Estimation techniques for seismic recurrence parameters for incomplete catalogues

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In probabilistic seismic hazard analysis, assessment of the three recurrence parameters, namely the mean activity rate λ, Gutenberg–Richter b-value, and maximum possible seismic event magnitude m_{max}, is paramount. Over the years, several assessment procedures have been developed, each with its advantages and disadvantages. Typically, estimation techniques for the mean activity rate λ and the Gutenberg–Richter b-value are discussed and evaluated separately from those designed for the maximum possible event magnitude m_{max}. Yet, the three parameters are typically defined in terms of joint distributions for λ, b, and m_{max}. In this study, we focused on systematically constructing joint distributions for the three recurrence parameters for considering complete and incomplete seismic event catalogues. The Bayesian formalism is introduced to constrain the parameter estimates with independent *a priori* information. Further, we discuss an iterative technique to solve the systems of equations sequentially. The procedures are compared and illustrated using Monte Carlo simulation and a seismic event catalogue for Cape Town, South Africa.

Keyword: complete and incomplete catalogues, maximum likelihood estimation, Bayesian estimation, iterative estimation.

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1. Introduction

In 1966, Allin Cornell, a Stanford University PhD graduate and already professor at the Massachusetts Institute of Technology (MIT), participated in a consulting project to assess an appropriate earthquake design ground motion for the Alibey Dam, north of Istanbul, Turkey. It was anticipated that the future dam would be affected by strong seismic activity caused by the nearby Anatolian fault.

Not surprisingly, to solve the expected problems, Cornell chose probabilistic techniques. Such techniques were close to his heart, as his PhD dissertation was based on probabilistic concepts of distributions of elements that affect engineering decisions. Moreover, quite recently, he had returned from Mexico, where he had met Dr Luis Esteva at the National Autonomous University of Mexico. Esteva was another enthusiast of applying probabilistic techniques in earthquake seismology. Esteva and his research group were studying probabilistic aspects of earthquake-generated ground motions, their dependence on magnitude and distance (currently called ground motion models), and the relationships between the frequency of earthquake occurrence and the frequency of occurrence of specified ground motion at a site. Esteva had published the first seismogenic-zone-based probabilistic seismic hazard maps of Mexico, which were expressed in terms of the modified Mercalli intensity (MMI), associated peak ground acceleration (PGA), and mean return periods. These maps were based on quite simple probabilistic notions of earthquake occurrence and are the first such maps ever computed.

In 1968, Cornell published one of his best-known works on the principles of probabilistic seismic hazard analysis (PSHA) (Cornell 1968). In this approach, he considered two types of ground motion, namely the descriptive MM intensity I and that recorded instrumentally, as, e.g. PGA. In addition, he made two fundamental assumptions, namely, first, about the nature of attenuation of seismic waves and second, about the earthquake magnitude distribution and the distribution of seismic events in time. He assumed that earthquake magnitudes are distributed according to the frequency-magnitude Gutenberg–Richter relation (Gutenberg and Richter 1942; 1956), and that the number of seismic events in time follow the Poisson distribution.

However, his approach excluded considering the variability of the predicted ground motion. Already in 1964, it was known (Rosenblueth 1964) that, based on the functional form of GMM, predicting the exact value of ground motion at a site is virtually impossible. The expected value of the ground motion could probably be predicted, but the ground motion at the site, at best, could be described by its distribution. Esteva (1969, 1970) incorporated the variability of ground motion into Cornell's seismic hazard formalism. Esteva proposed integration across the ground motion scatter, where the variability of ground motion is described by Gaussian distribution. In this manner, Esteva could quantify the ground motion uncertainty, and he was probably the first to call it "aleatory uncertainty". Integration across the ground motion scatter from the lognormally distributed residuals in the attenuation equation is standard today.

Unfortunately, none of these mentioned publications is as well known as the original work by Cornell (1968). However, Allen Cornell and Luis Esteva should be credited for the PSHA formalism known today (Alamilla *et al*., 2020).

In addition, the widespread use of PSHA, as formulated by Esteva and Cornell, is also owed to Dr Robin McGuire, who, in 1976, developed the freely available computer code EQRISK (McGuire, 1976). Before the introduction of OpenQuake (Pagani 2014), McGuire's computer code was probably the most frequently used software for PSHA. Accordingly, seismic hazard formalism, as developed by Esteva and Cornell, should be called the Esteva–Cornell, Cornell–McGuire (Atkinson 2004; Bommer and Abrahamson 2006; McGuire 2008), or the Esteva–Cornell–McGuire procedure.

The Esteva–Cornell–McGuire formalism is most often used in PSHA, our brief review of probabilistic hazard assessment is focused entirely on the approach of these pioneering authors.

It is important to note that none of the authors of the Esteva–Cornell–McGuire PSHA procedure discussed the assessment of the seismic hazard recurrence parameters required by their formalism. These parameters are the mean activity rate λ, the b-value of the frequency-magnitude Gutenberg–Richter relation, and the area-characteristic maximum possible seismic event magnitude m_{max}. The Esteva–Cornell–McGuire PSHA technique cannot be applied when these parameters are not known. Assessing these key parameters is particularly important when seismic event catalogues are incomplete and uncertain.

Over the years, several researchers have focused on assessing the recurrence parameters for both complete and incomplete seismic event catalogues. Amongst others, these researchers are Aki (1965), Utsu (1965), Molchan *et al*. (1970), Weichert (1980), Kijko and Sellevoll (1989, 1992), Rosenblueth (1986), Rosenblueth and Ordaz (1987), Marzocchi and Sandri (2003), Kijko and Smit (2012), Kijko *et al*. (2016), Ordaz and Giraldo (2018), and Vermeulen and Kijko (2019). The focus of these studies was the assessment of the mean activity rate λ and the Gutenberg–Richter b-value.

The assessment of m_{max} is often addressed separately, using techniques differing from those for assessing λ and the b-value, owing to the complex nature of the area-characteristic maximum possible seismic event magnitude. Several studies have been published on this topic, including those of Cooke (1979), Pisarenko (1991), Pisarenko *et al*. (1996) and references therein, Kijko (2004), Wheeler (2009), Kijko and Singh (2011), Pisarenko *et al*. (2014) and references therein, Vermeulen and Kijko (2017), Pisarenko and Rodkin (2017), and Beirlant *et al*. (2019). Some degree of debate remains amongst authors, e.g. Holshneider *et al*. (2011), who consider the parameter ill-defined, whereas others (e.g. Kagan, 2002a, b and references therein; Raschke, 2015) propose specific modifications to magnitude distribution, with one of the distribution parameters being called the "soft" maximum earthquake magnitude. Beyond the value of such m_{max}, the distribution decays much faster than that indicated by the classical Gutenberg–Richter relation. However, this implies that a "soft"' cut-off allows earthquake magnitudes larger than those defined by m_{max}. To avoid confusion, in the current study, the maximum earthquake magnitude m_{max} is defined as the upper limit of magnitude for a given seismogenic zone or entire region. This terminology assumes a sharp cut-off magnitude at the maximum magnitude, implying, by definition, that no earthquakes are possible with a magnitude exceeding m_{max}.

