One Shape Fits All? A Comprehensive Examination of Cryptocurrency Return Distributions

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Abstract: We perform the most comprehensive test of cryptocurrency return distributions to date. We fit 58 hypothetical distributions to 15 major cryptocurrencies to establish which of these best describes cryptocurrency returns. The answer is: "It depends." A sharp-peaked Cauchy distribution is the most likely distribution for the majority of return series. Specific distributions are definitively identified for only a handful of cryptocurrencies. The best fitting distributions are peaked and thick-tailed, with some possessing variable shape parameters. Our findings have implications for financial modelling and its applications, such as risk measurement and risk management.

Keywords: cryptocurrencies, return distributions, bitcoin, skewness, kurtosis, thick-tails.

JEL codes: C49, C59, G10

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1. Introduction

Following the pioneering works of Mandelbrot (1963) and Fama (1965), equity returns are widely acknowledged to be non-normally distributed. Newly emerging asset classes pose a further challenge for financial modelling. The proliferation of cryptocurrencies, following the introduction of Bitcoin in 2010, serves as a perfect example of this new challenge. While distributional assumptions are essential for academics and market practitioners for financial modelling and investment decision making, little is known about the specific distributions that describe returns on this novel asset class. The primary aim of this study is to fill this lacuna.

Asset pricing literature offers a limited number of studies on cryptocurrency distributions. Existing studies focus on relatively few cryptocurrencies and consider relatively few hypothetical distributions. Chan, Chu, Nadarajah and Osterrieder (2017) fit nine parametric distributions to seven cryptocurrencies and document that they are best described by the generalized hyperbolic, normal inverse Gaussian, generalized *t*, and Laplace distributions. Zhang, Wang, Li and Shen (2018) show that returns on eight cryptocurrencies exhibit thick tails. They do not investigate which specific distributions best fit returns. Bariviera, Basgall, Hasperué and Naiouf (2017) reject the normality assumption for Bitcoin, finding that returns are negatively skewed and highly leptokurtic. Specific distributions are not considered. Finally, Phillip, Chan and Peiris (2018) analyse the statistical properties of five cryptocurrencies (Bitcoin, Ethereum, Ripple, NEM and Dash) and reject the normality assumption for each return series. They report kurtosis coefficients ranging between 7 and 40 and returns exhibiting positive and negative skewness of varying degrees.

We are the first to perform a comprehensive examination of cryptocurrency return distributions. We fit a total of 58 potential distributions to a sample of 15 cryptocurrencies. While the literature cited above decisively suggests that cryptocurrency movements are not normally distributed, the question as to which distributions best describe these movements remains open. We resolve this puzzle by applying goodness-of-fit tests to identify the most appropriate distributions.

Our findings demonstrate that there is no single distribution that best represents all cryptocurrencies. Specific distributions are definitively identified for a handful of cryptocurrencies, namely Cardano, Tron, Chainlink and Iota. For the remainder, no distribution is definitive although the most likely and frequent candidate is the Cauchy distribution. The best fitting distributions are peaked and thick-tailed, with some possessing variable shape parameters.

Our findings not only provide new insights into the behaviour of cryptocurrency prices. They also matter for financial modelling and could be applicable in risk measurement and management or asset pricing, including the valuation of potential cryptocurrency derivatives.

2. Data and Methods

The sample comprises 15 largest cryptocurrencies by market capitalization as of 2 July 2019: Binance, Bitcoin, Bitcoin Cash, Cardano, Chainlink, Dash, EOS, Ethereum, IOTA, Litecoin, Monero, NEO, Ripple, Stellar, and Tron. Data are obtained from CoinMarketCap.com. Three cryptocurrencies – Tether, Bitcoin SV and UNUS SED LEO – are excluded on account of incomplete data and/or excessively short data series. All prices are in U.S. dollars and returns are estimated by taking logarithmic differences. The study period for daily returns runs from 28 April 2013 to 1 July 2019 and is dictated by data availability. Table A1 in the Online Appendix reports the descriptive statistics of the cryptocurrency returns. Notably, it also reports results of the Shapiro-Wilk (SW) test, commonly employed to verify the normality assumption when the distribution is unknown (Yap & Sim 2011). All distributions are non-normal, highly leptokurtic with excess kurtosis ranging between 3.33

(Chainlink) and 68.11 (Ethereum). Most series are positively skewed, with the exception of Bitcoin and Ethereum.

