Aspects of Matroid Connectivity and Uniformity

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Abstract

In approaching a combinatorial problem, it is often desirable to be armed with a notion asserting that some objects are more highly structured than others. In particular, focusing on highly structured objects may avoid certain degeneracies and allow for the core of the problem to be addressed. In matroid theory, the principle notion fulfilling this role of "structure" is that of connectivity. This thesis proves a number of results furthering the knowledge of matroid connectivity and also introduces a new structural notion, that of generalised uniformity.

The first part of this thesis considers 3-connected matroids and the presence of elements which may be deleted or contracted without the introduction of any non-minimal 2-separations. Principally, a Wheels-and-Whirls Theorem and then a Splitter Theorem is established, guaranteeing the existence of such elements, provided certain well-behaved structures are not present.

The second part of this thesis generalises the notion of a uniform matroid by way of a 2-parameter property capturing "how uniform" a given matroid is. Initially, attention is focused on matroids representable over some field. In particular, a finiteness result is established and a specific class of binary matroids is completely determined. The concept of generalised uniformity is then considered more broadly by an analysis of its relevance to a number of established matroid notions and settings. Within that analysis, a number of equivalent characterisations of generalised uniformity are obtained.

Lastly, the third part of the thesis considers a highly structured class of matroids whose members are defined by the nature of their circuits. A characterisation is achieved for the regular members of this class and, in general, the infinitely many excluded series minors are determined.

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Chapter 1

Introduction

Matroids fulfil something of a unifying role in combinatorics, bringing the study of seemingly disparate combinatorial objects (such as graphs, matrices, and geometric lattices) together by way of an axiomatic approach. A hallmark of this approach is that it is often possible to abstract a notion from one particular combinatorial setting to achieve a notion that holds for all matroids. One such notion is that of matroid connectivity, which arose from the eponymous notion for graphs, and has proven indispensable to matroid theorists since its introduction by Tutte [35] in 1966.

Let M be a matroid with ground set E and rank function r. The *connectivity function* λ_M of M is defined on all subsets X of E by

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

A subset X or a partition $(X, E - X)$ of E is *k-separating* if $\lambda_M(X) \leq k - 1$. A k-separating partition $(X, E - X)$ is a *k-separation* or *separation of order k* if $\min\{|X|, |E - X|\} \geq k$. A matroid is *$n$-connected* if it has no k-separations for all $k < n$. Matroid 1-separations and 2-separations are incredibly tame, and numerous well-studied matroid properties (for example, representability over a given field) are closed under direct sums and 2-sums. Furthermore, by a result of Cunningham and Edwards [10], every matroid may be decomposed into 3-connected components, each a minor of the original, such that a number of direct sums and 2-sums retrieves the original matroid. Thus, 3-connected matroids are, in a very natural sense, the fundamental building blocks of matroids.

The archetypal result concerning 3-connected matroids is Tutte's Wheels-and-Whirls Theorem [35] which states that if M is a non-empty 3-connected matroid that is neither a wheel nor a whirl, then it has an element e such that M/e or $M\backslash e$ is 3-connected. Seymour [32] extended this result into his equally celebrated Splitter Theorem, guaranteeing an element whose removal preserves 3-connectivity while additionally keeping a specified 3-connected minor. Such results allow for the inductive arguments on 3-connected matroids that are the mainstay of structural matroid theory. Over the years, multiple extensions and analogous results to Tutte's Wheels-and-Whirls Theorem and Seymour's Splitter Theorem have been established (see, for example [8, 14, 37]) and the main theorems of the first part of this thesis continue this tradition.

A k-separation (X, Y) is *minimal* if $\min\{|X|, |Y|\} = k$. In a 2-connected matroid, minimal 2-separations correspond to parallel and series pairs of elements. Thus, a 2-connected matroid whose only 2-separations are minimal is very "close" to being 3-connected. It is a widely used result of Bixby [2], that for every element e of a 3-connected matroid M, at least one of M/e or $M\backslash e$ has no non-minimal 2-separations. Inspired by this, the first part of this thesis is dedicated to the existence of elements e of a 3-connected matroid M for which neither M/e nor $M\backslash e$ has any non-minimal 2-separations, calling such elements *elastic*. In Part I, it is shown that elastic elements can be reliably found and the only obstructions to such elements are extremely well behaved. Specifically, analogues of Tutte's Wheels-and-Whirls Theorem and Seymour's Splitter Theorem are established.

A rank-r matroid is *uniform* if its bases coincide with its r-element subsets. Thus, there is a unique uniform matroid $U_{r,n}$ of rank r and size $n \geq r$, and, in a natural sense, this is the most well-behaved matroid of that size and rank. Indeed, taking connectivity to its extreme, it is easily shown that the only matroids with no separations of any order are the uniform matroids whose rank and corank differ by at most one. The second part of this thesis generalises the notion of a uniform matroid by the introduction of a 2-parameter property of matroids that captures how "close" to uniform a given matroid is. This concept of generalised uniformity turns out to be particularly intriguing in the case of matroids representable over some finite field, as we show in Part II that, for any prime power q, there are only finitely many $GF(q)$-representable matroids of a prescribed "uniformity". Indeed, we give tight bounds on the rank and corank of such matroids. Based on these bounds, we explicitly determine the binary matroids that are two "steps" removed from being uniform. The precise terminology and statement of these results are given in the introduction to Part II.

As is the case with all matroid notions, the concept of generalised uniformity has a number of equivalent characterisations and is of greater consequence in some settings than it is in others. The latter half of Part II explores the relevance of generalised uniformity to a number of select matroid notions and settings. This is a first treatment and is designed as a primer for any researcher looking to exploit this notion in different contexts. One application that we will comment on here is that this theory of generalised uniformity places the well-studied class of paving matroids in a wider context. A rank-r matroid is *paving* if every rank-$(r-2)$ flat is independent. It is a well known conjecture of Crapo and Rota [9] that paving matroids "predominate" amongst all matroids, and multiple recent papers ([28], [29]) have supported the more precise conjecture of Mayhew et. al [27] that asymptotically every matroid is paving. In the context of our theory of generalised uniformity, paving matroids are, informally, one "step" removed from uniform. As such, the results of Part II have immediate consequences for these matroids.

Finally, two classes of matroids with an undeniable high degree of structure are the classes of binary and regular matroids, with the first being the class of matroids representable over $GF(2)$, and the second being the class of matroids representable over all fields. It is easily seen that every graphic matroid is regular and thus binary. Moreover, a famous decomposition result of Seymour [32] states that every regular matroid can be obtained by direct sums, 2-sums and 3-sums starting with matroids each of which is either graphic, cographic, or a copy of a particular matroid R_{10}. Thus, regular matroids can be thought of as being very close to graphic. It is by now a well-beaten track to take a statement that holds for graphs and seek to determine if an analogous property holds for all binary matroids, or failing this, for regular matroids. The work of the last part of this thesis is such an endeavour and is motivated by the characterisation of binary matroids that, for every pair of distinct intersecting circuits C_1, C_2, their symmetric difference $C_1 \cup C_2 - C_1 \cap C_2$ is a disjoint union of circuits. Part III considers those matroids for which the symmetric difference of every pair of intersecting circuits is itself a circuit, dubbing these matroids *circuit-difference*. A clean characterisation exists for the graphic members of this class. The main result of Part III is that this characterisation extends to the regular members. In the general binary case, for which the aforementioned characterisation fails, the infinitely many excluded series minors are determined.

1.1 Overview

This thesis is partitioned into three parts. The first part concerns the existence of elements in 3-connected matroids whose deletion and contraction are 3-connected up to series and parallel pairs respectively. The main result of Chapter 2 is a Wheels-and-Whirls Theorem for such elements and the main result of Chapter 3 is the extension of that result to a Splitter Theorem. Much of the material of Chapter 2 has been published in *The Electronic Journal of Combinatorics* [12].

The second part of the thesis develops a theory of generalised uniformity for matroids. Chapter 4 considers the role of this notion in matroids representable over a given finite field, proving a finiteness result and completely determining a specific class of such matroids. Chapter 5 considers uniformity in a wider context, detailing its relevance to a number of established matroid notions and settings, and proving a number of equivalent characterisations. The majority of Chapter 4 has been published in *Advances in Applied Mathematics* [13].

Lastly, the third part of the thesis concerns those matroids that are circuit-difference. A characterisation is achieved for the regular members of this class, and, more generally, the infinitely many excluded series minors are fully determined. The work of Sections 6.2, 6.3 and 6.4 has been published in *The Electronic Journal of Combinatorics* [11].

Throughout this thesis, we will assume a working knowledge of matroid theory. We refer the unfamiliar reader to Oxley's excellent treatise [26]. The notation and terminology of this thesis will follow that work unless otherwise specified.

Part I
Elastic Elements in 3-connected Matroids

A result widely used in the study of 3-connected matroids is due to Bixby [2]: if e is an element of a 3-connected matroid M, then either $M\backslash e$ or M/e has no non-minimal 2-separations, in which case, $\mathrm{co}(M\backslash e)$, the cosimplification of $M\backslash e$, or $\mathrm{si}(M/e)$, the simplification of M/e, is 3-connected. This result is commonly referred to as Bixby's Lemma. Thus, although an element e of a 3-connected matroid M may have the property that neither $M\backslash e$ nor M/e is 3-connected, Bixby's Lemma says that at least one of $M\backslash e$ and M/e is close to being 3-connected in a very natural way. In this part of the thesis, we are interested in whether or not there are elements e in M such that both $\mathrm{co}(M\backslash e)$ and $\mathrm{si}(M/e)$ are 3-connected, in which case, we say e is *elastic*.

In general, a 3-connected matroid M need not have any elastic elements. For example, all wheels and whirls of rank at least four have no elastic elements. The reason for this is that every element of such a matroid is in a 4-element fan and, geometrically, every 4-element fan is positioned in a certain way relative to the rest of the elements of the matroid. Moreover, 4-element fans are not the only obstacles to M having elastic elements.

Let $n \geq 3$, and let $Z = \{z_1, z_2, \ldots, z_n\}$ be a basis of $PG(n-1, \mathbb{R})$. Suppose that L is a line that is freely placed relative to Z. For each $i \in \{1, 2, \ldots, n\}$, let w_i be the unique point of L contained in the hyperplane spanned by $Z - \{z_i\}$. Let $W = \{w_1, w_2, \ldots, w_n\}$, and let Θ_n denote the restriction of $PG(n-1, \mathbb{R})$ to $W \cup Z$. Note that Θ_n is 3-connected and Z is a corank-2 subset of Θ_n. For all $i \in \{1, 2, \ldots, n\}$, we denote the matroid $\Theta_n\backslash w_i$ by Θ_n^-. The matroid Θ_n^- is well defined as, up to isomorphism, $\Theta_n\backslash w_i \cong \Theta_n\backslash w_j$ for all $i, j \in \{1, 2, \ldots, n\}$.

For the interested reader, the matroid Θ_n underlies the matroid operation of segment-cosegment exchange [22] which generalises the operation of delta-wye exchange. A more formal definition of Θ_n is given in Section 2.4.

If $n = 3$, then Θ_3 is isomorphic to $M(K_4)$. However, for all $n \geq 4$, the matroid Θ_n has no 4-element fans and, also, no elastic elements. Furthermore, for all $n \geq 3$, the set W is a modular flat of Θ_n [22]. Thus, if M is a matroid and W is a subset of $E(M)$ such that $M|W \cong U_{2,n}$, then the generalised parallel connection $P_W(\Theta_n, M)$ of Θ_n and M exists. In particular, it is straightforward to construct 3-connected matroids having no 4-element fans and no elastic elements. For example, take $U_{2,n}$ and repeatedly use the generalised parallel connection to "attach" copies of Θ_k, where $4 \leq k \leq n$, to any k-element subset of the elements of $U_{2,n}$.

Let M be a 3-connected matroid, and let A and B be rank-2 and corank-2 subsets of $E(M)$. We say that $A \cup B$ is a Θ-*separator* of M if $r(M) \geq 4$ and $r^*(M) \geq 4$, and either $M|(A \cup B)$ or $M^*|(A \cup B)$ is isomorphic to one of the matroids Θ_n and Θ_n^- for some $n \geq 3$. We will show in Section 2.4 that if S is a Θ-separator of M, then S contains at most one elastic element. Note that if $r(M) = 3$, then $\text{si}(M/e)$ is 3-connected for all $e \in E(M)$, while if $r^*(M) = 3$, then $\text{co}(M \backslash e)$ is 3-connected for all $e \in E(M)$. The main theorem of Chapter 2 is that, alongside 4-element fans, Θ-separators are the only obstacles to elastic elements in 3-connected matroids.

A 3-separation (A, B) of a matroid is *vertical* if $\min\{r(A), r(B)\} \geq 3$. Now, let M be a matroid and let $(X, \{e\}, Y)$ be a partition of $E(M)$. We say that $(X, \{e\}, Y)$ is a *vertical* 3-*separation* of M if $(X \cup \{e\}, Y)$ and $(X, Y \cup \{e\})$ are both vertical 3-separations and $e \in \text{cl}(X) \cap \text{cl}(Y)$. Furthermore, $Y \cup \{e\}$ is *maximal* in this separation if there exists no vertical 3-separation $(X', \{e'\}, Y')$ of M such that $Y \cup \{e\}$ is a proper subset of $Y' \cup \{e'\}$. Essentially, all of the work of Chapter 2 goes into establishing the following theorem.

Theorem 1.1.1. *Let M be a 3-connected matroid with a vertical 3-separation $(X, \{e\}, Y)$ such that $Y \cup \{e\}$ is maximal. Then at least one of the following holds:*

(i) *X contains at least two elastic elements;*

(ii) *$X \cup \{e\}$ is a 4-element fan; or*

(iii) *X is contained in a Θ-separator.*

The instances of Theorem 1.1.1 in which $X \cup \{e\}$ is a 4-element fan or X is contained in a Θ-separator are handled in more detail in Section 2.2 and Section 2.4 respectively. The following is our Wheels-and-Whirls Theorem for elastic elements and is an almost immediate consequence of Theorem 1.1.1. Its proof appears in Section 2.5.

Theorem 1.1.2. *Let M be a 3-connected matroid. If $|E(M)| \geq 7$, then M has at least four elastic elements provided M has no 4-element fans and no Θ-separators. Moreover, if $|E(M)| \leq 6$, then every element of M is elastic.*

The condition in Theorem 1.1.2 that M has no 4-element fans and no Θ-separators is not necessarily that restrictive. For example, if M is an excluded minor for $GF(q)$-representability (or, more generally, for $\mathbb{P}$-representability, where $\mathbb{P}$ is a partial field), then M has no 4-element fans and no Θ-separators. The fact that M has no 4-element fans is well known and straightforward to show. To see that M has no Θ-separators, suppose that M has a Θ-separator. By duality, we may assume that M has rank-2 and corank-2 sets W and Z, respectively, such that $M|(W \cup Z)$ is isomorphic to either Θ_n or Θ_n^-, for some $n \geq 3$. Say $M|(W \cup Z)$ is isomorphic to Θ_n. Then the matroid M' obtained from M by a cosegment-segment exchange on Z is isomorphic to the matroid obtained from M by deleting Z and, for each $w \in W$, adding an element in parallel to w. It is shown in [22, Theorem 1.1] that the class of excluded minors for $GF(q)$-representability (or, more generally, $\mathbb{P}$-representability) is closed under the operation of cosegment-segment exchange, and so M' is also an excluded minor for $GF(q)$-representability. But M' contains elements in parallel, a contradiction. The same argument holds if $M|(W \cup Z)$ is isomorphic to Θ_n^- except that, in applying a cosegment-segment exchange, we additionally add an element freely in the span of W.

Chapter 3 extends the study of elastic elements to those whose removal also keeps a specified 3-connected minor. Let M be a 3-connected matroid and let N be a 3-connected minor of M. We say that an element e of M is *N-elastic* if both $\text{si}(M/e)$ and $\text{co}(M\backslash e)$ are 3-connected and have an N-minor. In contrast, we say that an element e of M is *N-revealing* if one of the matroids $\text{si}(M/e)$ or $\text{co}(M\backslash e)$ has an N-minor and is not 3-connected.

Now suppose that W is a rank-2 subset and Z is a corank-2 subset of $E(M)$ such that $S = W \cup Z$ is a Θ-separator of M. Letting $n = \max\{|W|, |Z|\}$, we say that S *reveals* the minor N in M if either

(i) $M|(W \cup Z) \in \{\Theta_n, \Theta_n^-\}$ and at least one element of Z is N-revealing in M; or dually,

(ii) $M^*|(W \cup Z) \in \{\Theta_n, \Theta_n^-\}$ and at least one element of W is N^*-revealing in M^*.

The following is our Splitter Theorem for elastic elements and is the main result of Chapter 3.

Theorem 1.1.3. *Let M be a 3-connected matroid with no 4-element fans and let N be a 3-connected minor of M such that M has no Θ-separators revealing N. If M has at least one N-revealing element, then M has at least two N-elastic elements.*

The requirement that M has at least one N-revealing element is a necessary one (consider, for example when M and N have the same rank), however, this is no great ask. Equivalently, Theorem 1.1.3 guarantees that either M has at least two N-elastic elements, or whenever $\mathrm{si}(M/e)$ has an N-minor, then $\mathrm{si}(M/e)$ is 3-connected, and whenever $\mathrm{co}(M\backslash e)$ has an N-minor, then $\mathrm{co}(M\backslash e)$ is 3-connected; an extremely strong condition.

Theorem 1.1.3 follows largely from the following result: the extension of Theorem 1.1.1 to N-elastic elements, proved in Section 3.2.

Theorem 1.1.4. *Let M be a 3-connected matroid and let N be a 3-connected minor of M. Let $(X, \{e\}, Y)$ be a vertical 3-separation of M such that M/e has an N-minor and $|X \cap E(N)| \leq 1$. If $(X', \{e'\}, Y')$ is a vertical 3-separation of M such that $Y \cup \{e\} \subseteq Y' \cup \{e'\}$ and $Y' \cup \{e'\}$ is maximal, then at least one of the following holds:*

(i) *X' contains at least two N-elastic elements;*

(ii) *$X' \cup \{e'\}$ is a 4-element fan; or*

(iii) *X' is contained in a Θ-separator revealing N.*

Having established lower bounds on the number of elastic and N-elastic elements, it is natural to consider the matroids with the minimum number of such elements. Let M be a matroid. An exactly 3-separating partition (X, Y) of $E(M)$ is a *sequential 3-separation* if there is an ordering $(e_1, e_2, \ldots, e_k)$ of X or Y such that $\{e_1, e_2, \ldots, e_i\}$ is 3-separating for all $i \in \{1, 2, \ldots, k\}$. A matroid has *path-width three* if its groundset is sequential; that is, there is an ordering $(e_1, e_2, \ldots, e_n)$ of its groundset such that $\{e_1, e_2, \ldots, e_i\}$ is 3-separating for all $i \in \{1, 2, \ldots, n\}$. The matroids of path-width three are extremely well behaved and been thoroughly characterised [17, 25]. The proofs of the next two theorems appear in Section 3.3.

Theorem 1.1.5. *Let M be a 3-connected matroid with no 4-element fans or Θ-separators. If M has exactly four elastic elements, then M has path-width three.*

Theorem 1.1.6. *Let M be a 3-connected matroid with no 4-element fans and let N be a 3-connected minor of M such that $|E(N)| \geq 4$ and M has no Θ-separators revealing N. Let K be the set of N-revealing elements of M. If M has exactly two N-elastic elements s_1 and s_2, then $(K \cup \{s_1, s_2\}, E(M) - K \cup \{s_1, s_2\})$ is a sequential 3-separation.*

Our study of elastic elements has strong links to the study of maintaining 3-connectivity relative to a fixed basis [4, 24, 38]. Let M be a 3-connected matroid. Suppose M is representable over some field $\mathbb{F}$ and we are given an $\mathbb{F}$-representation of M in standard form relative to some basis B. Often, we wish to be able to remove elements from M while keeping the information displayed by its representation. In particular, we wish to avoid pivoting. One way to achieve this is to contract elements only from basis B and delete elements only from $E(M) - B$. It is also desirable to do so while maintaining 3-connectivity. In [38], Whittle and Williams gave a Wheels-and-Whirls type result by showing that if $|E(M)| \geq 4$ and M has no 4-element fans, then M has at least four elements e such that either $e \in B$ and $\mathrm{si}(M/e)$ is 3-connected, or $e \in E(M) - B$ and $\mathrm{co}(M\backslash e)$ is 3-connected. Brettell and Semple [4] extended this to a Splitter Theorem type result. In Section 3.4, we show that both results are implied by our work. We also resolve a question posed in [38].

This part of the thesis is organised as follows. Chapter 2 considers the presence of elastic elements in 3-connected matroids. Section 2.1 consists of some preliminaries, while Sections 2.2, 2.3 and 2.4 concern elastic elements in fans, segments and Θ-separators respectively. Finally, Section 2.5 consists of the proofs of Theorem 1.1.1 and Theorem 1.1.2. Chapter 3 is dedicated to N-elastic elements. Section 3.1 consists of some further preliminaries, while in Section 3.2, we prove Theorem 1.1.3 and Theorem 1.1.4. Section 3.3 considers the matroids with the minimum possible number of elastic and N-elastic elements, and includes the proofs of Theorem 1.1.5 and Theorem 1.1.6. Lastly, in Section 3.4, we show that a number of established fixed-basis results are consequences of the presence of elastic elements.

Chapter 2

A Wheels-and-Whirls Theorem for elastic elements

In this chapter, we prove Theorem 1.1.1 and obtain our Wheels-and-Whirls analogue, Theorem 1.1.2, as a corollary. The chapter is structured as follows. The next section contains some necessary preliminaries on connectivity that are used throughout this part of the thesis, while Section 2.2 determines exactly when elements of a fan are elastic. Section 2.3 establishes two results concerning when an element in a rank-2 restriction of a 3-connected matroid is deletable or contractible, and Section 2.4 considers Θ-separators, and determines the elasticity of the elements of those sets. Lastly, Section 2.5 consists of the proofs of Theorem 1.1.1 and Theorem 1.1.2.

2.1 Preliminaries

Connectivity

The following lemma, due to Bixby [2], is typically referred to as Bixby's Lemma.

Lemma 2.1.1. *Let e be an element of a 3-connected matroid M. Then either $M\backslash e$ or M/e has no non-minimal 2-separations, in which case,* $\mathrm{co}(M\backslash e)$ *or* $\mathrm{si}(M/e)$ *is 3-connected, respectively.*

Let e be an element of a 3-connected matroid M. We say e is *deletable* if $\mathrm{co}(M\backslash e)$ is 3-connected, and e is *contractible* if $\mathrm{si}(M/e)$ is 3-connected. Thus, e is elastic if it is both deletable and contractible.

Two k-separations (X_1, Y_1) and (X_2, Y_2) *cross* if each of the intersections $X_1 \cap Y_1$, $X_1 \cap Y_2$, $X_2 \cap Y_1$, $X_2 \cap Y_2$ are non-empty. The next lemma is a standard tool for dealing with crossing separations. It is a straightforward consequence of the fact that the connectivity function λ of a matroid M is submodular, that is,

$$\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$$

for all $X, Y \subseteq E(M)$. An application of this lemma will be referred to as *by uncrossing.*

Lemma 2.1.2. *Let M be a k-connected matroid, and let X and Y be k-separating subsets of $E(M)$.*

(i) *If $|X \cap Y| \geq k-1$, then $X \cup Y$ is k-separating.*

(ii) *If $|E(M) - (X \cup Y)| \geq k-1$, then $X \cap Y$ is k-separating.*

The next five lemmas are used frequently throughout part of the thesis. The first follows from orthogonality, while the second follows from the first. The third follows from the first and second. A proof of the fourth and fifth can be found in [37] and [4], respectively.

Lemma 2.1.3. *Let e be an element of a matroid M, and let X and Y be disjoint sets whose union is $E(M) - \{e\}$. Then $e \in \mathrm{cl}(X)$ if and only if $e \notin \mathrm{cl}^*(Y)$.*

Lemma 2.1.4. *Let X be an exactly 3-separating set in a 3-connected matroid M, and suppose that $e \in E(M) - X$. Then $X \cup \{e\}$ is 3-separating if and only if $e \in \mathrm{cl}(X) \cup \mathrm{cl}^*(X)$.*

Lemma 2.1.5. *Let (X, Y) be an exactly 3-separating partition of a 3-connected matroid M, and suppose that $|X| \geq 3$ and $e \in X$. Then $(X - \{e\}, Y \cup \{e\})$ is exactly 3-separating if and only if e is in exactly one of $\mathrm{cl}(X - \{e\}) \cap \mathrm{cl}(Y)$ and $\mathrm{cl}^*(X - \{e\}) \cap \mathrm{cl}^*(Y)$.*

Lemma 2.1.6. *Let C^* be a rank-3 cocircuit of a 3-connected matroid M. If $e \in C^*$ has the property that $\mathrm{cl}(C^*) - \{e\}$ contains a triangle of M/e, then $\mathrm{si}(M/e)$ is 3-connected.*

Lemma 2.1.7. *Let (X, Y) be a 3-separation of a 3-connected matroid M. If $X \cap \mathrm{cl}(Y) \neq \emptyset$ and $X \cap \mathrm{cl}^*(Y) \neq \emptyset$, then $|X \cap \mathrm{cl}(Y)| = |X \cap \mathrm{cl}^*(Y)| = 1$.*

Vertical connectivity

A k-separation (X, Y) of a matroid M is *vertical* if $\min\{r(X), r(Y)\} \geq k$. As noted in the introduction to this part of the thesis, we say a partition $(X, \{e\}, Y)$ of $E(M)$ is a *vertical* 3*-separation* of M if $(X \cup \{e\}, Y)$ and $(X, Y \cup \{e\})$ are both vertical 3-separations of M and $e \in \mathrm{cl}(X) \cap \mathrm{cl}(Y)$. Furthermore, $Y \cup \{e\}$ is *maximal* if there is no vertical 3-separation $(X', \{e'\}, Y')$ of M such that $Y \cup \{e\}$ is a proper subset of $Y' \cup \{e'\}$. A k-separation (X, Y) of M is *cyclic* if both X and Y contain circuits. The next lemma gives a duality link between the cyclic k-separations and vertical k-separations of a k-connected matroid.

Lemma 2.1.8. *Let (X, Y) be a partition of the ground set of a k-connected matroid M. Then (X, Y) is a cyclic k-separation of M if and only if (X, Y) is a vertical k-separation of M^*.*

Proof. Suppose that (X, Y) is a cyclic k-separation of M. Then (X, Y) is a k-separation of M^*. Since (X, Y) is a k-separation of a k-connected matroid, (X, Y) is exactly k-separating, and so $r(X) + r(Y) - r(M) = k - 1$. Therefore, as $r^*(X) = r(Y) + |X| - r(M)$, it follows that

$$r^*(X) = ((k-1) - r(X) + r(M)) + |X| - r(M) = (k-1) + |X| - r(X).$$

As X contains a circuit, X is dependent, so $|X| - r(M) \geq 1$. Hence $r^*(X) \geq k$. By symmetry, $r^*(Y) \geq k$, and so (X, Y) is a vertical k-separation of M^*. A similar argument establishes the converse. □

Following Lemma 2.1.8, we say a partition $(X, \{e\}, Y)$ of the ground set of a 3-connected matroid M is a *cyclic* 3*-separation* if $(X, \{e\}, Y)$ is a vertical 3-separation of M^*. Of the next two results, the first combines Lemma 2.1.8 with a straightforward strengthening of [24, Lemma 3.1] and, in combination with Lemma 2.1.8, the second follows easily from Lemma 2.1.5.

Lemma 2.1.9. *Let M be a* 3*-connected matroid, and suppose that $e \in E(M)$. Then $\mathrm{si}(M/e)$ is not* 3*-connected if and only if M has a vertical* 3*-separation $(X, \{e\}, Y)$. Dually, $\mathrm{co}(M \backslash e)$ is not* 3*-connected if and only if M has a cyclic* 3*-separation $(X, \{e\}, Y)$.*

Lemma 2.1.10. *Let M be a* 3*-connected matroid. If $(X, \{e\}, Y)$ is a vertical* 3*-separation of M, then $(X - \mathrm{cl}(Y), \{e\}, \mathrm{cl}(Y) - e)$ is also a vertical* 3*-separation of M. Dually, if $(X, \{e\}, Y)$ is a cyclic* 3*-separation of M, then $(X - \mathrm{cl}^*(Y), \{e\}, \mathrm{cl}^*(Y) - \{e\})$ is also a cyclic* 3*-separation of M.*

Note that an immediate consequence of Lemma 2.1.10 is that if $(X, \{e\}, Y)$ is a vertical 3-separation such that $Y \cup \{e\}$ is maximal, then $Y \cup \{e\}$ must be closed. We will make repeated use of this fact.

2.2 Fans

Let M be a 3-connected matroid. A subset F of $E(M)$ with at least three elements is a *fan* if there is an ordering $(f_1, f_2, \ldots, f_k)$ of F such that

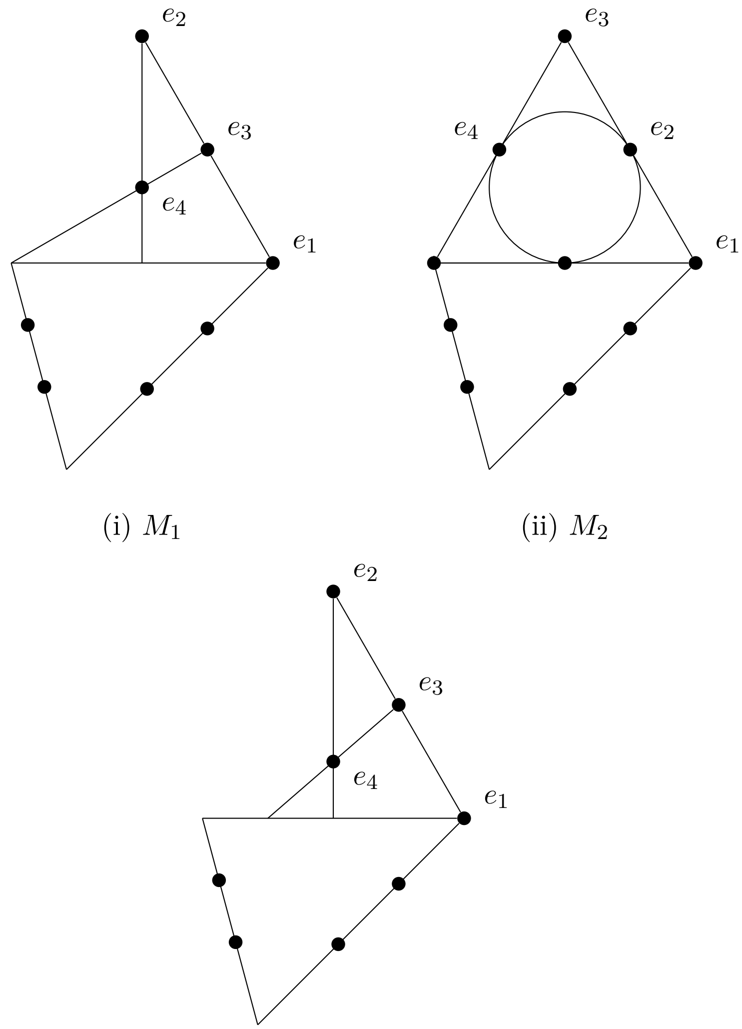
(i) for all $i \in \{1, 2, \ldots, k-2\}$, the triple $\{f_i, f_{i+1}, f_{i+2}\}$ is either a triangle or a triad, and

(ii) for all $i \in \{1, 2, \ldots, k-3\}$, if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triangle, then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triad, while if $\{f_i, f_{i+1}, f_{i+2}\}$ is a triad, then $\{f_{i+1}, f_{i+2}, f_{i+3}\}$ is a triangle.