The reason for assessing m_{max} separately is that the range of observations used by the sample likelihood function (earthquake magnitudes) depends on the unknown parameter m_{max}. This dependence violates the regularity conditions of the likelihood function (Cheng and Taylor 1995; LeCam 1970; Eadie *et al*. 1971; Davison 2003). The estimation process will, therefore, reach a maximum at the maximum *observed* event magnitude m_{max}^{obs} , and not the required maximum *possible* event magnitude m_{max}. Consequently, other estimation techniques have been applied (Cooke 1979; Pisarenko 1991; Pisarenko *et al*. 1996; Kijko 2004; Kijko and Singh 2011). These methods allow assessment for m_{max} when a significant amount of data is available. However, such an amount of data is not obtainable when the available catalogue is small or the area under investigation is not prone to significant seismic activity (e.g. Chinnery 1979; Bender 1988). The estimation of m_{max} could be improved by including prehistoric (paleo) and historical earthquakes (e.g. Kijko *et al*. 2016). Additional, independent information from tectonic, geological, and geophysical considerations could also be included using the Bayesian formalism (Coppersmith 1994; Cornell 1994; Kijko 2012; Johnston 1994). Although Bayesian formalism is a powerful tool, careful consideration of the applied technique, quality of the *a priori* information, and the seismic event catalogue is required to avoid biased estimates.

This study aims to provide a uniform, stepwise approach for the assessment of the three seismic hazard recurrence parameters, namely λ, b, and m_{max} that is applicable to complete and incomplete seismic event catalogues, and could utilise *a priori* information through Bayesian formalism.

In Section 2.1, we review assessment methodologies for the three recurrence parameters in the case of a complete seismic event catalogue. The case of an incomplete seismic event catalogues is discussed in Section 2.2. The Bayesian formalism, allowing for the incorporation of *prior* information, is presented in Section 2.3. Application examples are provided in Section 3, and the conclusions in Section 4.

2. Methodology

2.1. Complete seismic event catalogues

This section focuses on assessing the seismic hazard recurrence parameters for a single, complete seismic event catalogue. First, we assumed that the catalogue spans t years and contains magnitudes of seismic events that are each equal to or exceeding the known level of completeness m_C. According to the Gutenberg–Richter frequency-magnitude relation (Gutenberg and Richter 1942, 1956), the number of seismic events n, that have a magnitude equal to or larger than m, can be expressed by the equation

$$\log(n) = a - bm, \tag{1}$$

where parameter a is a measure of the seismicity level, and parameter b describes the ratio between the number of small and large events. Parameter b is known as the b-value of the Gutenberg–Richter relation.

Equation (1) was first established empirically by Ishimoto and Iida (1939) and promoted by Gutenberg and Richter (1942, 1956). The equation plays a central role in seismic studies. It is used to describe both tectonic and anthropogenic seismicity, can be applied in different time scales, and holds over a considerable range of seismic event magnitudes. Cornell (1968), citing Isacks and Oliver (1964), noted that the b-value for tectonic-origin seismicity typically varies between 0.65 and 1.0. Previously, it was generally accepted that $b \cong 1$ for areas of tectonic seismicity (Cornell 1968).

Second, we assumed that the magnitudes of seismic events are independent and identically distributed random variables, continuous in the interval $[m_c, m_{max}]$, with m_c representing the level of completeness of the seismic event catalogue.

For the frequency-magnitude Gutenberg–Richter relation (1), the probability distribution function (PDF) and cumulative distribution function (CDF) of earthquake magnitude are exponential, shifted-truncated, and equal to

$$f_M(m) = \begin{cases} \dfrac{\beta \exp[-\beta(m - m_C)]}{1 - \exp[-\beta(m_{\max} - m_C)]} & \text{for } m_C \leq m \leq m_{\max} \\ 0 & \text{for } m > m_{\max} \end{cases} \tag{2}$$

and

$$F_M(m) = \begin{cases} \dfrac{1 - \exp[-\beta(m - m_C)]}{1 - \exp[-\beta(m_{\max} - m_C)]}, & \text{for } m_C \leq m \leq m_{\max} \\ 1 & \text{for } m > m_{\max}. \end{cases} \tag{3}$$

(Page 1968; Cosentino *et al*. 1977), where $\beta = b\ln(10)$, and b is the parameter of the frequency-magnitude Gutenberg–Richter relation.

Following Aki (1965), knowledge of earthquake magnitude distribution (2) enables estimation of the β-value by the maximum likelihood (ML) method. The ML estimate of parameter β, $\hat{\beta}$, is defined as the value of β that, for specified n, m_C, and $m_{\max}$, maximises the sample likelihood function

$$L(\mathbf{m}\,|\,\beta) = \prod_{i=1}^{n} f_M(m_i), \tag{4a}$$

or equivalently

$$L(\mathbf{m}\,|\,\beta) = \prod_{i=1}^{n} \left[\frac{\beta \exp[-\beta(m_i - m_C)]}{1-\exp[-\beta(m_{\max} - m_C)]}\right], \tag{4b}$$

where $\mathbf{m} = (m_1, m_2, \dots, m_n)$ denotes magnitudes of n seismic events, each equal to or exceeding the level of completeness m_C. The maximisation of Eq. 4b, with respect to β, leads to equation (Hamilton 1967; Page 1968)

$$\frac{1}{\beta} = \bar{m} - m_C + \frac{(m_{max} - m_c)\exp[-\beta(m_{max} - m_c)]}{1 - \exp[-\beta(m_{max} - m_c)]}, \tag{5}$$

where $\bar{m}$ is the sample mean of earthquake magnitudes $\mathbf{m}$. As the β parameter appears on both sides of Eq. 5, the ML estimate $\hat{\beta}$ is obtained from an iterative solution of the equation.

When the magnitude range $[m_c, m_{max}]$ is large (in practice, this must be only a few units of magnitude), m_{max} could be assumed to be at infinity, and Eq. 5 takes the well-known form of the ML estimator of β (Aki 1965; Utsu 1965)

$$\hat{\beta}_{AU} = \frac{1}{\bar{m} - m_c}. \tag{6}$$

Utsu (1965) was the first to derive Eq. 6 by utilising the method of moments, i.e. comparing the first population moment with an equivalent sample moment. In the same year, Aki (1965) derived the same equation by applying the ML procedure. Further on in this paper, Eq. 6 is referred to as the Aki–Utsu estimator of the β-value, and is denoted as $\hat{\beta}_{AU}$.