In our empirical tests we consider 58 potential distributions. We provide a list in Table A2 of the Online Appendix. To investigate which distribution fits a specific series or most closely describes returns, we follow Bakouch, Khan, Hussain, and Chesneau (2019) and – for robustness - apply three goodness-of-fit tests. We apply the Kolmogorov-Smirnov (*KS*), Anderson-Darling (*AD*), and chi-squared (χ^2) tests, with the respective test statistics outlined in Equations (1)-(3):

$$KS = max\left\{\frac{i}{n} - z_i, z_i - \frac{i-1}{n}\right\}, \tag{1}$$

$$AD = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[ln\,(z_i\,(1 - z_i(x_{n-1+1}))], \tag{2}$$

$$\chi^2 = \sum_{i=1}^{n}\frac{(O_i - E_i)}{E_i}\,. \tag{3}$$

In Equations (1) and (2) n denotes the number of observations, $z_i = F_X(x_i)$ is the cumulative density function (CDF) of the distribution being tested, and i represents observations in ascending order. In Equation (3), O_i is the observed frequency for a given bin and E_i is an expected frequency for a given bin under a specified distribution. The number of bins is determined as $k = 1 + log_2 n$ where n is the sample size and k is the number of bins. The shared basis of these tests is that they compare the distance between hypothesized distributions and empirical distributions (Lemeshko, Lemeshko, and Postovalov 2010; Franke, Ho, and Christie 2012). Distributional parameters are estimated using four methods, namely the method of moments, maximum likelihood estimates, least squares estimates, and the method of *L*-moments (for an overview see Lloyd 1952; Hoskin 1990; Christopeit 1994; Everitt 2014). When estimating parameters, we select the least computationally intensive method for each distribution. Table A2 in the Online Appendix lists estimation methods for each distribution.

Once we have estimated statistics for the tests above, we rank distributions on the basis of these statistics and test the null hypothesis that returns conform to a specified distribution. If the null hypothesis is rejected for each hypothesized distribution, then no single distribution best describes returns. In such cases, goodness-of-fit tests indicate the best fitting distribution but not a specific distribution.

3. Results

Figure 1 plots the return distributions of the cryptocurrencies in our sample. A visual inspection confirms departures from normality in the form of excess kurtosis and thick-tails. The respective kernel density plots, estimated using the Epanechikov (1969) kernel, differ from the imposed normal distribution curve.

[Insert Figure 1 here]

Table 1 reports the results of the analysis of return distributions.[1] The null hypothesis of the empirical distribution approximating a hypothesized distribution is not rejected across all three goodness-of-fit tests for Cardano, Tron, Chainlink and Iota. Four-parameter Dagum and Burr distributions are fitted to Cardano and Chainlink returns respectively. These distributions possess shape parameters which permit varying levels of skewness, kurtosis and tail-thickness (Gomes-Silva, da Silva, Percontini, Ramos & Cordeiro 2017). The Dagum distribution captures positive asymmetry and is a good descriptor of extremes (Tahir, Cordeiro, Mansoor, Zubair & Alizadeh 2014). Similarly, the (slightly) positively skewed Burr distribution, by accommodating a wide range of skewness and kurtosis levels, can fit a wide range of empirical data (Tadikamalla 1980). Both distributions are thick-tailed. The Cauchy distribution, also a leptokurtic distribution with thick-tails, is fitted to Tron returns. This distribution is considered a pathological case; mean and variance are undefined, implying that

[1] For a more detailed description of the best fitting distributions, see Table A3 in the Online Appendix.

both moments are dominated by arbitrarily large observations and that instability of the moments is inherent to the data (Mahdizadeh & Zamanzade 2019).

[Insert Table 1 here]

Interestingly, two distributions can be fitted to Iota returns, the Error (KS and χ^2 tests) and Laplace (AD test) distributions. The Error distribution is a symmetric bell-shaped distribution which permits varying levels of the sharpness of the peakedness. A *k* parameter of 1 denotes sharp peakedness, transforming this distribution into the thick-tailed Laplace distribution and explaining why the Laplace distribution also fits Cardano returns. For all remaining cryptocurrencies, the Cauchy distribution appears to be the best fitting distribution although the null hypothesis that returns conform to this distribution is rejected for each cryptocurrency with the exception of Ethereum, Bitcoin Cash, EOS and Binance. For these currencies, the null hypothesis is not rejected by the KS test but is rejected by the other two tests. Neo returns are an exception. The Johnson SU distribution is the best fitting distribution. This is a flexible unbounded distribution which, depending upon its shape parameters, can capture asymmetry, excess kurtosis, and extreme tails (Choi & Nam 2008). Other candidate distributions, ranked on the basis of the KS, AD and χ^2 tests [columns (5), (6) and (7) in Table 1] are the log-logistic and hypersecant distributions.

