	6.822	6.822	11.409	6.822	6.822
6.591	6.591	6.591	6.591	10.985	6.591	6.822	6.822	6.822	6.822	11.409	6.822
6.591	6.591	6.591	6.591	6.591	10.985	6.822	6.822	6.822	6.822	6.822	11.409
5			$\boldsymbol{\Sigma}_5 = 1.3 \times \boldsymbol{\Sigma}_4$			5			$\boldsymbol{\Sigma}_5 = 1.304725 \times \boldsymbol{\Sigma}_4$		
14.281	8.568	8.568	8.568	8.568	8.568	14.885	8.901	8.901	8.901	8.901	8.901
8.568	14.281	8.568	8.568	8.568	8.568	8.901	14.885	8.901	8.901	8.901	8.901
8.568	8.568	14.281	8.568	8.568	8.568	8.901	8.901	14.885	8.901	8.901	8.901
8.568	8.568	8.568	14.281	8.568	8.568	8.901	8.901	8.901	14.885	8.901	8.901
8.568	8.568	8.568	8.568	14.281	8.568	8.901	8.901	8.901	8.901	14.885	8.901
8.568	8.568	8.568	8.568	8.568	14.281	8.901	8.901	8.901	8.901	8.901	14.885

generated from $N(0, \boldsymbol{\Sigma}_i)$ distributions where

$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} \alpha & \beta & \beta & \beta & \beta & \beta \\ \beta & \alpha & \beta & \beta & \beta & \beta \\ \beta & \beta & \alpha & \beta & \beta & \beta \\ \beta & \beta & \beta & \alpha & \beta & \beta \\ \beta & \beta & \beta & \beta & \alpha & \beta \\ \beta & \beta & \beta & \beta & \beta & \alpha \end{pmatrix} \text{ and } \boldsymbol{\Sigma}_i = \boldsymbol{\Sigma}_{i-1} + c1_6 1_6' + d\boldsymbol{I}_6, \quad i = 2, 3, 4, 5 \qquad (11)$$

These small samples result in the unbiased singular sample covariance matrix estimates $\frac{1}{2}\boldsymbol{Y}_i\boldsymbol{Y}_i'$, $i = 1, 2, 3, 4, 5$ and the ith estimate has rank 2.

3.2.1. Constraints

The compound symmetry structure within the covariance matrices is again specified by the restriction $\boldsymbol{g}_1(\boldsymbol{m})$ given by (8) i n Section 3.1. The constraints specifying linear growth in vari-ances and in covariances are, respectively,

$$\begin{aligned}\boldsymbol{g}_d = \boldsymbol{Q}_X(\boldsymbol{I}_5 \otimes \boldsymbol{C}_\alpha[1,\,])\boldsymbol{m} = \boldsymbol{Q}_X \begin{pmatrix} \boldsymbol{\Sigma}_1[1,1] \\ \boldsymbol{\Sigma}_2[1,1] \\ \boldsymbol{\Sigma}_3[1,1] \\ \boldsymbol{\Sigma}_4[1,1] \\ \boldsymbol{\Sigma}_5[1,1] \end{pmatrix} \text{ and} \\ \boldsymbol{g}_c = \boldsymbol{Q}_X(\boldsymbol{I}_5 \otimes \boldsymbol{C}_\beta[1,\,])\boldsymbol{m} = \boldsymbol{Q}_X \begin{pmatrix} \boldsymbol{\Sigma}_1[1,2] \\ \boldsymbol{\Sigma}_2[1,2] \\ \boldsymbol{\Sigma}_3[1,2] \\ \boldsymbol{\Sigma}_4[1,2] \\ \boldsymbol{\Sigma}_5[1,2] \end{pmatrix}\end{aligned} \tag{12}$$

where $\boldsymbol{\Sigma}_i[k,l]$ is the covariance of the kth and lth variable measured in the ith sample and

$$\boldsymbol{Q}_X = (\boldsymbol{I}_5 - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}') \text{ with } \boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix}.$$

The corresponding derivatives are given by

$$\boldsymbol{G}_d = \boldsymbol{Q}_X(\boldsymbol{I}_5 \otimes \boldsymbol{C}_\alpha[1,\,]) \text{ and } \boldsymbol{G}_c = \boldsymbol{Q}_X(\boldsymbol{I}_5 \otimes \boldsymbol{C}_\beta[1,\,]).$$

These constraints allow for different linear trends in variances and covariances, respectively.

3.2.2. Results

For 1000 simulated samples with $\boldsymbol{\Sigma}_1$ given in (11), linear growth specified according to (12) a n d c o m p o u n d s y m m e t r y a c c o r d i n g t o (8), the MLE of $\boldsymbol{\Sigma}_1$ was calculated. On the left-hand side in Table2, the population values for the covariance matrices and linear growth factors are given. The average of the MLEs of the covariance matrices and linear growth factors calculated for 1000 simulations are given on the right-hand side in Table 2.

3.3. Simulation of linear proportional growth

In this subsection, the MLEs obtained in the presence of linear proportional growth in covari-ance matrices with compound symmetry structure are given. Random samples $\boldsymbol{Y}_i : 6 \times 2$ for $i = 1, 2, 3, 4, 5$ were generated from $N(0, \boldsymbol{\Sigma}_i)$ distributions where

$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} \alpha & \beta & \beta & \beta & \beta & \beta \\ \beta & \alpha & \beta & \beta & \beta & \beta \\ \beta & \beta & \alpha & \beta & \beta & \beta \\ \beta & \beta & \beta & \alpha & \beta & \beta \\ \beta & \beta & \beta & \beta & \alpha & \beta \\ \beta & \beta & \beta & \beta & \beta & \alpha \end{pmatrix} \text{ and } \boldsymbol{\Sigma}_i = \rho_i \boldsymbol{\Sigma}_{Av} \tag{13}$$

where $\boldsymbol{\Sigma}_{Av} = \frac{1}{5}\sum_{i=1}^{5} \boldsymbol{\Sigma}_i$ and the ρ_i's are restricted to be on a straight line. This implies that the corresponding elements of the covariance matrices $\boldsymbol{\Sigma}_i$ for $i = 1, 2, \ldots, 5$ are on straight

Table 2. Simulation of linear growth. $n_1 = 2, n_2 = 2, n_3 = 2, n_4 = 2, n_5 = 2$.