If $k \geq 4$, then the elements f_1 and f_k are the *ends* of F. Furthermore, if $\{f_1, f_2, f_3\}$ is a triangle, then f_1 is a *spoke-end*; otherwise, f_1 is a *rim-end*. Observe that if F is a 4-element fan (f_1, f_2, f_3, f_4), then either f_1 or f_4 is the unique spoke-end of F depending on whether $\{f_1, f_2, f_3\}$ or $\{f_2, f_3, f_4\}$ is a triangle, respectively. The proof of the next lemma is straightforward and omitted.

Lemma 2.2.1. *Let M be a 3-connected matroid, and suppose that $F = (f_1, f_2, f_3, f_4)$ is a 4-element fan of M with spoke-end f_1. Then $(\{f_2, f_3, f_4\}, \{f_1\}, E(M) - F)$ is a vertical 3-separation of M provided $r(M) \geq 4$, in which case, $E(M) - \{f_2, f_3, f_4\}$ is maximal.*

We end this section by determining when an element in a fan of size at least four is elastic. Consider the rank-4 matroids M_1, M_2 and M_3 for which geometric representations are shown in Fig. 2.1. For each $i \in \{1, 2, 3\}$, the tuple $F = (e_1, e_2, e_3, e_4)$ is a 4-element fan of M_i and $(F - \{e_1\}, \{e_1\}, E(M_i) - F)$ is a vertical 3-separation of M_i. In M_1 and M_2, none of e_2, e_3, and e_4 are elastic, while in M_3, both e_2 and e_3 are elastic. The essence of the next result is that the configuration of the elements of F present in M_1 and M_2 are the only ways in which a 4-element fan does not contain elastic elements.



(i) M_1 (ii) M_2

(iii) M_3

Figure 2.1: For each $i \in \{1, 2, 3\}$, the tuple (e_1, e_2, e_3, e_4) is a 4-element fan and the partition $(\{e_2, e_3, e_4\}, \{e_1\}, E(M_i) - \{e_1, e_2, e_3, e_4\})$ of $E(M_i)$ is a vertical 3-separation of M_i. Furthermore, in M_1 and M_2, none of e_2, e_3, and e_4 are elastic, while in M_3, both e_2 and e_3 are elastic.

For subsets X and Y of a matroid, the *local connectivity* between X and Y, denoted $\sqcap(X, Y)$, is defined by

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y).$$

Let M be a 3-connected matroid and let k be a positive integer. A *flower* Φ of M is an (ordered) partition $(P_1, P_2, \ldots, P_k)$ of $E(M)$ such that each P_i has at least two elements and is 3-separating, and each $P_i \cup P_{i+1}$ is 3-separating, where all subscripts are interpreted modulo k. If $k \geq 4$, we say Φ is *swirl-like* if $\bigcup_{i \in I} P_i$ is exactly 3-separating for all proper subsets I of $\{1, 2, \ldots, k\}$ whose members form a consecutive set in the cyclic order $(1, 2, \ldots, k)$, and

$$\sqcap(P_i, P_j) = \begin{cases} 1, & \text{if } P_i \text{ and } P_j \text{ are consecutive;} \\ 0, & \text{if } P_i \text{ and } P_j \text{ are not consecutive} \end{cases}$$

for all distinct $i, j \in \{1, 2, \ldots, k\}$. For further details of swirl-like flowers and, more generally flowers, we refer the reader to [23].

Lemma 2.2.2. *Let M be a 3-connected matroid such that $r(M), r^*(M) \geq 4$, and let $F = (f_1, f_2, \ldots, f_n)$ be a maximal fan of M.*

(i) *If $n \geq 6$, then F contains no elastic elements of M.*

(ii) *If $n = 5$, then F contains either exactly one elastic element, namely f_3, or no elastic elements of M.*

(iii) *If $n = 4$, then F contains either exactly two elastic elements, namely f_2 and f_3, or no elastic elements of M.*

Moreover, if $n \in \{4, 5\}$ and F contains no elastic elements, then, up to duality, M has a swirl-like flower $(A, \{f_1, f_2\}, F - \{f_1, f_2\}, B)$ as shown geometrically in Fig. 2.2, or $n = 5$ and there is an element g such that $M|(F \cup \{g\}) \cong M(K_4)$.

Proof. It follows by Lemma 2.2.1 that the ends of a 4-element fan in M are not elastic. Thus, if $n \geq 6$, then, as every element of F is the end of a 4-element fan, F contains no elastic elements, and if $n = 5$, then, as every element of F, except f_3, is the end of a 4-element fan, F contains no elastic elements except possibly f_3. Thus (i) and (ii) hold, and we assume that $n \in \{4, 5\}$. By applying the dual argument if needed, we may also assume that $\{f_1, f_2, f_3\}$ is a triangle.

2.2.2.1. *If f_3 is contractible, then f_3 is elastic unless $n = 5$ and there is an element g such that $M|(F \cup \{g\}) \cong M(K_4)$, or $n = 4$ and f_2 is not contractible.*

Suppose that f_3 is contractible. If f_3 is not elastic, then $\mathrm{co}(M \backslash f_3)$ is not 3-connected. First assume that $n = 5$. Then, as f_2 is the end of a 4-element fan, $\mathrm{co}(M\backslash f_2)$ is not 3-connected, and so, by Bixby's Lemma, $\mathrm{si}(M/f_2)$ is 3-connected. By orthogonality, $\{f_2, f_3, f_4\}$ is the unique triad containing f_3, and so $\mathrm{co}(M\backslash f_3) \cong M/f_2\backslash f_3$. But then $\mathrm{co}(M\backslash f_3)$ is 3-connected unless there is an element g such that $\{f_2, f_4, g\}$ is a triangle of M, in which case $M|(F \cup \{g\}) \cong M(K_4)$. Now assume that $n = 4$. If f_3 is contained in a triad T^* other than $\{f_2, f_3, f_4\}$, then, by orthogonality, either f_1 or f_2 is contained in T^*. If $f_1 \in T^*$, then F is not maximal, a contradiction. Thus $f_2 \in T^*$. But then $T^* \cup \{f_4\}$ has corank 2 and so, as M is 3-connected, $(T^* \cup \{f_4\}) - \{f_2\}$ is a triad, contradicting orthogonality. Thus, as F is maximal, $\{f_2, f_3, f_4\}$ is the unique triad containing f_3. Hence $\mathrm{co}(M\backslash f_3) \cong M/f_2\backslash f_3$. Thus $\mathrm{co}(M\backslash f_3) \cong \mathrm{si}(M/f_2)$ and so, as $\mathrm{co}(M\backslash f_3)$ is not 3-connected, f_2 is not contractible. This completes the proof of (2.2.2.1).

Since (f_1, f_3, f_2, f_4) is also a fan ordering for F if $n = 4$, it follows by (2.2.2.1) that we may now assume $\mathrm{si}(M/f_3)$ is not 3-connected. We next complete the proof of the lemma for when $n = 4$. The remaining part of the lemma for when $n = 5$ is proved similarly and is omitted.

As $\mathrm{si}(M/f_3)$ is not 3-connected, it follows by Lemma 2.1.9 that

$$(A \cup \{f_1, f_2\}, \{f_3\}, B \cup \{f_4\})$$

is a vertical 3-separation of M, where $|A| \geq 1$ and $|B| \geq 2$. Say $|A| = 1$, where $A = \{f_0\}$. Then $A \cup \{f_1, f_2\}$ is a triad, and so $(f_0, f_1, f_2, f_3, f_4)$ is a 5-element fan, contradicting the maximality of F. Thus $|A| \geq 2$. Since $A \cup B$ and $B \cup \{f_4\}$ are 3-separating in M, it follows by uncrossing that B is 3-separating in M. Similarly, A is 3-separating in M. Hence

$$(A, \{f_1, f_2\}, \{f_3, f_4\}, B)$$

is a flower Φ. Since $\sqcap(\{f_1, f_2\}, \{f_3, f_4\}) = 1$, it follows by [23, Theorem 4.1] that

$$\sqcap(A, \{f_1, f_2\}) = \sqcap(\{f_3, f_4\}, B) = \sqcap(A, B) = 1.$$

To show that Φ is a swirl-like flower, it remains to show that

$$\sqcap(\{A, \{f_3, f_4\}) = \sqcap(B, \{f_1, f_2\}) = 0.$$

If $f_1 \notin \mathrm{cl}(A)$, then, as $f_2 \notin \mathrm{cl}(A \cup \{f_1\})$, it follows that $r(A \cup \{f_1, f_2\}) = r(A) + 2$. But then $\sqcap(A, \{f_1, f_2\}) = 0$, a contradiction. Thus $f_1 \in \mathrm{cl}(A)$. Furthermore,

$f_3 \notin \text{cl}(A)$. Assume that $f_4 \in \text{cl}(A \cup \{f_3\})$. Then, as $\sqcap(\{f_3, f_4\}, B) = 1$,

$$\begin{aligned} 1 &= r_{M/f_3}(A \cup \{f_1, f_2\}) + r_{M/f_3}(B \cup \{f_4\}) - r(M/f_3) \\ &= r_{M/f_3}(A \cup \{f_1, f_2, f_4\}) + r_{M/f_3}(B) - r(M/f_3) \\ &= r(A \cup F) - 1 + r(B) - (r(M) - 1) \\ &= r(A \cup F) + r(B) - r(M), \end{aligned}$$

and so B is 2-separating in M, a contradiction. Thus $f_4 \notin \text{cl}(A \cup \{f_3\})$, and so $\sqcap(A, \{f_3, f_4\}) = 0$. To see that $\sqcap(B, \{f_1, f_2\}) = 0$, first assume that $f_1 \in \text{cl}(B)$. Then, as $f_1 \in \text{cl}(A)$,

$$\begin{aligned} 1 &= r_{M/f_3}(A \cup \{f_1, f_2\}) + r_{M/f_3}(B \cup \{f_4\}) - r(M/f_3) \\ &= r_{M/f_3}(A) + r_{M/f_3}(B \cup \{f_1, f_2, f_4\}) - r(M/f_3) \\ &= r(A) + r(B \cup F) - 1 - (r(M) - 1) \\ &= r(A) + r(B \cup F) - r(M), \end{aligned}$$

and so A is 2-separating in M. This contradiction implies that $f_1 \notin \text{cl}(B)$. It follows that $r(B \cup \{f_1, f_2\}) = r(B) + 2$, that is $\sqcap(B, \{f_1, f_2\}) = 0$. We deduce that $(A, \{f_1, f_2\}, \{f_3, f_4\}, B)$ is a swirl-like flower. Lastly, as $f_1 \in \text{cl}(A)$ and $\sqcap(B, \{f_3, f_4\}) = 1$, it follows that $(A \cup \{f_1\}, \{f_2\}, B \cup \{f_3, f_4\})$ is a cyclic 3-separation of M, and so $\text{co}(M \backslash f_2)$ is not 3-connected, that is, f_2 is not elastic. Hence (iii) holds. □

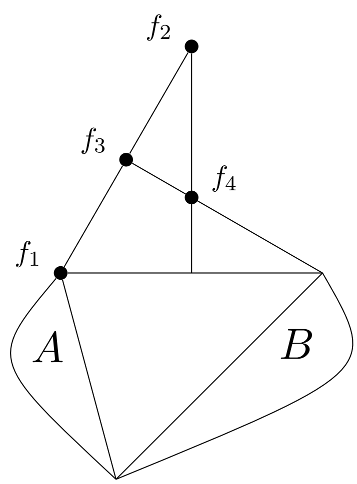


Figure 2.2: The swirl-like flower $(A, \{f_1, f_2\}, F - \{f_1, f_2\}, B)$ of Lemma 2.2.2 where, if $|F| = 5$, then f_5 is an element in B.

2.3 Elastic elements in segments

Let M be a matroid. A subset L of $E(M)$ of size at least two is a *segment* if $M|L$ is isomorphic to a rank-2 uniform matroid. In this section we consider when an element in a segment is deletable or contractible. We begin with the following elementary lemma.

Lemma 2.3.1. *Let L be a segment of a 3-connected matroid M. If L has at least four elements, then $M\backslash \ell$ is 3-connected for all $\ell \in L$.*

In particular, Lemma 2.3.1 implies that, in a 3-connected matroid, every element of a segment with at least four elements is deletable. We next determine the structure which arises when elements of a segment in a 3-connected matroid are not contractible.

Lemma 2.3.2. *Let M be a 3-connected matroid, and suppose that $L \cup \{w\}$ is a rank-3 cocircuit of M, where L is a segment. If two distinct elements y_1 and y_2 of L are not contractible, then there are distinct elements w_1 and w_2 of $E(M)-(L\cup\{w\})$ such that $(\mathrm{cl}(L)-\{y_i\})\cup\{w_i\}$ is a cocircuit for each $i \in \{1,2\}$.*

Proof. Let y_1 and y_2 be distinct elements of L that are not contractible. For each $i \in \{1,2\}$, it follows by Lemma 2.1.9 that there exists a vertical 3-separation $(X_i, \{y_i\}, Y_i)$ of M such that $y_j \in Y_i$, where $\{i,j\} = \{1,2\}$. By Lemma 2.1.10, we may assume $Y_i \cup \{y_i\}$ is closed, in which case, $L - \{y_i\} \subseteq Y_i$. Furthermore, for each $i \in \{1,2\}$, we may also assume, amongst all such vertical 3-separations of M, that $|Y_i|$ is minimised. If $w \in Y_i$, then, as $L \cup \{w\}$ is a cocircuit, X_i is contained in the hyperplane $E(M) - (L \cup \{w\})$, and so $y_i \notin \mathrm{cl}(X_i)$. This contradiction implies that $w \in X_i$. Thus, for each $i \in \{1,2\}$, we deduce that M has a vertical 3-separation

$$(U_i \cup \{w\}, \{y_i\}, V_i \cup (L - \{y_i\})),$$

where $U_i \cup \{w\} = X_i$ and $V_i \cup (L - \{y_i\}) = Y_i$. Next we show the following.

2.3.2.1. *For each $i \in \{1,2\}$, we have $w \in \mathrm{cl}_M(U_i \cup \{y_i\}) - \mathrm{cl}_M(U_i)$.*

Since $L \cup \{w\}$ is a cocircuit, the elements $y_i, w \notin \mathrm{cl}_M(U_i)$. But $y_i \in \mathrm{cl}_M(U_i \cup \{w\})$, and so $y_i \in \mathrm{cl}_M(U_i \cup \{w\}) - \mathrm{cl}_M(U_i)$. Thus, by the MacLane-Steinitz exchange property, $w \in \mathrm{cl}_M(U_i \cup \{y_i\}) - \mathrm{cl}_M(U_i)$.

2.3.2.2. *For each $i \in \{1,2\}$, we have $y_i \notin \mathrm{cl}_M(U_j \cup \{w\})$, where $\{i,j\} = \{1,2\}$.*

By Lemma 2.1.10,

$$(\mathrm{cl}(U_j \cup \{w\}) - \{y_j\}, \{y_j\}, (V_j \cup (L - \{y_j\})) - \mathrm{cl}(U_j \cup \{w\}))$$

is a vertical 3-separation of M. If $y_i \in \mathrm{cl}(U_j \cup \{w\})$, then, as $y_j \in \mathrm{cl}(U_j \cup \{w\})$, the segment L is contained in $\mathrm{cl}(U_j \cup \{w\})$. Therefore $L \cup \{w\} \subseteq \mathrm{cl}(U_j \cup \{w\})$, and so $(V_j \cup (L - \{y_j\})) - \mathrm{cl}(U_j \cup \{w\}) = V_j - \mathrm{cl}(U_j \cup \{w\})$. Since $V_j - \mathrm{cl}(U_j \cup \{w\})$ is contained in the hyperplane $E(M) - (L \cup \{w\})$, it follows that $y_j \notin V_j - \mathrm{cl}(U_j \cup \{w\})$, a contradiction. Thus (2.3.2.2) holds.

Since M is 3-connected and $(U_i \cup \{w\}, \{y_i\}, V_i \cup (L - \{y_i\}))$ is a vertical 3-separation, it follows by (2.3.2.1) that

$$r(U_i) + r(V_i \cup L) - r(M\backslash w) = r(U_i \cup \{w\}) - 1 + r(V_i \cup L) - r(M) = 1.$$

Thus $(U_i, V_i \cup L)$ is a 2-separation of $M\backslash w$ for each $i \in \{1,2\}$. We next show that

2.3.2.3. $|U_1 \cap V_2| = |U_2 \cap V_1| = 1$.

Let $\{i,j\} = \{1,2\}$. If $U_i \subseteq U_j$, then

$$y_i \in \mathrm{cl}(U_i \cup \{w\}) \subseteq \mathrm{cl}(U_j \cup \{w\}),$$

contradicting (2.3.2.2). Therefore, for $\{i,j\} = \{1,2\}$, we have $|U_i \cap V_j| \geq 1$. Consider the 2-connected matroid $M\backslash w$. Since $|U_j \cap V_i| \geq 1$, it follows by uncrossing that $U_i \cup (V_j \cup L)$ is 2-separating in $M\backslash w$. But, by (2.3.2.1), $w \in \mathrm{cl}_M(U_i \cup L)$ and so $U_i \cup V_j \cup (L \cup \{w\})$ is 2-separating in M. Since M is 3-connected, it follows that $|U_j \cap V_i| \leq 1$. Thus (2.3.2.3) holds.

Let w_1 and w_2 be the unique elements of $U_2 \cap V_1$ and $U_1 \cap V_2$, respectively. Now $|(U_1 \cup \{w\}) \cap (U_2 \cup \{w\})| \geq 2$ and so, by uncrossing, $V_1 \cup L$ and $V_2 \cup L$, as well as $V_1 \cup L$ and $V_2 \cup (L - \{y_1\})$, we see that $(V_1 \cap V_2) \cup L$ and $(V_1 \cap V_2) \cup (L - \{y_1\})$ are 3-separating in M. So

$$(U_1 \cup U_2 \cup \{w\}, \{y_1\}, (V_1 \cap V_2) \cup (L - \{y_1\}))$$

is a vertical 3-separation of M unless $r((V_1 \cap V_2) \cup (L - \{y_1\}) = 2$. Since $V_1 \cup L$ and $V_2 \cup L$ are closed, $(V_1 \cap V_2) \cup L$ is closed. Furthermore,

$$|(V_1 \cap V_2) \cup (L - \{y_1\})| < |V_1 \cup (L - \{y_1\})|,$$

and so, by the minimality of $|Y_1|$, we have $r((V_1 \cap V_2) \cup (L - \{y_1\}) = 2$. Therefore, as $(U_1 \cup \{w\}, \{y_1\}, V_1 \cup (L - \{y_1\}))$ and $(U_2 \cup \{w\}, \{y_2\}, V_2 \cup (L - \{y_2\}))$ are both vertical 3-separations, and

$$(V_1 \cap V_2) \cup (L - \{y_i\}) \cup \{w_i\} = V_i \cup (L - \{y_i\}),$$

it follows that $(V_1 \cap V_2) \cup (L - \{y_i\}) \cup \{w_i\}$ is a cocircuit for each $i \in \{1, 2\}$. Since $y_1 \in \mathrm{cl}((V_1 \cap V_2) \cup (L - \{y_1\}))$, we have $(V_1 \cap V_2) \cup L = \mathrm{cl}(L)$, thereby completing the proof of the lemma. □

2.4 Theta separators

We begin this section by formally defining, for all $n \geq 2$, the matroid Θ_n. Let $n \geq 2$, and let M be the matroid whose ground set is the disjoint union of $W = \{w_1, w_2, \ldots, w_n\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$, and whose circuits are as follows:

(i) all 3-element subsets of W;

(ii) all sets of the form $(Z - \{z_i\}) \cup \{w_i\}$, where $i \in \{1, 2, \ldots, n\}$; and

(iii) all sets of the form $(Z - \{z_i\}) \cup \{w_j, w_k\}$, where i, j, and k are distinct elements of $\{1, 2, \ldots, n\}$.

It is shown in [22, Lemma 2.2] that M is indeed a matroid, and we denote this matroid by Θ_n. If $n = 2$, then Θ_2 is isomorphic to the direct sum of $U_{1,2}$ and $U_{1,2}$, while if $n = 3$, then Θ_3 is isomorphic to $M(K_4)$. Also, for all n, the matroid Θ_n is self-dual under the map that interchanges w_i and z_i for all i [22, Lemma 2.1], and the rank of Θ_n is n. For all i, we say w_i and z_i are *partners*. Furthermore, it is easily checked that, for all $i, j \in \{1, 2, \ldots, n\}$, we have $\Theta_n \backslash w_i \cong \Theta_n \backslash w_j$. Up to isomorphism, we denote the matroid $\Theta_n \backslash w_i$ by Θ_n^-. Observe that if $n = 3$, then Θ_3^- is a 5-element fan. We refer to the elements in W and Z as the *segment elements* and *cosegment elements*, respectively, of Θ_n and Θ_n^-.

Recalling the definition of a Θ-separator, the next lemma considers the elasticity of elements in a Θ-separator when $n \geq 4$. The analogous lemma for when $n = 3$ is covered by Lemma 2.2.2. Observe that, if M is 3-connected and S is a Θ-separator of M such that $M|S \cong \Theta_n$ for some $n \geq 3$, then

$$r(M) = r(M \backslash S) + n - 2.$$

Lemma 2.4.1. *Let M be a 3-connected matroid, and let $n \geq 4$. Suppose that S is a Θ-separator of M. If $M|S \cong \Theta_n$, then S contains no elastic elements of M. Furthermore, if $M|S \cong \Theta_n^-$, then S contains exactly one elastic element, namely the unique cosegment element of $M|S$ with no partner, unless there is an element w of $\mathrm{cl}(S) - S$ such that $M|(S \cup \{w\}) \cong \Theta_n$.*

Proof. Suppose that $M|S \cong \Theta_n$, where $n \geq 4$. Without loss of generality, we may assume that S is the disjoint union of $W = \{w_1, w_2, \dots, w_n\}$ and $Z = \{z_1, z_2, \dots, z_n\}$, where W and Z are as defined in the definition of Θ_n. Let $i \in \{1, 2, \dots, n\}$. As $M|S \cong \Theta_n$, the set $C_i = (Z - \{z_i\}) \cup \{w_i\}$ is a circuit of M. Now, as Z has corank 2, the circuit C_i has corank 3, and so

$$\lambda(C_i) = r(C_i) + r^*(C_i) - |C_i| = (|C_i| - 1) + 3 - |C_i| = 2.$$

So C_i is 3-separating. Furthermore, $z_i \in \mathrm{cl}^*(C_i)$ and, by Lemma 2.1.3, $z_i \notin \mathrm{cl}(E(M) - (C_i \cup \{z_i\})$. Thus, by Lemma 2.1.5, $z_i \in \mathrm{cl}^*(E(M) - (C_i \cup \{z_i\})$ and so, as $E(M) - (C_i \cup \{z_i\})$ contains a triangle in $W - \{w_i\}$,

$$(C_i, \{z_i\}, E(M) - (C_i \cup \{z_i\}))$$

is a cyclic 3-separation of M. Therefore, by Lemma 2.1.9, z_i is not deletable. Moreover, as

$$(Z - \{z_i\}, \{w_i\}, E(M) - ((Z - \{z_i\}) \cup \{w_i\}))$$

is a vertical 3-separation of M, it follows by Lemma 2.1.9 that w_i is not contractible. Thus S contains no elastic elements of M.

Now suppose that $M|S \cong \Theta_n^-$, where $n \geq 4$. Without loss of generality, let S be the disjoint union of $W - \{w_j\}$ and Z, where $W = \{w_1, w_2, \dots, w_n\}$ and $Z = \{z_1, z_2, \dots, z_n\}$ are as defined in the definition of Θ_n. Let $z_i \in Z - \{z_j\}$. Then the argument in the last paragraph shows that

$$((Z - \{z_i\}) \cup \{w_i\}, \{z_i\}, E(M) - (Z \cup \{w_i\})$$

is a cyclic 3-separation of M provided $E(M) - (Z \cup \{w_i\})$ contains a circuit. If $n \geq 5$, then $|W| \geq 4$, and so $E(M) - (Z \cup \{w_i\})$ contains a circuit. Assume that

$n = 4$. Then, as $r^*(M) \geq 4$, we have $|E(M) - (Z \cup \{w_i\})| \geq 3$. Therefore, as $w_k \in \mathrm{cl}(Z \cup \{w_i\})$, where $w_k \in W - \{w_i, w_j\}$, and $Z \cup \{w_i\}$ is exactly 3-separating, it follows by Lemma 2.1.5 that $w_k \in \mathrm{cl}(E(M) - (Z \cup \{w_i, w_k\})$. In particular, $E(M) - (Z \cup \{w_i\})$ contains a circuit. Hence z_i is not deletable. Furthermore, the argument in the previous paragraph shows that if $w_i \in W - \{w_j\}$, then w_i is not contractible.

We complete the proof of the lemma by considering the elasticity of z_j. Since $|Z| \geq 4$, it follows by Lemma 2.3.1 that z_j is contractible. Assume that z_j is not deletable. Let $i \in \{1, 2, \ldots, n\}$ such that $i \neq j$. Then $C_i = (Z - \{z_i\}) \cup \{w_i\}$ is a circuit of M. Furthermore,

$$\begin{aligned} r^*((Z - \{z_i\}) \cup \{w_i\}) &= (r(M) - (|C_i| - 3)) + |C_i| - r(M) \\ &= 3. \end{aligned}$$

Therefore, as $z_j \in Z - \{z_i\}$ and all elements of $Z - \{z_i\}$ are not deletable, the dual of Lemma 2.3.2 implies that there is an element w such that $(Z - \{z_j\}) \cup \{w\}$ is a circuit. But then, as $w \in \mathrm{cl}(Z) - Z$, it follows that $w \in \mathrm{cl}(W - \{w_j\})$, and it is easily checked that $M|(S \cup \{w\}) \cong \Theta_n$, thereby completing the proof of the lemma. □

2.5 The existence of elastic elements

In this section, we prove Theorem 1.1.1 and Theorem 1.1.2. However, almost all of the section consists of the proof of Theorem 1.1.1. The proof of this theorem is essentially partitioned into two lemmas: Lemma 2.5.2 and Lemma 2.5.3. Let M be a 3-connected matroid with a vertical 3-separation $(X, \{e\}, Y)$ such that $Y \cup \{e\}$ is maximal. Lemma 2.5.2 establishes Theorem 1.1.1 for when X contains at least one non-contractible element, while Lemma 2.5.3 establishes the theorem for when every element in X is contractible.

To prove Lemma 2.5.2, we will make use of the following technical result which is extracted from the proof of Lemma 3.2 in [24].

Lemma 2.5.1. *Let M be a 3-connected matroid with a vertical 3-separation $(X_1, \{e_1\}, Y_1)$ such that $Y_1 \cup \{e_1\}$ is maximal. Suppose that $(X_2, \{e_2\}, Y_2)$ is a vertical 3-separation of M such that $e_2 \in X_1$, $e_1 \in Y_2$, and $Y_2 \cup \{e_2\}$ is closed. Then each of the following holds:*

(i) *None of* $X_1 \cap X_2$*,* $X_1 \cap Y_2$*,* $Y_1 \cap X_2$*, and* $Y_1 \cap Y_2$ *are empty.*

(ii) $r((X_1 \cap X_2) \cup \{e_2\}) = 2$*.*

(iii) *If* $|Y_1 \cap X_2| = 1$*, then* X_2 *is a rank-*3 *cocircuit.*

(iv) *If* $|Y_1 \cap X_2| \geq 2$*, then* $r((X_1 \cap Y_2) \cup \{e_1, e_2\}) = 2$*.*

Lemma 2.5.2. *Let* M *be a* 3*-connected matroid with a vertical* 3*-separation* $(X_1, \{e_1\}, Y_1)$ *such that* $Y_1 \cup \{e_1\}$ *is maximal. Suppose that at least one element of* X_1 *is not contractible. Then at least one of the following holds:*

(i) X_1 *has at least two elastic elements;*

(ii) $X_1 \cup \{e_1\}$ *is a* 4*-element fan; or*

(iii) X_1 *is contained in a* Θ*-separator* S*.*

Moreover, if (iii) holds, then X_1 *is a rank-*3 *cocircuit,* $M^*|S$ *is isomorphic to either* Θ_n *or* Θ_n^-*, where* $n = |X_1 \cup \{e_1\}| - 1$*, and there is a unique element* $x \in X_1$ *such that* x *is a segment element of* $M^*|S$ *and* $(X_1 - \{x\}) \cup \{e_1\}$ *is the set of cosegment elements of* $M^*|S$*.*

Proof. Let e_2 be an element of X_1 that is not contractible. Then, by Lemma 2.1.9, there exists a vertical 3-separation $(X_2, \{e_2\}, Y_2)$ of M. Without loss of generality, we may assume $e_1 \in Y_2$. Furthermore, by Lemma 2.1.10, we may also assume that $Y_2 \cup \{e_2\}$ is closed. By Lemma 2.5.1, each of $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, and $Y_1 \cap Y_2$ is non-empty. The proof is partitioned into two cases depending on the size of $Y_1 \cap X_2$. Both cases use the following:

2.5.2.1. *If* $X_1 \cap X_2$ *contains two contractible elements, then either* X_1 *has at least two elastic elements, or* $|X_1 \cap X_2| = 2$ *and there exists a triangle* $\{x, y_1, y_2\}$*, where* $x \in X_1 \cap X_2$*,* $y_1 \in Y_1 \cap X_2$*, and* $y_2 \in X_1 \cap Y_2$*.*

By Lemma 2.5.1(ii), $r((X_1 \cap X_2) \cup \{e_2\}) = 2$. Let x_1 and x_2 be distinct contractible elements of $X_1 \cap X_2$. If $|X_1 \cap X_2| \geq 3$, then, by Lemma 2.3.1 each of x_1 and x_2 is elastic. Thus we may assume that $|X_1 \cap X_2| = 2$ and that either x_1 or x_2, say x_1, is not deletable. Let (U, V) be a 2-separation of $M\backslash x_1$ such that neither $r^*(U) = 1$ nor $r^*(V) = 1$. Since x_1 is not deletable, such a separation exists. Furthermore, $|U|, |V| \geq 3$ as U and V each contain a cycle. If $x_1 \in \mathrm{cl}(U)$ or $x_1 \in \mathrm{cl}(V)$, then either $(U \cup \{x_1\}, V)$ or $(U, V \cup \{x_1\})$, respectively, is a 2-separation of M, a contradiction. So $\{x_2, e_2\} \nsubseteq U$ and $\{x_2, e_2\} \nsubseteq V$. Therefore, without loss of generality, we may assume $x_2 \in U - \mathrm{cl}(V)$ and $e_2 \in V - \mathrm{cl}(U)$. Since (U, V) is a 2-separation of $M\backslash x_1$ and $x_2 \notin \mathrm{cl}(V)$, we deduce that $(U - \{x_2\}, V \cup \{x_1\})$ is a 2-separation of M/x_2. Thus, as x_2 is contractible, $\mathrm{si}(M/x_2)$ is 3-connected, and so $r(U) = 2$. In turn, as $Y_1 \cup \{e_1\}$ and $Y_2 \cup \{e_2\}$ are both closed, this implies that $|U \cap (Y_1 \cup \{e_1\})| \leq 1$ and $|U \cap (Y_2 \cup \{e_2\})| \leq 1$; otherwise, $U \subseteq Y_1 \cup \{e_1\}$ or $U \subseteq Y_2 \cup \{e_2\}$. Thus $|U| = 3$ and, in particular, U is the desired triangle. Hence (2.5.2.1) holds.