Our findings are in general agreement with those of Chan *et al.* (2017), Zhang *et al.* (2018), Bariviera *et al.* (2017) and Phillip *et al*. (2018) who find that cryptocurrency returns exhibit excess levels of kurtosis and thick-tails. A number of cryptocurrencies considered by Chan *et al.* (2017), who similarly fit specific distributions to cryptocurrencies, also form part of our sample. Chan *et al.* (2017) fit the generalized hyperbolic distribution to Bitcoin and Litecoin returns and the normal inverse distribution to Dash, Monero and Ripple returns. Our results differ and we identify the Cauchy as the most appropriate distribution for the former three cryptocurrencies and for Ripple. The Error and Laplace distributions are fitted to Monero returns. A potential explanation is that the wider universe of distributions considered in this study leads to the identification of more appropriate and better fitting distributions.

4. Conclusions

We fit the broadest sample ever of 58 potential distributions to returns on 15 cryptocurrencies. We identify specific distributions for a handful of cryptocurrencies, namely Cardano, Tron, Chainlink and Iota. For the remainder, we identify the best fitting distribution, (decisively) the Cauchy distribution. We confirm findings of prior studies by showing that cryptocurrency returns are not normally distributed. Relatedly, the best fitting distributions are highly peaked and thick-tailed distributions.

Our findings provide new insights into the behaviour of cryptocurrencies that are informative and of interest to both researchers and investors. Furthermore, our results are important for the modelling of cryptocurrency returns and for applications to risk assessment and management, and asset pricing.