Population covariance matrices Diagonal factor 5 off-diagonal factor 3						Estimated covariance matrices Estimated diagonal factor 4.837 Estimated off-diagonal factor 2.857					
1		$\boldsymbol{\Sigma}_1$				1		$\boldsymbol{\Sigma}_1$			
5.000	3.000	3.000	3.000	3.000	3.000	5.326	3.331	3.331	3.331	3.331	3.331
3.000	5.000	3.000	3.000	3.000	3.000	3.331	5.326	3.331	3.331	3.331	3.331
3.000	3.000	5.000	3.000	3.000	3.000	3.331	3.331	5.326	3.331	3.331	3.331
3.000	3.000	3.000	5.000	3.000	3.000	3.331	3.331	3.331	5.326	3.331	3.331
3.000	3.000	3.000	3.000	5.000	3.000	3.331	3.331	3.331	3.331	5.326	3.331
3.000	3.000	3.000	3.000	3.000	5.000	3.331	3.331	3.331	3.331	3.331	5.326
2		$\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_1 + 3\mathbf{1}_6\mathbf{1}_6' + 2\boldsymbol{I}_6$				2		$\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_1 + 2.857\mathbf{1}_6\mathbf{1}_6' + 1.98\boldsymbol{I}_6$			
10.000	6.000	6.000	6.000	6.000	6.000	10.163	6.188	6.188	6.188	6.188	6.188
6.000	10.000	6.000	6.000	6.000	6.000	6.188	10.163	6.188	6.188	6.188	6.188
6.000	6.000	10.000	6.000	6.000	6.000	6.188	6.188	10.163	6.188	6.188	6.188
6.000	6.000	6.000	10.000	6.000	6.000	6.188	6.188	6.188	10.163	6.188	6.188
6.000	6.000	6.000	6.000	10.000	6.000	6.188	6.188	6.188	6.188	10.163	6.188
6.000	6.000	6.000	6.000	6.000	10.000	6.188	6.188	6.188	6.188	6.188	10.163
3		$\boldsymbol{\Sigma}_3 = \boldsymbol{\Sigma}_2 + 3\mathbf{1}_6\mathbf{1}_6' + 2\boldsymbol{I}_6$				3		$\boldsymbol{\Sigma}_3 = \boldsymbol{\Sigma}_2 + 2.857\mathbf{1}_6\mathbf{1}_6' + 1.98\boldsymbol{I}_6$			
15.000	9.000	9.000	9.000	9.000	9.000	15.000	9.045	9.045	9.045	9.045	9.045
9.000	15.000	9.000	9.000	9.000	9.000	9.045	15.000	9.045	9.045	9.045	9.045
9.000	9.000	15.000	9.000	9.000	9.000	9.045	9.045	15.000	9.045	9.045	9.045
9.000	9.000	9.000	15.000	9.000	9.000	9.045	9.045	9.045	15.000	9.045	9.045
9.000	9.000	9.000	9.000	15.000	9.000	9.045	9.045	9.045	9.045	15.000	9.045
9.000	9.000	9.000	9.000	9.000	15.000	9.045	9.045	9.045	9.045	9.045	15.000
4		$\boldsymbol{\Sigma}_4 = \boldsymbol{\Sigma}_3 + 3\mathbf{1}_6\mathbf{1}_6' + 2\boldsymbol{I}_6$				4		$\boldsymbol{\Sigma}_4 = \boldsymbol{\Sigma}_3 + 2.857\mathbf{1}_6\mathbf{1}_6' + 1.98\boldsymbol{I}_6$			
20.000	12.000	12.000	12.000	12.000	12.000	19.837	11.901	11.901	11.901	11.901	11.901
12.000	20.000	12.000	12.000	12.000	12.000	11.901	19.837	11.901	11.901	11.901	11.901
12.000	12.000	20.000	12.000	12.000	12.000	11.901	11.901	19.837	11.901	11.901	11.901
12.000	12.000	12.000	20.000	12.000	12.000	11.901	11.901	11.901	19.837	11.901	11.901
12.000	12.000	12.000	12.000	20.000	12.000	11.901	11.901	11.901	11.901	19.837	11.901
12.000	12.000	12.000	12.000	12.000	20.000	11.901	11.901	11.901	11.901	11.901	19.837
5		$\boldsymbol{\Sigma}_5 = \boldsymbol{\Sigma}_4 + 3\mathbf{1}_6\mathbf{1}_6' + 2\boldsymbol{I}_6$				5		$\boldsymbol{\Sigma}_5 = \boldsymbol{\Sigma}_4 + 2.857\mathbf{1}_6\mathbf{1}_6' + 1.98\boldsymbol{I}_6$			
25.000	15.000	15.000	15.000	15.000	15.000	24.673	14.758	14.758	14.758	14.758	14.758
15.000	25.000	15.000	15.000	15.000	15.000	14.758	24.673	14.758	14.758	14.758	14.758
15.000	15.000	25.000	15.000	15.000	15.000	14.758	14.758	24.673	14.758	14.758	14.758
15.000	15.000	15.000	25.000	15.000	15.000	14.758	14.758	14.758	24.673	14.758	14.758
15.000	15.000	15.000	15.000	25.000	15.000	14.758	14.758	14.758	14.758	24.673	14.758
15.000	15.000	15.000	15.000	15.000	25.000	14.758	14.758	14.758	14.758	14.758	24.673

lines and that the covariance matrices are proportional. The assumption of proportionality implies equal corresponding correlation coeffici ts.

3.3.1 Constraints

The compound symmetry structure within the covariance matrices is again specified by the restriction $\boldsymbol{g}_1(\boldsymbol{m})$ (cf., (8)). Proportional covariance matrices are implied by the restriction $\boldsymbol{g}_2(\boldsymbol{m})$ (cf., (9)). The linear structure between covariance matrices is in this case implied by the constraints $\boldsymbol{g}_r(\boldsymbol{m})$, w h e r e

$$\boldsymbol{g}_r(\boldsymbol{m}) = \boldsymbol{Q}_X \boldsymbol{\rho} \tag{14}$$

with derivative $\boldsymbol{G}_r(\boldsymbol{m}) = \boldsymbol{Q}_X \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{m}}$ and $\boldsymbol{Q}_X$ as defin d by (12) in Section 3.2.

Table 3. Simulation of linear proportional growth. $n_1 = 2, n_2 = 2, n_3 = 2, n_4 = 2, n_5 = 2$.

Population covariance matrices						Estimated covariance matrices					
1		$\mathbf{\Sigma_1 = 0.5\Sigma_{Av}}$				1		$\mathbf{\Sigma_1 = 0.510\Sigma_{Av}}$			
5.000	3.000	3.000	3.000	3.000	3.000	5.107	3.070	3.070	3.070	3.070	3.070
3.000	5.000	3.000	3.000	3.000	3.000	3.070	5.107	3.070	3.070	3.070	3.070
3.000	3.000	5.000	3.000	3.000	3.000	3.070	3.070	5.107	3.070	3.070	3.070
3.000	3.000	3.000	5.000	3.000	3.000	3.070	3.070	3.070	5.107	3.070	3.070
3.000	3.000	3.000	3.000	5.000	3.000	3.070	3.070	3.070	3.070	5.107	3.070
3.000	3.000	3.000	3.000	3.000	5.000	3.070	3.070	3.070	3.070	3.070	5.107
2		$\mathbf{\Sigma_2 = 0.75\Sigma_{Av}}$				2		$\mathbf{\Sigma_2 = 0.755\Sigma_{Av}}$			
7.500	4.500	4.500	4.500	4.500	4.500	7.562	4.545	4.545	4.545	4.545	4.545
4.500	7.500	4.500	4.500	4.500	4.500	4.545	7.562	4.545	4.545	4.545	4.545
4.500	4.500	7.500	4.500	4.500	4.500	4.545	4.545	7.562	4.545	4.545	4.545
4.500	4.500	4.500	7.500	4.500	4.500	4.545	4.545	4.545	7.562	4.545	4.545
4.500	4.500	4.500	4.500	7.500	4.500	4.545	4.545	4.545	4.545	7.562	4.545
4.500	4.500	4.500	4.500	4.500	7.500	4.545	4.545	4.545	4.545	4.545	7.562
3		$\mathbf{\Sigma_3 = \Sigma_{Av}}$				3		$\mathbf{\Sigma_3 = 1.000\Sigma_{Av}}$			
10.000	6.000	6.000	6.000	6.000	6.000	10.017	6.019	6.019	6.019	6.019	6.019
6.000	10.000	6.000	6.000	6.000	6.000	6.019	10.017	6.019	6.019	6.019	6.019
6.000	6.000	10.000	6.000	6.000	6.000	6.019	6.019	10.017	6.019	6.019	6.019
6.000	6.000	6.000	10.000	6.000	6.000	6.019	6.019	6.019	10.017	6.019	6.019
6.000	6.000	6.000	6.000	10.000	6.000	6.019	6.019	6.019	6.019	10.017	6.019
6.000	6.000	6.000	6.000	6.000	10.000	6.019	6.019	6.019	6.019	6.019	10.017
4		$\mathbf{\Sigma_4 = 1.25\Sigma_{Av}}$				4		$\mathbf{\Sigma_4 = 1.245\Sigma_{Av}}$			
12.500	7.500	7.500	7.500	7.500	7.500	12.472	7.493	7.493	7.493	7.493	7.493
7.500	12.500	7.500	7.500	7.500	7.500	7.493	12.472	7.493	7.493	7.493	7.493
7.500	7.500	12.500	7.500	7.500	7.500	7.493	7.493	12.472	7.493	7.493	7.493
7.500	7.500	7.500	12.500	7.500	7.500	7.493	7.493	7.493	12.472	7.493	7.493
7.500	7.500	7.500	7.500	12.500	7.500	7.493	7.493	7.493	7.493	12.472	7.493
7.500	7.500	7.500	7.500	7.500	12.500	7.493	7.493	7.493	7.493	7.493	12.472
5		$\mathbf{\Sigma_5 = 1.50\Sigma_{Av}}$				5		$\mathbf{\Sigma_5 = 1.490\Sigma_{Av}}$			
15.000	9.000	9.000	9.000	9.000	9.000	14.927	8.967	8.967	8.967	8.967	8.967
9.000	15.000	9.000	9.000	9.000	9.000	8.967	14.927	8.967	8.967	8.967	8.967
9.000	9.000	15.000	9.000	9.000	9.000	8.967	8.967	14.927	8.967	8.967	8.967
9.000	9.000	9.000	15.000	9.000	9.000	8.967	8.967	8.967	14.927	8.967	8.967
9.000	9.000	9.000	9.000	15.000	9.000	8.967	8.967	8.967	8.967	14.927	8.967
9.000	9.000	9.000	9.000	9.000	15.000	8.967	8.967	8.967	8.967	8.967	14.927
Population correlation matrix						Estimated correlation matrix					
1.00	0.60	0.60	0.60	0.60	0.60	1.000	0.601	0.601	0.601	0.601	0.601
0.60	1.00	0.60	0.60	0.60	0.60	0.601	1.000	0.601	0.601	0.601	0.601
0.60	0.60	1.00	0.60	0.60	0.60	0.601	0.601	1.000	0.601	0.601	0.601
0.60	0.60	0.60	1.00	0.60	0.60	0.601	0.601	0.601	1.000	0.601	0.601
0.60	0.60	0.60	0.60	1.00	0.60	0.601	0.601	0.601	0.601	1.000	0.601
0.60	0.60	0.60	0.60	0.60	1.00	0.601	0.601	0.601	0.601	0.601	1.000