The ML estimator for the mean activity rate λ_C, for a single, complete catalogue that spans a time interval t, and contains n events equal to or exceeding m_C, has the form (e.g. Benjamin and Cornell 2014)

$$\hat{\lambda}_C = \frac{n}{t}. \tag{7}$$

By introducing the upper limit of the distribution m_{max}, the activity rate for any given magnitude *m*, within interval $[m_C, m_{max}]$, becomes

$$\hat{\lambda}_m \equiv \lambda(m) = \lambda_C\, P[M \geq m] = \lambda_C\, [1 - F_M(m)]. \tag{8}$$

Following the approach applied in the assessment of the β-value and λ, the natural choice would be to determine the maximum possible event magnitude $m_{\max}$ using the ML method, i.e. by maximisation of the sample likelihood function

$$L(\mathbf{m}|m_{\max}) = \prod_{i=1}^{n} f_M(m_{\mathrm{i}}), \tag{9}$$

where $f_M(m)$ denotes the PDF of the earthquake magnitude in Eq. 2. Unfortunately, the sample likelihood function for m_{max} in Eq. 9 is defined such that its range depends on the unknown m_{max}, thereby violating the regularity condition of likelihood functions (Cheng and Traylor 1995; LeCam 1970). The ML estimate of Eq. 9 reaches its maximum at the maximum *observed* earthquake magnitude $m_{\max}^{obs}$, and not at the required maximum *possible* magnitude $m_{\max}$ (Pisarenko 1991; Cornell 1994). Consequently, an unbiased estimate of $\widehat{m}_{max}$ cannot be obtained by maximisation of the likelihood function (9). This fact illustrates Figure 1, where values of $\widehat{m}_{\max}$ smaller than $m_{\max}^{obs}$ are not allowed; however, values larger than $m_{\max}^{obs}$ would produce a smaller likelihood.

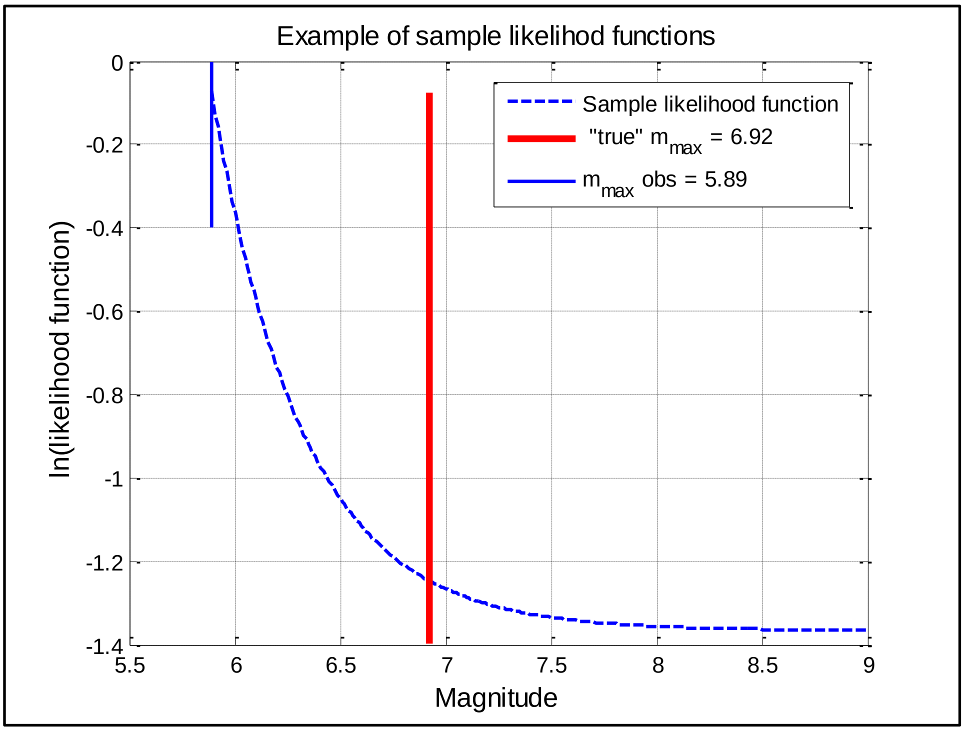

Fig. 1. Illustration of the sample likelihood function $L(\mathbf{m}|m_{\max})$. As the range of observed magnitudes depends on the unknown $\widehat{m}_{\max}$, the function reaches its maximum at the *maximum observed* magnitude $m_{\max}^{obs}$, and not, as expected, at the *maximum possible* $m_{\max}$. (After Kijko 2012).

This obstacle could be overcome by introducing a condition (Cooke 1979; Pisarenko 1991; Pisarenko *et al*. 1996), i.e. that the largest *observed* earthquake magnitude $m_{\max}^{obs}$, within the time span t^*, during which the largest event occurred, is equal to the largest *expected* earthquake magnitude $\boldsymbol{E}[m_{\max}^{obs}; t^*]$. It must be noted that time interval t^*, during which the largest event of $m_{\max}^{obs}$ took place, is not necessarily equal to the duration of the seismic event catalogue *t*. It could occur, and often does, that $m_{\max}^{obs}$ does not form part of the complete seismic event catalogue used in the estimation process but occurred before the start of the catalogue. In such situations $t^* \geq t$. It is essential to include such an event in the estimation process to prevent underestimation of the maximum possible magnitude m_{max}. In turn, not remembering the difference between $t^*$and t leads to an overestimation of m_{max}.

Replacing the expected value of the largest observed magnitude $\boldsymbol{E}[m_{\max}^{obs}; t^*]$ by the largest observed magnitude $m_{\max}^{obs}$ leads to an equation of the form

$$\widehat{m}_{\max} = m_{\max}^{obs} + \Delta, \tag{10}$$

where Δ is a correction factor. Following a mathematical formalism derived by Tate (1959) and, when applied to estimating the upper limit of the seismic event magnitude, Δ takes the form (Pisarenko 1991; Pisarenko *et al*. 1996)

$$\Delta = \frac{1}{n f_M(m_{\max}^{obs})}, \tag{11}$$

or alternatively, for the PDF in Eq. 2 as

$$\Delta = \frac{1 - \exp[-\beta(m_{\max}^{obs} - m_C)]}{n \exp[-\beta(m_{\max}^{obs} - m_C)]}. \quad (12)$$

The second formula describing Δ is of the form (Cooke 1979)

$$\Delta = \int_{m_C}^{m_{\max}} [F_M(m)]^n dm, \quad (13)$$

where $F_M(m)$ denotes a CDF. If applied to the CDF of Eq. 3, it takes the form (Kijko 2004)

$$\Delta = \frac{E_1(n_2) - E_1(n_1)}{\beta \exp(-n_2)}, \quad (14)$$

where $n_1 = \lambda_C t^*/\{1 - \exp[-\beta(m_{\max} - m_C)]\}$, $n_2 = n_1 \exp[-\beta(m_{\max} - m_C)]$, and $E_1(\cdot)$ denotes an exponential integral function.

