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On the equiconvergence of the spectral decomposition of the distributions connected with elliptic differential operators on the torus with Fourier integral

On the equiconvergence of the spectral decomposition of the distributions connected with elliptic differential operators on the torus with Fourier integral

Abdulkasim Akhmedov[a,d], Zuki Salleh[b] and Abdumalik Rakhimov[c]

[a,b]Centre for Mathematical Sciences, College of Computing & Applied Sciences, Universiti Malaysia Pahang, Lebuhraya Tun Razak, 26300 Gambang, Pahang, Malaysia
[c]Kulliyah of Engineering, International Islamic University of Malaysia, IIUM, 53100 Kuala Lumpur, Malaysia
[d]Faculty of Mathematics and Science, Paragon International University, No 8, St 315, Boeng Kak 1, Toul Kork, 12151, Phnom Penh, Cambodia

Email: [a]aakhmedov@paragoniu.edu.kh, [b]zuki@ump.edu.my, [c]abdumalik@iium.edu.my

Abstract. In this paper, we deal with the problems of the expansions of the periodic distributions. We obtained sufficient conditions for the equiconvergence of the spectral decompositions of the distributions connected with the elliptic differential operator on the torus with Fourier integral in the classes of the Sobolev.

1. Introduction

The problems of engineering sciences can be modelled using equations of mathematical physics. Mathematical models for many systems that are encountered in engineering, physics, and other applied sciences are often developed by applying various laws that describe the conservation of mass, momentum, and energy. These models are usually given as a single or set of ordinary or partial differential equations along with appropriate initial and boundary conditions which apply over the rectangular region. Solution of these equations using appropriate analytical methods provides local numerical values for the dependent variables of interest, such as fluid velocity, pressure, species concentration, temperature, force and electric potential. When an engineering process occur on the plate, the solution of the equations of mathematical physics can be solved by separation method but spectral expansions of the solution does not converge to the boundary function. This difficulty can be overcome by regularization of the spectral expansions. In this paper, we investigate sufficient conditions for equiconvergence of the spectral expansions of distributions connected with elliptic differential operators on the torus with Fourier integrals. The equiconvergence of the spectral expansion depends on the initial and/or boundary data. Regularization of the divergent series solution is accurate numerical interpretations of the solutions of the problems. The equiconvergence of the Fourier series and integral of the linear continuous functionals (distributions) in the case of spherical summation is studied in [1]. The method for localization of spectral decomposition of distributions for the first time was studied by Sh.A. Alimov [2]. Further results in latter expanded to the more general spectral expansions in [3]-[10]. We note that the results on the summability of the spectral decomposition connected with Fourier-Laplace series are obtained in [11].

In this paper, we obtain a result on the equiconvergence of the multiple Fourier series and integrals connected with distributions from Sobolev space. A precise equiconvergence relation between order of Riesz means of the spectral expansions and power of the singularity of the periodic distributions in summation associated with the elliptic differential operator.

2. Distributions on $\mathbb{T}^n$

We deal with the spectral expansions of the function defined on the torus $\mathbb{T}^n$, which can be defined by $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. The torus $\mathbb{T}^n$ often is identified with the cube $[-\pi,\pi)^n \subset \mathbb{R}^n$, where the measure on the torus is identified with the restriction of the Euclidean measure on the cube. Functions on $\mathbb{T}^n$ are the functions on $\mathbb{R}^n$ that are 2π-periodic in each of the coordinates.
We may identify functions on $\mathbb{T}^n$ with $2\pi\mathbb{Z}^n$ −periodic functions on $\mathbb{R}^n$. Let $\mathcal{F}_{\mathbb{R}^n}$ be the Euclidian Fourier transforms defined by

$$(\mathcal{F}_{\mathbb{R}^n})f(\xi) \coloneqq (2\pi)^{-\mathrm{N}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

and its inverse $\mathcal{F}_{\mathbb{R}^n}^{-1}$ is given by

$$f(x) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} (\mathcal{F}_{\mathbb{R}^n} f)(\xi)d\xi,$$