3.3.2 Results

On the left-hand side in Table 3, the population values for the covariance matrices and growth factor are given. The average of the MLEs of the covariance matrices and growth factor calcu-lated for 1000 simulations are given on the right-hand side in Table 3. Note that the matrix $\mathbf{\Sigma}_{Av}$ also equals the average of any two estimated $\mathbf{\Sigma}$'s which are on equal distances from the center of the five estimated $\mathbf{\Sigma}$'s. For example, $\mathbf{\Sigma}_{Av} = 0.5(\mathbf{\Sigma}_1 + \mathbf{\Sigma}_5)$. The corresponding population and estimated correlation matrices are also given.

4. Seemingly unrelated regression model

In this section, the flexibility and potential of the methodology given in Section 2 are practically illustrated in the case of longitudinal data with a specific focus on the SUR model (Greene 2012). Greene (2012) used a subset of the Grunfeld data set consisting of data of four firms over 20 years. The investment equation is given by:

$$\boldsymbol{I}_{ji} = \boldsymbol{\beta}_{1j} + \boldsymbol{\beta}_{2j}\boldsymbol{F}_{ji} + \boldsymbol{\beta}_{3j}\boldsymbol{C}_{ji} + \boldsymbol{e}_{ji},\ j = 1, 2, 3, 4 \text{ and } i = 1, 2, \ldots 20 \tag{15}$$

where

$\boldsymbol{I}_{ji}$ is the real gross investment for the jth firm in year i,
$\boldsymbol{F}_{ji}$ is the real value of the firm-shares outstanding,
$\boldsymbol{C}_{ji}$ is the value of the capital stock,
$\boldsymbol{\beta}_{1j}$, $\boldsymbol{\beta}_{2j}$, and $\boldsymbol{\beta}_{3j}$ are the respective regression constant and coeffici ts, and
$\boldsymbol{e}_{ji}$ is a disturbance term.

The SUR model can be formulated in terms of 20 independent multivariate observations. Let

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 : 4 \times 1 \\ \vdots \\ \boldsymbol{y}_{20} : 4 \times 1 \end{pmatrix} \text{ and } Cov(\boldsymbol{y}) = \begin{pmatrix} \mathbf{\Sigma}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{\Sigma}_{20} \end{pmatrix} \tag{16}$$

where $\boldsymbol{y}$ represents 20 independent multivariate observations that fall into the estimation framework described in Section 2. Note that each observation needs to be considered as a four-dimensional multivariate (normal) sample of size one.

In the following subsections, the same regression models are fit ed to the mean vectors, i.e., four regression models are fitted over time. However, it is argued that the usual assumption of equal covariance matrices is too restrictive. Constraints are used to impose relations within and between the covariance matrices. MLEs for both the mean vectors and covariance matrices are obtained using the general methodology described in Section 2 and illustrated in the following subsections. The effect of the different covariance structures on the estimates of the mean vectors is illustrated clearly.

4.1. Estimating mean vectors and regression parameters

The SUR model in the example referred to is fitted to the four elements of the 20 independent vector observations. Since a single multivariate observation from each population is observed, the iterative process (5) is initialized with

$$vec(\widehat{\mathbf{\Sigma}}_i) = \boldsymbol{t}_{i2} - vec[\boldsymbol{A}\boldsymbol{t}_{i1}(\boldsymbol{A}\boldsymbol{t}_{i1})'] \neq 0, \quad i = 1, \ldots, 20$$

where $\boldsymbol{A} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ and $\boldsymbol{X} = block(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \boldsymbol{X}_4)$, a block-diagonal matrix with $\boldsymbol{X}_j = (1_{20}\ \boldsymbol{F}_j\ \boldsymbol{C}_j)$.

This results in an initial estimate of $\mathbf{\Sigma}_i$ of rank 1 instead of the usual estimate $\boldsymbol{t}_{i2} - vec(\boldsymbol{t}_{i1}\boldsymbol{t}'_{i1})$, which is equal to zero as a starting value.

The constraints implied by the four regression models require the defin tion of the following matrices:

$$D_1 = I_{20} \otimes (I_4 \; 0_{p,p^2}), \tag{17}$$

$$C_1 = \begin{pmatrix} I_{20} & \otimes & I_4[1,] \\ I_{20} & \otimes & I_4[2,] \\ I_{20} & \otimes & I_4[3,] \\ I_{20} & \otimes & I_4[4,] \end{pmatrix} \tag{18}$$

where $I_4[j,]$ is the jth row of the 4×4 identity matrix, D_1 selects the mean vectors $\mu_1, \mu_2, \ldots, \mu_{20}$ from m, and C_1 selects successively the firs , second, third, and fourth elements of the 20 mean vectors μ_i, $i = 1, 2, \ldots 20$.

The corresponding set of constraints specifying the multiple regression is given by

$$g_1(m) = Q_1 C_1 D_1 m \text{ and } G_1(m) = Q_1 C_1 D_1 \text{ where}$$
$$Q_1 = I_{80} - X(X'X)^* X'. \tag{19}$$

4.2. Estimating covariance matrices

Estimability is obtained traditionally by assuming equal covariance matrices—an assumption which is generally too restrictive and usually yields unacceptable estimates. In the following subsections, more general and acceptable assumptions regarding the covariance matrices, which provide more reasonable estimates, are illustrated. In the first of these subsections, equality of covariance matrices is assumed, just to illustrate the estimation procedure under constraints. In the subsequent sections, the MLEs of regression parameters under the following covariance structures are given:

- Proportional growth with a single growth factor
- Proportional growth with two different growth factors, i.e., allowing a change in growth
- Linear growth in proportional covariance matrices
- Linear growth in proportional covariance matrices using two splines

These scenarios illustrate the versatility of the proposed methodology and provide the MLEs of the regression parameters under more realistic covariance assumptions.