Both procedures attempt to correct the bias of the classical ML estimator $\hat{m}_{max} = m_{max}^{obs}$. The first procedure (Eqs 11–12) is quite straightforward, and can be derived purely intuitively by employing the well-known properties of the uniform distribution (Kijko and Graham 1989). The procedure was probably first derived by Tate (1959). It was used by Pisarenko *et al.* (1996), who quoted it without deriving it, after Kendall and Stuart (1967), and applied for estimating maximum regional magnitude $m_{\max}$. The original derivation by Tate is very complex, and understanding it requires an advanced background in theoretical statistics. The advantage of the formula is that it does not require extensive calculations.

The second procedure (Eq. 14) can be derived by integrating parts of the CDF of the largest expected magnitude and replacing of it by the largest observed magnitude m_{max}^{obs} (Cooke 1979). This procedure is numerically more demanding than the first one; however, based on simulation analysis, it provides an estimate that is more accurate with a small number of observations and has a lower mean squared error (Kijko and Graham 1989). It is important to note that for large n, the two formulas are asymptotically equivalent (Vermeulen and Kijko 2017).

After replacing, in Eq. 5, $\bar{m} - m_C$ by $1/\hat{\beta}_{AU}$, the solution of the system of two equations

$$\begin{cases} \dfrac{1}{\beta} = \dfrac{1}{\hat{\beta}_{AU}} + \dfrac{(m_{max} - m_c)\exp[-\beta(m_{max} - m_c)]}{1 - \exp[-\beta(m_{max} - m_c)]} \\ \\ m_{\max} = m_{\max}^{obs} + \Delta \end{cases} \quad (15)$$

provides estimates for the required $\hat{\beta}$ and $\hat{m}_{max}$. The ML estimate of the mean activity rate λ is defined by Eq. 7.

From a numerical point of view, the simultaneous assessment of $\hat{\beta}$ and $\hat{m}_{\max}$ could be challenging. Following the approach applied to solve a similar problem (Kijko and Sellevoll 1992; Kijko *et al*. 2016), the system of Eq. 15 could be split into two equations and solved sequentially by iteration. Based on the $\hat{\beta}$-estimate, the new $\hat{m}_{max}$ is obtained from the solution of the second equation (Eq. 15). This procedure is repeated until the corrections to $\hat{\beta}$ and $\hat{m}_{max}$ are negligibly small. Application of this strategy to various data sets has shown that the procedure is efficient and fast. In most cases, the estimates of $\hat{\beta}$ and $\hat{m}_{max}$ are obtained within three interactions (Kijko and Sellevoll 1989).

The application of any iterative scheme requires determining starting points for the unknown parameters. The Aki–Utsu estimate (Eq. 6) could be used as the starting point for parameter β, whereas $m_{max} = m_{\max}^{obs} + 0.5$ could be used for the starting point of m_{max}. Such choice for a starting point for m_{max} is often used in engineering applications (e.g. Wheeler 2009).

An alternative, approximate solution for the first equation in the system of equations (Eq. 15) is the non-iterative estimate of $\hat{\beta}$, which takes the form (Gibowicz and Kijko 1994)

$$\hat{\beta} = \hat{\beta}_{AU}(1 - \hat{C}_f), \tag{16}$$

where the correction for the finite value of the upper limit m_{max} is

$$\hat{C}_f = \frac{(m_{max} - m_c)\exp\left[-\hat{\beta}_{AU}(m_{max} - m_c)\right]}{1 - \exp\left[-\hat{\beta}_{AU}(m_{max} - m_c)\right]}. \tag{17}$$

2.2 Incomplete seismic event catalogues

Assessment of the three seismic hazard recurrence parameters λ, β, and m_{max} becomes more complex when seismic event catalogues are incomplete. Here, an incomplete catalogue is defined as a seismic event database that is divided into several complete sub-catalogues, each complete but starting from a different magnitude level m_C (Fig. 2).

The first attempts to estimate the recurrence parameters λ and β in the case of incomplete catalogues were probably made by Molchan *et al*. (1970) and Stepp (1972). In addition to estimates for λ and β, Stepp's technique describes the assessment of the level of completeness m_C. The use of incomplete catalogues has also been discussed by Rosenblueth (1986) and Rosenblueth and Ordaz (1987); however, the most elegant, straightforward, and best-known is the procedure derived by Weichert (1980). The latest attempt to solve the problem was made by Kijko and Smit (2012). Their approach provides two simple equations for the assessment of λ and β. However, it does not allow the evaluation of m_{max}. In this section of the current paper, we outline the approach when, in addition to λ and β, the area-characteristic maximum possible seismic event magnitude m_{max} is estimated.

Assume that the seismic event catalogue is divided into s sub-catalogues, each with known, but different levels of completeness $m_C^{(1)}, m_C^{(2)}, \ldots, m_C^{(s)}$. Each of these sub-catalogues spans t_j years, and contains a record of n_j $(j = 1,2,\ldots,s)$ events with known magnitudes, as shown in Fig. 2.

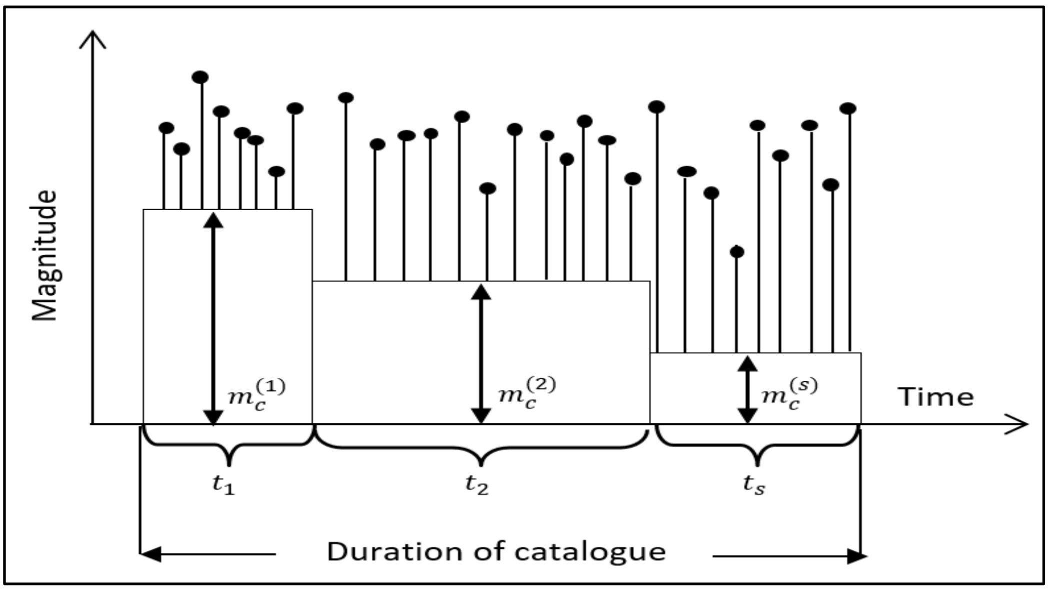

Fig. 2. Schematic illustration of a typical incomplete seismic event catalogue with different levels of completeness $m_c^{(j)}$ for $j = 1,..,s$, with s the total number of complete subsets of the catalogue. (After Kijko and Smit 2012).

