Hamiltonicity of Locally Hamiltonian and Locally Traceable Graphs

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Abstract

If $\mathcal{P}$ is a given graph property, we say that a graph G is *locally* $\mathcal{P}$ if $\langle N(v)\rangle$ has property $\mathcal{P}$ for every $v \in V(G)$ where $\langle N(v)\rangle$ is the induced graph on the open neighbourhood of the vertex v. We consider the complexity of the Hamilton Cycle Problem for locally traceable and locally hamiltonian graphs with small maximum degree. The problem is fully solved for locally traceable graphs with maximum degree 5 and also for locally hamiltonian graphs with maximum degree 6 [S. A. van Aardt, M. Frick, O. Oellermann and J. P. de Wet, Global cycle properties in locally connected, locally traceable and locally hamiltonian graphs, *Discrete Applied Mathematics*, 2016]. We show that the Hamilton Cycle Problem is NP-complete for locally traceable graphs with maximum degree 6 and for locally hamiltonian graphs with maximum degree 10. We also show that there exist regular connected nonhamiltonian locally hamiltonian graphs with connectivity 3, thus answering two questions posed by Pareek and Skupien [C. M. Pareek and Z. Skupień, On the smallest non-Hamiltonian locally Hamiltonian graph, *J. Univ. Kuwait (Sci.)*, 10:9 - 17, 1983].

Keywords: locally traceable, locally hamiltonian, nonhamiltonian, nontraceable

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order $n(G)$ and size $e(G)$ of G are the cardinalities of $V(G)$ and $E(G)$, respectively. If $X \subseteq V(G)$ then $\langle X\rangle$ denotes the subgraph of G induced by X. If $v \in V(G)$ then $N(v)$ denotes the open neighbourhood of v in G. A *Hamilton path* in a graph G is a is a path that contains all the vertices in $V(G)$, and a graph that contains a Hamilton path is called *traceable*. A *Hamilton cycle* is a cycle that contains all the vertices in $V(G)$ and a graph that contains a Hamilton cycle is called a *hamiltonian graph*. For undefined concepts we refer the reader to [7].

In 1965, Skupień [23] defined a graph to be *locally hamiltonian* if $\langle N(v)\rangle$ is hamiltonian for every vertex v in G. In general, if $\mathcal{P}$ is a given graph property, we say that G is *locally* $\mathcal{P}$ if $\langle N(v)\rangle$ has property $\mathcal{P}$ for every $v \in V(G)$. Locally connected, locally traceable and locally hamiltonian graphs have been intensively studied - see for example [1, 2, 5, 6, 8, 9, 11, 12, 15, 17, 18, 19, 20, 21, 22, 23, 25, 26]. We abbreviate locally connected, locally traceable and locally hamiltonian to LC, LT and LH, respectively.

The Hamilton Cycle Problem (abbreviated HCP) is the problem of deciding whether a given graph is hamiltonian. Akiyama, Nishizeki and Saito [3] proved the following.

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Theorem 1.1. [3] *The HCP is NP-complete for 2-connected cubic planar bipartite graphs.*

The HCP for LC graphs with maximum degree at most 4 was fully solved by Chartrand and Pippert [9]. They proved that $K_{1,1,3}$ is the only nonhamiltonian connected LC graph with maximum degree at most 4. Gordon, Orlovich, Potts and Strusevich [15] showed that the HCP is NP-complete for LC graphs with maximum degree 7 and they conjectured that it is polynomially solvable for LC graphs with maximum degree 6 or less. However, van Aardt, Burger, Frick, Thomassen and de Wet [1] showed that the HCP is NP-complete for LC graphs with maximum degree 5 or greater.

Van Aardt, Frick, Oellerman and de Wet [2] showed that the HCP for LT graphs with maximum degree at most 5 is fully solved. In Section 3 we show that there exist connected nonhamiltonian LT graphs of order n with maximum degree 6 for every $n \geq 7$ and we prove that the HCP for LT graphs with maximum degree at least 6 is NP-complete.

In Section 4 we investigate the hamiltonicity of LH graphs. Pareek and Skupień [21] showed that the smallest connected nonhamiltonian LH graph has order 11. This graph has maximum degree 8, and this led Pareek to speculate that every connected LH graph with maximum degree at most 7 is hamiltonian, and he published a proof for this [20]. However, we claim that his proof is not valid, and we explain the reasons for our claim. Nevertheless, it follows from Pareek's work that every connected LH graph with maximum degree 6 is hamiltonian. We show that for every $n \geq 11$ there exist connected nonhamiltonian LH graphs with maximum degree at most 9, but to date we have found only finitely many with maximum degree 8. We prove that the HCP for LH graphs with maximum degree 10 is NP-complete.

In Section 5 we respond to two questions concerning nonhamiltonian LH graphs posed by Pareek and Skupień [21], namely

- Does there exist a nonhamiltonian connected LH graph that is regular?
- Is K_4 the only regular LH graph that is not 4-connected?

We show by construction that the answer to the first question is positive. The constructed graphs are 3-connected, so this answers the second question in the negative.

Entringer and MacKendrick [11] established an upper bound for $f(n)$, the largest integer such that every connected LH graph of order n contains a path of length $f(n)$. Their results imply that $\lim_{n\to\infty} f(n)/n = 0$. In Section 6 we show that if $p(n,\Delta)$ is the largest integer such that every connected planar LH graph of order n with maximum degree Δ contains a path of length $p(n,\Delta)$, then $\lim_{n\to\infty} p(n,\Delta)/n = 0$ for $\Delta \geq 11$.

In the next section we present techniques that are used in subsequent sections to construct LT graphs and LH graphs with certain properties.

2. Construction techniques for LT and LH graphs

Van Aardt and de Wet [26] provided the following procedure, called *edge identification of two LT graphs*, which will be used in Section 3 to construct LT graphs with certain properties. For edge identification, a *suitable edge* in a graph G is defined as an edge uv such that there is a Hamilton path in $\langle N(u)\rangle$ that ends at v and a Hamilton path in $\langle N(v)\rangle$ that ends at u.

Construction 2.1. [26] *Let G_1 and G_2 be two LT graphs such that $E(G_i)$ contains a suitble edge u_iv_i, $i = 1,2$. Now create a graph G of order $n(G_1) + n(G_2) - 2$ by identifying the vertices u_1 and u_2 and the vertices v_1 and v_2 and call the resulting vertices u and v, while retaining all the edges present in the original two graphs* (*see Figure 1*). We say that G is obtained from G_1 and G_2 by identifying suitable edges.

Our next result shows that certain properties are retained when graphs are combined by means of edge identification. Parts (a) and (b) were proved in [26].

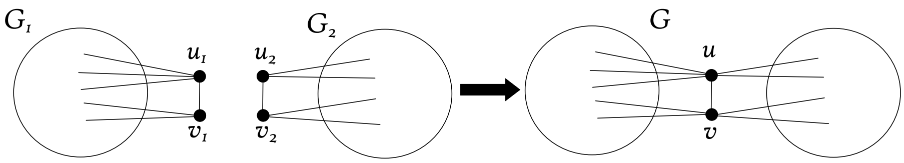

Figure 1: The edge identification procedure.

Lemma 2.2. *Let G_1 and G_2 be two LT graphs, each with order at least 3, and let G be a graph obtained from G_1 and G_2 by identifying suitable edges. Then the following hold.*

(a) G is LT.

(b) If G_1 and G_2 are planar, then G is planar.

(c) If G is hamiltonian, so are both G_1 and G_2.

Proof. (c) We will use the same notation as in Construction 2.1. Since $\{v, u\}$ is a cutset in G, it follows that no Hamilton cycle in G can include the edge vu. This implies that any Hamilton cycle in G has the form vQ_1uQ_2v where $v_1Q_1u_1$ is a Hamilton path in G_1 and $v_2Q_2u_2$ is a Hamilton path in G_2. Since $v_iu_i \in E(G_i)$ for $i = 1, 2$ it follows that each of G_1 and G_2 has a Hamilton cycle.

The following observation from [26] will be useful for selecting suitable edges to use in edge identification.

Observation 2.3. *[26] Let v be a vertex of degree two in a LT graph. Then any edge incident with v is suitable for use in edge identification.*

Van Aardt and De Wet also described the following procedure, called *triangle identification of two LH graphs*, which will be used in Sections 4, 5 and 6 to construct *LH* graphs with certain properties. For triangle identification, a *suitable triangle* in a graph G is defined as a triangle X such that for each vertex $x \in V(X)$, there is a Hamilton cycle of $\langle N_G(x)\rangle$ that contains the edge $X - x$.

Construction 2.4. *For $i = 1, 2$, let G_i be an LH graph that contains a suitable triangle X_i with $V(X_i) = \{u_i, v_i, w_i\}$. Now create a graph G of order $n(G_1) + n(G_2) - 3$ by identifying the vertices u_i, $i = 1, 2$ to a single vertex u, and similarly the vertices v_i, $i = 1, 2$ to v and w_i, $i = 1, 2$ to w, while retaining all the edges present in the original two graphs (see Figure 2).* We say that G is obtained from G_1 and G_2 by identifying suitable triangles.

