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Contents

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Generalizations of Lindelöf spaces via hereditary classes

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Abstract. In this paper by using hereditary classes [6], we define the notion of γ-Lindelöf modulo hereditary classes called $\gamma\mathcal{H}$-Lindelöf and obtain several properties of $\gamma\mathcal{H}$-Lindelöf spaces.

1 Introduction

Let (X, τ) be a topological space and $\mathcal{P}(X)$ the power set of X. In 1991, Ogata [13] introduced the notions of γ-operations and γ-open sets and investigated the associated topology τ_γ and weak separation axioms γ-T_i ($i = 0, 1/2, 1, 2$). In 2011, Noiri [10] defined an operation on an m-structure with property $\mathcal{B}$ (the generalized topology in the sense of Lugojan [8]). The operation is defined as a function $\gamma : m \to \mathcal{P}(X)$ such that $U \subseteq \gamma(U)$ for each $U \in m$ and is called a γ-operation on m. Then, it terns out that the operation is an unified form of several operations (for example, semi-γ-operation [7], pre-γ-operation [4]) defined on the family of generalized open sets. Moreover, he obtained some characterizations of γ-compactness and suggested some generalizations of compact spaces by using recent modifications of open sets in a topological space.

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In this paper by using hereditary classes [6], we define the notion of γ-Lindelöf modulo hereditary classes called $\gamma\mathcal{H}$-Lindelöf and obtain several properties of $\gamma\mathcal{H}$-Lindelöf spaces. Recently papers [1, 2, 3] have introduced some new classes of sets via hereditary classes.

2 Preliminaries

First we state the following: in [11], a minimal structure $\mathfrak{m}$ is defined as follows: $\mathfrak{m}$ is called a minima structure if $\emptyset, X \in \mathfrak{m}$. However, in this paper, we define as follows:

Definition 1 *Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily $\mathfrak{m}$ of $\mathcal{P}(X)$ is called a minimal structure (briefly $\mathfrak{m}$-structure) on X if $\mathfrak{m}$ satisfies the following conditions:*

1. $\emptyset, X \in \mathfrak{m}$.
2. *The union of any family of subsets belonging to $\mathfrak{m}$ belongs to $\mathfrak{m}$.*

A set X with an $\mathfrak{m}$-structure is called an $\mathfrak{m}$-space and denoted by $(X, \mathfrak{m})$. Each member of $\mathfrak{m}$ is said to be $\mathfrak{m}$-open and the complement of an $\mathfrak{m}$-open set is said to be $\mathfrak{m}$-closed.

Definition 2 [9] *Let X be a nonempty set and $\mathfrak{m}$ an $\mathfrak{m}$-structure on X. For a subset A of X, the $\mathfrak{m}$-closure of A is defined as follows: $\mathfrak{m}cl(A) = \cap\{F : A \subseteq F, X \setminus F \in \mathfrak{m}\}$.*

Lemma 1 [9] *Let X be a nonempty set and $\mathfrak{m}$ an $\mathfrak{m}$-structure on X. For the $\mathfrak{m}$-closure, the following properties hold, where A and B are subsets of X:*

1. $A \subseteq \mathfrak{m}cl(A)$,
2. $\mathfrak{m}cl(\emptyset) = \emptyset$, $\mathfrak{m}cl(X) = X$,
3. *If* $A \subseteq B$, *then* $\mathfrak{m}cl(A) \subseteq \mathfrak{m}cl(B)$,
4. $\mathfrak{m}cl(\mathfrak{m}cl(A)) = \mathfrak{m}cl(A)$.

Lemma 2 [14] *Let $(X, \mathfrak{m})$ be an $\mathfrak{m}$-space and A a subset of X. Then $x \in \mathfrak{m}cl(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in \mathfrak{m}$ containing x.*

Lemma 3 [15] *Let* (X, m) *be an* m*-space and* A *a subset of* X*. Then, the following properties hold:*

1. A *is* m*-closed if and only if* $mcl(A) = A$*,*

2. $mcl(A)$ *is* m*-closed.*

Definition 3 [10] *Let* (X, m) *be an* m*-space and* γ *an operation on* m*. A subset* A *of* X *is said to be* γ*-open if for each* $x \in A$ *there exists* $U \in m$ *such that* $x \in U \subseteq \gamma(U) \subseteq A$*. The complement of a* γ*-open set is said to be* γ*-closed. The family of all* γ*-open sets of* (X, m) *is denoted by* $\gamma(X)$*.*

3 $\gamma\mathcal{H}$-Lindelöf spaces

First, we recall the definition of a hereditary class used in the sequel. A subfamily $\mathcal{H}$ of the power set $\mathcal{P}(X)$ is called a hereditary class on X [6] if it satisfies the following property: $A \in \mathcal{H}$ and $B \subseteq A$ implies $B \in \mathcal{H}$.

Definition 4 *Let* $(X, m, \mathcal{H})$ *be a hereditary* m*-space and* γ *an operation on* m*, where* $\mathcal{H}$ *is a hereditary class on* X*. Then* m*-space* (X, m) *is said to be* $\gamma\mathcal{H}$*-Lindelöf (resp.* $\mathcal{H}$*-Lindelöf) if every cover* $\{U_\alpha : \alpha \in \Delta\}$ *of* X *by* m*-open sets, there exists a countable subset* Δ_0 *of* Δ *such that* $X \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$ *(resp.* $X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$*).*

Theorem 1 *Let* $(X, m, \mathcal{H})$ *be a hereditary* m*-space and* γ *an operation on* m*, where* $\mathcal{H}$ *is a hereditary class. Then the following properties are equivalent:*

1. $(X, \gamma(X))$ *is* $\mathcal{H}$*-Lindelöf;*

2. *For every family* $\{F_\alpha : \alpha \in \Delta\}$ *of* γ*-closed sets such that* $\cap\{F_\alpha : \alpha \in \Delta_0\} \notin \mathcal{H}$ *for every countable subfamily* Δ_0 *of* Δ*,* $\cap\{F_\alpha : \alpha \in \Delta\} \neq \emptyset$*.*

Proof. (1) $\Rightarrow$ (2): Let $(X, \gamma(X))$ be $\mathcal{H}$-Lindelöf. Suppose that $\cap\{F_\alpha : \alpha \in \Delta\} = \emptyset$, where F_α is γ-closed set. Then $X \setminus F_\alpha$ is γ-open for each $\alpha \in \Delta$ and $\cup_{\alpha\in\Delta}(X \setminus F_\alpha) = X \setminus \cap_{\alpha\in\Delta}(F_\alpha) = X$. By (1), there exists a countable subfamily Δ_0 of Δ such that $X \setminus \cup_{\alpha\in\Delta_0}(X \setminus F_\alpha) = \cap\{F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. This is a contradiction.
(2) $\Rightarrow$ (1): Suppose that $(X, \gamma(X))$ is not $\mathcal{H}$-Lindelöf. There exists a cover $\{U_\alpha : \alpha \in \Delta\}$ of X by γ-open sets such that $X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \notin \mathcal{H}$ for

every countable subset Δ_0 of Δ. Since $X \setminus U_\alpha$ is γ-closed for each $\alpha \in \Delta$ and $\cap\{(X \setminus U_\alpha) : \alpha \in \Delta_0\} \notin \mathcal{H}$ for every countable subset Δ_0 of Δ. By (2), we have $\cap\{(X \setminus U_\alpha) : \alpha \in \Delta\} \neq \emptyset$. Therefore, $X \setminus \cup\{U_\alpha : \alpha \in \Delta\} \neq \emptyset$. This is contrary that $\{U_\alpha : \alpha \in \Delta\}$ is a γ-open cover of X.

□

Lemma 4 [10] *Let (X, m) be an m-space. For $\gamma(X)$, the following properties hold:*

1. $\emptyset$, $X \in \gamma(X)$,
2. *If $A_\alpha \in \gamma(X)$ for each $\alpha \in \Lambda$, then $\cup_{\alpha\in\Lambda} A_\alpha \in \gamma(X)$,*
3. $\gamma(X) \subseteq m$.

Definition 5 [10] *An m-space (X, m) is said to be γ-regular if for each $x \in X$ and each $U \in m$ containing x, there exists $V \in m$ such that $x \in V \subseteq \gamma(V) \subseteq U$.*

Lemma 5 [10] *For an m-space (X, m), the following properties are equivalent:*

1. $m = \gamma(X)$;
2. (X, m) *is γ-regular;*
3. *For each $x \in X$ and each $U \in m$ containing x, there exists $W \in \gamma(X)$ such that $x \in W \subseteq \gamma(W) \subseteq U$.*

Theorem 2 *Let $(X, m, \mathcal{H})$ be a hereditary m-space and γ an operation on m, where $\mathcal{H}$ is a hereditary class. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ hold. If (X, m) is γ-regular, then the following properties are equivalent:*

1. (X, m) *is $\mathcal{H}$-Lindelöf;*
2. (X, m) *is $\gamma\mathcal{H}$-Lindelöf;*
3. $(X, \gamma(X))$ *is $\mathcal{H}$-Lindelöf;*
4. $(X, \gamma(X))$ *is $\gamma\mathcal{H}$-Lindelöf.*

Proof. (1) $\Rightarrow$ (2): Let (X, m) be $\mathcal{H}$-Lindelöf. For any cover $\{U_\alpha : \alpha \in \Delta\}$ of X by m-open sets, there exists a countable subset Δ_0 of Δ such that $X \setminus \cup\{\gamma(U\alpha) : \alpha \in \Delta_0\} \subseteq X \setminus \cup\{U\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Therefore, (X, m) is $\gamma\mathcal{H}$-Lindelöf.

(2) $\Rightarrow$ (3): Let (X, m) be $\gamma\mathcal{H}$-Lindelöf and $\{U_\alpha : \alpha \in \Delta\}$ a cover of X by γ-open sets. For each $x \in X$ there exists $\alpha(x) \in \Delta$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is γ-open, there exists $V_{\alpha(x)} \in m$ such that $x \in V_{\alpha(x)} \subseteq \gamma(V_{\alpha(x)}) \subseteq U_{\alpha(x)}$. Since the family $\{V_{\alpha(x)} : x \in X\}$ is a cover of X by m-open sets and (X, m) is $\gamma\mathcal{H}$-Lindelöf, there exists a countable number of points, say, $x_1, x_2, x_3, \cdots \in X$ such that $X \setminus \cup_{i=1}^{\infty}\gamma(V_{\alpha(x_i)}) \in \mathcal{H}$ and hence $X \setminus \cup_{i=1}^{\infty}U_{\alpha(x_i)} \in \mathcal{H}$. This shows that $(X, \gamma(X))$ is $\mathcal{H}$-Lindelöf.

(3) $\Rightarrow$ (4): By Lemma 4, $\gamma(X)$ is an m-structure and it follows that the same argument as (1) $\Rightarrow$ (2) that $(X, \gamma(X))$ is $\gamma\mathcal{H}$-Lindelöf.

(4) $\Rightarrow$ (1): Suppose that (X, m) is γ-regular. Let $(X, \gamma(X))$ be $\gamma\mathcal{H}$-Lindelöf. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of X by m-open sets. For each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $x \in U_{\alpha(x)}$. Since (X, m) is γ-regular, by Lemma 5 there exists $V_{\alpha(x)} \in \gamma(X)$ such that $x \in V_{\alpha(x)} \subseteq \gamma(V_{\alpha(x)}) \subseteq U_{\alpha(x)}$. Since $\{V_{\alpha(x)} : x \in X\}$ is a cover of X by γ-open sets and $(X, \gamma(X))$ is $\gamma\mathcal{H}$-Lindelöf, there exist a countable number of points, say, $x_1, x_2, x_3, \cdots \in X$ such that $X \setminus \cup_{i=1}^{\infty}\gamma(V_{\alpha(x_i)}) \in \mathcal{H}$; and hence $X \setminus \cup_{i=1}^{\infty}U_{\alpha(x_i)} \in \mathcal{H}$. This shows that (X, m) is $\mathcal{H}$-Lindelöf. □

Definition 6 *Let $(X, m, \mathcal{H})$ be a hereditary m-space. A subset A of X is said to be $\gamma\mathcal{H}$-Lindelöf relative to X if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by m-open sets of X, there exists a countable subset Δ_0 of Δ such that $A \setminus \cup\{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$.*

Theorem 3 *Let $(X, m, \mathcal{H})$ be a hereditary m-space. If A is γ-closed and B is $\gamma\mathcal{H}$-Lindelöf relative to X, then $A \cap B$ is $\gamma\mathcal{H}$-Lindelöf relative to X.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of $A \cap B$ by m-open subsets of X. Then $\{V_\alpha : \alpha \in \Delta\} \cup \{X \setminus A\}$ is a cover of B by m-open sets. Since $X \setminus A$ is γ-open, for each $x \in X \setminus A$, there exists an m-open set V_x such that $x \in V_x \subseteq \gamma(V_x) \subseteq X \setminus A$. Thus $\{V_\alpha : \alpha \in \Delta\} \cup \{V_x : x \in X \setminus A\}$ is a cover of B by m-open sets of X. Since B is $\gamma\mathcal{H}$-Lindelöf relative to X, there exist a countable subset Δ_0 of Δ and a countable points, says $x_1, x_2, \cdots \in X \setminus A$ such that $B \subseteq [(\cup_{\alpha\in\Delta_0}\gamma(V_\alpha)) \cup (\cup_{i=1}^{\infty}\gamma(V_{x_i}))] \cup H_0 \in \mathcal{H}$, where $H_0 \in \mathcal{H}$. Hence $A \cap B \subseteq [(\cup_{\alpha\in\Delta_0}\gamma(V_\alpha) \cap A) \cup (\cup_{i=1}^{\infty}\gamma(V_{x_i}) \cap A)] \cup (A \cap H_0) \subseteq \cup_{\alpha\in\Delta_0}\gamma(V_\alpha) \cup H_0$. Therefore, $A \cap B \setminus (\cup_{\alpha\in\Delta_0}\gamma(V_\alpha)) \subseteq H_0 \in \mathcal{H}$. Hence $A \cap B$ is $\gamma\mathcal{H}$-Lindelöf relative to X. □

Corollary 1 *If a hereditary m-space $(X, m, \mathcal{H})$ is $\gamma\mathcal{H}$-Lindelöf space, then every γ-closed subset of X is $\gamma\mathcal{H}$-Lindelöf relative to X.*

Proof. The proof is obvious by Theorem 3. □

Lemma 6 [12] *For a hereditary* $\mathfrak{m}$*-space* $(X, \mathfrak{m}, \mathcal{H})$*, the following properties hold:*

1. $\mathfrak{m}_H^*$ *is an* $\mathfrak{m}$*-structure on* X *such that* $\mathfrak{m}_H^*$ *has property* $\mathcal{B}$ *and* $\mathfrak{m} \subseteq \mathfrak{m}_H^*$.

2. $\beta(\mathfrak{m}, \mathcal{H}) = \{U \setminus H : U \in \mathfrak{m}, H \in \mathcal{H}\}$ *is a basis for* $\mathfrak{m}_H^*$*. such that* $\mathfrak{m} \subseteq \beta(\mathfrak{m}, \mathcal{H})$.

Theorem 4 *Let* $(X, \mathfrak{m}, \mathcal{H})$ *be a hereditary* $\mathfrak{m}$*-space. Then the following properties hold:*

1. *If* $(X, \mathfrak{m}_H^*, \mathcal{H})$ *is* $\mathcal{H}$*-Lindelöf, then* $(X, \mathfrak{m}, \mathcal{H})$ *is* $\mathcal{H}$*-Lindelöf.*

2. *If* $(X, \mathfrak{m}, \mathcal{H})$ *is* $\mathcal{H}$*-Lindelöf and* $\mathcal{H}$ *is closed under countable union, then* $(X, \mathfrak{m}_H^*, \mathcal{H})$ *is* $\mathcal{H}$*-Lindelöf.*

Proof. (1): The proof follows directly from the fact that every $\mathfrak{m}$-closed set is $\mathfrak{m}_H^*$-closed.

(2): Suppose that $\mathcal{H}$ is closed under countable union and X is $\mathcal{H}$-Lindelöf. Let $\{U_\alpha : \alpha \in \Delta\}$ be an $\mathfrak{m}_H^*$-open cover of X, then for each $x \in X$, $x \in U_{\alpha(x)}$ for some $\alpha(x) \in \Delta$. By Lemma 6 there exist $V_{\alpha(x)} \in \mathfrak{m}$ and $H_{\alpha(x)} \in \mathcal{H}$ such that $x \in V_{\alpha(x)} \setminus H_{\alpha(x)} \subseteq U_{\alpha(x)}$. Since $\{V_{\alpha(x)} : \alpha(x) \in \Delta\}$ is an $\mathfrak{m}$-open cover of X, there exists a countable subset Δ_0 of Δ such that $X \setminus \cup\{V_{\alpha(x)} : \alpha(x) \in \Delta_0\} = H \in \mathcal{H}$. Since $\mathcal{H}$ is closed under countable union, then $\cup\{H_{\alpha(x)} : \alpha(x) \in \Delta_0\} \in \mathcal{H}$. Hence, $H \cup \left[\cup\{H_{\alpha(x)} : \alpha(x) \in \Delta_0\}\right] \in \mathcal{H}$. Observe that $X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \subseteq H \cup \left[\cup\{H_{\alpha(x)} : \alpha(x) \in \Delta_0\}\right] \in \mathcal{H}$. By the heredity property of $\mathcal{H}$ we have $X \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$ and therefore, $(X, \mathfrak{m}_H^*, \mathcal{H})$ is $\mathcal{H}$-Lindelöf. □

4 Strongly $\mathcal{H}$-Lindelöf spaces

Definition 7 *A subset* A *of a hereditary* $\mathfrak{m}$*-space* $(X, \mathfrak{m}, \mathcal{H})$ *is said to be:*

1. *Strongly* $\mathcal{H}$*-Lindelöf relative to* X *if for every family* $\{V_\alpha : \alpha \in \Delta\}$ *of* $\mathfrak{m}$*-open sets such that* $A \setminus \cup_{\alpha \in \Delta} V_\alpha \in \mathcal{H}$*, there exists a countable subset* Δ_0 *of* Δ *such that* $A \setminus \cup_{\alpha \in \Delta_0} V_\alpha \in \mathcal{H}$*. If* $A = X$*, then* $(X, \mathfrak{m}, \mathcal{H})$ *is said to be Strongly* $\mathcal{H}$*-Lindelöf.*

2. *Strongly $\gamma\mathcal{H}$-Lindelöf relative to* X *if for every family* $\{V_\alpha : \alpha \in \Delta\}$ *of* m*-open sets such that* $A \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$*, there exists a countable subset* Δ_0 *of* Δ *such that* $A \setminus \cup_{\alpha\in\Delta_0}\gamma(V_\alpha) \in \mathcal{H}$*. If* $A = X$*, then* $(X, m, \mathcal{H})$ *is said to be Strongly* $\gamma\mathcal{H}$*-Lindelöf.*

Theorem 5 *Let* $(X, m, \mathcal{H})$ *be a hereditary* m*-space. Then the following properties hold:*

1. *If* $(X, m^*_H, \mathcal{H})$ *is Strongly* $\mathcal{H}$*-Lindelöf, then* $(X, m, \mathcal{H})$ *is Strongly* $\mathcal{H}$*-Lindelöf.*
2. *If* $(X, m, \mathcal{H})$ *is Strongly* $\mathcal{H}$*-Lindelöf and* $\mathcal{H}$ *is closed under countable union, then* $(X, m^*_H, \mathcal{H})$ *is Strongly* $\mathcal{H}$*-Lindelöf.*

Theorem 6 *Let* $(X, m, \mathcal{H})$ *be a hereditary* m*-space. Then the following properties are equivalent:*

1. $(X, m, \mathcal{H})$ *is Strongly* $\mathcal{H}$*-Lindelöf;*
2. *If* $\{F_\alpha : \alpha \in \Delta\}$ *is a family of* m*-closed sets such that* $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$*, then there exists a countable subfamily* Δ_0 *of* Δ *such that* $\cap\{F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$*.*

Proof. Suppose that $(X, m, \mathcal{H})$ is Strongly $\mathcal{H}$-Lindelöf. Let $\{F_\alpha : \alpha \in \Delta\}$ be a family of m-closed sets such that $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Then $\{X \setminus F_\alpha : \alpha \in \Delta\}$ is a family of m-open sets of X. Let $H = \cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$. Let $X \setminus H = X \setminus \cap\{F_\alpha : \alpha \in \Delta\} = \cup\{X \setminus F_\alpha : \alpha \in \Delta\}$. Since $(X, m, \mathcal{H})$ is Strongly $\mathcal{H}$-Lindelöf, there exists a countable Δ_0 of Δ such that $X \setminus \cup\{X \setminus F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. This implies that $\cap\{F_\alpha : \alpha \in \Delta\} \in \mathcal{H}$.

Conversely, let $\{V_\alpha : \alpha \in \Delta\}$ be any family of m-open sets of X such that $X \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. Then $\{X \setminus V_\alpha : \alpha \in \Delta\}$ is a family of m-closed sets of X. By assumption we have $\cap\{X \setminus V_\alpha : \alpha \in \Delta\} \in \mathcal{H}$ and there exists a countable subset Δ_0 of Δ such that $\cap\{X \setminus V_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. This implies that $X \setminus \cup\{V_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. This shows that $(X, m, \mathcal{H})$ is Strongly $\mathcal{H}$-Lindelöf.

□

Definition 8 *A subset* A *of a hereditary* m*-space* $(X, m, \mathcal{H})$ *is said to be* $m\mathcal{H}_g$*-closed if for every* $U \in m$ *with* $A \setminus U \in \mathcal{H}$*,* $mcl(A) \subseteq U$*.*

Proposition 1 *Let* $(X, m, \mathcal{H})$ *be a hereditary* m*-space. If* $(X, m, \mathcal{H})$ *is Strongly* $\mathcal{H}$*-Lindelöf and* $A \subseteq X$ *is* $m\mathcal{H}_g$*-closed, then* A *is Strongly* $\mathcal{H}$*-Lindelöf relative to* X*.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a family of m-open subsets of X such that $A \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. Since A is $m\mathcal{H}_g$-closed, $mcl(A) \subseteq \cup_{\alpha\in\Delta} V_\alpha$. Then $(X \setminus mcl(A)) \cup [\cup_{\alpha\in\Delta} V_\alpha]$ is an m-open cover of X and so $X \setminus [(X \setminus mcl(A)) \cup [\cup_{\alpha\in\Delta} V_\alpha]] \in \mathcal{H}$. Since X is Strongly $\mathcal{H}$-Lindelöf, there exists a countable subset Δ_0 of Δ such that $X \setminus [(X \setminus mcl(A)) \cup [\cup_{\alpha\in\Delta_0} V_\alpha]] \in \mathcal{H}$. $X \setminus [(X \setminus mcl(A)) \cup [\cup_{\alpha\in\Delta_0} V_\alpha]] = mcl(A) \cap (X \setminus \cup_{\alpha\in\Delta_0} V_\alpha) \supseteq A \setminus \cup_{\alpha\in\Delta_0} V_\alpha$. Therefore, $A \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$. Thus, A is Strongly $\mathcal{H}$-Lindelöf relative to X. □

Theorem 7 *Let $(X, m, \mathcal{H})$ be a hereditary m-space. Let A be an $m\mathcal{H}_g$-closed set such that $A \subseteq B \subseteq mcl(A)$. Then A is Strongly $\mathcal{H}$-Lindelöf elative to X if and only if B is Strongly $\mathcal{H}$-Lindelöf relative to X.*

Proof.
Suppose that A is Strongly $\mathcal{H}$-Lindelöf elative to X and $\{V_\alpha : \alpha \in \Delta\}$ is a family of m-open sets of X such that $B \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. By the heredity property, $A \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$ and A is Strongly $\mathcal{H}$-Lindelöf elative to X and hence there exists a countable subset Δ_0 of Δ such that $A \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$. Since A is $m\mathcal{H}_g$-closed, $mcl(A) \subseteq \cup_{\alpha\in\Delta_0} V_\alpha$ and so $mcl(A) \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$. This implies that $B \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$.

Conversely, suppose that B is Strongly $\mathcal{H}$-Lindelöf elative to X and $\{V_\alpha : \alpha \in \Delta\}$ is a family of m-open subsets of X such that $A \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. Given that A is $m\mathcal{H}_g$-closed, $mcl(A) \setminus \cup_{\alpha\in\Delta} V_\alpha = \emptyset \in \mathcal{H}$ and this implies $B \subseteq \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. Since B is Strongly $\mathcal{H}$-Lindelöf elative to X, there exists a countable subset Δ_0 of Δ such that $B \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$. Hence $A \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$. □

5 (γ, δ)-continuous functions

Definition 9 *Let (X, m) and (Y, n) be minimal spaces and γ (resp. δ) be an operation on m (resp. n). Then a function $f : (X, m) \to (Y, n)$ is said to be (γ, δ)-continuous if for each $x \in X$ and each $V \in n$ containing $f(x)$, there exists $U \in m$ containing x such that $f(\gamma(U)) \subseteq \delta(V)$.*

Lemma 7 *Let $f : X \to Y$ be a function.*

1. *If $\mathcal{H}$ is a hereditary class on X, then $f(\mathcal{H})$ is a hereditary class on Y.*
2. *If $\mathcal{H}$ is a hereditary class on Y, then $f^{-1}(\mathcal{H})$ is a hereditary class on X.*

Proof. (1): This is due to Lemma 3.8 of [5].
(2): Let $A \subseteq f^{-1}(H)$, where $H \in \mathcal{H}$. Then $f(A) \subseteq f(f^{-1}(H)) \subseteq H$. Hence $f(A) \in \mathcal{H}$ and $A \subseteq f^{-1}(f(A)) \in f^{-1}(\mathcal{H})$ and hence $A \in f^{-1}(\mathcal{H})$. □

Theorem 8 *Let* (X, m) *and* (Y, n) *be minimal spaces and* γ *(resp.* δ*) be an operation on* m *(resp.* n*) and* $\mathcal{H}$ *be a hereditary class on* X*. If* $(X, m, \mathcal{H})$ *is* $\gamma\mathcal{H}$*-Lindelöf and* $f : (X, m, \mathcal{H}) \to (Y, n)$ *is a* (γ, δ)*-continuous surjection, then* $(Y, n, f(\mathcal{H}))$ *is* $\delta f(\mathcal{H})$*-Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of Y by n-open sets. For each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since f is (γ, δ)-continuous, there exists $U_{\alpha(x)} \in m$ containing x such that $f(\gamma(U_{\alpha(x)})) \subseteq \delta(V_{\alpha(x)})$. Since $\{U_{\alpha(x)} : x \in X\}$ is a cover of X by m-open sets and $(X, m, \mathcal{H})$ is $\gamma\mathcal{H}$-Lindelöf, there exist a countable points $x_1, x_2, x_3, \cdots \in X$ such that $X \setminus \cup_{i=1}^{\infty}\gamma(U_{\alpha(x_i)}) = H_0$, where $H_0 \in \mathcal{H}$. Therefore, we have $Y \subseteq f(\cup_{i=1}^{\infty}\gamma(U_{\alpha(x_i)})) \cup f(H_0) \subseteq \cup_{i=1}^{\infty}\delta(V_{\alpha(x_i)}) \cup f(H_0)$. Hence $(Y, n, f(\mathcal{H}))$ is $\delta f(\mathcal{H})$-Lindelöf. □

Definition 10 [11] *A function* $f : (X, m) \to (Y, n)$ *is said to be* M*-closed if for each* m*-closed set* F *of* X*,* $f(F)$ *is* n*-closed in* Y*.*

Theorem 9 *Let* $f : (X, m) \to (Y, n, \mathcal{H})$ *be an* M*-closed surjective function. If for every* $y \in Y$*,* $f^{-1}(y)$ *is Strongly* $f^{-1}(\mathcal{H})$*-Lindelöf in* X*, then* $f^{-1}(A)$ *is Strongly* $f^{-1}(\mathcal{H})$*-Lindelöf relative to* X *whenever* A *is Strongly* $\mathcal{H}$*-Lindelöf relative to* Y *and* $A \setminus U \in \mathcal{H}$ *for every* $U \in n$*.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a family of m-open subsets of X such that $f^{-1}(A) \setminus \cup\{V_\alpha : \alpha \in \Delta\} \in f^{-1}(\mathcal{H})$. For each $y \in A$ there exists a countable subset $\Delta_0(y)$ of Δ such that $f^{-1}(y) \setminus \cup\{V_\alpha : \alpha \in \Delta_0(y)\} \in f^{-1}(\mathcal{H})$. Let $V_y = \cup\{V_\alpha : \alpha \in \Delta_0(y)\}$. Each V_y is an m-open set in (X, m) and $f^{-1}(y) \setminus V_y \in f^{-1}(\mathcal{H})$.

Now each set $f(X - V_y)$ is n-closed in Y and hence, $U(y) = Y - f(X - V_y)$ is n-open in Y. Note that $f^{-1}(U(y)) \subseteq V_y$. Thus, $\{U(y) : y \in A\}$ is a family of n-open subsets of Y such that $A \setminus \cup\{U(y) : y \in A\} \in \mathcal{H}$. Since A is Strongly $\mathcal{H}$-Lindelöf relative to Y, there exists a countable subset $\{U(y_i) : i \in \mathbb{N}\}$ such that $A \setminus \cup\{U(y_i) : i \in \mathbb{N}\} \in \mathcal{H}$ and hence $f^{-1}[A \setminus \cup\{U(y_i) : i \in \mathbb{N}\}] = f^{-1}(A) \setminus \cup\{f^{-1}(U(y_i)) : i \in \mathbb{N}\} \in f^{-1}(\mathcal{H})$. Since $f^{-1}(A) \setminus \cup\{V_{y_i} : i \in \mathbb{N}\} \subseteq f^{-1}(A) \setminus \cup\{f^{-1}(U(y_i)) : i \in \mathbb{N}\}$, then $f^{-1}(A) \setminus \cup\{V_{y_i} : i \in \mathbb{N}\} = f^{-1}(A) \setminus \cup\{V_\alpha : \alpha \in \Delta_0(y_i), i \in \mathbb{N}\} \in f^{-1}(\mathcal{H})$. Hence, $f^{-1}(A)$ is Strongly $f^{-1}(\mathcal{H})$-Lindelöf relative to X. □

A subset K of an m-space is said to be m-compact [14] if every cover of K by m-open sets of X has a finite subcover.

Theorem 10 *Let* $f : (X, m) \to (Y, n, \mathcal{H})$ *be an* M*-closed surjective function. If for every* $y \in Y$*,* $f^{-1}(y)$ *is* m*-compact in* X*, then* $f^{-1}(A)$ *is* $f^{-1}(\mathcal{H})$*-Lindelöf relative to* X *whenever* A *is* $\mathcal{H}$*-Lindelöf relative to* Y*.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of $f^{-1}(A)$ by $\mathfrak{m}$-open sets of X. For each $y \in A$ there exists a finite subset $\Delta_0(y)$ of Δ such that $f^{-1}(y) \subseteq \cup\{V_\alpha : \alpha \in \Delta_0(y)\}$. Let $V_y = \cup\{V_\alpha : \alpha \in \Delta_0(y)\}$. Each V_y is an $\mathfrak{m}$-open set in $(X, \mathfrak{m})$ and $f^{-1}(y) \subseteq V_y$. Since f is M-closed, by Theorem 3.1 of [11] there exists an $\mathfrak{n}$-open set U_y containing y such that $f^{-1}(U_y) \subseteq V_y$. The collection $\{U_y : y \in A\}$ is a cover of A by $\mathfrak{n}$-open sets of Y. Hence, there exists a countable subcollection $\{U_{y(i)} : i \in \mathbb{N}\}$ such that $A \setminus \cup\{U_{y(i)} : i \in \mathbb{N}\} \in \mathcal{H}$. Then $f^{-1}(A \setminus \cup\{U_{y(i)} : i \in \mathbb{N}\}) = f^{-1}(A) \setminus \cup\{f^{-1}(U_{y(i)}) : i \in \mathbb{N}\} \in f^{-1}(\mathcal{H})$. Since $f^{-1}(A) \setminus \cup\{V_{y(i)} : i \in \mathbb{N}\} \subseteq f^{-1}(A) \setminus \cup\{f^{-1}(U_{y(i)}) : i \in \mathbb{N}\})$, then $f^{-1}(A) \setminus \cup\{V_{y(i)} : i \in \mathbb{N}\} \in f^{-1}(\mathcal{H})$. Thus, $f^{-1}(A)$ is $f^{-1}(\mathcal{H})$-Lindelöf relative to X. □

References

[1] A. Al-Omari and T. Noiri, On operators in ideal minimal spaces, *Mathematica*, **58** (81), No 1-2, (2016), 3–13.

[2] A. Al-Omari and T. Noiri, Operators in minimal spaces with hereditary classes, *Mathematica*, **61** (84), No 2, (2019), 101–110.

[3] A. Al-Omari and T. Noiri, Properties of $\gamma\mathcal{H}$-compact spaces with hereditary classes, *Atti della Accademia Peloritana dei Pericolanti, Classe di Scienze Fisiche, Matematiche e Naturali*, **98** No. 2 (2020), A4 [11 pages].

[4] T. V. An, D. X. Cuong and H. Maki, On operation-preopen sets in topological spaces, *Sci. Math. Japon.*, **68** (2008), 11–30.

[5] C. Carpintero, E. Rosas, M. Salas-Brown and J. Sanabria, μ-Compactness with respect to a hereditary class, *Bol. Soc. Paran. Mat.* (3s), **34**(2) (2016), 231–236.

[6] Á. Császár, Modification of generalized topologies via hereditary classes, *Acta Math. Hungar.*, **115** (1-2) (2007), 29–35.

[7] G. Sai Sundara Krishnan, M. Ganster and K. Balachandran, Operation approaches on semi-open sets and applications, *Kochi J. Math.*, **2** (2007), 21–33.

[8] S. Lugojan, Generalized topology, *Stud. Cerc. Mat.*, **34** (1982), 348–360.

[9] H. Maki, K. C. Rao and A. Nagoor Gani, On generalization semi-open and preopen sets, *Pure Appl. Math.*, **49** (2011), 17–29.

[10] T. Noiri, A unifed theory for generalizations of compact spaces, *Ann. Univ. Sci. Budapest.*, **54** (2011), 79–96.

[11] T. Noiri and V. Popa, A unified theory of closed functions, *Bull. Math. Soc. Sci. Math. Roumanie*, **49**(**97**)(4) (2006), 371–382.

[12] T. Noiri and V. Popa, Generalizations of closed sets in minimal spaces with hereditary classes, *Ann. Univ. Sci. Budapest.*, **61** (2018), 69-83.

[13] H. Ogata, Operations on topological spaces and associated topology, *Math. Japan.*, **36** (1991), 175–184.

[14] V. Popa and T. Noiri, On M-continuous functions, *An. Univ. "Dunarea de Jos" Galati, Ser. Mat. Fiz. Mec. Teor.* (2), **18(23)** (2000), 31–41.

[15] V. Popa and T. Noiri, On weakly $(\tau, \mathfrak{m})$-continuous functions, *Rend. Circ. Mat. Palermo* (2), **51** (2002), 295–316.

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On expansive homeomorphism of uniform spaces

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Abstract. We study the notion of expansive homeomorphisms on uniform spaces. It is shown that if there exists a topologically expansive homeomorphism on a uniform space, then the space is always a Hausdorff space and hence a regular space. Further, we characterize orbit expansive homeomorphisms in terms of topologically expansive homeomorphisms and conclude that if there exist a topologically expansive homeomorphism on a compact uniform space then the space is always metrizable.

1 Introduction

A homeomorphism $h : X \longrightarrow X$ defined on metric space X is said to be an *expansive homeomorphism* provided there exists a real number $c > 0$ such that whenever $x, y \in X$ with $x \neq y$ then there exists an integer n (depending on x, y) satisfying $d(h^n(x), h^n(y)) > c$. Constant c is called an *expansive constant* for h. In 1950, Utz, [18], introduced the concept of expansive homeomorphisms

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with the name unstable homeomorphisms. The examples discussed in this paper on compact spaces were sub dynamics of shift maps, thus one can say that the theory of expansive homeomorphisms started based on symbolic dynamics but it quickly developed by itself.

Much attention has been paid to the existence / non–existence of expansive homeomorphisms on given spaces. Each compact metric space that admits an expansive homeomorphism is finite-dimensional [13]. The spaces admitting expansive homeomorphisms include the Cantor set, the real line/half-line, all open $n-$cells, $n \geq 2$ [12]. On the other hand, spaces not admitting expansive homeomorphisms includes any Peano continuum in the plane [9], the 2-sphere the projective plane and the Klein bottle [8].

Another important aspects of expansive homeomorphism is the study of its various generalizations and variations in different setting. The very first of such variation was given by Schwartzman, [16], in 1952 in terms of positively expansive maps, wherein the points gets separated by non–negative iterates of the continuous map. In 1970, Reddy, [14], studied point–wise expansive maps whereas $h-$expansivity was studied by R. Bowen, [4]. Kato defined and studied the notion of continuum–wise expansive homeomorphism [10]. Shah studied notion of positive expansivity of maps on metric $G-$spaces [17] whereas Barzanouni studied finite expansive homeomorphisms [2]. Tarun Das *et al.* [7] used the notion of expansive homeomorphism on topological space to prove the Spectral Decomposition Theorem on non–compact spaces. Achigar *et al.* studied the notion of orbit expansivity on non–Hausdorff space [1]. Authors in [3] studied expansivity for group actions. In this paper we study expansive homeomorphisms on uniform spaces.

In Section 2 we discuss preliminaries regarding uniform spaces and expansive homeomorphisms on metric /topological space required for the content of the paper. The notion of expansive homeomorphisms on topological spaces was first studied in [7] whereas on uniform spaces was first studied in [6] in the form of positively topological expansive maps. In Section 3 of this paper we define and study expansive homeomorphism on uniform spaces. Through examples it is justified that topologically expansive homeomorphism is weaker than metric expansive homeomorphism whereas stronger than expansive homeomorphism defined on topological space. Further, we show that if a uniform space admits a topologically expansive homeomorphism then the space is always a Hausdorff space and hence a regular space. The notion of orbit expansivity was first introduced in [1]. A characterization of orbit expansive homeomorphism on compact uniform spaces is obtained in terms of topologically expansive homeomorphism. As a consequence of this we conclude that if there is a topologically expansive homeomorphism on a compact uniform space then the space is always metrizable.

2 Preliminaries

In this Section we discuss basics required for the content of the paper.

2.1 Uniform spaces

Uniform spaces were introduced by A. Weil [19] as a generalization of metric spaces and topological groups. Recall, in a uniform space X, the closeness of a pair of points is not measured by a real number, like in a metric space, but by the fact that this pair of points belong or does not belong to certain subsets of the cartesian product, $X \times X$. These subsets are called the *entourages* of the uniform structure.

Let X be a non-empty set. A relation on X is a subset of $X \times X$. If U is a relation, then the *inverse* of U is denoted by U^{-1} and is a relation given by

$$U^{-1} = \{(y, x) : (x, y) \in U\}.$$

A relation U is said to be *symmetric* if $U = U^{-1}$. Note that $U \cap U^{-1}$ is always a symmetric set. If U and V are relations, then the *composite of* U and V is denoted by $U \circ V$ and is given by

$$U \circ V = \{(x, z) \in X \times X : \exists\, y \in X \text{ such that } (x, y) \in V \ \&\ (y, z) \in U\}.$$

The set, denoted by $\triangle_X$, given by $\triangle_X = \{(x, x) : x \in X\}$ is called the *identity relation* or *the diagonal of* X. For every subset A of X the set $U[A]$ is a subset of X and is given by $U[A] = \{y \in X : (x, y) \in U, \text{ for some } x \in A\}$. In case if $A = \{x\}$ then we denote it by $U[x]$ instead of $U[\{x\}]$. We now recall the definition of uniform space.