For a specified value of m_{max}, overall, the ML estimate of the β-value could be obtained by applying the multiplicative property of likelihood functions (Rao 1973). If applied to our problem, the joint sample likelihood function, which utilises information on the magnitudes of all seismic events that occurred within the entire time span of the catalogue, takes the form

$$L(\mathbf{m} \mid \beta) = \prod_{j=1}^{s} L(\mathbf{m}^j \mid \beta), \tag{19}$$

where $L(\mathbf{m}^j \mid \beta)$ represents the sample likelihood function for sub-catalogue j, based on magnitudes $\mathbf{m}^j$ observed within j^{th} sub-catalogue, $\mathbf{m} = (\mathbf{m}^1, \mathbf{m}^2, \dots, \mathbf{m}^s)$ and $j = 1,2,\dots,s$.

Following the same principles as discussed in Section 2.1, the ML estimates $\hat{\beta}$ and $\hat{m}_{max}$ are derived under the condition that the largest *observed* earthquake magnitude $m_{\max}^{obs}$, within time period t^*, is equal to the largest *expected* earthquake magnitude $\boldsymbol{E}[m_{\max}^{obs}; t^*]$. Utilising the same approach as applied by Kijko and Smit (2012), and under the assumption of a specified value for $m_{\max}$, the maximisation of the sample likelihood function (Eq. 19), with respect to β, leads to

$$\frac{1}{\beta} = \sum_{j=1}^{s} r_j \left[\frac{1}{\hat{\beta}_{AU}^j} + \frac{(m_{\max} - m_C^j)\, exp[-\beta(m_{\max} - m_C^j)]}{1 - exp[-\beta(m_{\max} - m_C^j)]} \right], \tag{20}$$

or equivalently

$$\frac{1}{\beta} = \frac{1}{\hat{\beta}_{AUE}} + \sum_{j=1}^{s} r_j \; C_f^j, \tag{21}$$

where the correction factors for the specified, finite value of m_{max} are of the form

$$C_f^j = \frac{(m_{\max} - m_C^j)\, exp[-\beta(m_{\max} - m_C^j)]}{1 - exp[-\beta(m_{\max} - m_C^j)]}, \tag{22}$$

and $\hat{\beta}_{AUE}$ is the Aki–Utsu extended (AUE) estimator of the β value (Kijko and Smit 2012)

$$\hat{\beta}_{AUE} = \left(\frac{r_1}{\hat{\beta}_{AU}^1} + \frac{r_2}{\hat{\beta}_{AU}^2} + \cdots + \frac{r_s}{\hat{\beta}_{AU}^s} \right)^{-1}. \tag{23}$$

In Eq. 23, $r_j = n_j/n$, n_j is the number of seismic events in the j^{th} sub-catalogue and $n = \sum_{j=1}^{s} n_j$ is the total number of events with magnitudes equal to or exceeding the level of completeness m_C^j. The $\hat{\beta}_{AU}^j$ estimate is a classical Aki–Utsu estimator (Eq. 6), calculated for each of the sub-catalogues j, ($j = 1, \dots, s$).

Similar to the case of a single seismic event catalogue, estimates of β and $m_{\max}$ from a catalogue with different levels of completeness is obtained from sequentially solving system 24

$$\begin{cases} \dfrac{1}{\beta} = \dfrac{1}{\hat{\beta}_{AUE}} + \displaystyle\sum_{j=1}^{s} r_j \; C_f^j \\ m_{\max} = m_{\max}^{obs} + \Delta \end{cases}, \tag{24}$$

by iteration, in the same manner as system 15. Following Kijko and Smit (2012), the mean activity rate λ_C, takes the form

$$\hat{\lambda}_C = \frac{n}{\sum_{j=1}^{s} t_j \frac{\exp\left[-\hat{\beta}\left(m_C^j - m_C\right)\right]}{1-\ \exp\left[-\hat{\beta}(m_{max} - m_C)\right]}}. \tag{25}$$

Estimate $\hat{\lambda}_C$ denotes the mean seismic activity rate for events with magnitudes equal to or exceeding any arbitrarily selected earthquake magnitude m_C, where $m_C \le m_C^j$ $(j = 1, \dots s)$. As expected, for $s = 1$; $m_C^1 = m_C^2 = \cdots = m_C^s = m_C$; $t = t_1$; $t_2 = t_3 = \cdots = 0$ and $n = n_1$ with $n_2 = n_3 = \cdots = 0$, the system of equations, Eq. 24, takes the form of Eq. 15, i.e. analogical equations derived for a single, complete catalogue. In addition, for $m_{max} \to +\infty$, Eq. 24 reduces to the classical Aki–Utsu estimator (Eq. 6), and λ_C takes the standard form $\lambda = n/t$ in Eq. 7.

Tests on synthetic and real data have shown that the iterative, sequential solutions for the parameters in the system of equations (Eqs 15 and 24) are obtained after not more than five iterations.

2.3 Account of *a priori* information

In most cases, the area-characteristic maximum possible seismic event magnitude m_{max} is the most challenging parameter to estimate. Its assessment could be improved by including additional, independent information. Fortunately, for several areas under investigation, the information available on the expected value of m_{max} was obtained independently from the seismic event catalogue. This section shows how the additional information could be incorporated into the estimation schemes of λ, β, and m_{max} defined in Sections 2.1 and 2.2.

The estimators for $\hat{m}_{max}$ (Cooke 1979; Pisarenko 1991), as introduced in Section 2.1, have several attractive properties and are applicable to a broad range of magnitude distributions. They can also be used when the number of earthquakes n is not known. In such instance, the number of earthquakes could be replaced by λt. This replacement is equivalent to the assumption that the number of earthquakes occurring in unit time conforms to a Poisson distribution with parameter λ, where t is the time span during which the largest observed event m_{max}^{obs}, was recorded. Furthermore, both estimators provide a value of $\hat{m}_{max}$ that is never less than the largest observed magnitude m_{max}^{obs}.

Although procedures based on the formalisms developed by Cooke (1979) and Pisarenko (1991) provide powerful tools for the evaluation of m_{max}, they have a significant weak point, namely most of the catalogues of tectonic-origin seismicity are too short and provide insufficient information for reliable estimation of m_{max}. In these cases, additional information, such as local geological conditions, similarity of tectonic environments, regime of stresses, worldwide database for maximum recorded events, geophysical data, and historical and pre-historical (paleo) seismicity could serve as supplementary data to help constrain m_{max} estimates. Bayesian formalism is the natural choice for including *a priori* data and improving the assessment of this parameter.