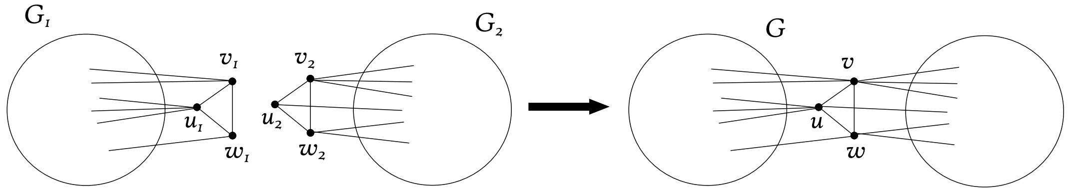

Figure 2: The triangle identification procedure.

Our next result shows that certain properties are retained when two graphs are combined by means of triangle identification. Parts (a) and (b) were proved in [26].

Lemma 2.5. *Let G_1 and G_2 be two LH graphs, each of order at least 4, and let G be a graph obtained from G_1 and G_2 by identifying suitable triangles. Then*

(a) G is LH.

(b) If G_1 and G_2 are planar, then so is G.

(c) If G is hamiltonian, so are both G_1 and G_2.

Proof. (c) We will use the same notation as in Construction 2.4. First note that since $\{u, v, w\}$ is a cutset, it follows that no Hamilton cycle in G includes more than one edge between vertices in $\{u, v, w\}$. Figure 3 shows the only possible patterns that a Hamilton cycle in G can follow. It follows that if G is hamiltonian, then so are both G_1 and G_2.

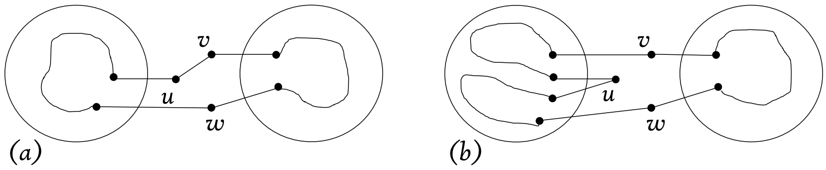

Figure 3: The possible Hamilton cycles through G.

We will also need the following procedure, called *triangle identification within an LH graph.*

Construction 2.6. *Let G be an LH graph that contains disjoint triangles X_1 and X_2 such that $N(X_1) \cap N(X_2) = \emptyset$ and for each $x \in N(X_i)$ there is a Hamilton cycle of $\langle N(x) \rangle$ that contains the edge $X_i - x$, $i = 1, 2$. Let $V(X_i) = \{u_i, v_i, w_i\}, i = 1, 2$. Now create a graph G' of order $n(G) - 3$ from G by identifying u_i, $i = 1, 2$ to a single vertex u, and similarly the vertices v_i, $i = 1, 2$ to v and w_i, $i = 1, 2$ to w, while retaining all the edges present in the original graph.* We say that G' is obtained from G by identifying suitable triangles within G.

Lemma 2.7. *If G' is a graph obtained from an LH graph G by identifying two suitable triangles within G, then G' is LH.*

Proof. Let X_1 and X_2 be two suitable triangles in G. We use the same notation as in Construction 2.6. In G' only the neighbourhoods of u, v, w need to be considered, as the neighbourhoods of all other vertices remain unchanged in the triangle identification procedure (except for possible label changes). Let C_i be a Hamilton cycle of $\langle N_G(u_i) \rangle$ containing the edge $v_i w_i$, $i = 1, 2$. Then in G', the cycles C_1 and C_2 have only the edge vw in common, since $N_G(u_1) \cap N_G(u_2) = \emptyset$. Hence $C_1 - vw$ and $C_2 - vw$ can be combined to form a Hamilton cycle of $\langle N_{G'}(u) \rangle$. Similarly, we can prove that $\langle N_{G'}(v) \rangle$ and $\langle N_{G'}(w) \rangle$ are hamiltonian. Hence G' is LH.

Our final result in this section will be used in Section 6.

Lemma 2.8. *In an LH graph G, any vertex of degree 3 can be used three times in triangle identification, once in combination with each distinct subset of two of its three neighbours.*

Proof. Let $v_1 \in V(G)$ such that $N(v_1) = \{v_2, v_3, v_4\}$ and note that $\langle N[v_1] \rangle \cong K_4$. Since $d(v_1) = 3$, each triangle $\langle N[v_1] - v_i \rangle$, $i = 2, 3, 4$, is suitable for triangle identification. There exists paths P_2, P_3 and P_4 such that the following are Hamilton cycles of $\langle N_G(v_i) \rangle$, $i = 1, 2, 3, 4$:
In $\langle N_G(v_1) \rangle$: $v_2v_3v_4v_2$

In $\langle N_G(v_2)\rangle$: $v_3v_1v_4P_2v_3$
In $\langle N_G(v_3)\rangle$: $v_2v_1v_4P_3v_2$
In $\langle N_G(v_4)\rangle$: $v_2v_1v_3P_4v_2$.

Let G_1 be an LH graphs with a suitable triangle $X = \langle\{x_1, x_2, x_3\}\rangle$. For each $i = 1, 2, 3$, let Q_i be the path in the Hamilton cycle of $\langle N_{G_1}(x_i)\rangle$ between the end vertices of the edge $X - x_i$. Now use triangle identification to combine G with G_1 to form the graph H_1 by identifying the triangle $\langle\{v_1, v_2, v_3\}\rangle\rangle$ with the triangle $\langle\{x_1, x_2, x_3\}\rangle$. Let the identified vertices retain the labels v_1, v_2, v_3. By Lemma 2.5, H_1 is LH and the following are Hamilton cycles of $\langle N_{H_1}(v_i)\rangle$, $i = 1, 2, 3, 4$:
In $\langle N_{H_1}(v_1)\rangle$: $C_{H_1,v_1} = v_2Q_1v_3v_4v_2$
In $\langle N_{H_1}(v_2)\rangle$: $C_{H_1,v_2} = v_3Q_2v_1v_4P_2v_3$
In $\langle N_{H_1}(v_3)\rangle$: $C_{H_1,v_3} = v_2Q_3v_1v_4P_3v_2$
In $\langle N_{H_1}(v_4)\rangle$: $C_{H_1,v_4} = v_2v_1v_3P_4v_2$.

The triangle $\langle\{v_1, v_2, v_4\}\rangle$ in H_1 is now suitable for triangle identification, since v_2v_4, v_1v_4, v_1v_2 are edges in C_{H_1,v_1}, C_{H_1,v_2}, C_{H_1,v_4} respectively.

Next, let G_2 be an LH graph with a suitable triangle $Y = \langle\{y_1, y_2, y_4\}\rangle$. For $i = 1, 2, 4$, let R_i be the path on the Hamilton cycle of $\langle N_{G_2}(y_i)\rangle$ between the end vertices of the edge $Y - y_i$. Now use triangle identification to combine H_1 with G_2 to form the graph H_2 by identifying the triangles $\langle\{v_1, v_2, v_4\}\rangle$ and $\langle\{y_1, y_2, y_4\}\rangle$. Let the identified vertices retain the lables v_1, v_2, v_4. By Lemma 2.5, H_2 is LH and the following are Hamilton cycles of $\langle N_{H_2}(v_i)\rangle$, $i = 1, 2, 3, 4$:
In $\langle N_{H_2}(v_1)\rangle$: $C_{H_2,v_1} = v_2Q_1v_3v_4R_1v_2$
In $\langle N_{H_2}(v_2)\rangle$: $C_{H_2,v_2} = v_3Q_2v_1R_2v_4P_2v_3$
In $\langle N_{H_2}(v_3)\rangle$: $C_{H_2,v_3} = v_2Q_3v_1v_4P_3v_2$
In $\langle N_{H_2}(v_4)\rangle$: $C_{H_2,v_4} = v_2R_4v_1v_3P_4v_2$.

Since v_3v_4, v_1v_4 and v_1v_3 are edges in C_{H_2,v_4}, C_{H_2,v_3}, C_{H_2,v_4}, respectively, the triangle $\langle\{v_1, v_3, v_4,\}\rangle$ in H_2 is now suitable for triangle identification, so a third triangle identification, using this triangle, may be performed.

3. Hamiltonicity of locally traceable graphs

A graph G is called *outerplanar* if there is a planar embedding of G such that every vertex of G lies on the boundary of the external region. Moreover, G is *maximally outerplanar* if the addition of any edge results in a non-outerplanar graph. Maximal outerplanar graphs are examples of hamiltonian LT graphs. Results of Asratian and Oksimets [6] imply the following.

Theorem 3.1. [6] *Suppose G is a connected LT graph of order $n \geq 3$. Then $e(G) \geq 2n - 3$. Moreover, $e(G) = 2n - 3$ if and only if G is a maximal outerplanar graph.*

In [2] a *magwheel* M_k is defined as the graph of order $2k+1$ obtained from the wheel W_k by adding, for each edge e on the rim of W_k, a vertex v_e and joining it to the two ends of the edge e. Magwheels are examples of connected nonhamiltonian LT graphs. The three magwheels with maximum degree 5 are shown in Figure 4.