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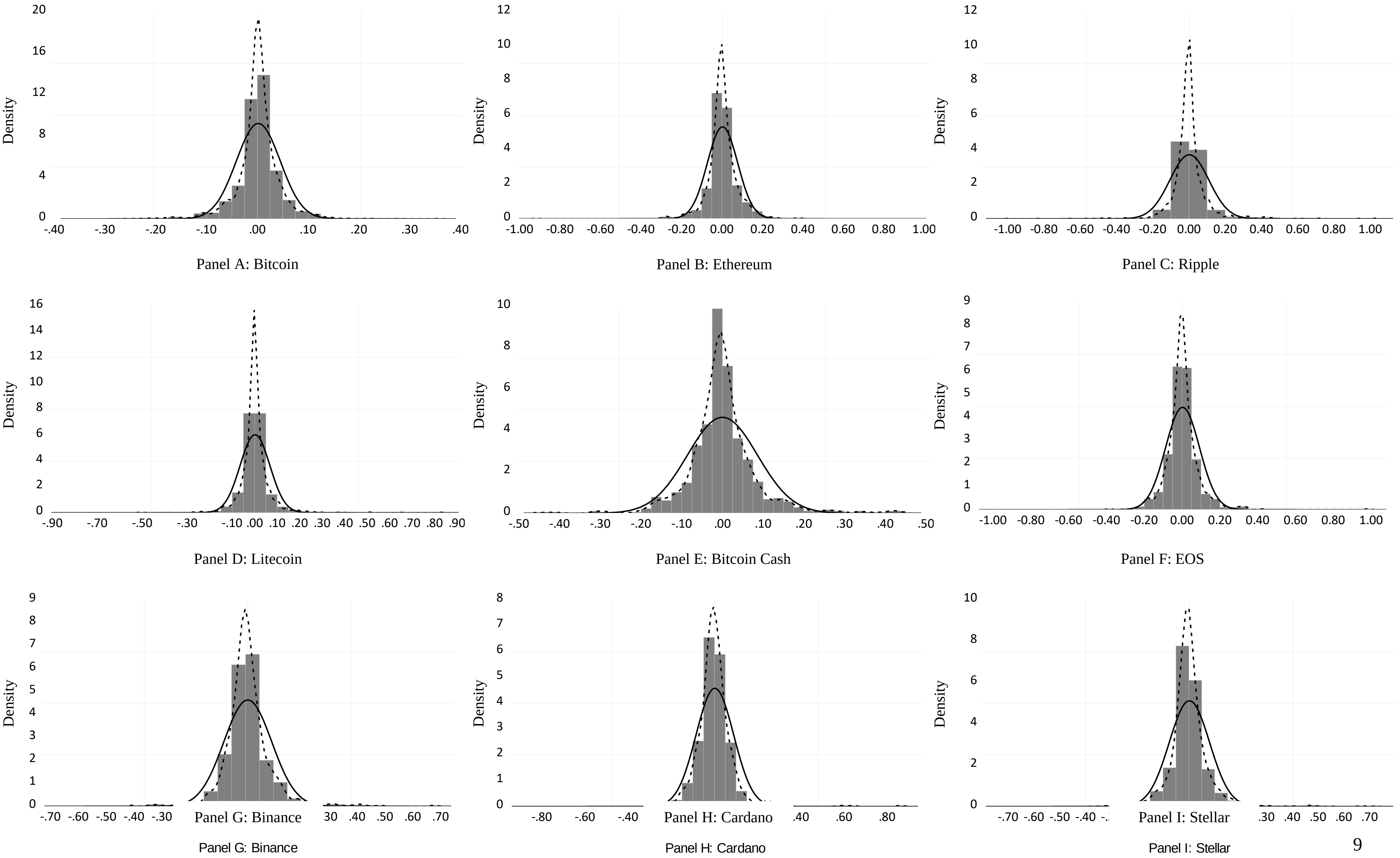
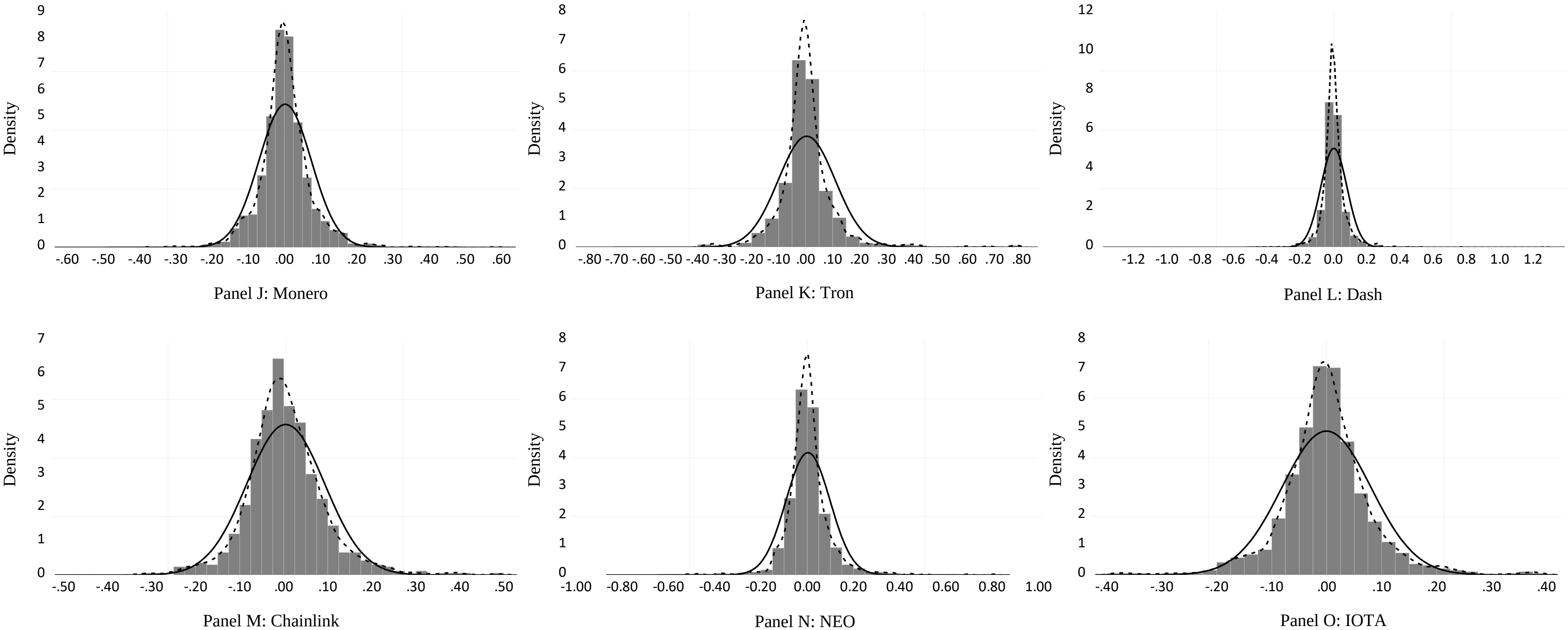
Figure 1. Overview of cryptocurrency returns





This figure reports histograms of returns for each of the cryptocurrencies in the sample. A kernel density function (dashed line - - -) is fitted to each series to describe the empirical distribution for comparison against a hypothetical normal distribution (sold line ——).