Cornell (1994) pointed out that the blind employment of the Bayesian formalism provides a flawed answer that disqualifies the procedure. In the following part of this section, we briefly recall the main points of Bayesian formalism, as Cornell formulated and examined the source of the problem. Subsequently, we show that a correct estimate of $m_{\max}$ could be obtained if the Cornell Bayesian formulation were modified.

The Bayesian formalism combines the prior distribution of $m_{\max}$, $\pi(m_{\max})$, and the sample likelihood function $L(\mathbf{m}|m_{\max})$ (Eq. 9) of seismic event magnitudes $\mathbf{m} = (m_1, m_2, \dots, m_n)$. The resulting distribution, which summarises both the *a priori* knowledge of $m_{\max}$, and the information coming from the recorded event magnitudes $\mathbf{m}$, is known as the posterior distribution of $m_{\max}$ and is of the form

$$p_{m_{\max}}(m_{\max}|\mathbf{m}) = \frac{\pi(m_{\max})L(\mathbf{m}|m_{\max})}{\int \pi(m_{\max})L(\mathbf{m}|m_{\max})\mathrm{d}m_{\max}}. \tag{26}$$

The posterior distribution, Eq. 26, replaces the sample likelihood function $L(\mathbf{m}|m_{\max})$ as the expression that incorporates all the available information on $m_{\max}$. At least three Bayesian estimators are discussed in the literature, namely the *maximum a posterior*, the *posterior mean*, and the *posterior median* (Mood *et al.* 1974). The *maximum a posterior*, known as the MAP estimate, corresponds to the magnitude at which the posterior distribution Eq. 26 achieves its maximum. The *posterior mean* is defined as the mean value of the posterior distribution, and the median value of the distribution determines the *posterior median*.

A properly built sample likelihood function is expected to be such that the probability of observing magnitudes $\mathbf{m}$ reaches its maximum at the 'true' $m_{\max}$ (Mood *et al.* 1974). As shown in Section 2.1, the sample likelihood function defined in Eq. 9 does not fulfil this condition, as it reaches its maximum at the maximum *observed* magnitude $m_{\max}^{obs}$, and not, as required, at the maximum *possible* magnitude $m_{\max}$ (Fig. 1).

Given that the sample likelihood function (Eq. 9) is part of the posterior distribution (Eq. 26), the MAP estimator has the same flaw and its estimate will be underestimated systematically. The degree of underestimation depends on the quantity of information provided by each of the two components of the posterior distribution. The underestimation of $\hat{m}_{\max}$ is minimal when the *a priori* function $\pi(m_{max})$ is well defined and accurate, and $L(\mathbf{m}|m_{\max})$ is based on a short catalogue with only a few weak events. In such instance, most information informing $m_{\max}$ will derive from $\pi(m_{max})$. However, when the sample likelihood function contains a significant amount of information (e.g. it is based on a long catalogue containing many events, a long record of historical earthquakes, and the like), the posterior distribution would be dominated by the sample likelihood function $L(\mathbf{m}|m_{\max})$. In this case, the MAP estimate $\hat{m}_{max}$ would be located somewhere between the maximum observed, m_{max}^{obs} and the 'true' m_{max}, i.e. $m_{\max}^{obs} < \hat{m}_{\max} < m_{\max}$. This would lead to a biased and underestimated value of $\hat{m}_{\max}$. For this reason, Cornell (1994) 'disqualified' his own suggested procedure.

The above discussion explores only the MAP estimate of $m_{\max}$. However, conclusions regarding the shortfalls of the Cornell (1994) Bayesian formalism are applicable to all Bayesian estimators utilising the sample likelihood function Eq. 9. For example, based on simulated data, the *posterior mean*, in contrast with the MAP, overestimates the value of $m_{\max}$. This overestimation could reach a value of one unit of magnitude (Kijko 2012).

A correction for the bias in the Cornell (1994) Bayesian procedure for m_{max} could be made in several ways. However, here, a simple, palliative solution is presented that is based on knowledge of the bias correction factor Δ (Eqs 12 and 14) and the replacement of the sample likelihood function $L(\mathbf{m}|m_{\max})$ by $L(\mathbf{m}+\Delta|m_{\max})$. Such transformation assures that the new likelihood function reaches its maximum at the maximum *possible* magnitude $m_{\max}$, and not before at the maximum *observed* magnitude $m_{\max}^{obs}$.

The assessment of $\hat{\beta}$ and $\hat{m}_{max}$ could be done in a way similar to that for several sub-catalogues (Eq. 24) each with a different level of completeness. Therefore, the Bayesian MAP estimation of parameters β and $m_{\max}$, in the presence of *a priori* information on $m_{\max}$, is obtained from the (conditional) maximisation of the posterior distribution of $m_{\max}$

$$MAX[p_{m_{\max}}(m_{\max}|\mathbf{m}+\ \Delta)], \tag{27}$$

under the condition $\frac{1}{\beta} = \frac{1}{\hat{\beta}_{AUE}} + \sum_{j=1}^{s} r_j\ C_f^j$.
The numerical maximisation of Eq. 27 is relatively straightforward. If $m_{\max}$ is estimated with the help of the Bayesian *posterior mean*, Eq. 27 is replaced by

$$\hat{m}_{max} = \int p_{m_{\max}}(\zeta|\mathbf{m}+\ \Delta)\mathrm{d}\zeta. \tag{28}$$

Finally, the choice of the *posterior median* estimator of $m_{\max}$, leads to the replacement of Eq. 27 by

$$\int_{m_C}^{\hat{m}_{max}} p_{m_{\max}}(\zeta|\mathbf{m}+\Delta)\mathrm{d}\zeta = 1/2. \tag{29}$$

The formalism described above is not limited to the maximum possible magnitude $m_{\max}$. The methodology is easily adaptable to include *a priori* information for the two other seismic hazard recurrence parameters λ and β. To illustrate the application of the Bayesian formalism to the estimation of β, it is assumed that some independent *a priori* information for this parameter and its standard error are available from previous investigations. These two new parameters are denoted β_0 and σ_{β_0}. Further, assuming that β_0 follows the Gaussian distribution with mean β_0 and standard deviation σ_{β_0}, the prior distribution of β takes the form

$$\pi(\beta) = \frac{1}{\sqrt{2\pi\sigma_{\beta_0}}} \exp\left[-\frac{(\beta-\beta_0)^2}{{\sigma_{\beta_0}}^2}\right]. \tag{30}$$

The ML assessment of β using Eqs 5 and 21 could then be replaced by the Bayesian formalism with the posterior distribution

$$p_\beta(\beta|\mathbf{m}) = \frac{\pi(\beta)L(\mathbf{m}|\beta)}{\int \pi(\beta)L(\mathbf{m}|\beta)\mathrm{d}\beta}, \tag{31}$$

where $L(\mathbf{m} \mid \beta)$ is defined by equation (19).