We now distinguish two cases depending on the size of $Y_1 \cap X_2$:

(I) $|Y_1 \cap X_2| = 1$; and

(II) $|Y_1 \cap X_2| \geq 2$.

Consider (I). Let w be the unique element in $Y_1 \cap X_2$. By Lemma 2.5.1, $(X_1 \cap X_2) \cup \{e_2\}$ is a segment of at least three elements and $(X_1 \cap X_2) \cup \{w\}$ is a rank-3 cocircuit. Let $L_1 = (X_1 \cap X_2) \cup \{e_2\}$. As $|Y_1 \cap X_2| = 1$, we may assume that L_1 is closed.

2.5.2.2. *At most one element of $X_1 \cap X_2$ is not contractible.*

Suppose that at least two elements in $X_1 \cap X_2$ are not contractible, and let x be such an element. Then, by Lemma 2.3.2, there is an element w' distinct from w such that $(L_1 - \{x\}) \cup \{w'\}$ is a rank-3 cocircuit. If $w' \in Y_1$, then $\{w, w'\} \subseteq \mathrm{cl}^*(X_1)$ and $e_1 \in \mathrm{cl}(X_1)$, contradicting Lemma 2.1.7. Thus $w' \in X_1$. Since $w' \in \mathrm{cl}^*(L_1 - \{x\})$, it follows by Lemma 2.1.4 that each of $(L_1 - \{x\}) \cup \{w'\}$ and $L_1 \cup \{w'\}$ are exactly 3-separating. Furthermore, as $x \in \mathrm{cl}((L_1 - \{x\}) \cup \{w'\})$, it follows by Lemma 2.1.5 that $x \notin \mathrm{cl}^*((L_1 - \{x\}) \cup \{w'\})$. Therefore

$$((L_1 - \{x\}) \cup \{w'\}, \{x\}, E(M) - (L_1 \cup \{w'\}))$$

is a vertical 3-separation of M. But then, as $L_1 \cup \{w'\} \subseteq X_1$, we contradict the maximality of $Y_1 \cup \{e_1\}$. Hence (2.5.2.2) holds.

If $|L_1| \geq 4$, then, by Lemma 2.3.1 and (2.5.2.2), $L_1 - \{e_2\}$, and more particularly X_1, contains at least two elastic elements. Thus, as $|Y_1 \cap X_2| = 1$, we may assume $|L_1| = 3$, and so $(L_1 - \{e_2\}) \cup \{w\}$ is a triad. Let $L_1 = \{x_1, x_2, e_2\}$ and let $\{i, j\} = \{1, 2\}$.

2.5.2.3. *For each $i \in \{1, 2\}$, the element x_i is contractible.*

If x_i is not contractible, then, by Lemma 2.1.9, M has a vertical 3-separation $(U_i, \{x_i\}, V_i)$, where $e_1 \in V_i$. By Lemma 2.1.10, we may assume that $V_i \cup x_i$ is closed. By Lemma 2.5.1, $Y_1 \cap U_i$ is non-empty and $r((X_1 \cap U_i) \cup \{x_i\}) = 2$. First assume that $|Y_1 \cap U_i| = 1$. Then $|(X_1 \cap U_i) \cup \{x_i\}| \geq 3$, and so x_i is contained in a triangle $T \subseteq (X_1 \cap U_i) \cup \{x_i\}$. If $x_j \in V_i$, then, as $V_i \cup \{x_i\}$ is closed, $e_2 \in V_i$. Thus $x_j, e_2 \notin T$ and so, by orthogonality, as $\{x_i, x_j, w\}$ is a triad, $w \in T$. This contradicts $w \in Y_1$. It now follows that $x_j \in X_1 \cap U_i$ and so $e_2 \in X_1 \cap U_i$. Thus, as L_1 is closed and $L_1 \subseteq (X_1 \cap U_i) \cup \{x_i\}$, we have $|(X_1 \cap U_i) \cup \{x_i\}| = 3$, and therefore $T = \{x_1, x_2, e_2\}$. Let z be the unique element in $Y_1 \cap U_i$. Then, by Lemma 2.5.1 again, $\{x_j, e_2, z\}$ is a triad, and so $z \in \text{cl}^*(X_1)$. Furthermore, $w \in \text{cl}^*(X_1)$ and $e_1 \in \text{cl}(X_1)$, and so, by Lemma 2.1.7, we deduce that $z = w$. This implies that $Y_2 = V_i$. But then $\text{cl}(Y_2 \cup \{e_2\})$ contains x_i, contradicting that $Y_2 \cup \{e_2\}$ is closed. Now assume that $|Y_1 \cap U_i| \geq 2$. By Lemma 2.5.1, $r((X_1 \cap V_i) \cup \{x_i, e_1\}) = 2$. If $x_j \in V_i$, then, as $V_i \cup \{x_i\}$ is closed, $e_2 \in X_1 \cap V_i$, and so $\{x_j, e_1, e_2\}$ is a triangle. Since $\{x_1, x_2, w\}$ is a triad, this contradicts orthogonality. Thus $x_j \in U_i$. Also, $e_2 \in U_i$; otherwise, as $V_i \cup \{x_i\}$ is closed, $x_j \in V_i$, a contradiction. By Lemma 2.5.1, $X_1 \cap V_i$ is non-empty, and so M has a triangle $T' = \{x_i, e_1, y\}$, where $y \in X_1 \cap V_i$. As $\{x_i, x_j, w\}$ is a triad, T' contradicts orthogonality unless $y = w$. But $w \in Y_1$ and therefore cannot be in $X_1 \cap V_i$. Hence x_i is contractible, and so (2.5.2.3) holds.

Since x_1 and x_2 are both contractible, it follows by (2.5.2.1) that either X_1 contains two elastic elements or w is in a triangle with two elements of X_1. If the latter holds, then $w \in \text{cl}(X_1)$. As $\{x_1, x_2, w\}$ is a triad and $(Y_1 \cup \{e_1\}) - \{w\}$ is contained in $Y_2 \cup \{e\}_2$, it follows that $w \notin \text{cl}((Y_1 \cup \{e_1\}) - \{w\})$. Therefore

$$(X_1 \cup \{w\}, (Y_1 \cup \{e_1\}) - \{w\})$$

is a 2-separation of M, a contradiction. Thus X_1 contains two elastic elements. This concludes (I).

Now consider (II). Let $L_1 = (X_1 \cap X_2) \cup \{e_2\}$ and $L_2 = (X_1 \cap Y_2) \cup \{e_1, e_2\}$. By parts (ii) and (iv) of Lemma 2.5.1, L_1 and L_2 are both segments. Since M is 3-connected, X_1 is 3-separating, and $Y_1 \cup \{e_1\}$ is closed, it follows that X_1 is a rank-3 cocircuit of M and L_2 is closed.

First assume that $|L_2| \geq 4$. Since X_1 is a rank-3 cocircuit of M, we have $r(Y_1) + 1 = r(M)$. Therefore, as $|L_2| \geq 4$ and $|X_1 \cap X_2| \geq 1$, it follows that $r^*(M) \geq 4$. Now, Lemma 2.3.1 implies that each element of L_2 is deletable. If

$|L_1| \geq 3$, then, by Lemma 2.1.6, each element of $L_2 - \{e_1, e_2\}$ is contractible, and so each element of $L_2 - \{e_1, e_2\}$ is elastic. Since $|L_2| \geq 4$, it follows that X_1 has at least two elastic elements. Thus we may assume that $|L_1| = 2$, that is $|X_1 \cap X_2| = 1$. We may also assume that $X_1 \cap Y_2$ contains at most one contractible element; otherwise, X_1 contains at least two elastic elements. Let $e_3, e_4, \dots, e_n$ denote the elements in $L_1 - \{e_1, e_2\}$. Without loss of generality, we may assume that if $X_1 \cap Y_2$ contains a contractible element, then it is e_n. Let $m = n - 1$ if e_n is contractible; otherwise, let $m = n$. Furthermore, let w_1 denote the unique element in $X_1 \cap X_2$. Since $(L_2 - \{e_1\}) \cup \{w_1\}$ is a rank-3 cocircuit, and at most one element of $L_2 - \{e_1\}$ is contractible, it follows by Lemma 2.3.2 that, for all $i \in \{2, 3, \dots, m\}$, there are distinct elements $w_2, w_3, \dots, w_m$ of Y_1 such that $(L_2 - \{e_i\}) \cup \{w_i\}$ is a cocircuit. Let $W = \{w_1, w_2, \dots, w_m\}$. As W is in the coclosure of the 3-separating set L_2, we have $r^*(W) = 2$. It follows that $(L_2 - \{e_i\}) \cup \{w_j, w_k\}$ is a cocircuit of M for all distinct elements $i, j, k \in \{1, 2, \dots, m\}$. By a comparison of the circuits of Θ_n, it is straightforward to deduce that $M^*|(W \cup L_2)$ is isomorphic to either Θ_n if no element of $X_1 \cap Y_2$ is contractible, or Θ_n^- if e_n is contractible. Hence X_1 is contained in a Θ-separator of M as described in the statement of the lemma.

We may now assume that $|L_2| = 3$. Let $L_2 = \{e_2, a, e_1\}$. If $|X_1 \cap X_2| = 1$, then $|X_1| = 3$, and so X_1 is a triad. In turn, this implies that $X_1 \cup \{e_1\}$ is a 4-element fan. Thus $|X_1 \cap X_2| \geq 2$. Let x_1 and x_2 be distinct elements in $X_1 \cap X_2$. Since $\{e_1, a, e_2\}$ is a triangle in M/x_i for each $i \in \{1, 2\}$, it follows by Lemma 2.1.6 that x_i is contractible for each $i \in \{1, 2\}$. Thus, by (2.5.2.1), either X_1 contains two elastic elements, or $X_1 \cap X_2 = \{x_1, x_2\}$ and a is in a triangle with two elements of X_2. The latter implies that $a \in \mathrm{cl}(X_2 \cup \{e_2\})$. As $a \notin \mathrm{cl}(Y_1 \cup \{e_1\})$ and $Y_2 - \{a\}$ is contained in $Y_1 \cup \{e_1\}$, it follows that $a \notin \mathrm{cl}(Y_2 - \{a\})$. Hence, as

$$r(X_2 \cup \{e_2\}) + r(Y_2) - r(M) = 2,$$

we have $r(X_2 \cup \{e_2, a\}) + r(Y_2 - \{a\}) + 1 - r(M) = 2$, and so

$$(X_2 \cup \{a, e_2\}, Y_2 - \{a\})$$

is a 2-separation of M, a contradiction. Thus X_1 contains two elastic elements. This concludes (II) and the proof of the lemma. □

Lemma 2.5.3. *Let M be a 3-connected matroid with a vertical 3-separation $(X_1, \{e_1\}, Y_1)$ such that $Y_1 \cup \{e_1\}$ is maximal. Suppose that every element of X_1 is contractible. Then at least one of the following holds:*

(i) *X_1 has at least two elastic elements;*

(ii) *$X_1 \cup \{e_1\}$ is a 4-element fan; or*

(iii) *X_1 is contained in a Θ-separator S.*

Moreover, if (iii) holds, then $X_1 \cup \{e_1\}$ is a circuit, $M|S$ is isomorphic to either Θ_n or Θ_n^- for some $n \in \{|X_1|, |X_1| + 1\}$, and X_1 is a subset of the cosegment elements of $M|S$.

Proof. First suppose that X_1 is independent. Then, as $r(X_1) = |X_1|$ and $\lambda(X_1) = r(X_1) + r^*(X_1) - |X_1|$, we have $r^*(X_1) = 2$. That is, X_1 is a segment in M^*. As $r^*(X_1) = 2$, it follows that either $(X_1 - \{x\}) \cup \{e_1\}$ is a circuit for some $x \in X_1$, or $X_1 \cup \{e_1\}$ is a circuit. If $(X_1 - \{x\}) \cup \{e_1\}$ is a circuit, then either $X_1 \cup \{e_1\}$ is a 4-element fan, or it is easily checked that $(X_1 - \{x\}, \{e_1\}, Y_1 \cup \{x\})$ is a vertical 3-separation, contradicting the maximality of $Y_1 \cup \{e_1\}$. Thus we may assume that $X_1 \cup \{e_1\}$ is a circuit of M. Now, if two elements of X_1 are deletable, then X_1 contains at least two elastic elements, so we may assume that at most one element of X_1 is deletable. Assume first that X_1 is coclosed, and let $X_1 = \{z_1, z_2, \ldots, z_n\}$. Without loss of generality, we may assume that if X_1 contains a deletable element, then it is z_n. Let $m = n - 1$ if z_n is deletable; otherwise, let $m = n$. Since $X_1 \cup \{e_1\}$ has corank 3 and X_1 is coclosed, it follows by the dual of Lemma 2.3.2 that, for all $i \in \{1, 2, \ldots, m\}$, there are distinct elements $w_1, w_2, \ldots, w_m$ such that $(X_1 - \{z_i\}) \cup \{w_i\}$ is a circuit. Let $W = \{w_1, w_2, \ldots, w_m\}$. Since X_1 is 3-separating and $W \subseteq \mathrm{cl}(X_1)$, it follows that $r(W) = 2$. As every 3-element subset of X_1 is a cocircuit, it follows by orthogonality that $(X_1 - \{z_i\}) \cup \{w_j, w_k\}$ is a circuit for all distinct $i, j, k \in \{1, 2, \ldots, m\}$. By a comparison with the circuits of Θ_n, it is easily checked that $M|(W \cup X_1)$ is isomorphic to Θ_n if $m = n$, and $M|(W \cup X_1)$ is isomorphic to Θ_n^- if $m = n - 1$, and so X_1 is contained in a Θ-separator of M as described in the statement of the lemma. Now assume that X_1 is not coclosed. Then, as $X_1 \cup \{e_1\}$ is a corank-3 circuit, $|\mathrm{cl}^*(X_1) - X_1| = 1$. Let $\{z_1\} = \mathrm{cl}^*(X_1) - X_1$, and denote the elements of X_1 as $z_2, z_3, \ldots, z_n$. Applying the previous argument to $X_1 \cup \{z_1\}$ and recalling that $X_1 \cup \{e_1\}$ is a circuit, we deduce that X_1 is again contained in a Θ-separator of M as described in the statement of the lemma.

Now suppose that X_1 is dependent, and let C be a circuit in X_1. As M is 3-connected, $|C| \geq 3$. If every element in C is deletable, then X_1 contains at least two elastic elements. Thus we may assume that there is an element, say g, in C that is not deletable. By Lemma 2.1.9, there exists a cyclic 3-separation $(U, \{g\}, V)$ in M, where $e_1 \in V$. By Lemma 2.1.10, we may also assume that $V \cup \{g\}$ is coclosed. Note that, as $(U, \{g\}, V)$ is a cyclic 3-separation, $r^*(U) \geq 3$, and so $|U| \geq 3$.

We next show that

2.5.3.1. $|X_1 \cap U|, |X_1 \cap V| \geq 2$.

If either $C - \{g\} \subseteq U$ or $C - \{g\} \subseteq V$, then $g \in \mathrm{cl}(U)$ or $g \in \mathrm{cl}(V)$, respectively, in which case either $(U \cup \{g\}, V)$ or $(U, V \cup \{g\})$ is a 2-separation of M, a contradiction. Thus $C \cap (X_1 \cap U)$ and $C \cap (X_1 \cap V)$ are both non-empty, and so $|X_1 \cap U|, |X_1 \cap V| \geq 1$. Say $X_1 \cap U = \{g'\}$, where $g' \in C$. Since C is a circuit, $g \in \mathrm{cl}_{M/g'}(V)$. Therefore, as $Y_1 \cup \{e_1\}$ is closed and so $g' \notin \mathrm{cl}(Y_1)$, and (U, V) is a 2-separation of $M \backslash g$, we have

$$\begin{aligned}
\lambda_{M/g'}(U \cap Y_1) &= r_{M/g'}(U \cap Y_1) + r_{M/g'}(V \cup \{g\}) - r(M/g') \\
&= r_M(U \cap Y_1) + r_M(V) - (r(M) - 1) \\
&= r_M(U \cap Y_1) + r_M(V) - r(M \backslash g) + 1 \\
&= r_M(U) - 1 + r_M(V) - r(M \backslash g) + 1 \\
&= r_M(U) + r_M(V) - r(M \backslash g) \\
&= 1.
\end{aligned}$$

Thus $(U \cap Y_1, V \cup \{g\})$ is a 2-separation of M/g'. Since every element in X_1 is contractible, g' is contractible, and so $r(U) = 2$. Since $|U| \geq 3$, it follows that $|U \cap Y_1| \geq 2$, and so $g' \in \mathrm{cl}(Y_1 \cup \{e_1\})$, a contradiction as $Y_1 \cup \{e_1\}$ is closed. Hence $|X_1 \cap U| \geq 2$. An identical argument interchanging the roles of U and V establishes that $|X_1 \cap V| \geq 2$, thereby establishing (2.5.3.1).

Say $|Y_1 \cap U| \geq 2$. It follows by two application of uncrossing that each of $(X_1 \cap V) \cup \{g\}$ and $(X_1 \cap V) \cup \{g, e_1\}$ is 3-separating. Since $|X_1 \cap V| \geq 2$ and M is 3-connected, $(X_1 \cap V) \cup \{g\}$ and $(X_1 \cap V) \cup \{g, e_1\}$ are exactly 3-separating. Therefore, by Lemma 2.1.4, $e_1 \in \mathrm{cl}((X_1 \cap V) \cup \{g\})$ or $e_1 \in \mathrm{cl}^*((X_1 \cap V) \cup \{g\})$.

Since $e_1 \in \mathrm{cl}(Y_1)$, it follows by Lemma 2.1.3 that $e_1 \notin \mathrm{cl}^*((X_1 \cap V) \cup \{g\})$. So $e_1 \in \mathrm{cl}((X_1 \cap V) \cup \{g\})$. Thus, if $r((X_1 \cap V) \cup \{g\}) \geq 3$, then $((X_1 \cap V) \cup \{g\}, \{e_1\}, Y_1 \cup U)$ is a vertical 3-separation, contradicting the maximality of $Y_1 \cup \{e_1\}$. Therefore $r((X_1 \cap V) \cup \{e_1, g\}) = 2$. But then $g \in \mathrm{cl}(V \cap X_1) \subseteq \mathrm{cl}(V)$, a contradiction.

Now assume that $|Y_1 \cap U| \leq 1$. Say $Y_1 \cap U$ is empty. Then $U \subseteq X_1$. Let $(U', \{h\}, V')$ be a cyclic 3-separation of M such that $V \cup \{g\} \subseteq V' \cup \{h\}$ with the property that there is no other cyclic 3-separation $(U'', \{h'\}, V'')$ in which $V' \cup \{h\}$ is a proper subset of $V'' \cup \{h'\}$. Observe that such a cyclic 3-separation exists as we can choose $(U, \{g\}, V)$ if necessary. If every element in U' is deletable, then, as $U' \subseteq X_1$ and $|U'| \geq 3$, it follows that X_1 has at least two elastic elements. Thus we may assume that there is an element in U' that is not deletable. By the dual of Lemma 2.5.2, either U', and thus X_1, contains at least two elastic elements or $U' \cup \{h\}$ is a 4-element fan, or U' is contained in a Θ-separator. If $U' \cup \{h\}$ is a 4-element fan, then, by Lemma 2.2.1,

$$((U' \cup \{h\}) - \{f\}, \{f\}, E(M) - (U' \cup \{h\}))$$

is a vertical 3-separation, where f is the spoke-end of the 4-element fan $U' \cup \{h\}$. But then, as $X_1 \cap V$ is non-empty, $Y_1 \cup \{e_1\}$ is properly contained in $E(M) - (U' \cup \{h\})$, contradicting maximality. If U' is contained in a Θ-separator, then, by the dual of Lemma 2.5.2, U' is a circuit and there is an element w of U' such that $(U' - \{w\}) \cup \{h\}$ is a cosegment. But then

$$((U' \cup \{h\}) - \{w\}, \{w\}, E(M) - (U' \cup \{h\}))$$

is a vertical 3-separation of M, contradicting the maximality of $Y_1 \cup \{e_1\}$ as $Y_1 \cup \{e_1\}$ is properly contained in $E(M) - (U' \cup \{h\})$. Hence we may assume that $|Y_1 \cap U| = 1$.

Let $Y_1 \cap U = \{y\}$. Since $|Y_1 \cap U| = 1$, we have $|Y_1 \cap V| \geq 2$ and so, by two applications of uncrossing, $X_1 \cap U$ and $(X_1 \cap U) \cup \{g\}$ are both 3-separating. Since M is 3-connected and $|X_1 \cap U| \geq 2$, these sets are exactly 3-separating. If $y \notin \mathrm{cl}(X_1 \cap U)$, then, by Lemma 2.1.3, $y \in \mathrm{cl}^*(V \cup \{g\})$. But then $V \cup \{g\}$ is not coclosed, a contradiction. Thus $y \in \mathrm{cl}(X_1 \cap U)$, and so $y \in \mathrm{cl}((X_1 \cap U) \cup \{g\})$. Now $y \notin \mathrm{cl}^*(V \cup \{g\})$, and so $y \notin \mathrm{cl}^*(V)$. Hence as $(X_1 \cap U) \cup \{g\}$ and, therefore, the complement $V \cup \{y\}$ is 3-separating, Lemma 2.1.4 implies that $y \in \mathrm{cl}(V)$. Therefore, as $(X_1 \cap U) \cup \{g\}$ and V each have rank at least three, it follows that $((X_1 \cap U) \cup \{g\}, \{y\}, V)$ is a vertical 3-separation of M. Note that $r(V) \geq 3$;

otherwise, $(X_1 \cap V) \subseteq \mathrm{cl}(\{y, e_1\})$, in which case, $Y_1 \cup \{e_1\}$ is not closed. But $(X_1 \cap U) \cup \{g\}$ is a proper subset of X_1, a contradiction to the maximality of $Y_1 \cup \{e_1\}$. This last contradiction completes the proof of the lemma. □

We now combine Lemmas 2.5.2 and 2.5.3 to prove Theorem 1.1.1.

Proof of Theorem 1.1.1. Let $(X, \{e\}, Y)$ be a vertical 3-separation of M, where $Y \cup \{e\}$ is maximal, and suppose that $X \cup \{e\}$ is not a 4-element fan and X is not contained in a Θ-separator. If at least one element in X is not contractible, then, by Lemma 2.5.2, X contains at least two elastic elements. On the other hand if every element in X is contractible, then by Lemma 2.5.3, X again contains at least two elastic elements. This completes the proof of the theorem. □

We end this chapter with the proof of Theorem 1.1.2.

Proof of Theorem 1.1.2. Let M be a 3-connected matroid. If every element of M is elastic, then the theorem holds. Therefore suppose that M has at least one non-elastic element, e say. Up to duality, we may assume that $\mathrm{si}(M/e)$ is not 3-connected. Then, by Lemma 2.1.9, M has a vertical 3-separation $(X, \{e\}, Y)$. As $r(X), r(Y) \geq 3$, this implies that $|E(M)| \geq 7$, and so we deduce that every element in a 3-connected matroid with at most six elements is elastic. Now, suppose that M has no 4-element fans and no Θ-separators, and let $(X', \{e'\}, Y')$ be a vertical 3-separation such that $Y' \cup \{e'\}$ is maximal and contains $Y \cup \{e\}$. Then it follows by Theorem 1.1.1 that X', and hence X, contains at least two elastic elements. Interchanging the roles of X and Y, an identical argument gives us that Y also contains at least two elastic elements. Thus, M contains at least four elastic elements. □

Chapter 3

A Splitter Theorem for elastic elements

This chapter concerns the existence of elastic elements whose removal also preserves a given 3-connected minor. The chapter is structured as follows. Section 3.1 consists of some necessary preliminaries, while the main results of this chapter, Theorems 1.1.3 and 1.1.4, are proved in Section 3.2. Section 3.3 considers the matroids that possess the minimum possible number of elastic or N-elastic elements, and includes the proofs of Theorems 1.1.5 and 1.1.6. Lastly, Section 3.4 considers the applications of our work to the study of maintaining 3-connectivity relative to a fixed basis.

3.1 Preliminaries

We begin this chapter's preliminaries with the following elementary lemma of which we will make repeated use without explicit reference.

Lemma 3.1.1. *Let M be a 3-connected matroid and let N be a 3-connected matroid of M. If $|E(N)| \geq 4$, then $\mathrm{si}(M)$ has an N-minor.*

The next two lemmas concern 3-connected minors across 2-separations. The first is elementary and the second is a slight strengthening of [4, Lemma 4.5] and follows the proof of that lemma.

Lemma 3.1.2. *Let (X, Y) be a 2-separation of a connected matroid M and let N be a 3-connected minor of M. Then $\{X, Y\}$ has a member U such that $|E(N) \cap U| \leq 1$. Moreover, if $u \in U$, then*

(i) *M/u has an N-minor if M/u is connected, and*

(ii) *$M\backslash u$ has an N-minor if $M\backslash u$ is connected.*

Lemma 3.1.3. *Let M be a 3-connected matroid with a vertical 3-separation $(X, \{e\}, Y)$ such that $Y \cup \{e\}$ is closed, and let N be a 3-connected minor of M/e such that $|X \cap E(N)| \leq 1$. Then M/x has an N-minor for every element x of X, and there is at most one element of X, say x', such that $M\backslash x'$ has no N-minor. Moreover, if such an element x' exists, then $x' \in \text{cl}^*(Y)$ and $e \in \text{cl}(X - \{x'\})$.*

Let M be a 3-connected and let N be a 3-connected minor of M. Recall from the introduction to this part of the thesis, that an element e of M is *N-revealing* if either of $\text{si}(M/e)$ or $\text{co}(M\backslash e)$ has an N-minor and is not 3-connected. Furthermore, if S is a Θ-separator of M, then S is said to *reveal* the minor N in M if, up to duality, $M|S \in \{\Theta_n, \Theta_n^-\}$ for some $n \geq 3$ and at least one of the cosegment elements of $M|S$ is N-revealing in M.

The next lemma gives a number of equivalent conditions for a Θ-separator to reveal a given 3-connected minor.

Lemma 3.1.4. *Let M be a 3-connected matroid such that $r(M), r^*(M) \geq 4$ and let N be a 3-connected minor of M. Let W be a rank-2 subset and Z be a corank-2 subset of $E(M)$ such that $M|(W \cup Z) \in \{\Theta_n, \Theta_n^-\}$ for some $n \geq 3$. Then the following are equivalent:*

(i) *At least one element of Z is N-revealing in M.*

(ii) *$\text{co}(M\backslash z)$ has an N-minor for at least two elements $z \in Z$.*

(iii) *Both $\text{si}(M/z)$ and $\text{co}(M\backslash z)$ have an N-minor for all $z \in Z$ and $\text{co}(M\backslash w)$ has an N-minor for all $w \in W$.*

Moreover, if $|E(N)| \leq 3$, then all of (i)-(iii) *hold.*

Proof. Certainly, (iii) implies (ii). Now let $\{z_1, z_2, \ldots, z_n\}$ be a labelling of Z and let $\{w_1, w_2, \ldots, w_k\}$ be a labelling of W such that $(Z - \{z_i\}) \cup \{w_i\}$ is a circuit of M for all $i \in \{1, \dots, k\}$. Note that $k \in \{n, n-1\}$. Letting $i \in \{1, \dots, k\}$, it is then straightforward to observe that, as $r(M), r^*(M) \geq 4$, the partition

$$((Z - \{z_i\}) \cup \{w_i\}, \{z_i\}, E(M) - Z \cup \{w_i\})$$

of $E(M)$ is a cyclic 3-separation of M. Thus, by Lemma 2.1.9, $\mathrm{co}(M \backslash z_i)$ is not 3-connected, and hence, (ii) implies (i).