Definition 1 *A* uniform structure (or uniformity) *on a set* X *is a non–empty collection* $\mathcal{U}$ *of subsets of* $X \times X$ *satisfying the following properties:*

1. *If* $U \in \mathcal{U}$, *then* $\triangle_X \subset U$.
2. *If* $U \in \mathcal{U}$, *then* $U^{-1} \in \mathcal{U}$.
3. *If* $U \in \mathcal{U}$, *then* $V o V \subseteq U$, *for some* $V \in \mathcal{U}$.
4. *If* U *and* V *are elements of* $\mathcal{U}$, *then* $U \cap V \in \mathcal{U}$.
5. *If* $U \in \mathcal{U}$ *and* $U \subseteq V \subseteq X \times X$, *then* $V \in \mathcal{U}$.

The pair $(X,\mathcal{U})$ *(or simply* X*) is called as a* uniform space*.*

Obviously every metric on a set X induces a uniform structure on X and every uniform structure on a set X defines a topology on X. Further, if the uniform structure comes from a metric, the associated topology coincides with the topology obtained by the metric. Also, there may be several different uniformities on a set X. For instance, the largest uniformity on X is the collection of all subsets of $X\times X$ which contains $\triangle_X$ whereas the smallest uniformity on X contains only $X\times X$. For more details on uniform spaces one can refer to [11].

Example 1 *Consider* $\mathbb{R}$ *with usual metric* d*. For every* $\epsilon>0$*, let*

$$U^d_\epsilon := \left\{(x,y)\in\mathbb{R}^2 : d(x,y)<\epsilon\right\}$$

Then the collection

$$\mathcal{U}_d = \left\{E\subseteq\mathbb{R}^2 : U^d_\epsilon\subseteq E, \quad \textit{for some } \epsilon>0\right\}$$

is a uniformity on $\mathbb{R}$*. Further, let* ρ *be an another metric on* $\mathbb{R}$ *given by* $\rho(x,y)=|e^x-e^y|, \;\; x,y\in\mathbb{R}$*. If for* $\epsilon>0$*,*

$$U^\rho_\epsilon := \left\{(x,y)\in\mathbb{R}^2 : \rho(x,y)<\epsilon\right\}$$

then the collection

$$\mathcal{U}_\rho = \left\{E\subseteq\mathbb{R}^2 : U^\rho_\epsilon\subseteq E \quad \textit{for some } \epsilon>0\right\}$$

is also a uniformity on $\mathbb{R}$*. Note that these two uniformities are distinct as the set* $\{(x,y):|x-y|<1\}$ *is in* $\mathcal{U}_d$ *but it is not in* $\mathcal{U}_\rho$*.*

Let X be a uniform space with uniformity $\mathcal{U}$. Then, the natural topology, $\tau_{\mathcal{U}}$, on X is the family of all subsets T of X such that for every x in T, there is $U\in\mathcal{U}$ for which $U[x]\subseteq T$. Therefore, for each $U\in\mathcal{U}$, $U[x]$ is a neighborhood of x. Further, the interior of a subset A of X consists of all those points y of X such that $U[y]\subseteq A$, for some $U\in\mathcal{U}$. For the proof of this, one can refer to [11, Theorem 4, P.178]. With the product topology on $X\times X$, it follows that every member of $\mathcal{U}$ is a neighborhood of Δ_X in $X\times X$. However, converse need not be true in general. For instance, in Example 1 every element of $\mathcal{U}_d$ is a neighborhood of $\Delta_{\mathbb{R}}$ in $\mathbb{R}^2$ but $\left\{(x,y):|x-y|<\frac{1}{1+|y|}\right\}$ is a neighborhood of $\Delta_{\mathbb{R}}$ but not a member of $\mathcal{U}_d$. Also, it is known that if X is a compact uniform space, then $\mathcal{U}$ consists of all the neighborhoods of the diagonal Δ_X [11]. Therefore for compact Hausdorff spaces the topology generated by different uniformities is unique and hence the only uniformity on X in this case is the natural uniformity. Proof of the following Lemma can be found in [11].

Lemma 1 *Let X be a uniform space with uniformity $\mathcal{U}$. Then the following are equivalent:*

1. *X is a T_1-space.*
2. *X is a Hausdorff space.*
3. *$\bigcap\{U : U \in \mathcal{U}\} = \Delta_X$.*
4. *X is a regular space.*

2.2 Various kind of expansivity on metric/topological spaces

Let X be a metric space with metric d and let $f : X \longrightarrow X$ be a homeomorphism. For $x \in X$ and a positive real number c, set

$$\Gamma_c(x, f) = \{y : d(f^n(x), f^n(y)) \leq c, \forall n \in \mathbb{Z}\}.$$

$\Gamma_c(x, f)$ is known as *the dynamical ball of x of size c*. Note that for each c, $\Gamma_c(x, f)$ is always non–empty. We recall the definition expansive homeomorphism defined by Utz [18].

Definition 2 *Let X be a metric space with metric d and let $f : X \longrightarrow X$ be a homeomorphism. Then f is said to be a* metric expansive homeomorphism, *if there exists $c > 0$ such that $\Gamma_c(x, f) = \{x\}$, for all $x \in X$. Constant c is known as an expansive constant for f.*

In the following we give some known example of metric expansive homeomorphisms.

Example 2
1. *Consider the set of real numbers $\mathbb{R}$ with usual metric d. For $\alpha \in \mathbb{R} \setminus \{0, 1, -1\}$, define $f_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ by $f_\alpha(x) = \alpha x$. Then f_α is a metric expansive homeomorphism with any positive real number c as an expansive constant.*
2. *Consider $X = \left\{\pm\frac{1}{n}, \pm\left(1 - \frac{1}{n}\right)\right\}$ with the metric d given by $d(x, y) = |x - y|$. Let $f : X \longrightarrow X$ be a map which fixes $0, 1, -1$ and takes any*

element $x \in X \setminus \{0, 1, -1\}$ *to its immediate right element. Then* f *is a metric expansive homeomorphism with expansive constant* c*, where* $0 < c < \frac{1}{6}$.

The notion of metric expansive homeomorphism is independent of the choice of metric if the space is compact but not the expansive constant. If the space is non–compact, then the notion of metric expansivity depends on the choice of metric even if the topology induced by different metrics are equivalent. For instance, see Example 4. Different variants and generalizations of expansivity are studied. We study few of them in this section.

Let (X, τ) be a topological space. For a subset $A \subseteq X$ and a cover $\mathcal{U}$ of X we write $A \prec \mathcal{U}$ if there exists $C \in \mathcal{U}$ such that $A \subseteq C$. If $\mathcal{V}$ is a family of subsets of X, then $\mathcal{V} \prec \mathcal{U}$ means that for each $A \in \mathcal{V}$, $A \prec \mathcal{U}$. If, in addition $\mathcal{V}$ is a cover of X, then $\mathcal{V}$ is said to be *refinement of* $\mathcal{U}$. Join of two covers $\mathcal{U}$ and $\mathcal{V}$ is a cover given by $\mathcal{U} \wedge \mathcal{V} = \{U \cap V | U \in \mathcal{U}, V \in \mathcal{V}\}$. Every open cover $\mathcal{U}$ of cardinality k can be refined by an open cover $\mathcal{V} = \bigwedge_{i=1}^{k} \mathcal{U}$ such that $\mathcal{V} \prec \mathcal{U}$ and $\mathcal{V} \bigwedge \mathcal{V} = \mathcal{V}$. The notion for orbit expansivity for homeomorphisms was first defined in [1]. We recall the definition.

Definition 3 *Let* $f : X \longrightarrow X$ *be a homeomorphism defined on a topological space* X*. Then* f *is said to be* an orbit expansive homeomorphism *if there is a finite open cover* $\mathcal{U}$ *of* X *such that if for each* $n \in \mathbb{Z}$*, the set* $\{f^n(x), f^n(y)\} \prec \mathcal{U}$*, then* $x = y$*. The cover* $\mathcal{U}$ *of* X *is called* an orbit expansive covering of f.

It can be observed that if f is an orbit expansive homeomorphism on a compact metric space and $\mathcal{U}$ is an orbit expansive covering of f, then $\mathcal{U}$ is a generator for f and therefore f is an expansive homeomorphism. Conversely, every expansive homeomorphism on a compact metric space has a generator $\mathcal{U}$, which is also an orbit expansive covering of f. Hence on compact metric space expansivity is equivalent to orbit expansivity. Another generalization of expansivity was defined and studied in [7]. We recall the definition.

Definition 4 *Let* X *be a topological space. Then a homeomorphism* $f : X \longrightarrow X$ *is said to be an* expansive homeomorphism *if there exists a closed neighborhood* N *of* Δ_X *such that for any two distinct* $x, y \in X$*, there is* $n \in \mathbb{Z}$ *satisfying* $(f^n(x), f^n(y)) \notin N$*. Neighborhood* N *is called* an expansive neighborhood *for* f.

Note that the term used in [7] is topologically expansive but we used the term expansive in above definition to differentiate it from our definition of expansivity on uniform spaces. Obviously, metric expansivity implies expansivity. Through examples it was justified in [7], that in general expansivity need not imply metric expansivity. Also, similar to proof of [15, Theorem 4], one can show that on a locally compact metric space X, if f is expansive with expansive neighborhood N, then for every $\epsilon > 0$ we can construct a metric d compatible with the topology of X such that f is a metric expansive with expansive constant $\epsilon > 0$.

3 Topologically expansive homeomorphism

In this section we study expansive homeomorphisms on uniform spaces. The notion was first defined in [6]. Let X be an uniform space with uniformity $\mathcal{U}$ and $f : X \longrightarrow X$ be a homeomorphism. For an entourage $D \in \mathcal{U}$ let

$$\Gamma_D(x, f) = \{y : (f^n(x), f^n(y)) \in D, \ \forall n \in \mathbb{Z}\}.$$

Definition 5 *Let X be an uniform space with uniformity $\mathcal{U}$. A homeomorphism $f : X \longrightarrow X$ is said to be a* topologically expansive homeomorphism, *if there exists an entourage $A \in \mathcal{U}$, such that for every $x \in X$,*

$$\Gamma_A(x, f) = \{x\}$$

Entourage A is called an expansive entourage.

Since every entourage $A \in \mathcal{U}$ contains some closed neighborhood F of Δ_X, it follows that every topologically expansive homeomorphism is an expansive homeomorphism. But in general converse need not be true as we can observe from the following Example:

Example 3 *Consider $\mathbb{R}$ with the uniformity $\mathcal{U}_d$ as given in Example 1. Then the translation T defined on $\mathbb{R}$ by $T(x) = x+1$ is an expansive homeomorphism with an expansive neighbourhood $N = \{(x, y) \in \mathbb{R}^2 : |x - y| \leq e^{-x}\}$. Note that $N \notin \mathcal{U}$. In fact, it is easy to observe that T is not topologically expansive.*

Example 4 *Consider $\mathbb{R}$ with uniformities $\mathcal{U}_\rho$ and $\mathcal{U}_d$ as given in Example 1. Define a homeomorphism $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = x + \ln 2$. Then it can be easily verified that f is topologically expansive for a closed entourage $A \in \mathcal{U}_\rho$ but not for any closed entourage $D \in \mathcal{U}_d$. Further, observe that f is metric expansivity with respect to metric ρ but is not metric expansive with respect to metric d.*

From Example 3, it can be observed that topologically expansivity is stronger than expansivity whereas from Example 4, it can be concluded that it is weaker than metric expansivity. Also, from Example 4, it can be concluded that the notion of topological expansivity depends on the choice of uniformity on the space and the notion of metric expansivity depends on the metric of the space. In spite of expansivity, in the following Proposition we show that if a uniform space admits a topologically expansive homeomorphism, the space is always Hausdorff space.

Proposition 1 *Let* X *be a uniform space with uniformity* $\mathcal{U}$ *and let* $f : X \longrightarrow X$ *be a topologically expansive homeomorphism. Then* X *is always a Hausdorff space.*

Proof. Let D be an expansive entourage of f. Since $\mathcal{U}$ is a uniformity on X there exists a symmetric set $E \in \mathcal{U}$, such that

$$E o E \subseteq D.$$

Given two distinct points x and y of X, by topological expansivity of f, there exists n in $\mathbb{Z}$, such that $(f^n(x), f^n(y)) \notin D$. But this implies

$$(f^n(x), f^n(y)) \notin E \circ E.$$

Let $U = f^{-n}(E[f^n(x)])$ and $V = f^{-n}(E[f^n(y)])$. Then $\operatorname{int}(U)$ and $\operatorname{int}(V)$ are open subsets of X with $x \in \operatorname{int}(U)$ and $y \in \operatorname{int}(V)$. Further, $U \cap V = \emptyset$. For, if $t \in U \cap V$, then $f^n(t) \in E[f^n(x)] \cap E[f^n(y)]$. But this implies that $(f^n(x), f^n(y)) \in E \circ E$, which is a contradiction. Hence X is a Hausdorff space. □

Following Corollary is a consequence of just Proposition 1 and Lemma 1.

Corollary 1 *If uniform space* X *admits a topological expansive homeomorphism then* X *is a regular space.*

Recall, for a compact Hausdorff space X, all uniformities generates a same topology on the space and therefore it is sufficient to work with the natural uniformity on X. Hence as consequence of Proposition 1, we can conclude the following:

Corollary 2 *Topological expansivity on a compact Hausdorff uniform space does not depend on choice of uniformity on the space.*

Since every compact metric space admits a unique uniform structure, it follows that on compact metric space: metric expansivity, topological expansivity and expansivity are equivalent.

Let X be a uniform space with uniformity $\mathcal{U}$. A cover $\mathcal{A}$ of a space X is a uniform cover if there is $U \in \mathcal{U}$ such that $U[x]$ is a subset of some member of the cover for every $x \in X$, equivalently, $\{U[x] : x \in X\} \prec \mathcal{A}$. It is known that every open cover of a compact uniform space is uniform cover. For instance, see Theorem 33 in [11].

Let X be a topological space and $f : X \longrightarrow X$ be an orbit expansive homeomorphism with an orbit expansive covering $\mathcal{A}$. Equivalently, f is orbit expansive if for every subset B of X, $f^n(B) \prec \mathcal{A}$ for all $n \in \mathbb{Z}$, then B is singleton. In the following we show that on compact uniform space, topological expansivity is equivalent to orbit expansivity:

Theorem 1 *Let X be a compact uniform space with uniformity $\mathcal{U}$. Then $f : X \longrightarrow X$ is a topologically expansive homeomorphism if and only if it is an orbit expansive homeomorphism.*

Proof. Let f be a topologically expansive homeomorphism with an expansive entourage D, $D \in \mathcal{U}$. Choose $E \in \mathcal{U}$ such that $E o E \subseteq D$. Now, $E \in \mathcal{U}$ and $\mathcal{U}$ is a uniformity. Therefore E contains diagonal and hence the collection $\{E[x] : x \in X\}$ is a cover of X by neighbourhoods. But X is compact. Let $\mathcal{A}$ be a finite subcover of $\{E[x] : x \in X\}$. We show that $\mathcal{A}$ is an orbit expansive covering for f. For $x, y \in X$ suppose that for each $n \in \mathbb{Z}$, $\{f^n(x), f^n(y)\} \prec \mathcal{A}$. But this implies that for each $n \in \mathbb{Z}$,

$$(f^n(x), f^n(y)) \in E o E \subseteq D.$$

Since D is expansive entourage, it follows that $x = y$. Hence $\mathcal{A}$ is an orbit expansive covering.

Conversely, let $\mathcal{A}$ be an orbit expansive covering of f. Since X is a compact uniform space, $\mathcal{A}$ is a uniform cover. Therefore there exists $U \in \mathcal{U}$ such that $\{U[x] : x \in X\} \prec \mathcal{A}$. Since the family of closed members of a uniformity $\mathcal{U}$ is a basis of $\mathcal{U}$, there is a closed member $D \in \mathcal{U}$ such that $D \subseteq U$. We claim that D is an expansive entourage of f. For $x, y \in X$ and for all $n \in \mathbb{Z}$, suppose

$$(f^n(x), f^n(y)) \in D.$$

Therefore, for each $n \in \mathbb{Z}$,

$$\{f^n(x), f^n(y)\} \subseteq U[f^n(x)].$$

This further implies that

$$\{f^n(x), f^n(y)\} \prec \{U[t] : t \in X\} \prec \mathcal{A}.$$

But $\mathcal{A}$ is an orbit expansive covering of f and therefore $x = y$. Hence f is topologically expansive with expansive entourage D. □

In [1, Theorem 2.7] authors showed that if a compact Hausdorff topological space admits an orbit expansive homeomorphism then it is metrizable. Therefore by Proposition 1 and Proposition 1, we have:

Corollary 3 *If a compact uniform space admits a topologically expansive homeomorphism, then it is always metrizable.*

Again as a consequence of Corollary 3, it follows that topological expansivity is equivalent with metric expansivity and it does not depend uniformity. However the following example shows that Corollary 3, is false for non-compact Hausdorff uniform spaces.

Example 5 *Consider* $\mathbb{R}$ *with the topology* $\tau_{\mathbb{R}}$ *whose base consists of all intervals* $[x, r)$*, where* x *is a real number,* r *is a rational number and* $x < r$*. Then* $\mathbb{R}$ *with topology* $\tau_{\mathbb{R}}$ *is a non–compact, paracompact, Hausdorff and not metrizable space. Also, it is known that every paracompact Hausdorff space, admits the uniform structure* $\mathcal{U}$*, consisting of all neighborhood of the diagonal. For instance, see* [11, Page 208]*. Hence if*

$$D = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| < 1\},$$

then $D \in \mathcal{U}$*. Define* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $f(x) = 3x$*. Then it is easy to see that* f *is topologically expansive with expansive entourage* D*. Note that* $\mathbb{R}$ *with uniformity* $\mathcal{U}$ *is a non-compact Hausdorff space.*

In the following Remark, we observe certain results related to topological expansivity as a consequence of expansivity.

Remark 1 *Let* X *be a uniform space with uniformity* $\mathcal{U}$ *and let* $f : X \longrightarrow X$ *be a homeomorphism.*

1. *Suppose* X *is a locally compact, paracompact uniform space. Since every topologically expansive homeomorphism is an expansive homeomorphism, it follows from Lemma 9 of [7], that there is a proper expansive neighborhood for* f*. Note that this neighborhood need not be an entourage. Recall, a set* $M \subseteq X \times X$ *is proper if for every compact subset* A *of* X*, the set* $M[A]$ *is compact.*

2. *Let* f *be topologically expansive homeomorphism. Then by Proposition 13 of* [7], *it follows that for each* $n \in \mathbb{N}$, f^n *is expansive. Note that this* f^n *need not be in general topologically expansive. For instance, let* $\mathcal{U}$ *be the usual uniformity on* $[0, \infty)$ *and* $f : [0, \infty) \longrightarrow [0, \infty)$ *be as homeomorphism constructed by Bryant and Coleman in* [5]. *Then it is easy to verify that* f *is topologically expansive but* f^n *is not topologically expansive, for any* $n > 1$.

3. *Let* X *be a uniform space with uniformity* $\mathcal{U}$ *and* Y *be a uniform space with uniformity* $\mathcal{V}$. *Suppose* $f : X \longrightarrow X$ *is topologically expansive and* $h : X \longrightarrow Y$ *is a homeomorphism. Then by Proposition 13 of* [7], *it follows that* $h \circ f \circ h^{-1}$ *is expansive on* Y. *However, the homeomorphism* $h \circ f \circ h^{-1}$ *need not be topologically expansive. For instance, let* $\mathcal{U}_\rho$ *and* $\mathcal{U}_d$ *be uniformities on* $\mathbb{R}$ *as defined in Example 1. Consider the identity homeomorphism* $h : \mathbb{R} \longrightarrow \mathbb{R}$, *where the domain* $\mathbb{R}$ *is considered with uniformity* $\mathcal{U}_\rho$ *whereas co-domain* $\mathbb{R}$ *is considered with the uniformity* $\mathcal{U}_d$. *Then as observed in Example 4,* $f(x) = x + \ln(2)$ *is topologically expansive with respect to* $\mathcal{U}_\rho$ *but* $h \circ f \circ h^{-1}$ *is not topologically expansive with respect to* $\mathcal{U}_d$.

Observe here that in each of the above Example, f is not uniformly continuous. In the following we show that Remarks above are true if the maps are uniformly continuous. Recall, a map $f : X \longrightarrow X$ is uniformly continuous relative to the uniformity $\mathcal{U}$ if for every entourage $V \in \mathcal{U}$, $(f \times f)^{-1}(V) \in \mathcal{U}$.

Proposition 2 1. *Let* X *be a uniform space with uniformity* $\mathcal{U}$. *Suppose both* f *and* f^{-1} *are uniformly continuous relative to* $\mathcal{U}$. *Then* f *is topologically expansive if and only if* f^n *is topologically expansive, for all* $n \in \mathbb{Z} \backslash \{0\}$.

2. *Let* X *be a uniform space with uniformity* $\mathcal{U}$ *and* Y *be a uniform space with uniformity* $\mathcal{V}$. *Suppose* $h : X \longrightarrow Y$ *is a homeomorphism such that both* h *and* h^{-1} *are uniformly continuous. Then* f *is topologically expansive on* X *if and only if* $h \circ f \circ h^{-1}$ *is topologically expansive on* Y.

Since the proof of the Proposition 2 is similar to the proof of Proposition 13 in [7], we omit the proof.

References

[1] M. Achigar, A. Artigue, I. Monteverde. Expansive homeomorphisms on non–Hausdorff spaces, *Top. & its Appl.*, Vol. **207** (2016), 109–122.

[2] A. Barzanouni, Finite expansive homeomorphisms, *Top. & its Appl.*, Vol. **253** (2019), 95–112.

[3] A. Barzanouni, M. S. Divandar, E. Shah, *On Properties of Expansive Group Actions*, to appear in Acta Mathematica Vietnamica.

[4] R. Bowen, Entropy–expansive maps, *Trans. of the AMS*, Vol. **164** (1972), 323–331.

[5] B. F. Bryant, D. B. Coleman, Some expansive homeomorphisms of the reals, *Am. Math. Mon.*, Vol. **73** (1966) 370–373.

[6] P. Das, T. Das, Various types of shadowing and specification on uniform spaces, *J. Dyn. Control Syst.*, Vol. **24** (2018), No. 2, 253–267.

[7] T. Das, K. Lee, D. Richeson, J. Wiseman. Spectral decomposition for topologically Anosov homeomorphisms on noncompact and non-metrizable spaces, *Top. & its Appl.*, Vol. **160**(2013), 149–158.

[8] K. Hiraide, Expansive homeomorphisms of compact surfaces are pseudo-Anosov, *Osaka J. Math.*, **27** (1990), no. 1, 117–162.

[9] H. Kato, The nonexistence of expansive homeomorphisms of Peano continua in the plane, *Top. & its Appl.*, Vol. **34** (1990), no. 2, 161–165.

[10] H. Kato, Continuum–wise expansive homeomorphisms, *Canad. J. Math.*, Vol. **45** (1993), 576–598.

[11] J. L. Kelley, *General Topology*, D. Van Nostrand Company, Inc., Toronto, New York, London, 1955.

[12] M. Kouno, On expansive homeomorphisms on manifolds, *J. Math. Soc. Japan*, Vol. **33** (1981), No. 3, 533–538.

[13] R. Mañé, Expansive homeomorphisms and topological dimension, *Trans. Amer. Math. Soc.*, Vol. **252** (1979), 313–319.

[14] W. L. Reddy, Point–wise expansion homeomorphisms, *J. Lond. Math. Soc.*, Vol. **2** (1970), 232–236.

[15] D. Richeson, J. Wiseman, Positively expansive dynamical systems, *Top. & its Appl.*, Vol. **154** (2007), 604–613.

[16] S. Schwartzman, *On Transformation Groups*, Dissertation, Yale University, 1952.

[17] E. Shah, Positively expansive maps on G−spaces, *J. Indian Math. Soc.*, Vol. **72** (2005), 91–97.

[18] W.R. Utz, Unstable homeomorphisms, *Proc. Amer. Math. Soc.*, Vol. **1** (1950), 769–774.

[19] A. Weil, *Sur les espaces à structure uniforme et sur la topolgie gènèrale*, Hermann, Paris, 1938.

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A common generalization of convolved (u, v)-Lucas first and second kinds p-polynomials

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Abstract. In this note the convolved (u, v)-Lucas first kind and the convolved (u, v)-Lucas second kind p-polynomials are introduced and study some of their properties. Several identities related to the common generalization of convolved (u, v)-Lucas first and second kinds p-polynomials are also presented.

1 Introduction

Buschman [2] introduced the homogeneous linear second order difference equation with constant coefficients as

$$U_0;\ U_1;\ U_{n+1} = aU_n + bU_{n-1},\ \text{for } n \geq 1, \tag{1}$$

that generalizes almost all numbers and polynomials sequences. The Lucas sequence of first and second kinds $U = U(a, b)$ and $V = V(a, b)$ can be recovered from (1) by taking $U_0 = 0$, $U_1 = 1$ and $V_0 = 2, V_1 = a$ respectively. These two kinds sequences comprise Fibonacci numbers, generalized Fibonacci

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numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, balancing polynomials, chebyshev polynomials etc. The interested reader may look [1, 3, 4, 5] for a detail review.

Şahin and Ramírez [6] introduced the convolved (p, q)-Fibonacci polynomials (convolved generalized Lucas polynomials) by $g^{(r)}_{p,q}(t) = \big(1 - p(x)t - q(x)t^2\big)^{-r} = \sum_{n=0}^{\infty} F^{(r)}_{p,q,n+1}(x)t^n,\ r \in \mathbb{Z}^+$. In [7], Ye and Zhang gave a common generalization of convolved generalized Fibonacci and Lucas polynomials and are given by $\sum_{n=0}^{\infty} T^{(r,m)}_{h,n}(x)t^n = \frac{\left(h(x)+2t\right)^m}{\left(1-h(x)t-t^2\right)^r},\ r \geq m$ and $r, m \in \mathbb{Z}^+$. They obtained some recurrence relations and identities of these polynomials.

In this study we introduce convolved (u, v)-Lucas first kind and second kind p-polynomials and derive some of their identities. Further the common generalization of these two polynomials is presented and some related results are discussed.

2 Convolved (u, v)-Lucas first and second kinds p-polynomials

In this section, we introduce convolved (u, v)-Lucas first kind p-polynomials and convolved (u, v)-Lucas second kind p-polynomials and present some of their properties.

Definition 1 *Let p be any non-negative integer. The (u, v)-Lucas first kind p-polynomials $\{L^p_{u,v,j}(x)\}_{j\geqslant p+1}$ are defined recursively by*

$$L^p_{u,v,j}(x) = u(x)L^p_{u,v,j-1}(x) + v(x)L^p_{u,v,j-p-1}(x)$$

with initials $L^p_{u,v,0}(x) = 0$ and $L^p_{u,v,j}(x) = (u(x))^{j-1}$ for $j = 1 \dots p$ and $u(x)$ and $v(x)$ are polynomials with real coefficients.

Let $g^p_{u,v}(t)$ be the generating function of $L^p_{u,v,j+1}(x)$. Then it is easy to see $g^p_{u,v}(t) = \sum_{j=0}^{\infty} L^p_{u,v,j+1}(x)t^j = \frac{1}{1-u(x)t-v(x)t^{p+1}}$. The finding result is the criterion to define the convolved (u, v)-Lucas first kind p-polynomials.

Definition 2 *Let $u(x)$ and $v(x)$ be polynomials with real coefficients. Then the convolved (u, v)-Lucas first kind p-polynomials $\{L^{(p,r)}_{u,v,j}(x)\}_{j\in\mathbb{N}}$ for $p \geqslant 1$ are defined by*

$$g^{(p,r)}_{u,v}(t) = \sum_{j=0}^{\infty} L^{(p,r)}_{u,v,j+1}(x)t^j = \left(1 - u(x)t - v(x)t^{p+1}\right)^{-r},\ r \in \mathbb{Z}^+. \tag{2}$$

Further simplification of relation (2) gives the following explicit formula

$$L_{u,v,j+1}^{(p,r)}(x) = \sum_{k=0}^{\lfloor \frac{j}{p+1} \rfloor} \frac{(r)_{j-pk}}{(j-(p+1)k)!k!} u^{j-(p+1)k}(x)v^k(x). \tag{3}$$

Consideration of formula (3) for different measures of (p, r) with $r = 4$ yield some values of convolved (u, v)-Lucas first kind p-polynomials which are listed in Table 1.

Table 1: Convolved (u, v)-Lucas first kind p-polynomials

j	$(p,r) = (1,4)$	$(p,r) = (2,4)$	$(p,r) = (3,4)$	$(p,r) = (4,4)$
0	1	1	1	1
1	$4u(x)$	$4u(x)$	$4u(x)$	$4u(x)$
2	$10u^2(x) + 4v(x)$	$10u^2(x)$	$10u^2(x)$	$10u^2(x)$
3	$20u^3(x) + 20u(x)v(x)$	$20u^3(x) + 4v(x)$	$20u^3(x)$	$20u^3(x)$
4	$35u^4(x) + 60u^2(x)v(x) + 10v^2(x)$	$35u^4(x) + 20u(x)v(x)$	$35u^4(x) + 4v(x)$	$35u^4(x)$
5	$56u^5(x) + 140u^3(x)v(x) + 60u(x)v^2(x)$	$56u^5(x) + 60u^2(x)v(x)$	$56u^5(x) + 20u(x)v(x)$	$56u^5(x) + 4v(x)$
6	$84u^6(x) + 280u^4(x)v(x) + 210u^2(x))v^2(x) + 20v^3(x)$	$84u^6(x) + 140u^3(x)v(x) + 10v^2(x)$	$84u^6(x) + 60u^2(x)v(x)$	$84u^6(x)) + 20u(x)v(x)$

Theorem 1 *The convolved (u, v)-Lucas first kind p-polynomials $L_{u,v,j+1}^{(p,r)}(x)$ satisfies the following relation*

$$u(x)L_{u,v,j-1}^{(p,r)}(x) + v(x)L_{u,v,j-p-1}^{(p,r)}(x) + L_{u,v,j}^{(p,r-1)}(x) = L_{u,v,j}^{(p,r)}(x), \tag{4}$$

with parameters $r > 1$ and $j > 1$.

Proof. Using the explicit formula (3) on the left-hand side of (4), we get

$$\frac{(r)_{j-2-pk}}{(j-2-(p+1)k)!k!} u^{j-1-(p+1)k}(x)v^k(x) + \frac{(r)_{j-p-2-pk}}{(j-p-2-(p+1)k)!k!}$$

$$\times u^{j-p-2-(p+1)k}(x)v^{k+1}(x)+\frac{(r-1)_{j-1-pk}}{(j-1-(p+1)k)!k!}u^{j-1-(p+1)k}(x)v^k(x)$$
$$=\frac{(r)_{j-2-pk}}{(j-2-(p+1)k)!k!}u^{j-1-(p+1)k}(x)v^k(x)+\frac{(r)_{j-2-pk}}{(j-1-(p+1)k)!(k-1)!}$$
$$\times u^{j-1-(p+1)k}(x)v^k(x)+\frac{(r-1)_{j-1-pk}}{(j-1-(p+1)k)!k!}u^{j-1-(p+1)k}(x)v^k(x)$$
$$=\frac{u^{j-1-(p+1)k}(x)v^k(x)}{(j-1-(p+1)k)!k!}\left[\frac{(j-1-(p+1)k)(r)_{j-1-pk}}{(r+j-2-pk)}+\frac{k(r)_{j-1-pk}}{(r+j-2-pk)}\right.$$
$$\left.+\frac{(r-1)(r)_{j-1-pk}}{(r+j-2-pk)}\right]$$
$$=\frac{(r)_{j-1-pk}}{(j-1-(p+1)k)!k!}u^{j-1-(p+1)k}(x)v^k(x)$$
$$=L^{(p,r)}_{u,v,j}(x).$$

This completes the proof. □

Theorem 2 *The following relation*

$$\sum_{k=1}^{r}L^{(p,k)}_{u,v,j}(x)=\frac{1}{u(x)}\left[L^{(p,r)}_{u,v,j+1}(x)-v(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-p}(x)\right] \tag{5}$$

holds for $j\geqslant 1$ *with* $L^{(p,0)}_{u,v,j+1}=0$.

Proof. Taking summation over 1 to r in relation (4), we have

$$\sum_{k=1}^{r}L^{(p,k)}_{u,v,j}(x)=u(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-1}(x)+v(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-p-1}(x)+\sum_{k=1}^{r}L^{(p,k-1)}_{u,v,j}(x)$$
$$=u(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-1}(x)+v(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-p-1}(x)+\sum_{k=1}^{r-1}L^{(p,k)}_{u,v,j}(x).$$

It follows that

$$L^{(p,r)}_{u,v,j}(x)=u(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-1}(x)+v(x)\sum_{k=1}^{r}L^{(p,k)}_{u,v,j-p-1}(x),$$

we get the desired result by replacing $j+1$ instead of j. □

Theorem 3 *For* $j \geqslant p+1$ *and* $L^{(p,r)}_{u,v,-j}(x) = 0$, *the polynomial* $L^{(p,r)}_{u,v,j+1}(x)$ *holds the following relation*

$$\begin{aligned} &u(x)\sum_{i=0}^{j-1} L^{(p,r)}_{u,v,i+1}(x) + \sum_{i=0}^{j} L^{(p,r-1)}_{u,v,i+1}(x) \\ &= \big(1-v(x)\big)\sum_{i=0}^{j-p-1} L^{(p,r)}_{u,v,i+1}(x) + \sum_{i=j-p}^{j} L^{(p,r)}_{u,v,i+1}(x). \end{aligned} \tag{6}$$

Proof. Consider $j = 1, 2, \dots$ in relation (4), follows

$$\begin{aligned} &u(x)L^{(p,r)}_{u,v,0}(x) + v(x)L^{(p,r)}_{u,v,-p}(x) + L^{(p,r-1)}_{u,v,1}(x) = L^{(p,r)}_{u,v,1}(x) \\ &u(x)L^{(p,r)}_{u,v,1}(x) + v(x)L^{(p,r)}_{u,v,-p+1}(x) + L^{(p,r-1)}_{u,v,2}(x) = L^{(p,r)}_{u,v,2}(x) \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &u(x)L^{(p,r)}_{u,v,p}(x) + v(x)L^{(p,r)}_{u,v,0}(x) + L^{(p,r-1)}_{u,v,p+1}(x) = L^{(p,r)}_{u,v,p+1}(x) \\ &u(x)L^{(p,r)}_{u,v,p+1}(x) + v(x)L^{(p,r)}_{u,v,1}(x) + L^{(p,r-1)}_{u,v,p+2}(x) = L^{(p,r)}_{u,v,p+2}(x) \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &u(x)L^{(p,r)}_{u,v,j-1}(x) + v(x)L^{(p,r)}_{u,v,j-p-1}(x) + L^{(p,r-1)}_{u,v,j}(x) = L^{(p,r)}_{u,v,j}(x) \\ &u(x)L^{(p,r)}_{u,v,j}(x) + v(x)L^{(p,r)}_{u,v,j-p}(x) + L^{(p,r-1)}_{u,v,j+1}(x) = L^{(p,r)}_{u,v,j+1}(x). \end{aligned}$$

Summation of these equalities yields the desired result. □

Definition 3 *Let* p *be any non-negative integer and* $u(x)$ *and* $v(x)$ *are polynomials of real coefficients. Then the* (u, v)*-Lucas second kind* p*-polynomials* $\{M^{p}_{u,v,j}(x)\}_{j\geqslant p+1}$ *are defined recursively by*

$$M^{p}_{u,v,j}(x) = u(x)M^{p}_{u,v,j-1}(x) + v(x)M^{p}_{u,v,j-p-1}(x),$$

with initials $M^{p}_{u,v,0}(x) = (p+1)\frac{M_1(x)}{u(x)}$ *and* $M^{p}_{u,v,j}(x) = M_1(x)u^{j-1}(x)$ *for* $j = 1 \dots p$ *and* $M_1(x)$ *is the first term of Lucas second kind like polynomial sequences.*

Many well-known polynomial sequences are special cases of (u, v)-Lucas second kind p-polynomials. For example, for $p = 1$, when $\big(u(x), v(x)\big) = (x, 1)$ and $M_1 = x$, $\big(u(x), v(x)\big) = (2x, 1)$ and $M_1 = 2x$, $\big(u(x), v(x)\big) = (6x, -1)$ and $M_1 = 3x$, $\big(u(x), v(x)\big) = (1, 2x)$ and $M_1 = 1$, $\big(u(x), v(x)\big) = (3x, -2)$

and $M_1 = 3x$ etc. the (u, v)-Lucas second kind p-polynomials turn into classical Lucas polynomials, Pell-Lucas polynomials, Lucas-balancing polynomials, Jacobsthal-Lucas polynomials, Fermat-Lucas polynomials respectively.

Let $h^p_{u,v}(t)$ denotes the generating function of $M^p_{u,v,j+1}(x)$. Then

$$h^p_{u,v}(t) = \sum_{j=0}^{\infty} M^p_{u,v,j+1}(x)t^j = \frac{M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p}{1 - u(x)t - v(x)t^{p+1}}.$$

The generating function $h^p_{u,v}(t)$ is more precious to define convolved (u, v)-Lucas second kind p-polynomials.

Definition 4 *Let p be any positive integer. Then the convolved (u, v)-Lucas second kind p-polynomials $\{M^{(p,r)}_{u,v,j}(x)\}_{j\in\mathbb{N}}$ are defined by*

$$h^{(p,r)}_{u,v}(t) = \sum_{j=0}^{\infty} M^{(p,r)}_{u,v,j+1}(x)t^j = \frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1 - u(x)t - v(x)t^{p+1}\right)^r}, \ r \in \mathbb{Z}^+. \quad (7)$$

Expression (7) reduces to the explicit formula

$$M^{(p,r)}_{u,v,j+1}(x) = \sum_{k=0}^{\min\{r,j\}} \sum_{i=0}^{\lfloor \frac{j-pk}{p+1} \rfloor} \binom{r}{k} \frac{(r)_{j-pk-pi}}{i!(j-pk-(p+1)i)!} M^r_1(x)(p+1)^k v^{k+i}(x) \times u^{j-(p+1)k-(p+1)i}(x). \quad (8)$$

Consideration of formula (8) for different measures of (p, r) gives some values of convolved (u, v)-Lucas second kind p-polynomials which are listed in Table 2.

Theorem 4 *The convolved (u, v)-Lucas second kind p-polynomials $M^{(p,r)}_{u,v,j+1}(x)$ satisfies following relation*

$$\begin{aligned} M^{(p,r)}_{u,v,j}(x) =& u(x)M^{(p,r)}_{u,v,j-1}(x) + v(x)M^{(p,r)}_{u,v,j-1-p}(x) \\ &+ M_1(x)M^{(p,r-1)}_{u,v,j}(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)M^{(p,r-1)}_{u,v,j-p}(x), \end{aligned} \quad (9)$$

with parameters $r > 1$ and $j > 1$.