A graph G is *fully cycle extendable* if every vertex in G lies on a 3-cycle and, for every nonhamiltonian cycle C of G, there exists a cycle C' in G that contains all the vertices of C plus a single new vertex. Van Aardt, Frick, Oellermann and de Wet [2] proved the following:

Theorem 3.2. [2] *Suppose G is a connected LT graph with $n(G) \geq 3$ and $\Delta(G) \leq 5$. Then G is fully cycle extendable if and only if $G \notin \{M_3, M_4, M_5\}$.*

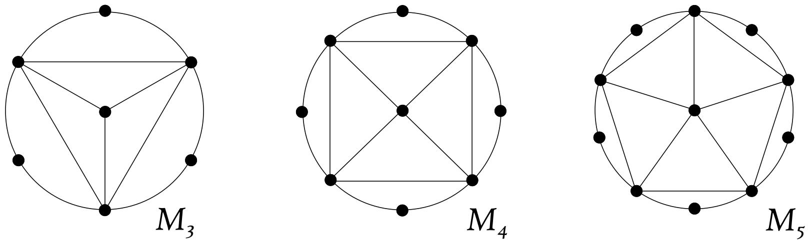

Figure 4: The magwheels M_3, M_4 and M_5.

Theorem 3.2 implies that there are only three nonhamiltonian connected LT graphs with maximum degree 5. For LT graphs with maximum degree 6 we now prove the following.

Theorem 3.3. *For any $n \geq 8$ there exists a nonhamiltonian planar connected LT graph G that has order n and maximum degree 6.*

Proof. Let G_7 be the graph M_3, depicted in Figure 4. For each $n \geq 8$, let G_n be the graph of order n obtained by combining G_{n-1} with a K_3 by means of edge identification, starting with the edge v_1v_2, and each time using one of the last edges added, choosing the edge such that the same vertex is never used more than twice, and specifically v_1 is only used once, as shown in Figure 5.

It follows from repeated application of Lemma 2.2 that for $n \geq 7$, the graph G_n is a connected planar nonhamiltonian LT graph of order n and it is clear from Figure 5 that it has maximum degree 6 if $n \geq 8$.

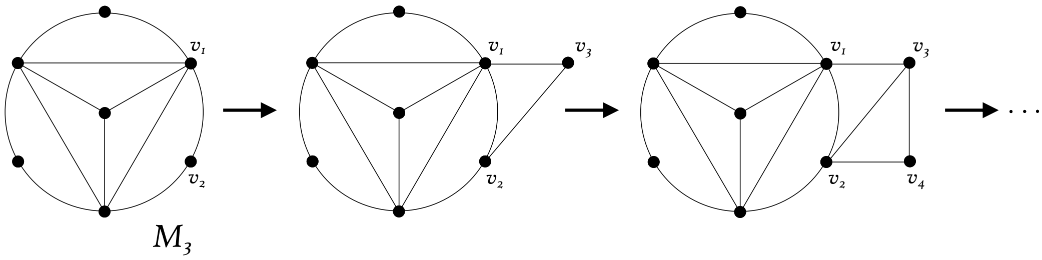

Figure 5: Constructing planar nonhamiltonian LT graphs with $\Delta(G) = 6$.

Theorem 3.1 implies that every nonhamiltonian connected LT graph of order n has at least $2n - 2$ edges. Since the graph G_n defined in the proof of Theorem 3.3 has $2n - 2$ edges, we have the following corollary.

Corollary 3.4. *For each $n \geq 7$, the minimum size of a nonhamiltonian connected LT graph of order n is $2n - 2$.*

By Theorem 3.2, the HCP for LT graphs with maximum degree 5 is fully solved. However, for maximum degree 6 we now prove the following.

Theorem 3.5. *The Hamilton Cycle Problem for connected planar LT graphs with maximum degree 6 is NP-complete.*

Proof. According to Theorem 1.1 the HCP for cubic (i.e., 3-regular) planar bipartite graphs is NP-complete. Now consider any 2-connected planar cubic bipartite graph G'. We shall show that G' can be transformed in polynomial time to a planar LT graph G with $\Delta(G) = 6$ such that G is hamiltonian if and only if G' is hamiltonian.

Let the partite sets of G' be $\mathcal{A}'$ and $\mathcal{B}'$. To create G, replace each vertex in $\mathcal{A}'$ with a triangle to create the set of graphs $\mathcal{A}$, and replace each vertex in $\mathcal{B}'$ with the nontraceable LT graph B (shown in Figure 6) to create the set of graphs $\mathcal{B}$. For convenience, the graphs in $\mathcal{A}$ and $\mathcal{B}$ will be referred to as “nodes” of G.

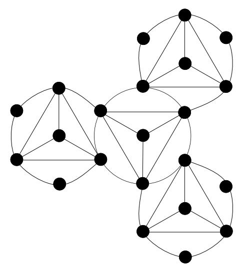

Figure 6: The graph B.

Each edge in G' is replaced by two “parallel” edges and one “diagonal” edge between them. Figure 8 demonstrates how a node $B \in \mathcal{B}$ is connected to its three neighbouring nodes in $\mathcal{A}$.

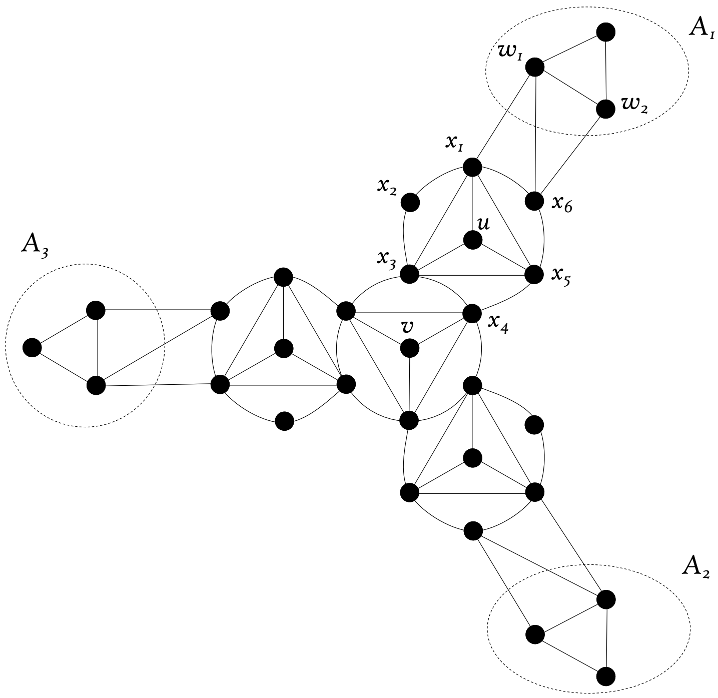

Figure 7: A node B in $\mathcal{B}$ connected to 3 nodes A_1, A_2, A_3 in $\mathcal{A}$.

Since G' is planar, G is also planar, and it is routine to confirm that G is LT and $\Delta(G) = 6$.

The heavy lines in Figure 8 show how paths on a Hamilton cycle of G' translate to paths on a Hamilton cycle of G. It is therefore clear that if G' is hamiltonian, then so is G.

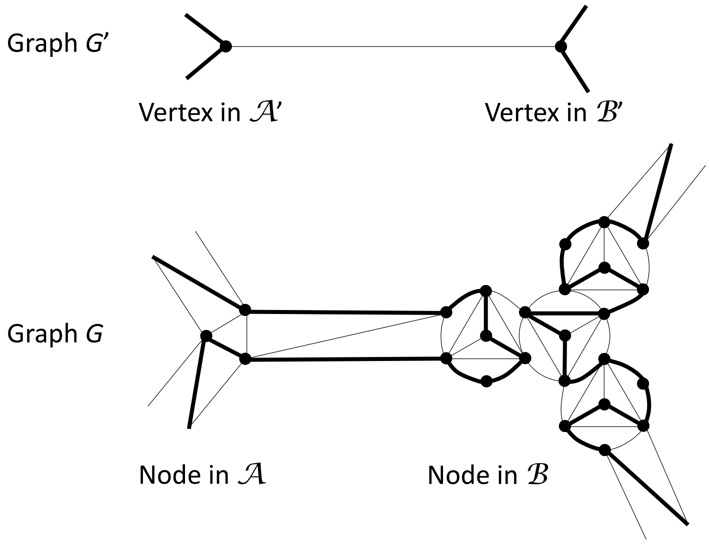

Figure 8: Illustrating how a Hamilton cycle in G' translates to a Hamilton cycle in G.

Now suppose G has a Hamilton cycle C. Then $|\mathcal{A}| = |\mathcal{B}|$, so to show that C translates to a Hamilton cycle of G', it is sufficient to show the following: for each node $B \in \mathcal{B}$, the cycle C contains exactly one path that enters B from a neighbouring node in $\mathcal{A}$ and then exits B to a different neighbouring node in $\mathcal{A}$.

We label the nodes and vertices as shown in Figure 7. We observe that C contains the path $x_1x_2x_3$, and hence also the edge x_5u. This implies the following.

Claim 1. If C contains exactly two of the three edges that join A_1 to B, then C contains either the path $w_1x_6w_2$ or the path $w_1x_1x_2x_3ux_5x_6w_2$. Analogous results hold with respect to A_2 and A_3.

Since $v \in V(C)$, Claim 1 implies that for at least one i, the cycle C contains either only one or all three edges incident with A_i. In either case C contains a path that exits B to a different node in $\mathcal{A}$ from which it entered. It also follows from Claim 1 that C contains at most one such path.

We conclude that G is hamiltonian if and only if G' is hamiltonian. This completes the proof.

4. Hamiltonicity of locally hamiltonian graphs

Skupień [22] observed that any triangulation of a closed surface is LH. In particular, triangulations of the plane (maximal planar graphs) are LH. He also proved the following useful result.