Suppose first that $|E(N)| \leq 3$. Letting $z \in Z$, we note that both $\mathrm{si}(M/z)$ and $\mathrm{co}(M \backslash z)$ are connected. Furthermore, as $r^*(M) \geq 4$, the matroid $\mathrm{co}(M \backslash z)$ has corank at least 3. Now note that, as z is in at least one circuit of the form $(Z - \{z_i\}) \cup \{w_i\}$, we have by orthogonality that any triad containing z must either be contained in Z or contain an element of W. If Z spans M, then $\mathrm{cl}(W)$ is a segment of size at four and, by orthogonality with this segment, it follows that z is in no triad with an element outside of Z. If Z does not span, then $r(E(M) - W \cup Z) \geq 3$, in which case, it is easily observed that z is in at most one triad with an element of $E(M) - W \cup Z$. In either case, it follows that $\mathrm{co}(M \backslash z)$ has rank at least two. Similarly, as Z is a cosegment, z is in at most two triangles. Thus, $\mathrm{si}(M/z)$, which has rank at least three, has corank at least two. We deduce that $\mathrm{si}(M/z)$ and $\mathrm{co}(M \backslash z)$ have $U_{1,3}$ and $U_{2,3}$-minors and thus an N-minor. Now let $i \in \{1, \dots, k\}$ and consider w_i. By orthogonality with both the segment W and the circuit $(Z - \{z_i\}) \cup \{w_i\}$, w_i is in at most two triads. As $r(M), r^*(M) \geq 4$, it follows that $\mathrm{co}(M \backslash w_i)$ has rank at least 2 and corank at least 3. Thus, $\mathrm{co}(M \backslash w_i)$ has a $U_{1,3}$ and a $U_{2,3}$-minor. We conclude that if $|E(N)| \leq 3$, then (iii), (ii) and (i) hold.

We may now assume that $|E(N)| \geq 4$. To complete the proof, we show that (i) implies (iii). Let $i \in \{1, \dots, k\}$ and suppose that $\mathrm{co}(M \backslash z_i)$ has an N-minor. As N is cosimple and $Z - \{z_i\}$ is a series class of $M \backslash z_i$, we have that $|(Z \cup \{w_i\}) \cap E(N)| \leq 1$. We then apply the dual of Lemma 3.1.3 to see that $M \backslash w_i$ has an N-minor and that both $M \backslash z_j$ and M/z_j have an N-minor for all $j \in \{1, \dots, n\} - \{i\}$. In particular, z_j is N-revealing for all $j \in \{1, \dots, k\}$. Thus, the choice of $i \in \{1, \dots, k\}$ was arbitrary. It follows that both M/z and $M \backslash z$ have an N-minor for all $z \in Z$ and $M \backslash w$ has an N-minor for all $w \in W$. Thus, (iii) is satisfied. □

It follows Lemma 3.1.4 that every Θ-separator reveals the empty matroid and all of $U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}$, and $U_{2,3}$. We will make free use of this fact.

3.2 The existence of N-elastic elements

In this section, we prove Theorem 1.1.3 and Theorem 1.1.4. We begin with three lemmas. The first two lemmas concern elastic elements in matroids with rank and corank at least four.

Lemma 3.2.1. *Let M be a 3-connected matroid such that $r(M), r^*(M) \geq 4$, and let N be a 3-connected minor of M with at most three elements. Then every elastic element of M is N-elastic.*

Proof. First note that, as $|E(N)| \leq 3$, N is a minor of either $U_{1,3}$ or $U_{2,3}$. Let x be an elastic element of M. Then $\mathrm{si}(M/x)$ and $\mathrm{co}(M\backslash x)$ are both 3-connected. Furthermore, as $r(M), r^*(M) \geq 4$, we have that $\mathrm{si}(M/x)$ has rank at least three and $\mathrm{co}(M\backslash x)$ has corank at least three. Thus, as $\mathrm{si}(M/x)$ is 3-connected, $\mathrm{si}(M/x)$ contains a circuit, but $\mathrm{si}(M/x)$ is not a circuit, and so $\mathrm{si}(M/x)$ has a $U_{2,3}$- and a $U_{1,3}$-minor. Similarly, as $\mathrm{si}(M^*/x)$, the dual of $\mathrm{co}(M\backslash x)$, is 3-connected and has rank at least three, $\mathrm{si}(M^*/x)$ has a $U_{2,3}$- and a $U_{1,3}$-minor. That is, $\mathrm{co}(M\backslash x)$ has a $U_{2,3}$- and a $U_{1,3}$-minor. This completes the proof of the lemma. □

Lemma 3.2.2. *Let M be a 3-connected matroid of corank at least four, and let N be a 3-connected minor of M. Let $(X, \{e\}, Y)$ be a vertical 3-separation of M such that M/e has an N-minor and $|X \cap E(N)| \leq 1$. If $Y \cup \{e\}$ is closed, then every elastic element in X is N-elastic.*

Proof. Let x be an elastic element of X. If $|E(N)| \leq 3$, then, by Lemma 3.2.1, x is N-elastic. Thus we may assume that $|E(N)| \geq 4$. In particular, N is simple and cosimple, and so if M/x or $M\backslash x$ has an N-minor, then $\mathrm{si}(M/x)$ and $\mathrm{co}(M\backslash x)$ has an N-minor, respectively. Therefore, by Lemma 3.1.3, x is N-elastic unless x is the unique exception in the statement of Lemma 3.1.3, in which case, $x \in \mathrm{cl}^*(Y)$ and $e \in \mathrm{cl}(X - \{x\})$. Suppose x is this unique exception. Then, as $x \in \mathrm{cl}^*(Y)$, it follows by Lemma 2.1.3, that $x \notin \mathrm{cl}(X - \{x\})$. Therefore, as $(Y \cup \{e\}, X)$ is a 3-separation of M, we have

$$\begin{aligned} 2 &= r(Y \cup \{e\}) + r(X) - r(M) \\ &= r(Y \cup \{e\}) + r(X - \{x\}) + 1 - r(M). \end{aligned}$$

In particular,

$$1 = r(Y \cup \{e\}) + r(X - \{x\}) - r(M\backslash x),$$

and so $(Y \cup \{e\}, X - \{x\})$ is a 2-separation of $M\backslash x$. Since $e \in \mathrm{cl}(X - \{x\})$, the partition $(Y, (X - \{x\}) \cup \{e\})$ is also a 2-separation of $M\backslash x$. Now, as x is elastic, $\mathrm{co}(M\backslash f)$ is 3-connected, and so at least one of $Y \cup \{e, f\}$ and X has corank 2, and at least one of $Y \cup \{f\}$ and $X \cup \{e\}$ has corank 2. By Lemma 2.1.3, $e \notin \mathrm{cl}^*(X) \cup \mathrm{cl}^*(Y)$, and we deduce that $r^*(X) = r^*(Y \cup \{f\}) = 2$. But then, as M is 3-connected, $r^*(M) = 3$, contradicting the assumption that M has corank at least four. Hence x is not the exception and so the lemma holds. □

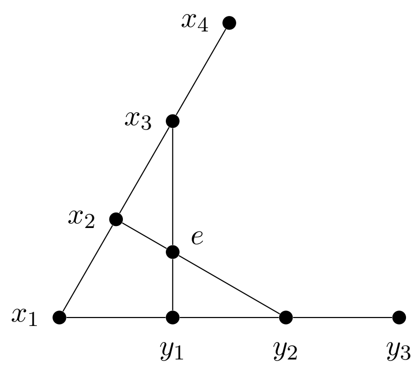


Figure 3.1: The 3-connected matroid L_8.

The condition in the statement of Lemma 3.2.2 that M has corank at least four is necessary. To see this, let L_8 denote the 3-connected rank-3 matroid for which a geometric representation is shown in Fig. 3.1. Let $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3\}$. Then $(X, \{e\}, Y)$ is a cyclic 3-separation of L_8, and $L_8\backslash e$ has a $U_{2,4}$-minor whose ground set contains Y. The element x_1 of L_8 is elastic but it is not $U_{2,4}$-elastic. Do note however, that every element of $X - \{x_1\}$ is $U_{2,4}$-elastic. The next lemma captures this last observation and is the corank three analogue of Lemma 3.2.2.

Lemma 3.2.3. *Let M be a 3-connected rank-3 matroid, and let N be a 3-connected minor of M. Let $(X, \{e\}, Y)$ be a cyclic 3-separation $(X, \{e\}, Y)$ of M such that $M\backslash e$ has an N-minor and $|E(N) \cap X| \leq 1$. If $X \cup \{e\}$ is not a 4-element fan, then there is at most one element of X that is not N-elastic. Moreover, if such an element x exists, then $x \in \mathrm{cl}(Y)$*

Proof. Since M is 3-connected and $(X, \{e\}, Y)$ is a cyclic 3-separation of M, we have $r(X) = r(Y) = 2$, and $|X|, |Y| \geq 3$. Furthermore, as $M\backslash e$ has an

N-minor and $|E(N) \cap X| \leq 1$, it follows that N is a minor of $U_{2,n}$ where $n = |\mathrm{cl}(Y)| \leq |Y| + 1$. Let $x \in X - \mathrm{cl}(Y)$. As $X \cup \{e\}$ is not a 4-element fan, $|X - \mathrm{cl}(Y)| \geq 3$, and so $\mathrm{co}(M\backslash x) = M\backslash x$ and $M\backslash x$ is 3-connected. Furthermore, $\mathrm{si}(M/x) \in \{U_{2,n}, U_{2,n+1}\}$ depending on whether or not e is in a triangle with x. In particular, $\mathrm{si}(M/x)$ and $\mathrm{co}(M\backslash x)$ are both 3-connected with N-minors. This completes the proof of the lemma. □

We now prove Theorem 1.1.4.

Proof of Theorem 1.1.4. Let $(X, \{e\}, Y)$ be a vertical 3-separation of M such that M/e has an N-minor and $|X \cap E(N)| \leq 1$. Without loss of generality, we may assume that $Y \cup \{e\}$ is closed. Now let $(X', \{e'\}, Y')$ be a vertical 3-separation of M such that $Y \cup \{e\} \subseteq Y' \cup \{e'\}$ and $Y' \cup \{e'\}$ is maximal, and suppose that $X' \cup \{e'\}$ is not a 4-element fan. If $r^*(M) = 3$, then, by Lemma 3.2.3, X' contains at least two N-elastic elements. Thus, we may assume that $r^*(M) \geq 4$. Suppose now, that X' is contained in a Θ-separator S. By Lemmas 2.5.2 and 2.5.3, there is a partition (W, Z) of S such that $r(W) = r^*(Z) = 2$ and, letting $n = \max\{|W|, |Z|\}$, either

(i) $M^*|S \in \{\Theta_n, \Theta_n^-\}$, $e' \in W$ and there is an element $z \in Z$ such that $X' = (W - \{e'\}) \cup \{z\}$; or

(ii) $M|S \in \{\Theta_n, \Theta_n^-\}$, $X' \cup \{e'\}$ is a circuit, and either $X' = Z$ or $X' = Z - \{z\}$ for some $z \in Z$.

If $|E(N)| \leq 3$, then S reveals N. Suppose $|E(N)| \geq 4$. Then, in case (i), it follows by Lemma 3.1.3, that M/w, and hence $\mathrm{si}(M/w)$, has an N-minor for all $w \in W - \{e\}$. In case (ii), it follows Lemma 3.1.3 that $M\backslash z$, and hence $\mathrm{co}(M\backslash z)$, has an N-minor for at least two elements $z \in Z$. We deduce by Lemma 3.1.4, that S reveals N. Thus, we may assume that X' is not contained in any Θ-separator. Then, as $Y' \cup \{e'\}$ is maximal, it follows by Theorem 1.1.1 that X' contains at least two elastic elements. By Lemma 3.2.2, each of these elastic elements is N-elastic, thereby completing the proof of the theorem. □

We remark here that the question of whether X' contains N-elastic elements in the instances of Theorem 1.1.4 in which $X' \cup \{e'\}$ is a 4-element fan or X' is contained in a Θ-separator is handled by combining Lemma 2.2.2 or Lemma 2.4.1 respectively with Lemma 3.2.2. We end this section by using Theorem 1.1.4 to prove Theorem 1.1.3.

Proof of Theorem 1.1.3. Let e be an N-revealing element of M. Then, up to duality, $\text{si}(M/e)$ has an N-minor and is not 3-connected. It follows by Lemma 2.1.9 and Lemma 3.1.2 that M has a vertical 3-separation $(X, \{e\}, Y)$ such that $|E(N) \cap X| \leq 1$. Choosing $(X - \text{cl}(Y), \{e\}, \text{cl}(Y) - \{e\})$ if necessary, M has a vertical 3-separation $(X', \{e'\}, Y')$ such that $Y \cup \{e\} \subseteq Y' \cup \{e'\}$ and $Y' \cup \{e'\}$ is maximal. Since M has no 4-element fans or Θ-separators revealing N, we deduce by Theorem 1.1.1 that X' contains at least two N-elastic elements, completing the proof of the theorem. □

3.3 Matroids with the smallest number of elastic elements

In this section, we prove three results regarding matroids with the smallest number of elastic or N-elastic elements.

Let M be a matroid. Recall that an exactly 3-separating partition (X, Y) of M is a *sequential* 3-*separation* if there is an ordering $(e_1, e_2, \ldots, e_k)$ of X or Y such that $\{e_1, e_2, \ldots, e_i\}$ is 3-separating for all $i \in \{1, 2, \ldots, k\}$. A *path of* 3-*separations* in M is an ordered partition $(P_0, P_1, \ldots, P_k)$ of $E(M)$ with the property that $P_0 \cup P_1 \cup \cdots \cup P_i$ is exactly 3-separating for all $i \in \{0, 1, \ldots, k-1\}$. Of the next two lemmas, the first follows easily from the definitions and the second is [4, Lemma 6.3].

Lemma 3.3.1. *A partition (X, Y) of a matroid M such that $|X|, |Y| \geq 2$ is a sequential 3-separation if and only if for some $U \in \{X, Y\}$, there is a path of 3-separations $(P_0, P_1, \ldots, P_k, U)$ in M such that $|P_0| = 2$ and $|P_i| = 1$ for all $i \in \{1, 2, \ldots, k\}$.*

Lemma 3.3.2. *Let M be a 3-connected matroid with distinct elements s_1 and s_2. Let Z be a subset of $E(M) - \{s_1, s_2\}$ such that $|E(M) - (Z \cup \{s_1, s_2\})| \geq 2$. If, for each $z \in Z$, there is a path of 3-separations $(X_z, \{z\}, Y_z)$ in M such that $\{s_1, s_2\} \subseteq X_z \subseteq Z \cup \{s_1, s_2\}$, then there is an ordering $(z_1, z_2, \ldots, z_k)$ of Z such that*

$$(\{s_1, s_2\}, \{z_1\}, \{z_2\}, \ldots, \{z_k\}, E(M) - (Z \cup \{s_1, s_2\}))$$

is a path of 3-separations in M.

Recall that a matroid has path-width three if there is an ordering $(e_1, e_2, \ldots, e_n)$ of its groundset such that $\{e_1, e_2, \ldots, e_i\}$ is 3-separating for all

$i \in \{1, 2, \ldots, n\}$. We next prove Theorem 1.1.5; that is, if a 3-connected matroid with no 4-element fans or Θ-separators has exactly four elastic elements, then it has path-width three.

Proof of Theorem 1.1.5. By Lemmas 3.3.1, 3.3.2 and 2.1.9 it suffices to show that there is a partition $(\{f_1, f_2\}, \{g_1, g_2\})$ of the four elastic elements of M such that every vertical 3-separation or cyclic 3-separation of M is of the form $(X, \{e\}, Y)$, where $\{f_1, f_2\} \subset X$ and $\{g_1, g_2\} \subset Y$. Suppose that this fails. By Theorem 1.1.1, each side of a vertical or cyclic 3-separation of M has at least two elastic elements. Thus, there must be a pair of partitions $(X_1, \{e_1\}, Y_2)$, $(X_2, \{e_2\}, Y_2)$ of $E(M)$, each a vertical or a cyclic 3-separation of M, such that each of the intersections $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$ and $Y_1 \cap Y_2$ contains a unique elastic element. Without loss of generality, we may assume that $e_1 \in Y_2$, $e_2 \in X_1$, $f_1 \in X_1 \cap X_2$, $f_2 \in X_1 \cap Y_2$, $g_1 \in Y_1 \cap X_2$ and $g_2 \in Y_1 \cap Y_2$, and that, up to duality, $(X_1, \{e_1\}, Y_1)$ is a vertical 3-separation. By uncrossing $X_2 \cup \{e_2\}$ with both X_1 and $X_1 \cup \{e_1\}$, we have that both $Y_1 \cap Y_2$ and $(Y_1 \cap Y_2) \cup \{e_1\}$ are 3-separating. If $r(Y_1 \cap Y_2) \geq 3$, it follows that $(Y_1 \cap Y_2, \{e_1\}, X_1 \cup X_2)$ is a vertical 3-separation of M and thus, by Theorem 1.1.1 that $Y_1 \cap Y_2$ contains at least two elastic elements, a contradiction. We deduce that $r((Y_1 \cap Y_2) \cup \{e_1\}) = 2$. Now, if $Y_1 \cap X_2 = \{g_1\}$, then either $Y_1 \cup \{e_1\}$ is a 4-element fan, a contradiction, or $(Y_1 - \{g_1\}, \{g_1\}, X_1 \cup \{e_1\})$ is a cyclic 3-separation of M, contradicting the fact that g_1 is elastic. Thus, $|Y_1 \cap X_2| \geq 2$. We then uncross Y_1 with both X_2 and $X_2 \cup \{e_2\}$ to see that both $(X_1 \cap Y_2) \cup \{e_2\}$ and $(X_1 \cap Y_2) \cup \{e_1, e_2\}$ are exactly 3-separating. If $r((X_1 \cap Y_2) \cup \{e_2\}) \geq 3$, then $((X_1 \cap Y_2) \cup \{e_2\}, \{e_1\}, X_2 \cup Y_1)$ is a vertical 3-separation of M and it follows Theorem 1.1.1 that $X_1 \cap Y_2$ contains at least two elastic elements, a contradiction. We deduce that $r((X_1 \cap Y_2) \cup \{e_1, e_2\}) = 2$. Now, if $Y_1 \cap Y_2 = \{g_2\}$, then either $Y_2 \cup \{e_2\}$ is a 4-element fan or $(Y_2 - \{g_2\}, \{g_2\}, X_2 \cup \{e_2\})$ is a cyclic 3-separation of M, a contradiction. Thus, $|Y_1 \cap Y_2| \geq 2$. Finally, if $X_1 \cap Y_2 = \{f_2\}$, then $(Y_2 - \{f_2\}, \{f_2\}, X_2 \cup \{e_2\})$ is a cyclic 3-separation of M a contradiction. Thus, $|X_1 \cap Y_2| \geq 2$ and the set $(X_1 \cap Y_2) \cup \{e_1, e_2\}$ is a segment of at least four elements. It follows that Y_2 is a rank-3 cocircuit and, by Lemmas 2.3.1 and 2.1.6, that every element of $X_1 \cap Y_2$ is elastic. This final contradiction completes the proof. □

The next lemma concerns N-elastic elements when $|E(N)| \leq 3$ and can be viewed as a small extension of Theorem 1.1.2 and Theorem 1.1.5.

Lemma 3.3.3. *Let M be a 3-connected matroid with no 4-element fans or Θ-separators and let $N \in \{U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}, U_{2,3}\}$. If $r(M), r^*(M) \geq 3$ and $|E(M)| \geq 8$, then M has at least four N-elastic elements. Moreover, if M has exactly four N-elastic elements, then M has path-width three.*

Proof. By Theorem 1.1.2, M has at least four elastic elements. If every elastic element of M is N-elastic, then the result follows Theorem 1.1.5. Otherwise, M has an elastic element e which is not N-elastic. Then, by Lemma 3.2.1, we may assume, up to duality, that $r(M) = 3$. We also note that $r^*(M) \geq 5$ as $|E(M)| \geq 8$. It follows that the 3-connected matroid $\text{co}(M \backslash e)$ has corank at least four and rank at least two. As $\text{co}(M \backslash e)$ is connected, it must then have both $U_{1,3}$ and $U_{2,3}$-minors, and thus, an N-minor. Now, as e is not N-elastic, we deduce that $\text{si}(M/e)$ has no N-minor. This is only possible if $\text{si}(M/e) \cong U_{2,3}$ and $N \cong U_{1,3}$. Moreover, in this case, M is comprised of a triangle $\{e_1, e_2, e_3\}$ and three segments L_1, L_2 and L_3 meeting at e such that $e_i \in L_i$ for all $i \in \{1, 2, 3\}$. As M has at least eight elements, at least one of these segments, say L_1, has at least four elements. It is then easily seen that every element of $E(M) - L_1$ and at least one element of L_1 is $U_{2,4}$-elastic and thus, $U_{1,3}$-elastic. Thus, M has at least five N-elastic elements, completing the proof of the lemma. □

To see that the requirement of Lemma 3.3.3 that M have rank and corank at least three is necessary, consider the case when M is $U_{2,5}$ and N is $U_{1,3}$. If $e \in E(M)$, then M/e, which is isomorphic to $U_{1,4}$, has a $U_{1,3}$-minor but $\text{si}(M/e)$, which is isomorphic to $U_{1,1}$, has no $U_{1,3}$-minor. Thus, in this case, M has no N-elastic elements. To see that the requirement $|E(M)| \geq 8$ is necessary, consider the case when M is F_7 and N is $U_{1,3}$. If $e \in E(M)$, then M/e has a $U_{1,3}$-minor but $\text{si}(M/e)$, which is isomorphic to $U_{2,3}$, has no $U_{1,3}$-minor. Thus, again, M has no N-elastic elements.

We next prove our main result regarding matroids with the minimum number of N-elastic elements. Theorem 1.1.6 is a direct consequence by Lemma 3.3.1.

Theorem 3.3.4. *Let M be a 3-connected matroid with no 4-element fans and let N be a 3-connected minor of M such that $|E(N)| \geq 4$ and M has no Θ-separators revealing N. Let K be the set of N-revealing elements of M. If M has exactly two N-elastic elements s_1 and s_2, then K has an ordering $(e_1, e_2, \ldots, e_n)$ such that $(\{s_1, s_2\}, \{e_1\}, \{e_2\}, \ldots, \{e_n\}, E(M) - K \cup \{s_1, s_2\})$ is a path of 3-separations in M and, for all $i < n$, both M/e_i and $M \backslash e_i$ have an N-minor.*

Proof. We first ensure that $|E(M)-(K\cup\{s_1,s_2\})|\geq 2$. If M has no Θ-separators, then this follows Theorem 1.1.2, as no element of K is elastic. Otherwise, let W be a rank-2 subset and Z a corank-2 subset of $E(M)$ such that $W\cup Z$ is a Θ-separator of M. Note that $\min\{|W|,|Z|\}\geq 3$, as M has no 4-element fans. By Lemma 2.4.1, at most one elastic element of $W\cup Z$ is elastic. Thus, if $|E(M)-(K\cup\{s_1,s_2\})|\leq 1$, then at least one element of W and at least one element of Z is N-revealing. As M has no Θ-separators revealing N, we deduce that, indeed, $|E(M)-(K\cup\{s_1,s_2\})|\geq 2$.

Next, for each $e\in K$, we select a suitable path of 3-separations $(X_e,\{e\},Y_e)$. Let $e\in K$. Up to duality, we may assume that $\mathrm{si}(M/e)$ has an N-minor and is not 3-connected. Then, by Lemmas 2.1.9, 2.1.10 and 3.1.2, there is a vertical 3-separation $(X,\{e\},Y)$ of M such that $|E(N)\cap X|\leq 1$ and $Y\cup\{e\}$ is closed. By Theorem 1.1.1, X contains at least two N-elastic elements. Thus $\{s_1,s_2\}\subseteq X$. Furthermore, by Lemma 3.1.3, M/x has an N-minor for all $x\in X$ and there is at most one element, x', for which $M\backslash x'$ has no N-minor. If there is no such element x', then let $X_e=X$ and $Y_e=Y$. Otherwise, we note by Lemma 3.1.3, that $x'\in \mathrm{cl}^*(Y)$. It follows by Lemma 2.1.4, that $(X-\{x'\},\{e\},Y\cup\{x'\})$ is a path of 3-separations. In this case, let $X_e=X-\{x'\}$ and $Y_e=Y\cup\{x'\}$. Observe that, by our selection process, we have that $\{s_1,s_2\}\in X_e$ and, for all $x\in X_e$, both M/x and $M\backslash x$ have an N-minor. Moreover, as $|E(N)|\geq 4$, the latter property implies that $X_e\subseteq K\cup\{s_1,s_2\}$.

Let $Y=E(M)-K\cup\{s_1,s_2\}$. By an application of Lemma 3.3.2, there is an ordering $(e_1,e_2,\dots,e_n)$ of K such that $(\{s_1,s_2\},\{e_1\},\dots,\{e_n\},Y)$ is a path of 3-separations. It remains to show that there is such an ordering for which both M/e_i and $M\backslash e_i$ have an N-minor for all $i<n$.

For all $i\in\{1,2,\dots,n\}$, let $A_i=\{s_1,s_2,e_1,\dots,e_{i-1}\}$ and let $B_i=E(M)-A_i\cup\{e_i\}$. Observe that, $(A_i,\{e_i\},B_i)$ is a path of 3-separations and, by Lemma 2.1.5, e_i is in exactly one of $\mathrm{cl}(A_i)\cap\mathrm{cl}(B_i)$ or $\mathrm{cl}^*(A_i)\cap\mathrm{cl}^*(B_i)$. We next show the following:

3.3.4.1. *Let $j\in\{1,2,\dots,n\}$. If $e_j\in\mathrm{cl}(A_j)\cap\mathrm{cl}(B_j)$ and $r(B_j)\geq 3$, or $e_j\in\mathrm{cl}^*(A_j)\cap\mathrm{cl}^*(B_j)$ and $r^*(B_j)\geq 3$, then both M/e_i and $M\backslash e_i$ have an N-minor for all $i<j$.*

Assume that $e_j\in\mathrm{cl}(A_j)\cap\mathrm{cl}(B_j)$, applying a dual argument otherwise. If A_j is a segment or a cosegment, then the result is easily seen to hold. Thus, we may assume that $r(A_j), r^*(A_j) \geq 3$. Letting $L = \text{cl}(B_j) - B_j$, we then note that

$$r(L) = r(\text{cl}(B_j) \cap (A_j \cup \{e_j\})) \leq r(B_j) + r(A_j \cup \{e_j\}) - r(M) = \lambda(B_j) = 2.$$

Thus, L is either the singleton $\{e_j\}$, or it is a segment. Letting $\ell \in L$, we see that $(A_j \cup \{e_j\} - \{\ell\}, \{\ell\}, B_j)$ is a vertical 3-separation of M and, by Lemma 2.1.10, so is $(A_j - L, \{\ell\}, B_j \cup (L - \{\ell\}))$. In particular, $\text{si}(M/\ell)$ is not 3-connected by Lemma 2.1.9. Thus, by the definition of K, $\text{si}(M/\ell)$, and hence M/ℓ has an N-minor. As N is simple and the choice of ℓ was arbitrary, it follows easily that, if $|L| \geq 2$, then M/ℓ and $M\backslash\ell$ have an N-minor for all $\ell \in L$. Now let N' be an N-minor of M/e_j. If $|E(N') \cap B_j| \leq 1$, then B_j contains two N-elastic elements by Theorem 1.1.1, a contradiction as $\{s_1, s_2\} \subset A_j$. Thus, $|E(N') \cap A_j| \leq 1$. Now, if M/e and $M\backslash e$ have an N-minor for all $e \in A_j - L$, then we are done. Otherwise, it follows Lemma 3.1.3 that there is an element x of $A_j - L$ such that $M\backslash x$ does not have an N-minor and, furthermore, $x \in \text{cl}^*(B_j \cup (L - \{e_j\}))$. As $\text{co}(M\backslash x)$ has no N-minor, we have by the definition of K, that $\text{si}(M/x)$ is not 3-connected, and thus, by Bixby's Lemma, $\text{co}(M\backslash x)$ is 3-connected. Observing that $B_j \cup (L - \{e_j\})$ is 2-separating in $M\backslash x$, it follows that either $(A_j - L) \cup \{e_j\}$ or $B_j \cup (L - \{e_j\}) \cup \{x\}$ has corank-2. The first implies that $e_j \in \text{cl}^*(A_j) \cap \text{cl}(A_j)$, a contradiction by Lemma 2.1.5. The second implies that $\text{si}(M/x)$ is 3-connected by Lemma 2.3.1. As this is a further contradiction, we deduce that there is no such element x, and thus, (3.3.4.1) holds.

Now to complete the proof. Suppose first that $r(Y) = 2$. In this case, letting $L = \text{cl}(Y) - Y$, we have that $Y \cup L$ is a segment. It then follows easily from the definition of K that, if $|L| \geq 2$, then M/ℓ and $M\backslash\ell$ have an N-minor for all $\ell \in L$. If $K \subseteq L$, then we are done. Otherwise, let j be the largest index such that $e_j \notin L$. By (3.3.4.1), $M\backslash e_i$ and M/e_i have an N-minor for all $i < j$. If M/e_j and $M\backslash e_j$ both have an N-minor, then we are done. Otherwise, for every element ℓ of L, we observe that $e_j \notin X_\ell$ and thus, $(X_\ell - L) \subseteq A_j$. It follows that $L \subset \text{cl}(A_j)$ and, consequently, there is an ordering $(e'_1, e'_2, \ldots, e'_n)$ of K such that $e'_n = e_j$ and $(\{s_1, s_2\}, \{e'_1\}, \ldots, \{e'_n\}, Y)$ is a path of 3-separations in M. If $r^*(Y) = 2$, then either $(Y, E(M) - Y)$ is a 2-separation, or $Y \cup \{e_j, e_n\}$ is a 4-element fan. As both of these are contradictions, we deduce that $r^*(Y) = 3$. Thus, by switching to the dual and taking this new ordering of K, we may have assumed that $r(Y) \geq 3$. Letting $j = n$, the theorem then follows by (3.3.4.1). $\square$

We end this section with some examples exhibiting the fact the minimum number of elastic elements and N-elastic elements is obtained.

Recalling the labelling of the matroid Θ_n given in Section 2.4, let Θ'_n denote the matroid achieved from Θ_n by relabelling every element z_i of Z as z'_i. For our first example, consider the matroid $P_W(\Theta_n, \Theta'_n)\backslash\{w_1, w_2\}$ where $n \geq 4$. This matroid has no Θ-separators or 4-element fans and has exactly four elastic elements, namely $\{z_1, z_2, z'_1, z'_2\}$. Moreover, these elastic elements are N-elastic for all $N \in \{U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}, U_{2,3}\}$.