Table 2: Convolved (u, v)-Lucas second kind p-polynomials

j	$(p,r)=(2,1)$	$(p,r)=(2,2)$	$(p,r)=(2,3)$	$(p,r)=(2,4)$
0	$M_1(x)$	$M_1^2(x)$	$M_1^3(x)$	$M_1^4(x)$
1	$M_1(x)u(x)$	$2M_1^2(x)u(x)$	$3M_1^3(x)u(x)$	$4M_1^4(x)u(x)$
2	$M_1(x)u^2(x)+\frac{3M_1(x)v(x)}{u(x)}$	$3M_1^2(x)u^2(x)+\frac{6M_1^2(x)v(x)}{u(x)}$	$6M_1^3(x)u^2(x)+\frac{9M_1^3(x)v(x)}{u(x)}$	$10M_1^4(x)u^2(x)+\frac{12M_1^4(x)v(x)}{u(x)}$
3	$M_1(x)u^3(x)+4M_1(x)v(x)$	$4M_1^2(x)u^3(x)+14M_1^2(x)v(x)$	$10M_1^3(x)u^3(x)+30M_1^3(x)v(x)$	$20M_1^4(x)u^3(x)+52M_1^4(x)v(x)$
4	$M_1(x)u^4(x)+5M_1(x)u(x)v(x)$	$5M_1^2(x)u^4(x)+24M_1^2(x)u(x)v(x)+\frac{9M_1^2(x)v^2(x)}{u^2(x)}$	$15M_1^3(x)u^4(x)+66M_1^3(x)u(x)v(x)+\frac{27M_1^3(x)v^2(x)}{u^2(x)}$	$35M_1^4(x)u^4(x)+140M_1^4(x)u(x)\times v(x)+\frac{54M_1^4(x)v^2(x)}{u^2(x)}$
5	$M_1(x)u^5(x)+6M_1(x)u^2(x)v(x)+\frac{3M_1(x)v^2(x)}{u(x)}$	$6M_1^2(x)u^5(x)+36M_1^2(x)u^2(x)\times v(x)+\frac{30M_1^2(x)v^2(x)}{u(x)}$	$21M_1^3(x)u^5(x)+120M_1^3(x)u^2(x)\times v(x)+\frac{108M_1^3(x)v^2(x)}{u(x)}$	$56M_1^4(x)u^5(x)+300M_1^4(x)u^2(x)\times v(x)+\frac{264M_1^4(x)v^2(x)}{u(x)}$

Proof. Using (7) on the right-hand side of relation (9), we get

$$\begin{aligned}
&u(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t^2+v(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}\\
&\times t^{p+2}+M_1(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}t+(p+1)\frac{v(x)}{u(x)}M_1(x)\\
&\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}t^{p+1}\\
&=\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t\left[u(x)t+v(x)t^{p+1}\right]\\
&+\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}t\left[M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right]\\
&=\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}t\left[\frac{u(x)t+v(x)t^{p+1}}{1-u(x)t-v(x)t^{p+1}}+1\right]
\end{aligned}$$

$$= \frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1 - u(x)t - v(x)t^{p+1}\right)^r}t,$$

which proves the result. □

Theorem 5 *The polynomial* $M^{(p,r)}_{u,v,j+1}(x)$ *obey the following relation*

$$\begin{aligned} M^{(p,r)}_{u,v,j}(x) &= u(x)\sum_{k=1}^{r} M^{(p,k)}_{u,v,j-1}(x) + v(x)\sum_{k=1}^{r} M^{(p,k)}_{u,v,j-1-p}(x) \\ &+\left(M_1(x) - 1\right)\sum_{k=1}^{r-1} M^{(p,k)}_{u,v,j}(x) + (p+1)\frac{v(x)}{u(x)}M_1(x)\sum_{k=1}^{r-1} M^{(p,k)}_{u,v,j-p}(x), \end{aligned} \tag{10}$$

with parameters $r > 1$, $j \geqslant 2$, $p \geqslant 1$ *and* $M^{(p,0)}_{u,v,j+1}(x) = 0$.

Proof. Consider $r = 1, 2, \dots$ in relation (9) which follows

$$\begin{aligned} M^{(p,1)}_{u,v,j}(x) &= u(x)M^{(p,1)}_{u,v,j-1}(x) + v(x)M^{(p,1)}_{u,v,j-1-p}(x) + M_1(x)M^{(p,0)}_{u,v,j}(x) \\ &\quad + (p+1)\frac{v(x)}{u(x)}M_1(x)M^{(p,0)}_{u,v,j-p}(x) \\ M^{(p,2)}_{u,v,j}(x) &= u(x)M^{(p,2)}_{u,v,j-1}(x) + v(x)M^{(p,2)}_{u,v,j-1-p}(x) + M_1(x)M^{(p,1)}_{u,v,j}(x) \\ &\quad + (p+1)\frac{v(x)}{u(x)}M_1(x)M^{(p,1)}_{u,v,j-p}(x) \\ M^{(p,3)}_{u,v,j}(x) &= u(x)M^{(p,3)}_{u,v,j-1}(x) + v(x)M^{(p,3)}_{u,v,j-1-p}(x) + M_1(x)M^{(p,2)}_{u,v,j}(x) \\ &\quad + (p+1)\frac{v(x)}{u(x)}M_1(x)M^{(p,2)}_{u,v,j-p}(x) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ M^{(p,r)}_{u,v,j}(x) &= u(x)M^{(p,r)}_{u,v,j-1}(x) + v(x)M^{(p,r)}_{u,v,j-1-p}(x) + M_1(x)M^{(p,r-1)}_{u,v,j}(x) \\ &\quad + (p+1)\frac{v(x)}{u(x)}M_1(x)M^{(p,r-1)}_{u,v,j-p}(x). \end{aligned}$$

Summation of these bunch equalities yields the desired result. □

In order to verify the result (10), assume $j = 5$ with $(p, r) = (2, 3)$, gives

$$M^{(2,3)}_{u,v,5} = u(x)\sum_{k=1}^{3} M^{(2,k)}_{u,v,4}(x) + v(x)\sum_{k=1}^{3} M^{(2,k)}_{u,v,2}(x)$$

$$+\big(M_1(x)-1\big)\sum_{k=1}^{2} M_{u,v,5}^{(2,k)}(x)+3\frac{v(x)}{u(x)}M_1(x)\sum_{k=1}^{2} M_{u,v,3}^{(2,k)}(x).$$

Simplification of right-hand side gives

$$\begin{aligned}
&u(x)\big[M_{u,v,4}^{(2,1)}(x)+M_{u,v,4}^{(2,2)}(x)+M_{u,v,4}^{(2,3)}(x)\big]+v(x)\big[M_{u,v,2}^{(2,1)}(x)+M_{u,v,2}^{(2,2)}(x)\\
&\quad+M_{u,v,2}^{(2,3)}(x)\big]+\big(M_1(x)-1\big)\big[M_{u,v,5}^{(2,1)}(x)+M_{u,v,5}^{(2,2)}(x)\big]\\
&\quad+3\frac{v(x)}{u(x)}M_1(x)\big[M_{u,v,3}^{(2,1)}(x)+M_{u,v,3}^{(2,2)}(x)\big]=15M_1^3(x)u^4(x)\\
&\quad+66M_1^3(x)u(x)v(x)+27M_1^3(x)\frac{v^2(x)}{u^2(x)}=M_{u,v,5}^{(2,3)}(x),
\end{aligned}$$

and the result is verified.

3 Common generalization of convolved (u,v)-Lucas first and second kinds p-polynomials

In this section we give the common generalization of convolved (u,v)-Lucas first and second kinds p-polynomials and obtain some recurrence relations of these polynomials.

Definition 5 *Let p, r and m be all positive integers. Then the common generalization of convolved (u,v)-Lucas first and second kinds p-polynomials $\{E_{u,v,j}^{(p,r,m)}(x)\}_{j\in\mathbb{N}}$ are defined by*

$$\sum_{j=0}^{\infty} E_{u,v,j}^{(p,r,m)}(x)t^j=\frac{\Big(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\Big)^m}{\big(1-u(x)t-v(x)t^{p+1}\big)^r},\quad r\geqslant m. \tag{11}$$

Assumption of $m=0$ and $m=r$ reduces the expression (11) to convolved (u,v)-Lucas first kind p-polynomials and convolved (u,v)-Lucas second kind p-polynomials respectively.

Theorem 6 *The common generalization of convolved (u,v)-Lucas first and second kinds p-polynomials has the following explicit formula*

$$\begin{aligned}
E_{u,v,j}^{(p,r,m)}(x)=&\sum_{k=0}^{\min\{m,j\}}\sum_{i=0}^{\lfloor\frac{j-pk}{p+1}\rfloor}\binom{m}{k}\frac{(r)_{j-pk-pi}}{\big(j-pk-(p+1)i\big)!i!}M_1^m(x)(p+1)^k v^{k+i}(x)\\
&\times u^{j-(p+1)k-(p+1)i}(x).
\end{aligned}$$

Proof. We run the proof by taking right-hand side of the expression (11)

$$\begin{aligned}
&\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}\\
&=\sum_{k=0}^{m}\binom{m}{k}M_1^{m-k}(x)\left((p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^k\sum_{j=0}^{\infty}\binom{-r}{j}(-t)^j\left(u(x)+v(x)t^p\right)^j\\
&=\sum_{k=0}^{m}\binom{m}{k}M_1^{m-k}(x)(p+1)^k\frac{v^k(x)}{u^k(x)}M_1^k(x)t^{pk}\sum_{j=0}^{\infty}\frac{(r)_j}{j!}t^j\sum_{i=0}^{j}\binom{j}{i}u^{j-i}(x)v^i(x)t^{pi}\\
&=\sum_{j=0}^{\infty}\sum_{k=0}^{\min\{m,j\}}\sum_{i=0}^{\lfloor\frac{j-pk}{p+1}\rfloor}\binom{m}{k}M_1^m(x)(p+1)^k\frac{v^k(x)}{u^k(x)}t^{pk}\frac{(r)_{j-pk-pi}}{(j-pk-pi)!}t^{j-pk-pi}\\
&\quad\times\frac{(j-pk-pi)!}{(j-pk-(p+1)i)!i!}u^{j-pk-(p+1)i}(x)v^i(x)t^{pi}\\
&=\sum_{j=0}^{\infty}\sum_{k=0}^{\min\{m,j\}}\sum_{i=0}^{\lfloor\frac{j-pk}{p+1}\rfloor}\binom{m}{k}M_1^m(x)(p+1)^kv^{k+i}(x)u^{j-(p+1)k-(p+1)i}(x)\\
&\quad\times\frac{(r)_{j-pk-pi}}{(j-pk-(p+1)i)!i!}t^j.
\end{aligned}$$

Comparing the left-hand side of expression (11), we get the required result. □

Theorem 7 *The common generalization of convolved* (u,v)*-Lucas first and second kinds* p*-polynomials obeys the following relations*

(i) $E^{(p,r,m)}_{u,v,j}(x)=u(x)E^{(p,r,m)}_{u,v,j-1}(x)+v(x)E^{(p,r,m)}_{u,v,j-(p+1)}(x)+E^{(p,r-1,m)}_{u,v,j}(x)$;

(ii) $E^{(p,r,m+1)}_{u,v,j}(x)=M_1(x)E^{(p,r,m)}_{u,v,j}(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)E^{(p,r,m)}_{u,v,j-p}(x)$;

(iii) $$E^{(p,r,m)}_{u,v,j}(x)=\frac{1}{u(x)}\left\{\frac{j+1}{r-1}E^{(p,r-1,m)}_{u,v,j+1}(x)-\frac{p(p+1)M_1(x)m}{r-1}\frac{v(x)}{u(x)}E^{(p,r-1,m-1)}_{u,v,j+1-p}(x)\right.$$
$$\left.-(p+1)v(x)E^{(p,r,m)}_{u,v,j-p}(x)\right\}.$$

Proof. Using the expression (11), we have

$$\sum_{j=0}^{\infty}E^{(p,r,m)}_{u,v,j}(x)t^j=\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}$$

$$
\begin{aligned}
&= \frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m\left(u(x)t+v(x)t^{p+1}\right)}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}\\
&\quad+\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}\\
&= u(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t\\
&\quad+v(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t^{p+1}\\
&\quad+\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}},
\end{aligned}
$$

and the expression (i) follows. The proof of (ii) is analogous to (i). In order to proof (iii), we proceed as follows:

The common generalization of convolved (u, v)-Lucas first and second kinds p-polynomials can be written undoubtedly as

$$
\begin{aligned}
&(r-1)\left(u(x)-(p+1)v(x)t^p\right)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}\\
&= \frac{d}{dt}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}\\
&\quad-p(p+1)\frac{v(x)}{u(x)}M_1(x)m\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{m-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r-1}}t^{p-1},
\end{aligned}
$$

and if and only if

$$
\begin{aligned}
(r-1)\left[u(x)E^{(p,r,m)}_{u,v,j}(x)+(p+1)v(x)E^{(p,r,m)}_{u,v,j-p}(x)\right] &= (j+1)E^{(p,r-1,m)}_{u,v,j+1}(x)\\
&\quad-p(p+1)\frac{v(x)}{u(x)}M_1(x)mE^{(p,r-1,m-1)}_{u,v,j+1-p}(x).
\end{aligned}
$$

This follows the result. □

The following corollary is an immediate consequence of Theorem 7.

Corollary 1 *Let* r *and* m *be any positive integers with* $r \geq m$*. Then, the* *following relations*

$$\sum_{k=1}^{r-m}\left[u(x)E^{p,m+k,m}_{u,v,j-1}(x)+v(x)E^{(p,m+k,m)}_{u,v,j-(p+1)}(x)\right]=E^{(p,r,m)}_{u,v,j}(x)-E^{(p,m,m)}_{u,v,j}(x) \quad (12)$$

and

$$\begin{aligned}&\sum_{k=0}^{m}M_1^k(x)E^{(p,r,m-k)}_{u,v,j-p}(x)\\&\qquad=\frac{u(x)}{v(x)(p+1)M_1(x)}\left[E^{(p,r,m+1)}_{u,v,j}(x)-M_1^{m+1}(x)E^{(p,r,0)}_{u,v,j}(x)\right]\end{aligned} \quad (13)$$

hold.

The following are some examples to understand the above corollary.

Example 1 *Consider* $r=4$ *and* $m=3$ *on the left of relation* (12)*, gives*

$$\begin{aligned}&\sum_{k=1}^{1}\left[u(x)E^{(p,3+k,3)}_{u,v,j-1}(x)+v(x)E^{(p,3+k,3)}_{u,v,j-(p+1)}(x)\right]\\&=u(x)E^{(p,4,3)}_{u,v,j-1}(x)+v(x)E^{(p,4,3)}_{u,v,j-(p+1)}(x)\\&=u(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^4}t\\&\quad+v(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^4}t^{p+1}\\&=\left(u(x)t+v(x)t^{p+1}-1\right)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^4}\\&\quad+\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^4}\\&=\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^4}-\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^3}{\left(1-u(x)t-v(x)t^{p+1}\right)^3}\\&=E^{(p,4,3)}_{u,v,j}(x)-E^{(p,3,3)}_{u,v,j}(x).\end{aligned}$$

Example 2 *Consider* $r = 5$ *and* $m = 2$ *in relation* (13), *we have*

$$\sum_{k=0}^{2} M_1^k(x) E_{u,v,j-p}^{(p,5,2-k)}(x) = \frac{u(x)}{v(x)(p+1)M_1(x)} \left[E_{u,v,j}^{(p,5,3)}(x) - M_1^3(x) E_{u,v,j}^{(p,5,0)}(x)\right]. \tag{14}$$

Expansion of left side of (14) *gives*

$$\begin{aligned}
&E_{u,v,j-p}^{(p,5,2)}(x) + M_1(x) E_{u,v,j-p}^{(p,5,1)}(x) + M_1^2(x) E_{u,v,j-p}^{(p,5,0)}(x) \\
&= \frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)} M_1(x) t^p\right)^2}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} t^p + M_1(x) \\
&\quad \times \frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)} M_1(x) t^p\right)}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} t^p + M_1^2(x) \frac{1}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} t^p \\
&= \frac{3M_1^2(x) + (p+1)^2 \frac{v^2(x)}{u^2(x)} M_1^2(x) t^{2p} + 3(p+1)\frac{v(x)}{u(x)} M_1^2(x) t^p}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} t^p.
\end{aligned}$$

On the other hand, expansion of right side of (14) *gives*

$$\begin{aligned}
&\frac{u(x)}{v(x)(p+1)M_1(x)} \left[E_{u,v,j}^{(p,5,3)}(x) - M_1^3(x) E_{u,v,j}^{(p,5,0)}(x)\right] \\
&= \frac{u(x)}{v(x)(p+1)M_1(x)} \left[\frac{\left(M_1(x) + (p+1)\frac{v(x)}{u(x)} M_1(x) t^p\right)^3}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} \right. \\
&\quad \left. - M_1^3(x) \frac{1}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} \right] \\
&= \frac{3M_1^2(x) + (p+1)^2 \frac{v^2(x)}{u^2(x)} M_1^2(x) t^{2p} + 3(p+1)\frac{v(x)}{u(x)} M_1^2(x) t^p}{\left(1 - u(x)t - v(x)t^{p+1}\right)^5} t^p.
\end{aligned}$$

Theorem 8 *The following identity*

$$\begin{aligned}
E_{u,v,j+1-p}^{(p,r,m)}(x) &= \frac{u(x)}{v(x)(p+1)M_1(x)(mp-j)} \left[M_1(x)(j+1) E_{u,v,j+1}^{(p,r,m)}(x) \right. \\
&\quad \left. - r u(x) E_{u,v,j}^{(p,r+1,m+1)}(x) - r(p+1) v(x) E_{u,v,j-p}^{(p,r+1,m+1)}(x)\right]
\end{aligned}$$

holds for every non-negative integers r *and* m.

Proof. It is observed that

$$\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r} = \frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r} \times\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-m}.$$

Differentiating both sides gives

$$\begin{aligned}\frac{d}{dt}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r} &= \left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-m}\\ &\times\frac{d}{dt}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}\\ &+(r-m)p(p+1)\frac{v(x)}{u(x)}M_1(x)t^{p-1}\\ &\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^r},\end{aligned}$$

and we have

$$\begin{aligned}\frac{d}{dt}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^r} &= rp(p+1)\frac{v(x)}{u(x)}M_1(x)\\ \times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t^{p-1}&\\ +ru(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}+r(p+1)v(x)&\\ \times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}t^p.&\end{aligned}$$

Now, we get

$$\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-m}\frac{d}{dt}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}$$

$$+(r-m)p(p+1)\frac{v(x)}{u(x)}M_1(x)t^{p-1}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}$$

$$= rp(p+1)\frac{v(x)}{u(x)}M_1(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{r-1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t^{p-1}+ru(x)$$
$$\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}+r(p+1)v(x)$$
$$\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^r}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}t^p.$$

It follows that

$$\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)\frac{d}{dt}\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}$$
$$-mp(p+1)\frac{v(x)}{u(x)}M_1(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^m}{\left(1-u(x)t-v(x)t^{p+1}\right)^r}t^{p-1}$$
$$=ru(x)\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{m+1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}+r(p+1)v(x)$$
$$\times\frac{\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)^{m+1}}{\left(1-u(x)t-v(x)t^{p+1}\right)^{r+1}}t^p.$$

This implies

$$\left(M_1(x)+(p+1)\frac{v(x)}{u(x)}M_1(x)t^p\right)\frac{d}{dt}E^{(p,r,m)}_{u,v,j}(x)-mp(p+1)\frac{v(x)}{u(x)}$$
$$\times M_1(x)E^{(p,r,m)}_{u,v,j+1-p}(x)=ru(x)E^{(p,r+1,m+1)}_{u,v,j}(x)+r(p+1)v(x)E^{(p,r+1,m+1)}_{u,v,j-p}(x).$$

Further simplification gives

$$(p+1)\frac{v(x)}{u(x)}M_1(x)(j-mp)E^{(p,r,m)}_{u,v,j+1-p}(x)=ru(x)E^{(p,r+1,m+1)}_{u,v,j}(x)+r(p+1)$$
$$\times v(x)E^{(p,r+1,m+1)}_{u,v,j-p}(x)-M_1(x)(j+1)E^{(p,r,m)}_{u,v,j+1}(x),$$

and the result follows. □

References

[1] M. Anđelić, Z. Du, C. M. da Fonseca and E. Kiliç, A matrix approach to some second-order difference equations with sign-alternating coefficients, *J. Differ. Equ. Appl.*, **26** (2020), 149–162.

[2] R. G. Buschman, Fibonacci numbers, Chebyshev polynomials, Generalizations and differential equations, *Fibonacci Quart.*, **1** (1963), 1–8, 19.

[3] W. M. Abd-Elhameed, Y.H. Youssri, N. El-Sissi and M. Sadek, New hypergeometric connection formulae between Fibonacci and chebyshev polynomials, *Ramanujan J.*, **42** (2017), 347–361.

[4] C. M. da Fonseca, Unifying some Pell and Fibonacci identities, *Appl. Math. Comput.*, **236** (2014), 41–42.

[5] A. F. Horadam, Basic properties of a certain generalized sequence of numbers, *Fibonacci Quart.*, **3** (1965), 161–176.

[6] A. Şahin and J. L. Ramírez, Determinantal and permanental representations of convolved Lucas polynomials, *Appl. Math. Comput.*, **281** (2016), 314–322.

[7] X. Ye and Z. Zhang, A common generalization of convolved generalized Fibonacci and Lucas polynomials and its applications, *Appl. Math. Comput.*, **306** (2017), 31–37.

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Multiplicity of solutions for Robin problem involving the $p(x)$-laplacian

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Abstract. This paper is concerned with the existence and multiplicity of solutions for $p(x)$-Laplacian equations with Robin boundary condition. Our technical approach is based on variational methods.

1 Introduction

The purpose of this paper is to study the existence and multiplicity of solutions for the following Robin problem involving the $p(x)$-Laplacian

$$\begin{cases} \text{-div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) & \text{in} \quad \Omega \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu} = \beta(x)|u|^{p(x)-2}u & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

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where Ω is an open bounded subset of $\mathbb{R}^N (N \geq 2)$, with smooth boundary, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, $\beta \in L^\infty(\Omega)$, with $\operatorname{ess}\inf_{\Omega} \beta > 0$ and $p \in C_+(\overline{\Omega})$ with

$$1 < p^- := \inf_{\overline{\Omega}} p(x) \leq p^+ := \sup_{\overline{\Omega}} p(x) < +\infty.$$

Recently, there has been an increasing interest in studying of problems (1). This great interest may be justified by their various physical applications, we can for example refer to [3, 2, 6, 9, 16, 19, 23, 24, 25, 27, 30, 32]. In fact, there are applications concerning elastic mechanics [33], electrorheological fluids [28, 29], image restoration [12], dielectric breakdown, electrical resistivity and polycrystal plasticity and continuum mechanics [4]. We refer to [18] for an overview of this subject and to [11, 14] for the $p(x)-$Laplacian equations.

From the variational point of view, by using a theorem obtained by B. Ricceri in [5], the work [2] shows the existence of at least three solutions for a Navier problem involving the fourth order operator.

The authors in [6] obtained the existence of three distinct weak solutions of $p(x)-$Laplacian Dirichlet problems as applications of critical point theorem obtained by G. Bonanno and S.A. Marano in [7]. In the same breath, the authors in [30] consider the $p(x)-$Laplacian-like problem (originated from a capillary phenomenon) which the main tool is a general critical point theorem in [8].

In the statement of problem (1), $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function verifying (F_0) such that

(F_0) There exists a constant $c_1 \geq 0$ such that

$$| f(x,t) | \leq c_1(1 + |t|^{q(x)-1}),$$

for all $(x,t) \in \Omega \times \mathbb{R}$ where $q \in C_+(\overline{\Omega})$, $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

Where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } \ p(x) < N \\ +\infty & \text{if } \ p(x) \geq N. \end{cases}$$

Motivated by the references mentioned above, we establish the existence and multiplicity of solutions for problem (1). It is known that the extension $p(x)$-Laplace operator possesses more complicated structure than the p-Laplacian. For example, it is inhomogeneous and usually it does not have the so-called first eigenvalue, since the infimum of its spectrum is zero. This provokes some mathematical difficulties which makes the study of such a problems particulary interesting.

Now, we formulate our main results as follows.

Theorem 1 *Assume that* (F_0) *and the following assumptions hold.*

(F_1) $0 < \lim_{t\to 0} \frac{F(x,t)}{|t|^{p^-}} < \frac{1}{p^-}$, *for* $|t| > \delta$, *with* $\delta > 0$,

(F_2) $\lim_{|t|\to+\infty} \frac{p(x)F(x,t)}{|t|^{p^-}} \le 0$ *a.e* $x \in \Omega$,

(F_3) $\lim_{|t|\to+\infty} \int_\Omega F(x,t)dx = -\infty$,

Then the problem (1) *has two weak solutions.*

Theorem 2 *Assume that* (F_0) *and the following conditions hold.*

(F_4) $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^{p^-}} = 0$,

(F_5) $f(x,t)t > 0$ *for all* $(x,t) \in \Omega \times \mathbb{R}$,

(F_6) $\lim_{|t|\to+\infty} [f(x,t)t - p^+F(x,t)] = -\infty$.

Then the problem (1) *has at least one weak solution.*

Through taking the same methods of this paper, results similar to Theorems 1–2 can also be proven for Neumann and Steklov problems.

Our paper is organized as follows. We first present some necessary preliminary results on variable exponent Sobolev spaces. Next, we give the proof of the main results about the existence of weak solutions.

2 Preliminaries

In the sequel, let $p(x) \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \left\{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\right\}.$$

The variable exponent Lebesgue space is defined by

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < +\infty\}$$

furnished with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\sigma > 0 : \int_\Omega |\frac{u(x)}{\sigma}|^{p(x)}\,dx \leq 1\},$$

and the variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 1 [21] *The spaces* $L^{p(x)}(\Omega)$ *and* $W^{1,p(x)}(\Omega)$ *are separable, uniformly convex, reflexive Banach spaces. The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$, *where* $q(x)$ *is the conjugate function of* $p(x)$; *i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. *For* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$ *we have*

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}.$$

Proposition 2 [21] *For* $p, r \in C_+(\overline{\Omega})$ *such that* $r(x) \leq p^*(x)$ $(r(x) < p^*(x))$ *for all* $x \in \overline{\Omega}$, *there is a continuous (compact) embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega),$$

where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \textit{if} \quad p(x) < N \\ +\infty & \textit{if} \quad p(x) \geq N. \end{cases}$$

Proposition 3 [15] *For* $p \in C_+(\overline{\Omega})$ *and such* $r \in C_+(\partial\Omega)$ *that* $r(x) \leq p^\partial(x)$ $(r(x) < p^\partial(x))$ *for all* $x \in \overline{\Omega}$, *there is a continuous (compact) embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega),$$

where

$$p^\partial(x) = (p(x))^\partial := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \textit{if} \quad p(x) < N \\ +\infty & \textit{if} \quad p(x) \geq N. \end{cases}$$

Proposition 4 [17], [Theorem 2.1] *For any* $u \in W^{1,p(x)}(\Omega)$, *let*

$$\| u \|_{\partial} := |u|_{L^{p(x)}(\partial\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Then $\| u \|_{\partial}$ *is a norm on* $W^{1,p(x)}(\Omega)$ *which is equivalent to*

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Now, for any $u \in X := W^{1,p(x)}(\Omega)$ define

$$\|u\| := \inf\left\{\sigma > 0 : \int_{\Omega}\left(\left|\frac{\nabla u(x)}{\sigma}\right|^{p(x)} dx + \int_{\partial\Omega} \beta(x)\left|\frac{u(x)}{\sigma}\right|^{p(x)}\right) d\sigma_x \leq 1\right\}.$$

Where $\beta \in L^{\infty}(\Omega)$ and $d\sigma_x$ is the measure on the boundary $\partial\Omega$.Then by (4), $\|.\|$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|.\|_{W^{1,p(x)}(\Omega)}$ and $\|.\|_{\partial}$. Now, we introduce the modular $\rho : X \to \mathbb{R}$ defined by

$$\rho(u) = \int_{\Omega} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \beta(x)|u(x)|^{p(x)} d\sigma_x$$

for all $u \in X$. Here, we give some relations between the norm $\|.\|$ and the modular ρ.

Proposition 5 [21] *For* $u \in X$ *we have*

(i) $\|u\| < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;

(ii) *If* $\|u\| < 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;

(iii) *If* $\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$.

Lemma 1 [26] *Let* $X = X_1 \oplus X_2$, *where* X *is a real Banach space and* $X_2 \neq 0$, *and is finite dimensional. Suppose that* $\phi \in C^1(X, R)$ *satisfies Cerami condition (C) with the following assertions:*

(i) *There is a constant* α *and a bounded neighborhood* D *of* 0 *in* X_2 *such that* $\phi|\partial D \leq \alpha$.

(ii) *There is a constant* $\beta > \alpha$ *such that* $\phi|X_1 \geq \beta$.

Then ϕ *possesses a critical value* c, *moreover,* c *can be characterized as*

$$c = \inf_{h\in\Gamma} \max_{u\in\overline{D}} \phi(h(u))$$

where

$$\Gamma = \{h \in C(D, X) | h = id \text{ on } \partial D\}.$$

Lemma 2 [10] *Let $X = X_1 \oplus X_2$, where X is a real Banach space and $X_2 \neq 0$, and is finite dimensional. Suppose that $\phi \in C^1(X, R)$ satisfies Palais-smalle condition (PS) with the following assertions for some $r > 0$:*

(i) $\phi(u) \leq 0$, *for* $u \in X_1$, $\|u\| \leq r$.

(ii) $\phi(u) \geq 0$, *for* $u \in X_2$, $\|u\| \leq r$.

Assume also that ϕ is bounded below and $\inf_X \phi < 0$. Then ϕ has at least two nonzero critical points.

Definition 1 *We say that $u \in X$ is a weak solution of (1) if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2} u v d\sigma_x = \int_\Omega f(x,u) v dx$$

for all $v \in X$.

The functional associated to (1) is given by

$$\phi(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{1}{p(x)}\beta(x)|u|^{p(x)} dx - \int_\Omega F(x,u) dx \quad (2)$$

It should be noticed that under the condition (F_0) the functional ϕ is of class $C^1(X, \mathbb{R})$ and

$$\phi'(u).v = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2} u v d\sigma_x - \int_\Omega f(x,u) v dx, \quad \forall\, u, v \in X.$$

Then, we can see that the weak solution of (1) corresponds to critical point of the functional ϕ.

3 Proof of main result

We recall that ϕ satisfies Palais-smale condition (PS) in X, if any sequence (u_n) such that $\phi(u_n)$ is bounded and $\phi'(u_n) \to 0$ as $n \to +\infty$, has convergent subsequence.

Proof of Theorem 1
Let start by the following lemma.

Lemma 3 *Any bounded sequence (PS) of ϕ has a strongly convergent subsequence.*

Proof. Let $(u_n) \subset X$ be a sequence bounded (PS) sequence of ϕ. Up to a subsequence, we may find $u \in X$ such that $u_n \rightharpoonup u$.

From the growth condition (F_0) and Sobolev embedding, we have that $\int_\Omega f(x,u)(u_n - u)dx \to 0$, since $\phi^{'}(u_n)(u_n - u) \to 0$ then

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u \nabla((u_n - u))dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)-2}u_n(u_n - u)d\sigma_x \to 0.$$

As the mapping $A : W^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\langle Au, v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv d\sigma_x$$

for all $u, v \in X$ is of type (S_+), so $u_n \to u$ in $W^{1,p(x)}(\Omega)$. $\square$

Lemma 4 *The functional ϕ is coercive, that is, $\phi(u) \to +\infty$ when $\|u\| \to +\infty$.*

Proof. Suppose that there exist $(u_n) \subset X$ and a positive constant C such that

$$\|u_n\| \to +\infty, \quad \phi(u_n) \leq C.$$

Putting $v_n = \frac{u_n}{\|u_n\|}$, so we may find $v \in X$ and a subsequence of (v_n) still denoted by (v_n) such that $v_n \rightharpoonup v$ in X and $v_n \to v$ in $L^{p(x)}(\Omega)$.

By (F_1), for any $\epsilon > 0$, $\exists L > 0$ such that

$$F(x,t) \leq \frac{\varepsilon}{p(x)}|t|^{p^-} \ \forall |t| > L \ a.e \ x \in \Omega,$$

thus, we may find a positive constant C such that

$$F(x,t) \leq \frac{\varepsilon}{p(x)}|t|^{p^-} + C \ \forall t \in \mathbb{R} \ a.e \ x \in \Omega.$$

Therefore,

$$\frac{C}{\|u_n\|^{p^-}} \geq \frac{\phi(u_n)}{\|u_n\|^{p^-}} \geq \frac{1}{p^+}\frac{1}{\|u_n\|^{p^-}}\left[\int_\Omega |\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)}d\sigma_x\right]$$

$$- \frac{\epsilon}{p^+}\int_\Omega |v_n|^{p^-} dx - \frac{C|\Omega|}{\|u_n\|^{p^-}}$$
$$\geq \frac{1}{p^+} - \frac{\epsilon}{p^+}\int_\Omega |v_n|^{p^-} dx - \frac{C|\Omega|}{\|u_n\|^{p^-}}.$$

Consequently, choosing ϵ, such that $\int_\Omega |v_n|^{p^-} dx > C_0$, where C_0 is the best constant in the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$.

On the other hand, because $\|v\|_{W^{1,p(x)}(\Omega)} \leq \liminf \|v_n\| = 1$ by

$$\int_\Omega |\nabla v|^{p^-} dx + \int_\Omega |v|^{p^-} dx \leq C_0,$$

so we get $\int_\Omega \nabla |v|^{p^-} \, dx = 0$, which means that $v = \mathtt{constant} \neq 0$.

We obtain

$$\lim_{|u_n| \to \infty} \int_\Omega F(x, u_n)\, dx \to -\infty.$$

When $\|u_n\| \to +\infty$, $|u_n| \to +\infty$, thereby,

$$\begin{aligned} C &\geq \frac{1}{p^+}\left[\int_\Omega |\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)} d\sigma_x\right] - \int_\Omega F(x, u_n) dx \\ &\geq \frac{\|u_n\|^{p^-}}{p^+} - \frac{\epsilon}{p^+}\int_\Omega |v_n|^{p^-} dx - \frac{C|\Omega|}{\|u_n\|^{p^-}}, \end{aligned}$$

which implies that ϕ is coercive and bounded from below.

Now verifying the conditions (i) and (ii) in Lemma 2.

The same idea from [1] and Chung [13],we have $W^{1,p(x)}(\Omega) = W_0 \oplus \mathbb{R}$. If $u \in \mathbb{R}$, for $\|u\| < \rho$, $\rho > 0$ and by (2)

$$\begin{aligned} \phi(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} \frac{1}{p(x)}\beta(x)|u|^{p(x)} d\sigma_x - \int_\Omega F(x, u) dx \\ &= -\int_\Omega F(x, u) dx \\ &\leq 0 \end{aligned}$$

If $u \in W_0 = \left\{z \in W^{1,p(x)}(\Omega)/ \int_\Omega z dx = 0\right\}$, from (F_0) and (F_1)

$$F(x, t) \leq \left(\frac{1}{p^-} - \epsilon\right)|u|^{p^-} + C\int_\Omega |u|^{q(x)} dx.$$

In virtue of the the continuous embedding X into $L^{p^-}(\Omega)$ and $L^{p^+}(\Omega)$,

$$\begin{aligned}\phi(u) &\geq \frac{1}{p^+}\|u\|^{p^+} - \int_\Omega F(x,u)dx \\ &\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{1}{p^-}\int_\Omega |u|^{p^-}dx + \epsilon\int_\Omega |u|^{p^-}dx - C\int_\Omega |u|^{q(x)}dx \\ &\geq C(\epsilon)\|u\|^{p^+} - C\|u\|^{q^+} - C\|u\|^{q^-},\end{aligned}$$

for $\|u\| = \rho$ small enough then $\phi(u) \geq 0$ for $\|u\| \leq \rho$ $\forall u \in W_0$. On the other side, when $\inf_X \phi(u) = 0$ then $\forall u \in \mathbb{R}$ is a minimum of ϕ, that means ϕ admits infinite critical points.
When $u \in X$ with $\inf_X \phi(u) < 0$, by applying Lemma 2, ϕ has at least two nontrivial critical points, then the problem (1) has two nontrivial solutions in X. □

***Proof of Theorem** 2* We recall the following important inequality (cf.[22])

Lemma 5 *(Poincaré-Writingers inequality) There exists a positive constant* $\underline{C}$ *such that for any* $u \in W_0$ *we have*

$$|u|_{p(x)} \leq \underline{C}|\nabla u|_{p(x)}.$$

Lemma 6 *Suppose that the conditions* (F_0), (F_4) *and* (F_6) *are hold. Then* ϕ *verifies the Cerami condition* $(C)_c$.

Proof. Let $K \in \mathbb{R}$ such that

$$|\phi(u_n)| \leq K$$

and

$$(1 + \|u_n\|)\phi'(u_n) \to 0 \text{ in } X^*. \tag{3}$$

Suppose that $\|u_n\| \to +\infty$ as $n \to +\infty$. Taking $v_n = \frac{u_n}{\|u_n\|}$, so

$$v_n \rightharpoonup v \text{ in } X.$$

Thus,

$$v_n(x) \to v(x) \ \text{ a.e } \ x \in \Omega$$

and

$$v_n \to v \text{ in } L^{p(x)}(\Omega).$$

Let $h \in X$, according to (3) we have that,

$$\left|\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla h dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)-2}u_n h d\sigma_x - \int_\Omega f(x,u_n)h dx\right| \leq \frac{\epsilon_n\|h\|}{1+\|u_n\|}. \tag{4}$$

Dividing (4) by $\|u_n\|^{p^- -1}$ we have

$$\frac{1}{\|u_n\|^{p^- -1}}\left|\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla h dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)-2}u_n h d\sigma_x - \int_\Omega f(x,u_n)h dx\right| \leq \frac{\epsilon_n\|h\|}{(\|u_n\|^{p^- -1})(1+\|u_n\|)}.$$

Then

$$\frac{1}{\|u_n\|^{p^- -1}}\left|\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla h dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)-2}u_n h d\sigma_x - \int_\Omega f(x,u_n)h dx\right| \leq \frac{\epsilon_n\|h\|}{1+\|u_n\|}. \tag{5}$$

Since $\|u_n\|^{p(x)-1} \geq \|u_n\|^{p^- -1} > 1$,

$$\begin{aligned}
&\frac{1}{\|u_n\|^{p^- -1}}\left|\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla h dx\right. \\
&\quad \left. + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)-2}u_n h d\sigma_x - \int_\Omega f(x,u_n)h dx\right| \\
&\geq \frac{1}{\|u_n\|^{p^- -1}}\left|\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla h dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)-2}u_n h d\sigma_x\right| \\
&\quad - \frac{1}{\|u_n\|^{p^- -1}}\left|\int_\Omega f(x,u_n)h dx\right| \\
&\geq \left|\int_\Omega |\nabla v_n|^{p(x)-2}\nabla v_n \nabla h dx + \int_{\partial\Omega} \beta(x)|v_n|^{p(x)-2}v_n h d\sigma_x\right|
\end{aligned}$$

$$-\frac{1}{\|u_n\|^{p^--1}}\left|\int_\Omega f(x,u_n)h\,dx\right|.$$

Consequently

$$\left|\int_\Omega |\nabla v_n|^{p(x)-2}\nabla v_n\nabla h\,dx+\int_{\partial\Omega}\beta(x)|v_n|^{p(x)-2}v_n h\,d\sigma_x\right| \\ -\frac{1}{\|u_n\|^{p^--1}}\left|\int_\Omega f(x,u_n)h\,dx\right|\le\frac{\epsilon_n\|h\|}{1+\|u_n\|}, \tag{6}$$

with $\epsilon_n\to 0$ and $h\in X$.