4.2.1 Variances and covariances constant over time

In order to express the assumption of equal covariance matrices, i.e., $\Sigma_1 = \Sigma_2 = \cdots = \Sigma_{20} = \Sigma$ in terms of constraints, the following matrices are used in the sequel:

$$D_2 = I_{20} \otimes (\mathbf{0}_{16\times 4} \; I_{16}) \text{ and } Q_2 = (I_{20} - \mathbf{1}_{20}\mathbf{1}'_{20}/20) \otimes I_{16}. \tag{20}$$

The matrix D_2 selects $vec(\Sigma_1 + \mu_1\mu'_1), \ldots, vec(\Sigma_{20} + \mu_{20}\mu'_{20})$ from m and Q_2 equates the covariance matrices. Consequently, the constraint which implies equality of covariance matrices is given by

$$g_2(m) = Q_2 \begin{pmatrix} vec(\Sigma_1) \\ \vdots \\ vec(\Sigma_{20}) \end{pmatrix} = Q_2[D_2 m - (D_1 m) \otimes (D_1 m)']$$

$$= \boldsymbol{Q}_2 \begin{pmatrix} \boldsymbol{m}_{12} - vec(\boldsymbol{m}_{11}\boldsymbol{m}_{11}') \\ \vdots \\ \boldsymbol{m}_{20,2} - vec(\boldsymbol{m}_{20,1}\boldsymbol{m}_{20,1}') \end{pmatrix}. \tag{21}$$

Imposing the constraints $\boldsymbol{g}_1$ (cf., (19)) and $\boldsymbol{g}_2$ given in (21) on $\boldsymbol{m}$ yields the traditional MLEs of the parameters (Greene 2012).

4.2.2 Proportional growth in covariance matrices

In many situations, the assumption of proportional covariance matrices over time periods is acceptable, which implies equal correlations of corresponding variates. This assumption then compensates for the lack of information due to an insufficient number of observations at any point in time. Different growth factors are allowed over specific time periods.

4.2.2.1. Proportional growth with a single growth factor. The first model considered is that of a constant proportional increase or decrease in covariance matrices, stipulated by $\boldsymbol{\Sigma}_i = \rho\boldsymbol{\Sigma}_{i-1}$, as illustrated in Table 4. Constraints are specified similarly to those in Section 3.1. A typical subvector of the constraints $\boldsymbol{g}_2(\boldsymbol{m})$ is given by:

$$\boldsymbol{g}_{2i}(\boldsymbol{m}) = vec(\boldsymbol{\Sigma}_i) - \rho.vec(\boldsymbol{\Sigma}_{i-1}) = 0, \quad i = 2, \ldots, 20 \tag{22}$$

with derivative

$$\boldsymbol{G}_{2i}(\boldsymbol{m}) = \frac{\partial vec(\boldsymbol{\Sigma}_i)}{\partial \boldsymbol{m}} - \rho\frac{\partial vec(\boldsymbol{\Sigma}_{i-1})}{\partial \boldsymbol{m}} - vec(\boldsymbol{\Sigma}_{i-1})\frac{\partial \rho}{\partial \boldsymbol{m}}$$

where

$$\rho = (\text{ sum of elements of } \boldsymbol{\Sigma}_{11}) \,/\, (\text{ sum of elements of } \boldsymbol{\Sigma}_{10}) = \boldsymbol{a}'\boldsymbol{m}/\boldsymbol{b}'\boldsymbol{m} = S_{11}/S_{10}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are appropriate selection vectors and

$$\frac{\partial \rho}{\partial \boldsymbol{m}} = (S_{10}\boldsymbol{a}' - S_{11}\boldsymbol{b}')/S_{10}^2.$$

4.2.2.2. Proportional growth with two different growth factors, i.e., allowing a change in growth. The previous model is extended and illustrated in Table 6 by using different proportional growth factors

$$\rho_1 = (\boldsymbol{\Sigma}_{10}[1,1]/\boldsymbol{\Sigma}_1[1,1])^{\frac{1}{9}} \text{ and } \rho_2 = (\boldsymbol{\Sigma}_{20}[1,1]/\boldsymbol{\Sigma}_{11}[1,1])^{\frac{1}{9}} \tag{23}$$

over the first 10 and over the last 10 covariance matrices. The notation $\boldsymbol{\Sigma}_i[1,1]$ represents the variance of the first variable measured in the ith sample, $i = 1, \ldots, 20$. As can be seen from Table 5, the first growth factor ρ_1 indicates a positive growth of 1.0729 per year in variances over the first 10 years, while the second growth factor ρ_2 represents a slight negative growth in variances of 0.9915 per year over the last 10 years. The model in this case is fitted making use of the additional restriction that $\boldsymbol{\Sigma}_{10} = \boldsymbol{\Sigma}_{11}$. Equal corresponding correlation coeffici ts of all the covariance matrices can be seen again.

4.2.3 Linear growth in proportional covariance matrices

4.2.3.1. Linear growth in proportional covariance matrices. The assumption of proportional covariance matrices over time periods may be extended to accommodate a linear

Table 4. Estimated growth in proportional covariance matrices with a single growth factor $\rho = 1.023052$.