For our second example, we start with F_7 but note that any sufficiently structured 3-connected matroid would do. Let T be a triangle of F_7 and, letting $n \geq 5$, freely add $n-3$ points to the line $\text{cl}(T)$. Relabel this line as $W = \{w_1, \ldots, w_n\}$ in such a way that $\{w_1, w_2\} \cap T = \emptyset$. Call the resulting matroid F_7^n. Then consider the matroid $P_W(F_7^n, \Theta_n)\backslash\{w_1, w_2\}$. This matroid has no Θ-separators or 4-element fans and has exactly two F_7-elastic elements, namely, z_1 and z_2.

Our third example demonstrates why the inequality $i < n$ of Theorem 3.3.4 is strict. Let M_f be the 11-element matroid of Figure 3.2 and let Θ_4^+ be the matroid achieved from Θ_4 by adding an element f freely to the line W. Letting $W^+ = \{w_1, w_2, w_3, w_4, f\}$, consider the matroid $M = P_{W^+}(M_f, \Theta_4^+)\backslash\{w_1, w_2\}$. This matroid has no 4-element fans and no Θ-separators. Moreover, it has precisely two F_7-elastic elements, z_1 and z_2, and three F_7-revealing elements, z_3, z_4 and f. Although M/z_3, $M\backslash z_3$, $M\backslash z_4$, M/z_4 and M/f all have an F_7-minor, $M\backslash f$ has no such minor.

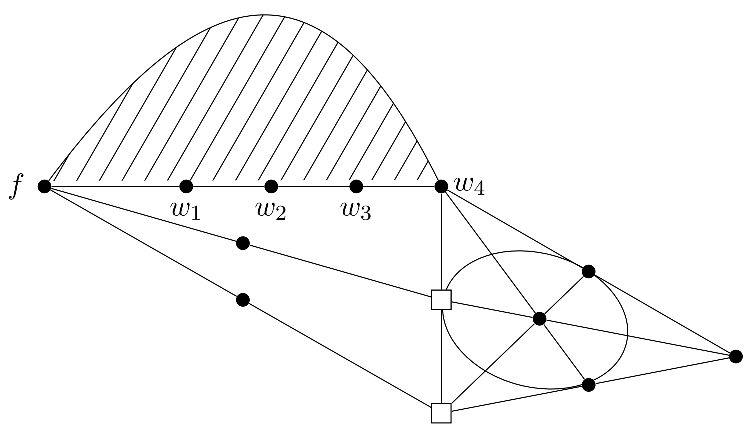


Figure 3.2: A geometric representation of the matroid M_f when the hatched area is omitted. Including the hatched area gives a schematic diagram of the rank-6 matroid $P_{W^+}(M_f, \Theta_4^+)$ with the elements $\{z_1, z_2, z_3, z_4\}$ suppressed.

3.4 Applications to fixed-basis theorems

In this section, we show that a number of established results regarding maintaining 3-connectivity relative to a fixed basis are consequences of the presence of elastic elements and Θ-separators. Let M be a 3-connected matroid and let B be a basis of M. Following [38], we say that an element e of M is *removable with respect to* B if either

(i) $e \in B$ and $\mathrm{si}(M/e)$ is 3-connected, or

(ii) $e \in E(M) - B$ and $\mathrm{co}(M\backslash e)$ is 3-connected.

One easily observes that a 5-element fan may have no removable elements with respect to a given basis B. However, removable elements are abundant in all larger Θ-separators. The straightforward proof of the following is omitted.

Lemma 3.4.1. *Let M be a 3-connected matroid and let B be a basis of M. Let W be a rank-2 subset and let Z be a corank-2 subset of M such that $W \cup Z$ is a Θ-separator of M with at least six elements. Then,*

(i) $|\mathrm{cl}(W) - B| \geq |\mathrm{cl}(W)| - 2$ *and* $\mathrm{co}(M\backslash w)$ *is 3-connected for all* $w \in \mathrm{cl}(W)$, *and*

(ii) $|B \cap \mathrm{cl}^*(Z)| \geq |\mathrm{cl}^*(Z)| - 2$ *and* $\mathrm{si}(M/z)$ *is 3-connected for all* $z \in \mathrm{cl}^*(Z)$.

One may also immediately observe that if an element is elastic, then it is removable with respect to any basis. We now show that the main theorem of [38] follows from Theorem 1.1.2, Theorem 1.1.5, and a treatment of Θ-separators.

Theorem 3.4.2 ([38], Theorem 1.1)**.** *Let M be a 3-connected matroid with no 4-element fans where $|E(M)| \geq 4$. Let B be a basis of M. Then M has at least four elements that are removable with respect to B. Moreover, if M has exactly four removable elements with respect to B, then M has path-width three.*

Proof. Suppose first that there is a rank-2 subset W and a corank-2 subset Z of $E(M)$ such that $W \cup Z$ is a Θ-separator of M. Then, $r(M), r^*(M) \geq 4$ and, up to duality, $M|(W \cup Z) \in \{\Theta_n, \Theta_n^-\}$, where $n \geq 4$ as M has no 4-element fans. By Lemma 3.4.1, at least $|Z| - 2$ elements of Z and at least $|W| - 2$ elements of W are removable with respect to B. If Z spans M, then $|\mathrm{cl}(W)| \geq 4$ and, as $|B \cap \mathrm{cl}(W)| \leq 2$, it follows by Lemma 2.3.1 that M has at least four elements that are removable with respect to B. Moreover, in this instance, M has path-width three as any ordering of $E(M)$ progressing first through the elements of Z will be sequentially 3-separating. Otherwise, Z does not span M. Then, for any $w \in W$, the partition $(W \cup Z - \{w\}, \{w\}, E(M) - W \cup Z)$ is a vertical 3-separation of M. Let $(U, \{e\}, V)$ be a vertical 3-separation of M such that $V \cup \{e\}$ is maximal containing $W \cup Z$. Then, by Theorem 1.1.1, U has at least two elastic elements, or is contained in a Θ-separator. In particular, by Lemmas 2.5.2, 2.5.3 and 3.4.1, the set $E(M) - W \cup Z$ has at least two elements that are removable with respect to B, bringing the total to at least five. To complete the proof, we may now assume that M has no Θ-separators. In this case, we have by Theorem 1.1.2, that M has at least four elastic elements. Moreover, if M has exactly four removable elements with respect to B, then these are precisely the elastic elements of M and thus, M has path-width three by Theorem 1.1.5. □

In [38], Whittle and Williams asked if there exists a 3-connected matroid M with no 4-element fans such that for every basis B of M, there are exactly four elements of M which are removable with respect to B. We can answer this in the negative.

Proposition 3.4.3. *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 4$. Then there exists a basis B of M such that M has at least five removable elements with respect to B.*

Proof. Suppose first that $r(M), r^*(M) \geq 4$ and that M has a rank-2 subset W and a corank-2 subset Z such that $M|(W \cup Z) \in \{\Theta_n, \Theta_n^-\}$, where $n \geq 4$. Then, letting B be any basis of M containing the independent set Z, we have by Lemma 3.4.1, that every element of Z and at least one element of W is removable with respect to B. This gives a total of at least five such elements. Thus, by applying a dual argument, we may assume that M has no Θ-separators. Then, by Theorem 1.1.2, M has at least four elastic elements. Moreover, as elastic elements are removable with respect to any basis, we may assume that M has exactly four elastic elements. The only 3-connected matroid on four elements is $U_{2,4}$. Thus, M must have at least five elements. Let e be some non-elastic element of M. As M has no coloops, M has bases both containing and avoiding e. If e is not removable with respect to any basis, it follows that the matroids $\mathrm{si}(M/e)$ and $\mathrm{co}(M\backslash e)$ are not 3-connected, a contradiction to Bixby's Lemma. Thus, e is removable with respect to some basis B and the result follows. □

Let M be a 3-connected matroid, let N be a 3-connected minor of M and let B be a basis of M. Following [4], an element e of M is called (N, B)-*robust* if either

(i) $e \in B$ and M/e has an N-minor, or

(ii) $e \in E(M) - B$ and $M \backslash e$ has an N-minor.

Furthermore, such an element is called (N, B)-*strong* if either

(i) $e \in B$ and $\mathrm{si}(M/e)$ is 3-connected with an N-minor, or

(ii) $e \in E(M) - B$ and $\mathrm{co}(M \backslash e)$ is 3-connected with an N-minor.

The next lemma follows by combining Lemma 3.4.1 with Lemma 3.1.4.

Lemma 3.4.4. *Let M be a 3-connected matroid, let N be a 3-connected minor of M and let B be a basis of M. Let S be a Θ-separator of M with at least six elements. If S reveals N in M, then at least $|S| - 4$ elements of S are (N, B)-strong.*

Evidently, an N-elastic element of M is (N, B)-strong for every basis B of M. We end this part of the thesis by showing that the two main theorems of [4] follow from Theorem 1.1.3, Theorem 3.3.4, and a treatment of Θ-separators.

Theorem 3.4.5 ([4], Theorems 1.1 and 1.2)**.** *Let M be a 3-connected matroid with no 4-element fans such that $|E(M)| \geq 5$, let N be a 3-connected minor of M and let B be a basis of M. If M has two distinct (N, B)-robust elements, then M has two distinct (N, B)-strong elements. Moreover, letting P denote the set of (N, B)-robust elements of M, if M has precisely two (N, B)-strong elements, then $(P, E(M) - P)$ is a sequential 3-separation.*

Proof. If M has rank or corank at most two, then the result follows easily from the fact that $|E(M)| \geq 5$. Likewise, the result is easy when $|E(M)| \in \{6, 7\}$. Thus, we may assume that $r(M), r^*(M) \geq 3$ and $|E(M)| \geq 8$. Furthermore, as M has no 4-element fans, any Θ-separator of M has at least seven elements. Thus, if M has a Θ-separator revealing N, then M has at least three (N, B)-strong elements by Lemma 3.4.4. We may therefore assume that M has no such Θ-separators. Now, if $|E(N)| \leq 3$, then M has at least four N-elastic by Lemma 3.3.3 and we are done. Thus, for the remainder of the proof, we may assume that $|E(N)| \geq 4$. In this case, every (N, B)-robust element is either (N, B)-strong or N-revealing. It follows that either each of the two guaranteed (N, B)-robust elements are (N, B)-strong or, by Theorem 1.1.3, M has at least two N-elastic elements. In either case, M has at least two (N, B)-strong elements, thus concluding the proof of the first part of the theorem. Now suppose that M has precisely two (N, B)-strong elements $\{s_1, s_2\}$. If M has no N-revealing elements, then $P = \{s_1, s_2\}$ and $(P, E(M)-P)$ is trivially a sequential 3-separation. Otherwise, M has at least one N-revealing element. In this case, it follows Theorem 1.1.3 that s_1 and s_2 are N-elastic and that M has no further N-elastic elements. Now let K be the set of N-revealing elements of M. Note that $P - \{s_1, s_2\} \subseteq K$. By Theorem 3.3.4, K has an ordering $(e_1, e_2, \ldots, e_n)$ such that $(\{s_1, s_2\}, \{e_1\}, \{e_2\}, \ldots, \{e_n\}, E(M) - K \cup \{s_1, s_2\})$ is a path of 3-separations and, for all $i < n$, both M/e_i and $M\backslash e_i$ have an N-minor. In particular, e_i is (N, B)-robust for all $i < n$, and consequently, P is either $K \cup \{s_1, s_2\}$ or $K \cup \{s_1, s_2\} - \{e_n\}$. Thus, by Lemma 3.3.1, $(P, E(M)-P)$ is a sequential 3-separation, completing the proof. □

Part II
Generalised Uniformity in Matroids

A matroid M is *paving* if every rank-$(r(M)-2)$ flat is independent, or equivalently, if $M|H$ is uniform for every hyperplane H of M. Thus, in a natural sense, paving matroids are close to being uniform. In this part of the thesis, we generalise this observation and describe a two-parameter property of matroids that captures just how close to uniform a given matroid is.

For positive integers k and ℓ, we define a matroid to be (k,ℓ)-*uniform* if it has no minor isomorphic to $U_{k,k} \oplus U_{0,\ell}$. It is easy to show that a matroid is $(1,1)$-uniform precisely if it is uniform and is $(2,1)$-uniform precisely if it is paving. It is also evident that all matroids are (k,ℓ)-uniform for some (k,ℓ) pair and that if M is (k,ℓ)-uniform, then it is (k',ℓ')-uniform for all $k' \geq k$ and $\ell' \geq \ell$. Furthermore, an easy duality argument shows that a matroid is (k,ℓ)-uniform if and only if its dual is (ℓ,k)-uniform. All of these facts are used freely throughout this part of the thesis.

This generalised notion of uniformity is of particular consequence when we restrict our attention to matroids representable over a given finite field. Letting q be a prime power, it is easy to show that if a uniform matroid is $GF(q)$-representable, then it has corank at most 1 or rank at most $q-1$. In Section 4.1, we extend this observation and give explicit bounds on the rank and corank of a (k,ℓ)-uniform $GF(q)$-representable matroid when k and ℓ are arbitrary positive integers. A consequence of this is the following result:

Theorem 3.4.6. *Let (k,ℓ) be a pair of positive integers and let q be a prime power. Then only finitely many simple cosimple $GF(q)$-representable matroids are (k,ℓ)-uniform.*

Note that both the simple and cosimple requirements in this theorem are necessary, as the uniform matroids $U_{1,n}$ and $U_{n-1,n}$ are representable over every field for all $n \geq 1$. This finiteness result has an interesting corollary regarding Rota's Conjecture:

Corollary 3.4.7. *For every prime power* q*, the set of excluded minors for* $GF(q)$*-representability is finite if and only if for some fixed pair* (k_q, ℓ_q) *of positive integers, every such excluded minor is* (k_q, ℓ_q)*-uniform.*

To illustrate, for $q \leq 4$, every excluded minor of $GF(q)$-representability is $(2, 1)$-uniform, that is, paving. As Geelen, Gerards and Whittle have announced a proof of Rota's Conjecture [15], it would seem that such (k_q, ℓ_q) pairs exist for all q. If well behaved, these bounds may offer improved methods for explicitly determining the excluded minors of $GF(q)$-representability.

By applying duality to the lists of binary $(2, 1)$-uniform and $(3, 1)$-uniform matroids of Acketa [1] and Rajpal [31] respectively, one may explicitly list all binary $(1, 2)$-uniform and $(1, 3)$-uniform matroids. These results concern binary (k, ℓ)-uniform matroids such that $k + \ell \leq 4$. We complete this picture in Sections 4.2 and 4.3 by determining the binary $(2, 2)$-uniform matroids. The most difficult part of the characterisation is in establishing the following result:

Theorem 3.4.8. *The* 3*-connected binary* $(2, 2)$*-uniform matroids are precisely the* 3*-connected minors of* $Z_5\backslash t, P_{10}, AG(4, 2)$*, and* $AG(4, 2)^*$*.*

Here, $Z_5\backslash t$ is the tipless binary 5-spike, $AG(4, 2)$ is the rank-5 binary affine geometry, and P_{10} is the rank-5 binary matroid represented by the matrix of Figure 3.3. It is easily seen that P_{10} is self-dual and that $P_{10}/5\backslash 10 \cong M(\mathcal{W}_4)$. Moreover, by pivoting, one can show that $P_{10}/8 \cong Z_4$. A further description of P_{10} is given in Section 4.3.

$$
\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix}
$$
$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
$$

Figure 3.3: A binary representation of P_{10}.

The *nullity* of a set X in a matroid M is $|X| - r_M(X)$. The following characterisation of (k, ℓ)-uniform matroids in terms of the nullity of certain flats will be treated as an alternate definition.

Proposition 3.4.9. *A matroid M is (k, ℓ)-uniform if and only if every rank-$(r(M) - k)$ flat of M has nullity less than ℓ.*

Proof. Suppose first that M is not (k, ℓ)-uniform. Then M has an independent set X and coindependent set Y such that $M/X\backslash Y \cong U_{k,k} \oplus U_{0,\ell}$ and $r_M(X) = r(M) - k$. Letting Z denote the ℓ loops of $M/X\backslash Y$, every element of Z must be in the closure of X in M. Thus, $\mathrm{cl}_M(X)$ is a rank-$(r(M) - k)$ flat of M with nullity at least ℓ. For the converse, suppose M has a rank-$(r(M) - k)$ flat F of nullity at least ℓ. Contracting any basis for F achieves a rank-k matroid with at least ℓ loops. An appropriate restriction then yields a $U_{k,k} \oplus U_{0,\ell}$-minor. □

Having defined a notion that generalises that of a uniform matroid, it is natural to explore which of the special properties enjoyed by uniform matroids are in fact consequences of having the highest possible "uniformity", and if such properties fall away in a predictable fashion as a matroid's "distance" to being uniform increases. This is the focus of Chapter 5. In that chapter, we further formalise the notion of uniformity and consider its relevance to various other matroid notions and settings. In particular, a number of equivalent characterisations of (k, ℓ)-uniformity are identified.

This part of the thesis is organised as follows. Chapter 4 concerns the representability of (k, ℓ)-uniform matroids. First, in Section 4.1, we prove tight bounds on the rank and corank of a (k, ℓ)-uniform matroid representable over a specific finite field. Then, in Sections 4.2 and 4.3, we explicitly determine all binary $(2, 2)$-uniform matroids. Chapter 5 considers (k, ℓ)-uniformity in a more general setting. First, Section 5.1 details the links between matroid uniformity and linear codes. Section 5.2 then introduces the notion of the *uniform-distance* of a matroid and details the impact of weak and strong maps on this invariant. Finally, Section 5.3 considers uniformity's relevance to two notions associated with Tutte: the Tutte connectivity and Tutte polynomial of a matroid.

Chapter 4

Uniformity over finite fields

This chapter concerns the uniformity of matroids representable over a finite field. In particular, the finiteness result Theorem 3.4.6 is established, and the binary $(2,2)$-uniform matroids are fully determined. The chapter is structured as follows. First, Section 4.1 proves best-possible bounds on the rank and corank of a (k,ℓ)-uniform $GF(q)$-representable matroid, an immediate consequence being Theorem 3.4.6. Section 4.2 then characterises the class of $(2,2)$-uniform matroids and gives an explicit list of those that are binary. Finally, Section 4.3 consists of the proof of Theorem 3.4.8, thus completing the determination of the binary $(2,2)$-uniform matroids.

4.1 Tight bounds on rank and corank

In this section, we present tight upper bounds on the rank and corank of a $GF(q)$-representable (k,ℓ)-uniform matroid. We will make use of the following elementary lemma.

Lemma 4.1.1. *Let M be a $GF(q)$-representable matroid and let $k \leq r(M)$. Then every rank-$(r(M)-k)$ flat of M is contained in at most $(q^k-1)/(q-1)$ flats of rank $r(M)-k+1$.*

Proof. It suffices to consider simple matroids. The result then follows easily from the fact that every simple rank-r $GF(q)$-representable matroid is a restriction of the projective geometry $PG(r-1,q)$, for which the result holds. □

We now prove the following:

Proposition 4.1.2. *Let M be a $GF(q)$-representable (k,ℓ)-uniform matroid. If $r(M) > k$, then*

$$r^*(M) \leq \ell\left(\frac{q^{k+1}-1}{q-1}\right) - (k+1). \tag{4.1.1}$$

Furthermore, if M is not $(k+1, \ell-i)$-uniform for some $0 < i < \ell$, then

$$r^*(M) \leq i\left(\frac{q^{k+1}-1}{q-1}\right) + (\ell - i) - (k+1). \tag{4.1.2}$$

Proof. Letting $r = r(M)$, consider an independent set I of M with size $r-(k+1)$. By Lemma 4.1.1, I is contained in at most $(q^{k+1}-1)/(q-1)$ flats of rank $r-k$. Furthermore, as M is (k,ℓ)-uniform, every such flat has nullity at most $\ell-1$, and thus, has at most ℓ elements not in I. Hence,

$$|E(M)| \leq (r-k-1) + \ell\left(\frac{q^{k+1}-1}{q-1}\right).$$

Moreover, if for some positive integer $i < \ell$, M is not $(k+1, \ell-i)$-uniform, then we may choose I such that $\text{cl}(I)$ has nullity at least $\ell - i$. In this case, each rank-$(r-k)$ flat containing I has at most i elements not in $\text{cl}(I)$. Thus,

$$|E(M)| \leq (r-k-1) + (\ell - i) + i\left(\frac{q^{k+1}-1}{q-1}\right).$$

Both bounds then follow the fact that $|E(M)| = r(M) + r^*(M)$. □

Note that in Proposition 4.1.2, the condition $r(M) > k$ is necessary to avoid the infinitely many $GF(q)$-representable non-simple matroids that are "trivially" (k,ℓ)-uniform by virtue of either having rank less than k, or by having rank k and less than ℓ loops. However, as there are only finitely many simple $GF(q)$-representable matroids up to a certain rank, Theorem 3.4.6 is an immediate consequence.

To see that the bounds of Proposition 4.1.2 are tight, consider first the matroid achieved from $PG(k,q)$ by adding $(\ell-1)$ elements in parallel with every point. This matroid is (k,ℓ)-uniform and has corank meeting bound (4.1.1). Next, for any $i \in \{1,\ldots,\ell-1\}$, consider the matroid achieved from $PG(k,q)$ by first adding $(i-1)$ elements in parallel to every point and then adding $(\ell-i)$ loops. This matroid is (k,ℓ)-uniform but is not $(k+1,\ell-i)$-uniform. Moreover, it has corank meeting bound (4.1.2).

The following is the dual statement of Proposition 4.1.2 and follows the fact that a matroid is (k, ℓ)-uniform if and only if its dual is (ℓ, k)-uniform. As such, the bounds given are also the best possible.

Corollary 4.1.3. *Let M be a $GF(q)$-representable (k, ℓ)-uniform matroid. If $r^*(M) > \ell$, then*

$$r(M) \leq k\left(\frac{q^{\ell+1}-1}{q-1}\right) - (\ell + 1). \tag{4.1.3}$$

Furthermore, if M is not $(k - i, \ell + 1)$-uniform for some $0 < i < k$, then

$$r(M) \leq i\left(\frac{q^{\ell+1}-1}{q-1}\right) + (k - i) - (\ell + 1). \tag{4.1.4}$$

The next proposition concerns (k, ℓ)-uniform matroids for which neither k nor ℓ is 1 and considers rank and corank concurrently. We will make use of the subsequent corollary in Section 4.3 when determining the 3-connected binary $(2, 2)$-uniform matroids.

Proposition 4.1.4. *Let M be a (k, ℓ)-uniform matroid such that $\min\{k, \ell\} \geq 2$. If M is $GF(q)$-representable, then either*

$$r(M) \leq (k-1)\left(\frac{q^{\ell+1}-1}{q-1}\right) - \ell \tag{4.1.5}$$

or,

$$r^*(M) \leq \left(\frac{q^{k+1}-1}{q-1}\right) + (\ell - 1) - (k + 1) \tag{4.1.6}$$

Proof. If either $r(M) \leq k$ or $r^*(M) \leq \ell$, then the result is easily seen to hold. Thus, we may assume that $r(M) > k$ and $r^*(M) > \ell$. Now, if M is $(k - 1, \ell)$-uniform, then, by Corollary 4.1.3,

$$r(M) \leq (k-1)\left(\frac{q^{\ell+1}-1}{q-1}\right) - (\ell + 1)$$

and (4.1.5) holds. Likewise, if M is not $(k + 1, \ell - 1)$-uniform, then (4.1.6) holds by Proposition 4.1.2. Thus, we assume that M is $(k + 1, \ell - 1)$-uniform but not $(k-1, \ell)$-uniform. The latter implies that M has a rank-$(r-k+1)$ flat F of nullity at least ℓ, and the former implies that $M|F$ is $(2, \ell - 1)$-uniform. By another application of Corollary 4.1.3, we have that $r(M|F) \leq 2(q^\ell - 1)/(q - 1) - \ell$, and thus, $r(M) \leq 2(q^\ell - 1)/(q - 1) - \ell + (k - 1)$. It is then routine to show that (4.1.5) holds. □

Corollary 4.1.5. *Let M be a binary matroid. If M is $(2,2)$-uniform, then* $\min\{r(M), r^*(M)\} \leq 5$.

4.2 The binary $(2,2)$-uniform matroids that are not 3-connected

In this section we describe all $(2,2)$-uniform matroids which are not 3-connected and explicitly list those that are binary. The following results contain some redundancy but have been chosen for their clarity and to emphasise links to paving matroids. A matroid M is *sparse paving* if both M and M^* are paving, or equivalently, if M is both $(2,1)$- and $(1,2)$-uniform.

It is easily observed that a matroid M has the property that every rank-$(r(M)-2)$ flat has nullity less than 2 if and only if the union of any pair of circuits of M has rank at least $r(M)-1$. Thus, the latter condition is a further characterisation of $(2,2)$-uniform matroids. We will make repeated of this fact, referring to it as the $(2,2)$-*uniform circuit property.*

Proposition 4.2.1. *Let M be a disconnected matroid. Then M is $(2,2)$-uniform if and only if*

(i) *M or M^* is paving; or*

(ii) *$M \cong M_p \oplus U_{0,1}$ or $M \cong M_p^* \oplus U_{1,1}$, where M_p is a paving matroid; or*

(iii) *$M \cong M_p \oplus U_{1,2}$, where M_p is a sparse paving matroid.*

Proof. The disconnected matroids of type (i), (ii) and (iii) are easily seen to be $(2,2)$-uniform. To see that there are no others, let M be a disconnected $(2,2)$-uniform matroid. If M has a loop l, then $M\backslash l$ is certainly paving and (ii) holds. Otherwise, by duality, we may assume that M has no loops or coloops. It follows that if $r(M) \leq 2$ or $r^*(M) \leq 2$, then (i) holds. Hence, we may also assume that $r(M), r^*(M) \geq 3$. Now, if every component of M has rank, corank at least two, then each component contains at least two circuits and the union of any two such circuits has rank less than $r(M)-1$, a contradiction to the $(2,2)$-uniform circuit property. Thus, up to duality, M has at least one rank-1 component M_1. If $|E(M_1)| \geq 3$, then by the $(2,2)$-uniform circuit property, $r(M) \leq 2$, a contradiction. Thus, $M_1 \cong U_{1,2}$. It then follows easily from the $(2,2)$-uniform circuit property that $M\backslash E(M_1)$ is both $(2,1)$-uniform and $(1,2)$-uniform. In particular, (iii) is satisfied. □

Recall that $P(M_1, M_2)$ denotes the parallel connection of matroids M_1 and M_2 across some common basepoint.

Proposition 4.2.2. *Let M be a connected matroid that is not 3-connected. Then M is $(2,2)$-uniform if and only if*

(i) *M or M^* is paving; or*

(ii) *M or M^* has rank 3 and no parallel class of size more than two; or*

(iii) *M has a parallel or series pair $\{p, p'\}$ such that $M\backslash p/p'$ is sparse paving; or*

(iv) *$M = P(N, U_{2,4})\backslash p$, where N is a connected matroid such that N/p and N^*/p are paving.*

Proof. It is straightforward to show that all matroids of type (i)-(iv) are $(2,2)$-uniform. To see that this list is complete, let M be a connected $(2,2)$-uniform matroid that is not 3-connected. If M has rank or corank at most 3, then it is easily seen to satisfy (i) or (ii). Thus, we may assume that $r(M), r^*(M) \geq 4$. Suppose now that, up to duality, M has a parallel pair $\{p, p'\}$ and let $N = M\backslash p/p'$. If there exists a circuit C of N of rank at most $r(N) - 2$, then as C or $C \cup p'$ is a circuit of M, it follows that $C \cup \{p, p'\}$ contains two circuits of M whose union has rank at most $r(N) - 1 = r(M) - 2$. Similarly, if there exists a pair of circuits C_1, C_2 of N such that $r_N(C_1 \cup C_2) \leq r(N) - 1$, then $C_1 \cup C_2 \cup \{p, p'\}$ contains two circuits of M whose union has rank at most $r(M)-2$. Both situations contradict the fact that M is $(2,2)$-uniform. Hence, N is sparse paving and (iii) holds. Otherwise, M has no parallel or series pairs and we may assume that $M = P(M_1, M_2)\backslash p$, for some connected matroids M_1, M_2 each having at least three elements and rank, corank at least two. If $r(M_1), r(M_2) \geq 3$, then by the $(2,2)$-uniform circuit property, each of $M_1\backslash p$ and $M_2\backslash p$ contains at most one circuit. As each M_i is connected, it follows that for $i \in \{1,2\}$, $M_i\backslash p$ is a circuit and $r^*(M) \leq 3$, a contradiction. Thus, without loss of generality, $r(M_1) = 2$ and $r(M) = r(M_2) + 1$. If $|E(M_1)| \geq 5$, then $E(M_1) - p$ contains two triangles of M, and by the $(2,2)$-uniform circuit property, $r(M) \leq 3$, a contradiction. Thus, $M_1 \cong U_{2,4}$. Now let $T = E(M_1) - p$. By the $(2,2)$-uniform circuit property, $r_M(C \cup T) \geq r(M) - 1 = r(M_2)$ for every circuit C of M_2. It follows that every circuit of M_2 containing p must have rank at least $r(M_2) - 1$ and every circuit avoiding p has rank at least $r(M_2) - 2$. Thus, M_2/p is paving. Also by the $(2,2)$-uniform circuit property, every pair of circuits of $M_2\backslash p$ must span. We conclude that M_2/p and $M_2^*/p = (M_2\backslash p)^*$ are paving and that (iv) holds. □

Restricting our attention to binary matroids, we may ignore case (iv) of Proposition 4.2.2 as such matroids have a $U_{2,4}$-minor. We then achieve the following list by combining Propositions 4.2.1 and 4.2.2 with Acketa's list [1] of binary paving matroids. Note that, as $M(K_4), F_7, F_7^*$ and $AG(3,2)$ have transitive automorphism groups, any parallel connections of these matroids and $U_{2,3}$ are free of reference to a specific basepoint. The matroid S_8 is isomorphic to the unique non-tip deletion of the binary 4-spike Z_4.