By (F_0), (F_4) and (F_6) we conclude that $\frac{f(x,u_n)}{\|u_n\|^{p^--1}}$ is bounded in $(L^{p^-}(\Omega))^*$ which is separable and reflexive space, then up to a subsequence denoted also $\left(\frac{f(x,u_n)}{\|u_n\|^{p^--1}}\right)$, we have $\frac{f(x,u_n)}{\|u_n\|^{p^--1}}\rightharpoonup\widetilde{f}$, in $(L^{p^-}(\Omega))^*$. Since $\frac{f(x,u_n)}{\|u_n\|^{p^--1}}\to 0$ a.e $x\in\Omega$, hence

$$\frac{f(x,u_n)}{\|u_n\|^{p^--1}}\rightharpoonup 0 \text{ in } (L^{p^-}(\Omega))^*$$

Therefore, taking $h=v_n-v\in X$, in (6)

$$\int_\Omega |\nabla v_n|^{p(x)-2}\nabla v_n\nabla(v_n-v)\,dx+\int_{\partial\Omega}\beta(x)|v_n|^{p(x)-2}v_n(v_n-v)\,d\sigma_x\to 0.$$

By (S_+) type of the operator

$$L(u).v=\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx+\int_{\partial\Omega}\beta(x)|u|^{p(x)-2}uv\,d\sigma_x,$$

we have $v_n\to v$ in X, so $v\neq 0$. Since $|\phi(u)|\le K$ we obtain

$$p^+\phi(u)\ge -p^+K \tag{7}$$

Taking $h=u_n$, in (4)

$$-\int_\Omega |\nabla u_n|^{p(x)}dx+\int_{\partial\Omega}\beta(x)|u_n|^{p(x)}d\sigma_x+\int_\Omega f(x,u_n)u_n\,dx\ge-\epsilon_n$$

Then

$$-p^-\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}dx+\int_{\partial\Omega}\beta(x)\frac{|u_n|^{p(x)}}{p(x)}d\sigma_x+\int_\Omega f(x,u_n)u_n\,dx\ge-\epsilon_n. \tag{8}$$

Adding (7)to (8), we obtain

$$\int_\Omega f(x,u_n)u_n dx - p^+ \int_\Omega F(x,u_n)dx \geq C \tag{9}$$

Obviously, this is contradiction and then the proof of Lemma 6 is reached. □

Lemma 7 *Suppose that the conditions* (F_5) *and* (F_6) *hold, then the function* $\phi/\mathbb{R}$ *is anti-coercive.*

Proof. From (F_6), for all $K > 0$ there exists $R > 0$ such that $p^+F(x,u) \geq f(x,u)u \geq K$ for a.e $x \in \Omega$, $u \in \mathbb{R}$ and thus for all $u \in \mathbb{R}$,

$$\int_\Omega F(x,u)dx \geq \frac{1}{p^+}K|\Omega| - c|\Omega|,$$

hence

$$\int_\Omega F(x,u)dx \to +\infty \text{ when } |u| \to +\infty.$$

By (2) and K is arbitrary

$$\phi(u) = \int_{\partial\Omega} \beta(x)\frac{|u|^{p(x)}}{p(x)}d\sigma_x - \int_\Omega F(x,u)dx \geq -\int_\Omega F(x,u)dx$$

Then

$$\phi(u) \to -\infty \text{ when } |u| \to +\infty.$$

□

Lemma 8 *Under the hypothesis* (F_4), *we have* $\inf_{W_0} \phi > -\infty$.

Proof. Let $u \in W_0$ with $\|u\| > 1$ By (F_5), for $\epsilon > 0$, we may find $K(\epsilon) > 0$ such that $F(x,u) \leq \epsilon|u|^{p^-} + K(\epsilon)$, for a.e $x \in \Omega$ and for all $u \in \mathbb{R}$.Hence,

$$\begin{aligned} F(x,u) &\leq \epsilon\int_\Omega |u|^{p^-} + K(\epsilon)|\Omega| \\ &\leq \epsilon C\|u\|^{p^-} + K(\epsilon)|\Omega| \end{aligned} \tag{10}$$

Then, when $u \in W_0$ we have

$$\begin{aligned} \phi(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} + \int_{\partial\Omega} \frac{1}{p(x)}\beta(x)|u|^{p(x)}d\sigma_x - \int_\Omega F(x,u)dx \\ &\geq \overline{C}\|u\|^{p^+} - \epsilon C\|u\|^{p^-} - K(\epsilon)|\Omega| \\ &\geq -K(\epsilon)|\Omega|. \end{aligned} \tag{11}$$

It follows that $\inf_{W_0} \phi > -\infty$.

According to previous Lemmas 6, 7 and 8, the assumptions of Lemma 1 are satisfied and then the proof of Theorem 2 is achieved. □

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References

[1] M. Allaoui, A. El Amrouss and A. Ourraoui, On a Class of Nonlocal $p(x)$-Laplacian Neumann Problems, *Thai Journal of Mathematics*, Volume **15** (2017) Number 1, 91–105.

[2] A. Allaoui, A. El Amrouss, A. Ourraoui, Three Solutions For A Quasi-Linear Elliptic Problem, *Applied Mathematics E-Notes*, **13** (2013), 51–59.

[3] G. A. Afrouzi, A. Hadjian, S. Heidarkhani, Steklove problems involving the $p(x)$-Laplacian, *Electronic Journal of Differential Equations*, Vol. 2014 (2014) No. 134, 1–11.

[4] S. Antontsev, S. Shmarev, Handbook of Differential Equations, Stationary Partial Differ. Equ. 3 (2006). Chap. 1

[5] B. Ricceri, A three critical points theorem revisited, *Nonlinear Anal.*, **70** (2009), 3084–3089.

[6] G. Bonanno, A. Chinnì, Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent, *J. Math. Anal. Appl.*, **418** (2014), 812–827.

[7] G. Bonanno, S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.*, 89 (2010), 1–10.

[8] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.*, **2009** (2009), 1–20.

[9] M. M. Boureanu, Infinitely many solutions for a class of degenerate anisotropic elliptic problems with variable exponent, *Taiwanese Journal of Mathematics*, 15 (5), 2291–2310.

[10] H. Brezis, L. Nirenberg, Remarks on finding critical points, *Commun Pure Appl Math.*, **44** (1991), 939–963. doi:10.1002/ cpa.3160440808.

[11] J. Chabrowski, Y. Fu, Existence of solutions for $p(x)$-Laplacian problems on a bounded domain, *J. Math. Anal. Appl.*, **306** (2005), 604–618.

[12] Y. Chen, S. Levine, R. Rao, Variable exponent, linear growth functionals in image restoration, *SIAMJ. Appl. Math.*, **66** (2006), 1383–1406.

[13] N. T. Chung, Multiple solutions for a class of $p(x)$-Kirchhoff type problems with Neumann boundary con- ditions, *Advances in Pure Appl. Math.*, **4** (2) (2013), 165–177.

[14] S.-G. Deng, A local mountain pass theorem and applications to a double perturbed $p(x)$-Laplacian equations, *Appl. Math. Comput.*, **211** (2009), 234–241.

[15] S.-G. Deng, Eigenvalues of the $p(x)$-Laplacian Steklov problem, *J. Math. Anal. Appl.*, **339** (2008), 925–937.

[16] G. Dai, Infinitely many solutions for a $p(x)$-Laplacian equation in $\mathbb{R}^N$, *Nonlinear Anal.*, **71** (2009), 1133–1139.

[17] S.-G. Deng,Positive solutions for Robin problem involving the $p(x)$-Laplacian, *J. Math. Anal. Appl.*, **360** (2009), 548–560.

[18] L. Diening, P. Hästö, A. Nekvinda, *Open problems in variable exponent Lebesgue and Sobolev spaces*, in: P. Drábek, J. Rákosník (Eds.), Proceedings, Milovy, Czech Republic, 2004, pp. 38–58.

[19] X. Fan, X. Han, Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Anal.*, **59** (2004), 173–188.

[20] M. Ferrara, S. Heidarkhani, Multiple solutions for perturbed p-Laplacian boundary-value problems with impulsive effects, *Electronic journal of differential equations*, n. 2014 (2014), 1–14, ISSN: 1072–6691.

[21] X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ *J. Math. Anal. Appl.*, **263** (2001), 424–446.

[22] X. Fan. Eigenvalues of the $p(x)$-Laplacian Neumann problems, *Nonlinear Anal.*, 67:2982-2992, 2007. http://dx.doi.org/10.1016/j.na.2006.09.052.

[23] Y. Fu, X. Zhang, A multiplicity result for $p(x)$-Laplacian problem in $\mathbb{R}^N$, *Nonlinear Anal.*, **70** (2009), 2261–2269.

[24] A. Ourraoui, Multiplicity results for Steklov problem with variable exponent, *Applied Mathematics and Computation*, **277** (2016), 34–43.

[25] A. Ourraoui, Some Results For Robin Type Problem Involving $p(x)-$Laplacian, (preprint).

[26] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations. In CBMS Regional Conference Series in Mathemat- ics, Vol. 65, American Mathematical Soceity, Providence (1986).

[27] M. A. Ragusa, A. Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, *Adv. Nonlinear Anal.*, **9** (2020), 710–728.

[28] M. Růžička, Flow of shear dependent electrorheological fluids, *CR Math. Acad. Sci. Paris*, **329** (1999), 393–398.

[29] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory, Springer-Verlag, Berlin, 2002.

[30] S. Shokooh, A. Neirameh, Existence results of infinitely many weak solutions for $p(x)-$Lapalacian-Like operators, *U.P.B. Sci. Bull.*, Series A, Vol. **78**, Iss. 4, 2016.

[31] E. Zeidler, Nonlinear Functional Analysis and its Applications, vol. II/B, Berlin, Heidelberg, New York, 1985.

[32] Q. H. Zhang, Existence and asymptotic behavior of positive solutions for variable exponent elliptic systems, *Nonlinear Anal.*, **70** (2009), 305–316.

[33] V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, *Math. USSR Izv.*, **29** (1987), 33–66.

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Stability result of the Lamé system with a delay term in the internal fractional feedback

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Abstract. In this article, we consider a Lamé system with a delay term in the internal fractional feedback. We show the existence and uniqueness of solutions by means of the semigroup theory under a certain condition between the weight of the delay term in the fractional feedback and the weight of the term without delay. Furthermore, we show the exponential stability by the classical theorem of Gearhart, Huang and Pruss.

1 Introduction

In this article, we consider the initial boundary value problem for the Lamé system given by:

$$\begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\mathrm{div}\ u) & \\ +a_1\partial_t^{\sigma,\kappa}u(x,t-\tau) + a_2u_t(x,t) = 0 & \text{in } \Omega\times(0,+\infty), \\ u = 0 & \text{in } \Gamma\times(0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega\times(0,\tau), \end{cases} \tag{P}$$

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where μ, λ are Lamé constants, $u = (u_1, u_2, ..., u_n)^T$. Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$. Moreover, $a_1 > 0$, $a_2 > 0$ and the constant $\tau > 0$ is the time delay. The notation $\partial_t^{\sigma,\kappa}$ stands for the exponential fractional derivative operator of order σ. It is defined by

$$\partial_t^{\sigma,\kappa} w(t) = \frac{1}{\Gamma(1-\sigma)} \int_0^t (t-s)^{-\sigma} e^{-\kappa(t-s)} \frac{dw}{ds}(s)\, ds \qquad 0 < \sigma < 1, \qquad \kappa > 0.$$

Delay effects arise in many applications and pratical problems because, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [1], [17], and references therein. In many cases it was shown that delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used.The stability issue of systems with delay is, therefore, of theoretical and practical importance. In particular, consider the wave equation with homogeneous Dirichlet boundary condition

$$\begin{cases} u''(x,t) - \Delta_x u(x,t) + \mu_1 u'(x,t) & \\ +\mu_2 u'(x,t-\tau) = 0 & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ u(x,0) = u_0(x), u'(x,0) = u_1(x) & \text{in } \Omega, \\ u'(x,t-\tau) = f_0(x,t-\tau) & \text{in } \Omega \times (0,\tau). \end{cases} \tag{PW}$$

For instance in [13] the authors studied the problem (PW). They determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (PW) will be instable if $\mu_2 > \mu_1$. The main approach used in [13] is an observability inequality obtained with a Carleman estimate.

Noting that the case of the wave equation with internal fractional feedback (without delay) have treated in [8] where it is proven global existence and uniqueness results. As far as we are concerned, this is the first work in the literature that takes into account the uniform decay rates for Lamé system with delay term in the internal fractional feedback.

The remainder of the paper falls into five sections. In Section **2**, we show that the above system can be replaced by an augmented one obtained by coupling an equation with a suitable diffusion, and we study of energy functional associated to system. In section **3**, we state a well-posedness result for problem (P). In section **4**, we prove the strong asymptotic stability of solutions. In section **5** we show the exponential stability using the Gearhart-Huang-Pruss theorem.

2 Preliminary

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 1 (see [12]) *Let ω be the function:*

$$\omega(\xi) = |\xi|^{(2\sigma-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \sigma < 1. \tag{1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t\psi(\xi,t)+(\xi^2+\kappa)\psi(\xi,t)-U(t)\omega(\xi) = 0, \quad -\infty < \xi < +\infty, \kappa > 0, t > 0, \tag{2}$$

$$\psi(\xi,0) = 0, \tag{3}$$

$$O(t) = (\pi)^{-1}\sin(\sigma\pi)\int_{-\infty}^{+\infty}\omega(\xi)\psi(\xi,t)\,d\xi \tag{4}$$

is given by

$$O = I^{1-\sigma,\kappa}U = D^{\sigma,\kappa}U, \tag{5}$$

where

$$[I^{\sigma,\kappa}f](t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}e^{-\kappa(t-s)}f(s)\,ds.$$

Proof. From (2) and (3), we have

$$\psi(\xi,t) = \int_0^t \omega(\xi)e^{-(\xi^2+\kappa)(t-s)}U(s)ds. \tag{6}$$

Hence, by using (4), we get

$$O(t) = (\pi)^{-1}\sin(\sigma\pi)e^{-\kappa t}\int_0^t\left[2\int_0^{+\infty}|\xi|^{2\sigma-1}e^{-\xi^2(t-s)}d\xi\right]e^{\kappa s}U(s)ds. \tag{7}$$

Thus,

$$\begin{aligned} O(t) &= (\pi)^{-1}\sin(\sigma\pi)e^{-\kappa t}\int_0^t \left[(t-s)^{-\sigma}\Gamma(\sigma)\right] e^{\kappa s}U(s)ds \\ &= (\pi)^{-1}\sin(\sigma\pi)\int_0^t \left[(t-s)^{-\sigma}\Gamma(\sigma)\right] e^{-\kappa(t-s)}U(s)ds \end{aligned} \tag{8}$$

which completes the proof. Indeed, we know that $(\pi)^{-1}\sin(\sigma\pi) = \dfrac{1}{\Gamma(\sigma)\Gamma(1-\sigma)}$. □

Lemma 1 (see [5]) *If* $\lambda \in D_\kappa = \mathbb{C}\backslash]-\infty, -\kappa]$ *then*

$$\int_{-\infty}^{+\infty} \frac{\omega^2(\xi)}{\lambda+\kappa+\xi^2}\, d\xi = \frac{\pi}{\sin\sigma\pi}(\lambda+\kappa)^{\sigma-1}.$$

We make the following hypotheses on the damping and the delay functions:

$$a_1\kappa^{\sigma-1} < a_2. \tag{9}$$

We are now in a position to reformulate system (P). As in [13], we introduce the new variable

$$z(x,\rho,t) = u_t(x, t-\rho\tau), \quad x\in\Omega,\ \rho\in(0,1),\ t>0.$$

Then the above variable z satisfies

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x\in\Omega,\ \rho\in(0,1),\ t>0.$$

Consequently, by using Theorem 1, the system (P) is equivalent to

$$\begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div}\ u) & \\ \quad +\zeta\displaystyle\int_{-\infty}^{+\infty}\omega(\xi)\psi(x,\xi,t)\,d\xi + a_2u_t(t) = 0 & \text{in } \Omega\times(0,+\infty), \\ \psi_t(x,\xi,t) + (\xi^2+\kappa)\psi(x,\xi,t) & \\ \quad -z(x,1,t)\omega(\xi) = 0 & \text{in } \Omega\times(-\infty,\infty)\times(0,+\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } \Omega\times(0,1)\times(0,+\infty), \\ u(x,t) = 0l & \text{on } \Gamma\times(0,+\infty), \\ z(x,0,t) = u_t(x,t), & \text{in } \Omega\times(0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{on } \Omega, \\ \psi(x,\xi,0) = 0 & \text{on } \Omega\times(-\infty,\infty), \\ z(x,\rho,0) = f_0(x,-\rho\tau) & \text{in } \Omega\times(0,1), \end{cases} \tag{P'}$$

where $\zeta = (\pi)^{-1}\sin(\sigma\pi)a_1$.

We define the energy of the solution by:

$$\begin{aligned} E(t) = \frac{1}{2}\sum_{j=1}^{n}\left(\|u_{jt}\|^2_{L^2(\Omega)} + \mu\|\nabla u_j\|^2_{L^2(\Omega)} + \zeta\int_\Omega\int_{-\infty}^{+\infty}|\psi_j(x,\xi,t)|^2\,d\xi dx\right) \\ +\frac{\nu}{2}\sum_{j=1}^{n}\int_\Omega\int_0^1|z_j(x,\rho,t)|^2\,d\rho dx + \frac{(\mu+\lambda)}{2}\|\mathrm{div}\ u\|^2_{L^2(\Omega)}. \end{aligned} \tag{10}$$

where ν is a positive constant verifying

$$\tau\zeta\left(\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\right) < \nu < \tau\left(2a_2 - \zeta\left(\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\right)\right). \tag{11}$$

Remark 1 *Using Lemma 1, the condition (11) means that*

$$\tau\kappa^{\sigma-1} < \nu < \tau(2a_2 - a_1\kappa^{\sigma-1}).$$

Lemma 2 *Let (u,ψ,z) be a regular solution of the problem (P'). Then there exists a positive constant C such that the energy functional defined by (10) satisfies*

$$E'(t) \le -C\sum_{j=1}^{n}\int_\Omega\left(u_t^2 + z(x,1,t)^2\right)dx. \tag{12}$$

Proof. Multiplying the first equation in (P) by $\overline{u}_{jt}$, integrating over Ω and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|u_{jt}\|_2^2 - \mu\Re\int_\Omega\Delta u_j\overline{u}_{jt}\,dx - (\mu+\lambda)\Re\int_\Omega\frac{\partial}{\partial x_j}(\mathrm{div}\ u)\overline{u}_{jt}\,dx$$

$$+\zeta\int_\Omega\overline{u}_{jt}\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx + a_2\int_\Omega|u_{jt}(t)|^2\,dx = 0.$$

Then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\sum_{j=1}^{n}\left(\|u_{jt}\|^2_{L^2(\Omega)} + \mu\|\nabla u_j\|^2_{L^2(\Omega)}\right) + \frac{(\mu+\lambda)}{2}\|\mathrm{div}\ u\|^2_{L^2(\Omega)} \\ + a_2\sum_{j=1}^{n}\|u_{jt}\|^2_{L^2(\Omega)} + \zeta\Re\sum_{j=1}^{n}\int_\Omega\overline{u}_{jt}\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx = 0. \end{aligned} \tag{13}$$

Multiplying the second equation in (P') by $\zeta\overline{\psi}_j$ and integrating over $\Omega \times (-\infty,+\infty)$, we obtain:

$$\begin{aligned} \frac{\zeta}{2}\frac{d}{dt}\sum_{j=1}^{n}\|\psi_j\|^2_{L^2(\Omega\times(-\infty,+\infty))} + \zeta\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,d\xi\,dx \\ - \zeta\Re\sum_{j=1}^{n}\int_\Omega z_j(x,1,t)\int_{-\infty}^{+\infty}\omega(\xi)\overline{\psi}_j(x,\xi,t)\,d\xi\,dx = 0. \end{aligned} \tag{14}$$

Multiplying the third equation in (P') by $\nu\overline{z}_j$ and integrating over $\Omega\times(0,1)$, we get:

$$\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{n}\|z_j\|^2_{L^2(\Omega\times(0,1))} + \frac{\tau^{-1}}{2}\sum_{j=1}^{n}\int_\Omega\left(z_j^2(x,1,t)-u_{jt}^2(x,t)\right)=0. \tag{15}$$

From (10), (13) and (15) we get

$$\begin{aligned} \mathcal{E}'(t) = &- a_2\sum_{j=1}^{n}\|u_{jt}\|^2_{L^2} - \zeta\sum_{j=1}^{n}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,d\xi\,dx \\ &- \zeta\Re\sum_{j=1}^{n}\int_\Omega\overline{u}_{jt}\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx \\ &+ \zeta\Re\sum_{j=1}^{n}\int_\Omega z_j(x,1,t)\int_{-\infty}^{+\infty}\omega(\xi)\overline{\psi}_j(x,\xi,t)\,d\xi\,dx \\ &+ \frac{\nu\tau^{-1}}{2}\sum_{j=1}^{n}\int_\Omega u_t^2(x,t)\,dx - \frac{\nu\tau^{-1}}{2}\sum_{j=1}^{n}\int_\Omega z_j^2(x,1,t)\,dx. \end{aligned} \tag{16}$$

Moreover, we have

$$\left|\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\right| \le \left(\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,d\xi\right)^{\frac{1}{2}}.$$

Then

$$\begin{aligned} &\left|\int_\Omega z_j(x,1,t)\int_{-\infty}^{+\infty}\omega(\xi)\overline{\psi}_j(x,\xi,t)\,d\xi\,dx\right| \\ &\le \left(\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\right)^{\frac{1}{2}}\|z_j(x,1,t)\|_{L^2(\Omega)}\left(\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,dx\,d\xi\right)^{\frac{1}{2}} \end{aligned}$$

and

$$\left|\int_\Omega \overline{u}_{jt}(x,t)\int_{-\infty}^{+\infty}\omega(\xi)\psi_j(x,\xi,t)\,d\xi\,dx\right|$$

$$\leq\left(\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\right)^{\frac{1}{2}}\|u_{jt}(x,t)\|_{L^2(\Omega)}\left(\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\kappa)|\psi_j(x,\xi,t)|^2\,dx\,d\xi\right)^{\frac{1}{2}}.$$

Applying the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}E'(t)\leq&\left(-a_2+\frac{\zeta I}{2}+\frac{\nu\tau^{-1}}{2}\right)\sum_{j=1}^n\int_\Omega u_{jt}^2(x,t)\,dx\\&+\left(\frac{\zeta I}{2}-\frac{\nu\tau^{-1}}{2}\right)\sum_{j=1}^n\int_\Omega z_j^2(x,1,t)\,dx,\end{aligned}$$

where $I=\displaystyle\int_{-\infty}^{+\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi$, which implies

$$E'(t)\leq -C\sum_{j=1}^n\int_\Omega\left(u_{jt}^2(x,t)+z_j^2(x,1,t)\right)\,dx$$

with

$$C=\min\left\{\left(a_2-\frac{\zeta I}{2}-\frac{\nu\tau^{-1}}{2}\right),\left(-\frac{\zeta I}{2}+\frac{\nu\tau^{-1}}{2}\right)\right\}.$$

Since ν is chosen satisfying assumption (11), the constant C is positive. This completes the proof of the lemma. □

3 Well-posedness

In this section, we give the existence and uniqueness result for system (P') using the semigroup theory. Let us denote $U=(u,v,\psi,z)^T$, where $v=u_t$. The system (P') can be rewrite as follows:

$$\begin{cases}U'=\mathcal{A}U, & t>0,\\ U(0)=(u_0,u_1,\psi_0,f_0),\end{cases}\tag{17}$$

where $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ is the linear operator defined by

$$\mathcal{A}\begin{pmatrix}u\\v\\\psi\\z\end{pmatrix}=\begin{pmatrix}v\\ \mu\Delta u+(\mu+\lambda)\nabla(\operatorname{div}u)-\zeta\displaystyle\int_{-\infty}^{+\infty}\omega(\xi)\psi(x,\xi)\,d\xi-a_2v\\ -(\xi^2+\kappa)\psi+z(x,1)\omega(\xi)\\ -\tau^{-1}z_\rho(x,\rho)\end{pmatrix}\tag{18}$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H} = \left(H_0^1(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n \times \left(L^2(\Omega \times (-\infty,+\infty))\right)^n \times \left(L^2(\Omega \times (0,1))\right)^n.$$

For any $U = (u, v, \psi, z)^T \in \mathcal{H}, \tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\psi}, \tilde{z})^T \in \mathcal{H}$, we equip $\mathcal{H}$ with the inner product defined by

$$< U, \tilde{U} >_{\mathcal{H}} = \sum_{j=1}^{n} \int_{\Omega} \left(v_j \bar{\tilde{v}}_j + \mu \nabla u_j \nabla \bar{\tilde{u}}_j\right) dx + (\mu + \lambda) \int_{\Omega} (\text{div } u)(\text{div } \bar{\tilde{u}})\, dx$$
$$+ \ \zeta \sum_{j=1}^{n} \int_{\Omega} \int_{-\infty}^{+\infty} \psi_j(x, \xi) \bar{\tilde{\psi}}_j(x, \xi) d\xi dx + \zeta \sum_{j=1}^{n} \int_{\Omega} \int_{0}^{1} z(x, \rho) \bar{\tilde{z}}_j(x, \rho) d\rho dx.$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \psi, z)^T \text{ in } \mathcal{H} : u \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^n, v \in \left(H^1(\Omega)\right)^n, \\ -(\xi^2 + \kappa)\psi + z(x, 1, t)\omega(\xi) \in \left(L^2(\Omega \times (-\infty, +\infty))\right)^n, \\ z \in \left(L^2\left(\Omega; H^1(0, 1)\right)\right)^n, \\ |\xi|\psi \in \left(L^2(\Omega \times (-\infty, +\infty))\right)^n, v = z(., 0) \text{ in } \Omega \end{array} \right\}. \quad (19)$$

We show that the operator $\mathcal{A}$ generates a C_0 semigroup in $\mathcal{H}$. We prove that $\mathcal{A}$ is a maximal dissipative operator. For this we need the following two Lemmas.

Lemma 3 *The operator $\mathcal{A}$ is dissipative and satisfies for any $U \in D(\mathcal{A})$,*

$$\Re\langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq -C \sum_{j=1}^{n} \int_{\Omega} \left(v^2 + z(x, 1)^2\right) dx. \quad (20)$$

Proof. For any $U = (u, v, \psi, z) \in D(\mathcal{A})$, using (17), (12) and the fact that

$$E(t) = \frac{1}{2} \|U\|_{\mathcal{H}}^2, \quad (21)$$

estimate (20) easily follows. □

Lemma 4 *The operator $(\tilde{\lambda} I - \mathcal{A})$ is surjective for $\tilde{\lambda} > 0$.*

Proof. For any $G = (G_1, G_2, G_3, G_4)^T \in \mathcal{H}$, where $G_i = (g_i^1, g_i^2, ..., g_i^n)^T$, we show that there exists $U \in D(\mathcal{A})$ satisfying

$$(\tilde{\lambda} I - \mathcal{A})U = G. \quad (22)$$

Equation (22) is equivalent to

$$\begin{cases} \tilde{\lambda} u - v = G_1(x), \\ \tilde{\lambda} v - \mu \Delta u - (\mu + \lambda)\nabla(\operatorname{div} u) + \zeta \displaystyle\int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi)\, d\xi + a_2 v = G_2(x), \\ \tilde{\lambda}\psi + (\xi^2 + \kappa)\psi - z(x,1)\omega(\xi) = G_3(x,\xi), \\ \tilde{\lambda} z(x,\rho) + \tau^{-1} z_\rho(x,\rho) = G_4(x,\rho) \end{cases} \tag{23}$$

Suppose u is found with the appropriate regularity. Then, $(23)_1$ and $(23)_3$ yield

$$v = \tilde{\lambda} u - G_1(x) \in \left(H^1(\Omega)\right)^n \tag{24}$$

and

$$\psi = \frac{G_3(x,\xi) + \omega(\xi) z(x,1)}{\xi^2 + \kappa + \tilde{\lambda}}. \tag{25}$$

We note that the last equation in (23) with $z(x,0) = v(x)$ has a unique solution given by

$$z(x,\rho) = v(x)e^{-\tilde{\lambda}\rho\tau} + \tau e^{-\tilde{\lambda}\rho\tau} \int_0^\rho G_4(x,r)e^{\tilde{\lambda} r\tau} dr. \tag{26}$$

Inserting (24) in (26), we get

$$z(x,\rho) = \tilde{\lambda} u(x)e^{-\tilde{\lambda}\rho\tau} - G_1(x)e^{-\tilde{\lambda}\rho\tau} + \tau e^{-\tilde{\lambda}\rho\tau} \int_0^\rho G_4(x,r)e^{\tilde{\lambda} r\tau} dr, \quad x \in \Omega, \rho \in (0,1). \tag{27}$$

In particular,

$$z(x,1) = \tilde{\lambda} u(x)e^{-\tilde{\lambda}\tau} + z_0(x), \quad x \in \Omega \tag{28}$$

with $z_0 \in L^2(\Omega)$ defined by

$$z_0(x) = -G_1(x)e^{-\tilde{\lambda}\tau} + \tau e^{-\tilde{\lambda}\tau} \int_0^1 G_4(x,r)e^{\tilde{\lambda} r\tau} dr, \quad x \in \Omega. \tag{29}$$

Inserting (24) in $(23)_2$, we get

$$\begin{aligned} &(\tilde{\lambda}^2 + \tilde{\lambda} a_2)u - \mu\Delta u - (\mu + \lambda)\nabla(\operatorname{div} u) + \gamma a_1 \int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi)\, d\xi \\ &= G_2(x) + (\tilde{\lambda} + a_2)G_1(x). \end{aligned} \tag{30}$$

Solving system (30) is equivalent to finding $u \in \left(H^2 \cap H_0^1(\Omega)\right)^n$ such that

$$\sum_{j=1}^{n} \int_\Omega \left((\tilde{\lambda}^2 + \tilde{\lambda} a_2) u_j \overline{w}_j - \mu \Delta u_j \overline{w}_j \right) dx - (\mu + \lambda) \int_\Omega \frac{\partial}{\partial x_j} (\operatorname{div} u) \overline{w}_j dx$$
$$+ \zeta \sum_{j=1}^{n} \int_\Omega \overline{w}_j \int_{-\infty}^{+\infty} \omega(\xi) \psi_j(x, \xi) \, d\xi dx = \sum_{j=1}^{n} \int_\Omega (g_2^j(x) + (\tilde{\lambda} + a_2) g_1^j(x)) \overline{w}_j dx \tag{31}$$

for all $w \in \left(H_0^1(\Omega)\right)^n$. Inserting (25) in (31) and using (28), we obtain that

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \int_\Omega \left((\tilde{\lambda}^2 + \tilde{\lambda} a_2) u_j \overline{w}_j + \mu \nabla u_j \nabla \overline{w}_j \, dx \right) + (\mu + \lambda) \int_\Omega (\operatorname{div} u)(\operatorname{div} \overline{w}) \, dx \\ \displaystyle + \tilde{\lambda} \theta \sum_{j=1}^{n} \int_\Omega u_j \overline{w}_j e^{-\tilde{\lambda}\tau} \, dx = \sum_{j=1}^{n} \int_\Omega \left(g_2^j(x) + (\tilde{\lambda} + a_2) g_1^j(x) \right) \overline{w}_j \, dx \\ \displaystyle - \zeta \sum_{j=1}^{n} \int_\Omega \overline{w}_j \left(\int_{-\infty}^{\infty} \frac{\omega(\xi) g_3^j(x, \xi)}{\xi^2 + \kappa + \tilde{\lambda}} \, d\xi \right) dx - \theta \sum_{j=1}^{n} \int_\Omega \overline{w}_j z_0(x) \, dx. \end{cases} \tag{32}$$

where $\theta = \zeta \displaystyle\int_{-\infty}^{+\infty} \frac{\omega^2(\xi)}{\xi^2 + \kappa + \tilde{\lambda}} \, d\xi$. Problem (32) is of the form

$$\mathcal{B}(u, w) = \mathcal{L}(w), \tag{33}$$

where $\mathcal{B} : \left(H_0^1(\Omega)\right)^n \times \left(H_0^1(\Omega)\right)^n \to \mathbb{C}$ is the sesquilinear form defined by

$$\mathcal{B}(u, w) = \sum_{j=1}^{n} \int_\Omega \left((\tilde{\lambda}^2 + \tilde{\lambda} a_2) u_j \overline{w}_j + \mu \nabla u_j \nabla \overline{w}_j \, dx \right)$$
$$+ (\mu + \lambda) \int_\Omega (\operatorname{div} u)(\operatorname{div} \overline{w}) \, dx + \tilde{\lambda} \theta \sum_{j=1}^{n} \int_\Omega u_j \overline{w}_j e^{-\tilde{\lambda}\tau} \, dx$$

and $\mathcal{L} : \left(H_0^1(\Omega)\right)^n \to \mathbb{C}$ is the antilinear functional given by

$$\mathcal{L}(w) = \sum_{j=1}^{n} \int_\Omega \left(g_2^j(x) + (\tilde{\lambda} + a_2) g_1^j(x) \right) \overline{w}_j \, dx$$
$$- \zeta \sum_{j=1}^{n} \int_\Omega \overline{w}_j \left(\int_{-\infty}^{\infty} \frac{\omega(\xi) g_3^j(x, \xi)}{\xi^2 + \kappa + \tilde{\lambda}} \, d\xi \right) dx - \theta \sum_{j=1}^{n} \int_\Omega \overline{w}_j z_0(x) \, dx.$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Consequently, by the Lax-Milgram theorem, we conclude that for all $w \in \left(H_0^1(\Omega)\right)^n$, the system (33) has a unique solution $u \in \left(H_0^1(\Omega)\right)^n$. By the regularity theory for the linear elliptic equations, it follows that $u \in \left(H^2(\Omega)\right)^n$. Therefore, the operator $(\tilde{\lambda} I - \mathcal{A})$ is surjective for any $\tilde{\lambda} > 0$. Consequently, using Hille-Yosida theorem, we have the following existence result:

Theorem 2 (Existence and uniqueness)

(1) *If* $U_0 \in D(\mathcal{A})$, *then system* (17) *has a unique strong solution*

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) *If* $U_0 \in \mathcal{H}$, *then system* (17) *has a unique weak solution*

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

$\square$

4 Strong stability

One simple way to prove the strong stability of (17) is to use the following theorem due to Arendt-Batty and Lyubich-Vũ (see [2] and [10]).

Theorem 3 ([2]-[10]) *Let* X *be a reflexive Banach space and* $(T(t))_{t\geq 0}$ *be a* C_0-*semigroup generated by* A *on* X. *Assume that* $(T(t))_{t\geq 0}$ *is bounded and that no eigenvalues of* A *lie on the imaginary axis. If* $r(A) \cap iR$ *is countable, then* $(T(t))_{t\geq 0}$ *is stable.*

Our main result is the following theorem

Theorem 4 *The* C_0*-semigroup* $e^{t\mathcal{A}}$ *is strongly stable in* $\mathcal{H}$*; i.e, for all* $U_0 \in \mathcal{H}$, *the solution of* (17) *satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 4, we need the following two lemmas.

Lemma 5 *$\mathcal{A}$ does not have eigenvalues on* $i\mathbb{R}$.

Proof.
Case 1: We will argue by contraction. Let us suppose that there $\tilde{\lambda} \in \mathbb{R}, \tilde{\lambda} \neq 0$ and $U \neq 0$, such that

$$\mathcal{A}U = i\tilde{\lambda}U. \tag{34}$$

Then, we get

$$\begin{cases} i\tilde{\lambda}u - v = 0, \\ i\tilde{\lambda}v - \mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) + \zeta\displaystyle\int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi)\, d\xi + a_2 v = 0, \\ i\tilde{\lambda}\psi + (\xi^2+\kappa)\psi - z(x,1)\omega(\xi) = 0, \\ \\ i\tilde{\lambda}z(x,\rho) + \tau^{-1}z_\rho(x,\rho) = 0. \end{cases} \tag{35}$$

Then, from (20) we have

$$v = 0, \quad z(x,1) = 0. \tag{36}$$

Hence, from $(35)_3$ and (36) we obtain

$$u \equiv 0, \quad \psi \equiv 0. \tag{37}$$

Note that $(35)_4$ gives us $z = ve^{-i\tilde{\lambda}\rho\tau} = 0$ as the unique solution of $(35)_4$. Hence $U \equiv 0$.
Now if $\tilde{\lambda} = 0$, inserting $(35)_1$ into $(35)_2$, we deduce that

$$\{-\mu\Delta u - (\mu+\lambda)\nabla(\text{div } u) = 0, u = 0 \text{ in } \Gamma. \tag{38}$$

Multiplying by $\overline{u}$, integrating over Ω we have

$$\|\nabla u\|^2_{L^2(\Omega)} + \|\text{div } u\|^2_{L^2(\Omega)} = 0. \tag{39}$$

Hence $u = 0$. Then $U \equiv 0$. □

Lemma 6 *We have*

$$i\mathbb{R} \subset \rho(\mathcal{A}).$$

Proof. We will prove that the operator $i\tilde{\lambda}I - \mathcal{A}$ is surjective for $\tilde{\lambda} \neq 0$. For this purpose, let $G = (G_1, G_2, G_3, G_4)^T \in \mathcal{H}$, we seek $X = (u, v, \psi, z)^T \in D(\mathcal{A})$ of solution of the following equation

$$(i\tilde{\lambda}I - \mathcal{A})X = G. \tag{40}$$

Equivalently, we have

$$\begin{cases} i\tilde{\lambda}u - v = G_1, \\ i\tilde{\lambda}v - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \zeta\displaystyle\int_{-\infty}^{+\infty}\omega(\xi)\psi(x,\xi)\,d\xi + a_2 v = G_2, \\ i\tilde{\lambda}\psi + (\xi^2+\kappa)\psi - z(x,1)\omega(\xi) = G_3, \\ i\tilde{\lambda}z(x,\rho) + \tau^{-1}z_\rho(x,\rho) = G_4. \end{cases} \tag{41}$$

From $(41)_1$ and $(41)_2$, we have

$$-\tilde{\lambda}^2 u - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \zeta\int_{-\infty}^{+\infty}\omega(\xi)\psi(x,\xi)\,d\xi + a_2 v(t) = (G_2 + i\tilde{\lambda}G_1) \tag{42}$$

with $u_{|\Gamma} = 0$. Solving system (42) is equivalent to finding $u \in (H^2 \cap H_0^1(\Omega))^n$ such that

$$\begin{cases} \displaystyle\sum_{j=1}^n \int_\Omega \left(-(\tilde{\lambda}^2 + i\tilde{\lambda}a_2)u_j\overline{w}_j + \mu\nabla u_j\nabla\overline{w}_j\,dx\right) + (\mu+\lambda)\int_\Omega(\operatorname{div} u)(\operatorname{div}\overline{w})\,dx \\ \displaystyle +i\tilde{\lambda}\theta\sum_{j=1}^n\int_\Omega u_j\overline{w}_j e^{-\tilde{\lambda}\tau}\,dx = \sum_{j=1}^n\int_\Omega\left(g_2^j(x) + (i\tilde{\lambda}+a_2)g_1^j(x)\right)\overline{w}_j\,dx \\ \displaystyle -\zeta\sum_{j=1}^n\int_\Omega\overline{w}_j\left(\int_{-\infty}^{\infty}\frac{\omega(\xi)g_3^j(x,\xi)}{\xi^2+\kappa+i\tilde{\lambda}}\,d\xi\right)dx - \theta\sum_{j=1}^n\int_\Omega\overline{w}_j z_0(x)\,dx \end{cases} \tag{43}$$

for all $w \in (H_0^1(\Omega))^n$. We can rewrite (43) as

$$-\left(L_{\tilde{\lambda}}u, w\right)_{((H_0^1(\Omega))^n,((H_0^1(\Omega))')^n)} + a_{(H_0^1(\Omega))^n}(u,w) = l(w) \tag{44}$$

with the sesquilinear form defined by

$$a_{(H_0^1(\Omega))^n}(u,w) = \mu\sum_{j=1}^n\int_\Omega\nabla u_j\nabla\overline{w}_j\,dx + i\tilde{\lambda}a_2\sum_{j=1}^n\int_\Omega u_j\,\overline{w}_j\,dx$$
$$+ i\tilde{\lambda}\theta\sum_{j=1}^n\int_\Omega u_j\overline{w}_j e^{-\tilde{\lambda}\tau}\,dx$$

and

$$(L_{\tilde{\lambda}} u, w)_{((H_0^1(\Omega))^n, ((H_0^1(\Omega))')^n)} = \sum_{j=1}^{n} \int_\Omega \tilde{\lambda}^2 u_j \overline{w}_j \, dx.$$

Using the compactness of the embedding from $L^2(\Omega)$ into $H^{-1}(\Omega)$ and from $H_0^1(\Omega)$ into $L^2(\Omega)$ we deduce that the operator $L_{\tilde{\lambda}}$ is compact from $(L^2(\Omega))^n$ into $(L^2(\Omega))^n$. Consequently, by the Fredholm alternative, proving the existence of a solution u of (44) reduces to proving that there is not a nontrivial solution for (44) for $l \equiv 0$. Indeed if there exists $u \neq 0$, such that

$$(L_\lambda u, w)_{((H_0^1(\Omega))^n, ((H_0^1(\Omega))')^n)} = a_{(H_0^1(\Omega))^n}(u, w) \quad \forall w \in (H_0^1(\Omega))^n, \tag{45}$$

then $i\tilde{\lambda}$ is an eigenvalue of $\mathcal{A}$. Therefore from Lemma 5 we deduce that $u = 0$.