Mean	Covariance matrix				Mean	Covariance matrix			
1	$\boldsymbol{\Sigma}_1$				11	$\boldsymbol{\Sigma}_{11} = \rho\boldsymbol{\Sigma}_{10}$			
215.3	5751.4	−1582.6	527.5	−212.9	521.0	7223.6	−1987.7	662.5	−267.4
225.8	−1582.6	6381.0	1015.4	332.0	356.7	−1987.7	8014.3	1275.4	417.0
33.1	527.5	1015.4	562.9	2.1	93.1	662.5	1275.4	707.0	2.7
33.7	−212.9	332.0	2.1	117.7	75.1	−267.4	417.0	2.7	147.8
2	$\boldsymbol{\Sigma}_2 = \rho\boldsymbol{\Sigma}_1$				12	$\boldsymbol{\Sigma}_{12} = \rho\boldsymbol{\Sigma}_{11}$			
419.0	5884.0	−1619.1	539.7	−217.8	580.5	7390.1	−2033.5	677.8	−273.5
282.8	−1619.1	6528.1	1038.9	339.7	351.3	−2033.5	8199.0	1304.8	426.6
64.7	539.7	1038.9	575.9	2.2	103.9	677.8	1304.8	723.3	2.8
60.7	−217.8	339.7	2.2	120.4	87.5	−273.5	426.6	2.8	151.2
3	$\boldsymbol{\Sigma}_3 = \rho\boldsymbol{\Sigma}_2$				13	$\boldsymbol{\Sigma}_{13} = \rho\boldsymbol{\Sigma}_{12}$			
543.4	6019.6	−1656.4	552.1	−222.8	557.5	7560.5	−2080.4	693.4	−279.9
425.1	−1656.4	6678.5	1062.8	347.5	371.0	−2080.4	8388.0	1334.8	436.5
95.1	552.1	1062.8	589.2	2.2	98.5	693.4	1334.8	740.0	2.8
71.3	−222.8	347.5	2.2	123.2	71.0	−279.9	436.5	2.8	154.7
4	$\boldsymbol{\Sigma}_4 = \rho\boldsymbol{\Sigma}_3$				14	$\boldsymbol{\Sigma}_{14} = \rho\boldsymbol{\Sigma}_{13}$			
260.8	6158.4	−1694.6	564.8	−228.0	587.2	7734.8	−2128.4	709.4	−286.3
369.8	−1694.6	6832.5	1087.3	355.5	366.2	−2128.4	8581.4	1365.6	446.5
72.5	564.8	1087.3	602.8	2.3	108.2	709.4	1365.6	757.0	2.9
47.9	−228.0	355.5	2.3	126.0	82.6	−286.3	446.5	2.9	158.3
5	$\boldsymbol{\Sigma}_5 = \rho\boldsymbol{\Sigma}_4$				15	$\boldsymbol{\Sigma}_{15} = \rho\boldsymbol{\Sigma}_{14}$			
436.0	6300.4	−1733.7	577.9	−233.2	676.5	7913.1	−2177.4	725.8	−292.9
412.1	−1733.7	6990.0	1112. 4	363.7	390.1	−2177.4	8779.2	1397.1	456.8
82.5	577.9	1112.4	616.6	2.3	111.9	725.8	1397.1	774.5	2.9
67.3	−233.2	363.7	2.3	128.9	87.4	−292.9	456.8	2.9	161.9
6	$\boldsymbol{\Sigma}_6 = \rho\boldsymbol{\Sigma}_5$				16	$\boldsymbol{\Sigma}_{16} = \rho\boldsymbol{\Sigma}_{15}$			
476.0	6445.6	−1773.6	591.2	−238.6	713.1	8095.5	−2227.6	742.5	−299.7
419.9	−1773.6	7151.1	1138.0	372.1	394.3	−2227.6	8981.6	1429.3	467.4
79.9	591.2	1138.0	630.9	2.4	122.2	742.5	1429.3	792.3	3.0
71.3	−238.6	372.1	2.4	131.9	99.0	−299.7	467.4	3.0	165.6
7	$\boldsymbol{\Sigma}_7 = \rho\boldsymbol{\Sigma}_6$				17	$\boldsymbol{\Sigma}_{17} = \rho\boldsymbol{\Sigma}_{16}$			
483.5	6594.2	−1814.5	604.8	−244.1	880.3	8282.1	−2279.0	759.6	−306.6
446.2	−1814.5	7316.0	1164.2	380.7	468.0	−2279.0	9188.6	1462.2	478.1
73.6	604.8	1164.2	645.4	2.5	133.0	759.6	1462.2	810.6	3.1
68.4	−244.1	380.7	2.5	134.9	119.0	−306.6	478.1	3.1	169.5
8	$\boldsymbol{\Sigma}_8 = \rho\boldsymbol{\Sigma}_7$				18	$\boldsymbol{\Sigma}_{18} = \rho\boldsymbol{\Sigma}_{17}$			
349.6	6746.2	−1856.3	618.8	−249.7	976.1	8473.0	−2331.5	777.1	−313.6
434.0	−1856.3	7484.6	1191.1	389.5	493.7	−2331.5	9400.4	1496.0	489.2
73.6	618.8	1191.1	660.3	2.5	149.8	777.1	1496.0	829.3	3.2
52.2	−249.7	389.5	2.5	138.0	140.8	−313.6	489.2	3.2	173.4
9	$\boldsymbol{\Sigma}_9 = \rho\boldsymbol{\Sigma}_8$				19	$\boldsymbol{\Sigma}_{19} = \rho\boldsymbol{\Sigma}_{18}$			
428.9	6901.7	−1899.1	633.0	−255.5	1262.2	8668.3	−2385.2	795.1	−320.9
411.2	−1899.1	7657.2	1218.5	398.5	551.9	−2385.2	9617.1	1530.4	500.4
83.7	633.0	1218.5	675.5	2.6	170.2	795.1	1530.4	848.4	3.2
62.3	−255.5	398.5	2.6	141.2	175.8	−320.9	500.4	3.2	177.4
10	$\boldsymbol{\Sigma}_{10} = \rho\boldsymbol{\Sigma}_9$				20	$\boldsymbol{\Sigma}_{20} = \rho\boldsymbol{\Sigma}_{19}$			
443.0	7060.8	−1942.9	647.6	−261.4	1358.2	8868.1	−2440.2	813.4	−328.3
379.3	−1942.9	7833.7	1246.6	407.6	582.2	−2440.2	9838.8	1565.7	512.0
81.6	647.6	1246.6	691.1	2.6	196.1	813.4	1565.7	868.0	3.3
67.4	−261.4	407.6	2.6	144.5	178.0	−328.3	512.0	3.3	181.5

Table 5. Estimated growth in proportional covariance matrices with two different growth factors $\rho_1 = 1.072856$ and $\rho_2 = 0.991520$.

Mean	Covariance matrix				Mean	Covariance matrix			
1	$\mathbf{\Sigma}_1$				11	$\mathbf{\Sigma}_{11} = \mathbf{\Sigma}_{10}$			
213.6	4642.2	−1323.0	408.6	−159.7	513.9	8741.8	−2491.3	769.4	−300.8
230.6	−1323.0	5126.4	736.0	288.9	358.5	−2491.3	9653.5	1386.0	544.1
32.5	408.6	736.0	409.8	1.2	91.6	769.4	1386.0	771.7	2.3
35.2	−159.7	288.9	1.2	94.1	75.2	−300.8	544.1	2.3	177.3
2	$\mathbf{\Sigma}_2 = \rho_1\mathbf{\Sigma}_1$				12	$\mathbf{\Sigma}_{12} = \rho_2\mathbf{\Sigma}_{11}$			
410.8	4980.5	−1419.4	438.4	−171.4	574.2	8667.6	−2470.2	762.9	−298.3
286.8	−1419.4	5499.9	789.6	310.0	353.7	−2470.2	9571.6	1374.3	539.5
62.3	438.4	789.6	439.7	1.3	102.1	762.9	1374.3	765.2	2.2
60.9	−171.4	310.0	1.3	101.0	87.4	−298.3	539.5	2.2	175.7
3	$\mathbf{\Sigma}_3 = \rho_1\mathbf{\Sigma}_2$				13	$\mathbf{\Sigma}_{13} = \rho_2\mathbf{\Sigma}_{12}$			
533.0	5343.3	−1522.8	470.3	−183.9	560.2	8594.1	−2449.3	756.4	−295.7
426.6	−1522.8	5900.6	847.2	332.6	372.2	−2449.3	9490.4	1362.6	534.9
91.1	470.3	847.2	471.7	1.4	98.4	756.4	1362.6	758.7	2.2
71.3	−183.9	332.6	1.4	108.3	71.7	−295.7	534.9	2.2	174.3
4	$\mathbf{\Sigma}_4 = \rho_1\mathbf{\Sigma}_3$				14	$\mathbf{\Sigma}_{14} = \rho_2\mathbf{\Sigma}_{13}$			
262.1	5732.6	−1633.8	504.6	−197.3	592.6	8521.3	−2428.5	750.0	−293.2
371.0	−1633.8	6330.5	908.9	356.8	367.2	−2428.5	9410.0	1351.0	530.4
70.2	504.6	908.9	506.1	1.5	108.6	750.0	1351.0	752.3	2.2
49.2	−197.3	356.8	1.5	116.2	83.0	−293.2	530.4	2.2	172.8
5	$\mathbf{\Sigma}_5 = \rho_1\mathbf{\Sigma}_4$				15	$\mathbf{\Sigma}_{15} = \rho_2\mathbf{\Sigma}_{14}$			
430.6	6150.3	−1752.8	541.3	−211.6	680.7	8449.0	−2407.9	743.6	−290.7
412.4	−1752.8	6791.7	975.1	382.8	390.4	−2407.9	9330.2	1339.6	525.9
79.9	541.3	975.1	543.0	1.6	113.0	743.6	1339.6	745.9	2.2
67.9	−211.6	382.8	1.6	124.7	88.0	−290.7	525.9	2.2	171.3
6	$\mathbf{\Sigma}_6 = \rho_1\mathbf{\Sigma}_5$				16	$\mathbf{\Sigma}_{16} = \rho_2\mathbf{\Sigma}_{15}$			
469.2	6598.3	−1880.5	580.8	−227.0	717.8	8377.4	−2387.5	737.3	−288.3
420.5	−1880.5	7286.5	1046.2	410.7	394.5	−2387.5	9251.0	1328.2	521.4
77.6	580.8	1046.2	582.5	1.7	123.1	737.3	1328.2	739.6	2.2
71.7	−227.0	410.7	1.7	133.8	99.3	−288.3	521.4	2.2	169.9
7	$\mathbf{\Sigma}_7 = \rho_1\mathbf{\Sigma}_6$				17	$\mathbf{\Sigma}_{17} = \rho_2\mathbf{\Sigma}_{16}$			
477.5	7079.1	−2017.5	623.1	−243.6	881.2	8306.3	−2367.2	731.1	−285.8
446.4	−2017.5	7817.3	1122.4	440.6	467.3	−2367.2	9172.6	1317.0	517.0
72.1	623.1	1122.4	624.9	1.8	133.6	731.1	1317.0	733.3	2.2
69.0	−243.6	440.6	1.8	143.5	118.8	−285.8	517.0	2.2	168.4
8	$\mathbf{\Sigma}_8 = \rho_1\mathbf{\Sigma}_7$				18	$\mathbf{\Sigma}_{18} = \rho_2\mathbf{\Sigma}_{17}$			
349.7	7594.8	−2164.5	668.5	−261.3	978.5	8235.9	−2347.2	724.9	−283.4
434.1	−2164.5	8386.9	1204.2	472.7	491.8	−2347.2	9094.8	1305.8	512.6
72.9	668.5	1204.2	670.5	2.0	150.0	724.9	1305.8	727.1	2.1
53.6	−261.3	472.7	2.0	154.0	140.7	−283.4	512.6	2.1	167.0
9	$\mathbf{\Sigma}_9 = \rho_1\mathbf{\Sigma}_8$				19	$\mathbf{\Sigma}_{19} = \rho_2\mathbf{\Sigma}_{18}$			
425.2	8148.1	−2322.2	717.2	−280.4	1261.8	8166.0	−2327.3	718.7	−281.0
411.6	−2322.2	8997.9	1291.9	507.1	547.9	−2327.3	9017.7	1294.7	508.3
82.8	717.2	1291.9	719.3	2.1	170.2	718.7	1294.7	720.9	2.1
63.1	−280.4	507.1	2.1	165.2	174.7	−281.0	508.3	2.1	165.6
10	$\mathbf{\Sigma}_{10} = \rho_1\mathbf{\Sigma}_9$				20	$\mathbf{\Sigma}_{20} = \rho_2\mathbf{\Sigma}_{19}$			
437.3	8741.8	−2491.3	769.4	−300.8	1364.4	8096.8	−2307.5	712.6	−278.6
380.2	−2491.3	9653.5	1386.0	544.1	577.5	−2307.5	8941.2	1283.7	503.9
80.9	769.4	1386.0	771.7	2.3	195.7	712.6	1283.7	714.8	2.1
67.9	−300.8	544.1	2.3	177.3	177.6	-278.6	503.9	2.1	164.2