In this way, e.g. the MAP estimate of $\hat{\beta}$ and $\hat{m}_{max}$, in the presence of *a priori* information for $m_{\max}$ and β is obtained from the simultaneous maximisation of the posterior distributions of $m_{\max}$ and β

$$\begin{cases} MAX[p_{m_{\max}}(m_{\max}|\mathbf{m}+\Delta)] \\ MAX\left[p_\beta(\beta|\mathbf{m})\right]. \end{cases} \tag{32}$$

or equivalently

$$MAX[p_{m_{\max}}(m_{\max}|\mathbf{m}+\Delta)],$$

under the condition $\frac{1}{\beta} = \frac{1}{\hat{\beta}_{AUE}} + \sum_{j=1}^{s} r_j \; C_f^j - \frac{\beta-\beta_0}{n\,\sigma_0^2}$.

The posterior distribution for parameter λ could be built in a manner similar to that for β. However, it has to be noted that although application of the normal distribution as *a prior* is probably quite common, it is far from optimal. It would be significantly better to replace the Gaussian *prior* by the Gamma *prior*. Because of the conjugtivity property of Gamma distribution (Johnson *et al.*, 1994), the posterior distribution would provide a quite elegant, close-form solution derived by Esteva (Newmark and Rosenblueth, 1971) and would be more sensible than using Gaussian *priors* (Iervolino and Giorgio, 2015). The Gaussian distribution does not have correct domain, as the likelihood of having negative values of β and $\boldsymbol{\lambda}$ is not zero.

3. Applications

In this section, some of the estimators discussed in Section 2 are compared with those most often referred to in literature. The Bayesian MAP estimators for β and m_{max}, defined in the system Eq. 32, are used to estimate the seismic hazard recurrence parameters for Cape Town, South Africa.

3.1. Comparison of three b-value estimation procedures from catalogues with different levels of completeness

In Section 2, several solutions were provided for effectively solving the same problem. However, the question arises which of these is the best.

In this section, we compare the performance of three procedures for estimating β based on a theoretical seismic event catalogue with different levels of completeness. In this comparison, we selected the three methods most often used according to literature, which are those of Weichert (1980), Kijko and Sellevoll (1989), and Kijko and Smit (2012). We used the Monte Carlo simulation technique in our comparison process.

The procedure by Weichert (1980) provides a reliable estimation of β and λ by using magnitude–binned data. The procedure offers solutions that are computed sequentially and are easy to apply. Ordaz and Giraldo (2018) contend that such sequential search for the two parameters is a weak point of the approach. The solution for one of the parameters will be optimal only if the answer to the other is already known. However, the problem could be resolved by a sequential solution of each equation, repeated until corrections to the unknown parameters are negligibly small.

The second procedure is described in Kijko and Sellevoll (1989). The authors discuss the ML estimation of λ, β, and m_{max} for incomplete catalogues, as well as the inclusion of historical events and their uncertainties. The two parameters λ and β are assessed simultaneously. The difference between the procedures of Kijko and Sellevoll (1989) and Weichert (1980) is the manner in which the seismic event magnitudes are treated. Kijko and Sellevoll (1989) assume a continuous magnitude scale, whereas the Weichert (1980) approach is based on binned magnitude classes. However, a significant number of observations have indicated that the two procedures provide the same results (Weichert and Kijko 1989).

The third method we applied is the extended ML Aki–Utsu estimator, as defined in Eq. 23. Similar to the Kijko and Sellevoll (1989) procedure, the seismic event magnitudes are treated as continuous data. The difference between the two estimators is the construction of the sample likelihood function, which results in an estimator, and is notably easier to manage.

The Monte Carlo simulation technique was applied to investigate the performance of the three procedures. A hypothetical seismic event catalogue, with a low annual activity rate characteristic of areas of low seismicity, was divided into four sub-catalogues, each with a time span of 50 years. Each sub-catalogue was assigned a level of completeness of $m_C^1 = 4.2$, $m_C^2 = 4.0$, $m_C^3 = 3.6$, and $m_C^4 = 3.0$, respectively. The seismic event magnitudes were generated according to the PDF defined in Eq. 2, with parameters $m_{\mathrm{max}} = 7.0$, mean activity rate $\lambda_{3.0} = 10$, and $\beta = 2.303$, which is equivalent to the Gutenberg–Richter b-value equal to 1. Each simulation was repeated 10 000 times. The results of the experiment are shown in Table 1.

Table 1. Results of the Weichert (1980), Kijko and Sellevoll (1989) and updated, and the extended Aki–Utsu (Eq. 23) estimation procedures for the $\boldsymbol{\beta}$ parameter. The estimates are based on 10 000 simulated catalogues, each divided into four 50-year-duration sub-catalogues, and with a level of completeness of $\boldsymbol{m_C^1 = 4.2}$, $\boldsymbol{m_C^2 = 4.0}$, $\boldsymbol{m_C^3 = 3.6}$, and $\boldsymbol{m_C^4 = 3.0}$, respectively. The earthquake magnitudes were generated using the PDF defined in Eq. 2, with parameters $\boldsymbol{m_{\mathrm{max}} = 7.0}$, mean activity rate $\boldsymbol{\lambda_{3.0} = 10}$, and $\boldsymbol{\beta = 2.303}$, (equivalent to the Gutenberg––Richter $\boldsymbol{b}$-value equal to 1.0). The table is modified after Vermeulen (2020).

$\boldsymbol{\beta}$-parameter descriptive statistics	**Weichert (1980)**	**Kijko and Sellevoll (1989)**	**Kijko and Smit (2012) Eq. 23**
Mean	2.005	2.182	2.190
Standard deviation	0.034	0.027	0.026
Mean square error	0.090	0.015	0.013
Bias	-0.298	-0.121	-0.112
95% confidence interval	[1.950, 2.061]	[2.137, 2.226]	[2.146, 2.233]

The results of the simulations summarised in Table 1 show that the performance of all three estimators is similar and they provide a β-value that is significantly undesimated. The source of this undestimation is the lack in the simulated catalogue of events with magnitudes close to the upper limit of magnitude $m_{max} = 7.0$. Moreover, neither of the two applied procedures (Kijko and Sellevoll 1998; Kijko and Smit 2012) could provide a reasonable value of $m_{max} = 7.0$. Judging from the estimated mean values, standard deviations, and mean square errors, the updated, extended Aki–Utsu estimator (Eq. 23) and the Kijko and Sellevoll procedures appear slightly superior. Further, some more-advanced comparisons have indicated that the Weichert (1980) estimator is biased at lower activity rates (relating to fewer observations) (Vermeulen 2020). The above example is concerning, as the assessment of m_{max} for areas of weak seismicity based only on the seismic event catalogue provides biased (underestimated) b-values that are, therefore, not applicable for the assessment of m_{max}. More-trustworthy assessments of these two parameters could be conducted by incorporating additional, independent information and applying the Bayesian formalism.

3.2. Seismic hazard recurrence parameters for Cape Town, South Africa

The Bayesian formalism for β and m_{max}, defined in Eq. 32, was applied for the assessment of seismic hazard recurrence parameters of Cape Town, South Africa.