Theorem 4.1. [22] *Suppose G is a connected LH graph of order $n \geq 3$. Then $e(G) \geq 3n - 6$. Moreover, $e(G) = 3n - 6$ if and only if G is a maximal planar graph.*

Goldner and Harary [14] showed that the graph $G11A$, depicted in Figure 9, is the smallest nonhamiltonian maximal planar graph. Pareek and Skupień [21] showed that $G11A$ is also the smallest connected non-hamiltonian LH graph. A computer search showed that the four graphs in Figure 9 are the only connected nonhamiltonian LH graphs of order 11. Note that $G11A$ is a maximal planar graph and has size 27, while the other three graphs have size 30 each and are therefore not planar. Also note that all four graphs have maximum degree 8.

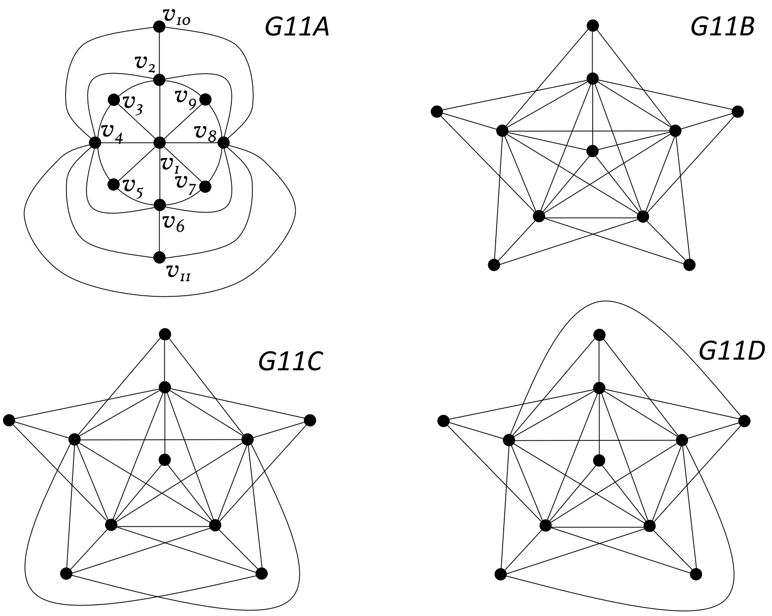

Figure 9: The nonhamiltonian LH graphs of order 11.

In 1983 Pareek [20] published a paper claiming that every connected LH graph with maximum degree less than 8 is hamiltonian. However, the proof in his paper omits several special cases, and some of the claims that he makes on which he bases further results are false.

Pareek's proof will not be set out in detail. Rather, we will focus on the main reasons why we believe it is not valid. Pareek considers a longest cycle $C = v_1v_2 \ldots v_tv_1$ in an LH graph G with $\Delta(G) \leq 7$. He shows that if G is not hamiltonian, then C contains a vertex v_1 of degree at least 7 that has six neighbours on C and one neighbour x in $G - V(C)$. Let $\{x, v_2, v_m, v_i, v_j, v_k, v_l\} \subseteq N(v_1)$. Since $\langle N(v_1)\rangle$ is hamiltonian, x has two neighbours in $N(v_1)$, say v_i and v_k. It suffices to consider the following three cases (Figure 10). The possibility that a graph may belong to both Case 1 and Case 2 is not explicitly considered, but does not affect the logic of the argument.

Case 1. $v_{k+1} \in N(v_1)$.
Case 2. $v_{k-1} \in N(v_1)$
Case 3. $N(v_1) \cap \{v_{i-1}, v_{i+1}, v_{k-1}, v_{k+1}\} = \emptyset$.

Since $\langle N(v_k)\rangle$ is hamiltonian, v_k and x have a common neighbour $v_p \neq v_1$ on C.

We agree up to this point. But then Pareek claims that Case 3 converts to either Case 1 or Case 2 and we do not agree with that. Pareek argues that in Case 3, the fact that the neighbourhoods of v_1, v_i, v_k, v_j, v_l and v_p induce hamiltonian graphs implies that $d_C(v_p) = 6$ and that v_p has a neighbour in $\{v_{k-1}, v_{k+1}\}$. By relabelling the vertices so that v_p becomes v_1, it would then follow that this case converts to either Case 1 or Case 2. However, Figure 11 (a) shows an example of such a situation where the neighbourhoods of v_1, v_i, v_k, v_j, v_l and v_p induce hamiltonian graphs, but neither v_k nor v_i has consecutive neighbours on C. This case does therefore not convert to Case 1 or Case 2. (We have illustrated the case where $v_p = v_i$, as this leads to the simplest example, but even if v_p and v_i are distinct, the same kind of counterexample is possible.)

The next step in Pareek's proof is to show that if Case 1 occurs, then so does Case 2. We do not agree with this either. The graph in Figure 11 (b) is a counterexample: the neighbourhoods of v_1, v_i and v_k induce hamiltonian graphs, but Case 2 does not occur (it is also possible to find Hamilton cycles in the graphs

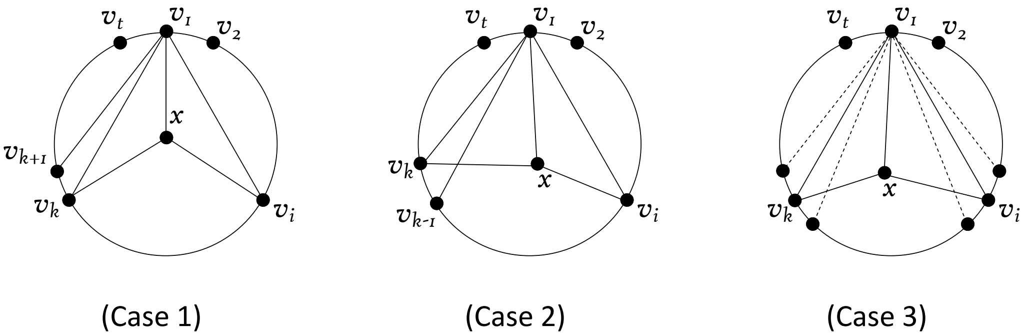

Figure 10: The three cases used in Pareek's proof.

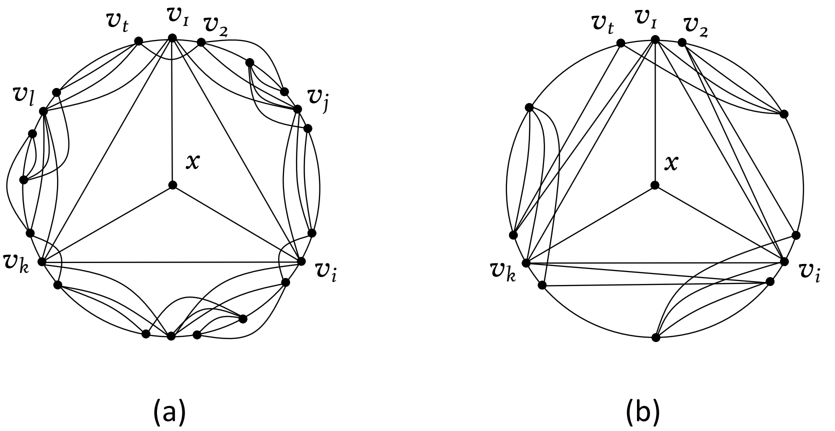

Figure 11: Counterexamples to Pareek's Claims.

induced by the neighbourhoods of the unlabeled vertices in the figure, but for the sake of clarity these are not shown).

Pareek's final step is to show that Case 2 is not possible. However, he omits some of the possible subcases of Case 2, but more seriously, the proof fails if $k < p < t$.

We therefore regard the problem as to whether there exists a nonhamiltonian connected LH graph with maximum degree 7 as unsolved. Nevertheless, it follows from the correct part of Pareek's proof that every connected LH graph with maximum degree at most 6 is hamiltonian. Moreover, by adapting the technique that Pareek had used, van Aardt et al. [2] proved the following.

Theorem 4.2. [2] *Let G be a connected LH graph with $n(G) \geq 3$ and $\Delta(G) \leq 6$. Then G is fully cycle extendable.*

Theorem 4.2 extends the result of Altshuler [4] that any 6-regular triangulation of the torus is hamiltonian.

It is easy to construct a planar LH graph of any order $k \geq 4$ with maximum degree at most 6 that contains a triangle with vertices u_1, u_2 and u_3 of degrees at most 3, 4 and 5 respectively (see [26] for an explicit construction). We are now in a position to prove the following.

Theorem 4.3. *For every $n \geq 11$ there exists a connected planar nonhamiltonian LH graph G with $\Delta(G) \leq 9$.*

Proof. For any $k \geq 4$, let H_k be a planar LH graph of order k with $\Delta(H_k) \leq 6$ such that H_k contains a triangle with vertices u_1, u_2 and u_3 of degrees at most 3, 4 and 5 respectively. Now combine H_k with the graph $G11A$ in Figure 9 using triangle identification by identifying u_1 with v_1, u_2 with v_2 and u_3 with v_3. Then the resulting graph G is a connected graph with $\Delta(G) = 9$ and $n(G) = 11 + k - 3$ and, by Lemma 2.5, G is both planar and nonhamiltonian.

We have found nonhamiltonian connected LH graphs with maximum degree 8 and order 11, 13, 14 and as large as 34, but we do not know whether there are such graphs of infinitely large order. The following theorem implies that there are none of order 12. The proof is long and uninteresting, and can be found in [25].