Now, if $\tilde{\lambda} = 0$, the system (41) is reduced to the following system

$$\begin{cases} v = -G_1, \\ -\mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \zeta \displaystyle\int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi)\, d\xi + a_2 v = G_2, \\ (\xi^2+\kappa)\psi - z(x,1)\omega(\xi) = G_3, \\ \tau^{-1} z_\rho(x,\rho) = G_4. \end{cases} \tag{46}$$

Solving system (46) is equivalent to finding $u \in (H^2 \cap H_0^1(\Omega))^n$ such that

$$\begin{aligned} &\mu \sum_{j=1}^{n} \int_\Omega \nabla u_j \nabla \overline{w}_j \, dx + (\mu+\lambda) \int_\Omega (\operatorname{div} u)(\operatorname{div} \overline{w})\, dx = \sum_{j=1}^{n} \int_\Omega g_2^j \overline{w}_j \, dx \\ &+ \left(\zeta \int_{-\infty}^{\infty} \frac{\omega^2(\xi)}{\xi^2+\kappa}\, d\xi + a_2 \right) \sum_{j=1}^{n} \int_\Omega g_1^j \overline{w}_j \, dx \\ &- \tau\zeta \int_{-\infty}^{\infty} \frac{\omega^2(\xi)}{\xi^2+\kappa}\, d\xi \sum_{j=1}^{n} \int_\Omega \int_0^1 g_4^j(x,s)\, ds \overline{w}_j \, dx \\ &- \zeta \sum_{j=1}^{n} \int_\Omega \overline{w}_j \int_{-\infty}^{\infty} \frac{\omega(\xi) g_3^j(x,\xi)}{\xi^2+\kappa}\, d\xi \, dx. \end{aligned} \tag{47}$$

for all $w \in (H_0^1(\Omega))^n$. Consequently, problem (47) is equivalent to the problem

$$\mathcal{B}(u, w) = \mathcal{L}(w), \tag{48}$$

where the sesquilinear form $\mathcal{B} : (H_0^1(\Omega))^n \times (H_0^1(\Omega))^n \to \mathbb{C}$ and the antilinear form $\mathcal{L} : (H_0^1(\Omega))^n \to \mathbb{C}$ are defined by

$$\mathcal{B}(u, w) = \mu \sum_{j=1}^{n} \int_\Omega \nabla u_j \nabla \overline{w}_j \, dx + (\mu+\lambda) \int_\Omega (\operatorname{div} u)(\operatorname{div} \overline{w})\, dx \tag{49}$$

and

$$\begin{aligned}\mathcal{L}(w) = &\sum_{j=1}^{n}\int_{\Omega} g_2^j\overline{w}_j\,dx + \left(\zeta\int_{-\infty}^{\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi + a_2\right)\sum_{j=1}^{n}\int_{\Omega} g_1^j\overline{w}_j\,dx\\ &- \tau\zeta\int_{-\infty}^{\infty}\frac{\omega^2(\xi)}{\xi^2+\kappa}\,d\xi\sum_{j=1}^{n}\int_{\Omega}\int_0^1 g_4^j(x,s)\,ds\overline{w}_j\,dx \\ &- \zeta\sum_{j=1}^{n}\int_{\Omega}\overline{w}_j\int_{-\infty}^{\infty}\frac{\omega(\xi)g_3^j(x,\xi)}{\xi^2+\kappa}\,d\xi\,dx.\end{aligned} \tag{50}$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in (H_0^1(\Omega))^n$ problem (48) admits a unique solution $u \in (H_0^1(\Omega))^n$. Applying the classical elliptic regularity, it follows from (47) that $u \in (H^2(\Omega))^n$. Therefore, the operator $\mathcal{A}$ is surjective. □

5 Exponential stability

The necessary and suficient conditions for the exponential stability of the C_0-semigroup of contractions on a Hilbert space were obtained by Gearhart [7] and Huang [9] independently, see also Pruss [15]. We will use the following result due to Gearhart.

Theorem 5 ([15]- [9]) *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{51}$$

and

$$\overline{\lim_{|\beta|\to\infty}}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{52}$$

Our main result is as follows.

Theorem 6 *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ generated by $\mathcal{A}$ is exponentially stable.*

Proof. We will need to study the resolvent equation $(i\tilde{\lambda} - \mathcal{A})U = G$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\tilde{\lambda}u - v = G_1, \\ i\tilde{\lambda}v - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \zeta\displaystyle\int_{-\infty}^{+\infty} \omega(\xi)\psi(x,\xi)\,d\xi + a_2 v = G_2, \\ i\tilde{\lambda}\psi + (\xi^2+\kappa)\psi - z(x,1)\omega(\xi) = G_3, \\ i\tilde{\lambda}z(x,\rho) + \tau^{-1}z_\rho(x,\rho) = G_4, \end{cases} \tag{53}$$

where $G = (G_1, G_2, G_3, G_4)^T$. Taking inner product in $\mathcal{H}$ with U and using (20) we get

$$|\mathrm{Re}\langle\mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}. \tag{54}$$

This implies that

$$\sum_{j=1}^{n}\int_\Omega v_j^2(x)\,dx, \quad \sum_{j=1}^{n}\int_\Omega z_j^2(x,1)\,dx \leq C\|U\|_{\mathcal{H}}\|G\|_{\mathcal{H}}. \tag{55}$$

From $(53)_3$, we obtain

$$\psi = \frac{z(x,1)\omega(\xi) + G_3}{i\tilde{\lambda} + \xi^2 + \kappa}. \tag{56}$$

Then

$$\begin{aligned} \|\psi\|_{L^2(\Omega\times(-\infty,+\infty))} &\leq \left\|\frac{\omega(\xi)}{i\tilde{\lambda}+\xi^2+\kappa}\right\|_{L^2(-\infty,+\infty)} \|z(x,1)\|_{L^2(\Omega)} \\ &\quad + \left\|\frac{G_3}{i\tilde{\lambda}+\xi^2+\kappa}\right\|_{L^2(\Omega\times(-\infty,+\infty))} \\ &\leq \left(2(1-\sigma)\frac{\pi}{\sin\sigma\pi}(|\tilde{\lambda}|+\kappa)^{\sigma-2}\right)^{\frac{1}{2}} \|z(x,1)\|_{L^2(\Omega)} \\ &\quad + \frac{\sqrt{2}}{|\tilde{\lambda}|+\kappa}\|G_3\|_{L^2(\Omega\times(-\infty,+\infty))}. \end{aligned} \tag{57}$$

Similarly, we have

$$\begin{aligned} \|\xi\psi\|_{L^2(\Omega\times(-\infty,+\infty))} &\leq \left\|\frac{\xi\omega(\xi)}{i\tilde{\lambda}+\xi^2+\kappa}\right\|_{L^2(-\infty,+\infty)} \|z(x,1)\|_{L^2(\Omega)} \\ &\quad + \left\|\frac{\xi G_3}{i\tilde{\lambda}+\xi^2+\kappa}\right\|_{L^2(\Omega\times(-\infty,+\infty))} \\ &\leq \left(2\sigma\frac{\pi}{\sin\sigma\pi}(|\tilde{\lambda}|+\kappa)^{\sigma-1}\right)^{\frac{1}{2}} \|z(x,1)\|_{L^2(\Omega)} \\ &\quad + \frac{\sqrt{2}}{\sqrt{|\tilde{\lambda}|+\kappa}}\|G_3\|_{L^2(\Omega\times(-\infty,+\infty))}. \end{aligned} \tag{58}$$

Let us introduce the following notation

$$\mathcal{I}_u(x) = \sum_{j=1}^{n} \left(|v_j(x)|^2 + \mu |\nabla u_j(x)|^2 \right) + (\mu+\lambda)|\text{div } u(x)|^2$$

and

$$\mathcal{E}_u = \int_\Omega \mathcal{I}_u(x)\, dx.$$

□

Lemma 7 *We have that*

$$\mathcal{E}_u \leq c\|G\|_{\mathcal{H}}^2 + c'\|G\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{59}$$

for positive constants c *and* c'.

Proof. Multiplying the equation $(53)_2$ by $\bar{u}$, integrating on Ω we obtain

$$\begin{aligned} &-\int_\Omega v_j(\overline{i\tilde{\lambda}u_j})\, dx + \mu \int_\Omega |\nabla u_j|^2\, dx + (\mu+\lambda)\int_\Omega (\text{div } u)\frac{\partial \bar{u}_j}{\partial x_j}\, dx \\ &+ \zeta \int_\Omega \bar{u}_j \left(\int_{-\infty}^{+\infty} \omega(\xi)\psi_j(x,\xi)\, d\xi \right) dx + a_2 \int_\Omega \bar{u}_j v_j\, dx = \int_\Omega \bar{u} g_2^j\, dx. \end{aligned} \tag{60}$$

From $(53)_1$, we have $i\tilde{\lambda}u_j = v_j + g_1^j$. Then

$$\begin{aligned} &-\int_\Omega |v_j|^2\, dx + \mu \int_\Omega |\nabla u_j|^2\, dx + (\mu+\lambda)\int_\Omega (\text{div } u)\frac{\partial \bar{u}_j}{\partial x_j}\, dx \\ &+ \zeta \int_\Omega \bar{u}_j \left(\int_{-\infty}^{+\infty} \omega(\xi)\psi_j(x,\xi)\, d\xi \right) dx + a_2 \int_\Omega \bar{u}_j v_j\, dx \\ &= \int_\Omega \bar{u}_j g_2^j\, dx + \int_\Omega v_j \bar{g}_1^j\, dx. \end{aligned} \tag{61}$$

Hence

$$\begin{aligned} &-\sum_{j=1}^{n}\int_\Omega |v_j|^2\, dx + \mu \sum_{j=1}^{n}\int_\Omega |\nabla u_j|^2\, dx + (\mu+\lambda)\int_\Omega |\text{div } u|^2\, dx \\ &+ \zeta \sum_{j=1}^{n}\int_\Omega \bar{u}_j \left(\int_{-\infty}^{+\infty} \omega(\xi)\psi_j(x,\xi)\, d\xi \right) dx + a_2 \sum_{j=1}^{n}\int_\Omega \bar{u}_j v_j\, dx \\ &= \sum_{j=1}^{n}\int_\Omega \bar{u}_j g_2^j\, dx + \sum_{j=1}^{n}\int_\Omega v_j \bar{g}_1^j\, dx. \end{aligned} \tag{62}$$

We can estimate

$$\left|\int_\Omega \bar{u}_j \left(\int_{-\infty}^{+\infty} \omega(\xi)\psi_j(x,\xi)\, d\xi\right) dx\right|$$

$$\leq \|u_j\|_{L^2(\Omega)} \left(\int_{-\infty}^{+\infty} \frac{\omega^2(\xi)}{\xi^2+\kappa}\, d\xi\right)^{\frac{1}{2}} \left(\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\kappa)|\psi_j(x,\xi)|^2\, d\xi\, dx\right)^{\frac{1}{2}}$$

$$\leq \frac{\varepsilon}{2} \left(\int_{-\infty}^{+\infty} \frac{\omega^2(\xi)}{\xi^2+\kappa}\, d\xi\right) \|u_j\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\kappa)|\psi_j(x,\xi)|^2\, d\xi\, dx$$

$$\leq \frac{\varepsilon}{2} C(\Omega) \left(\int_{-\infty}^{+\infty} \frac{\omega^2(\xi)}{\xi^2+\kappa}\, d\xi\right) \|\nabla u_j\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\kappa)|\psi_j(x,\xi)|^2\, d\xi\, dx,$$

$$\left|\int_\Omega \bar{u}_j v_j\, dx\right| \leq \|u_j\|_{L^2(\Omega)} \|v_j\|_{L^2(\Omega)} \leq \frac{\varepsilon}{2} C(\Omega) \|\nabla u_j\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \|v_j\|^2_{L^2(\Omega)},$$

$$\left|\int_\Omega \bar{u}_j g_2^j\, dx\right| \leq \frac{\varepsilon}{2} C(\Omega) \|\nabla u_j\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \|g_2^j\|^2_{L^2(\Omega)},$$

$$\left|\int_\Omega v_j \bar{g}_1^j\, dx\right| \leq \frac{\varepsilon}{2} \|v_j\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \|g_1^j\|^2_{L^2(\Omega)}.$$

Choosing ε small enough, we conclude (59).

Moreover, the equation $(53)_4$ has a unique solution

$$\begin{aligned} z(x,\rho) &= e^{-i\tau\tilde{\lambda}\rho} z(x,0) + \tau e^{-i\tau\tilde{\lambda}\rho} \int_0^\rho e^{-i\tau\tilde{\lambda}r} G_4(x,r) \\ &= e^{-i\tau\tilde{\lambda}\rho} v(x) + \tau e^{-i\tau\tilde{\lambda}\rho} \int_0^\rho e^{-i\tau\tilde{\lambda}r} G_4(x,r)\, dr. \end{aligned}$$

Then

$$\|z(x,\rho)\|_{L^2(\Omega\times(0,1))} \leq \|v(x)\|_{L^2(\Omega)} + \tau \|G_4(x,\rho)\|_{L^2(\Omega\times(0,1))}. \tag{63}$$

Finally, (57), (59) and (63) imply that

$$\|U\|_{\mathcal{H}} \leq C\|G\|_{\mathcal{H}}$$

for a positive constant C. The conclusion then follows by applying Theorem 5. □

Remark 2 *We can extend the results of this paper to more general measure density instead of* (1)*, that is ω is an even nonnegative measurable function such that*

$$\int_{-\infty}^{\infty} \frac{\omega(\xi)^2}{1+\xi^2}\, d\xi < \infty. \tag{64}$$

References

[1] C. Abdallah, P. Dorato, J. Benitez-Read & R. Byrne, *Delayed positive feedback can stabilize oscillatory system*, ACC, San Francisco, (1993), 3106–3107.

[2] W. Arendt & C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. Am. Math. Soc.*, **306** (1988), 837–852.

[3] R. L. Bagley & P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, *J. Rheology.*, **27** (1983), 201–210.

[4] R. L. Bagley & P. J. Torvik, On the appearance of the fractional derivative in the behavior of real material, *J. Appl. Mech.*, **51** (1983), 294–298.

[5] A. Benaissa, S. Gaouar, Asymptotic stability for the Lamé system with fractional boundary damping, *Comput. Math. Appl.*, **77** (2019)-5, 1331–1346.

[6] H. Brézis, *Operateurs Maximaux Monotones et semi-groupes de contractions dans les espaces de Hilbert,* Notas de Matemàtica (50), Universidade Federal do Rio de Janeiro and University of Rochester, North-Holland, Amsterdam, (1973).

[7] L. Gearhart, Spectral theory for contraction semigroups on Hilbert spaces, *Trans. Amer. Math. Soc.* **236** (1978), 385–394.

[8] H. Haddar, J. R Li and D. Matignon, Efficient solution of a wave equation with fractional-order dissipative terms, *J. Comput. Appl. Math.*, **234** (2010)-6, 2003–2010.

[9] F. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Differ. Equ.*, **1** (1985), 43–55.

[10] I. Lyubich Yu & V. Q. Phóng, Asymptotic stability of linear differential equations in Banach spaces, *Studia Mathematica*, **88** (1988)-(1), 37–42.

[11] F. Mainardi & E. Bonetti, The applications of real order derivatives in linear viscoelasticity, *Rheol. Acta*, **26** (1988), 64–67.

[12] B. Mbodje, Wave energy decay under fractional derivative controls, *IMA Journal of Mathematical Control and Information.,* **23** (2006), 237–257.

[13] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.*, **45** (2006)-5, 1561–1585.

[14] I. Podlubny, Fractional differential equations, *Mathematics in Science and Engineering,* **198** (1999), Academic Press.

[15] J. Prüss, On the spectrum of C_0-semigroups, *Transactions of the American Mathematical Society.* **284** (1984)-2, 847–857.

[16] S. G. Samko, A. A. Kilbas, O. I. Marichev, emphFractional integrals and derivatives, Amsterdam: Gordon and Breach (1993) [Engl. Trans. from the Russian (1987)].

[17] I. H. Suh & Z. Bien, Use of time delay action in the controller design, *IEEE Trans. Autom. Control*, **25** (1980) 600–603.

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Vertex Turán problems for the oriented hypercube

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Abstract. In this short note we consider the oriented vertex Turán problem in the hypercube: for a fixed oriented graph $\overrightarrow{F}$, determine the maximum cardinality $ex_v(\overrightarrow{F}, \overrightarrow{Q_n})$ of a subset U of the vertices of the oriented hypercube $\overrightarrow{Q_n}$ such that the induced subgraph $\overrightarrow{Q_n}[U]$ does not contain any copy of $\overrightarrow{F}$. We obtain the exact value of $ex_v(\overrightarrow{P_k}, \overrightarrow{Q_n})$ for the directed path $\overrightarrow{P_k}$, the exact value of $ex_v(\overrightarrow{V_2}, \overrightarrow{Q_n})$ for the directed cherry $\overrightarrow{V_2}$ and the asymptotic value of $ex_v(\overrightarrow{T}, \overrightarrow{Q_n})$ for any directed tree $\overrightarrow{T}$.

1 Introduction

One of the most studied problems in extremal combinatorics is the so-called Turán problem originated in the work of Turán [15] (for a recent survey see

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[9]). A basic problem of this sort asks for the maximum possible number of edges $ex(F, G)$ in a subgraph G' of a given graph G that does not contain F as a subgraph.

Much less attention is paid to the vertex version of this problem. This problem can be formalized as follows: what is the the maximum cardinality $ex_v(F, G)$, of a subset U of vertices of a given graph G such that $G[U]$ does not contain F as a subgraph.

We will consider Turán type problems for the *n-dimensional hypercube* Q_n, the graph with vertex set $V_n = \{0, 1\}^n$ corresponding to subsets of an n-element set and edges between vertices that differ in exactly one coordinate.

Edge-Turán problems in the hypercube have attracted a lot of attention. This research was initiated by Erdős [6], who conjectured that $ex(C_4, Q_n) = (1 + o(1))n2^{n-1}$, i.e., any subgraph of Q_n having significantly more than half of the edges of Q_n must contain a copy of C_4. This problem is still unsolved. Conlon [5] showed, extending earlier results due to Chung [3] and Füredi and Özkahya [7, 8], that $ex(C_{2k}, Q_n) = o(n2^n)$ for $k \neq 2, 3, 5$.

Concerning the vertex Turán problem in the hypercube Q_n, it is obvious that we can take half of the vertices of Q_n such that they induce no edges. Kostochka [14] and later, independently, Johnson and Entringer [12] showed $ex_v(C_4, Q_n) = \max_j\{\sum_{i \not\equiv j \mod 3} \binom{n}{i}\}$. Johnson and Talbot [11] proved a local stability version of this result. Chung, Füredi, Graham, Seymour [4] proved that if U contains more than 2^{n-1} vertices, then there is a vertex of degree at least $\frac{1}{2}\log n - \frac{1}{2}\log\log n + \frac{1}{2}$ in $Q_n[U]$. This shows that for any star S_k with k fixed, we have $ex_v(S_k, Q_n) = 2^{n-1}$ for large enough n. Alon, Krech, and Szabó [1] investigated the function $ex_v(Q_d, Q_n)$.

Let us note that there is a simple connection between the edge and the vertex Turán problems in the hypercube.

Proposition 1 $ex_v(F, Q_n) \leq 2^{n-1} + \frac{ex(F,Q_n)}{n}$.

Proof. If a subgraph G of Q_n contains more than $2^{n-1} + \frac{ex(F,Q_n)}{n}$ vertices, then it contains more than $\frac{ex(F,Q_n)}{n}$ edges in every direction, thus more than $ex(F, Q_n)$ edges altogether, hence G contains a copy of F. $\square$

For every tree T, this observation implies that $ex_v(T, Q_n) = \left(\frac{1}{2} + \mathcal{O}\left(\frac{1}{n}\right)\right) 2^n$, using the well-known result from Turán theory which states that $ex(n, T) = O(n)$ (and so $ex(F, Q_n) = \mathcal{O}(2^n)$). Also, together with Conlon's result on the cycles mentioned earlier, we obtain $ex_v(C_k, Q_n) = \left(\frac{1}{2} + o(1)\right) 2^n$ for $k \neq 2, 3, 5$.

In this paper, we consider an oriented version of this problem. There is a natural orientation of the edges of the hypercube. An edge uv means that u and v differ in only one coordinate; if u contains 1 and v contains 0 in this coordinate, then we direct the edge from v to u. We denote the hypercube Q_n with this orientation by $\overrightarrow{Q_n}$. With this orientation it is natural to forbid oriented subgraphs. We will denote by $ex_v(\overrightarrow{F}, \overrightarrow{Q_n})$ the maximum number of vertices that an $\overrightarrow{F}$-free subgraph of $\overrightarrow{Q_n}$ can have. As vertices of the hypercube correspond to sets, instead of working with subsets of the vertices of $\overrightarrow{Q}_n$ we will consider families $\mathcal{G} \subseteq 2^{[n]}$ of sets. We will say that $\mathcal{G} \subseteq 2^{[n]}$ is $\overrightarrow{F}$-free if for the corresponding subset U of vertices of $\overrightarrow{Q}_n$ the induced subgraph $\overrightarrow{Q}_n[U]$ is $\overrightarrow{F}$-free.

For example, there is only one orientation of C_4 that embeds into the hypercube, we will denote it by $\overrightarrow{C_4}$. Hence we have $ex_v(\overrightarrow{C_4}, \overrightarrow{Q_n}) = ex_v(C_4, Q_n)$, which is known exactly, due to the above mentioned result of Kostochka and Johnson and Entringer. However, there are three different orientations of P_3, according to how many edges go towards the middle vertex: $\overrightarrow{V_2}$ denotes the orientation with a source (i.e., $\overrightarrow{V_2}$ is the path abc such that the edge ab is directed from b to a and the edge bc is directed from b to c). The directed path $\overrightarrow{P_k}$ is a path on k vertices $v_1, \dots, v_k$ with edges going from v_i to v_{i+1} for every $i < k$. The *height* of a directed graph is the length of a longest directed path in it.

If we consider the hypercube as the Boolean poset, then each edge of the hypercube goes between a set A and a set $A \cup \{x\}$ for some $x \notin A$. Then in $\overrightarrow{Q_n}$ the corresponding directed edge goes from A to $A \cup \{x\}$. A directed acyclic graph $\overrightarrow{F}$ can be considered as a poset F; we will say that F is the *poset of* $\overrightarrow{F}$. The poset corresponding to a directed tree is said to be a *tree poset.* Forbidding copies of a poset in a family of sets in this order-preserving sense has an extensive literature (see [10] for a survey on the theory of forbidden subposets). We say $\mathcal{P} \subset 2^{[n]}$ is a *copy* of P if there exists a bijection $f : P \to \mathcal{P}$ such that $p < p'$ implies $f(p) \subset f(p')$. We say that $\mathcal{F} \subset 2^{[n]}$ is *P-free*, if there is no $\mathcal{P} \subset \mathcal{F}$ that is a copy of P. Observe that if P is the poset of the directed acyclic graph $\overrightarrow{F}$, then any P-free family is $\overrightarrow{F}$-free.

The oriented version of the vertex Turán problem in the hypercube corresponds to the following variant of the forbidden subposet problem. We say $\mathcal{P} \subset 2^{[n]}$ is a *cover-preserving copy* of P if there exists a bijection $f : P \to \mathcal{P}$ such that if p covers p' in P, then $f(p)$ covers $f(p')$ in the Boolean poset. Thus it is not surprising that we can use techniques and results from the theory of forbidden subposet problems in our setting.

In this paper, we consider Vertex Turán problems for directed trees. Our main result determines the asymptotic value of the vertex Turán number $ex_v(\overrightarrow{T}, \overrightarrow{Q_n})$ for any directed tree $\overrightarrow{T}$.

Theorem 1 *For any directed tree* $\overrightarrow{T}$ *of height* h*, we have*

$$ex_v(\overrightarrow{T}, \overrightarrow{Q_n}) = \left(\frac{h-1}{h} + o(1)\right) 2^n.$$

Below we obtain the exact value of the vertex Turán number for some special directed trees (namely $\overrightarrow{V_2}$ and $\overrightarrow{P_k}$).

Theorem 2

$$ex_v(\overrightarrow{V_2}, \overrightarrow{Q_n}) = 2^{n-1} + 1.$$

It would be natural to consider the following generalization of $\overrightarrow{V_2}$: let $\overrightarrow{V_r}$ denote the star with r leaves all edges oriented towards the leaves. Note that if one takes the elements of the r highest levels of the Boolean poset and every other level below them, then the corresponding family in $\overrightarrow{Q_n}$ will be $\overrightarrow{V_r}$-free. Computing the cardinality of this family we have $ex_v(\overrightarrow{V_r}, \overrightarrow{Q_n}) = 2^{n-1} + \Omega(n^{r-2})$. We conjecture that $ex_v(\overrightarrow{V_r}, \overrightarrow{Q_n}) = 2^{n-1} + \Theta(n^{r-2})$ holds for every $r \geq 3$.

Theorem 3 *For any pair* k, n *of integers with* $k \leq n$ *we have*

$$ex_v(\overrightarrow{P_k}, \overrightarrow{Q_n}) = \max_{j \in [k]} \left\{ \sum_{i \not\equiv j \ mod\ k} \binom{n}{i} \right\}.$$

After submitting this paper we learned that the above theorem was proved in a different context by Katona [13].

2 Proofs

2.1 Proof of Theorem 1

We follow the lines of a proof of Bukh [2] that shows that if T is a tree poset with $h(T) = k$ and $\mathcal{F} \subseteq 2^{[n]}$ is a T-free family of sets, then $|\mathcal{F}| \leq (k - 1 + O(\frac{1}{n}))\binom{n}{\lfloor n/2 \rfloor}$ holds. The proof of this theorem consists of several lemmas. Some of them we will state and use in their original form, some others we will state and prove in a slightly altered way so that we can apply them in our setting. First we need several definitions. For a family $\mathcal{F} \subseteq 2^{[n]}$, its *Lubell-function*

$$\lambda_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} = \frac{1}{n!} \sum_{F \in \mathcal{F}} |F|!(n-|F|)!$$

is the average number of sets in $\mathcal{F}$ that a maximal chain $\mathcal{C}$ in $2^{[n]}$ contains. A poset P is called *saturated* if all its maximal chains have length $h(P)$. For any poset T its *opposite poset* T' consists of the same elements as T with $t \leq_{T'} t'$ if and only if $t' \leq_T t$. For a family $\mathcal{F} \subseteq 2^{[n]}$ of sets, its complement family is $\overline{\mathcal{F}} = \{[n] \setminus F : F \in \mathcal{F}\}$. Clearly, $\mathcal{F}$ contains a copy of P if and only if $\overline{\mathcal{F}}$ contains a copy of P' and $\lambda_n(\mathcal{F}) = \lambda_n(\overline{\mathcal{F}})$.

Lemma 1 (Bukh [2]) *Every tree poset* T *is an induced subposet of a saturated tree poset* T' *with* $h(T) = h(T')$*.*

An *interval* in a poset P is a set of the form $[x, y] = \{z \in P : x \leq z \leq y\}$.

Lemma 2 (Bukh [2]) *If* T *is a saturated tree poset that is not a chain, then there exists* $t \in T$ *that is a leaf in* $H(T)$ *and there exists an interval* $I \subset T$ *containing* t *such that* $|I| < h(T)$ *holds, and* $T \setminus I$ *is a saturated tree poset with* $h(T) = h(T \setminus I)$*.*

From now on we fix a tree poset T and we denote its height by k. We say that a chain in $2^{[n]}$ is *fat* if it contains k members of $\mathcal{F}$.

Lemma 3 *If* $\mathcal{F} \subseteq \bigcup_{j=i}^{i+k-1} \binom{[n]}{j}$ *is a family with* $\lambda(\mathcal{F}) \geq (k-1+\varepsilon)$*, then there are at least* $(\varepsilon/k)n!$ *fat chains.*

Proof. Let C_i denote the number of maximal chains that contain exactly i sets from $\mathcal{F}$. As $\mathcal{F} \subseteq \bigcup_{j=i}^{i+k-1} \binom{[n]}{j}$, we have $C_i = 0$ for all $i > k$. Then counting the number of pairs $(F, \mathcal{C})$ with $\mathcal{C}$ being a maximal chain and $F \in \mathcal{F} \cap \mathcal{C}$, in two different ways, we obtain

$$\sum_{i=0}^{n} iC_i = \lambda(\mathcal{F})n! \geq (k-1+\varepsilon)n!.$$

This, and $\sum_i C_i = n!$ imply

$$kC_k = \sum_{i \geq k} iC_i \geq \sum_{i=0}^{n} iC_i - (k-1)\sum_{i<k} C_i \geq \varepsilon n!.$$

Therefore the number of fat chains in $\mathcal{F}$ is $C_k \geq (\varepsilon/k)n!$. □

Lemma 4 *Let T be a saturated tree poset of height k. Suppose $\mathcal{F} \subseteq \cup_{j=i}^{i+k-1} \binom{[n]}{j}$ is a family with $n/4 \leq i \leq 3n/4$. Moreover, suppose $\mathcal{L}$ is a family of fat chains with $|\mathcal{L}| > \frac{4\binom{|T|+1}{2}}{n} n!$. Then there is a copy of T in $\mathcal{F}$ that contains only sets that are contained in some fat chain in $\mathcal{L}$.*

Proof. We proceed by induction on $|T|$. If T is a chain, then the k sets in any element of $\mathcal{L}$ form a copy of T. In particular, it gives the base case of the induction. So suppose T is not a chain. Then applying Lemma 2, there exists a leaf t in T and interval $I \subseteq T$ containing t such that $h(T \setminus I) = k$ and $T \setminus I$ is a saturated tree poset. Our aim is to use induction to obtain a copy of $T \setminus I$ in $\mathcal{F}$ that can be extended to a copy of T. Finding a copy of $T \setminus I$ is immediate, but in order to be able to extend it, we need a copy satisfying some additional properties, described later.

By passing to the opposite poset T' of T and considering $\overline{\mathcal{F}}$, we may assume that t is a minimal element of T. There exists a maximal chain C in T that contains I, and we have $|C| = k$ as T is saturated. Then $s := |C \setminus I| = k - |I| \geq 1$.

We need several definitions. Let $F_1 \supset F_2 \supset \cdots \supset F_s$ be a chain with $|F_j| = i + k - j$ for $j = 1, \ldots, s$. Then this chain is a *bottleneck* if there exists a family $\mathcal{S} \subset \mathcal{F}$ with $|\mathcal{S}| < |T|$ such that for every fat chain $F_1 \supset F_2 \supset \cdots \supset F_s \supset F_{s+1} \supset \cdots \supset F_k$ in $\mathcal{L}$ we have $\mathcal{S} \cap \{F_{s+1}, \ldots, F_k\} \neq \emptyset$. Such an $\mathcal{S}$ is a *witness* to the fact that $F_1, \ldots, F_s$ is a bottleneck (and we assume all sets of the witness are contained in F_s). We say that a fat chain is *bad* if its top s sets form a bottleneck. A fat chain is *good* if it is not bad. Observe that if there is a copy $\mathcal{F}_{T\setminus I}$ of $T \setminus I$ consisting of sets of good fat chains, then we can extend $\mathcal{F}_{T\setminus I}$ to a copy of T. Indeed, as the sets $F'_1, \ldots, F'_s$ representing $C \setminus I$ in $\mathcal{F}_{T\setminus I}$ do not form a bottleneck and $|\mathcal{F}_{T\setminus I}| < |T|$, there must be a good fat chain $F'_1 \supset \cdots \supset F'_s \supset F'_{s+1} \supset \cdots \supset F'_k$ such that $F'_{s+1}, \ldots, F'_k \notin \mathcal{F}_{T\setminus I}$, therefore $\mathcal{F}_{T\setminus I} \cup \{F'_{s+1}, \ldots, F'_k\}$ is a copy of T. Therefore all we need to prove is that there are enough good fat chains to obtain a copy of $T \setminus I$ by induction.

Let us bound the number of bad fat chains. If $|\mathcal{C} \cap \mathcal{F}| < s$, then $\mathcal{C}$ cannot be bad. We partition maximal chains in $2^{[n]}$ according to their sth largest set F_s from $\mathcal{F}$. As the top s sets must form a bottleneck, there is a witness $\mathcal{S}$ to this fact. This means that if $\mathcal{C}$ is bad, then $\mathcal{C}$ must meet $\mathcal{S}$ whose elements are all contained in F_s. But as $|\mathcal{S}| < |T|$ and all sets of $2^{F_s} \cap \mathcal{F}$ have size between $n/4$ and $3n/4$, the proportion of those chains that do meet $\mathcal{S}$ is at most $4|T|/n$ (any proper non-empty subset of F_S is contained in at most $1/|F_s|$ proportion of chains going through F_s). This holds independently of the choice of F_s, thus the number of bad fat chains is at most $\frac{4|T|}{n}n!$. So the number of good fat chains is at least

$$|\mathcal{L}| - \frac{4|T|}{n}n! \geq \frac{4(\binom{|T|+1}{2} - |T|)}{n}n! = \frac{4\binom{|T|}{2}}{n}n!.$$

As $|T \setminus I| < |T|$, the induction hypothesis implies the existence of a copy of $T \setminus I$ among the sets contained in good fat chains, as required. □

The next lemma essentially states that if a a T-free family is contained in the union of k consecutive levels, then its cardinality is asymptotically at most the cardinality of the $k-1$ largest levels. Formally, let $b(i) = b_{k,n}(i) = \max\{\binom{n}{j} : i \leq j \leq i+k-1\}$. So if $i \leq n/2 - k + 1$, then $b(i) = \binom{n}{i+k-1}$, if $i \geq n/2$, then $b(i) = \binom{n}{i}$, while if $n/2 - k + 1 < i < n/2$, then $b(i) = \binom{n}{\lfloor n/2 \rfloor}$.

Lemma 5 *If T is a tree poset of height k, then there exists n_0 such that for $n > n_0$, $n/4 \leq i \leq 3n/4 - k$ any $\mathcal{F} \subset \bigcup_{j=i}^{i+k-1} \binom{[n]}{j}$ of cardinality at least $\left(k - 1 + \frac{k4|T|^2}{n}\right) b(i)$ contains a copy of T.*

Proof. By Lemma 1 we may suppose that T is a saturated tree poset. Assume $\mathcal{F} \subseteq \bigcup_{j=i}^{i+k-1} \binom{[n]}{j}$ is a T-free family that contains at least $\left(k - 1 + \frac{k4|T|^2}{n}\right) b(i)$ sets. Then $\mathcal{F} \subseteq \bigcup_{j=i}^{i+k-1} \binom{[n]}{j}$ implies that $\lambda_n(\mathcal{F}) \geq k - 1 + \frac{k4|T|^2}{n}$.

Let $\varepsilon = 4k|T|^2/n$. Then we can apply Lemma 3 to find $4|T|^2 n!/n$ fat chains. Then we can apply Lemma 4 with $k = h(T)$ to obtain a copy of T in $\mathcal{F}$, contradicting the T-free property of $\mathcal{F}$. □

With Lemma 5 in hand, we can now prove Theorem 1. Let us consider a $\overrightarrow{T}$-free family $\mathcal{F}$. Let T be the poset of $\overrightarrow{T}$ and let T^* be the saturated poset containing T with $h(T) = h(T^*) = k$ - guaranteed by Lemma 1. For any integer $0 \leq i \leq n - k + 1$, let $\mathcal{F}_i = \{F \in \mathcal{F} : i \leq |F| \leq i + k - 1\}$. Observe that the $\overrightarrow{T}$-free property of $\mathcal{F}$ implies that $\mathcal{F}_i$ is T^*-free for every i. Note that every $F \in \mathcal{F}$ belongs to exactly k families $\mathcal{F}_i$ unless $|F| < k - 1$ or $|F| > n - k + 1$. It is well-known that $\left|\binom{[n]}{\leq n/4} \cup \binom{[n]}{\geq 3n/4}\right| = o\left(\frac{1}{n}2^n\right)$, therefore using Lemma 5 we obtain

$$\begin{aligned} k|\mathcal{F}| - o\left(\frac{1}{n}2^n\right) &\leq \sum_{i=n/4}^{3n/4} |\mathcal{F}_i| \leq \left(k - 1 + \frac{k4|T|^2}{n}\right) \sum_{i=n/4}^{3n/4} b(i) \\ &\leq \left(k - 1 + \frac{k4|T|^2}{n}\right)\left(2^n + k\binom{n}{\lfloor n/2 \rfloor}\right). \end{aligned}$$

After rearranging, we get $|\mathcal{F}| \leq \left(\frac{k-1}{k} + o(1)\right) 2^n$.

2.2 Proof of Theorem 2

To prove the lower bound, we show a $\overrightarrow{V_2}$-free family in $\overrightarrow{Q_n}$ of cardinality $2^{n-1}+1$. Simply take every second level in the hypercube starting from the $(n-1)$st level and also take the vertex corresponding to $[n]$.

We prove the upper bound by induction on n (it is easy to see the base case $n = 2$). We will need the following simple claim.