Corollary 4.2.3. *The following matroids and their duals are all the binary $(2,2)$-uniform matroids that are not 3-connected.*

(i) *The matroids of rank at most 1 other than $U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}$;*

(ii) *the non-simple rank-2 binary matroids with at most one loop;*

(iii) *the loopless, non-simple rank-3 binary matroids with every parallel class of size at most 2;*

(iv) *$M_p \oplus U_{0,1}$ and $M_p \oplus U_{1,2}$, for M_p in $\{M(K_4), F_7, F_7^*, AG(3,2)\}$;*

(v) *$P(Z_4, U_{2,3})\backslash t$ and $P(S_8, U_{2,3})\backslash t$, where t is the tip of Z_4;*

(vi) *$P(F_7, U_{2,3})\backslash p$ and $P(AG(3,2), U_{2,3})\backslash p$; and*

(vii) *$P(M_p, U_{2,3})$ for M_p in $\{M(K_4), F_7, F_7^*, AG(3,2)\}$.*

4.3 The 3-connected binary $(2,2)$-uniform matroids

In this section we prove Theorem 3.4.8, and in doing so, complete the determination of the binary $(2,2)$-uniform matroids. We also remark that two of the important matroids of this section, P_9 and L_{10}, arise as graft matroids. A *graft*

[32] is a pair (G, γ) where G is a graph and γ is a subset of $V(G)$ thought of as the *coloured* vertices. The associated *graft matroid* is the vector matroid of the matrix obtained by adjoining the incidence vector of the set γ to the vertex-edge incidence matrix of G. We follow [21] in using P_9 to denote the simple binary extension of $M(\mathcal{W}_4)$ represented by the matrix of Figure 4.1. This is isomorphic to the graft of $\mathcal{W}_4$ in which the hub vertex and three of the four rim vertices are coloured. By considering the representation of the matroid P_{10} given in Figure 3.3, we see that P_{10} arises as a single-element coextension of P_9. In fact, it is routine (if tedious) to verify that P_{10} is the 3-sum of P_9 and F_7 across any of the four triangles of P_9 other than $\{1,4,8\}$ and $\{3,4,7\}$. Up to isomorphism, there are two other simple binary extensions of $M(\mathcal{W}_4)$, namely $M(K_5\backslash e)$ and $M^*(K_{3,3})$.

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

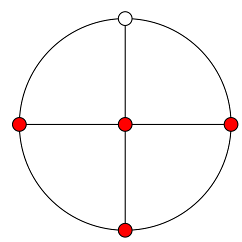


Figure 4.1: A binary representation of P_9 and P_9 as a graft of $\mathcal{W}_4$.

In proving Theorem 3.4.8, we will require the following characterisation of binary matroids with no $M(\mathcal{W}_4)$-minor due to Oxley [21, Theorem 2.1]. Here Z_r is the rank-r binary spike with tip t and y is some non-tip element of Z_r.

Lemma 4.3.1. *Let M be a binary matroid. Then M is 3-connected and has no $M(\mathcal{W}_4)$ minor if and only if*

(i) *$M \cong Z_r, Z_r^*, Z_r\backslash y$, or $Z_r\backslash t$ for some $r \geq 3$; or*

(ii) *$M \cong U_{0,0}, U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}$, or $U_{2,3}$.*

The flats of the rank-r binary spike are very well behaved and the straightforward proof of the following is omitted.

Lemma 4.3.2. *Let $r \geq 3$. The matroids Z_r and $Z_r\backslash y$ are $(2,2)$-uniform if and only if $r \leq 4$. The matroid $Z_r\backslash t$ is $(2,2)$-uniform if and only if $r \leq 5$.*

Now consider the rank-5 binary affine geometry $AG(4,2)$. As its rank-3 flats are all isomorphic to $U_{3,4}$, this matroid is certainly $(2,2)$-uniform. Viewing $AG(4,2)$ as the deletion of a hyperplane H from the projective geometry $PG(4,2)$, we see that every element of H is in a triangle with two elements of $AG(4,2)$. It follows that any rank-5 binary extension of $AG(4,2)$ must have a rank-3 flat of nullity at least 2 and hence fail to be $(2,2)$-uniform. Furthermore, by Corollary 4.1.5, $AG(4,2)$ has no binary $(2,2)$-uniform coextensions. Thus, $AG(4,2)$ is a maximal binary $(2,2)$-uniform matroid. The next lemma concerning binary affine matroids will be used in the proof of Theorem 3.4.8.

Lemma 4.3.3. *Let M be a simple rank-5 binary extension of $M(K_{3,3})$. Then M is $(2,2)$-uniform if and only if M is affine.*

Proof. If M is a simple rank-5 binary affine matroid, then it is a restriction of $AG(4,2)$ and thus is $(2,2)$-uniform. For the other direction, let M be a simple rank-5 binary extension of $M(K_{3,3})$ that is $(2,2)$-uniform. By uniqueness of binary representation, M may be represented by a binary matrix whose first nine columns are the representation of $M(K_{3,3})$ given in Figure 4.2. Let e label an extension column. It is easily seen that if the last entry of column e is zero, then e is in a triangle with two elements of $M(K_{3,3})$. But every pair of elements of $M(K_{3,3})$ are in a circuit of size four. Thus, if column e ends in zero, then e is in a rank-3 flat of M of nullity at least 2. This is a contradiction to the fact that M is $(2,2)$-uniform. We conclude that every extension column ends in 1 and that, consequently, M is affine. □

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

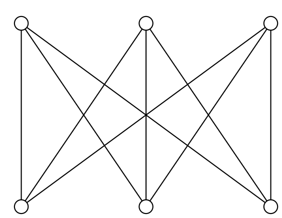


Figure 4.2: Binary and graphic representations for $M(K_{3,3})$.

Two of the four non-isomorphic simple rank-5 binary single-element extensions of $M(K_{3,3})$ are affine. These are the well-known regular matroid R_{10} and a matroid that we name L_{10}, a representation for which is given in Figure 4.3. In [32], R_{10} is identified as the graft matroid of $K_{3,3}$ in which every vertex is coloured. We remark here that L_{10} is the graft matroid of $K_{3,3}$ in which all but two vertices, both in the same partition, are coloured.

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

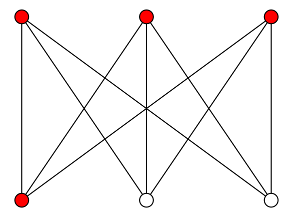


Figure 4.3: A binary representation of L_{10} and L_{10} as a graft of $K_{3,3}$.

In our final step before proving Theorem 3.4.8, we determine the binary $(2, 2)$-uniform coextensions of $M(K_5\backslash e)$ and P_9; geometric representations of which are given in Figure 4.7.

Lemma 4.3.4. *The sets of non-isomorphic binary* $(2, 2)$*-uniform coextensions of* $M(K_5\backslash e)$ *and* P_9*, respectively, are* $\{L_{10}\}$ *and* $\{P_{10}, L_{10}\}$*.*

Proof. Let M be a binary $(2, 2)$-uniform matroid with a subset $X \subseteq E(M)$ such that $M/X \cong N$ for N in $\{M(K_5\backslash e), P_9\}$. By uniqueness of binary representation, we may assume that M/X is represented by the binary matrix A given in Figure 4.4, where $\alpha \in \{0, 1\}$ depends on N.

$$\begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & \alpha \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Figure 4.4: Matrix A. $M[A]$ is isomorphic to $M(K_5\backslash e)$ when $\alpha = 0$ and P_9 when $\alpha = 1$, respectively.

The set $H = \{e_1, e_2, e_3, e_5, e_6, e_9\}$ is a hyperplane of $M[A]$ regardless of α. As M is $(2, 2)$-uniform, it follows that $H \cup X$ is a hyperplane of M of nullity

3. Moreover, $M|H \cup X$ is $(1,2)$-uniform and $(M|H \cup X)^*$ is $(2,1)$-uniform by duality. Thus, $(M|H \cup X)^*$ is a rank-3 simple matroid with $|X| + 6$ elements. It follows that $|X| = 1$. By appropriate row operations, one then sees that M may be represented by the 5×10 binary matrix B as given in Figure 4.5. It remains to determine the coefficients $\beta_5, \ldots, \beta_9$.

$$\begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & x \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 & 1 \end{bmatrix}$$

Figure 4.5: Matrix B. $M[B]/x$ is isomorphic to $M(K_5 \backslash e)$ when $\alpha = 0$ and P_9 when $\alpha = 1$, respectively.

$$\begin{array}{ccccccc} e_5 & e_6 & e_9 & e_1 & e_2 & e_3 & x \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & \beta_5 \\ 0 & 1 & 0 & 0 & 1 & 1 & \beta_6 \\ 0 & 0 & 1 & 1 & \alpha & 1 & \beta_9 \end{bmatrix}$$

$$\begin{array}{cccccc} e_1 & e_3 & e_4 & x & e_7 & e_8 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & \beta_7 & \beta_8 \end{bmatrix}$$

Figure 4.6: Matrices representing $(M|H \cup x)^*$ and $M|H' \cup x$.

A representation for $(M|H \cup x)^*$ is given in Figure 4.6. As this must be simple, we deduce that $\beta_5 = \beta_6 = 1$ and $\beta_9 = 1 - \alpha$. To determine β_7 and β_8, we consider the hyperplane $H' = \mathrm{cl}_M(\{e_1, e_3, e_4\})$ of $M[A]$. If $\alpha = 0$, then the hyperplane $H' \cup x$ of M contains e_9 and by an identical argument to before, $\beta_7 = \beta_8 = 1$. We conclude that if $N \cong M(K_5 \backslash e)$, then $M \cong L_{10}$. Otherwise $N \cong P_9$, $\alpha = 1$ and $H' = \{e_1, e_3, e_4, e_7, e_8\}$. Then $M|H' \cup x$ is represented by the rank-4 matrix of Figure 4.6. As this matroid must be $(1,2)$-uniform, it follows that either $\beta_7 = \beta_8 = 1$, in which case $M \cong L_{10}$, or precisely one of $\{\beta_7, \beta_8\}$ is zero, in which case, $M \cong P_{10}$. □

We now conclude this chapter with the proof of Theorem 3.4.8.

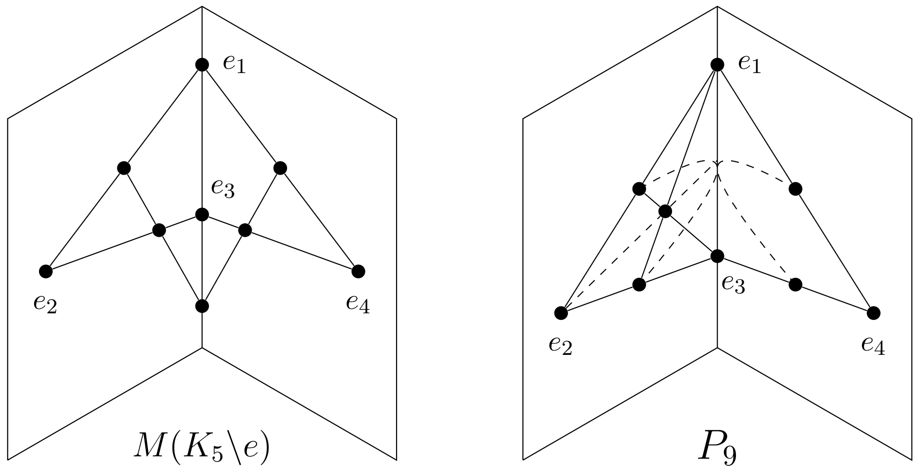


Figure 4.7: Geometric representations of $M(K_5\backslash e)$ and P_9.

Proof of Theorem 3.4.8. We first observe that a matroid is a binary 3-connected $(2,2)$-uniform matroid if and only if its dual is also. In particular, both $AG(4,2)$ and $AG(4,2)^*$ are minor-maximal such matroids. To complete our list, let M be a minor-maximal binary 3-connected $(2,2)$-uniform matroid. If $r(M) \leq 4$, or $r^*(M) \leq 4$, then M is a minor of either $AG(4,2)$ or $AG(4,2)^*$, a contradiction to maximality. Thus, $r(M), r^*(M) \geq 5$. Switching to the dual if necessary, we may then assume by Corollary 4.1.5 that $r(M) = 5$.

If M has no $M(\mathcal{W}_4)$ minor, then by Lemma 4.3.1, M is isomorphic to one of $Z_r, Z_r^*, Z_r\backslash t, Z_r\backslash y$ for some $r \geq 3$ and, by Lemma 4.3.2, $M \cong Z_5\backslash t$. Otherwise, we may assume that M does possess an $M(\mathcal{W}_4)$-minor. Then, as $r(M) = 5$, M is an extension of a single-element coextension N of $M(\mathcal{W}_4)$. As $M(\mathcal{W}_4)$ is self-dual, the matroid N^* is a binary $(2,2)$-uniform single-element extension of $M(\mathcal{W}_4)$. These are just the simple binary extensions of $M(\mathcal{W}_4)$, namely $M(K_5\backslash e)$, P_9 and $M^*(K_{3,3})$. Thus, $N \in \{M^*(K_5\backslash e), P_9^*, M(K_{3,3})\}$. If $N \cong M(K_{3,3})$, then by Lemma 4.3.3, M must be affine and thus, by maximality, $M \cong AG(4,2)$. Otherwise, $N \in \{M^*(K_5\backslash e), P_9^*\}$, in which case, by the dual of Lemma 4.3.4, M is isomorphic to either P_{10} or L_{10}^*. But, as L_{10} is affine, L_{10}^* is a minor of $AG(4,2)^*$. We conclude by maximality that, in this case, $M \cong P_{10}$. The theorem then follows by duality. □

Chapter 5

Uniformity in context

The purpose of this chapter is to highlight the fundamental nature of generalised uniformity by describing its relevance to several selected matroid notions and settings. The chapter is structured as follows. Firstly, Section 5.1 details the links between generalised uniformity and the study of linear codes. In Section 5.2, we further formalise the notion of "how uniform" a given matroid is by defining the *uniform-distance* of a matroid, before considering the relevance of this invariant to weak and strong maps. In particular, a characterisation of (k, ℓ)-uniform matroids by way of certain quotients and lifts is given. Lastly, Section 5.3 considers the role that uniformity plays in two notions associated Tutte: namely, Tutte connectivity and the Tutte polynomial. A characterisation of (k, ℓ)-uniformity in terms of the latter is given. We omit much of the rich background behind each topic and refer the interested reader to [19, 20] for a treatment of weak and strong maps, to [7] for a discussion on the Tutte polynomial, and to [16] for a discussion on linear codes and their links to matroids.

5.1 Linear codes

In this section, we consider the applications of our work thus far to the study of linear codes. For positive integers n, r and d, an $[n, r, d]$ *linear code* over a field $\mathbb{F}$ is a rank-r subspace of $V(n, \mathbb{F})$ such that the hamming distance between any two vectors (*codewords*) of this subspace is at least d. Let $\mathfrak{C}$ be an $[n, r, d]$ linear code over $\mathbb{F}$. A *generator matrix* for $\mathfrak{C}$ is any $r \times n$ matrix over $\mathbb{F}$ whose row space is $\mathfrak{C}$. It is an easy exercise to show that, up to labelling, every generator matrix for

$\mathfrak{C}$ gives rise to the same vector matroid, which we denote as $M_{\mathfrak{C}}$. The *dual code* $\mathfrak{C}^{\perp}$ of $\mathfrak{C}$ is the linear code of length n and rank $n-r$ consisting of all vectors $v \in V(n,q)$ such that $u \cdot v = 0$ for all $u \in \mathfrak{C}$. The generator matrices for $\mathfrak{C}^{\perp}$ are the *parity check matrices* for $\mathfrak{C}$ and it is easily checked that $M_{\mathfrak{C}^{\perp}} = (M_{\mathfrak{C}})^*$.

We call $\mathfrak{C}$ a (k,ℓ)-*uniform code* if the matroid $M_{\mathfrak{C}}$ is (k,ℓ)-uniform. By duality, $\mathfrak{C}$ is a (k,ℓ)-uniform code if and only if $\mathfrak{C}^{\perp}$ is an (ℓ,k)-uniform code. It is easy to show that the minimum distance between any two codewords of $\mathfrak{C}$ is the size of the smallest non-zero codeword of $\mathfrak{C}$, or equivalently, the size of the smallest circuit of $M_{\mathfrak{C}^{\perp}}$. As the matroid $M_{\mathfrak{C}^{\perp}}$ has rank $r^* = n-r$, the well-known *singleton bound* [31] $d \leq r^* + 1$ is an immediate consequence of this fact. In particular, the singleton bound is met if and only if $M_{\mathfrak{C}^{\perp}}$ (and hence $M_{\mathfrak{C}}$) is uniform. Thus, in the language of coding theory, the $(1,1)$-uniform codes are precisely the *maximum distance separable* (MDS) codes. A small extension of this observation is that the distance of a linear code is always determined by (k,ℓ)-uniformity where $k = 1$.

Lemma 5.1.1. *Let $\mathfrak{C}$ be an $[n,r,d]$ linear code over some field $\mathbb{F}$ and let $r^* = n-r$. Then $d \geq r^* - \ell + 2$ if and only if $\mathfrak{C}$ is a $(1,\ell)$-uniform code.*

Proof. The matroid $M_{\mathfrak{C}^{\perp}}$ has rank r^*. Thus, every circuit of $M_{\mathfrak{C}^{\perp}}$ has rank at least $r^* - \ell + 1$ if and only if $M_{\mathfrak{C}^{\perp}}$ is $(\ell,1)$-uniform. By duality, the latter occurs if and only if $M_{\mathfrak{C}}$ is $(1,\ell)$-uniform. □

Now, given any $[n,r,d]$ linear code, one may treat the integer

$$t = (n-r) + 1 - d$$

as that code's offset from obtaining the singleton bound. For example, the MDS codes are those with a zero such t-value. The next lemma uses the results of Section 4.1 to give a bound on the rank of a linear code in terms of t. The condition $(n-r) > 1$ is necessary as the linear code associated with the matroids $U_{n,n}$ or $U_{n-1,n}$ is MDS for any n.

Lemma 5.1.2. *Let $\mathfrak{C}$ be an $[n,r,d]$ linear code over $GF(q)$ such that $(n-r) > 1$ and let $t = (n-r) + 1 - d$. Then*

$$r \leq \frac{q^{t+2} - 1}{q - 1} - (t+2).$$

Proof. Letting $r^* = n - r$, we have that $d = r^* + 1 - t$. Now, as every circuit of $M_{\mathfrak{C}^{\perp}}$ has size at least d, every such circuit has rank at least $d - 1 = r^* - t$.

Equivalently, $M_{\mathfrak{C}^\perp}$ is $(t+1,1)$-uniform. The result then follows an application of Proposition 4.1.2. □

A routine expansion and rearrangement of the terms from Lemma 5.1.2 yields the following upper bound on the rank of a linear code of a fixed length and distance.

Corollary 5.1.3. *Let $\mathfrak{C}$ be an $[n,r,d]$ linear code over $GF(q)$ such that $(n-r) > 1$. Then,*

$$r \leq (n-d+3) - \log_q\left[(q-1)(n-d+3)+1\right].$$

An $[n,r,d]$ linear code over $GF(q)$ has been called *optimal* [36] if there exists no $[n,r',d]$ linear code over $GF(q)$ such that $r < r'$. Thus, Corollary 5.1.3 gives an upper bound of the rank of such a code.

The remainder of this section briefly considers the impact of matroid uniformity on linear codes more generally. We will make use of the following three lemmas. The first can be found in [26], while the second and third are both elementary consequences of Proposition 3.4.9 and their proofs are omitted. For any field $\mathbb{F}$, the *support* of a vector $\underline{v} = (v_1, v_2, \ldots, v_n)$ from $V(n,\mathbb{F})$ is the set $\{i : v_i \neq 0\}$.

Lemma 5.1.4 ([26], Proposition 9.2.4)**.** *Let A be an $m \times n$ matrix over a field $\mathbb{F}$ and let $M = M[A]$. Then the set of cocircuits of M coincides with the set of minimal non-empty supports of vectors from the row space of A.*

Lemma 5.1.5. *A matroid M is (k,ℓ)-uniform if and only if every subset $X \subseteq E(M)$ of size at least $r(M)-k+\ell$ has rank at least $r(M)-k+1$.*

Lemma 5.1.6. *A matroid M is (k,ℓ)-uniform if and only if every subset $X \subseteq E(M)$ of size at most $r^*(M)-\ell+k$ has corank at least $|X|-k+1$.*

The *weight* of a codeword is its number of non-zero entries, or equivalently, the size of its support. Lemma 5.1.1 establishes that an $[n,r,d]$ linear code is $(1,\ell)$-uniform if and only if it has no non-zero codewords of weight less than $(n-r)-\ell+2$. The following proposition, though perhaps unenlightening, is the natural extension of that result, and characterises (k,ℓ)-uniform codes.

Proposition 5.1.7. *Let $\mathfrak{C}$ be an $[n,r,d]$ linear code over some field $\mathbb{F}$ and let $r^* = n-r$. Then $\mathfrak{C}$ is (k,ℓ)-uniform if and only if every subset X of $\{1,2,\ldots,n\}$*

with size at most $r^*-\ell+k$ *has a subset of size* $|X|-k+1$ *containing no non-empty support of a codeword of* $\mathfrak{C}$.

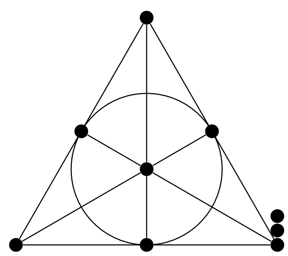
Proof. Let A be a generator matrix for $\mathfrak{C}$ with columns labelled $(1, 2, \ldots, n)$ in order. Suppose that $M[A]$ is (k, ℓ)-uniform and let X be a subset of $\{1, 2, \ldots, n\}$ with size at most $r^* - \ell + k$. Let X' be a cobasis for X in $M[A]$. By Lemma 5.1.6 it must be that $|X'| \geq |X| - k + 1$. If X' contains a support for a codeword of $\mathfrak{C}$, then by by Lemma 5.1.4, X' contains a cocircuit of $M[A]$, a contradiction. Thus X' contains no such support. Conversely suppose $M[A]$ is not (k, ℓ)-uniform. Then by Lemma 5.1.6, there is a a subset X of $\{1, 2 \ldots, n\}$ of size at most $r^* - \ell + k$ for which every subset of size $|X| - k + 1$ is codependent. Equivalently, by Lemma 5.1.4, every size $|X| - k + 1$ subset of X contains a support of a codeword of $\mathfrak{C}$. □

We end this section by observing the following property of $(2, 2)$-uniform linear codes.

Proposition 5.1.8. *Let* $\mathfrak{C}$ *be an* $[n, r, d]$ *linear code over some field* $\mathbb{F}$ *and let* $r^* = n - r$. *If* $\mathfrak{C}$ *is* $(2, 2)$*-uniform, then the supports of the codewords of* $\mathfrak{C}$ *with weight at most* r^* *are precisely the non-cospanning cocircuits of* $M_{\mathfrak{C}}$.

Proof. Let A be a generator matrix for $\mathfrak{C}$ with columns labelled $(1, 2, \ldots, n)$ in order. Let $\mathcal{W}$ be the set of supports of codewords of $\mathfrak{C}$ with weight at most r^*. By Lemma 5.1.4, every cocircuit of $M[A]$ is a support of a codeword of $\mathfrak{C}$. In particular, every non-cospanning cocircuit of $M[A]$ is in $\mathcal{W}$. Now observe that for all $X \in \mathcal{W}$, the set $E - X$ must be a flat of size at least r. If any such flat is not a hyperplane, then $\mathfrak{C}$ fails to be $(2, 2)$-uniform by Lemma 5.1.5. Thus, every such flat is a hyperplane and the result follows. □

To see that the converse of Proposition 5.1.8 does not hold, consider the binary matroid obtained from F_7 by the addition of two elements in parallel with some chosen point. Geometric and binary representations of this matroid are given in Figure 5.1. The associated binary code has the property that the codewords with weight at most $r^* = 6$ are precisely the incidence vectors of the non-cospanning cocircuits of the matroid. However, due to the 3-element parallel class, this matroid is not $(2, 2)$-uniform. Indeed, this is an example of the broader fact that, while the complement of the support of a codeword of $\mathfrak{C}$ must be a flat of $M_{\mathfrak{C}}$, there may exist flats of $M_{\mathfrak{C}}$ whose complements are not supports for codewords of $\mathfrak{C}$. As such, obtaining cleaner characterisations than Proposition 5.1.7 may prove resistive.



$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Figure 5.1: Geometric and binary representations of $P(F_7, U_{1,3})$.

5.2 Uniform-distance and maps

In this section, we introduce the uniform-distance of a matroid and consider the place of this invariant in the context of strong and weak maps.

Uniform-distance

We define the *uniform-distance* $\mathfrak{ud}(M)$ of a matroid M as

$$\mathfrak{ud}(M) = \min\{k + \ell : M \text{ is } (k, \ell)\text{-uniform}\} - 2.$$

Here, the "-2" term ensures that uniform matroids have a uniform-distance of zero. Evidently, this quantity is an invariant under isomorphism and duality. Moreover, it is non-increasing under minors. In particular, it is easy to show that, for any matroid M with element e,

$$\mathfrak{ud}(M) \geq \mathfrak{ud}(M \backslash e), \mathfrak{ud}(M/e) \geq \mathfrak{ud}(M) - 1.$$

Furthermore, as every matroid M is trivially $(r(M) + 1, 1)$-uniform, we have the upper bound

$$\mathfrak{ud}(M) \leq \min\{r(M), r^*(M)\}.$$

We next observe that uniform-distance increases linearly with rank for a few well-known and important classes of 3-connected matroids that arise frequently as examples or counterexamples in matroid problems of connectivity and representability. Defined for all $r \geq 3$, the rank-r *free swirl* Ψ_r is the matroid obtained from the rank-r whirl $\mathcal{W}^r$ by first freely adding an element to each 3-point line, then deleting the spoke elements.

Lemma 5.2.1. *Let $r \geq 3$. Then $\mathfrak{ud}(M(\mathcal{W}_r)) = \mathfrak{ud}(\mathcal{W}_r) = r-2$ and $\mathfrak{ud}(\Psi_r) = r-3$. Moreover, $M(\mathcal{W}_r)$ and $\mathcal{W}^r$ are (k,ℓ)-uniform if and only if $r \leq k+\ell$, and Ψ_r is (k,ℓ)-uniform if and only if $r \leq (k+\ell)+1$.*

Proof. The matroids $M(\mathcal{W}_r)$, $\mathcal{W}^r$ and Ψ_r are all simple, or equivalently, $(r-1,1)$-uniform. Now let $1 \leq k \leq r-2$. For both $M(\mathcal{W}_r)$ and $\mathcal{W}^r$, the rank-$(r-k)$ flats of maximum nullity are the closed rank-$(r-k)$ fans and these have nullity $r-k-1$. Considering the construction of Ψ_r from $\mathcal{W}^r$, the rank-$(r-k)$ flats of Ψ_r of maximum nullity are seen to be the sets of $2(r-k-1)$ elements in the span of $r-k$ consecutive spoke elements before deletion of those spokes. Such a flat has nullity $r-k-2$. The result then follows Proposition 3.4.9. □

Defined for all $r \geq 3$, an *r-spike with tip t* is a rank-r matroid whose groundset is the union of r triangles $L_1, L_2, \ldots, L_r$ called legs, all of which contain the element t, and, for all $1 \leq k \leq r-1$, the union of any k legs has rank at least $k+1$. Such a spike is *free* if its non-spanning circuits are all of the form L_i or $L_i \cup L_j - \{t\}$ where $i, j \in \{1, 2, \ldots, r\}$.

Lemma 5.2.2. *For $r \geq 3$, let S_r be an r-spike with tip t and non-tip element x. Then $\mathfrak{ud}(S_r) = \mathfrak{ud}(S_r \backslash x) = r-2$ and, provided S_r is not the free 3-spike, $\mathfrak{ud}(S_r \backslash t) = \max\{r-3, 1\}$. Moreover, S_r and $S_r \backslash x$ are (k,ℓ)-uniform if and only if $r \leq k+\ell$ and, if $r \geq 4$, then $S_r \backslash t$ is (k,ℓ)-uniform if and only if $r \leq (k+\ell)+1$.*

Proof. As S_r is simple, it is $(r-1,1)$-uniform. Suppose $r=3$. Then, as every line of S_r has at most three elements, S_r is both $(1,2)$ and $(2,1)$-uniform. Furthermore, $S_r \backslash t$ is $(1,1)$-uniform if and only if S_r is the free 3-spike. Now, suppose $r \geq 4$ and let $2 \leq s \leq r-1$. Let F be a collection of $s-1$ legs of S_r not containing x. In S_r and $S_r \backslash x$, F is a rank-s flat with maximum possible nullity, this being $s-1$. In $S_r \backslash t$, the set $F \backslash t$ is a rank-s flat with maximum possible nullity, $s-2$. The result then follows Proposition 3.4.9. □

Many of the results of the previous chapter may be expressed in terms of uniform-distance. We present one such result here, a straightforward strengthening of Corollary 3.4.7.

Theorem 5.2.3. *For every prime power q, the set of excluded minors for $GF(q)$-representability is finite if and only if, for some non-negative integer σ_q, every such excluded minor has uniform-distance at most σ_q.*

Strong and weak maps

This section concerns uniformity's role in maps between matroids of the same size. Let M_1 and M_2 be matroids on groundsets E_1 and E_2 respectively such that $|E_1| = |E_2|$. A bijection $\phi : E_1 \mapsto E_2$ is a *strong map* from M_1 to M_2 if for every flat F of M_2, its preimage $\phi^{-1}(F)$ is a flat in M_1. When $E_1 = E_2$ and the identity map is a strong map from M_1 to M_2, then M_2 is called a *quotient* of M_1. Equivalently, M_2 is a quotient of M_1 if there exists a matroid N with a subset X such that $N \backslash X = M_1$ and $N/X = M_2$. Such a quotient is *elementary* if $|X| = 1$. We first focus on a particular type of elementary quotient. For a matroid M of non-zero rank, the *truncation* $T(M)$ of M is achieved from M by first freely extending by an element then contracting this extension element. The collection of flats of $T(M)$ is easily seen to be all the flats of M other than the hyperplanes. Moreover, truncation has a pleasant interpretation in terms of the lattice of flats, as the geometric lattice for $T(M)$ is achieved from that of M by removing all copoints of the latter while ensuring that $E(M)$ remains the unique maximal element. For a matroid M of rank zero, we define $T(M)$ to be M. For any matroid M and positive integer i, the i'th truncation of M is defined inductively as $T^i(M) = T(T^{i-1}(M))$ where $T^0(M) = M$.