Theorem 4.4. [25] *Let G be a connected nonhamiltonian LH graph of order $n = 12$. Then $\Delta(G) = 9$.*

Chvátal [10] and Widgerson [27] independently proved that the Hamilton Cycle Problem for maximal planar graphs is NP-complete. Although neither author was interested in the minimum value of the maximum degree for which this is true, it is straightforward to manipulate the construction Chvátal used to show that the theorem holds for a maximum degree as low as 12. However, we shall make a further improvement for LH graphs (that is, if we drop the requirement that the graph be planar).

We shall make use of the graph H depicted in Figure 12(a), which is the smallest nontraceable maximal planar graph, discovered by Goodey [13]. (Figure 12 is a nonplanar depiction of H, designed to emphasize the symmetry in the graph.)

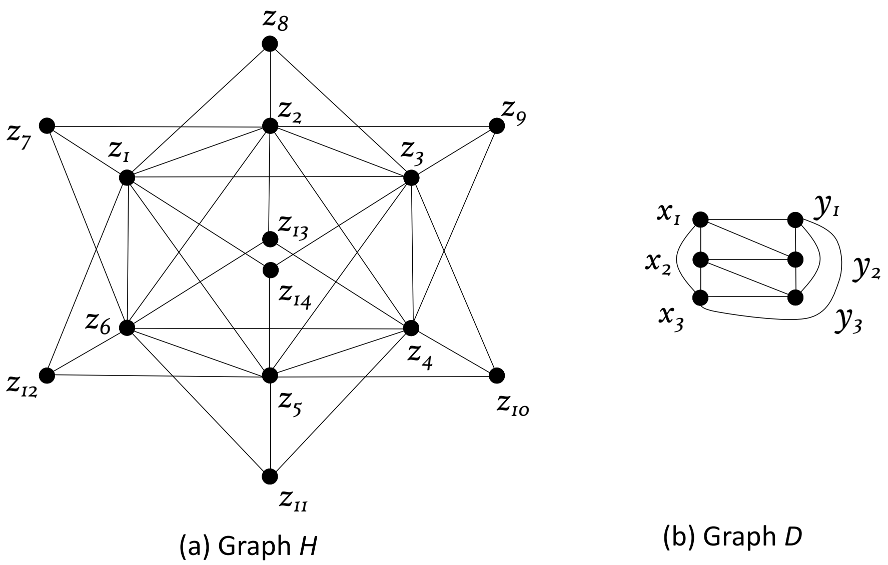

Figure 12: Graphs used in the proof of Theorem 4.6.

The graph F shown in Figure 13 is obtained by using triangle identification to combine the graph H with three distinct copies of the graph D, shown in Figure 12(b). Specifically, we identify each of the triangles $\langle\{z_1, z_2, z_7\}\rangle$, $\langle\{z_3, z_4, z_9\}\rangle$ and $\langle\{z_5, z_6, z_{11}\}\rangle$ in H with the triangle $\langle\{x_1, x_2, x_3\}\rangle$ in each copy of D. In each step we let the vertices of the identified triangles retain the labels of the vertices that were in H. In the i^{th} step, the vertices y_1, y_2, y_3 of D are relabelled u_i, v_i, w_i, respectively, $i = 1, 2, 3$, as shown in Figure 13. Now let $W_i = \{u_i, v_i, w_i\}$, $i = 1, 2, 3$ and let $W = W_1 \cup W_2 \cup W_3$.

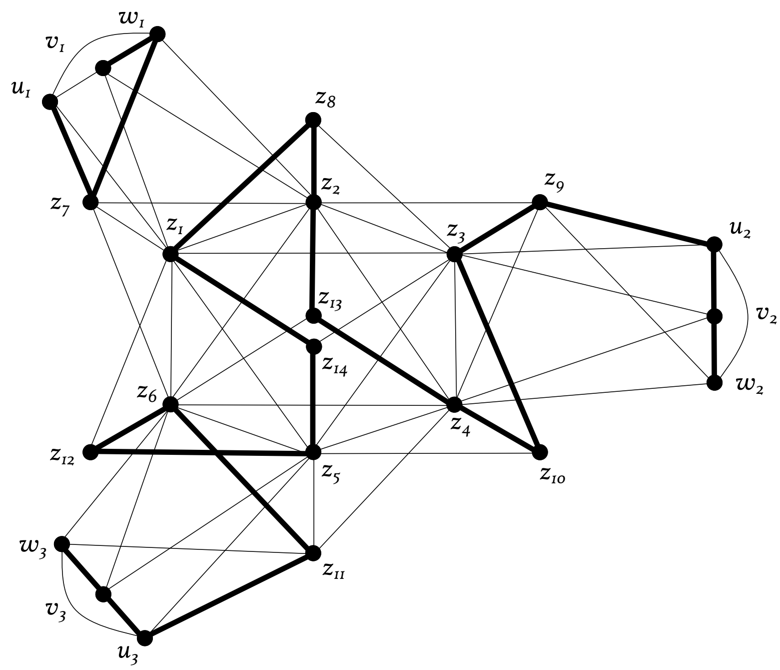

Figure 13: The graph F

If $\mathcal{P}$ is a set of disjoint paths in a graph G such that $\bigcup_{P\in\mathcal{P}} V(P) = V(G)$, then $\mathcal{P}$ is called a *path cover* of G. A path cover consisting of k paths is called a *k-path cover*. The following properties of the graph F play a crucial role in the proof of the subsequent theorem.

Lemma 4.5.

(*a*) *F is LH.*

(*b*) *F has a 2-path cover $\{P_1, P_2\}$ such that both end-vertices of P_i are in W for $i = 1, 2$.*

(*c*) *Let $\mathcal{P}$ be any path cover of F such that every path in $\mathcal{P}$ has both its end vertices in W. Then at most one of the paths in $\mathcal{P}$ has its two end-vertices in different members of the set $\{W_1, W_2, W_3\}$.*

Proof. (a) Since H is a maximal planar graph, it is LH. It is routine to check that D is LH and that in each step the triangles used for identification are suitable triangles. Hence, by Lemma 2.5, F is LH.

(b) The heavy lines in Figure 13 show a 2-path cover of F, with the end-vertices of both paths in W.

(c) Let $Z_1 = \{z_8, z_{10}, z_{12}, z_{13}, z_{14}\}$ and $Z_2 = \{z_1, z_2, z_3, z_4, z_5, z_6, \}$. Note that Z_1 is an independent set and $N(Z_1) = Z_2$. Since $|Z_1| = 5$ and $|Z_2| = 6$, it therefore follows that all the vertices in $Z_1 \cup Z_2$ lie on the same path in $\mathcal{P}$. Thus any path that has its end-vertices in different members of $\{W_1, W_2, W_3\}$ contains all the vertices in $Z_1 \cup Z_2$. The result follows.

We are now ready to prove the main result of this section.

Theorem 4.6. *The Hamilton Cycle Problem for LH graphs with maximum degree 10 is NP-complete.*

Proof. Starting with a cubic bipartite graph G', we will construct a connected LH graph G with $\Delta(G) = 10$ such that G is hamiltonian if and only if G' is hamiltonian.

Let the partite sets of G' be $\mathcal{A}'$ and $\mathcal{B}'$. The vertices in $\mathcal{A}'$ are replaced by copies of K_4 to create the set $\mathcal{A}$ of G. The vertices in $\mathcal{B}'$ are replaced by copies of the graph B, shown in Figure 14, to create the set $\mathcal{B}$ of G. The graph B is constructed as follows. Let H_a, H_b and H_c each be a copy of H, with vertices labeled

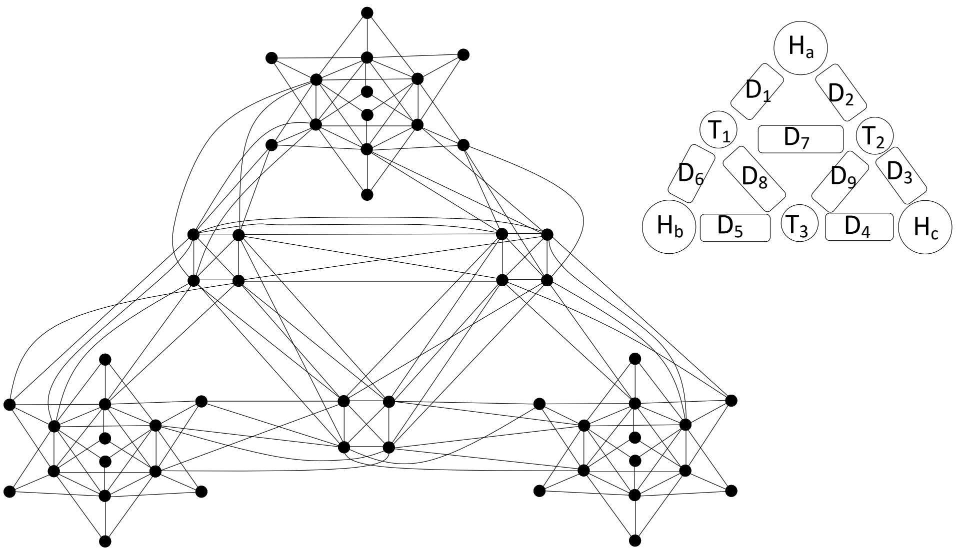

Figure 14: The graph B used in the proof of Theorem 4.6.