Table 6. Estimated linear growth in proportional covariance matrices.

Mean	Covariance matrix				Mean	Covariance matrix			
1	$\boldsymbol{\Sigma}_1 = \rho_1 \boldsymbol{\Sigma}_{Av}, \rho_1 = 0.7443$				11	$\boldsymbol{\Sigma}_{11} = \rho_{11} \boldsymbol{\Sigma}_{Av}, \rho_{11} = 1.0135$			
217.0	5386.3	−1430.4	496.8	−190.8	519.7	7334.4	−1947.8	676.5	−259.8
225.6	−1430.4	5952.5	938.2	312.2	356.3	−1947.8	8105.3	1277.5	425.1
33.0	496.8	938.2	522.2	2.0	92.7	676.5	1277.5	711.0	2.7
33.9	−190.8	312.2	2.0	109.4	75.1	−259.8	425.1	2.7	148.9
2	$\boldsymbol{\Sigma}_2 = \rho_2 \boldsymbol{\Sigma}_{Av}, \rho_2 = 0.7712$				12	$\boldsymbol{\Sigma}_{12} = \rho_{12} \boldsymbol{\Sigma}_{Av}, \rho_{12} = 1.0404$			
417.7	5581.2	−1482.2	514.8	−197.7	579.1	7529.2	−1999.5	694.5	−266.7
283.2	−1482.2	6167.8	972.1	323.5	351.6	−1999.5	8320.6	1311.4	436.4
64.2	514.8	972.1	541.0	2.0	103.4	694.5	1311.4	729.9	2.8
60.7	−197.7	323.5	2.0	113.3	87.5	−266.7	436.4	2.8	152.9
3	$\boldsymbol{\Sigma}_3 = \rho_3 \boldsymbol{\Sigma}_{Av}, \rho_3 = 0.7981$				13	$\boldsymbol{\Sigma}_{13} = \rho_{13} \boldsymbol{\Sigma}_{Av}, \rho_{13} = 1.0673$			
540.9	5776.0	−1533.9	532.8	−204.6	559.1	7724.0	−2051.2	712.5	−273.6
426.2	−1533.9	6383.1	1006.0	334.8	370.3	−2051.2	8535.9	1345.3	447.7
94.3	532.8	1006.0	559.9	2.1	98.4	712.5	1345.3	748.8	2.8
71.3	−204.6	334.8	2.1	117.3	71.0	−273.6	447.7	2.8	156.8
4	$\boldsymbol{\Sigma}_4 = \rho_4 \boldsymbol{\Sigma}_{Av}, \rho_4 = 0.8250$				14	$\boldsymbol{\Sigma}_{14} = \rho_{14} \boldsymbol{\Sigma}_{Av}, \rho_{14} = 1.0942$			
263.2	5970.8	−1585.6	550.7	−211.5	589.6	7918.8	−2103.0	730.4	−280.5
369.1	−1585.6	6598.4	1040.0	346.1	365.0	−2103.0	8751.2	1379.3	459.0
72.0	550.7	1040.0	578.8	2.2	108.2	730.4	1379.3	767.7	2.9
48.0	−211.5	346.1	2.2	121.2	82.6	−280.5	459.0	2.9	160.8
5	$\boldsymbol{\Sigma}_5 = \rho_5 \boldsymbol{\Sigma}_{Av}, \rho_5 = 0.8520$				15	$\boldsymbol{\Sigma}_{15} = \rho_{15} \boldsymbol{\Sigma}_{Av}, \rho_{15} = 1.1211$			
435.5	6165.6	−1637.4	568.7	−218.4	678.1	8113.6	−2154.7	748.4	−287.4
411.3	−1637.4	6813.7	1073.9	357.3	388.7	−2154.7	8966.4	1413.2	470.3
81.9	568.7	1073.9	597.7	2.3	112.0	748.4	1413.2	786.5	3.0
67.4	−218.4	357.3	2.3	125.2	87.5	−287.4	470.3	3.0	164.8
6	$\boldsymbol{\Sigma}_6 = \rho_6 \boldsymbol{\Sigma}_{Av}, \rho_6 = 0.8789$				16	$\boldsymbol{\Sigma}_{16} = \rho_{16} \boldsymbol{\Sigma}_{Av}, \rho_{16} = 1.1480$			
474.9	6360.4	−1689.1	586.7	−225.3	714.7	8308.4	−2206.4	766.4	−294.3
419.7	−1689.1	7028.9	1107.8	368.6	392.9	−2206.4	9181.7	1447.1	481.5
79.4	586.7	1107.8	616.6	2.3	122.3	766.4	1447.1	805.4	3.1
71.4	−225.3	368.6	2.3	129.2	99.0	−294.3	481.5	3.1	168.7
7	$\boldsymbol{\Sigma}_7 = \rho_7 \boldsymbol{\Sigma}_{Av}, \rho_7 = 0.9058$				17	$\boldsymbol{\Sigma}_{17} = \rho_{17} \boldsymbol{\Sigma}_{Av}, \rho_{17} = 1.1750$			
482.7	6555.2	−1740.8	604.7	−232.2	880.0	8503.2	−2258.2	784.3	−301.2
446.2	−1740.8	7244.2	1141.8	379.9	467.5	−2258.2	9397.0	1481.1	492.8
73.3	604.7	1141.8	635.5	2.4	133.0	784.3	1481.1	824.3	3.1
68.5	−232.2	379.9	2.4	133.1	118.9	−301.2	492.8	3.1	172.7
8	$\boldsymbol{\Sigma}_8 = \rho_8 \boldsymbol{\Sigma}_{Av}, \rho_8 = 0.9327$				18	$\boldsymbol{\Sigma}_{18} = \rho_{18} \boldsymbol{\Sigma}_{Av}, \rho_{18} = 1.2019$			
351.3	6750.0	−1792.6	622.6	−239.1	975.9	8698.0	−2309.9	802.3	−308.1
433.5	−1792.6	7459.5	1175.7	391.2	492.4	−2309.9	9612.3	1515.0	504.1
73.4	622.6	1175.7	654.3	2.5	149.7	802.3	1515.0	843.2	3.2
52.4	−239.1	391.2	2.5	137.1	140.7	−308.1	504.1	3.2	176.6
9	$\boldsymbol{\Sigma}_9 = \rho_9 \boldsymbol{\Sigma}_{Av}, \rho_9 = 0.9596$				19	$\boldsymbol{\Sigma}_{19} = \rho_{19} \boldsymbol{\Sigma}_{Av}, \rho_{19} = 1.2288$			
429.0	6944.8	−1844.3	640.6	−246.0	1259.9	8892.8	−2361.6	820.3	−315.0
410.5	−1844.3	7674.8	1209.6	402.5	549.5	−2361.6	9827.6	1548.9	515.4
83.5	640.6	1209.6	673.2	2.5	170.1	820.3	1548.9	862.1	3.3
62.4	−246.0	402.5	2.5	141.0	175.5	−315.0	515.4	3.3	180.6
10	$\boldsymbol{\Sigma}_{10} = \rho_{10} \boldsymbol{\Sigma}_{Av}, \rho_{10} = 0.9865$				20	$\boldsymbol{\Sigma}_{20} = \rho_{20} \boldsymbol{\Sigma}_{Av}, \rho_{20} = 1.2557$			
442.4	7139.6	−1896.0	658.6	−252.9	1357.6	9087.6	−2413.4	838.2	−321.9
378.5	−1896.0	7890.0	1243.6	413.8	579.7	−2413.4	10042.8	1582.9	526.7
81.4	658.6	1243.6	692.1	2.6	195.9	838.2	1582.9	881.0	3.3
67.4	−252.9	413.8	2.6	145.0	177.7	−321.9	526.7	3.3	184.5