Let

$$\mathcal{F}_{\mathbb{T}^n} = \left(f \mapsto \hat{f}\right): C^\infty(\mathbb{T}^n) \to S(\mathbb{Z}^n)$$

by the toroidal Fourier transform defined by

$$\hat{f}(m) \coloneqq (2\pi)^{-\mathrm{N}} \int_{\mathbb{T}^n} e^{-ix\cdot \mathrm{m}} f(x)dx.$$

Space $L^2(\mathbb{T}^n)$ is a Hilbert space with the inner product

$$(u,v)_{L^2(\mathbb{T}^n)} \coloneqq \int_{\mathbb{T}^n} u(x)\overline{v(x)}dx,$$

where $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$. The Fourier coefficients of $u \in L^2(\mathbb{T}^n)$ are

$$\hat{u}(m) = (2\pi)^{-\mathrm{N}} \int_{\mathbb{T}^n} e^{-im\cdot x} u(x)dx, \qquad m \in \mathbb{Z}^n$$

and they are well-defined for all $m \in \mathbb{Z}^n$ due to Hölder's inequality and compactness of $\mathbb{T}^n$.
The system of functions $\{e^{im\cdot x}: m \in \mathbb{Z}^n\}$ forms an orthonormal basis on $L^2(\mathbb{T}^n)$. Thus the partial sums of the Fourier series

$$\sum_{m\in\mathbb{Z}^n} \hat{u}(m)e^{im\cdot x}$$

converge to u in the L^2 −norm, so that we shall identify u with its Fourier series representation:

$$u(x) = \sum_{m\in\mathbb{Z}^n} \hat{u}(m)e^{im\cdot x}.$$

Plancherel's identity. If $u \in L^2(\mathbb{T}^n)$ then $\hat{u} \in l^2(\mathbb{Z}^n)$ and

$$\|\hat{u}\|_{l^2(\mathbb{Z}^n)} = \|u\|_{L^2(\mathbb{T}^n)}.$$

The space of linear functionals acting in $\mathbb{C}^\infty(\mathbb{T}^n)$ we denote by $\mathcal{D}'(\mathbb{T}^n)$, which is called the space of periodic distributions. For $u \in \mathcal{D}'(\mathbb{T}^n)$ and $\varphi \in \mathbb{C}^\infty(\mathbb{T}^n)$, we shall write

$$u(\varphi) = \langle u, \varphi\rangle.$$

For any $\psi \in \mathbb{C}^\infty(\mathbb{T}^n)$,

$$\varphi \mapsto \int_{\mathbb{T}^n} \varphi(x)\psi(x)dx$$

is a periodic distribution, which gives the embedding $\psi \in \mathbb{C}^\infty(\mathbb{T}^n) \subset \mathcal{D}'(\mathbb{T}^n)$. Due to the test function equality $\langle \partial^\alpha\psi, \varphi\rangle = \langle \psi, (-1)^{|\alpha|}\partial^\alpha\varphi\rangle$, it is natural to define distributional derivatives by

$$\langle \partial^\alpha f, \varphi\rangle \coloneqq \langle f, (-1)^{|\alpha|}\partial^\alpha\varphi\rangle.$$

The topology of $\mathcal{D}'(\mathbb{T}^n)$ is the weak-topology. For $u \in \mathcal{D}'(\mathbb{T}^n)$ and $s \in \mathbb{R}$ we define

$$\|u\|_{H^s(\mathbb{T}^n)} := \left(\sum_{m\in\mathbb{Z}^n} \langle m\rangle^{2s} |\hat{u}(m)|^2\right)^{1/2}$$

The *Sobolev space* $H^s(\mathbb{T}^n)$ is then the space of 2π-periodic distributions u for which $\|u\|_{H^s(\mathbb{T}^n)} < \infty$. For elements of the Sobolev space formally we write their Fourier series representation

$$\sum_{m\in\mathbb{Z}^n} \hat{u}(m) e^{im\cdot x}$$

2π-periodic Dirac delta δ is expressed by $\delta(x) = \sum_{m\in\mathbb{Z}^n} \hat{u}(m) e^{im\cdot x}$, or by $\left(\hat{\delta}(m)\right)_{m\in\mathbb{Z}^n}$, where $\hat{\delta}(m) \equiv 1$ and belongs to $H^{-s}(\mathbb{T}^n)$ if and only if $s > n/2$.