in the same way, except with an a,b, or c subscript added to indicate to which graph each vertex belongs (in H_a, the vertex corresponding to z_1 in H is labeled $z_{a,1}$, and so forth). Let T_i, $i = 1, 2, 3$, be three copies of K_4, with vertices labeled $t_{i,j}$, $j = 1, 2, 3, 4$. Let D_m, $m = 1, 2, \dots, 9$, be nine copies of D, with vertices labeled as in D, with appropriate subscripts added, as was done for the vertices in H_a, H_b and H_c. We now use Constructions 2.4 and 2.6 to create the graph B by identifying the following triangles:
$\{z_{a,1}, z_{a,2}, z_{a,7}\}$ and $\{x_{1,1}, x_{1,2}, x_{1,3}\}$;
$\{z_{a,3}, z_{a,4}, z_{a,9}\}$ and $\{x_{2,1}, x_{2,2}, x_{2,3}\}$;
$\{t_{1,1}, t_{1,2}, t_{1,3}\}$ and $\{y_{1,1}, y_{1,2}, y_{1,3}\}$;
$\{t_{2,1}, t_{2,2}, t_{2,3}\}$ and $\{y_{2,1}, y_{2,2}, y_{2,3}\}$;
$\{t_{1,1}, t_{1,2}, t_{1,4}\}$ and $\{x_{7,1}, x_{7,2}, x_{7,3}\}$;
$\{t_{2,1}, t_{2,2}, t_{2,4}\}$ and $\{y_{7,1}, y_{7,2}, y_{7,3}\}$;
$\{t_{1,1}, t_{1,3}, t_{1,4}\}$ and $\{y_{6,1}, y_{6,2}, y_{6,3}\}$;
$\{t_{1,2}, t_{1,3}, t_{1,4}\}$ and $\{x_{8,1}, x_{8,2}, x_{8,3}\}$;
$\{t_{2,2}, t_{2,3}, t_{2,4}\}$ and $\{x_{9,1}, x_{9,2}, x_{9,3}\}$;
$\{t_{2,1}, t_{2,3}, t_{2,4}\}$ and $\{y_{3,1}, y_{3,2}, y_{3,3}\}$;
$\{z_{b,1}, z_{b,2}, z_{b,7}\}$ and $\{x_{6,1}, x_{6,2}, x_{6,3}\}$;
$\{z_{b,3}, z_{b,4}, z_{b,9}\}$ and $\{x_{5,1}, x_{5,2}, x_{5,3}\}$;
$\{y_{5,1}, y_{5,2}, y_{5,3}\}$ and $\{t_{3,1}, t_{3,2}, t_{3,3}\}$;
$\{y_{4,1}, y_{4,2}, y_{4,3}\}$ and $\{t_{3,1}, t_{3,2}, t_{3,4}\}$;
$\{x_{4,1}, x_{4,2}, x_{4,3}\}$ and $\{z_{c,3}, z_{c,4}, z_{c,9}\}$;
$\{x_{3,1}, x_{3,2}, x_{3,3}\}$ and $\{z_{c,1}, z_{c,2}, z_{c,7}\}$;
$\{t_{3,1}, t_{3,3}, t_{3,4}\}$ and $\{y_{8,1}, y_{8,2}, y_{8,3}\}$;
$\{t_{3,2}, t_{3,3}, t_{3,4}\}$ and $\{y_{9,1}, y_{9,2}, y_{9,3}\}$.

The schematic in Figure 14 shows how the various graphs were combined to create B. It is a simple matter to confirm that K_4 can be combined four times in succession with four copies of the graph D using triangle identification (in such a way that each vertex in K_4 is used three times). It follows from Lemmas 2.5 and 2.7 that B is LH.

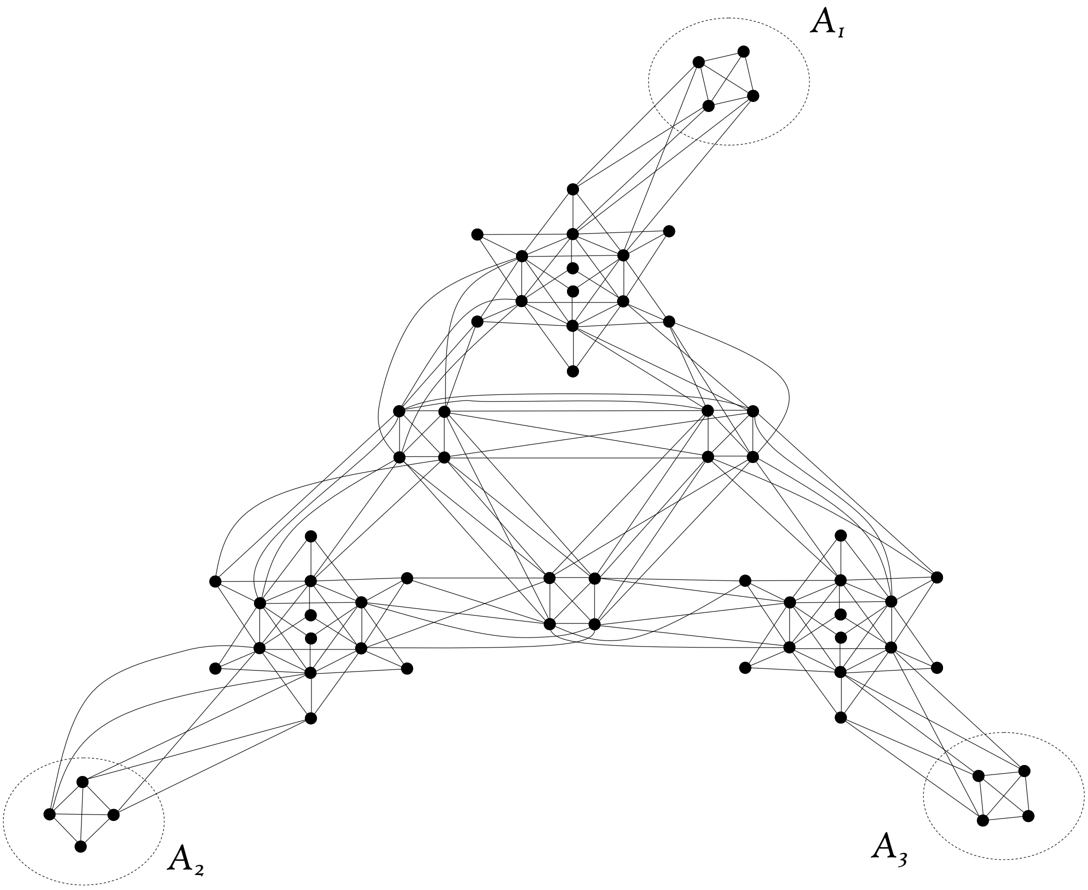

Figure 15: A node in $\mathcal{B}$ connected to three nodes in $\mathcal{A}$.

The elements of $\mathcal{A}$ and $\mathcal{B}$ will be called *nodes* of G. If there is an edge between two vertices in $\mathcal{A}'$ and $\mathcal{B}'$, then in G the corresponding nodes $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are joined via the graph D as follows: First identify one of the three triangles in A with the triangle $\langle\{x_1, x_2, x_3\}\rangle$ in D and then identify the triangle $\langle\{y_1, y_2, y_3\}\rangle$ in D with a triangle in B as indicated in Figure 15. A different triangle of A is used for connecting it to each of its neighbouring nodes in $\mathcal{B}$.

By Lemmas 2.5 and 2.7, the graph G is LH. It is routine to check that $\Delta(G) = 10$. A subsection of G' is shown in Figure 16 together with the corresponding subsection of G, with the heavy lines showing how a Hamilton cycle in G' translates to a Hamilton cycle in G. It follows that if G' is hamiltonian, then so is G.

Now suppose C is a Hamilton cycle in G. Note that G contains several copies of the graph F presented in Figure 13. One such copy is encircled by dotted lines in Figure 15. Using the property of F given by Lemma 4.5(c), it is now easy to see that C contains exactly one path that exits the node B to a different node in $\mathcal{A}$ from which it had entered B. Thus C translates to a Hamilton cycle in G'.