Lemma 5.2.4. *Let M be a matroid and let (k, ℓ) be a pair of positive integers. Then M is (k, ℓ)-uniform if and only if $T^i(M)$ is $(k - i, \ell)$-uniform for all $0 \leq i \leq k - 1$.*

Proof. If M has rank less than k, then, for all $0 \leq i \leq k - 1$, the matroid $T^i(M)$ has rank less than $k - i$ and the result holds trivially. Otherwise, $r(M) \geq k$. Letting $i \in \{0, \ldots, k-1\}$, the matroid $T^i(M)$ has rank $r' = r(M) - i$. Moreover, the rank-$(r' - k + i)$ flats of $T^i(M)$ are the rank $r(M) - k$ flats of M. The lemma is then a direct consequence of Proposition 3.4.9. □

Following [26], we define the *Higgs lift* $L(M)$ of a matroid M to be the matroid obtained from M by first taking the free coextension and then deleting the coextension element. Equivalently, this is the dual operation to truncation in the sense that $L(M) = (T(M^*))^*$. In direct analogy with truncation, the i'th Higgs lift is defined inductively as $L^i(M) = L(L^{i-1}(M))$ where $L^0(M) = M$. The next lemma is the dual of Lemma 5.2.4 and follows directly from the fact that a matroid M is (k, ℓ)-uniform if and only if M^* is (ℓ, k)-uniform.

Lemma 5.2.5. *Let M be a matroid and (k, ℓ) be a pair of positive integers. Then M is (k, ℓ)-uniform if and only if $L^i(M)$ is $(k, \ell - i)$-uniform for all $0 \leq i \leq \ell - 1$.*

Evidently, $T(U_{r,n}) = U_{r-1,n}$ whenever $n \geq r \geq 1$. As such, the truncation (or dually, Higgs lift) of a uniform matroid is also uniform. Indeed, the uniform matroids of size n were historically defined [6] as the matroids obtained from the free matroid $U_{n,n}$ by successive truncations. We now present the following characterisation of (k, ℓ)-uniform matroids in terms of truncations and Higgs lifts.

Proposition 5.2.6. *Let M be a matroid and let (k, ℓ) be a pair of positive integers. Then M is (k, ℓ)-uniform if and only if the matroid achieved from M after $k - 1$ truncations and $\ell - 1$ Higgs lifts is uniform.*

Proof. A matroid is uniform if and only if it is $(1, 1)$-uniform. By Lemma 5.2.4, if $k \geq 2$, then a matroid M is (k, ℓ)-uniform if and only if $T(M)$ is $(k-1, \ell)$-uniform. Dually, by Lemma 5.2.5, if $\ell \geq 2$, then a matroid M is (k, ℓ)-uniform if and only if $L(M)$ is $(k, \ell - 1)$-uniform. Repeated applications of these observations yields the result. □

Observe that in the statement of Proposition 5.2.6, no order is imposed on the sequence of truncations and Higgs lifts performed. This is in fact an instance of a more general phenomena. Letting M be a matroid of rank greater than k and corank greater than ℓ, it is routine to show that there is a unique matroid obtained from M after performing k truncations and ℓ Higgs lifts: namely, the matroid whose bases are all the sets of the form $(B - X) \cup Y$ where B is a basis of M, X is a k-element subset of B and Y is an ℓ-element subset of $E(M) - (B - X)$. In this sense, the operations of truncation and Higgs lifts commute.

An immediate consequence of Proposition 5.2.6 is the following characterisation of uniform-distance:

Corollary 5.2.7. *Let M be a matroid and let γ be a non-negative integer. Then $\mathfrak{ud}(M) \leq \gamma$ if and only there is a sequence of γ operations, each either a truncation or a Higgs lift, such that after starting with M and successively applying each operation, the resulting matroid is uniform.*

Now again, let M_1 and M_2 be matroids on groundsets E_1 and E_2 respectively such that $|E_1| = |E_2|$. A bijection $\phi : E_1 \mapsto E_2$ is a *weak map* [19] from M_1 to M_2 if for every independent set X of M_2, its preimage $\phi^{-1}(X)$ is independent in

M_1. It is easily seen that every strong map is a weak map. Furthermore, it is well known [19] that any weak map between two matroids of the same size can be uniquely factored into a number of truncations followed by a rank-preserving weak map. It is a consequence of the next two results that if such a weak map decreases uniform-distance, it must do so at the truncation stage, while if an increase in uniform-distance occurs, this must be forced by the respective rank-preserving weak map.

Lemma 5.2.8. *Let M be a matroid. Then*

$$\mathfrak{ud}(T(M)) \in \{\mathfrak{ud}(M), \mathfrak{ud}(M) - 1\}$$

Proof. By Lemma 5.2.4, $T(M)$ is (k, ℓ)-uniform if and only if M is $(k+1, \ell)$-uniform. Thus, $\mathfrak{ud}(T(M) = \mathfrak{ud}(M) - 1$, unless M is $(1, \ell)$-uniform for some positive integer ℓ such that M is not (k', ℓ')-uniform for any pair (k', ℓ') of positive integers such that $k' \geq 2$ and $k' + \ell' \leq \ell + 1$. However, as M is $(2, \ell)$-uniform, $T(M)$ is $(1, \ell)$-uniform. Thus, in this exceptional case, $\mathfrak{ud}(T(M)) = \mathfrak{ud}(M)$. □

Lemma 5.2.9. *Let M_1 and M_2 be matroids of the same size and rank such that M_2 is a weak map image of M_1. Then $\mathfrak{ud}(M_1) \leq \mathfrak{ud}(M_2)$. Moreover, for any pair (k, ℓ) of positive integers, if M_2 is (k, ℓ)-uniform, then M_1 is also (k, ℓ)-uniform.*

Proof. Let ϕ be a weak map from M_1 to M_2 and let $r = r(M_1) = r(M_2)$. Suppose that M_2 is (k, ℓ)-uniform and let F be a rank-$(r-k)$ flat of M_1. As ϕ is a weak map, $\phi(F)$ has rank at most $r - k$ in M_2. Then, as M_2 is (k, ℓ)-uniform, $\phi(F)$ has nullity less than ℓ in M_2 and thus has size less than $r - k + \ell$. Thus, F has nullity less than ℓ in M_1. As the choice of F was arbitrary, we conclude that M_1 is (k, ℓ)-uniform and the lemma holds. □

In [26] a matroid M_1 is said to be "freer" than M_2 if M_2 is a rank-preserving weak map image of M_1. We conclude this section by remarking that the last lemma supports the intuitive notion that if a matroid is freer than another, then it is at least as uniform.

5.3 Tutte connectivity and the Tutte polynomial

In this section, we detail the role that uniformity plays in two important matroid notions associated with Tutte: namely, Tutte connectivity and the Tutte polynomial.

Tutte Connectivity

We follow Oxley [26] in defining the *Tutte connectivity* $\tau(M)$ of a matroid M as $\tau(M) = \min\{j : M \text{ has a j-separation}\}$ provided M has a t-separation for some $t \geq 1$, or ∞ otherwise. It was observed in [18] that the only matroids having infinite Tutte connectivity are uniform. Furthermore, the connectivity of uniform matroids is easily determined.

Lemma 5.3.1. *If M is a uniform matroid, then*

$$\tau(M) = \begin{cases} r(M)+1 & \text{if } r^*(M) \geq r(M)+2 \\ r^*(M)+1 & \text{if } r^*(M) \leq r(M)-2 \\ \infty & \text{if } |r(M)-r^*(M)| < 2 \end{cases}$$

In particular, uniform matroids have no separations of order at most $\min\{r(M), r^*(M)\}$. We will consider the affect of uniformity on the connectivity of a matroid more generally. In proving the next proposition, we will make use of the following three lemmas. The proof of the first and third lemmas are elementary. The second follows from the first and Proposition 3.4.9. We denote the nullity and conullity of a subset X as $\text{null}(X)$ and $\text{null}^*(X)$ respectively.

Lemma 5.3.2. *If X is a subset of a matroid M with groundset E, then*

$$r(M) - r(X) = \text{null}^*(E - X)$$

Lemma 5.3.3. *A matroid M on groundset E is (k, ℓ)-uniform if and only if there is no subset $X \subseteq E$ such that* $\text{null}(X) \geq \ell$ *and* $\text{null}^*(E - X) \geq k$.

Lemma 5.3.4. *If X is a subset of a matroid M with groundset E, then*

$$\lambda(X) = r(M) - \text{null}^*(X) - \text{null}^*(E - X).$$

The next proposition captures the behaviour of the connectivity function in a (k, ℓ)-uniform matroid.

Proposition 5.3.5. *Let M be a (k, ℓ)-uniform matroid and let $\{X, E - X\}$ be a partition of the groundset E such that $|X| \leq |E - X|$. Then either*

(i) $\lambda(X) \geq r^*(M) - 2(\ell - 1)$,

(ii) $\lambda(X) \geq r(M) - 2(k - 1)$*; or*

(iii) $\mathrm{null}(X) < \ell \leq \mathrm{null}(E - X)$ *and* $\mathrm{null}^*(X) < k \leq \mathrm{null}^*(E - X)$.

Furthermore, if only (iii) holds, then

$$|X| - (k + \ell) + 2 \leq \lambda(X) \leq |E - X| - (k + \ell).$$

Proof. If $\mathrm{null}(X) < \ell$ and $\mathrm{null}(E - X) < \ell$, then (i) holds by Lemma 5.3.4. Dually, if $\mathrm{null}^*(X) < k$ and $\mathrm{null}^*(E - X) < k$ then (ii) holds. Otherwise, we may assume that $\mathrm{null}(A) \geq \ell$ for some $A \in \{X, E - X\}$. As M is (k, ℓ)-uniform, it follows Lemma 5.3.3 that $\mathrm{null}^*(E - A) < k$. By the above, this implies that $\mathrm{null}^*(A) \geq k$. A further application of Lemma 5.3.3 implies that $\mathrm{null}(E - A) < \ell$. Thus,

$$\mathrm{null}(E - A) < \ell \leq \mathrm{null}(A) \text{ and } \mathrm{null}^*(E - A) < k \leq \mathrm{null}^*(A).$$

Now, using the fact that $\lambda(Y) = |Y| - \mathrm{null}(Y) - \mathrm{null}^*(Y)$ for every subset $Y \subseteq E(M)$, one then achieves that

$$|A| - (k + \ell) \geq \lambda(A) = \lambda(E - A) \geq |E - A| - (k + \ell) + 2.$$

In particular, $|A| \geq |E - A| + 2$, from which we deduce that $A = E - X$ and the result follows. □

To see that the situation described in part (iii) Proposition 5.3.5 does occur and that the inequalities given are sharp, consider the 8-element rank-4 matroid shown in Figure 5.2 achieved as the parallel connection of $M(K_4)$ and $U_{2,3}$. As every line of this rank-4 matroid has nullity less than two, it is $(2, 2)$-uniform. Letting E be the groundset of this matroid and letting $X = \{x_1, x_2, x_3\}$, we see that $(X, E - X)$ is a 2-separation meeting the bounds of Proposition 5.3.5 part (iii). In particular, $\mathrm{null}(X) = \mathrm{null}^*(X) = 1$, $\mathrm{null}(E - X) = \mathrm{null}^*(E - X) = 2$ and $\lambda(X) = |X| - 2 = |E - X| - 4 = 1$.

For (k, ℓ)-uniform matroids where $k = 1$, we may exclude case (iii) of Proposition 5.3.5 by restricting to vertical connectivity.

Lemma 5.3.6. *If M is a $(1, \ell)$-uniform matroid, then it has vertical connectivity at least*

$$\min\{r(M) - 2, r^*(M) - 2(\ell - 1)\}$$

Proof. Assume to the contrary, that M has a vertical t separation (X, Y) for some $t < \min\{r^*(M) - 2(\ell - 1), r(M) - 2\}$. Then by Proposition 5.3.5, the smallest side of this separation is coindependent. This is a contradiction as both X and Y must contain a cocircuit. □

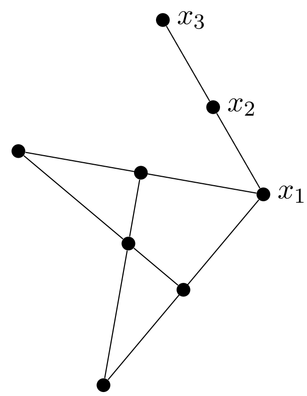


Figure 5.2: $P(M(K_4), U_{2,3})$

The main consequence of Proposition 5.3.5, articulated by the next lemma, is that if M is a (k, ℓ)-uniform matroid, then any separation of M of sufficiently low order with respect to its rank and corank must have a comparatively "small" side.

Lemma 5.3.7. *Let M be a (k, ℓ)-uniform matroid and let n be a positive integer such that $r(M) \geq n + 2(k-1)$ and $r^*(M) \geq n + 2(\ell - 1)$. Then any n-separation (X, Y) of M must satisfy*

$$n \leq \min\{|X|, |Y|\} < n + (k + \ell - 2).$$

Proof. Follows easily from Proposition 5.3.5 and the fact that $\lambda(X) \leq n - 1 < \min\{r(M) - 2(k-1), r^*(M) - 2(\ell - 1)\}$. □

We end our analysis with the following natural corollary regarding uniform-distance.

Corollary 5.3.8. *Let M be a matroid and let n be a positive integer. If $r(M), r^*(M) \geq n + 2 \cdot \mathfrak{ud}(M)$, then any n-separation (X, Y) of M must satisfy*

$$n \leq \min\{|X|, |Y|\} < n + \mathfrak{ud}(M).$$

The Tutte polynomial

The *Tutte polynomial* [7] of a matroid M is defined as

$$T(M;x,y) = \sum_{X \subseteq E(M)} (x-1)^{r(M)-r(X)}(y-1)^{|X|-r(X)}$$

The study of this polynomial is the core of matroid invariant theory and is associated with Tutte due to his extensive use of this polynomial, primarily in the setting of graphs [33]. Brylowski [5] showed that the Tutte polynomial is, in a very natural sense, *the* matroid invariant as any matroid invariant satisfying certain recursive behaviour (so called *Tutte-Grothendieck invariants* [7]) must be an evaluation of this polynomial. Many long-standing problems in matroid theory have formulations in terms of this polynomial; perhaps most notably, the problem of finding the *critical exponent* (see [26, 36]) of a matroid representable over some field. We refer the interested reader to Oxley and Brylowski's paper [7] for a detailed treatment of a number of such problems.

The Tutte polynomial of a uniform matroid is easily seen to be

$$T(U_{r,n}, x, y) = \sum_{i=0}^{r-1} \binom{n}{i}(x-1)^{r-i} + \binom{n}{r} + \sum_{i=r+1}^{n} \binom{n}{i}(y-1)^{i-r}$$

Furthermore, the situation is complicated only slightly in the case of matroids that are both $(1,2)$- and $(2,1)$-uniform, as it is routine to show that if M is a rank-r sparse paving matroid on n elements, then

$$T(M, x, y) = \sum_{i=0}^{r-1} \binom{n}{i}(x-1)^{r-i} + b + c \cdot (x-1)(y-1) + \sum_{i=r+1}^{n} \binom{n}{i}(y-1)^{i-r}$$

where b is the number of bases of M and $c = \binom{n}{r} - b$.

We will detail the link between uniformity and the Tutte polynomial by first describing uniformity's link to a closely related polynomial. The (Whitney) *rank-generating polynomial* $R(M;x,y)$ of a matroid M is defined as

$$R(M;x,y) = \sum_{X \subseteq E(M)} x^{r(M)-r(X)} y^{|X|-r(X)}.$$

The Tutte polynomial is then achieved as

$$T(M;x,y) = R(M;x-1,y-1).$$

As detailed by the next lemma, matroid uniformity corresponds directly to the absence of terms in the rank-generating polynomial.

Lemma 5.3.9. *Let M be a matroid and let $R(M;x,y) = \sum_i \sum_j b_{ij}x^i y^j$. Then the following are equivalent:*

(i) *M is (k,ℓ)-uniform.*

(ii) $b_{k\ell} = 0$

(iii) $b_{ij} = 0$ *for all $i \geq k$, $j \geq \ell$.*

Proof. By Proposition 3.4.9, a matroid M is (k,ℓ)-uniform if and only if M has no rank-$(r(M)-k)$ set of nullity ℓ. The latter occurs precisely when each subset of M with rank at most $r(M)-k$ has nullity less than ℓ. These properties correspond to conditions (i), (ii) and (iii) respectively. □

By a straightforward consideration of the change of variables that occurs between the rank-generating polynomial and the Tutte polynomial, one achieves the following characterisations of uniformity in terms of the latter.

Lemma 5.3.10. *Let M be a matroid and let $T(M;x,y) = \sum_i \sum_j a_{ij}x^i y^j$. Then the following are equivalent:*

(i) *M is (k,ℓ)-uniform.*

(ii) $\frac{\partial T(M;x,y)}{\partial^k x\ \partial^\ell y} = 0$.

(iii) $a_{ij} = 0$ *for all $i \geq k$, $j \geq \ell$.*

We remark that, unlike as is the case for the rank-generating polynomial, the (k,ℓ)'th coefficient of the Tutte polynomial may be zero for matroids that are not (k,ℓ)-uniform. To see this, consider the matroid $U_{m,m} \oplus U_{0,n}$, where m and n are positive integers not both equal to one. It is easily seen that

$$T(U_{m,m} \oplus U_{0,n}; x, y) = x^m y^n.$$

In particular, $a_{11} = 0$ but $U_{m,m} \oplus U_{0,n}$ is not $(1,1)$-uniform.

A function f from the class of matroids to some set Ω is a *Tutte invariant* [7] if $f(M) = f(N)$ whenever M and N have the same Tutte polynomial. It is an immediate consequence of Lemma 5.3.10 that (k,ℓ)-uniformity is a Tutte invariant.

Corollary 5.3.11. *For all matroids M, let*

$$f(M) = \{(k, \ell) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : M \text{ is } (k, \ell)\text{-uniform}\}.$$

If M_1 and M_2 are two matroids with the same Tutte polynomial, then $f(M_1) = f(M_2)$.

Indeed, one can relax Corollary 5.3.11 to any two matroids whose Tutte polynomials have the same zero coefficients. The next result regarding uniform-distance follows by combining Lemma 5.3.9 and Lemma 5.3.10.

Corollary 5.3.12. *Let M be a matroid. The following are equivalent:*

(i) $\mathfrak{ud}(M) \geq \gamma - 2$.

(ii) *The (k, ℓ)'th coefficient of $R(M; x, y)$ is positive for all $k + \ell < \gamma$.*

(iii) $\frac{\partial T(M;x,y)}{\partial^k x \, \partial^\ell y} \neq 0$ *for all $k + \ell < \gamma$.*

By the proceeding discussion, all of our results regarding (k, ℓ)-uniform matroids have interpretations in terms of the Tutte polynomial. We end this section and this part of the thesis by presenting one such result, a direct combination of Lemma 5.3.10 with Corollary 4.1.3 and Theorem 3.4.6.

Theorem 5.3.13. *For every pair (k, ℓ) of positive integers and every prime power q, there are only finitely many simple cosimple $GF(q)$-representable matroids M such that*

$$\frac{\partial T(M; x, y)}{\partial^k x \, \partial^\ell y} = 0.$$

Moreover, if M is such a matroid, then

$$r(M) \leq k\left(\frac{q^{\ell+1} - 1}{q - 1}\right) - (\ell + 1).$$

Part III
Structured Circuits in Matroids

We define a matroid M to be *circuit-difference* if $C_1 \triangle C_2$ is a circuit whenever C_1 and C_2 are distinct intersecting circuits of M. Evidently, all such matroids are binary. An example of such a matroid is the tipless binary r-spike, that is, the matroid whose binary representation is $[I_r \mid J_r - I_r]$, where J_r is the $r \times r$ matrix of all ones. The question of characterising the circuit-difference matroids was raised at a workshop proceeding the Oxley65 matroid theory conference at Louisiana State University in 2019. In particular, it was asked if these matroids are precisely the binary matroids for which no component contains a pair of skew circuits. Recall that subsets X and Y of $E(M)$ are *skew* in M if $r(X \cup Y) = r(X) + r(Y)$. It is easy to check that no two circuits of the tipless binary r-spike are skew. The following is the main result of this part of the thesis:

Theorem 5.3.14. *Let M be a connected regular matroid. Then M is a circuit-difference matroid if and only if it has no pair of skew circuits.*

To see that this theorem does not in fact extend to all binary matroids, consider the matroid S_8 for which a binary representation is shown in Figure 5.3. In this matroid, $\{1,4,7,8\}$ and $\{2,3,5,6,8\}$ are circuits whose symmetric difference is the disjoint union of the circuits $\{1,2,6\}$ and $\{3,4,5,7\}$. Thus S_8 is not a circuit-difference matroid. However, since $r(S_8) = 4$ and the only 3-circuits of S_8 contain 6, the matroid S_8 has no two skew circuits. Thus one implication of the last theorem fails for arbitrary connected binary matroids. However, as the next result shows, the other implication does hold in the more general context. The proof of this lemma will be given in Section 6.2.

Lemma 5.3.15. *Let M be a connected binary matroid. If M has a pair of skew circuits, then M is not circuit-difference.*

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Figure 5.3: A binary representation for S_8.

In [26], the term "series extension" for matroids is defined as the addition of an element e to a matroid M to create a matroid M' in which $\{e, f\}$ is a cocircuit where f is an element of M, and $M'/e = M$. It will be expedient here to use the term "series extension" more broadly. We shall call a matroid M' a *series extension* of M if it is obtained from M by a sequence of one-element series-extension moves. Pfeil [30] defined a connected matroid M to be *unbreakable* if M/F is connected for every flat F of M. He proved that a matroid is unbreakable if and only if its dual has no two skew circuits, and he determined all unbreakable regular matroids. Combining Pfeil's two results with Theorem 5.3.14 gives the following full characterisation of the regular circuit-difference matroids:

Theorem 5.3.16. *A regular matroid M is circuit-difference if and only if every component of M is a series extension of one of the following matroids: $U_{0,1}$, $U_{1,m}$ for some $m \geq 1$; $M^*(K_n)$ for some $n \geq 1$; $M(K_{3,3})$; or R_{10}.*

One can check explicitly, or deduce from this, that $M(K_4)$ is a circuit-difference matroid, but that $M(K_4)/e$ is not circuit-difference for each element e. Thus the class of circuit-difference matroids is not minor-closed. However, in Section 6.4, we shall show in that this class is closed under series minors and will characterize the infinitely many excluded series minors.

This part of the thesis is structured as follows. Section 6.1 concerns the graphic case of the circuit-difference problem, which was the original cause of motivation and whose resolution is particularly clean. In Section 6.2, we prove some auxiliary results that will be used in the proofs of the main results. Section 6.3 consists of the proof of Theorem 5.3.14. Lastly, in Section 6.4, we determine the excluded series minors for the class of circuit-difference matroids.

Chapter 6

Circuit-difference matroids

6.1 Motivation - The graphic case

The purpose of this section is to motivate the problem of characterising the circuit-difference matroids by a treatment of the naturally arising graphic case. Specifically, we prove the following theorem: the restriction of Theorem 5.3.14 to graphic matroids.

Theorem 6.1.1. *Let G be a 2-connected, loopless graph without isolated vertices and suppose that $|V(G)| \geq 3$. Then $M(G)$ is circuit-difference if and only if every two cycles of G share at least two vertices.*

We prove the forward direction of Theorem 6.1.1 as Lemma 6.1.2 and the converse as Lemma 6.1.3.

Lemma 6.1.2. *Let G be a 2-connected, loopless graph without isolated vertices and suppose that $|V(G)| \geq 3$. If G has a pair of cycles that share at most one vertex, then $M(G)$ is not circuit-difference.*

Proof. Let C_1,C_2 be a pair of cycles of G that share at most one vertex. Suppose firstly, that C_1 and C_2 are vertex disjoint. Then letting u_1, v_1 and u_2, v_2 be vertices in C_1 and C_2 respectively, we have that, as G is 2-connected, there exists disjoint paths P_u and P_v in G with ends u_1, u_2 and v_1, v_2 that otherwise avoid $C_1 \cup C_2$. We now note that there are two paths, from u_i to v_i in each cycle C_i. We may refer to these as the "clockwise" and "anticlockwise" u_i to v_i paths in each cycle. Let D_{cw} be the cycle in G formed by following the clockwise path from u_1 to v_1, followed by P_v, then by the clockwise path from v_2 to u_2 before finally returning along P_u. Similarly, let D_{ccw} be the cycle achieved by taking only counterclockwise paths. Together, the cycles D_{cw} and D_{ccw} use all the edges of cycles C_1 and C_2. Furthermore, the edges that these cycles share are precisely those of the paths P_u and P_v. Hence, the symmetric difference of the edge sets of these two cycles is simply the union of the edge sets of the two cycles C_1 and C_2. Therefore, $M(G)$ is not circuit-difference. It remains to consider when C_1 and C_2 meet at a single vertex. In this case, we follow the same argument as above by simply letting $u = u_1 = u_2$ and replacing all mention of the path P_u with the vertex u. Hence the result holds. □

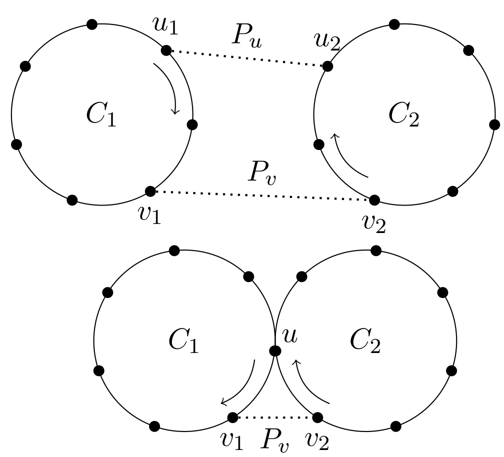


Figure 6.1: Illustrating the proof of Lemma 6.1.2. Arrows indicate the clockwise paths.

Lemma 6.1.3. *Let G be a 2-connected, loopless graph without isolated vertices and suppose that $|V(G)| \geq 3$. If $M(G)$ is not circuit difference, then G has a pair of cycles that share at most one vertex.*

Proof. Take two cycles in G that instantiate that G is not circuit-difference. Colour one red and the other blue. Let S be the set of edges common to both red and blue cycles and let P be a maximal path in S. If the end vertices of P are u and v then deleting P will leave a blue path, B and a red path R, both with end vertices u and v. Moreover, R and B cannot be internally vertex disjoint as then $R \cup B$ would be a cycle that is the symmetric difference of the red and blue cycles. Hence, R and B share at least one internal vertex. Let $w_1 = u$ and let R_1 be the subpath of R achieved by traversing R until the first shared vertex $w_2 \neq v$ is encountered. Continuing on to vertex v, there may then be a number of shared edges to traverse. Let R_2 be the red path from the end vertex of this shared path to the next shared vertex, w_3. Continuing in this fashion until $v = w_n$ is encountered, we now note that, corresponding to each red subpath R_i, there exists a disjoint blue subpath B_i with the same end vertices as R_i. Furthermore, all B_i paths are mutually disjoint, as otherwise a blue subcycle would exist. Choosing the pairs (R_1, B_1) and (R_2, B_2), we achieve two cycles, $R_1 \cup B_1$ and $R_2 \cup B_2$, that can share at most one vertex, w_2. Thus the result holds. □

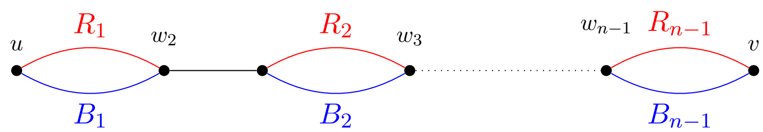


Figure 6.2: Illustrating the proof of Lemma 6.1.3

To achieve a list of graphs for which every two cycles share at least two vertices, we utilise a result of Bollobás [3]. A representation of each of the listed graphs is given in Figure 6.3. For a graph G, $\delta(G)$ denotes the minimum degree of a vertex of G.

Theorem 6.1.4 ([3], Theorem 2.2)**.** *Let G be a connected graph such that any two cycles share at least one vertex. Suppose that $\delta(G) \geq 3$ and G does not have a single vertex v that is used by all cycles. Then G is one of the following.*

1. *A three-vertex graph, where there can be multiple edges joining each.*
2. *K_4, where one triangle can have multiple edges*
3. *K_5*
4. *$K_5 \backslash e$, where edges adjacent to e can be multiple.*
5. *A wheel where the spokes can be multiple.*
6. *$K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, $K'''_{3,n}$, where the edges on the three vertices on one side of the bipartite graph can be multiple.*

We end this section with the following comprehensive list of graphic circuit-difference matroids. This is achieved by filtering out those of Bollobás' graphs that contain cycles meeting at a single vertex.