Table 1. Results of goodness-of-fit tests

In column (1), *KS*, *AD*, and χ^2 refer to distributions identified by the Kolmogorov-Smirnov, Anderson-Darling and chi-squared tests respectively. In column (4), the same convention is followed, with corresponding test statistics reported. The asterisks * and ** indicate statistical significance at the respective 5% and 1% levels of significance. Column (3) reports the respective scale, location and shape parameters, where applicable in the latter case. Distributions identified and ranked according to *KS*, *AD* and χ^2 test statistics are reported in column (5), (6) and (7). For the respective probability and cumulative density functions (PDF and CDFs), and location, scale and shape parameters for the best fitting distributions, see Table A3 in the Online Appendix. For a full list of the distributions tested and methods applied, scale and shape parameters, refer to Table A2 in the Online Appendix.

(1) Crypto.	(2) Best Fit (KS/AD/χ^2)	(3) Parameters	(4) Goodness-of-fit (KS/AD/χ^2)	(5) KS	(6) AD	(7) χ^2
Bitcoin	1. Cauchy 1. Cauchy 1. Cauchy	σ =0,01489 μ =0,00223	0,03025* 5,4355** 43,855**	2. Laplace 3. Error 4. Johnson SU 5. Log-Logistic (3P)	2. Laplace 3. Error 4. Johnson SU 5. Dagum (4P)	2. Error 3. Laplace 4. Johnson SU 5. Log-logistic (3P)
Ethereum	1. Cauchy 1. Cauchy 1. Cauchy	σ =0,02591 μ =-0,00157	0,04149 7,1984** 40,904**	2. Burr (4P) 3. Laplace 4. Error 5. Log-Logistic	2. Burr (4P) 3. Laplace 4. Error 5. Log-logistic (3P)	2. Error 3. Laplace 4. Burr (4P) 5. Log-logistic (3P)
Ripple	1. Cauchy 1. Cauchy 1. Cauchy	σ =0,02111 μ =-0,00268	0,02947* 4,528 ** 40,978**	2. Johnson SU 3. Log-Logistic (3P) 4. Dagum (4P) 5. Laplace	2. Johnson SU 3. Burr (4P) 4. Dagum (4P) 5. Log-logistic (3P)	2. Johnson SU 3. Burr (4P) 4. Dagum (4P) 5. Error
Litecoin	1. Cauchy 1. Cauchy 1. Cauchy	σ =0,01937 μ =0	0,0357** 5,0281** 54,546**	2. Log-Logistic (3P) 3. Dagum (4P) 4. Burr (4P) 5. Laplace	2. Johnson SU 3. Burr (4P) 4. Dagum (4P) 5. Log-logistic (3P)	2. Johnson SU 3. Error 4. Laplace 5. Burr (4P)
Bitcoin Cash	1. Cauchy 1. Cauchy 1. Cauchy	σ =0,0317 μ =-0,00505	0,04385 2,6096* 23,807**	2. Laplace 3. Error 4. Log-logistic (3P) 5. Dagum (4P)	2. Laplace 3. Error 4. Johnson SU 5. Burr (4P)	2. Error 3. Laplace 4. Burr (4P) 5. Johnson (SU)
EOS	1. Cauchy 1. Cauchy 1. Cauchy	σ =0,03234 μ =-0,00242	0,04126 2,642* 21,121*	2. Burr (4P) 3. Log-logistic (3P) 4. Dagum (4P) 5. Laplace	2. Laplace 3. Error 4. Burr (4P) 5. Dagum (4P)	2. Error 3. Laplace 4. Burr (4P) 5. Log-logistic
Binance	1. Cauchy 1. Cauchy 1. Cauchy	σ =0,03142 μ =5,5510E-4	0,04926 3,4931* 17,792*	2. Burr (4P) 3. Log-Logistic (3P) 4. Dagum (4P) 5. Johnson SU	2. Johnson SU 3. Burr (4P) 4. Log-logistic (3P) 5. Dagum (4P)	2. Johnson SU 3. Error 4. Laplace 5. Burr (4P)
Cardano	1. Burr (4P)	k =0,79473 α =22,139 β =0,77855 γ =-0,79404	0,04803 2,4485 20,494*	2. Log-logistic (3P) 3. Dagum (4P) 4. Cauchy 5. Laplace	2. Log-Logistic (3P) 3. Burr (4P) 4. Cauchy 5. Laplace	2. Burr (4P) 3. Log-logistic (3P) 4. Cauchy 5. Error
	1. Dagum (4P) 1. Dagum (4P)	k =0,76682 α =12,567 β =0,45671 γ =-0,44661	0,05107 2,201 15,297			
Stellar	1. Cauchy 1. Cauchy 1. Cauchy	σ =0,02792 μ =-0,00407	0,04236** 7,8608** 44,95**	2. Burr (4P) 3. Log-Logistic (3P) 4. Dagum (4P) 5. Laplace	2. Burr (4P) 3. Dagum (4P) 4. Log-logistic (3P) 5. Johnson SU	2. Burr (4P) 3. Error 4. Laplace 5. Dagum (4P)