Table 7. Estimated linear growth in proportional covariance matrices using two splines.

Mean	Covariance matrix				Mean	Covariance matrix			
1		$\mathbf{\Sigma}_1 = \rho_1 \mathbf{\Sigma}_{Av}, \rho_1 = 0.6014$			11		$\mathbf{\Sigma}_{11} = \rho_{11} \mathbf{\Sigma}_{Av}, \rho_{11} = 1.1575$		
214.1	4496.3	−1306.7	392.1	−155.8	514.8	8653.2	−2514.7	754.5	−299.9
231.0	−1306.7	4926.6	704.8	277.2	359.0	−2514.7	9481.2	1356.3	533.4
32.4	392.1	704.8	395.2	0.5	91.7	754.5	1356.3	760.6	1.0
35.2	−155.8	277.2	0.5	91.2	75.2	−299.9	533.4	1.0	175.4
2		$\mathbf{\Sigma}_2 = \rho_2 \mathbf{\Sigma}_{Av}, \rho_2 = 0.6632$			12		$\mathbf{\Sigma}_{12} = \rho_{12} \mathbf{\Sigma}_{Av}, \rho_{12} = 1.1493$		
411.9	4958.2	−1440.9	432.3	−171.8	575.0	8591.8	−2496.8	749.2	−297.8
287.0	−1440.9	5432.7	777.2	305.7	354.0	−2496.8	9414.0	1346.7	529.7
62.3	432.3	777.2	435.8	0.6	102.2	749.2	1346.7	755.2	1.0
61.0	-171.8	305.7	0.6	100.5	87.5	−297.8	529.7	1.0	174.2
3		$\mathbf{\Sigma}_3 = \rho_3 \mathbf{\Sigma}_{Av}, \rho_3 = 0.7250$			13		$\mathbf{\Sigma}_{13} = \rho_{13} \mathbf{\Sigma}_{Av}, \rho_{13} = 1.1411$		
534.2	5420.1	−1575.1	472.6	-187.8	560.2	8530.4	−2479.0	743.8	−295.6
426.6	−1575.1	5938.7	849.6	334.1	372.8	−2479.0	9346.7	1337.1	525.9
91.3	472.6	849.6	476.4	0.6	98.5	743.8	1337.1	749.8	1.0
71.4	−187.8	334.1	0.6	109.9	71.8	−295.6	525.9	1.0	172.9
4		$\mathbf{\Sigma}_4 = \rho_4 \mathbf{\Sigma}_{Av}, \rho_4 = 0.7868$			14		$\mathbf{\Sigma}_{14} = \rho_{14} \mathbf{\Sigma}_{Av}, \rho_{14} = 1.1329$		
262.3	5881.9	−1709.3	512.9	−203.8	592.3	8469.0	−2461.2	738.5	-293.5
371.6	−1709.3	6444.8	922.0	362.6	367.9	−2461.2	9279.4	1327.5	522.1
70.3	512.9	922.0	517.0	0.7	108.7	738.5	1327.5	744.4	1.0
49.2	−203.8	362.6	0.7	119.3	83.0	−293.5	522.1	1.0	171.7
5		$\mathbf{\Sigma}_5 = \rho_5 \mathbf{\Sigma}_{Av}, \rho_5 = 0.8486$			15		$\mathbf{\Sigma}_{15} = \rho_{15} \mathbf{\Sigma}_{Av}, \rho_{15} = 1.1246$		
431.4	6343.8	−1843.6	553.2	−219.9	680.6	8407.6	−2443.3	733.1	−291.4
413.0	−1843.6	6950.9	994.4	391.1	391.2	−2443.3	9212.1	1317.8	518.3
80.0	553.2	994.4	557.6	0.7	113.1	733.1	1317.8	739.0	1.0
68.0	−219.9	391.1	0.7	128.6	88.1	−291.4	518.3	1.0	170.5
6		$\mathbf{\Sigma}_6 = \rho_6 \mathbf{\Sigma}_{Av}, \rho_6 = 0.9104$			16		$\mathbf{\Sigma}_{16} = \rho_{16} \mathbf{\Sigma}_{Av}, \rho_{16} = 1.1164$		
470.1	6805.7	−1977.8	593.4	−235.9	717.6	8346.2	−2425.5	727.8	−289.2
420.9	−1977.8	7456.9	1066.8	419.5	395.2	-2425.5	9144.8	1308.2	514.5
77.7	593.4	1066.8	598.2	0.8	123.2	727.8	1308.2	733.6	1.0
71.8	−235.9	419.5	0.8	138.0	99.3	−289.2	514.5	1.0	169.2
7		$\mathbf{\Sigma}_7 = \rho_7 \mathbf{\Sigma}_{Av}, \rho_7 = 0.9721$			17		$\mathbf{\Sigma}_{17} = \rho_{17} \mathbf{\Sigma}_{Av}, \rho_{17} = 1.1082$		
478.4	7267.6	−2112.0	633.7	−251.9	881.3	8284.8	-2407.6	722.4	−287.1
446.7	−2112.0	7963.0	1139.2	448.0	467.8	−2407.6	9077.6	1298.6	510.7
72.1	633.7	1139.2	638.8	0.8	133.7	722.4	1298.6	728.2	1.0
69.1	−251.9	448.0	0.8	147.3	118.9	−287.1	510.7	1.0	168.0
8		$\mathbf{\Sigma}_8 = \rho_8 \mathbf{\Sigma}_{Av}, \rho_8 = 1.0339$			18		$\mathbf{\Sigma}_{18} = \rho_{18} \mathbf{\Sigma}_{Av}, \rho_{18} = 1.1000$		
350.0	7729.4	−2246.2	674.0	−267.9	978.5	8223.4	−2389.8	717.0	−285.0
434.5	−2246.2	8469.1	1211.6	476.5	492.5	−2389.8	9010.3	1289.0	506.9
72.9	674.0	1211.6	679.4	0.9	150.2	717.0	1289.0	722.8	1.0
53.6	−267.9	476.5	0.9	156.7	140.7	−285.0	506.9	1.0	166.7
9		$\mathbf{\Sigma}_9 = \rho_9 \mathbf{\Sigma}_{Av}, \rho_9 = 1.0957$			19		$\mathbf{\Sigma}_{19} = \rho_{19} \mathbf{\Sigma}_{Av}, \rho_{19} = 1.0918$		
425.8	8191.3	−2380.5	714.2	−283.9	1262.0	8162.0	−2371.9	711.7	−282.9
412.1	−2380.5	8975.2	1284.0	505.0	548.9	−2371.9	8943.0	1279.4	503.2
82.9	714.2	1284.0	720.0	1.0	170.4	711.7	1279.4	717.4	0.9
63.2	−283.9	505.0	1.0	166.1	174.7	−282.9	503.2	0.9	165.5
10		$\mathbf{\Sigma}_{10} = \rho_{10} \mathbf{\Sigma}_{Av}, \rho_{10} = 1.1575$			20		$\mathbf{\Sigma}_{20} = \rho_{20} \mathbf{\Sigma}_{Av}, \rho_{20} = 1.0836$		
438.1	8653.2	−2514.7	754.5	−299.9	1363.9	8100.6	−2354.1	706.3	−280.7
380.8	−2514.7	9481.2	1356.3	533.4	578.5	−2354.1	8875.7	1269.7	499.4
80.9	754.5	1356.3	760.6	1.0	196.0	706.3	1269.7	712.0	0.9
67.9	−299.9	533.4	1.0	175.4	177.6	−280.7	499.4	0.9	164.2