The local magnitude 6.3 Ceres–Tulbagh earthquake of 1969 remains the most destructive earthquake in South African history. The earthquake occurred on 29 September 1969 in the Ceres–Tulbagh area, ca. 90 km from Cape Town. The event was reportedly felt as far as Durban in KwaZulu-Natal, over 1 000 km away. More than 70% of the buildings in Tulbagh suffered damage, and over half of the local population was left homeless. Moreover, most local roads in the area were badly cracked. Large fires were ignited when electricity lines and boxes were damaged. Most of the Tulbagh community was evacuated, and many of the damaged houses were never rebuilt (www.stormchasing.co.za). Twelve people were killed and many more injured (Fig. 3). According to an AXCO Insurance Market Report, the insured loss of this event was US$7.4 million at that time. The total uninsured loss was 3.5 times as high (Davis and Kijko 2003).

Fig. 3. Damage observed at Tulbagh after the Ceres earthquake. Source: Pule *et al.* (2015). Adapted from Fernandez (1974).

The catalogue used in this study encompasses southern Africa and it is compiled and maintained by the Council for Geoscience (CGS) of South Africa. The catalogue incorporates databases provided by neighbouring countries, the Sub-Saharan Africa (SSA)-Global Earthquake Model (GEM) earthquake catalogue (Poggi *et al*. 2017), and data from the International Seismological Centre (ISC) in the United Kingdom and the United States Geological Survey (USGS). However, the database of seismic events for the investigated area is highly incomplete and uncertain, particularly regarding the size and location of the strongest historical events. All seismic event magnitudes were homogenised and expressed in terms of local magnitude M_L, which is quite close to moment magnitude M_W (Heaton *et al*. 1986; Deichmann 2006; Manzunzu *et al*. 2021).

The catalogue applied in our study spans ca. 120 years, from 1 January 1900 to 31 December 2019. The catalogue contains events located within a radius of 300 km from the epicentre of the Ceres–Tulbagh earthquake. Events prior to 1 January 1900 were not used in the calculations, as they are not considered a complete or reliable source of information. After a detailed investigation of completeness, the catalogue was divided into four parts, each with a different level of completeness (Table 2).

Table 2. Division of the tectonic-origin earthquake catalogue used in the analysis.

Type of catalogue	**Time span**	m_C	**SE**
Complete #1	1900/01/01–1964/12/31	4.2	0.3
Complete #2	1965/01/01–1990/12/31	3.8	0.2
Complete #3	1991/01/01–1995/12/31	3.1	0.2
Complete #4	1996/01/01–2019/12/31	3.0	0.1

m_C = level of completeness of the sub-catalogue for the specified time period
SE = standard error (assumed uncertainty in earthquake magnitude determination)

The estimation process was performed twice to demonstrate the effect of implementing additional information on the hazard recurrence parameters β and m_{max}, namely 1) without (Eq. 27) and 2) with an account of *a priori* information (Eq. 32). Following Fenton *et al*. (2006), we assumed that the *a priori* information for the β-value for the investigated area is equal to 2.12 ± 0.46. This is equivalent to $b = 0.92 \pm 0.20$.

The maximum possible seismic event magnitude m_{max} for different parts of the world has been discussed by several authors (Fenton *et al*. 2006; Calais *et al*. 2016; Stevens and Avouac 2017). Most of the assessments of m_{max} that are characteristic to areas similar to the stable continental region of South Africa oscillate around $m_{max} = 7.0$, with a standard error in the order of 0.5. Accordingly, for this study, an *a priori* value of $m_{max} = 7.0 \pm 0.5$ was assumed for the Cape Town area. The results of the estimated recurrence parameters using the system of equations, Eq. 27 and Eq. 32, are shown in Table 3.

Table 3. Comparison of seismic recurrence parameters $(\hat{\lambda}, \hat{b}, \widehat{m}_{\max})$ for the Cape Town area estimated without *a priori* information (Eq. 27) and with account of *a priori* information (Eq. 32).

Parameters	**Without including *a priori* information** Eq. 27	**Including *a priori* information** Eq. 32
Mean seismic activity rate $\hat{\lambda}$ (events/year) for $m_C = 3.0$	2.27 ± 0.44	2.41 ± 0.44
$\hat{\beta}$-value	1.84 ± 0.23	2.00 ± 0.16
Gutenberg–Richter $\hat{b}$-value	0.80 ± 0.10	0.87 ± 0.07
$\widehat{m}_{\max}$	6.56 ± 0.36	6.99 ± 0.75

It is important to note that during the calculations we assumed a maximum observed magnitude of 6.3, the strongest earthquake that occurred in the area within the time interval 1800–2020.

The results of the two assessments differ notably, particularly for m_{max}. The differences in the assessments of m_{max} have a significant effect on the estimated mean return periods. In the absence of any *a priori* information, the mean return period of a seismic event with a magnitude of 6.56 is 285 years. The mean return period for the same magnitude, but calculated with an account of *a priori* information for both the β and m_{max} parameters, is

1310 years, i.e. 4.6 times longer. The inclusion of *a priori* information in the calculation of $\hat{\beta}$ increased the estimate from 1.84 to 2.00 and reduced the standard error. The mean activity rate estimate $\hat{\lambda}_c$ also increased with the inclusion of additional information in the estimation of β and m_{max}.

4. Conclusions

In this study, we focused on the assessment of three key parameters, namely the mean activity rate λ, the Gutenberg–Richter b-value, and the maximum possible event magnitude m_{max} that are required by the probabilistic seismic hazard formalism of Esteva–Cornell–McGuire.

This study presents a uniform approach to the ML estimate of the recurrence parameters λ and $\beta(= b\ln(10))$ in the case of a complete seismic event catalogue, with a single level of completeness, and for the more common case where the catalogue has multiple levels of completeness. Two systems of equations were derived, which are applicable to complete (Eq. 15) and incomplete (Eq. 27) seismic event catalogues.

Although the mathematical formalism leading to the systems of equations (Eq. 15) and (Eq. 27) provide powerful tools for evaluating the three recurrence parameters, these systems of equations have a significant weak point, namely most of the catalogues of tectonic-origin seismicity are too short and provide insufficient information for reliable estimation of m_{max}. When additional and independent geological, geophysical, and seismological data are available, the application of Bayesian formalism is the natural choice to improve the assessment of this parameter. Therefore, we present the third system of equations (Eq. 32), which allows incorporating additional information. Further, we show how each of the three systems of equations could be solved sequentially by an iterative scheme.

For illustrative purposes, two applications of the derived formalisms are presented, namely a comparison of the estimation of β by Weichert (1980), Kijko and Sellevoll (1989), and the extended Aki–Utsu estimator (Eq. 27), as well as an assessment of the probabilistic seismic hazard recurrence parameters for the Cape Town region. These two examples demonstrate the importance of using an appropriate assessment procedure and the value-added effect of implementing additional information.

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