Lemma 6 *Let $\mathcal{F} \subset 2^{[n]}$ is a maximal $\overrightarrow{V_2}$-free family, then $\mathcal{F}$ contains the set $[n]$ and at least one set of size $n-1$.*

Proof.[Proof of lemma] First note that $[n]$ can be added to any $\overrightarrow{V_2}$-free family as there is only one subset of $[n]$ of size n. Also, if $\mathcal{F}$ does not contain any set of size $n-1$, then one such set S can be added to $\mathcal{F}$. Indeed, if we add S, no copy of $\overrightarrow{V_2}$ having sets of size $n-1$ and n will be created because $[n]$ is the only set of size n in $\mathcal{F} \cup \{S\}$. Furthermore, no copy of $\overrightarrow{V_2}$ having sets of size $n-2$ and $n-1$ will be created as S is the only set of size $n-1$ in $\mathcal{F} \cup \{S\}$. $\square$

Now we are ready to prove Theorem 2. Let $\mathcal{F} \subset 2^{[n]}$ be a $\overrightarrow{V_2}$-free family. For some $x \in [n]$, define

$$\mathcal{F}_x^- = \{F \mid F \in \mathcal{F},\ x \notin F\} \quad \text{and} \quad \mathcal{F}_x^+ = \{F \backslash \{x\} \mid F \in \mathcal{F},\ x \in F\}.$$

Then $\mathcal{F}_x^-, \mathcal{F}_x^+ \subset 2^{[n]\backslash\{x\}}$ and they are also $\overrightarrow{V_2}$-free. By induction, we have

$$|\mathcal{F}| = |\mathcal{F}_x^-| + |\mathcal{F}_x^+| \leq 2^{n-2} + 1 + 2^{n-2} + 1 = 2^{n-1} + 2.$$

Assume that $|\mathcal{F}| = 2^{n-1} + 2$. Then $|\mathcal{F}_x^-| = |\mathcal{F}_x^+| = 2^{n-2} + 1$ must hold for all $x \in [n]$. By Lemma 6, $|\mathcal{F}_x^-| = 2^{n-2} + 1$ implies that $[n]\backslash\{x\}$ and at least one set of size $n-2$ are in $\mathcal{F}$. This holds for all $x \in [n]$, so all sets of size $n-1$, and at least one set of size $n-2$ are in $\mathcal{F}$. However, these would form a forbidden $\overrightarrow{V_2}$ in $\mathcal{F}$, contradicting our original assumption on $\mathcal{F}$. This proves that $|\mathcal{F}| \leq 2^{n-1} + 1$.

2.3 Proof of Theorem 3

Let U be a set of vertices in Q_n such that the subgraph of Q_n induced by U (i.e., $Q_n[U]$) is $\overrightarrow{P_k}$-free. Let $\mathcal{F} \subset 2^{[n]}$ be a family of subsets corresponding to U.

First, we will introduce a weight function. For every $F \in \mathcal{F}$, let $w(F) = \binom{n}{|F|}$. For a maximal chain $\mathcal{C}$, let $w(\mathcal{C}) = \sum_{F \in \mathcal{C} \cap \mathcal{F}} w(F)$ denote the weight of $\mathcal{C}$. Let $\mathbf{C}_n$ denote the set of all maximal chains in $[n]$. Then

$$\frac{1}{n!} \sum_{\mathcal{C} \in \mathbf{C}_n} w(\mathcal{C}) = \frac{1}{n!} \sum_{\mathcal{C} \in \mathbf{C}_n} \sum_{F \in \mathcal{C} \cap \mathcal{F}} w(F) = \frac{1}{n!} \sum_{F \in \mathcal{F}} |F|!(n-|F|)! w(F) = |\mathcal{F}|.$$

This means that the average of the weights of the full chains equals the cardinality of $\mathcal{F}$. Thus, if we can find an upper bound that is valid for the weight of any chain, then this will be an upper bound on $|\mathcal{F}|$ too.

Our assumption that there is no $\overrightarrow{P_k}$ means that there are no k neighboring members of $\mathcal{F}$ in a chain. For a given chain $\mathcal{C}$, let $a_1, a_2, \dots a_t$ denote the sizes of those elements of $\mathcal{C}$ that are not in $\mathcal{F}$. Then $0 \leq a_1 < a_2 < \dots < a_t \leq n$, $a_1 \leq k-1$, $n-k+1 \leq a_t$ and $a_{i+1} - a_i \leq k$ for all $i = 1, 2, \dots t-1$. The weight of the chain $\mathcal{C}$ is

$$w(\mathcal{C}) = 2^n - \sum_{i=1}^{t} \binom{n}{a_i}.$$

We claim that this is maximized when the numbers $\{a_1, a_2, \dots a_t\}$ are all the numbers between 0 and n which give the same residue when divided by k.

Assume that $w(\mathcal{C})$ is maximized by a different kind of set $\{a_1, a_2, \dots a_t\}$. Then there is an index i such that $a_{i+1} - a_i < k$.

If $a_i \leq \frac{n}{2}$ then we can decrease the numbers $\{a_1, a_2, \dots a_i\}$ by 1. (If a_1 becomes -1 then we simply remove that number.) The resulting set of numbers will still satisfy the conditions and $w(\mathcal{C})$ increases. Otherwise, $a_{i+1} > \frac{n}{2}$ must hold. Similarly, we can increase the numbers $\{a_{i+1}, a_{i+2}, \dots a_n\}$ by 1 to achieve the same result. We proved that

$$w(\mathcal{C}) \leq 2^n - \min_{j \in [k]} \sum_{i \equiv j \bmod k} \binom{n}{i} = \max_{j \in [k]} \left\{ \sum_{i \not\equiv j \bmod k} \binom{n}{i} \right\}$$

holds for any full chain $\mathcal{C}$. Therefore the same upper bound holds for $|\mathcal{F}|$ as well.

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References

[1] N. Alon, A. Krech, and T. Szabó, Turán's theorem in the hypercube, *SIAM Journal on Discrete Mathematics*, **21**(1):66–72, 2007.

[2] B. Bukh, Set families with a forbidden subposet, *The Electronic Journal of Combinatorics*, **16**(1):142, 2009.

[3] F. R. Chung, Subgraphs of a hypercube containing no small even cycles, *Journal of Graph Theory*, **16**(3):273–286, 1992.

[4] F. R. Chung, Z. Füredi, R. L. Graham, and P. Seymour, On induced subgraphs of the cube, *Journal of Combinatorial Theory, Series A*, **49**(1):180–187, 1988.

[5] D. Conlon, An extremal theorem in the hypercube, *The Electronic Journal of Combinatorics*, **17**(1):R111, 2010.

[6] P. Erdős, On some problems in graph theory, combinatorial analysis and combinatorial number theory, *Graph Theory and Combinatorics (Cambridge, 1983), Academic Press, London*, pages 1–17, 1984.

[7] Z. Füredi and L. Özkahya, On 14-cycle-free subgraphs of the hypercube, *Combinatorics, Probability & Computing*, 18(5):725, 2009.

[8] Z. Füredi and L. Özkahya, On even-cycle-free subgraphs of the hypercube, *Journal of Combinatorial Theory Series A*, 118(6):1816–1819, 2011.

[9] Z. Füredi and M. Simonovits, The history of degenerate (bipartite) extremal graph problems. In *Erdős Centennial*, pages 169–264. Springer, 2013.

[10] J. R. Griggs and W.-T. Li. Progress on poset-free families of subsets. *in: Recent Trends in Combinatorics*, pages 317–338, 2016.

[11] J. R. Johnson and J. Talbot. Vertex Turán problems in the hypercube. *Journal of Combinatorial Theory, Series A*, 117(4):454–465, 2010.

[12] K. A. Johnson and R. Entringer, Largest induced subgraphs of the n-cube that contain no 4-cycles, *Journal of Combinatorial Theory, Series B*, 46(3):346–355, 1989.

[13] G. O. H. Katona, Families of subsets having no subset containing another one with small difference, *Nieuw Arch. Wiskunde*, 20(3):54–67, 1972.

[14] E. Kostochka, Piercing the edges of the n-dimensional unit cube, *Diskret. Analiz Vyp*, 28(223):55–64, 1976.

[15] P. Turán, On an external problem in graph theory, *Mat. Fiz. Lapok*, 48:436–452, 1941.

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Analysis and optimisation of a $M/M/1/WV$ queue with Bernoulli schedule vacation interruption and customer's impatience

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Abstract. In this investigation, we establish a steady-state solution of an infinite-space single-server Markovian queueing system with working vacation (WV), Bernoulli schedule vacation interruption, and impatient customers. Once the system becomes empty, the server leaves the system and takes a vacation with probability p or a working vacation with probability $1-p$, where $0 \leq p \leq 1$. The working vacation period is interrupted if the system is non empty at a service completion epoch and the server resumes its regular service period with probability $1-q$ or carries on with the working vacation with probability q. During vacation and working vacation periods, the customers may be impatient and leave the system. We use a probability generating function technique to obtain the expected number of customers and other system characteristics. Stochastic decomposition of the queueing model is given. Then, a cost function is constructed by considering different cost elements of the system states, in order to determine the optimal values of the service rate during regular

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busy period, simultaneously, to minimize the total expected cost per unit time by using a quadratic fit search method (QFSM). Further, by taking illustration, numerical experiment is performed to validate the analytical results and to examine the impact of different parameters on the system characteristics.

1 Introduction

Queueing modeling is being employed in a large variety of day-to-day congestion problems as well as in industrial scenarios, such as computer systems, call centers, web services, communication/telecommunication systems, etc. For nearly a century, many queueing models have been developed to analyze the characteristics of many systems and recommendations have been issued to suggest how to deal with congestion situations. In many queueing scenarios, when there is no job present in the system, the server may take a vacation (V) or may provide a service for a secondary job, known as working vacation (WV). Queueing systems with vacation and working vacation have been the subject of interest for the queueing theorists. A detailed surveys of the literature devoted to vacation queues are found in [9], [26], [27], and the references therein. Working vacation queue was first introduced by [24] in an $M/M/1$ queueing system. [17] analyzed a single server queue with batch arrivals and general service time distribution. [28] provided the analysis for an $M/G/1$ queueing model with multiple vacations and exhaustive service discipline at which the server works with different rate rather than completely stopping the service during vacation. [15] provided performance analysis of $GI/M/1$ queue with working vacations. Then, [23] analyzed the $M/M/1$ queue with single and multiple working vacation and impatient customers. They computed closed form solution and various performance measures with stochastic decomposition for both the working vacation policies. After that, a Markovian queueing system with two-stage working vacations has been considered by [25]. Recently, [18] examined an infinite-buffer multiserver queue with single and multiple synchronous working vacations.

In this investigation, we considered vacation interruption policy at which during working vacation period, the server may come back to the regular working period without completing the ongoing working vacation. The concept of vacation interruptions was introduced by [13]. After that, [14], [16], and [31] generated the vacation interruption model for $GI/Geo/1$, $GI/M/1$, and $M/G/1$ queueing models, respectively. Working vacation queueing system with service interruption and multi-optional repair was considered by [11]. Then,

[10] examined system performance measures for an $M/G/1$ queueing model with single working vacations and a Bernoulli interruption schedule. [29] studied the strategic behaviour in a discrete-time working vacation queue with a Bernoulli interruption schedule. [22] investigated a single server queueing model with multiple working vacations and vacation interruption where an arriving customer can balk the system at some particular times. Recently, a study of an infinite-space single server Markovian queue with working vacation and vacation interruption was established by [20].

Over recent decade, customer's impatience becomes the burning issue of private and government sector businesses. Thus, an increasing attention has been seen in queueing systems with impatient customers due to the absence (vacation) of the server. [1] gave the analysis of customers' impatience in different queues with server vacation. Then, vacation queueing models with impatient customers and a waiting server have been examined by [21]. [30] analyzed an $M/M/1$ queue with vacations and impatience timers which depends on the server's states. [8] examined a queueing model with feedback, reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption. Further, performance and economic analyzes of different queueing models with vacation/working vacation and customer's impatience have been treated by [5, 6], [2, 3], [4], [19, 7], and the references therein.

In this work, the main objective is to analyze the queueing performance of an infinite-space single-server working vacation queueing system with Bernoulli schedule vacation interruption at which whenever the system becomes empty, the server switches to the vacation period with a certain probability p and to the working vacation with a complementary probability $1-p$. During the vacation period, the customers are served at a lower service rate. During this period, at each service completion instant, if there are customers in the queue, the server either remains in the working vacation status with probability q, or switches to the regular service status with probability $1-q$. During vacation and working vacation periods, the customers may get impatient with different rates and leave the system . In this study, the probability generating function (PGF) is used to determine the stationary system and queue lengths. The stochastic decomposition of the queueing model is also provided. Further, the cost optimization analysis of the system is carried out using quadratic fit search method (QFSM) in order to minimize the total expected cost of the system with respect to the service rate during normal busy period.

The rest of the paper is organized as follows. Section 2 describes the queueing system by stating the requisite hypotheses and notations which are needed to develop the model. Section 3 is devoted to a practical application of the proposed queueing model. In Section 4, the steady-state equations governing the queueing model are constructed and the steady-state solution of the considered queueing system is obtained, using the probability generating function technique. In the Section 5, we focus on useful system characteristics in terms of state probabilities. Section 6 is devoted to the stochastic decomposition of the queueing system. In Section 7, we construct a cost function. Numerical analysis has been carried out in Section 8. Finally, we ended the paper with a conclusion in Section 9.

2 Model description

Consider an infinite-buffer single server Markovian queueing system where the arriving customers follow Poisson process with rate λ. During the regular service period, the customers are served with an exponential rate μ_b. The server begins a vacation with probability p or a working vacation with probability $1-p$, where $0 \leq p \leq 1$, at the instant when he finds the system empty. During the working vacation period, the server renders service to the customers with a lower rate $\mu_v(\mu_v < \mu_b)$. A new busy period starts if the system is non empty after the end of vacation period or working vacation period. Further, it is assumed that the working vacation period is interrupted if the system is non empty at a service completion instant and the server resumes the regular service period with probability $1-q$ or carries on with the working vacation with probability q. Vacation and working vacation periods are assumed to be exponentially distributed with rates θ and ϕ respectively.

Whenever a customer arrives to the system and realizes that the server is on vacation (resp. working vacation) he activates an exponentially distributed impatience timer T_1 (resp. T_2) with parameter ξ (resp. α), where $\alpha < \xi$. If the server comes back from his vacation or working vacation before the timer T_1 or T_2 expires, the customer remains in the system till the completion of his service. The customer leaves the system and never returns if T_1 or T_2 expires while the server is still on vacation or working vacation.

At time t, let $L(t)$ denote the total number of customers in the system and $J(t)$ denotes the state of the server with

$$J(t) = \begin{cases} 0, & \text{when the server is in working vacation period,} \\ 1, & \text{when the server is in vacation period,} \\ 2, & \text{when the server is in regular service period.} \end{cases}$$

Then, the pair $\{L(t), J(t), t \geq 0\}$ defines a two dimensional continuous time discrete state Markov chain with state space $E = \{((0,0) \cup (0,1)) \cup (i,j), i = 1,2,...,j = 0,1,2\}$. Let $P_{ij} = \lim_{t\to\infty} P\{L(t) = i, J(t) = j\}$ denote the stationary probabilities of the Markov process $\{L(t), J(t), t \geq 0\}$.

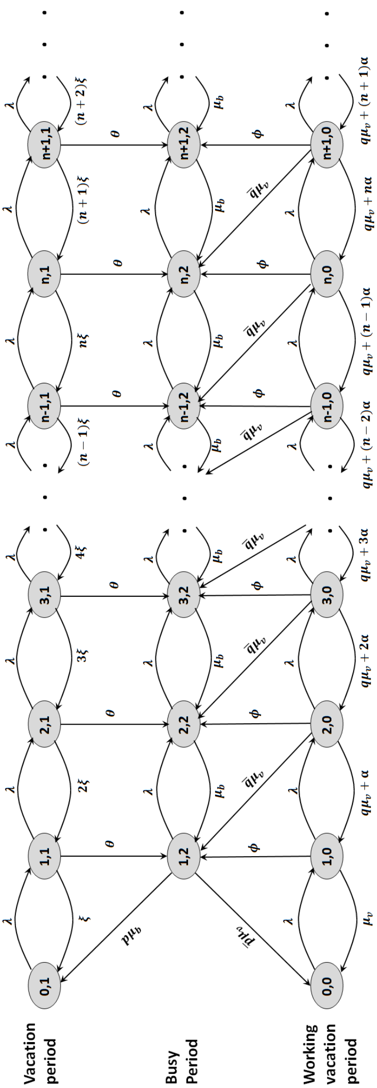

Figure 1: State-transition-rate diagram.

3 Practical application of the queueing model

Reducing energy costs is a major problem in modern information and communication technology (ICT) systems, as the inactive devices in modern ICT systems consume a significant amount of energy. We consider a ICT system with a single device, wherein jobs arrive according to a Poisson process with rate λ. The job processing time is exponentially distributed with rate μ_b. When the system work has been done, to reduce energy costs, the device switches either to off state with probability p or to a lower energy state with a complimentary probability $1-p$ wherein it keeps part of its capacity and processes the incoming jobs with a lower rate μ_v ($\mu_b > \mu_v$), which is also exponentially distributed. The lower energy state can be considered as the working vacation status of the device. In order to avoid the increasing workload and the prolonged job sojourn time, once a job arrives at an empty device, the device processes the job with the rate μ_v, and begins to move to the regular service period. The switching process takes time and the processing of the current job can not be interrupted. Then, at each time of service completion during the working vacation period, the device can remain in the working vacation period with probability q or switch to the regular service period with probability$1-q$.

If the device successfully switches to the regular service period and finds jobs online, it will process them with rate μ_b (the working vacation period is interrupted).

Moreover, we suppose that whenever a customer arrives to the system and finds that the device is on vacation (resp. working vacation) he activates an impatience timer T_1, (resp. T_1) exponentially distributed with parameter ξ (resp. α). If the device returns from its vacation/working vacation before the time expires, the customer stays in the system until his service is completed. However, if impatience timer expires while the server is still on vacation/working vacation, the customer abandons the queue, never to return.

4 Stationary Solution of the Model

Using the theory of Markov process, the stationary equations governing the system are as follows

$$\lambda P_{01} = \xi P_{11} + p\mu_b P_{12}, \tag{1}$$

$$(\lambda + \theta + n\xi)P_{n,1} = \lambda P_{n-1,1} + (n+1)\xi P_{n+1,1}, \qquad n \geq 1, \tag{2}$$

$$\lambda P_{00} = \mu_v P_{10} + (1-p)\mu_b P_{1,2}, \tag{3}$$

$$(\lambda + \mu_v + \phi + (n-1)\alpha)P_{n,0} = \lambda P_{n-1,0} + (q\mu_v + n\alpha)P_{n+1,0}, \qquad n \geq 1, \quad (4)$$

$$(\lambda + \mu_b)P_{12} = \mu_b P_{2,2} + \theta P_{1,1} + \phi P_{1,0} + (1-q)\mu_v P_{2,0}, \quad (5)$$

$$(\lambda + \mu_b)P_{n2} = \lambda P_{n-1,2} + \theta P_{n,1} + \phi P_{n,0} + \mu_b P_{n+1,2} + (1-q)\mu_v P_{n+1,0}, \ n \geq 2. \quad (6)$$

Define the Probability generating functions (PGFs) as

$$P_0(z) = \sum_{n=0}^{\infty} P_{n,0} z^n,$$

$$P_1(z) = \sum_{n=0}^{\infty} P_{n,1} z^n,$$

$$P_2(z) = \sum_{n=1}^{\infty} P_{n,2} z^n,$$

with $P_0(1) + P_1(1) + P_2(1) = 1$, $P_0'(z) = \sum_{n=1}^{\infty} n z^{n-1} P_{n,0}$, and $P_1'(z) = \sum_{n=1}^{\infty} n$ $z^{n-1} P_{n,1}$.
Multiplying equation (2) by z^n and summing over n, we get after using equation (1)

$$\xi(1-z)P_1'(z) - [\lambda(1-z) + \theta]P_1(z) + p\mu_b P_{12} + \theta P_{01} = 0. \quad (7)$$

Multiplying equation (4) by z^n and summing over n, we get after using equation (3)

$$\begin{aligned} &\alpha z(1-z)P_0'(z) - [(1-z)(\lambda z - \mu_v + \alpha) + \mu_v(1-q) + z\phi]P_0(z) \\ &+[z\phi - (1-z)(\mu_v - \alpha) + (1-q)(\lambda z + \mu_v)]P_{00} + q(1-p)z\mu_b P_{12} = 0. \end{aligned} \quad (8)$$

Remark 1 *If* $p = 1$, *equation* (7) *becomes*

$$\xi(1-z)P_1'(z) = [\lambda(1-z) + \theta]P_1(z) - (\mu_b P_{12} + \theta P_{01}),$$

which matches with the result given in [1].

Remark 2 *If* $q = 1$ *and* $p = 0$, *equation* (8) *becomes*

$$\alpha z(1-z)P_0'(z) - [(1-z)(\lambda z - \mu_v + \alpha) + z\phi]P_0(z) + [z\phi - (1-z)(\mu_v - \alpha)]P_{00} + \mu_b P_{12} z = 0.$$

This matches with the result done in [23].

Remark 3 *If $q = 1$, $p = 0$ and $\alpha = 0$, then equation (8) reduces to*

$$P_0(z) = \frac{\mu_v(1-z)P_{00} - z(\mu_b P_{12} + \phi P_{00})}{\lambda z^2 - z(\lambda + \mu_v + \phi) + \mu_v},$$

which is same as in [24].

Multiplying equation (6) by z^n and summing over n, we get after using equation (5)

$$\begin{aligned}(1-z)(\lambda z - \mu_b)P_2(z) &= (z\phi + (1-q)\mu_v)P_0(z) + \theta z P_1(z) \\ &\quad - [(\phi + (\lambda+\mu_v)(1-q))P_{00} + q(1-p)\mu_b P_{12}]z \\ &\quad - \mu_v(1-q)(1-z)P_{00} - (\theta P_{01} + p\mu_b P_{12}).\end{aligned} \tag{9}$$

Putting $z = 1$ into equations (7) and (8), we respectively get

$$\theta P_1(1) = p\mu_b P_{12} + \theta P_{01}, \tag{10}$$

and

$$[\phi + \mu_v(1-q)]P_0(1) = [\phi + (1-q)(\lambda+\mu_v)]P_{00} + (1-p)q\mu_b P_{12}. \tag{11}$$

4.1 Solution of differential equations

Equation (7) can be rewritten as

$$P_1'(z) - \left[\frac{\lambda}{\xi} + \frac{\theta}{\xi(1-z)}\right]P_1(z) + \frac{p\mu_b P_{12} + \theta P_{01}}{\xi(1-z)} = 0, \tag{12}$$

for $\xi \neq 0$ and $z \neq 1$.

To solve the linear differential equation (12), we multiple both sides of the equation by $I.F = e^{-\frac{\lambda}{\xi}z}(1-z)^{\frac{\theta}{\xi}}$ and integrating from 0 to z, we have

$$P_1(z) = e^{\frac{\lambda}{\xi}z}(1-z)^{-\frac{\theta}{\xi}}\left[P_1(0) - \left(\frac{p\mu_b P_{12} + \theta P_{01}}{\xi}\right)K(z)\right], \tag{13}$$

where

$$K(z) = \int_0^z e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\theta}{\xi}-1}dx.$$

Then, by letting $z \to 1$, we obtain

$$P_1(1) = e^{\frac{\lambda}{\xi}}\left[P_1(0) - \left(\frac{p\mu_b P_{12} + \theta P_{01}}{\xi}\right)K(1)\right]\lim_{z\to 1}(1-z)^{-\frac{\theta}{\xi}}.$$

Since $0 \leq P_1(1) = \sum_{n=0}^{\infty} P_{n,1} \leq 1$ and $\lim\limits_{z\to 1}(1-z)^{-(\frac{\theta}{\xi})} \to \infty$,we must have

$$P_{01} = P_1(0) = \left(\frac{p\mu_b P_{12} + \theta P_{01}}{\xi}\right) K(1), \tag{14}$$

which gives

$$P_{12} = T_0 P_{01}, \tag{15}$$

where $T_0 = \frac{\xi - \theta K(1)}{p\mu_b K(1)}$.

Then, substituting equation (15) into equations (10) and (13), we respectively get

$$P_1(1) = \frac{\xi}{\theta K(1)} P_{01}, \tag{16}$$

and

$$P_1(z) = e^{\frac{\lambda}{\xi}z}(1-z)^{-\frac{\theta}{\xi}}\left[1 - \frac{K(z)}{K(1)}\right] P_{00}. \tag{17}$$

From equations (1) and (15), we get

$$P_{11} = U_1 P_{01}, \tag{18}$$

where $U_1 = \frac{\lambda - p\mu_b T_0}{\xi}$.

From equations (2) (for $n = 1$) and (18), we get

$$P_{21} = U_2 P_{01}, \tag{19}$$

where $U_2 = g_1 U_1 - \frac{\lambda}{2\xi} U_0$, $g_1 = \frac{\lambda + \phi + \xi}{2\xi}$ and $U_0 = 1$.
From equations (2) (for $n = 2$) and (18)-(19), we get

$$P_{31} = U_3 P_{01}, \tag{20}$$

where $U_3 = g_2 U_2 - \frac{\lambda}{3\xi} U_1$ and $g_2 = \frac{\lambda + \phi + 2\xi}{3\xi}$.
Then, recursively, it yields

$$P_{n1} = U_n P_{01},$$

where

$$U_n = \begin{cases} \frac{\lambda - p\mu_b T_0}{\xi}, & \text{if } \ n = 1, \\ g_{n-1} U_{n-1} - \frac{\lambda}{n\xi} U_{n-2}, & \text{if } \ n \geq 2, \end{cases}$$

with

$$g_{n-1} = \frac{\lambda + \theta + (n-1)\xi}{n\xi}.$$

Next, equation (8) can be expressed as

$$P_0'(z) - \left\{ \frac{\lambda z-\mu_v+\alpha}{z\alpha} + \frac{\phi}{\alpha(1-z)} + \frac{\mu_v(1-q)}{\alpha z(1-z)} \right\} P_0(z) + \left\{ \frac{\phi}{\alpha(1-z)} - \frac{\mu_v-\alpha}{z\alpha} + \frac{(1-q)(z\lambda+\mu_v)}{\alpha z(1-z)} \right\} P_{00}$$
$$+\frac{q(1-p)\mu_b}{\alpha(1-z)} P_{12} = 0,$$

for $\alpha \neq 0$, $z \neq 0$, and $z \neq 1$.

Now, in order to solve the above differential equation we multiply it both sides by $I.F = e^{\frac{-\lambda}{\alpha}z} z^{\left(\frac{\mu_v q}{\alpha}-1\right)} (1-z)^{\frac{\phi+\mu_v(1-q)}{\alpha}}$ and integrating from 0 to z, we get

$$\begin{aligned} P_0(z) = z^{-\left(\frac{\mu_v q}{\alpha}-1\right)} (1-z)^{-\left(\frac{\phi+\mu_v(1-q)}{\alpha}\right)} \Bigg\{ \left(\frac{\mu_v}{\alpha}-1\right) P_{00} A(z) \\ - \frac{\mu_v(1-q)}{\alpha} P_{00} B(z) - \left(\frac{\phi+(1-q)\lambda}{\alpha} P_{00} + \frac{q(1-p)\mu_b}{\alpha} P_{12} \right) C(z) \Bigg\}, \end{aligned} \tag{21}$$

where

$$\begin{aligned} A(z) &= \int_0^z e^{\frac{\lambda}{\alpha}(z-x)} x^{\frac{\mu_v q}{\alpha}-2} (1-x)^{\frac{\phi+\mu_v(1-q)}{\alpha}} dx, \\ B(z) &= \int_0^z e^{\frac{\lambda}{\alpha}(z-x)} x^{\frac{\mu_v q}{\alpha}-2} (1-x)^{\frac{\phi+\mu_v(1-q)}{\alpha}-1} dx, \\ C(z) &= \int_0^z e^{\frac{\lambda}{\alpha}(z-x)} x^{\frac{\mu_v q}{\alpha}-1} (1-x)^{\frac{\phi+\mu_v(1-q)}{\alpha}-1} dx. \end{aligned}$$

Taking limit $z \to 1$ in equation (21), we get

$$\begin{aligned} P_0(1) = \Bigg\{ \left(\frac{\mu_v}{\alpha}-1\right) A(1) P_{00} - \frac{\mu_v(1-q)}{\alpha} B(1) P_{00} \\ - \left[\frac{(\phi+(1-q)\lambda)}{\alpha} P_{00} + \frac{q(1-p)\mu_b P_{12}}{\alpha} \right] C(1) \Bigg\} \lim_{z\to 1} (1-z)^{-\left(\frac{\phi+\mu_v(1-q)}{\alpha}\right)}. \end{aligned}$$

Since $0 \leq P_0(1) = \sum_{n=0}^{\infty} P_{n,0} \leq 1$ and $\lim_{z\to 1}(1-z)^{-\left(\frac{\phi+\mu_v(1-q)}{\alpha}\right)} \to \infty$,we must have

$$P_{12} = S_1 P_{00}, \tag{22}$$

where

$$S_1 = \left[\frac{(\mu_v-\alpha)\frac{A(1)}{C(1)} - \mu_v(1-q)\frac{B(1)}{C(1)} - (\phi+(1-q)\lambda)}{q(1-p)\mu_b} \right].$$

Substituting equation (22) into equation (21), we get

$$P_0(z) = P_{00}\Bigg\{\left(\frac{\mu_v}{\alpha}-1\right)\left[A(z)-\frac{A(1)}{C(1)}C(z)\right] - \frac{\mu_v(1-q)}{\alpha}\left[B(z)-\frac{B(1)}{C(1)}C(z)\right]\Bigg\} z^{-(\frac{\mu_v}{\alpha}q-1)}(1-z)^{-(\frac{\phi+\mu_v(1-q)}{\alpha})}.$$

Substituting equation (22) into equation (11), we obtain

$$P_0(1) = HP_{00}, \tag{23}$$

where

$$H = \left[\frac{(\mu_v-\alpha)\frac{A(1)}{C(1)} + \mu_v(1-q)\left(1-\frac{B(1)}{C(1)}\right)}{\phi+\mu_v(1-q)}\right].$$

From equations (15), (16), and (22), we find

$$P_1(1) = \frac{\xi S_1}{\theta K(1) T_0} P_{00}. \tag{24}$$

From equations (3) and (22), we get

$$P_{10} = V_1 P_{00}. \tag{25}$$

where $V_1 = \dfrac{\lambda-(1-p)\mu_b S_1}{\mu_v}$.

From equations (4)(for $n=1$) and (25), we obtain

$$P_{20} = V_2 P_{00}, \tag{26}$$

where $V_2 = f_0 V_1 - \frac{\lambda}{q\mu_v+\alpha}V_0$, $f_0 = \frac{\lambda+\mu_v+\phi}{q\mu_v+\alpha}$ and $V_0 = 1$.

From equations (4) (for $n=2$) and (25)-(26), we get

$$P_{30} = V_3 P_{00}, \tag{27}$$

where $V_3 = f_1 V_2 - \frac{\lambda}{q\mu_v+2\alpha}V_1$ and $f_1 = \frac{\lambda+\mu_v+\phi+\alpha}{q\mu_v+2\alpha}$.

From equations (4)(for $n=3$) and (26)-(27), we get

$$P_{40} = V_4 P_{00}, \tag{28}$$

where $V_4 = f_2 V_3 - \frac{\lambda}{q\mu_v+3\alpha} V_2$ and $f_2 = \frac{\lambda+\mu_v+\phi+2\alpha}{q\mu_v+3\alpha}$.

Then, recursively, it yields

$$P_{n0} = V_n P_{00},$$

where

$$V_n = \begin{cases} 1, & \text{if } \ n=0, \\ \frac{\lambda-(1-p)\mu_b S_1}{\mu_v}, & \text{if } \ n=1, \\ f_{n-2}V_{n-1} - \frac{\lambda}{q\mu_v+(n-1)\alpha} v_{n-2}, & \text{if } \ n \geq 2, \end{cases}$$

with

$$f_{n-2} = \frac{\lambda+\mu_v+\phi+(n-2)\alpha}{q\mu_v+(n-1)\xi}.$$

Next, substituting equations (10) and (11) into equation (9), we get

$$P_2(z) = \frac{(z\phi+(1-q)\mu_v)P_0(z)+\theta z P_1(z) - z(\phi+\mu_v(1-q))P_0(1) - z\theta P_1(1)}{(1-z)(\lambda z-\mu_b)} - \frac{\mu_v(1-q)}{\lambda z-\mu_b} P_{00}. \tag{29}$$

Applying L'Hospital's rule to equation (29), we get

$$P_2(1) = \frac{(\phi+\mu_v(1-q))P_0'(1)+\theta P_1'(1) - \mu_v(1-q)P_0(1)}{\mu_b-\lambda} + \frac{\mu_v(1-q)}{\mu_b-\lambda} P_{00}. \tag{30}$$

This implies

$$P_0'(1) = \frac{(\mu_b-\lambda)P_2(1)+\mu_v(1-q)(P_0(1)-P_{00}) - \theta P_1'(1)}{\phi+\mu_v(1-q)}. \tag{31}$$

Equation (7) can be rewritten as

$$P_1'(z) = \frac{[\lambda(1-z)+\theta]P_1(z) - p\mu_b P_{12} - \theta P_{01}}{\xi(1-z)}$$

Applying L'Hospital's rule, we have

$$P_1'(1) = \frac{\lambda}{\theta+\xi} P_1(1). \tag{32}$$

Further, equation (8) can be rewritten as

$$\begin{aligned} P_0'(z) = {} & \frac{1}{\alpha z(1-z)} \big([(1-z)(\lambda z-\mu_v+\alpha)+\mu_v(1-q)+z\phi]P_0(z) \\ & -[z\phi-(1-z)(\mu_v-\alpha)+(1-q)(\lambda z+\mu_v)]P_{00} - q(1-p)z\mu_b P_{11} \big). \end{aligned}$$

Applying L'Hospital's rule, we have

$$P_0'(1) = \frac{(\lambda+\alpha-\mu_v-\phi)P_0(1)+(\mu_v+\phi-\alpha+\lambda(1-q)+q\mu_b(1-p)S_1)P_{00}}{\alpha+\phi+\mu_v(1-q)}. \tag{33}$$

Next, substituting equations (32) and (33) into (30), we obtain

$$\begin{aligned} P_2(1) = &\left[\frac{(\phi+\mu_v(1-q))(\lambda+\alpha-\mu_v-\phi)}{(\alpha+\phi+\mu_v(1-q))(\mu_b-\lambda)} - \frac{\mu_v(1-q)}{\mu_b-\lambda}\right]P_0(1) \\ &+ \frac{\lambda\theta P_1(1)}{(\theta+\xi)(\mu_b-\lambda)} + (\phi+\mu_v(1-q)) \\ &\left[\frac{\mu_v+\phi-\alpha-\lambda(1-q)+q\mu_b(1-p)S_1}{(\alpha+\phi+\mu_v(1-q))(\mu_b-\lambda)} + \frac{\mu_v(1-q)}{\mu_b-\lambda}\right]P_{00}. \end{aligned} \tag{34}$$

Using equations (23)-(24) and (34), and normalization condition, we can get the value of P_{00}. Next, we need to write $P_{n,2}$ in terms of $P_{0,0}$.
Substituting equations (15), (18), (22), and (25)-(26) into equation (5), we get

$$P_{22} = S_2 P_{00}, \tag{35}$$

where $S_2 = (1+\rho)S_1 - \frac{\theta S_1}{\mu_b T_0}U_1 - \frac{\phi V_1 + V_2\mu_v(1-q)V_1}{\mu_b}$, $\rho = \frac{\lambda}{\mu_b}$.

Substituting equations (15), (19), (22), and (26)-(27) into equation (6) (for $n=2$), we obtain

$$P_{32} = S_3 P_{00}, \tag{36}$$

where $S_3 = (1+\rho)S_2 - \rho S_1 - \frac{\theta S_1}{\mu_b T_0}U_2 - \frac{\phi V_2 + \mu_v(1-q)V_3}{\mu_b}$.

Substituting equations (15), (20), (27)-(28), and (35)-(36) into equation (6) (for $n=3$), we find

$$P_{42} = S_4 P_{00},$$

where $S_4 = (1+\rho)S_3 - \rho S_2 - \frac{\theta S_1}{\mu_b T_0}U_3 - \frac{\phi V_3 + \mu_v(1-q)V_4}{\mu_b}$.

Then, recursively, it yields

$$P_{n2} = S_n P_{00},$$

where

$$S_n = \begin{cases} 1, & \text{if } n = 1, \\ (1+\rho)S_{n-1} - \rho S_{n-2} - \frac{\theta S_1}{\mu_b T_0}U_{n-1} - \frac{\phi V_{n-1} + \mu_v(1-q)V_n}{\mu_b}, & \text{if } n \geq 2, \end{cases}$$

with $S_0 = 0$.

5 Performance measures

As the steady-state probabilities are obtained one can easily derive the various performance measures of the model.
– The probability that the system is in working vacation $(P_0(1))$.

$$P_0(1) = \left[\frac{(\mu_v - \alpha)\frac{A(1)}{C(1)} + \mu_v(1-q)\left(1 - \frac{B(1)}{C(1)}\right)}{\phi + \mu_v(1-q)}\right] P_{00}.$$

– The probability that the system is in vacation period $(P_1(1))$.

$$P_1(1) = \frac{\xi S_1}{\theta K(1) T_0} P_{00}.$$

– The probability that the system is in busy period $(P_2(1))$.

$$\begin{aligned} P_2(1) = &\left[\frac{(\phi+\mu_v(1-q))(\lambda+\alpha-\mu_v-\phi)}{(\alpha+\phi+\mu_v(1-q))(\mu_b-\lambda)} - \frac{\mu_v(1-q)}{\mu_b-\lambda}\right] P_0(1) + \frac{\lambda\theta P_1(1)}{(\theta+\xi)(\mu_b-\lambda)} \\ &+ (\phi+\mu_v(1-q))\left[\frac{\mu_v+\phi-\alpha-\lambda(1-q)+q\mu_b(1-p)S_1}{(\alpha+\phi+\mu_v(1-q))(\mu_b-\lambda)} + \frac{\mu_v(1-q)}{\mu_b-\lambda}\right] P_{00} \end{aligned}$$

Substituting equation (23) into equation (33), we get the expected number of customers when the system is on working vacation period $(E(L_0))$.

$$E(L_0) = P_0'(1) = \left[\frac{(\lambda+\alpha-\mu_v-\phi)H + \mu_v+\phi-\alpha+\lambda(1-q)+q\mu_b(1-p)S_1}{\alpha+\phi+\mu_v(1-q)}\right] P_{00}.$$

Substituting equation (24) into equation (32), we get the expected number of customers when the system is on vacation period $(E(L_1))$.

$$E(L_1) = P_1'(1) = \frac{\lambda\xi S_1}{\theta(\theta+\xi)K(1)T_0} P_{00}.$$

Equation (9) can be rewritten as

$$\begin{aligned} P_2(z) = &\frac{(z\phi + (1-q)\mu_v)P_0(z) + \theta z P_1(z) - z(\phi+\mu_v(1-q))P_0(1) - z\theta P_1(1)}{((1-z)(\lambda z - \mu_b)} \\ &- \frac{\mu_v(1-q)}{\lambda z - \mu_b} P_{00}. \end{aligned}$$

Differentiating the above equation and applying L'Hospital's rule, we get

$$\begin{aligned} E(L_2) = P_2'(1) = &\tfrac{\phi+\mu_v(1-q)}{2(\mu_b-\lambda)} P_0''(1) + \tfrac{(\lambda\mu_v(1-q)+\mu_b\phi)}{(\mu_b-\lambda)^2} P_0'(1) + \tfrac{\theta}{2(\mu_b-\lambda)} P_1''(1) \\ &+ \tfrac{(\lambda\mu_v(1-q)+\mu_b\phi)}{(\mu_b-\lambda)^2} P_0'(1) + \tfrac{\lambda\mu_v(1-q)}{(\mu_b-\lambda)^2}(P_{00} - P_0(1)) + \tfrac{\lambda\mu_v(1-q)}{(\mu_b-\lambda)^2} P_{00}. \end{aligned} \quad (37)$$

Differentiating equation (7) twice with respect to z and letting $z = 1$, we obtain

$$\frac{P_1''(1)}{2} = \frac{\lambda}{\theta + 2\xi} P_1'(1). \tag{38}$$

Differentiating equation (8) twice with respect to z and letting $z = 1$, we obtain

$$\frac{P_0''(1)}{2} = \frac{(\lambda - \mu_v - \phi)P_0'(1) + \lambda P_0(1)}{\phi + 2\alpha + \mu_v(1-q)}. \tag{39}$$

Substituting equations (38) and (39) into equation (37), we get the expected number of customer when the server is busy ($E(L_2)$).

$$\begin{aligned} E(L_2) = \tfrac{1}{\mu_b-\lambda} &\left[\tfrac{(\phi+\mu_v(1-q))(\lambda-\mu_v-\phi)}{\phi+2\alpha+\mu_v(1-q)} + \tfrac{\lambda\mu_v(1-q)+\phi\mu_b}{\mu_b-\lambda}\right] P_0'(1) + \tfrac{1}{\mu_b-\lambda} \\ &\times \left[\tfrac{\lambda\theta}{\theta+2\xi} + \tfrac{\phi\mu_b}{\mu_b-\lambda}\right] P_1'(1) + \tfrac{\lambda\mu_v(1-q)}{(\mu_b-\lambda)^2} P_{00} + \tfrac{\lambda}{\mu_b-\lambda}\left[\tfrac{\phi+\mu_v(1-q)}{\phi+2\alpha+\mu_v(1-q)} - \tfrac{\mu_v(1-q)}{\mu_b-\lambda}\right] P_0(1). \end{aligned}$$

The expected number of customers in the system can be computed as $E(L) = E(L_0) + E(L_1) + E(L_2)$.