increase over the time slots by making use of the fact that if covariance matrices are proportional, then each covariance matrix is also proportional to the overall average covariance matrix $\boldsymbol{\Sigma}_{Av} = \frac{1}{20}\sum_{i=1}^{20}\boldsymbol{\Sigma}_i$, constituting 20 different proportionality constants ρ_i for $i = 1, 2, \ldots, 20$. The ρ_i's can now be restricted, for example, to be on one or more straight lines in order to ensure estimability. This linearization of the ρ_is also implies that the corresponding elements of the Σ_i's are on straight lines. In this way, equal correlation coefficients as well as a linear increase or decrease on the corresponding elements of the $\boldsymbol{\Sigma}_i$'s are accomplished. Formally

$$\rho_i = \alpha + i\beta$$

which again implies that

$$\boldsymbol{\Sigma}_i = (\alpha + i\beta)\boldsymbol{\Sigma}_{Av}. \tag{24}$$

Constraints are specified similarly to those in Section 3.1, where a typical subvector of the constraints $\boldsymbol{g}_2(\boldsymbol{m})$ (cf., (9)) is given by:

$$\boldsymbol{g}_{2i}(\boldsymbol{m}) = vec(\boldsymbol{\Sigma}_i) - \rho_i.vec(\boldsymbol{\Sigma}_{Av}) = 0, \quad i = 1, \ldots, 20 \tag{25}$$

with derivative

$$\boldsymbol{G}_{2i}(\boldsymbol{m}) = \frac{\partial vec(\boldsymbol{\Sigma}_i)}{\partial \boldsymbol{m}} - \rho_i \frac{\partial vec(\boldsymbol{\Sigma}_{Av})}{\partial \boldsymbol{m}} - vec(\boldsymbol{\Sigma}_{Av})\frac{\partial \rho_i}{\partial \boldsymbol{m}}$$

where

$$\rho_i = \boldsymbol{\Sigma}_i[1,1]/\boldsymbol{\Sigma}_{Av}[1,1] = \boldsymbol{a}'\boldsymbol{m}/\boldsymbol{b}'\boldsymbol{m}$$

where $\boldsymbol{\Sigma}_i[1,1]$ is the variance of the fi st variable measured in the ith sample, $\boldsymbol{\Sigma}_{Av}[1,1]$ is the variance of the first variable in $\boldsymbol{\Sigma}_{Av}$ and

$$\frac{\partial \rho_i}{\partial \boldsymbol{m}} = \left(\boldsymbol{b}'\boldsymbol{m}\boldsymbol{a}' - \boldsymbol{a}'\boldsymbol{m}\boldsymbol{b}'\right) / \left[(\boldsymbol{b}'\boldsymbol{m})^2\right]$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are appropriate selection vectors.

Linearity of the $\rho_i's$ implies the constraint $\boldsymbol{g}_3(\boldsymbol{m}) = \boldsymbol{Q}_\rho \boldsymbol{\rho}$ where

$$\boldsymbol{\rho} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_{20} \end{pmatrix}, \quad \boldsymbol{Q}_\rho = (\boldsymbol{I}_{20} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}') \text{ and } \boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & 20 \end{pmatrix}. \tag{26}$$

The MLEs under these constraints are illustrated in Table 6.

The estimated average covariance matrix for the model fitted in Table 6 is given by:

$$\boldsymbol{\Sigma}_{Av} = \begin{pmatrix} 7237.0 & -1921.9 & 667.5 & -256.3 \\ -1921.9 & 7997.7 & 1260.5 & 419.4 \\ 667.5 & 1260.5 & 701.6 & 2.7 \\ -256.3 & 419.4 & 2.7 & 147.0 \end{pmatrix}$$

Note that $\boldsymbol{\Sigma}_{Av}$ equals the average of any two estimated $\boldsymbol{\Sigma}$'s which are on equal distances from the center of the 20 estimated $\boldsymbol{\Sigma}$'s, e.g., $\boldsymbol{\Sigma}_{Av} = \frac{1}{2}(\boldsymbol{\Sigma}_5 + \boldsymbol{\Sigma}_{16})$.

4.2.3.2. Linear growth in proportional covariance matrices using two splines. The model can be reformulated similarly in terms of two growth splines separately fitted over the fi st 10 and last 10 years. The estimated average covariance matrix for this model fitted in Table 7 is given by:

$$\mathbf{\Sigma}_{Av} = \begin{pmatrix} 7475.82 & -2172.53 & 651.86 & -259.08 \\ -2172.53 & 8191.20 & 1171.80 & 460.86 \\ 651.86 & 1171.80 & 657.08 & 0.87 \\ -259.08 & 460.86 & 0.87 & 151.57 \end{pmatrix}$$

In this case, $\mathbf{\Sigma}_{Av}$ equals the average of any four estimated $\mathbf{\Sigma}$'s, two of which are on symmetric numbers in the first column and two of which are on symmetric numbers in the second column, e.g., $\mathbf{\Sigma}_{Av} = \frac{1}{4}(\mathbf{\Sigma}_1 + \mathbf{\Sigma}_{10} + \mathbf{\Sigma}_{11} + \mathbf{\Sigma}_{20})$ or $\mathbf{\Sigma}_{Av} = \frac{1}{4}(\mathbf{\Sigma}_4 + \mathbf{\Sigma}_7 + \mathbf{\Sigma}_{12} + \mathbf{\Sigma}_{19})$.

As can be expected, the results in Table 4 correspond largely to the results in Table 6, while the results in Table 5 correspond to the results in Table 7 to a large extent. It is, however, evident that as far as the means and the covariances are concerned, the growth patterns in the fi st 10 years differ substantially from that in the second 10 years. These examples stress the importance of taking different covariance patterns into consideration.

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