3. Main Results

Let $D_j = -i\frac{\partial}{\partial x_j}$ be the differential operator. For every multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, we define

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_N^{\alpha_N}$$

A differential operator of order 2ℓ is defined as follows

$$\wp(D) = \sum_{|\alpha|=2\ell} a_\alpha D^\alpha.$$

The symbol of the operator $\wp(D)$ is obtained by replacing the D to variable $\xi \in R^N$:

$$\wp(\xi) = \sum_{|\alpha|=2\ell} a_\alpha \xi^\alpha$$

Differential operator $\wp(D)$ is called elliptic if the symbol satisfies the condition:

$$\wp(\xi) \neq 0, \qquad \forall \xi \in R^N \setminus \{0\}.$$

Let use notation $\partial\Omega$ to denote the boundary of the following set $\Omega = \{\xi \in R^N : \wp(\xi) < 1\}$. In this paper we restrict our focus to considering the operators $\wp(D)$ such that $\partial\Omega$ is a C^∞ −hypersurface with strictly positive gaussian curvature. The closure of the operator $\mathfrak{H}(D)$ with the domain $C^\infty(T^N)$ is self-adjoint with the spectral function

$$\theta(x-y,\lambda) = (2\pi)^{-N} \sum_{\wp(n)<\lambda} e^{in(x-y)}$$

For any complex l with $\Re(l) \geq 0$, we introduce the Riesz means of the spectral function

$$\theta^l(x-y,\lambda) = \int_0^\lambda \left(1-\frac{t}{\lambda}\right)^l d_t\theta(x-y,t).$$

We denote the Riesz means $\sigma_\lambda^l(x,f)$ of the spectral decomposition of the distribution f from $H^{-s}(T^N)$ as follows

$$\sigma_\lambda^l(x,f) = \langle f, \theta^l(x-y,\lambda)\rangle = (2\pi)^{-N} \sum_{\wp(n)<\lambda} \left(1-\frac{\wp(n)}{\lambda}\right)^l \langle f, e^{in(x-y)}\rangle.$$

Let denote by $\theta_0^l(x,y,\lambda)$ Riesz means of the spectral function of the $\wp(D)$ with the domain $C_0^\infty(R^N)$:

$$\theta_0^l(x-y,\lambda) = (2\pi)^{-N} \int_{\wp(\xi)<\lambda} \left(1-\frac{\wp(\xi)}{\lambda}\right)^l e^{i(x-y)\xi} d\xi$$

and the Riesz means of the Fourier integral are defined with the help of the latter kernel as follows

$$E_\lambda^l(x,f) = < f, \theta_0^l(x,y,\lambda) > \qquad (1.1)$$

The main results of this paper are the following

Theorem. Let $s > 0$. If $l \geq \frac{N-1}{2} + s$, then for an arbitrary $f \in H^{-s}(T^N)$

$$\sigma_\lambda^l(x,f) - E_\lambda^l(x,f) = O(1)\|f\|_{H^{-s}(T^N)}.$$

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The proof of the Theorem is preceded by a few Lemmas.