– The average rate of abandonment of customers due to impatience (R_a).

$$R_a = \alpha \sum_{n=0}^{\infty} (n-1)P_{n,0} + \xi \sum_{n=0}^{\infty} nP_{n,1} = \alpha(E[L_0] - (P_0(1) - P_{00})) + \xi E[L_1].$$

6 Stochastic decomposition of the model

The stochastic decomposition structures for the mean queue length and mean waiting times at stationary state are expressed in the following Theorems.

Theorem 1 *If* $\lambda < \mu_b$*, the stationary queue length L can be decomposed into the sum of two independent random variables as* $L = L_0 + L_d$*, where* L_0 *is the stationary queue length of a classical* $M/M/1$ *queue without vacations and* L_d *is the additional queue length due to the effect of working vacation or vacation with its pgf as*

$$\begin{aligned} L_d(z) = \left(\tfrac{1}{1-\rho}\right) &\left\{ \left[1 - \rho z - \tfrac{(\phi z + \mu_v(1-q))}{\mu_b(1-z)}\right] P_0(z) + z\left[\tfrac{\phi+\mu_v(1-q)}{\mu_b(1-z)}\right] P_0(1) \right. \\ &\left. + \left[1 - \rho z - \tfrac{\theta z}{\mu_b(1-z)}\right] P_1(z) + \tfrac{\theta z}{\mu_b(1-z)} P_1(1) + \tfrac{\mu_v(1-q)}{\mu_b} P_{00} \right\}. \end{aligned} \tag{40}$$

Proof. Consider

$$
\begin{aligned}
L(z) &= P_0(z) + P_1(z) + P_2(z) \\
&= \left[1 + \frac{\phi z + \mu_v(1-q)}{(1-z)(\lambda z - \mu_b)}\right] P_0(z) + \left[1 + \frac{\theta z}{(1-z)(\lambda z - \mu_b)}\right] P_1(z) \\
&\quad - z\left[\frac{\phi + \mu_v(1-q)}{(1-z)(\lambda z - \mu_b)}\right] P_0(1) - \left[\frac{\theta z}{(1-z)(\lambda z - \mu_b)}\right] P_1(1) - \frac{\mu_v(1-q)}{\lambda z - \mu_b} P_{00} \\
&= \left(\frac{\mu_b - \lambda}{\mu_b - \lambda z}\right) \left\{ \left[\frac{\mu_b - \lambda z}{\mu_b - \lambda} - \frac{(\phi z + \mu_v(1-q))}{(\mu_b - \lambda)(1-z)}\right] P_0(z) \right. \\
&\quad + z\left[\frac{\phi + \mu_v(1-q)}{(1-z)(\mu_b - \lambda)}\right] P_0(1) + \left[\frac{\mu_b - \lambda z}{\mu_b - \lambda} - \frac{\theta z}{(\mu_b - \lambda)(1-z)}\right] P_1(z) \\
&\quad \left. + \left[\frac{\theta z}{(1-z)(\mu_b - \lambda)}\right] P_1(1) + \frac{\mu_v(1-q)}{\mu_b - \lambda} P_{00} \right\} = \frac{(1-\rho)}{1-\rho z} \times L_d(z),
\end{aligned}
$$

where $L_d(z)$ can be expressed in series expansion as

$$
\begin{aligned}
L_d(z) &= \left(\frac{1}{1-\rho}\right) \left\{ \left[1 - \rho z - \frac{(\phi z + \mu_v(1-q))}{\mu_b(1-z)}\right] P_0(z) + z\left[\frac{\phi + \mu_v(1-q)}{\mu_b(1-z)}\right] P_0(1) \right. \\
&\quad \left. + \left[1 - \rho z - \frac{\theta z}{\mu_b(1-z)}\right] P_1(z) + \frac{\theta z}{\mu_b(1-z)} P_1(1) + \frac{\mu_v(1-q)}{\mu_b} P_{00} \right\} \\
&= \frac{1}{1-\rho} \left\{ \sum_{n=0}^{\infty} P_{n,0} z^n - \rho \sum_{n=0}^{\infty} P_{n,0} z^{n+1} + \frac{\phi}{\mu_b} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} P_{n+k,0} z^n \right. \\
&\quad + \frac{\mu_v(1-q)}{\mu_b} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} P_{n+k+1,0} z^n + \sum_{n=0}^{\infty} P_{n,1} z^n - \rho \sum_{n=0}^{\infty} P_{n,1} z^{n+1} \\
&\quad \left. + \frac{\phi}{\mu_b} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} P_{n+k,1} z^n \right\} = \sum_{n=0}^{\infty} t_n z^n,
\end{aligned}
$$

such that $t_0 = \frac{1}{1-\rho}(P_{00} + P_{01})$, and

$$
\begin{aligned}
t_n = \frac{1}{1-\rho} \Bigg\{ & P_{n,0} - \rho P_{n-1,0} + \frac{\phi}{\mu_b} \sum_{k=0}^{\infty} P_{n+k,0} + \frac{\mu_v(1-q)}{\mu_b} \sum_{k=0}^{\infty} P_{n+k+1,0} \\
& + P_{n,1} - \rho P_{n-1,1} + \frac{\phi}{\mu_b} \sum_{k=0}^{\infty} P_{n+k,1} \Bigg\}, \quad n \geq 1.
\end{aligned}
$$

Now, we show that $\sum_{n=0}^{\infty} t_n = 1$ for $t_n \in [0,1]$.

$$\sum_{n=0}^{\infty} t_n = \frac{1}{1-\rho}\Bigg\{(1-\rho)\sum_{n=0}^{\infty} P_{n,0} + \frac{\phi}{\mu_b}\sum_{n=1}^{\infty} nP_{n,0} + \frac{(1-q)\mu_v}{\mu_b}\sum_{n=1}^{\infty}(n-1)P_{n,0}$$
$$+ (1-\rho)\sum_{n=0}^{\infty} P_{n,1} + \frac{\theta}{\mu_b}\sum_{n=1}^{\infty} nP_{n,1}\Bigg\}$$
$$= \frac{1}{1-\rho}\Bigg\{(1-\rho)\sum_{n=1}^{\infty} P_{n,0} + \left(\frac{\phi+\mu_v(1-q)}{\mu_b}\right)\sum_{n=1}^{\infty} nP_{n,0}$$
$$- \frac{\mu_v(1-q)}{\mu_b}\sum_{n=1}^{\infty}(n-1)P_{n,0} + (1-\rho)\sum_{n=0}^{\infty} P_{n,1} + \frac{\theta}{\mu_b}\sum_{n=1}^{\infty} nP_{n,1}\Bigg\}.$$

Applying equation (31), we get

$$\sum_{n=0}^{\infty} t_n = \frac{1}{1-\rho}\Bigg\{(1-\rho)\sum_{n=1}^{\infty} P_{n,0} - \frac{\mu_v(1-q)}{\mu_b}\sum_{n=1}^{\infty} P_{n,0} + (1-\rho)\sum_{n=0}^{\infty} P_{n,1} + \frac{\theta}{\mu_b}\sum_{n=1}^{\infty} nP_{n,1}$$
$$+ \left(\frac{\phi+\mu_v(1-q)}{\mu_b}\right)\left[\frac{(\mu_b-\lambda)P_2(1) + \mu_v(1-q)(P_0(1)-P_{00}) - \theta P_1'(1)}{\phi+\mu_v(1-q)}\right]\Bigg\} +$$
$$= \sum_{n=0}^{\infty} P_{n,0} + 1 - P_0(1) - P_1(1) - \frac{(1-q)\mu_v}{\mu_b(1-\rho)}P_{00} + \frac{(1-q)\mu_v}{\mu_b(1-\rho)}P_{00} + \sum_{n=0}^{\infty} P_{n,1} = 1.$$

Hence, $L_d(z)$ is a PGF of the additional queue length due to the Bernoulli schedule vacation interruption. □

Theorem 2 *If $\lambda < \mu_b$, the stationary waiting time can be decomposed into the sum of two independent random variables as $W = W_0 + W_d$, where W_0 is the waiting time of a customer corresponding to classical $M/M/1$ queue which has an exponential distribution with the parameter $\mu_b(1-\rho)$ and W_d is the additional delay due to due to the effect of working vacation or vacation with its Laplace-Stieltjes transform (LST).*

$$W_d^*(s) = \frac{1}{(\mu_b-\lambda)s}\Bigg\{[(\mu_b-\lambda+s)s - \phi(\lambda-s) - \lambda(1-q)\mu_v]\,P_0\left(1-\frac{s}{\lambda}\right)$$
$$+ [(\mu_b-\lambda+s)s - \theta(\lambda-s)]\,P_1\left(1-\frac{s}{\lambda}\right)$$
$$+(\lambda-s)(\phi+\mu_v(1-q))P_0(1) + (\lambda-s)\theta P_1(1) + (1-q)\mu_v s P_{00}\Bigg\}.$$

Proof. The relationship between the probability generating function L and LST of waiting time [12] is given by

$$L(z) = W^*(\lambda(1-z)).$$

Assume that $s = \lambda(1-z)$, so $z = 1 - \frac{s}{\lambda}$ and $1 - z = \frac{s}{\lambda}$. Applying the relations in equation (40), we obtain the desired result. $\square$

7 Cost model

Practically, queueing managers are interested in minimizing operating cost of unit time. In this part of paper, we first formulate a steady-state expected cost function per unit time, where the service rate (μ_b) is the decision variable. Our main goal is to determine the optimum value of μ_b in order to minimize the expected cost function. To this end, we have to define the following cost elements:

- C_1 : Cost per unit time when the server is on working during regular busy period.
- C_2 : Cost per unit time when the server is on vacation period.
- C_3 : Cost per unit time when the server is on busy period.
- C_4 : Cost per service per unit time during regular busy period.
- C_5 : Cost per service per unit time during working vacation period.
- C_6 : Cost per unit time when a customer reneges.
- C_7 : Holding cost per customer per unit time.

Let $\mathcal{T}_c$ be the total expected cost per unit time of the system:

$$\mathcal{T}_c = C_1P_0(1) + C_2P_1(1) + C_3P_2(1) + \mu_bC_4 + \mu_vC_5 + C_6R_{ren} + C_7E[L].$$

7.1 The optimization study

In this subsection we focus on the optimization of the service rate (μ_b) in different cases in order to minimize the cost function $\mathcal{T}_c$. We solve the stated optimization problem using QFSM method.

Given a 3-point pattern, we can fit a quadratic function through corresponding functional values that has a unique minimum, x^q, for the given objective function $\mathcal{T}_c(x)$. Quadratic fit uses this approximation to improve the current 3-point pattern by replacing one of its points with optimum x^q. The unique optimum x^q of the quadratic function agreeing with $\mathcal{T}_c(x)$ at 3-point operation (x^l, x^m, x^u) is given by

$$x^q \cong \frac{1}{2}\left[\frac{\mathcal{T}_c(x^l)((x^m)^2-(x^u)^2)+\mathcal{T}_c(x^m)((x^u)^2-(x^l)^2)+\mathcal{T}_c(x^u)((x^l)^2-(x^m)^2)}{\mathcal{T}_c(x^l)(x^m-x^u)+\mathcal{T}_c(x^m)(x^u-x^l)+\mathcal{T}_c(x^u)(x^l-x^m)}\right].$$

The optimization problem can be illustrated mathematically as:

Minimize: $\mathcal{T}_c(\mu_b) = C_1P_0(1)+C_2P_1(1)+C_3P_2(1)+\mu_bC_4+\mu_vC_5+C_6R_{ren}+C_7E[L]$.

Suppose that all system parameters have fixed values, and the only controlled parameter is the service rate (μ_b).

8 Numerical results

In this section, we provide numerical experiments to illustrate how different system parameters affect some system characteristics.

The system parameters chosen are presented in Tables and Figures given in the following items:

- Table 1 and Figure 2 : $\lambda = 2.4, \mu_v = 3.0, p = 0.3, q = 0.8, \theta = 1.8, \phi = 0.8, \alpha = 0.1$, and $\xi = 1.9$.
- Table 2 : $\mu_v = 2.6, p = 0.4, \theta = 1.4, \phi = 0.8, \alpha = 0.1$, and $\xi = 1.2$.
- Table 3 : $\lambda = 3.2, q = 0.6, \theta = 1.1, \phi = 0.7, \alpha = 0.3$, and $\xi = 1.7$.
- Table 4 : $\lambda = 3.0, q = 0.7, \theta = 0.8, \phi = 0.2, \mu_v = 2.4$, and $p = 0.4$.
- Table 5 : $\lambda = 2.8, q = 0.8, \alpha = 0.2, \xi = 1.5, \mu_v = 2.2$, and $p = 0.4$.
- Figure 3 : $\mu_b = 4.5, \mu_v = 2.6, \alpha = 0.1, \xi = 1.2, \phi = 0.8, p = 0.4$, and $\theta = 1.4$.
- Figure 4 : $\lambda = 3.4, \mu_v = 2.6, \alpha = 0.1, \xi = 1.2, \phi = 0.8$, and $\theta = 1.4$.

- Figure 5 : $\mu_b = 4.7, q = 0.9, \alpha = 0.2, \xi = 1.2, \phi = 0.3, \theta = 0.7$, and $p = 0.4$.
- Figures 6-8 : $\lambda = 3.0, \mu_b = 4.5, \mu_v = 2.6, q = 0.7, \xi = 1.2, \theta = 1.4$, and $p = 0.4$.
- Figures 7-9 : $\lambda = 3.0, \mu_b = 4.5, \mu_v = 2.6, q = 0.7, \alpha = 0.4, \phi = 0.6$, and $p = 0.5$.

Table 1: Search for the optimum service rate μ_b^* during regular busy period.

μ^l	μ^m	μ^u	$\mathcal{T}_c(\mu^l)$	$\mathcal{T}_c(\mu^m)$	$\mathcal{T}_c(\mu^u)$	μ^q	$\mathcal{T}_c(\mu^q)$
5.100000	5.400000	5.700000	410.484439	394.420852	391.963589	5.604179	391.910733
5.400000	5.604179	5.700000	394.420852	391.910733	391.963589	5.645648	391.857148
5.604179	5.645648	5.700000	391.910733	391.857148	391.963589	5.643959	391.856942
5.604179	5.643959	5.645648	391.910733	391.856942	391.857148	5.643089	391.856912
5.604179	5.643089	5.643959	391.910733	391.856912	391.856942	5.643048	391.856912
5.604179	5.643048	5.643089	391.910733	391.856912	391.856912	5.643033	391.856912
5.604179	5.643033	5.643048	391.910733	391.856912	391.856912	5.643032	391.856912
5.604179	5.643032	5.643033	391.910733	391.856912	391.856912	5.643031	391.856912

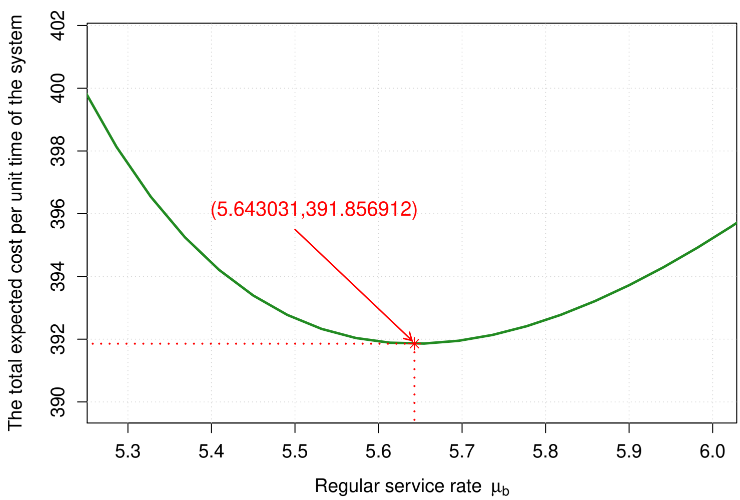

Figure 2: Effect of μ_b on $\mathcal{T}_c$.

Table 2: Optimal values of μ_b^* and $\mathcal{T}_c(\mu_b^*)$ for different values of λ and $\bar{q}$.

	$\lambda = 3.5$		$\lambda = 4.5$		$\lambda = 5.5$	
	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$
$\bar{q} = 0.3$	4.862986	352.655384	6.012295	410.465278	7.138987	466.276595
$\bar{q} = 0.6$	4.589849	333.020545	5.717871	387.784157	6.829078	440.910714
$\bar{q} = 0.9$	4.449532	323.932229	5.563229	377.161323	6.663980	429.046786

Table 3: Optimal values of μ_b^* and $\mathcal{T}_c(\mu_b^*)$ for different values of μ_v and p.

	$\mu_v = 2.2$		$\mu_v = 2.5$		$\mu_v = 2.8$	
	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$
$p = 0.3$	3.607634	279.308457	3.600697	286.217326	3.593235	293.371886
$p = 0.6$	3.313594	257.348468	3.310018	265.481915	3.306657	273.738369
$p = 0.9$	3.134604	243.821124	3.133657	252.623948	3.132829	261.457645

Table 4: Optimal values of μ_b^* and $\mathcal{T}_c(\mu_b^*)$ for different values of α and ξ.

	$\alpha = 0.1$		$\alpha = 0.4$		$\alpha = 0.7$	
	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$
$\xi = 0.5$	4.101791	324.408533	4.026891	319.266445	3.961865	315.540988
$\xi = 1.0$	4.126021	322.687741	4.045522	317.026212	3.975738	312.850216
$\xi = 1.5$	4.139667	322.657325	4.056594	316.842638	3.984583	312.525274

Table 5: Optimal values of μ_b^* and $\mathcal{T}_c(\mu_b^*)$ for different values of θ and ϕ.

	$\theta = 0.8$		$\theta = 1.4$		$\theta = 2.0$	
	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$	μ_b^*	$\mathcal{T}_c(\mu_b^*)$
$\phi = 0.4$	4.026534	307.781311	4.023896	302.709691	4.020944	300.922353
$\phi = 0.8$	4.114485	312.824843	4.078663	302.785848	4.071751	298.539176
$\phi = 1.2$	4.228526	319.734187	4.143652	305.561523	4.112019	299.095820

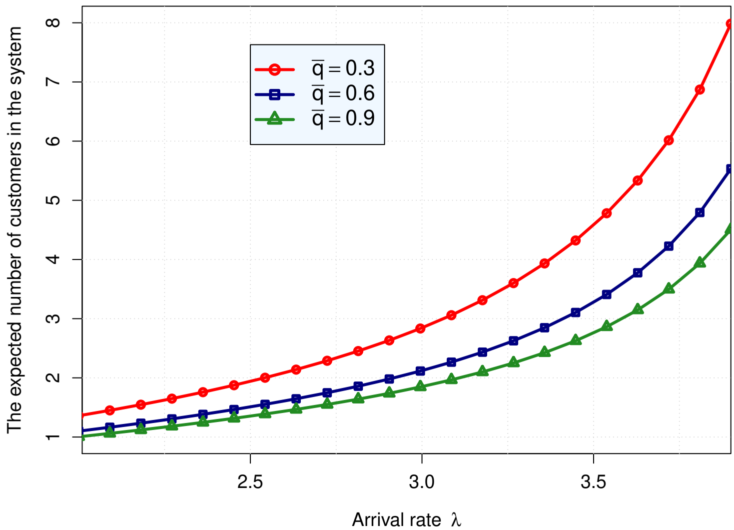

Figure 3: Effect of λ and $\bar{q}$ on $E[L]$.

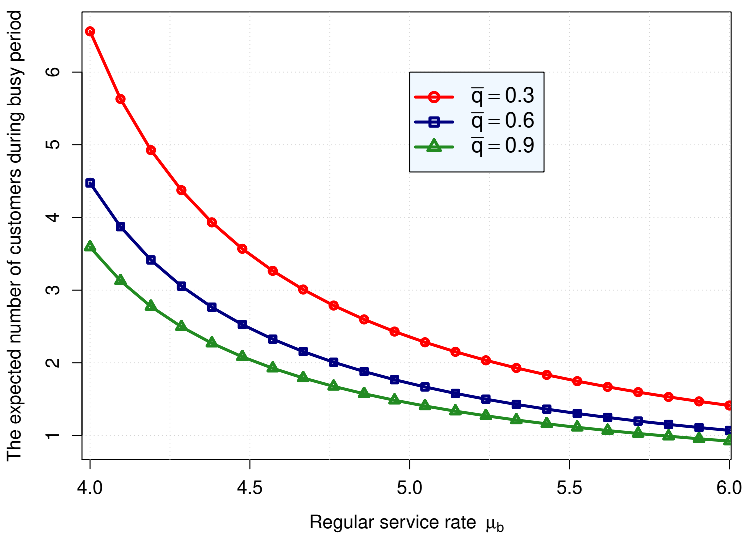

Figure 4: Effect of μ_b and $\bar{q}$ on $E[L_2]$.

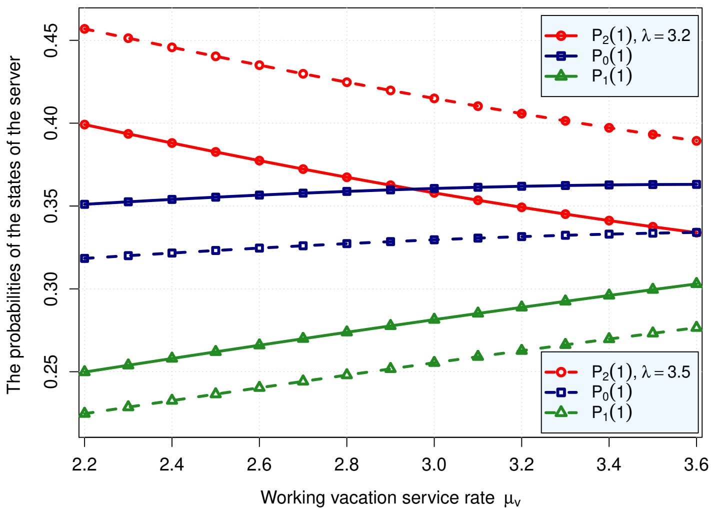

Figure 5: Effect of μ_v and λ on $P_0(1), P_1(1)$ and $P_2(1)$.

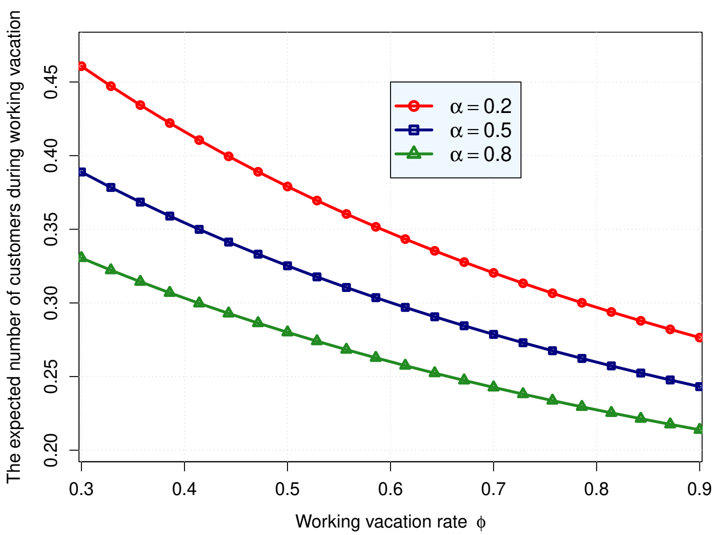

Figure 6: Effect of ϕ and α on $E[L_0]$.

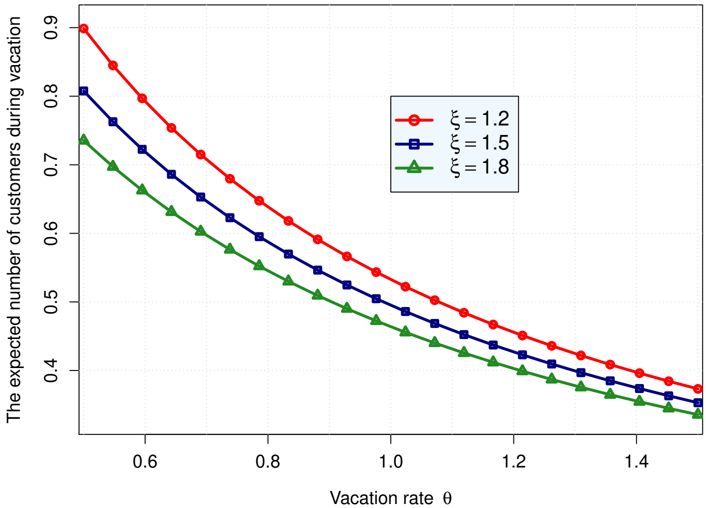

Figure 7: Effect of θ and ξ on $E[L_1]$.

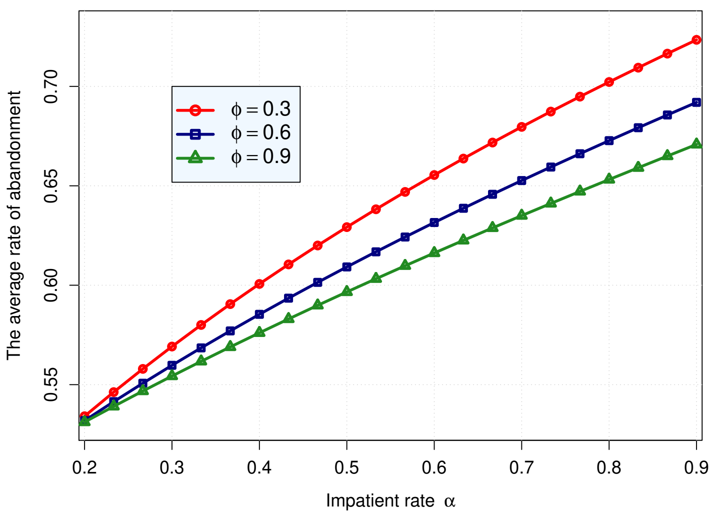

Figure 8: Effect of α and ϕ on R_a.

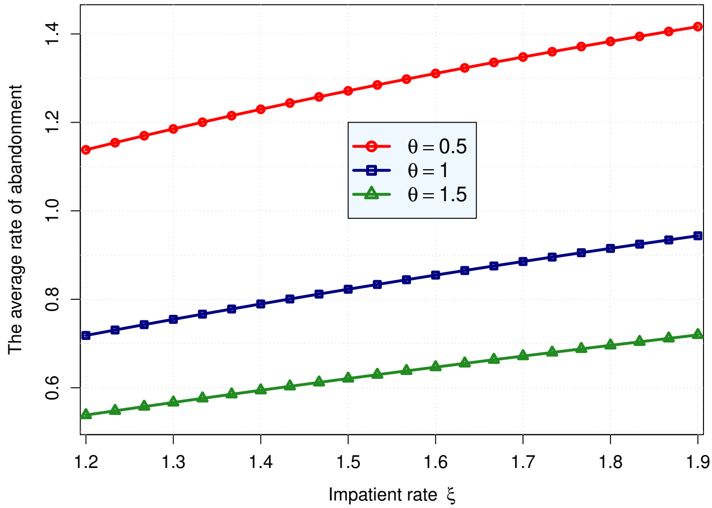

Figure 9: Effect of ξ and θ on R_a.

8.1 Discussion

– From Table 1 and Figure 2, we easily observe that the curve is convex. This proves that there exists some value of the service rate μ_b that minimizes the total expected cost function for the chosen set of model parameters. By adopting QFSM and choosing the initial 3-point pattern as $(\mu^l, \mu^m, \mu^u) = (5.10, 5.40, 5.70)$, and after finite iterations, we see that the minimum expected operating cost per unit time converges to the solution $\mathcal{T}_c = 391.856912$ at $\mu_b^* = 5.643031$.

– From Tables 2-5, we have:

– As intuitively expected, the optimum cost function $\mathcal{T}_c(\mu_b^*)$ increases with (λ), (μ_v), and (ϕ) and decreases with $(\overline{q})$, (p), (ξ), (α), and (θ). With the increasing of the arrival rate, the mean system size increases significantly. This increases significantly the optimum cost function $\mathcal{T}_c(\mu_b^*)$. Obviously, the increasing of the vacation rate increases the probability of the regular busy period which in turns decreases the mean system size. This results in the decreasing of the minimum expected cost. Further, the impatience rates either during vacation or working vacation periods lead to the decreasing of the mean number of customers in the systems which implies a decreasing in the optimal expected cost. Then, when the probability with which the server resumes its service during working vacation period to the regular service increases the customers are served faster. Consequently, $\mathcal{T}_c(\mu_b^*)$ decreases. The same when the probability that the server switches to the vacation period at which the customers may get impatient and leave the system. This yields to the decreasing of the mean number of customers in the system and consequently the total expected cost decreases accordingly. In addition, the decreasing of the optimum cost function $\mathcal{T}_c(\mu_b^*)$ with (ϕ) can be due to the choice of the system parameters.

– The average rate of abandonment (R_a) increases with (ξ) and (α) and decreases with (θ) and (ϕ). This is quite reasonable; the higher the impatience rate (resp. vacation and working vacation rate), the greater (resp. the lower) the average rate of reneging (R_a) and the smaller the mean number of customers in the system $(E(L_0))$ and $(E(L_1))$.

– With the increasing of (μ_b) and $(\bar{q})$, the mean number of customers in the system decreases. Obviously, the smaller (resp. greater) the mean service rate during regular busy period (resp. the probability that the server switches to the regular busy period), the higher the mean number of customers served and the smaller the mean system size during this period $(E(L_2))$.

– As it should be, the service rate (μ_v) decreases the probability that the server is in regular period $(P_2(1))$ and increases the probabilities that the server is on vacation and working vacation periods $(P_1(1))$ and $(P_0(1))$ respectively. Further, obviously, the increasing of the arrival rate (λ) increases $(P_0(1))$, $(P_1(1))$, and $(P_2(1))$.

9 Conclusion

The steady-state solution of an infinite-space single-server Markovian queueing system with working vacation (WV), Bernoulli schedule vacation interruption, and impatient customers has been presented. The proposed queueing system can be applied in diverse real life situations of day-to-day as well as industrial congestion problems including call centers, telecommunication networks, manufacturing system, and so on. The analytical results using probability generating function (PGF) technique are obtained. The performance indices derived may be helpful to the decision makers for improving the availability of the server.

References

[1] E. Altman, U. Yechiali, Analysis of customers' impatience in queues with server vacation, *Queueing Syst*, **52** (4) (2006), 261–279.

[2] A. A. Bouchentouf, A. Guendouzi, Sensitivity analysis of multiple vacation feedback queueing system with differentiated vacations, vacation interruptions and impatient customers, *International Journal of Applied Mathematics and Statistics*, **57** (6) (2018), 104–121.

[3] A. A. Bouchentouf, A. Guendouzi, Cost optimization analysis for an $\mathsf{M^X/M/c}$ vacation queueing system with waiting servers and impatient customers, *SeMA Journal*, **76** (2) (2019), 309–341.

[4] A. A. Bouchentouf, A. Guendouzi, The $\mathsf{M^X/M/c}$ Bernoulli feedback queue with variant multiple working vacations and impatient customers: Performance and economic analysis, *Arab. J. Math.*, **9** (2020), 309–327.

[5] A. A. Bouchentouf, A. Guendouzi, A. Kandouci, Performance and economic analysis of Markovian Bernoulli feedback queueing system with vacations, waiting server and impatient customers, *Acta Universitatis Sapientiae, Mathematica*, **10** (2) (2018), 218–241.

[6] A. A. Bouchentouf, A. Guendouzi, A. Kandouci, Performance and economic study of heterogeneous $\mathsf{M/M/2/N}$ feedback queue with working vacation and impatient customers, *ProbStat Forum*, **12** (1) (2019), 15–35.

[7] A. A. Bouchentouf, A. Guendouzi, S. Majid, On impatience in Markovian $\mathsf{M/M/1/N/DWV}$ queue with vacation interruption, *Croatian Operational Research Review*, **11** (1) (2020) 21–37.

[8] A. A. Bouchentouf, L. Yahiaoui, On feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption, *Arab. J. Math.*, **6** (1) (2017), 1–11.

[9] B. T. Doshi, Queueing systems with vacation-a survey, *Queueing Syst.*, **1** (1) (1986), 29–66.

[10] S. Gao, Z. Liu, An $\mathsf{M/G/1}$ queue with single working vacation and vacation interruption under Bernoulli schedule, *Applied Mathematical Modelling*, **37** (3) (2013), 1564–1579.

[11] M. Jain, G. C. Sharma, R. Sharma, Working vacation queue with service interruption and multi-optional repair, *Intentional Journal of Information and Management Science*, **22** (2011), 157– 175.

[12] J. Keilson, L. D. Servi, A distribution form of Little's law, *Operations Research Letters*, **7** (5) (1988), 223–227.

[13] J. Li, N. Tian, The $\mathsf{M/M/1}$ queue with working vacations and vacation interruptions, *Journal of Systems Science and Systems Engineering*, **16** (1) (2007), 121–127.

[14] J. Li, N. Tian, The discrete-time $\mathsf{GI/Geo/1}$ queue with working vacations and vacation interruption, *Applied Mathematics and Computation*, **185** (1) (2007), 1–10.

[15] J. H. Li, N. S. Tian, Z. Y. Ma, Performance analysis of $\mathsf{GI/M/1}$ queue with working vacations, *Operations Research Letters*, **35** (5) (2007), 595–600.

[16] J. Li, N. Tian, Z. Ma, Performance analysis of $\mathsf{GI/M/1}$ queue with working vacations and vacation interruption, *Applied Mathematical Modelling*, **32** (12) (2008), 2715–2730.

[17] K. C. Madan, M. Al-Rawwash, On the $\mathsf{M^X/G/1}$ queue with feedback and optional server vacations based on a single vacation policy, *Applied Mathematics and Computation*, **160** (3) (2005), 909–919.

[18] S. Majid, P. Manoharan, Analysis of a $\mathsf{M/M/c}$ queue with single and multiple synchronous working vacations, *Applications and Applied Mathematics*, **12** (2) (2017), 671–694.

[19] S. Majid, P. Manoharan, Impatient customers in an $\mathsf{M/M/c}$ queue with Single and Multiple Synchronous Working Vacations, *Pakistan journal statistics and operation research*, **XIV** (3) (2018), 571–594.

[20] S. Majid, P. Manoharan, Analysis of an $\mathsf{M/M/1}$ Queue with Working Vacation and Vacation Interruption, *Applications and Applied Mathematics*, **14** (1) (2019), 19–33.

[21] R. Padmavathy, K. Kalidass, K. Ramanath, Vacation queues with impatient customers and a waiting server, *Int. J. Latest Trends Softw. Eng.*, **1** (1) (2011), 10–19.

[22] P. Rajadurai, M. C. Saravanarajan, V. M. Chandrasekaran, A study on M/G/1 feedback retrial queue with subject to server breakdown and repair under multiple working vacation policy, *Alexandria Engineering Journal*, **57** (2) (2018), 947–962.

[23] N. Selvaraju, C. Goswami, Impatient customers in an M/M/1 queue with single and multiple working vacations, *Computers and Industrial Engineering*, **65** (2) (2013), 207–215.

[24] L. D. Servi, S. G. Finn, M/M/1 queues with working vacations (M/M/1/WV), *Performance Evaluation*, **50** (1) (2002), 41–52.

[25] W. Sun, S. Li, Q. Li: Equilibrium balking strategies of customers in Markovian queues with two-stage working vacations, *Applied Mathematics and Computation*, **248** (2014), 195– 214.

[26] H. Takagi, Queueing Analysis: A Foundation of Performance Evaluation, Volume 1: Vacation and Priority System, Elsevier, Amsterdam (1991).

[27] N. Tian, Z. G. Zhang, Vacation Queueing Models-Theory and Applications, Springer-Verlag, New York (2006).

[28] D. A. Wu, H. Takagi, An M/G/1 queues with multiple working vacations and exhaustive service discipline, *Performance Evaluation*, **63** (7) (2006), 654–681.

[29] M. Yu, A. S. Alfa, Strategic queueing behavior for individual and social optimization in managing discrete time working vacation queue with Bernoulli interruption schedule, *Computers &Operations Research*, **73** (C) (2016), 43–55.

[30] D. Yue, W. Yue, G. Zhao, Analysis of an M/M/1 queue with vacations and impatience timers which depends on the server's states, *Journal of Industrial & Management Optimization*, **12** (2) (2016), 653–666.

[31] Zhang, M., Hou, Z.: Performance analysis of M/G/1 queue with working vacations and vacation interruption, *Journal of Computational and Applied Mathematics*, **234** (10) (2010), 2977–2985.

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Some integral inequalities for a polynomial with zeros outside the unit disk

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Abstract. The goal of this paper is to generalize and refine some previous inequalities between the L^p- norms of the $s\text{th}$ derivative and of the polynomial itself, in the case when the zeros are outside of the open unit disk.

1 Introduction

Let $\mathbb{P}_n$ be the class of polynomials $P(z) := \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ of degree n and $P^{(s)}(z)$ is its s^{th} derivative. For $P \in \mathbb{P}_n$, we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \tag{1}$$

and for every $r \geq 1$,

$$\left\{ \int_0^{2\pi} \left|P'(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}} \leq n \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}}. \tag{2}$$

The inequality (1) is a classical result of Bernstein [10], where as the inequality (2) is due to Zygmund [14] who proved it for all trigonometric polynomials of

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degree n and not only for those of the form $P(e^{i\theta})$. Arestov [1] proved that (2) remains true for $0 < r < 1$ as well. If we let $r \to \infty$ in (2) we get (1).

The above two inequalities (1) and (2) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact, if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then (1) and (2) can be respectively replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{3}$$

and

$$\left\{ \int_0^{2\pi} \left| P'(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} \leq nC_r \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}, \tag{4}$$

where

$$C_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + e^{i\gamma} \right|^r d\gamma \right\}^{\frac{-1}{r}}. \tag{5}$$

The inequality (3) was conjectured by Erdös and later verified by Lax [9], where as (4) was proved by De-Bruijn [6] for $r \geq 1$. Further, Rahman and Schmeisser [12] have shown that (4) holds for $0 < r < 1$ as well. If we let $r \to \infty$ in inequality (4), we get (3).