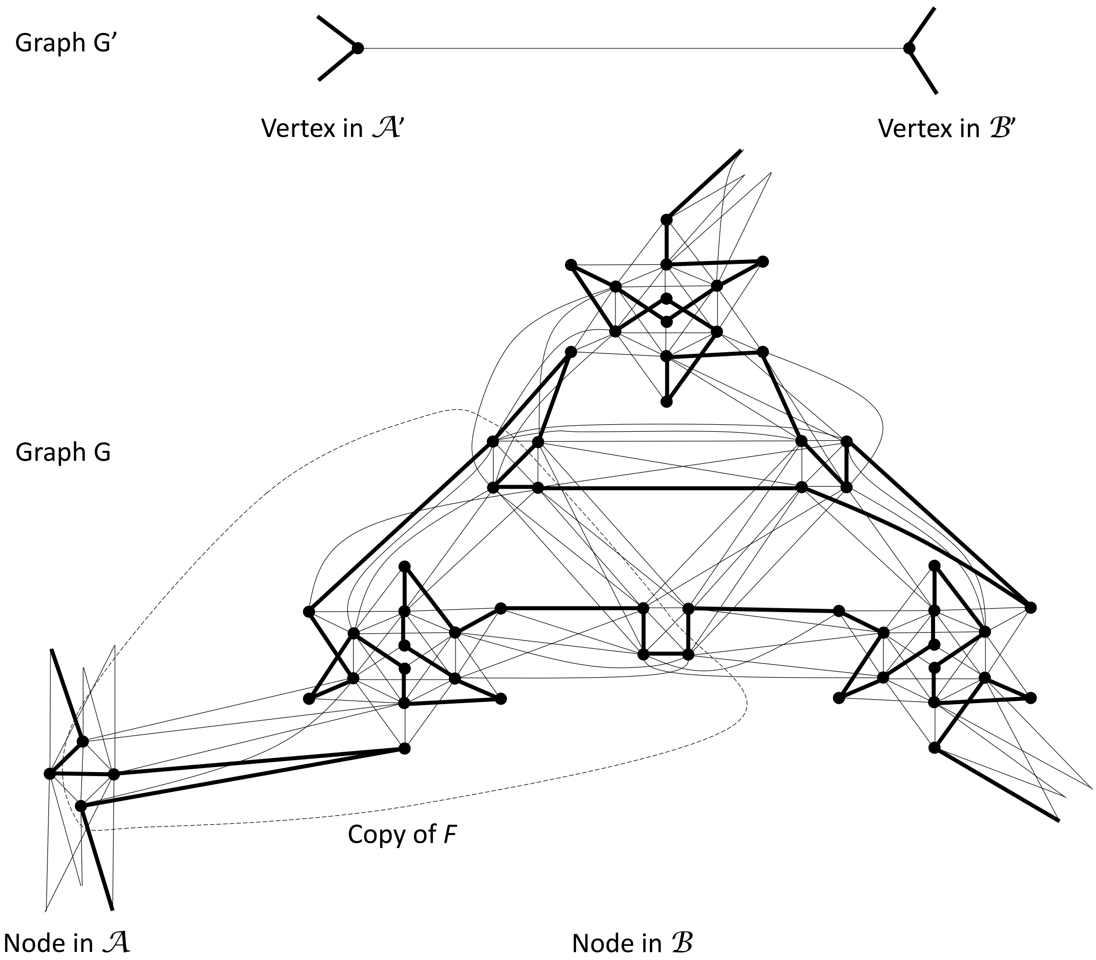

Figure 16: Illustrating how a Hamilton cycle in G' translates to a Hamilton cycle in G.

5. Regular connected nonhamiltonian *LH* graphs

Regular connected LH graphs have not yet received much attention in the literature, except in terms of 6-regular triangulations of the torus [4, 24]. The hamiltonicity of such graphs is readily implied by Theorem 4.2.

The two questions of Pareek and Skupien regarding regular LH graphs mentioned in Section 1 are both answered by the following theorem.

Theorem 5.1. *For every $r \geq 11$, there exists a nonhamiltonian connected r-regular graph with connectivity 3.*

Proof. To construct an 11-regular connected, nonhamiltonian LH graph R_{11} we start with the Goldner-Harary graph $G11$ shown in Figure 17 with the vertices labeled as shown. We then use triangle identification to combine $G11$ with other LH graphs that have the required degree sequences so that the resulting graph will be 11-regular. These graphs are shown as graphs $H11A$ and $H11B$ in Figure 18 and were constructed by starting with the triangle $\langle\{w_1, w_2, w_3\}\rangle$ and then adding edges linking it to a K_{12} or K_{13} as shown. To limit the degrees of the vertices making up the K_{12} or K_{13} subgraphs to 11, edges were removed between some of these vertices, as indicated in Figure 18. It is routine to confirm that these graphs are LH and that

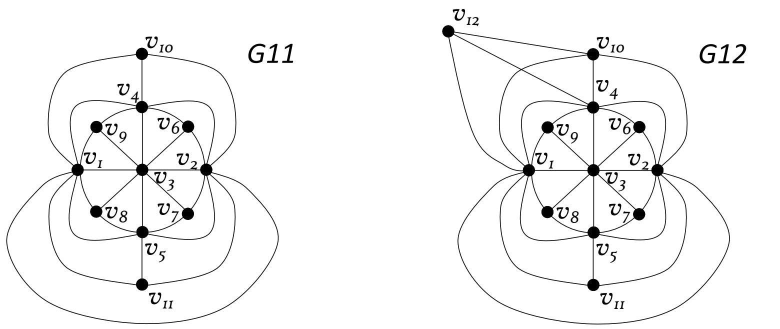

Figure 17: The graphs $G11$ and $G12$ used in to construct regular nonhamiltonian LH graphs.

the triangle $\langle\{w_1, w_2, w_3\}\rangle$ in each of these graphs is suitable for use in triangle identification. In particular we create the graph R_{11} by combining $G11$ with five copies of $H11A$ and one copy of $H11B$, each time identifying the vertices w_1, w_2, w_3 with appropriate vertices in $G11$. Note that in each step the degrees of the vertices in $G11$ that are identified with w_1, w_2, w_3 of $H11A$ increase by 1, 2, 8, respectively, while the degrees of those that are identified with w_1, w_2, w_3 of $H11B$ increase by $2, 2, 8$, respectively. The table below provides the details of the construction. The first column indicates the first, second and third vertices of the triangle in $G11$ that are identified, respectively, with the vertices w_1, w_2, w_3 of the graph in the second column.

Vertices in $G11$	Second graph
v_4, v_2, v_6	$H11A$
v_5, v_1, v_8	$H11A$
v_3, v_4, v_9	$H11A$
v_1, v_4, v_{10}	$H11A$
v_2, v_5, v_{11}	$H11A$
v_5, v_3, v_7	$H11B$

The resulting graph is 11-regular and by Lemma 2.5 is connected, nonhamiltonian, and LH. Since it was obtained by means of triangle identification, it has connectivity 3. This technique can easily be extended to create r-regular, connected, nonhamiltonian LH graphs for odd values of r greater than 11. Due to problems with vertex degree parity, the technique does not work for even values of r when starting with graph $G11$. For even values of r greater than or equal to 12 we can use graph $G12$ in Figure 17. To create a 12-regular, connected, nonhamiltonian LH graph R_{12} we combine $G12$ with two copies of $H12A$, three copies of $H12B$ and one copy of $H12C$. The details are given in Figure 18.

Vertices in $G12$	Name of second graph
v_3, v_5, v_7	$H12A$
v_2, v_5, v_{11}	$H12A$
v_5, v_3, v_8	$H12B$
v_4, v_2, v_6	$H12B$
v_4, v_1, v_9	$H12B$
v_4, v_{10}, v_{12}	$H12C$

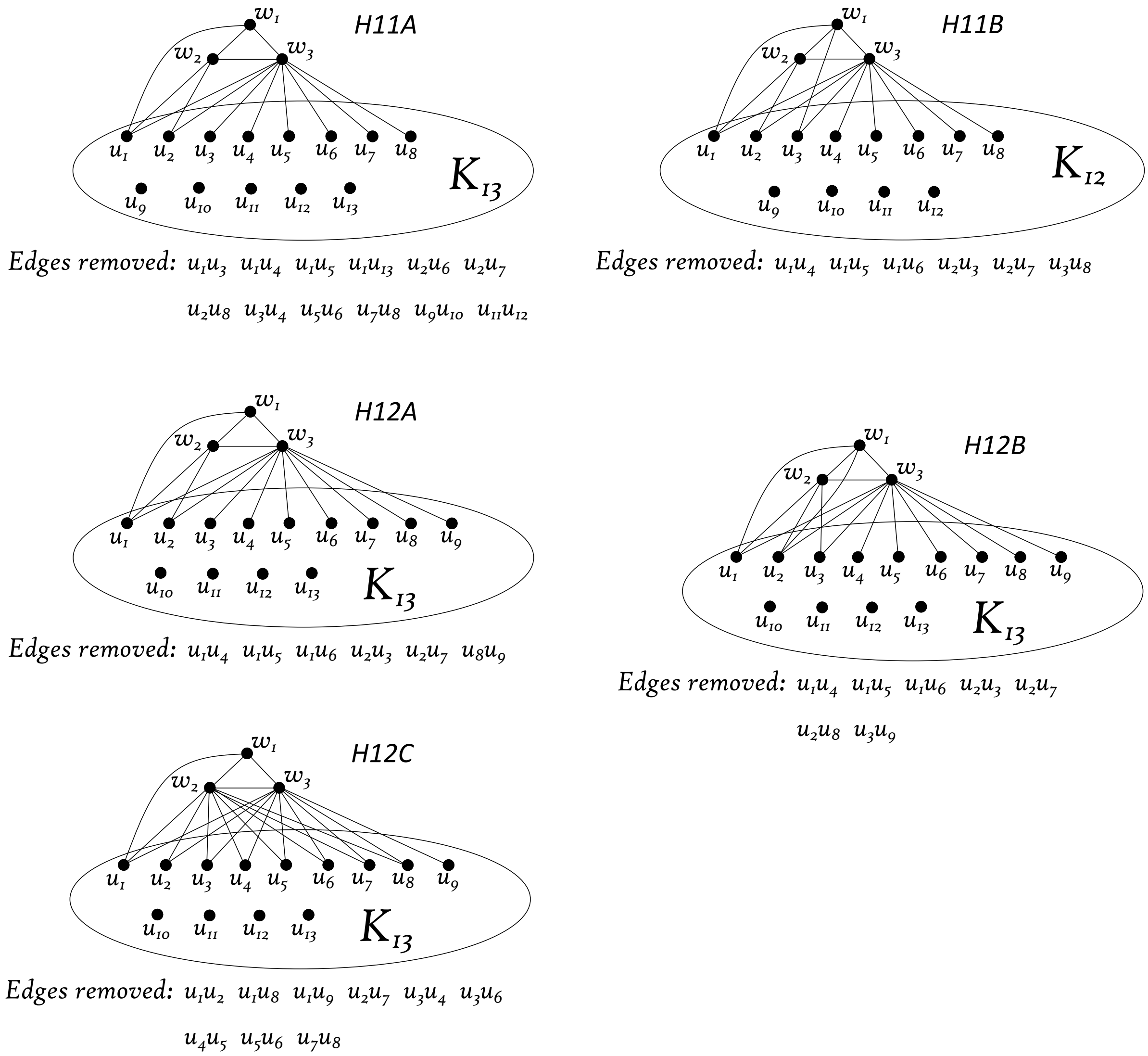

Figure 18: The graphs used to construct regular nonhamiltonian LH graphs in combination with $G11$ and $G12$.