Monero	1. Error	k =1,0 σ =0,07249 μ =0,00215	0,03335* 2,8561* 32,735**	2. Laplace 3. Johnson SU 4. Burr (4P) 5. Log-Logistic (3p)	2. Error 3. Johnson SU 4. Burr (4P) 5. Dagum (4P)	2. Laplace 3. Johnson SU 4. Cauchy 5. Burr (4P)
	1. Laplace	λ =19,509 μ =0,00215	0,03335* 2,8561* 32,735**			
	1. Error					
Tron	1. Cauchy 1. Cauchy 1. Cauchy	σ =0,03495 μ =-0,00367	0,04036 2,2061 9,7874	2. Log-logistic (3P) 3. Burr (4P) 4. Dagum (4P) 5. Laplace	2. Burr (4P) 3. Dagum (4P) 4. Log-logistic (3P) 5. Laplace	2. Burr (4P) 3. Log-logistic (3P) 4. Dagum (4P) 5. Error
Dash	1. Cauchy 1. Cauchy 1. Cauchy	σ =0,0265 μ =-0,00249	0,04049** 8,2858** 66,969**	2. Burr (4P) 3. Log-logistic (3P) 4. Dagum (4P) 5. Laplace	2. Burr (4P) 3. Johnson SU 4. Log-logistic (3P) 5. Dagum (4P)	2. Johnson SU 3. Burr (4P) 4. Dagum (4P) 5. Log-logistic (3P)
Chainlink	1. Burr (4P) 1. Burr (4P) 1. Burr (4P)	k =0,68203 α =227,47 β =8,9261 γ =-8,9488	0,02481 0,62275 6,0529	2. Log-logistic (3P) 3. Dagum (4P) 4. Johnson SU 5. Hypersecant	2. Johnson SU 3. Log-logistic (4P) 4. Dagum (4P) 5. Hypersecant	2. Johnson SU 3. Log-logistic (3P) 4. Dagum (4P) 5. Hypersecant
Neo	1. Johnson SU 1.Johnson SU 1.Johnson SU	γ =-0,27303 δ =1,2121 λ =0,07674 ξ =-0,02118	0,04687* 4,0881** 39,967**	2. Cauchy 3. Dagum (4P) 4. Log-logistic (3P) 5. Burr (4P)	2. Cauchy 3. Burr (4P) 4. Log-logistic (3P) 5. Dagum (4P)	2. Cauchy 3. Error 4. Laplace 5. Burr (4P)
Iota	1. Error	k =1,0 σ =0,08139 μ =-5,0180E-4	0,02846 0,44619 6,2261	2. Laplace 3. Johnson SU 4. Burr (4P) 5. Dagum (4P)	2. Error 3. Johnson SU 4. Burr (4P) 5. Dagum (4P)	2. Laplace 3. Johnson SU 4. Burr (4P) 5. Dagum (4P)
	1. Laplace	λ =17,376 μ =-5,0180E-4	0,02846 0,44619 6,2261			
	1. Error					