Lemma 1. Let $\varkappa(\omega)$ be the map which takes S^{N-1} homeomorphically onto $\{\xi \in R^N : \wp(\xi) = 1\}$, by sending $\omega \in S^{N-1}$ into the unique point on $\{\xi \in R^N : \wp(\xi) = 1\}$ at which the exterior normal to $\{\xi \in R^N : \wp(\xi) = 1\}$ has direction ω. For any complex l with $\Re(l) \geq 0$ the following asymptotical estimate has a place

$$\theta_0^l(x,y,\lambda) = O(1)\begin{cases} \lambda^{N/2\ell}, & \text{for } |x-y|\lambda^{\frac{1}{2\ell}} \leq 1, \\ \dfrac{\lambda^{\left(\frac{N-1}{2}-\Re(l)\right)\frac{1}{2\ell}}}{|x-y|^{\frac{N+1}{2}+s}}\left[K\left(\varkappa\left(\dfrac{x-y}{|x-y|}\right)\right) + K\left(\varkappa\left(\dfrac{y-x}{|x-y|}\right)\right)\right], & \text{for } |x-y|\lambda^{\frac{1}{2\ell}} > 1. \end{cases}$$

We refer for the proof to [12]. Let $\widehat{\Theta}_\lambda^s(\xi)$ be the Fourier transformation of the Bochner-Riesz kernel $\theta_0^l(x-y,\lambda)$. Then

$$\left|\widehat{\Theta}_\lambda^l(\xi)\right| \leq c(1+|\xi|)^{-N-l} \tag{1.2}$$

$$\left|\theta_0^l(x,\lambda)\right| \leq c(1+|x|)^{-N-l} \tag{1.3}$$

From the definition of the kernel $\theta_0^l(x-y,\lambda)$, it is apparent that

$$\widehat{\Theta}_\lambda^s(\xi) = \begin{cases} \left(1-\dfrac{\wp(\xi)}{\lambda}\right)^s, & \text{if } \wp(\xi) < \lambda \\ 0, & \text{if } \wp(\xi) \geq \lambda \end{cases}$$

The Poisson summation formula is valid if the function g(x) and its Fourier transformation $\hat{g}(\xi)$ satisfy the (1.2), (1.3). Then

$$\sum_{n\in Z^N} g(x+2\pi n) = (2\pi)^{-\frac{N}{2}} \sum_{n\in Z^N} \hat{g}(n)e^{inx}$$

Therefore, for the function $g(x) = \theta_0^l(x,\lambda)$ we obtain

$$\sum_{n\in Z^N} \theta_0^l(x+2\pi n,\lambda) = (2\pi)^{-\frac{N}{2}} \sum_{n\in Z^N} \widehat{\Theta}_\lambda^l(n)e^{inx}$$

The regularized Dirichlet kernel $\theta^l(x-y,\lambda)$ is the Riesz means of the partial sums of the Fourier series of the Dirac delta function can be written as:

$$\theta^l(x,\lambda) = (2\pi)^{-N} \sum_{\wp(n)<\lambda} \left(1-\frac{\wp(n)}{\lambda}\right)^l f_n e^{inx} \tag{1.4}$$

By reference of the latter formula and from the definition of $\theta_0^l(x,\lambda)$ we have

$$\theta^l(x,\lambda) = \sum_{n\in Z^N} \theta_0^l(x+2\pi n,\lambda)$$

Separating the right-hand side of the latter by $n = 0$, we obtain

$$\theta^l(x,\lambda) = \theta_0^l(x,\lambda) + \Theta_{*,\lambda}^s(x)$$

where the function $\Theta_{*,\lambda}^s(x)$ defined as

$$\Theta_{*,\lambda}^s(x) = \sum_{n\in Z^N, n\neq 0} \theta_0^l(x+2\pi n,\lambda) \tag{1.5}$$

Now, from formula *(1.5)*

$$\sigma_\lambda^l(x,f) - E_\lambda^l(x,f) = \langle f, \Theta_{*,\lambda}^s(x-y)\rangle$$

Equality *(1.3)* and the following lemma ensure the assertation of the Theorem 1.