Lemma 6.1.5. *Let G be a 2-connected, loopless graph without isolated vertices and suppose that $\delta(G) \geq 3$. Then $M(G)$ is circuit-difference precisely when G is one of the following.*

(i) *A two-vertex graph with at least 3 edges,*

(ii) K_4*,*

(iii) $K_{3,3}$

Proof. Let G be a graph satisfying the hypothesis. If $|V(G)| \leq 2$ then, as $\delta(G) \geq 3$, we have that $|V(G)| = 2$ and (i) holds. Otherwise $|V(G)| \geq 3$ and if G has no vertex v that is part of every cycle, then G must be in one of the classes of graphs as detailed by (1) to (6) in Theorem 6.1.4. It is then easy to check (see Figure 6.3) that only the specific cases of K_4 and $K_{3,3}$ have the property that every two cycles share at least two vertices. Now suppose that G does possess a vertex v that is part of every cycle. As G is circuit-difference, and $\delta(G) \geq 3$, we have by Theorem 6.1.1, that there exist two cycles C_1 and C_2 that meet at v and at least one other vertex u. We may partition each cycle C_i into two paths P_i and P_i' with endpoints u and v. As cycles C_1 and C_2 are distinct, we may suppose without loss of generality that paths P_1' and P_2' are distinct. If none of these paths contains any other vertices other than u and v, we are done. Otherwise, suppose without loss of generality, that P_1 contains an internal vertex w. Then, as $\delta(G) \geq 3$, we have that w must sit on a further cycle C_3. As C_3 necessarily passes through v, we may partition C_3 into two paths P_3 and P_3' with endpoints w and v. If C_3 includes u, we may redefine C_3 to be the subcyle of C_3 that avoids u (redefining paths P_3 and P_3' accordingly). Now, $C' = P_1'P_2'$ is a cycle containing u and v but avoiding w. Therefore, C' and C_3 share only one vertex v, contradicting Theorem 6.1.1. Hence, there exists no such vertex w and the result follows. □

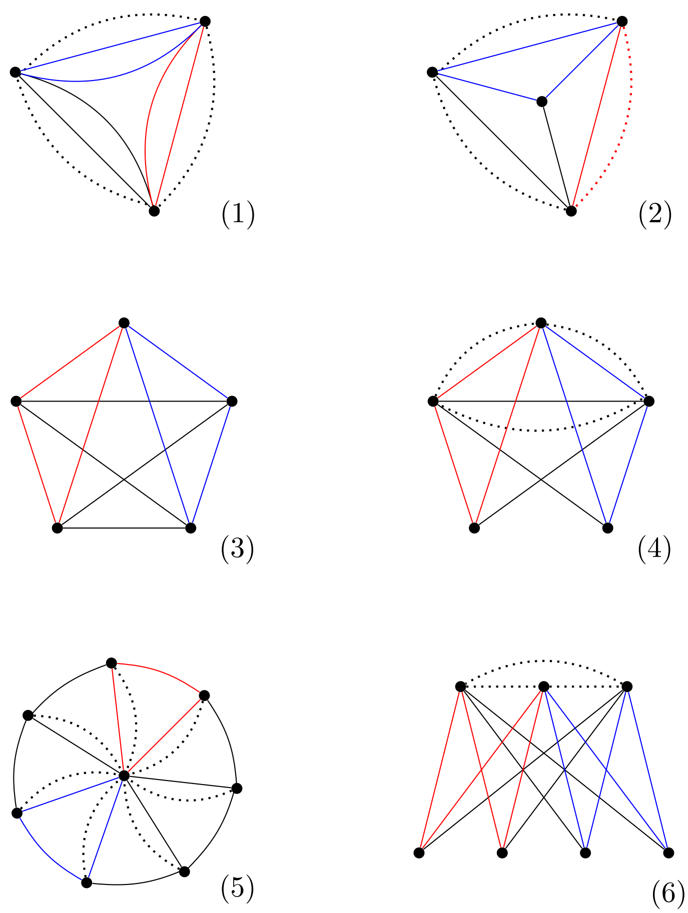


Figure 6.3: Bollabás' list of graphs with, where appropriate, an example of when such a graph contains a pair of cycles meeting at exactly one vertex. A dotted line signifies that there may be multiple edges joining the end vertices.

6.2 Auxiliary results

The results in this section will be used in the proof of Theorem 5.3.14 and in the determination of the excluded series minors for the class of circuit-difference matroids. We begin this section with the proof of Lemma 5.3.15

Proof of Lemma 5.3.15. Let C_1 and C_2 be skew circuits of M, and let D be a circuit that meets C_1 and C_2. Since C_1 and C_2 are skew, $|D - (C_1 \cup C_2)| > 0$.

As is easily checked,

$$D - (C_1 \cup C_2) = (C_1 \triangle D) \cap (C_2 \triangle D).$$

Thus $C_1 \triangle D$ meets $C_2 \triangle D$. As their symmetric difference is the disjoint union of the circuits C_1 and C_2, we deduce that M is not circuit-difference for either $C_1 \triangle D$ or $C_2 \triangle D$ is not a circuit, or both are circuits but their symmetric difference is not. □

The straightforward proof of the next lemma is omitted.

Lemma 6.2.1. *Let M be a matroid. If M has a pair of skew circuits, then so does every series extension of M.*

The next result makes repeated use of the fact that if a circuit in a matroid meets a 2-cocircuit, then it contains that 2-cocircuit.

Lemma 6.2.2. *Let M be a circuit-difference binary matroid and suppose that M' is obtained from M by adding an element e in series to an element f of M. Then M' is circuit-difference.*

Proof. Let D_1 and D_2 be an intersecting pair of circuits of M'. Suppose first that $e \in D_1 \cap D_2$. Then $f \in D_1 \cap D_2$ and $\{D_1 - e, D_2 - e\}$ is an intersecting pair of circuits of M'/e. Thus $(D_1 - e) \triangle (D_2 - e)$, which equals $D_1 \triangle D_2$, is a circuit of M'/e. Hence $D_1 \triangle D_2$ or $(D_1 \triangle D_2) \cup \{e\}$ is a circuit of M'. Because $f \notin D_1 \triangle D_2$, the latter cannot occur. Hence $D_1 \triangle D_2$ is a circuit of M'.

Assume next that $e \in D_1 - D_2$. Then $f \in D_1 - D_2$. Now, $D_1 - e$ is a circuit of M'/e. Moreover, D_2 is a circuit of M'/e as otherwise M' would have a circuit that contains e and is contained in $D_2 \cup \{e\}$. As such a circuit would avoid f, we have a contradiction. We now know that $(D_1 - e) \triangle D_2$ is a circuit of M'/e containing f, so $D_1 \triangle D_2$ is a circuit of M'.

Finally, assume that $e \notin D_1 \cup D_2$. Then $f \notin D_1 \cup D_2$ and so D_1 and D_2 are circuits of M'/e. Hence so is $D_1 \triangle D_2$. As this set avoids f, it must also be a circuit of M' and the lemma is proved. □

In the proof of Theorem 5.3.14, we will encounter a matroid with the property that the complement of every circuit is a circuit. We call such matroids *circuit-complementary*. Such matroids that are binary form an interesting subclass of the class of circuit-difference matroids and are crucial in Section 6.4 when considering the excluded series minors for the latter class.

Lemma 6.2.3. *Let M be a connected binary matroid that is circuit-complementary. Then M is a circuit-difference matroid.*

Proof. Let C_1 and C_2 be an intersecting pair of circuits of M. Then $C_1 \triangle C_2$ is a disjoint union of circuits. If there are at least two circuits in this union, then, since this union avoids $C_1 \cap C_2$, we violate the property that the complement of every circuit is a circuit. □

Again, the proof of the next result is elementary and is omitted.

Lemma 6.2.4. *Let M be a connected binary matroid that is circuit-complementary.*

(i) *If $\{e, f\}$ is a cocircuit of M, then M/e is circuit-complementary.*

(ii) *If M' is a series extension of M, then M' is circuit-complementary.*

Lemma 6.2.5. *Let M be a cosimple connected graphic matroid that is circuit-complementary. Then $M \cong U_{1,4}$.*

Proof. Let $M = M(G)$. By Lemmas 5.3.15 and 6.2.3, M has no two skew circuits. Let C be a cycle of G. Then $E(G) - C$ is a cycle C' of G. Now, C and C' must have exactly two common vertices, otherwise G is not 2-connected or M has two skew circuits. It follows that G has two vertices u and v that are joined by four internally disjoint paths where these paths use all of the edges of G. As $M(G)$ is cosimple, we deduce that $M \cong U_{1,4}$. □

The following lemma makes repeated use of the fact that in a loopless 2-connected graph, the set of edges meeting a vertex is a bond.

Lemma 6.2.6. *Let M be a cosimple connected cographic matroid that is circuit-complementary. Then $M \cong U_{1,4}$.*

Proof. Let $M = M^*(G)$. Then G is 2-connected and simple. Take a vertex v of G and let C_1 be the set of edges meeting v. Then C_1 is a bond in G and hence a circuit of M. Thus $E(G) - C$ is also a bond of G. Hence G has a vertex w that is not adjacent to v. Let C_2 be the set of edges meeting w. Then $E(G) = C_1 \cup C_2$ and G is isomorphic to $K_{2,n}$ for some $n \geq 2$. Let u be a vertex of G other than v or w. The complement of the set of edges meeting u is a bond of G. Thus $n = 2$ and G is a 4-cycle. Hence $M \cong U_{1,4}$. □

Lemma 6.2.7. *Let M be a connected cosimple regular matroid that is circuit-complementary. Then M is isomorphic to $U_{1,4}$ or R_{10}.*

Proof. If M is graphic or cographic, then, by Lemmas 6.2.5 and 6.2.6, $M \cong U_{1,4}$. Now assume that M is neither graphic nor cographic and is not isomorphic to R_{10}. Then, by Seymour's Regular Matroids Decomposition Theorem [32], as M is connected, it can be obtained from graphic matroids, cographic matroids and copies of R_{10} by a sequence of 2-sums and 3-sums. Moreover, each matroid that is used to build M occurs as a minor of M.

6.2.7.1. *M is 3-connected.*

If M is not 3-connected, then M has a 2-separation, (X, Y). Then M is the 2-sum, with basepoint p say, of matroids M_X and M_Y with ground sets $X \cup p$ and $Y \cup p$, respectively. As M is connected, so are M_X and M_Y. Suppose X is independent in M. Then M_X must be a circuit with at least three elements. Thus M is not cosimple, a contradiction. We may now assume that both X and Y contain circuits of M. Hence, by the circuit-complementary property, both X and Y are circuits of M. As $r(X) + r(Y) = r(M) + 1$, we see that $(|X|-1)+(|Y|-1) = r(M)+1$ and, consequently, $r^*(M) = |X|+|Y|-r(M) = 3$. Then M^* is a rank-3 simple binary connected matroid having X and Y as disjoint cocircuits. It follows that M^* is graphic, a contradiction. We conclude that (6.2.7.1) holds.

We may now assume that there are matroids M_1 and M_2 each with at least seven elements such that $E(M_1) \cap E(M_2)$ is a triangle T in both matroids and M is the 3-sum of M_1 and M_2 across this triangle. Moreover, M_1 and M_2 are both minors of M, and $E(M_i) - T$ spans T in M_i for each i. Let $X_i = E(M_i) - T$. Then (X_1, X_2) is a 3-separation of M. Suppose X_1 is independent in M. As X_1 spans T, it follows that M_1 has rank $|X_1|$, so M_1^* has rank three and has T as a triad. Since M is cosimple, no element of X_1 is in a 2-circuit of M_1^*. As M_1 is binary, it follows that $|X_1| \leq 3$ so $|E(M_1)| \leq 6$, a contradiction. We conclude by the circuit-complementary property that both X_1 and X_2 must be circuits of M. Then, as $r(X_1) + r(X_2) = r(M) + 2$, we have $r(M) = |X_1| + |X_2| - 4$ so M^* has rank four, has (X_1, X_2) as a 3-separation and has each of X_1 and X_2 as a cocircuit. Since M^* is a disjoint union of cocircuits, it is affine. As M^* is simple, $|E(M^*)| \leq 8$. But $|X_i| \geq 4$ for each i, so $|E(M^*)| = 8$ and $M^* \cong AG(3, 2)$. This contradicts the fact that M is regular and thereby completes the proof of the lemma. □

Lemma 6.2.8. *Let M be a connected regular matroid and let X be a series class in M. Then $M\backslash X$ and M/X cannot both be connected and circuit-complementary.*

Proof. Assume that the lemma fails. By Lemma 6.2.7, each of $M\backslash X$ and M/X is a series extension of $U_{1,4}$ or R_{10}. Note that X is independent in M. For every series extension M' of $U_{1,4}$ or R_{10}, we have that

$$(r(M'), r^*(M')) \in \{(k+1, 3), (k+5, 5) : k \geq 0\}.$$

Thus, for some non-negative integer m,

$$(r(M\backslash X), r^*(M\backslash X)) \in \{(m+1, 3), (m+5, 5)\}.$$

Now let $|X| = t$. Then $(r(M), r^*(M)) \in \{(m+t, 4), (m+4+t, 6)\}$. Thus $(r(M/X), r^*(M/X)) \in \{(m, 4), (m+4, 6)\}$, so M/X cannot be a series extension of $U_{1,4}$ or R_{10}, a contradiction. □

Although the following lemma is well known, we include a proof for completeness.

Lemma 6.2.9. *Let Y be a set in a connected matroid M such that $|Y| \geq 2$ and $M|Y$ is connected. Let W be a minimal non-empty subset of $E(M)-Y$ such that M has a circuit C such that $C \cap Y \neq \emptyset$ and $C - Y = W$. Then W is a series class of $M|(Y \cup W)$.*

Proof. This is certainly true if $|W| = 1$. Now, suppose that d_1 and d_2 are distinct elements of W that are not in series in $M|(Y \cup W)$. Then $M|(Y \cup W)$ has a circuit K containing w_1 and not w_2. As W is independent, K meets Y. But $K \cap W \subseteq W - w_2$. Thus we have a contradiction to the choice of W. We deduce that every two elements of W are in series in $M|(Y \cup W)$. Since $M|Y$ is connected, no element of Y is in series with an element of W. Thus W is indeed a series class of $M|(Y \cup W)$. □

6.3 Regular circuit-difference matroids

In this section, we prove Theorem 5.3.14.

Proof of Theorem 5.3.14. Let M be a regular connected matroid. By Lemma 5.3.15, if M has a pair of skew circuits, then M is not circuit-difference. To prove the converse, consider all connected regular matroids with no two skew circuits that are not circuit-difference and choose M to be such a matroid with the minimum number of elements. Then, by Lemma 6.2.2, M is cosimple. Let C_1 and C_2 be a pair of intersecting circuits of M such that $C_1 \triangle C_2$ is not a circuit and $|C_1 \cup C_2|$ is a minimum among such pairs. As $M|(C_1 \cup C_2)$ is connected, we must have that $E(M) = C_1 \cup C_2$ by our choice of M. Now, $C_1 \triangle C_2$ is a disjoint union of at least two circuits.

5.3.14.1. *If D is a circuit of M contained in $C_1 \triangle C_2$, then $(C_1 \triangle C_2) - D$ is a circuit of M.*

Clearly, D meets both $C_1 - C_2$ and $C_2 - C_1$ but contains neither of these sets. The choice of $\{C_1, C_2\}$ implies that $C_1 \triangle D$ is a circuit and hence that $(C_1 \triangle D) \triangle C_2$ is a circuit. The last set is $(C_1 \triangle C_2) - D$, so (5.3.14.1) holds.

Let $Z = C_1 \triangle C_2$. As M has no two skew circuits, $M|Z$ is connected and, by 5.3.14.1, it is circuit-complementary. Thus, by Lemma 6.2.7, $M|Z$ is a series extension of $U_{1,4}$ or of R_{10}. Let X be a minimal non-empty subset of $C_1 \cap C_2$ such that M has a circuit whose intersection with $C_1 \cap C_2$ is X. Then, by Lemma 6.2.9, X is a series class of $M|(Z \cup X)$. Thus every circuit of $M|(Z \cup X)$ that meets X must contain X.

5.3.14.2. *Every circuit of $M|Z$ is a circuit of $(M|(Z \cup X))/X$.*

Let D be a circuit of M that is contained in Z. Then D meets both $C_1 - C_2$ and $C_2 - C_1$ and, by 5.3.14.1, $Z - D$ is a circuit of M that also meets both $C_1 - C_2$ and $C_2 - C_1$. Assume that D is not a circuit of $(M|(Z \cup X))/X$. Then $M|(Z \cup X)$ has a circuit K such that $K \subseteq D \cup X$ and $K \cap D \neq D$. Thus K meets and so contains X. Hence $K - D = X$. As $|K \cup D| = |X \cup D| < |C_1 \cup C_2|$, it follows that $K \triangle D$ is a circuit of M and hence that $K \triangle D$ meets $C_1 - C_2$ and $C_2 - C_1$. As $|C_1 \cup K| < |C_1 \cup D| < |C_1 \cup C_2|$, the choice of $\{C_1, C_2\}$ implies that $C_1 \triangle K$ is a circuit C of M and that $C_1 \triangle (Z - D)$ is a circuit C' of M. As C and C' both contain the non-empty set $(D - K) \cap C_1$ and both avoid the non-empty set $(D-K)\cap C_2$, we see that $|C\cup C'|<|C_1\cup C_2|$ and $C\triangle C'$ is a circuit of M. This circuit is $[C_1\triangle K]\triangle[C_1\triangle(Z-D)]$, which equals $K\triangle(Z-D)$. But the last set is a disjoint union of two circuits, a contradiction. Thus (5.3.14.2) holds.

We know that $M|Z$ is connected and circuit-complementary. Moreover, the choice of X implies that $M|(Z\cup X)$ has a circuit that meets $C_1\cap C_2$ in X. Therefore $M|(Z\cup X)$ is connected. Moreover, by (5.3.14.2), $(M|(Z\cup X))/X$ is connected. It follows by Lemma 6.2.8 that $(M|(Z\cup X))/X$ is not circuit-complementary. Thus $(M|(Z\cup X))/X$ has a circuit J such that $Z-J$ is not a circuit of $(M|(Z\cup X))/X$. If J is a circuit of $M|Z$, then, as $M|Z$ is circuit-complementary, $Z-J$ is a circuit of $M|Z$. Thus, by (5.3.14.2), we obtain the contradiction that $Z-J$ is a circuit of $(M|(Z\cup X))/X$. We deduce that J is not a circuit of $M|Z$. Then $J\cup X'$ is a circuit K of $M|Z$ for some non-empty subset X' of X. By the choice of X, it follows that $X'=X$. Now, $Z=D\cup D'$ for some disjoint circuits D and D'. We deduce using (5.3.14.2) that K meets both D and D' but contains neither. Hence $|K\cup D|<|C_1\cup C_2|$, so $K\triangle D$ is a circuit of M. As $|D'\cup(K\triangle D)|<|C_1\cup C_2|$, we see that $D'\triangle(K\triangle D)$ is a circuit of M, that is, $(Z-K)\cup X$ is a circuit of M. Thus $Z-J$ is a circuit of $(M|(Z\cup X))/X$, a contradiction. □

6.4 Excluded series minors

In this section, we show that the class of circuit-difference matroids is closed under series minors, and we characterize the infinitely many excluded series minors for this class.

Lemma 6.4.1. *The class of circuit-difference matroids is closed under series minors.*

Proof. Let M be a circuit-difference matroid. Evidently, $M\backslash e$ is circuit-difference for all $e\in E(M)$. Now let $\{e,f\}$ be a cocircuit of M and consider M/e. A circuit C of M/e contains f if and only if $C\cup\{e\}$ is a circuit of M. Thus the collection $\mathcal{C}(M/e)$ of circuits of M/e is $\mathcal{C}(M\backslash e)\cup\{C-e : f\in C\in\mathcal{C}(M)\}$. It is now routine to check that M/e is a circuit-difference matroid. □

Let N_5 be the 5-element matroid that is obtained from a triangle by adding single elements in parallel to exactly two of its elements. This is easily seen to be an excluded series minor for the class of circuit-difference matroids. Although the next proposition is not needed for the proof of the main result of this section, it seems to be of independent interest.

Proposition 6.4.2. *A connected binary matroid M has a pair of skew circuits if and only if M has a series minor isomorphic to N_5.*

Proof. If M has a series minor isomorphic to N_5, then, by Lemma 6.2.1, as N_5 has a pair of skew circuits, so does M. For the converse, let C_1 and C_2 be a pair of skew circuits of M, and let D be a circuit meeting both such that $|D-(C_1 \cup C_2)|$ is a minimum. Let $M' = M|(C_1 \cup C_2 \cup D)$. Next we show the following.

6.4.2.1. *If $C_1 - D$ or $C_1 \cap D$ contains $\{x, y\}$, then $\{x, y\}$ is a cocircuit of M'.*

Suppose that this fails. Then M' has a circuit K that contains x but not y. Assume first that K meets C_2. Then, by the choice of D, we must have that $K - (C_1 \cup C_2) = D - (C_1 \cup C_2)$. Then $K \triangle D$ is a disjoint union of circuits that is contained in $(C_1 \cup C_2) - y$ or $(C_1 \cup C_2) - x$. But, for each z in C_1, the matroid $(M|(C_1 \cup C_2))\backslash z$ has C_2 as its only circuit. As $K \triangle D \neq C_2$, we have a contradiction. We deduce that K avoids C_2. As $y \notin K$, we must have that $K \cap (D - (C_1 \cup C_2))$ is non-empty. Then $K \triangle D$ is a disjoint union of circuits that does not contain $D - (C_1 \cup C_2)$. One such circuit must meet $C_2 \cap D$ and C_1. But this violates the choice of D. Thus (6.4.2.1) holds.

By (6.4.2.1) and symmetry, we can perform a sequence of series contractions in M', reducing each of the sets $C_1 - D$, $C_1 \cap D$, $C_2 \cap D$, and $C_2 - D$ to a single element. The resulting matroid is a series minor of M that has two disjoint 2-circuits such that deleting one element from each leaves a circuit with at least three elements. It follows that M has N_5 as a series minor. □

We call a matroid *hyperplane-complementary* if the complement of every hyperplane is a hyperplane. One such matroid is the binary affine geometry $AG(r-1, 2)$ of rank at least two. The next result determines all simple binary hyperplane-complementary matroids. For all k, every rank-k flat of $AG(r-1, 2)$ is isomorphic to $AG(k-1, 2)$.

Lemma 6.4.3. *A simple rank-r binary matroid M is hyperplane-complementary if and only if $r \geq 2$ and $M \cong AG(r-1, 2)\backslash X$ for some set X such that $AG(r-1, 2)|X$ does not contain a copy of $AG(r-3, 2)$.*

Proof. Suppose that M is hyperplane-complementary. Then $r \geq 2$. Moreover, $E(M)$ is a disjoint union of cocircuits, so every circuit of M has even cardinality. Hence we can view M as $AG(r-1,2)\backslash X$ for some set X. Let $E = E(AG(r-1,2))$. Then $E(M) = E - X$. Assume that $AG(r-1,2)|X$ contains a copy Z of $AG(r-3,2)$. For $y \in E - X$, consider the closure $\text{cl}_A(Z \cup y)$ of $Z \cup y$ in $AG(r-1,2)$. This closure is a rank-$(r-1)$ flat of $AG(r-1,2)$ and is thus isomorphic to $AG(r-2,2)$. Let $Y = \text{cl}_A(Z \cup y) \cap (E - X)$ and $W = (E - X) - Y$. Then Y is contained in some copy of $AG(r-3,2)$, and W is contained in some copy of $AG(r-2,2)$. Thus $r(Y) \leq r-2$ and $r(W) \leq r-1$. Hence W is contained in a hyperplane W' of M whose complement in $E(M)$ is not a hyperplane. Thus M is not hyperplane-complementary, a contradiction.

Now let $M = AG(r-1,2)\backslash X$ where $r \geq 2$ and $AG(r-1,2)|X$ does not contain a copy of $AG(r-3,2)$. Let H be a hyperplane of $AG(r-1,2)$. Then $AG(r-1,2)|H = AG(r-2,2)$. If $r(H - X) \leq r-2$, then $H - X$ is contained in some copy of $AG(r-3,2)$ that is contained in H and so, as $AG(r-2,2)$ is hyperplane-complementary, X contains a copy of $AG(r-3,2)$. This contradiction implies that the hyperplanes of M are all of the sets of the form $H - X$ where H is a hyperplane of $AG(r-1,2)$. As $AG(r-1,2)$ is hyperplane-complementary, so is M. $\square$

Recall that $AG(r-1,2)$ is obtained from the projective geometry $PG(r-1,2)$ by deleting a hyperplane, that is, by deleting a copy of $PG(r-2,2)$. It is a well-known consequence of the unique representability of binary matroids that if $PG(r-1,2)|E_1 \cong PG(r-1,2)|E_2$, then $PG(r-1,2)\backslash E_1 \cong PG(r-1,2)\backslash E_2$. Thus, as all single-element deletions of $PG(r-2,2)$ are isomorphic, there is, up to isomorphism, a unique simple binary rank-r single-element extension of $AG(r-1,2)$. We shall denote this extension by $AG(r-1,2)+e$.

Let $\mathbb{M}$ be the set of all matroids of rank at least three of the form $[AG(r-1,2)+e]\backslash X$ such that $AG(r-1,2)\backslash X$ is hyperplane-complementary of rank r. Thus N_5^* is the unique rank-3 member of $\mathbb{M}$ while its rank-4 members are the tipped binary 4-spike and a non-tip deletion thereof, that is, S_8. We now show that the duals of the matroids in $\mathbb{M}$ are precisely the excluded series minors for the class of circuit-difference matroids.

Theorem 6.4.4. *A binary matroid M is an excluded series minor for the class of circuit-difference matroids if and only if $M^* \in \mathbb{M}$.*

Proof. Let M be an excluded series minor for the class of circuit-difference matroids. By Lemma 6.2.2, M is cosimple. Let C_1 and C_2 be intersecting circuits of M such that $C_1 \triangle C_2$ is not a circuit and $|C_1 \cup C_2|$ is minimal.

6.4.4.1. $M^* \in \mathbb{M}$.

Evidently, $E(M) = C_1 \cup C_2$. Then C_1 and C_2 are the only circuits of M containing $C_1 - C_2$ and $C_2 - C_1$, respectively. Now, letting $x \in C_1 \cap C_2$, suppose that $(C_1 \cap C_2) - x$ contains an element y. Then, as x and y are not in series, M has a circuit D containing x but not y. As D meets both C_1 and C_2, we have, by the choice of $\{C_1, C_2\}$, that $C_1 \triangle D$, $C_2 \triangle D$, and hence $(C_1 \triangle D) \triangle (C_2 \triangle D)$ are circuits of M. This last circuit is $C_1 \triangle C_2$, so we have a contradiction. Thus $C_1 \cap C_2 = \{x\}$. To see that M/x is circuit-complementary, let $D \in \mathcal{C}(M/x)$ such that $D \notin \{C_1 - x, C_2 - x\}$. Then either D or $D \cup x$ is a circuit D' of M, and D' must meet $C_1 - C_2$ and $C_2 - C_1$. Hence, by the choice of $\{C_1, C_2\}$, we have that $C_1 \triangle D'$ and hence $(C_1 \triangle D') \triangle C_2$ is a circuit of M. Next we show that $(C_1 \triangle C_2) - D$ is a circuit of M/x. Suppose it is not. Then $x \notin D'$ and M has a circuit D'' containing x such that $D'' \subsetneqq (C_1 \triangle C_2) - D$. Using D'' in place of D' above, we see that $C_1 \triangle C_2 \triangle D''$ is a circuit of M that properly contains D', a contradiction. We conclude that $(C_1 \triangle C_2) - D$ is a circuit of M/x, so M/x is circuit-complementary. Therefore, $M^* \backslash x$ is hyperplane-complementary. As M is cosimple, M^* is simple. Moreover, M^* has $C_1 - x$ and $C_2 - x$ as hyperplanes, so M^* has the form $[AG(r-1, 2) + e] \backslash X$. Since M^* is connected, $r(M^*) \geq 2$. But if $r(M^*) = 2$, then $M^* \cong U_{2,3}$, so $M \cong U_{1,3}$ and M is circuit-difference, a contradiction. Thus $M^* \in \mathbb{M}$, so (6.4.4.1) holds.

To prove the converse, let $M^* = [AG(r-1,2)+e] \backslash X$ where $AG(r-1,2) \backslash X$ is hyperplane-complementary of rank r and $r \geq 3$. By Lemma 6.4.3, $AG(r-1,2)|X$ does not contain a copy of $AG(r-3,2)$. Consider $AG(r-1,2)+e$ and let H_0 be the hyperplane of $PG(r-1,2)$ whose deletion gives $AG(r-1,2)$. Take a rank-$(r-2)$ flat F of $PG(r-1,2)$ that is contained in H_0 and avoids e. Apart from H_0, there are exactly two hyperplanes, H_1 and H_2, of $PG(r-1,2)$ that contain F. Then $H_1 - H_0$ and $H_2 - H_0$ are hyperplanes of $AG(r-1,2)+e$, and $H_1 - (H_0 \cup X)$ and $H_2 - (H_0 \cup X)$ are hyperplanes of $[AG(r-1,2)+e] \backslash X$. The complements of these two hyperplanes are circuits C_1, C_2 of M that meet in the element e. We now note that $C_1 \triangle C_2$ is not a circuit of M otherwise $\{e\}$ is a hyperplane of M^* and we obtain the contradiction that $r(M^*) \leq 2$. Hence M is not circuit-difference.

6.4.4.2. *If D is a circuit of $M\backslash e$, then $e \notin \mathrm{cl}(D)$.*

Suppose that $e \in \mathrm{cl}(D)$ for some circuit D of $M\backslash e$. Then there is a partition $\{X_1, X_2\}$ of D such that X_i is a circuit of M/e for both i. As M/e is circuit-complementary, $X_1 \cup X_2 = E(M/e) = E(M) - e$. This is a contradiction as $X_1 \cup X_2 = D \subsetneqq E(M) - e$. Hence (6.4.4.2) holds.

6.4.4.3. *$M\backslash f$ is circuit-difference for all f in $E(M)$.*

Suppose some $M\backslash f$ is not circuit-difference. Then it has a pair of intersecting circuits D_1, D_2 such that $D_1 \triangle D_2$ contains a pair of disjoint circuits K_1, K_2. Suppose first that both D_1 and D_2 avoid e. Then so do K_1 and K_2. Thus, by (6.4.4.2), none of D_1, D_2, K_1, or K_2 has e is in its closure. Hence all of D_1, D_2, K_1, and K_2 are circuits of M/e, so M/e is not circuit-difference, a contradiction. Hence at least one of D_1 and D_2 must contain e, so $f \neq e$. Now suppose $e \in D_1 - D_2$ and $e \in K_1$. Then $D_1 - e$ and D_2 are intersecting circuits of M/e with circuits $K_1 - e$ and K_2 in their symmetric difference. This again contradicts the fact that M/e is circuit-difference. Hence, by symmetry, we must have that $e \in D_1 \cap D_2$. Consequently, K_1, K_2, $D_1 - e$, and $D_2 - e$ are circuits of M/e. If $D_1 \cap D_2 = \{e\}$, then $D_1 - e$ and $D_2 - e$ are disjoint. Thus their union is $E(M/e)$. But this union avoids f, a contradiction. Hence $(D_1 \cap D_2) - e$ must be non-empty. But then $D_1 - e$ and $D_2 - e$ are intersecting circuits of M/e so their symmetric difference, which equals $D_1 \triangle D_2$, is a circuit of M/e. However, this symmetric difference contains K_1 and K_2, which are circuits of M/e. This contradiction completes the proof of (6.4.4.3).

As M^* is simple, M is cosimple and hence no series contractions can be performed. Thus, by (6.4.4.3), every series minor of M is circuit-difference and the theorem holds. □

The next result, the last of this thesis, follows immediately by combining the last theorem with Tutte's excluded-minor characterization of binary matroids [34].

Corollary 6.4.5. *A matroid M is an excluded series minor for the class of circuit-difference matroids if and only if $M \cong U_{n,n+2}$ for some $n \geq 2$, or M^* can be obtained from $AG(r-1,2) + e$ for some $r \geq 3$ by deleting some set X such that $e \notin X$ and $AG(r-1,2)|X$ does not contain a copy of $AG(r-3,2)$.*

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