6. Longest paths in *LH* graphs

The title of this section comes from a paper by Entringer and MacKendrick [11]. For $n \geq 4$, they define $f(n)$ to be the largest integer such that every connected LH graph on n vertices contains a path of length $f(n)$. They established the following upper bound for $f(n)$.

Theorem 6.1. [11] $f(n) \leq 24\sqrt{n/3} + 4$ *for* $n \geq 4$.

Although Entringer and MacKendrick did not explicitly state it, the following corollary is an obvious implication of Theorem 6.1.

Corollary 6.2. $\lim_{n\to\infty} \frac{f(n)}{n} = 0$

In [26] we proved a theorem similar to Theorem 6.1:

Theorem 6.3. [26] *For any natural number* $k > 0$ *there exists a planar connected LH graph* G *with* $\Delta(G) \leq 14$ *such that the difference between the order* n *of* G *and the length of a longest path in* G *is at least* k.

This result holds for planar graphs, gives a smaller upper bound for $f(n)$ for small n than the result by Entringer and Mackendrick, and limits the maximum vertex degree to 14, but does not imply Corollary 6.2. The LH graphs constructed by Entringer and MacKendrick to provide the bound in Theorem 6.1 are nonplanar and there is no restriction on their maximum degree. However, it is possible to prove a result equivalent to Corollary 6.2 for planar graphs with bounded maximum vertex degree. We define $p(n, \Delta)$ to be the largest integer such that every connected planar LH graph of order n with maximum degree Δ contains a path of length $p(n, \Delta)$. We now prove the following result, which is stronger than Corollary 6.2.

Theorem 6.4. $\lim_{n\to\infty} p(n, \Delta)/n = 0$ *for every* $\Delta \geq 11$.

PROOF. Consider the order 23 graph G_0 shown in Figure 19. This graph is the Goldner-Harary graph shown in Figure 9 (a) and the first graph in Figure 20 with 12 vertices added using triangle identification with multiple copies of K_4. We see that $\Delta(G_0) = 11$ and by Lemma 2.5 G_0 is LH, planar and nonhamiltonian. Let the K_3 subgraphs of G_0 that are encircled in Figure 19 be labeled $H_1, H_2, \ldots, H_6$ as shown. The graph G_0 is traceable, but it should be noted that there is no Hamilton path that starts in H_i and ends in H_i, $i \in \{1, 2, 3, 4, 5, 6\}$. Now let the graphs $G_{0,1}, G_{0,2}, \ldots, G_{0,6}$ be six copies of G_0, each with the K_3 subgraphs labeled in the same way as in G_0. Use triangle identification to combine G_0 with $G_{0,i}$ by identifying H_i in G_0 with H_i in $G_{0,i}$, $i = 1, 2, 3, 4, 5, 6$, to create the graph G_1. We know this is possible, since each H_i contains a vertex that is of degree 3 in G_0 and in $G_{0,i}$. Also note that $\Delta(G_1) = 11$ and that G_1 is planar. Since each $G_{0,i}$ contains a vertex cutset of order 5, it follows that a longest path in G_1 omits one H_j subgraph in four of the subgraphs represented by $G_{0,i}$ so that the longest path in G_1 has length $23+2\times 20+4\times 17 = 131$, while $n(G_1) = 23+6\times 20 = 143$. One can now repeat the procedure by combining G_1 with 6×5 copies of G_0 in the same way to create the graph G_2. A longest path in G_2 will contain $23 + 2 \times 20 + 4 \times 17 + 2 \times 20 + 6 \times 4 \times 17 = 579$ vertices, while $n(G_2) = 23 + 6 \times 20 + 6 \times 5 \times 20 = 743$. This process can be continued indefinitely. By Lemma 2.5 the graph G_k is planar and $\Delta(G_k) = 11$, while the longest path in G_k contains $p_k = 23 + 2 \times 20 + 4 \times 17 + \sum_{i=2}^{k}(2 \times 20 + 6 \times 4^{i-1} \times 17)$ vertices, while $n(G_k) = 23 + \sum_{i=1}^{k} 6 \times 5^{i-1} \times 20$. It is then easy to show that $\lim_{k\to\infty} \frac{p_k}{n(G_k)} = 0$ and the theorem follows. The result can easily be extended to greater values for the maximum degree by combining the graph G_k with a planar graph with the required maximum degree by triangle identification with one of the outer triangle subgraphs.

Note that Entringer and MacKendrick's limit only implies the existence of connected nontraceable LH graphs of order greater than or equal to 200. However, it is shown in [26] that the smallest connected

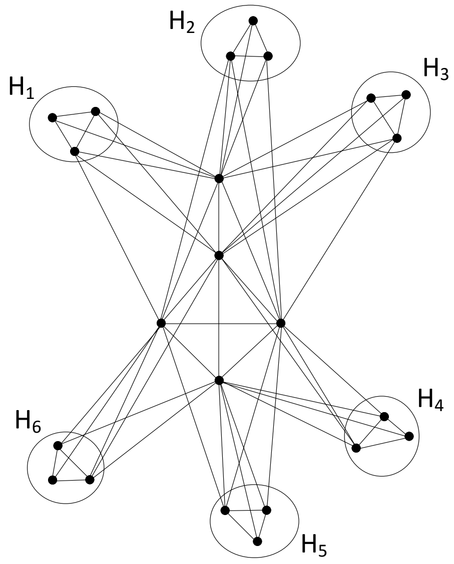

Figure 19: The graph G_0 used in Theorem 6.4

nontraceable LH graph has order 14, so there is much room for improvement for low values of n. Our next theorem provides an upper limit for $f(n)$ that is smaller than the one given by Entringer and MacKendrick for $n \leq 427$ and implies that $f(n) < n$ for every $n \geq 15$.

Theorem 6.5. $f(n) \leq \lceil (2/3)n \rceil + 4$.

Proof. Consider the graph G_0 shown in Figure 20. This is the Goldner-Harary graph shown in Figure 9 (a), redrawn to emphasize the fact that the six vertices of degree 3 are connected to each other by a cutset of 5 vertices. Now choose any vertex of degree 3, call it v_1, and using Lemma 2.5 use triangle identification to combine G_0 with three copies of K_4, each time using v_1 and two of its neighbours, to create the graph G_1. Thus G_1 now has a vertex cutset of order six (v_1 is now also in the cutset), the removal of which will result in eight components. In general, the graph G_{i-1} can be combined with three copies of K_4 using any vertex of degree 3 in $V(G_{i-1})$, call it v_{i-1}, to create the graph G_i. By Lemmas 2.8 and 2.5, G_i is LH and planar. Also, G_i has a vertex cutset of order $5+i$, the removal of which will result in a graph consisting of $6+2i$ isolated vertices. It follows that a longest path in G_i has no more than $2(5+i)+1$ vertices, and that $n(G_i) = 11 + 3i$. Let $q(n)$ be the number of vertices in a longest path in a graph on n vertices constructed in this way (where the last vertex v_i to be used in triangle identification may have been used once, twice, or three times). Then $q(n) \leq \lceil (2/3)n \rceil + 4$.

7. Open problems

1. Let Δ_H denote the largest integer such that every connected LH graph with maximum degree Δ_H is hamiltonian. We suspect that $\Delta_H = 7$, but it has only been proven to be at least 6.

2. Let Δ_P denote the largest integer such that the Hamilton Cycle Problem for LH graphs with maximum degree Δ_P is solvable in polynomial time. It follows from Theorems 4.2 and 4.6 that $6 \leq \Delta_P \leq 9$. We conjecture that $\Delta_P = 8$.

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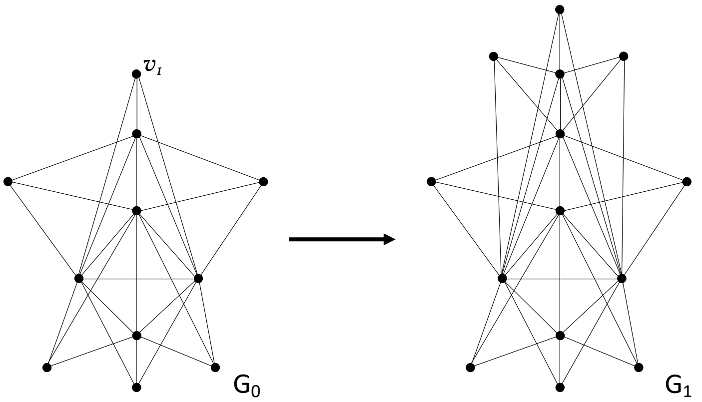

Figure 20: The graphs G_0 and G_1 used in Theorem 6.5

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