Lemma 2. Let $l > 0$, $f \in H^{-l}(T^N) \cap \varepsilon'(T^N)$ and let $\sup f \subset \Omega \subset\subset T^N$. Then uniformly in any compact set $K \subset T^N \backslash \overline{\Omega}$:

$$\langle f, \Theta_{*,\lambda}^s(x-y)\rangle \geq O(1)\|f\|_{-l},$$

Proof of Lemma 2: Let Ω_0 is a proper sub domain of the domain Ω. Then

$$\left|\langle f, \Theta^l_{*,\lambda}(x-y)\rangle\right| \leq \|f\|_{-s}\left\|\Theta^l_{*,\lambda}(x-y)\right\|_{s,0}, \tag{1.6}$$

where $\|.\|_{s,0}$ is a norm of in the space $H^{-s}(\Omega_0)$ via the variable $y \in \Omega_0$. When $|x-y| > c > 0$, we have

$$\left\|\Theta^l_{*,\lambda}(x-y)\right\|_0 = O\left(\lambda^{\frac{1}{2m}}\right)^{-\left(\frac{N-1}{2}-l\right)},$$

where $\|.\|_0$ is a norm in $L_2(\Omega_0)$. Then Lemma 2 follows from (1.6) and

$$\left\|\Theta^l_{*,\lambda}(\mathrm{x}-\mathrm{y})\right\|_{1,0} = O\left(\lambda^{\frac{1}{2m}}\right)^{-\left(\frac{N-1}{2}-l\right)}\left\|\Theta^l_{*,\lambda}(x-y)\right\|_0$$

The proof of the Theorem is consequence of the latter estimation.

4. Conclusion

The sufficient conditions which guarantee the equiconvergence of the multiple Fourier series and integrals corresponding to distributions from Sobolev classes are obtained by using spectral properties of elliptic differential operators. We note that a precise relation between order of Riesz means of the spectral expansions and power of the singularity of the periodic distributions in summation associated with the elliptic differential operator is established.

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References

[1] Rakhimov, A. A. (2015). On the Equiconvergence of the Fourier. *Journal of Applied Mathematics and Physics, 2*(3), 1361-1366.

[2] Alimov, S. A. (1993). On the Spectral Decompositions of Distributions. *Doklady Mathematics*(331), 661-662.

[3] Alimov, S. A., & Rakhimov, A. A. (1996). Localization of Spectral Expansions of Distributions. *Diff. Equations*(32), 798-802.

[4] Rakhimov, A. A., & Alimov, S. A. (1997). Localization of Spectral Expansions of Distributions in a Closed Domain. *Diff. Equations*(33), 80-82.

[5] Rakhimov, A. A. (2000). On the Localization of Multiple Trigonometric Series of Distributions. *Dokl. Math.*(62), 163-165.

[6] Rakhimov, A. A. (1996). Localization Conditions for Spectral Decompositions Related to Elliptic Operators from Class. *Mathematical Notes*(59), 298-302.

[7] Rakhimov, A. A., Ahmedov, A. A., & Hishamuddin, Z. (2012). On the Spectral Expansions of Distributions Connected with Schrodinger Operator. *Applied Mathematics Letter*(25), 921-924.

[8] Rakhimov, A. A. (1996). Spectral Decompositions of Distributions from Negative Sobolev Classes. *Diff. Equations*(32), 1011-1013.

[9] Rakhimov, A. A. (2017). On the uniform convergence of Fourier series on a closed domain. *Eurasian Mathematical Journal, 8*(3), 60-69.

[10] Rakhimov, A. A. (2016). On the uniform convergence of Fourier series. *Malaysian Journal of Mathematical Sciences, 10*(S), 55-60.

[11] Ahmedov, A. A., Nurullah, A. F., & Rakhimov, A. A. (2013). Localization of Fourier-Laplace Series of Distributions. *Malaysian Journal of Mathematical Sciences, 7*(2), 315-326.

[12] Ashurov, R. R. (1991). On the summability of multiple trigonometrical Fourier series. *Mathematical Notes, 49*(6), 563-568.