In the literature, there already exists various refinements and generalisations of (3) and (4), for example see Aziz [2], Aziz and Dawood [3], Mir and Baba [11], Zireh [13] etc.

2 Main results

In this paper, we shall use a parameter β and obtain certain generalisations and refinements of inequalities (3) and (4).

Theorem 1 *If* $P \in \mathbb{P}_n$ *and* $P(z) \neq 0$ *in* $|z| < 1$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \leq 1, 1 \leq s \leq n$ *and* $\gamma \geq 1$,

$$\begin{aligned} &\left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)...(n-s+1)}{2^s} P(e^{i\theta}) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \\ &\leq n(n-1)...(n-s+1) E_\gamma \left(1 + \frac{|\beta|}{2^{s-1}} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \end{aligned} \tag{6}$$

where

$$E_\gamma = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| 1 + e^{i\alpha} \right|^\gamma d\alpha \right\}^{\frac{-1}{\gamma}}. \tag{7}$$

Instead of proving Theorem 1, we prove the following more general result which includes not only Theorem 1 as a special case, but also leads to several interesting generalisations and refinements of (3) and (4).

Theorem 2 *If* $P \in \mathbb{P}_n$ *and* $P(z) \neq 0$ *in* $|z| < 1$, *then for every* $\beta, \delta \in \mathbb{C}$ *with* $|\beta| \leq 1, |\delta| \leq 1, 1 \leq s \leq n$ *and* $\gamma \geq 1$,

$$\begin{aligned} &\left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)...(n-s+1)}{2^s} P(e^{i\theta}) \right. \right. \\ &\quad \left. \left. + \frac{\delta m n(n-1)...(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \right|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \\ &\leq n(n-1)...(n-s+1) E_\gamma \left(1 + \frac{|\beta|}{2^{s-1}} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \end{aligned} \tag{8}$$

where $m = \min_{|z|=1} |P(z)|$ *and* E_γ *is defined by* (7).

Now we present and discuss some consequences of these results. First, we point out that inequalities involving polynomials in the Chebyshev norm on the unit circle in the complex plane are a special case of the polynomial inequalities involving the integral norm. For example if we let $\gamma \to \infty$ in (6), noting that $E_\gamma \to \frac{1}{2}$ we get the following result.

Corollary 1 *If* $P \in \mathbb{P}_n$ *and* $P(z) \neq 0$ *in* $|z| < 1$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \leq 1$ *and* $1 \leq s \leq n$,

$$\begin{aligned} &\max_{|z|=1} \left| z^s P^{(s)}(z) + \frac{\beta n(n-1)...(n-s+1)}{2^s} P(z) \right| \\ &\leq \frac{n(n-1)...(n-s+1)}{2} \left(1 + \frac{|\beta|}{2^{s-1}} \right) \max_{|z|=1} |P(z)|. \end{aligned} \tag{9}$$

If we take $s = 1$ *in* (9), *we get the following generalization of* (3).

Corollary 2 *If* $P \in \mathbb{P}_n$ *and* $P(z) \neq 0$ *in* $|z| < 1$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \leq 1$,

$$\max_{|z|=1}\left|zP'(z) + \frac{\beta n}{2}P(z)\right| \leq \frac{n}{2}(1 + |\beta|)\max_{|z|=1}|P(z)|. \tag{10}$$

Remark 1 *For* $\beta = 0$, (10) *reduces to* (3). *It should be noted that inequality* (10) *can also be obtained by simply applying the triangle inequality to the left hand side of it and estimating the first of the resulting terms directly by inequality* (3).

Next, we show that Theorem 2 implies other inequalities in the Chebyshev norm on the unit circle of a polynomial. If we let $\gamma \to \infty$ *in* (8) *and choose the argument of* δ *suitably with* $|\delta| = 1$, *we get the following result.*

Corollary 3 *If* $P \in \mathbb{P}_n$ *and* $P(z) \neq 0$ *in* $|z| < 1$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \leq 1$ *and* $1 \leq s \leq n$,

$$\begin{aligned}\max_{|z|=1}&\left|z^s P^{(s)}(z) + \frac{\beta n(n-1)...(n-s+1)}{2^s}P(z)\right| \\ &\leq \frac{n(n-1)...(n-s+1)}{2}\left\{\left(1 + \frac{|\beta|}{2^{s-1}}\right)\max_{|z|=1}|P(z)| \right. \\ &\qquad \left. - \left(\left|1 + \frac{\beta}{2^s}\right| - \left|\frac{\beta}{2^s}\right|\right)m\right\},\end{aligned} \tag{11}$$

where $m =$ $min_{|z|=1}|P(z)|$.

Taking $s = 1$ *in inequality* (11), *we get the following refinement of* (10).

Corollary 4 *If* $P \in \mathbb{P}_n$ *and* $P(z) \neq 0$ *in* $|z| < 1$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \leq 1$,

$$\max_{|z|=1}\left|zP'(z)+\frac{\beta n}{2}P(z)\right| \leq \frac{n}{2}\left\{(1+|\beta|)\max_{|z|=1}|P(z)| - \left(\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right)m\right\}, \tag{12}$$

where $m =$ $min_{|z|=1}|P(z)|$.

When $\beta = 0$, *the above inequality* (12) *recovers a result of Aziz and Dawood* [1]. *Several other interesting results easily follow from Theorem 2 and here, we mention a few of these. Taking* $\beta = 0$ *in* (8), *we immediately get the following result.*

Corollary 5 *c2.5 If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\delta \in \mathbb{C}$ with $|\delta| \leq 1, 1 \leq s \leq n$ and $\gamma \geq 1$,*

$$\begin{aligned}
&\left\{\int_0^{2\pi} \left|e^{is\theta}P^{(s)}(e^{i\theta}) + \frac{\delta m n(n-1)\dots(n-s+1)}{2}\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}} \\
&\quad \leq n(n-1)\dots(n-s+1)E_\gamma\left\{\int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta\right\}^{\frac{1}{\gamma}},
\end{aligned} \tag{13}$$

where $m =$ $\min_{|z|=1}|P(z)|$ and E_γ is defined by (7).

For $s = 1$ and $\delta = 0$, inequality (13) reduces to inequality (4).

Letting $\gamma \to \infty$ in (13) and choosing the argument of δ with $\delta = 1$, we get the following interesting generalization of a result of Aziz and Dawood [1].

Corollary 6 *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $1 \leq s \leq n$, we have*

$$\max_{|z|=1}|P^{(s)}(z)| \leq \frac{n(n-1)\dots(n-s+1)}{2}\left(\max_{|z|=1}|P(z)| - \min_{|z|=1}|P(z)|\right). \tag{14}$$

For the proof of Theorem 2, we need the following lemmas.

3 Lemmas

Lemma 1 *Let $F \in \mathbb{P}_n$ and $F(z)$ has all its zeros in $|z| \leq 1$. If $P(z)$ is a polynomial of degree n such that*

$$|P(z)| \leq |F(z)| \text{ for } |z| = 1,$$

then for any $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $1 \leq s \leq n$,

$$\begin{aligned}
&\left|z^s P^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s}P(z)\right| \\
&\leq \left|z^s F^{(s)}(z) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s}F(z)\right| \text{ for } |z| \geq 1.
\end{aligned}$$

The above lemma is due to Hans and Lal [8].

By applying Lemma 1 to polynomials $P(z)$ and $z^n \min_{|z|=1}|P(z)|$, we get the following result.

Lemma 2 *If* $P \in \mathbb{P}_n$ *and* $P(z)$ *has all its zeros in* $|z| \leq 1$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \leq 1, 1 \leq s \leq n$,

$$\begin{aligned}&\left|z^s P^{(s)}(z) + \frac{\beta n(n-1)...(n-s+1)}{2^s} P(z)\right| \\ &\qquad \geq n(n-1)...(n-s+1)|z|^n \left|1 + \frac{\beta}{2^s}\right| \min_{|z|=1} |P(z)| \text{ for } |z| \geq 1.\end{aligned}$$

Lemma 3 *If* $P \in \mathbb{P}_n$ *and* $P(z) \neq 0$ *in* $|z| < 1$, *then for every* $\beta \in \mathbb{C}$ *with* $|\beta| \leq 1, 1 \leq s \leq n$ *and* $|z| = 1$,

$$\begin{aligned}&\left|z^s P^{(s)}(z) + \frac{\beta n(n-1)...(n-s+1)}{2^s} P(z)\right| \\ &\quad \leq \left|z^s Q^{(s)}(z) + \frac{\beta n(n-1)...(n-s+1)}{2^s} Q(z)\right| \\ &\qquad\qquad - n(n-1)...(n-s+1)\left\{\left|1 + \frac{\beta}{2^s}\right| - \left|\frac{\beta}{2^s}\right|\right\} m,\end{aligned}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ *and* $m = \min_{|z|=1} |P(z)|$.

Proof of Lemma 3.3. If $P(z)$ has a zero on $|z| = 1$, then $m = 0$ and the result follows by Lemma 1. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| > 1$ and so $m > 0$, we have $|\lambda m| < |P(z)|$ on $|z| = 1$ for any λ with $|\lambda| < 1$. It follows by Rouche's theorem that the polynomial $G(z) = P(z) - \lambda m$ has no zeros in $|z| < 1$. Therefore, the polynomial $H(z) = z^n \overline{G(\frac{1}{\bar{z}})} = Q(z) - m\bar{\lambda} z^n$ will have all its zeros in $|z| \leq 1$. Also $|G(z)| = |H(z)|$ for $|z| = 1$. On applying Lemma 1, we get for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1, 1 \leq s \leq n$ and $|z| \geq 1$,

$$\begin{aligned}&\left|z^s G^{(s)}(z) + \frac{\beta n(n-1)...(n-s+1)}{2^s} G(z)\right| \\ &\qquad\qquad \leq \left|z^s H^{(s)}(z) + \frac{\beta n(n-1)...(n-s+1)}{2^s} H(z)\right|.\end{aligned}$$

Equivalently

$$\begin{aligned}&\left|z^s P^{(s)}(z) + \frac{\beta n(n-1)...(n-s+1)}{2^s} (P(z) - \lambda m)\right| \\ &\qquad\qquad \leq \left|\left(z^s Q^{(s)}(z) - \bar{\lambda} m n(n-1)...(n-s+1) z^n\right)\right.\end{aligned}$$

$$+\frac{\beta n(n-1)\dots(n-s+1)}{2^s}(Q(z)-\bar{\lambda}mz^n)\Bigg|.$$

This implies that

$$\begin{aligned}&\Bigg|\left(z^sP^{(s)}(z)+\frac{\beta n(n-1)\dots(n-s+1)}{2^s}P(z)\right)\\ &\quad-\frac{\beta n(n-1)\dots(n-s+1)}{2^s}\lambda m\Bigg|\\ &\le\Bigg|\left(z^sQ^{(s)}(z)+\frac{\beta n(n-1)\dots(n-s+1)}{2^s}Q(z)\right)\\ &\quad-\bar{\lambda}mn(n-1)\dots(n-s+1)z^n\left(1+\frac{\beta}{2^s}\right)\Bigg|.\end{aligned}\tag{15}$$

Since $Q(z)$ has all its zeros in $|z|\le 1$, therefore, by Lemma 3.2 we have for every $\beta\in\mathbb{C}$ with $|\beta|\le 1$ and $|z|\ge 1$,

$$\begin{aligned}&\left|z^sQ^{(s)}(z)+\frac{\beta n(n-1)\dots(n-s+1)}{2^s}Q(z)\right|\\ &\quad\ge n(n-1)\dots(n-s+1)|z|^n\left|1+\frac{\beta}{2^s}\right|\min_{|z|=1}|Q(z)|\\ &\quad= n(n-1)\dots(n-s+1)|z|^n\left|1+\frac{\beta}{2^s}\right|m.\end{aligned}\tag{16}$$

Now choosing a suitable argument of λ in the left-hand side of (15), in view of (16), we get for $|z|=1$,

$$\left|z^sP^{(s)}(z)+\frac{\beta n(n-1)\dots(n-s+1)}{2^s}P(z)\right|-|\lambda|n(n-1)\dots(n-s+1)\left|\frac{\beta}{2^s}\right|m$$

$$\begin{aligned}&\le\left|z^sQ^{(s)}(z)+\frac{\beta n(n-1)\dots(n-s+1)}{2^s}Q(z)\right|\\ &\qquad-n(n-1)\dots(n-s+1)\left|1+\frac{\beta}{2^s}\right||\lambda|m.\end{aligned}$$

Equivalently

$$\left|z^sP^{(s)}(z)+\frac{\beta n(n-1)\dots(n-s+1)}{2^s}P(z)\right|$$

$$\leq \left| z^s Q^{(s)}(z) + \frac{\beta n(n-1)...(n-s+1)}{2^s} Q(z) \right|$$
$$- n(n-1)...(n-s+1)\left(\left|1+\frac{\beta}{2^s}\right| - \left|\frac{\beta}{2^s}\right| \right)|\lambda| m.$$

Letting $|\lambda| \to 1$, we get for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $|z| = 1$,

$$\left| z^s P^{(s)}(z) + \frac{\beta n(n-1)...(n-s+1)}{2^s} P(z) \right|$$

$$\leq \left| z^s Q^{(s)}(z) + \frac{\beta n(n-1)...(n-s+1)}{2^s} Q(z) \right|$$
$$- n(n-1)...(n-s+1)\left(\left|1+\frac{\beta}{2^s}\right| - \left|\frac{\beta}{2^s}\right| \right) m,$$

which completes the proof of Lemma 3

The following lemma is due to Aziz and Rather [5].

Lemma 4 *If* A, B, C *are non-negative real numbers such that* $B + C \leq A$, *then for every real number* α,

$$\left|(A-C)e^{i\alpha} + (B+C)\right| \leq \left|Ae^{i\alpha} + B\right|.$$

Lemma 5 *If* $P \in \mathbb{P}_n$ *and* $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, *then for each* $\alpha, 0 \leq \alpha < 2\pi$ *and* $\gamma > 0$, *we have*

$$\int_0^{2\pi}\int_0^{2\pi} \left|P'(e^{i\theta}) + e^{i\alpha} Q'(e^{i\theta})\right|^\gamma d\theta d\alpha \leq 2\pi n^\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^\gamma d\theta.$$

The above lemma is due to Aziz and Rather [4].

4 Proof of the Theorem

Proof of Theorem 2.2. Since $P \in \mathbb{P}_n, P(z) \neq 0$ in $|z| < 1$ and $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, therefore, for each $\alpha, 0 \leq \alpha < 2\pi, F(z) = P(z) + e^{i\alpha} Q(z) \in \mathbb{P}_n$ and we have

$$F^{(s)}(z) = P^{(s)}(z) + e^{i\alpha} Q^{(s)}(z),$$

which is clearly a polynomial of degree $n-s, 1 \leq s \leq n$. By the repeated application of inequality (2), we have for each $\gamma \geq 1$,

$$\begin{aligned}
&\int_0^{2\pi} \left|P^{(s)}(e^{i\theta}) + e^{i\alpha} Q^{(s)}(e^{i\theta})\right|^\gamma d\theta \\
&\leq (n-s+1)^\gamma \int_0^{2\pi} \left|P^{(s-1)}(e^{i\theta}) + e^{i\alpha} Q^{(s-1)}(e^{i\theta})\right|^\gamma d\theta \\
&\vdots \\
&\leq (n-s+1)^\gamma (n-s+2)^\gamma \ldots (n-1)^\gamma \int_0^{2\pi} \left|P'(e^{i\theta}) + e^{i\alpha} Q'(e^{i\theta})\right|^\gamma d\theta.
\end{aligned} \tag{17}$$

Integrating (17) with respect to α on $[0, 2\pi]$ and using Lemma 4 we get

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} \left|P^{(s)}(e^{i\theta}) + e^{i\alpha} Q^{(s)}(e^{i\theta})\right|^\gamma d\theta d\alpha \\
&\leq 2\pi (n-s+1)^\gamma (n-s+2)^\gamma \ldots (n-1)^\gamma n^\gamma \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta.
\end{aligned} \tag{18}$$

Now by Lemma 3 for each $\theta, 0 \leq \theta < 2\pi$, $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $1 \leq s \leq n$,

$$\begin{aligned}
&\left|e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\ldots(n-s+1)}{2^s} P(e^{i\theta})\right| \\
&\qquad + \frac{mn(n-1)\ldots(n-s+1)}{2}\left(\left|1+\frac{\beta}{2^s}\right| - \left|\frac{\beta}{2^s}\right|\right) \\
&\qquad \leq \left|e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\ldots(n-s+1)}{2^s} Q(e^{i\theta})\right| \\
&\qquad - \frac{mn(n-1)\ldots(n-s+1)}{2}\left(\left|1+\frac{\beta}{2^s}\right| - \left|\frac{\beta}{2^s}\right|\right).
\end{aligned} \tag{19}$$

Taking

$$A = \left|e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\ldots(n-s+1)}{2^s} Q(e^{i\theta})\right|,$$

$$B = \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right|$$

and

$$C = \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right)$$

in Lemma 3.4, so that by (19),

$$B + C \le A - C \le A,$$

we get for every real α,

$$\begin{aligned}
&\Bigg| \Bigg\{ \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right| \\
&\qquad + \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \Bigg\} \\
&\qquad + e^{i\alpha} \Bigg\{ \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right| \\
&\qquad - \frac{mn(n-1)\dots(n-s+1)}{2} \left(\left| 1 + \frac{\beta}{2^s} \right| - \left| \frac{\beta}{2^s} \right| \right) \Bigg\} \Bigg| \\
&\le \Bigg| \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right| \\
&\qquad + e^{i\alpha} \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right| \Bigg|.
\end{aligned}$$

This implies for each $\gamma \ge 1$,

$$\begin{aligned}
\int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^\gamma &\le \int_0^{2\pi} \Bigg| \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} P(e^{i\theta}) \right. \\
&\quad + e^{i\alpha} \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)\dots(n-s+1)}{2^s} Q(e^{i\theta}) \right| \Bigg|^\gamma d\theta,
\end{aligned} \tag{20}$$

where

$$F(\theta)=\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n(n-1)...(n-s+1)}{2^s}P(e^{i\theta})\right| \\ +\frac{mn(n-1)...(n-s+1)}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right),$$

and

$$G(\theta)=\left|e^{is\theta}Q^{(s)}(e^{i\theta})+\frac{\beta n(n-1)...(n-s+1)}{2^s}Q(e^{i\theta})\right| \\ -\frac{mn(n-1)...(n-s+1)}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right).$$

Integrating both sides of(20) with respect to α from 0 to 2π, we get for each $\gamma\geq 1$,

$$\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^{\gamma}d\theta d\alpha$$

$$\leq\int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n(n-1)...(n-s+1)}{2^s}P(e^{i\theta})\right|\right.\right.$$

$$\left.\left.+e^{i\alpha}\left|e^{is\theta}Q^{(s)}(e^{i\theta})+\frac{\beta n(n-1)...(n-s+1)}{2^s}Q(e^{i\theta})\right|\right|^{\gamma}d\alpha\right\}d\theta$$

$$=\int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left(e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n(n-1)...(n-s+1)}{2^s}P(e^{i\theta})\right)\right.\right.$$

$$\left.\left.+e^{i\alpha}\left(e^{is\theta}Q^{(s)}(e^{i\theta})+\frac{\beta n(n-1)...(n-s+1)}{2^s}Q(e^{i\theta})\right)\right|^{\gamma}d\alpha\right\}d\theta$$

$$=\int_0^{2\pi}\left\{\int_0^{2\pi}\left|e^{is\theta}\left(P^{(s)}(e^{i\theta})+e^{i\alpha}Q^{(s)}(e^{i\theta})\right)\right.\right.$$

$$\left.\left.+\frac{\beta n(n-1)...(n-s+1)}{2^s}\left(P(e^{i\theta})+e^{i\alpha}Q(e^{i\theta})\right)\right|^{\gamma}d\alpha\right\}d\theta.$$

Therefore, it follows by Minkowski's inequality that for $\gamma \geq 1$,

$$
\begin{aligned}
&\left\{\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^{\gamma}d\theta d\alpha\right\}^{\frac{1}{\gamma}}\\
&\leq\left\{\int_0^{2\pi}\int_0^{2\pi}\left|e^{is\theta}\Big(P^{(s)}(e^{i\theta})+e^{i\alpha}Q^{(s)}(e^{i\theta})\Big)\right.\right.\\
&\quad\left.\left.+\frac{\beta n(n-1)...(n-s+1)}{2^s}\Big(P(e^{i\theta})+e^{i\alpha}Q(e^{i\theta})\Big)\right|^{\gamma}d\theta d\alpha\right\}^{\frac{1}{\gamma}} \qquad (21)\\
&\leq\left\{\int_0^{2\pi}\int_0^{2\pi}\left|P^{(s)}(e^{i\theta})+e^{i\alpha}Q^{(s)}(e^{i\theta})\right|^{\gamma}d\theta d\alpha\right\}^{\frac{1}{\gamma}}\\
&\quad+\frac{|\beta|n(n-1)...(n-s+1)}{2^s}\left\{\int_0^{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})+e^{i\alpha}Q(e^{i\theta})\right|^{\gamma}d\theta d\alpha\right\}^{\frac{1}{\gamma}}.
\end{aligned}
$$

For $0 \leq \theta < 2\pi$, it can be easily verified that

$$nP(e^{i\theta})-e^{i\theta}P'(e^{i\theta})=e^{i(n-1)\theta}\overline{Q'(e^{i\theta})}$$

and

$$nQ(e^{i\theta})-e^{i\theta}Q'(e^{i\theta})=e^{i(n-1)\theta}\overline{P'(e^{i\theta})}.$$

Hence

$$
\begin{aligned}
nP(e^{i\theta})+e^{i\alpha}nQ(e^{i\theta})&=e^{i\theta}P'(e^{i\theta})+e^{i(n-1)\theta}\overline{Q'(e^{i\theta})}\\
&\quad+e^{i\alpha}\big(e^{i\theta}Q'(e^{i\theta})+e^{i(n-1)\theta}\overline{P'(e^{i\theta})}\big),
\end{aligned}
$$

which gives

$$
\begin{aligned}
n|P(e^{i\theta})+e^{i\alpha}Q(e^{i\theta})|&\leq|P'(e^{i\theta})+e^{i\alpha}Q'(e^{i\theta})|+|\overline{Q'(e^{i\theta})}+e^{i\alpha}\overline{P'(e^{i\theta})}|\\
&=2|P'(e^{i\theta})+e^{i\alpha}Q'(e^{i\theta})|.
\end{aligned} \qquad (22)
$$

Using (18), (22) and Lemma in (21), we get for every $\gamma \geq 1$, $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $1 \leq s \leq n$,

$$\begin{aligned}\left\{\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^{\gamma}d\theta d\alpha\right\}^{\frac{1}{\gamma}} \\ \leq (2\pi)^{\frac{1}{\gamma}}n(n-1)\dots(n-s+1)\left(1+\frac{|\beta|}{2^{s-1}}\right)\left\{\int_0^{2\pi}|P(e^{i\theta})|^{\gamma}d\theta\right\}^{\frac{1}{\gamma}}.\end{aligned} \tag{23}$$

Now for every real α and $t \geq 1$, it is easy to verify that

$$|t+e^{i\alpha}| \geq |1+e^{i\alpha}|.$$

Observe that for every $\gamma \geq 1$ and $a, b \in \mathbb{C}$ such that $|b| \geq |a|$, we have

$$\int_0^{2\pi}|a+be^{i\alpha}|^{\gamma}d\alpha \geq |a|^{\gamma}\int_0^{2\pi}|1+e^{i\alpha}|^{\gamma}d\alpha. \tag{24}$$

Indeed, if $a = 0$, the above inequality (24) is obvious. In case of $a \neq 0$, we get

$$\begin{aligned}\int_0^{2\pi}\left|1+e^{i\alpha}\frac{b}{a}\right|d\alpha = \int_0^{2\pi}\left|1+e^{i\alpha}\left|\frac{b}{a}\right|\right|d\alpha = \int_0^{2\pi}\left|\left|\frac{b}{a}\right|+e^{i\alpha}\right|d\alpha \\ \geq \int_0^{2\pi}\left|1+e^{i\alpha}\right|^{\gamma}d\alpha.\end{aligned}$$

If we take

$$a = F(\theta),$$

$$b = G(\theta),$$

because $|b| \geq |a|$ from (19), we get from (24) that

$$\int_0^{2\pi}|F(\theta)+e^{i\alpha}G(\theta)|^{\gamma}d\alpha \geq |F(\theta)|^{\gamma}\int_0^{2\pi}|1+e^{i\alpha}|^{\gamma}d\alpha. \tag{25}$$

Integrating both sides of (25) with respect to θ from 0 to 2π, we get from (23), that

$$\left\{\int_0^{2\pi}|1+e^{i\alpha}|^\gamma d\alpha\int_0^{2\pi}\left(\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n(n-1)\ldots(n-s+1)}{2^s}P(e^{i\theta})\right|\right.\right.$$
$$\left.\left.+\frac{mn(n-1)\ldots(n-s+1)}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)\right)^\gamma d\theta\right\}^{\frac{1}{\gamma}} \tag{26}$$
$$\leq(2\pi)^{\frac{1}{\gamma}}n(n-1)\ldots(n-s+1)\left(1+\frac{|\beta|}{2^{s-1}}\right)\left\{\int_0^{2\pi}|P(e^{i\theta})|^\gamma d\theta\right\}^{\frac{1}{\gamma}}.$$

Now using the fact that for every $\delta\in\mathbb{C}$ with $|\delta|\leq 1$,

$$\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n(n-1)\ldots(n-s+1)}{2^s}P(e^{i\theta})\right.$$
$$\left.+\frac{\delta mn(n-1)\ldots(n-s+1)}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)\right|\leq\left|e^{is\theta}P^{(s)}(e^{i\theta})\right.$$
$$\left.+\frac{\beta n(n-1)\ldots(n-s+1)}{2^s}P(e^{i\theta})\right|$$
$$+\frac{mn(n-1)\ldots(n-s+1)}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right),$$

we get from (26) that for every $\gamma\geq 1$,

$$\left\{\int_0^{2\pi}\left|e^{is\theta}P^{(s)}(e^{i\theta})+\frac{\beta n(n-1)\ldots(n-s+1)}{2^s}P(e^{i\theta})\right.\right.$$
$$\left.\left.+\frac{\delta mn(n-1)\ldots(n-s+1)}{2}\left(\left|1+\frac{\beta}{2^s}\right|-\left|\frac{\beta}{2^s}\right|\right)\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}$$
$$\leq\frac{(2\pi)^{\frac{1}{\gamma}}n(n-1)\ldots(n-s+1)}{\left\{\int_0^{2\pi}|1+e^{i\alpha}|^\gamma d\alpha\right\}^{\frac{1}{\gamma}}}\left(1+\frac{|\beta|}{2^{s-1}}\right)\left\{\int_0^{2\pi}|P(e^{i\theta})|^\gamma d\theta\right\}^{\frac{1}{\gamma}},$$

which is equivalent to (8) and this completes the proof of Theorem 2

Note. Given a polynomial $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \in \mathbb{P}_n$, we associate with it the polynomial

$$\bar{P}(z) := \overline{P(\bar{z})} = \sum_{\nu=0}^{n} \overline{a_\nu} z^\nu$$

and

$$Q(z) := z^n \overline{P(\frac{1}{\bar{z}})} = \sum_{\nu=0}^{n} \overline{a_{n-\nu}} z^\nu.$$

If $P(z) \equiv \alpha Q(z)$, where $|\alpha| = 1$, then $P(z)$ is said to be self-inversive.
It was shown by Dewan and Govil [7] that the inequality (4) is still valid if the condition that $P(z) \neq 0$ in $|z| < 1$ is replaced by the condition that $P(z) = \alpha Q(z), |\alpha| = 1$. Here we present the following result for self-inversive polynomials.

Preposition. If $P \in \mathbb{P}_n$ is self-inverse, then for every $\beta \in \mathbb{C}, 1 \leq s \leq n$ and $\gamma \geq 1$,

$$\left\{ \int_0^{2\pi} \left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)...(n-s+1)}{2^s} P(e^{i\theta}) \right|^\gamma \right\}^{\frac{1}{\gamma}}$$
$$\leq n(n-1)...(n-s+1) E_\gamma \left(1 + \frac{|\beta|}{2^{s-1}} \right) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}},$$

where E_γ is defined by (7).
Proof. Since $P \in \mathbb{P}_n$ is a self-inversive polynomial, therefore, $P(z) \equiv \alpha Q(z)$, where $|\alpha| = 1$ and $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})} \in \mathbb{P}_n$. This implies for every $\beta \in \mathbb{C}$ and $1 \leq s \leq n$,

$$\left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)...(n-s+1)}{2^s} P(e^{i\theta}) \right|$$
$$= \left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)...(n-s+1)}{2^s} Q(e^{i\theta}) \right|,$$

for all $z \in \mathbb{C}$ so that

$$\frac{\left| e^{is\theta} P^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)...(n-s+1)}{2^s} P(e^{i\theta}) \right|}{\left| e^{is\theta} Q^{(s)}(e^{i\theta}) + \frac{\beta n(n-1)...(n-s+1)}{2^s} Q(e^{i\theta}) \right|} = 1.$$

Now proceeding similarly as in the proof of Theorem 2, the preposition follows.

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References

[1] V. V. Arestov, On integral inequalities for trigonometric polynomials and their derivatives, *Izv. Akad. Nauk. SSSR., Ser. Mat.*, **45** (1981), 3–22.

[2] A. Aziz, Inequalities for the polar derivative of a polynomial, *J. Approx. Theory*, **55** (1988), 183–193.

[3] A. Aziz and Q. M. Dawood, Inequalities for a polynomial and its derivative, *J. Approx. Theory*, **54** (1988), 306–313.

[4] A. Aziz and N. A. Rather, Some Zygmund type L^{q} inequalities for polynomials, *J. Math. Anal. Appl.*, **289** (2004), 14–29.

[5] A. Aziz and N. A. Rather, New L^{q} inequalities for polynomials, *Math. Ineq. Appl.*, **1** (1998), 177–191.

[6] N. G. De-Bruijn, Inequalities concerning polynomials in the complex domain, *Nederl. Akad. Wetnesch Proc.*, **50** (1947), 1265–1272.

[7] K. K. Dewan and N. K. Govil, An inequality for self-inverse polynomials, *J. Math. Anal. Appl.*, **95** (1983), 490.

[8] S. Hans and R. Lal, Generalization of some polynomial inequalities not vanishing in a disk, *Analysis Math.*, **40** (2014), 105–115.

[9] P. D. Lax, Proof of a conjecture of P. Erdös on the derivative of a Polynomial, *Bull. Amer. Math. Soc.*, **50** (1944), 509–513.

[10] G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, *Topics in polynomials, Extremal problems, Inequalities, Zeros,* World Scientific, Singapore, 1994.

[11] A. Mir and S. A. Baba, Some integral inequalities for the polar derivative of a polynomial, *Anal. Theory Appl.*, **27** (2011), 340–350.

[12] Q. I. Rahman and G. Schmeisser, L^p inequalities for polynomials, *J. Approx. Theory*, **53** (1988) 26–32.

[13] A. Zireh, Generalization of certain well-known inequalities for the derivative of polynomials, *Analysis Math.*, **41** (2015), 117–132.

[14] A. Zygmund, A remark on conjugate series, *Proc. London Math. Soc.*, **34** (1932), 392–400.

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Labels distance in bucket recursive trees with variable capacities of buckets

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Abstract. The bucket recursive tree is a natural multivariate structure. In this paper, we apply a trivariate generating function approach for studying of the depth and distance quantities in this tree model with variable bucket capacities and give a closed formula for the probability distribution, the expectation and the variance. We show as $j \to \infty$, limiting distributions are Gaussian. The results are obtained by presenting partial differential equations for moment generating functions and solving them.

1 Introduction

Trees are defined as connected graphs without cycles, and their properties are basics of graph theory. For example, a connected graph is a tree, if and only if the number of edges equals the number of nodes minus 1 [5]. Furthermore,

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each pair of nodes is connected by a unique path. A rooted tree is a tree with a countable number of nodes, in which a particular node is distinguished from the others and called the root node. A recursive tree with n nodes is an unordered rooted tree, where the nodes are labelled by distinct integers from $\{1, 2, 3, ..., n\}$ in such a way that the sequence of labels lying on the unique path from the root node to any node in the tree are always forming an increasing sequence. Call a random recursive tree a tree chosen uniformly at random from the $(n-1)!$ possible recursive trees on n nodes. A random recursive tree can also be constructed as follows. The node 1 is distinguished as the root. We imagine the nodes arriving one by one. For $k \geq 2$, node k attaches itself to a node chosen uniformly at random from $1, 2, ..., k-1$ (for more information and applications, see [11]) .

Mahmoud and Smythe introduced bucket recursive trees as a generalization of random recursive trees [10]. In this model the bucket is a node that can hold up to $b \geq 1$ labels. The capacity of a bucket v ($c = c(v)$) is defined by the number of its labels. They applied a probabilistic analysis for studying the height and depth of the largest label in these trees. Kuba and Panholzer analyzed these trees as a special instance of bucket increasing trees which is a family of some combinatorial objects [8]. They obtained exact and limiting distribution results for the parameters depth of a specified label, descendants of a specified label and degree of a specified label. A (probabilistic) description of random bucket recursive trees is given by a generalization of the stochastic growth rule for ordinary random recursive trees (which are the special instance $b = 1$), where a tree grows by progressive attraction of increasing integer labels: when inserting element $n+1$ into an existing bucket recursive tree containing n elements (i.e., containing the labels $\{1, 2, ..., n\}$) all n existing elements in the tree compete to attract the element $n+1$, where all existing elements have equal chance to recruit the new element. If the element winning this competition is contained in a node with less than b elements (an unsaturated bucket or node), element $n+1$ is added to this node, otherwise if the winning element is contained in a node with already b elements (a saturated bucket or node), element $n+1$ is attached to this node as a new bucket containing only the element $n+1$. Starting with a single bucket as root node containing only element 1 leads after $n-1$ insertion steps, where the labels $2, 3, ..., n$ are successively inserted according to this growth rule, to a so called random bucket recursive tree with n elements and maximal bucket-size b. In this paper we consider a model of bucket trees where the nodes are buckets with variable capacities labelled with integers $1, 2, \cdots, n$ (not the same capacities as bucket recursive trees).

Definition 1 [6] *A size-*n *bucket recursive tree* T_n *with variable bucket capacities and maximal bucket size* b *starts with the root labelled by 1. The tree grows by progressive attraction of increasing integer labels: when inserting label* $j+1$ *into an existing bucket recursive tree* T_j*, except the labels in the non-leaf buckets with capacity* $< b$ *all labels in the tree (containing label 1) compete to attract the label* $j+1$*. For the root node and buckets with capacity* b*, we always produce a new bucket* $j+1$*. But for a leaf with capacity* $c < b$*, either the label* $j+1$ *is attached to this leaf as a new bucket containing only the label* $j+1$ *or is added to that leaf and make a bucket with capacity* $c+1$*. This process ends with inserting the label* n *(i.e., the largest label) in the tree.*

Figure 1 illustrates such a tree of size 19 with $b = 3$.

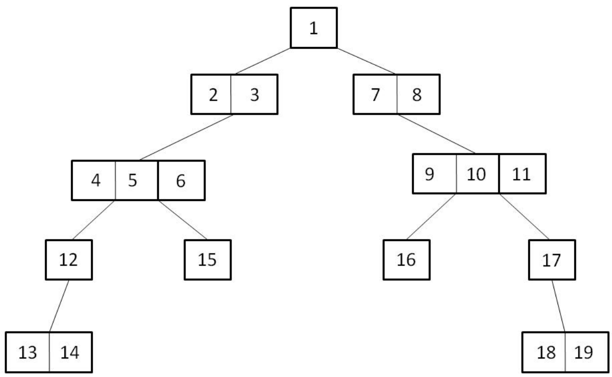

Figure 1: A bucket recursive tree with variable capacities of buckets with 19 elements and $b = 3$.

Bucket recursive trees with variable capacities of buckets are appeared in chemistry, social science, in some computer science applications and furthermore. They are appeared as a model for the spread of epidemics, for pyramid schemes, for the family trees of preserved copies of ancient texts. In the family trees, suppose males with the same ethical traits come together in each generation. Suppose up to 3 people are matched with the same attributes. Then a bucket recursive trees with variable capacities of buckets with maximal bucket size 3 is formed. In this case, and in a genealogy of n people, the distance between two specific individuals is the quantity examined in this article. For another example, if n atoms in a branching molecular structure are stochastically labelled with integers 1, 2, ..., n, then atoms in different functional groups can be considered as the labels of different buckets of a bucket recursive tree (the size of the largest functional group is b).

In passing, we give the combinatorial description of our model. Let $d(\nu)$ be the out-degree of node ν. It will be convenient to define for trees the size $|T|$ of a tree T via $|T| = \sum_{\nu} c(\nu)$. An increasing labelling of an ordered tree T is then a labelling of T, where the labels $\{1, 2, ..., |T|\}$ are distributed amongst the nodes of T. Then a class $\mathcal{T}$ of a new family of *bucket-increasing trees* can be defined in the following way: A sequence of non-negative numbers $(\alpha_k)_{k\geq 0}$ with $\alpha_0 > 0$ and a sequence of non-negative numbers $\beta_1, \beta_2, ..., \beta_{b-1}$ is used to define the weight $w(T)$ of any ordered tree T by $w(T) := \Pi_\nu w(\nu)$, where ν ranges over all nodes of T. It is natural that $w(\nu)$ must be dependent on $c(\nu)$ and $d(\nu)$. Thus the weight $w(\nu)$ of a node ν is given as follows:

$$w(\nu) := \begin{cases} \alpha_{d(\nu)}, & \nu \text{ is root or complete } (c(\nu) = b) \\ \beta_{c(\nu)}, & \nu \text{ is incomplete } (c(\nu) < b). \end{cases} \tag{1}$$

The above definition is reasonable because the root is the only incomplete node that has outdegree ≥ 1. Thus for complete nodes and root, the weight is dependent on the out-degree and described by the sequence α_k, whereas for incomplete nodes except of root the weights are dependent on the capacities.

Furthermore, $\mathcal{L}(\mathcal{T})$ denotes the set of different increasing labelings of the tree T with distinct integers $\{1, 2, ..., |T|\}$, where $L(T) := |\mathcal{L}(\mathcal{T})|$ denotes its cardinality. Then the family $\mathcal{T}$ consists of all trees T together with their weights $w(T)$ and the set of increasing labelings $\mathcal{L}(T)$. For a given degree-weight sequence $(\alpha_k)_{k\geq 0}$ with a degree-weight generating function $\varphi(t) := \sum_{k\geq 0} \alpha_k t^k$ and a bucket-weight sequence $\beta_1, \beta_2, ..., \beta_{b-1}$, we define the exponential generating function

$$T_{r,k_1,...k_r}(z) \ := \ \sum_{n=1}^{\infty} T_{n,b,r,k_1,...k_r} \frac{z^n}{n!}, \tag{2}$$

where $T_{n,b,r,k_1,...k_r} := \sum_{|T|=n} w(T) \cdot L(T)$ is the total weights. For this model,

$$\begin{aligned} T_{n,b,r,k_1,...k_r} &= \frac{(n-1)!(b!)^{n(1-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|)}}{b}, \quad n \geq 1 \\ \varphi(T_{r,k_1,...k_r}(z)) &= \frac{(b-1)!}{1 - b!^{1-\sum_{i=1}^{r} |\mathcal{P}_{k_i}|} z}, \end{aligned} \tag{3}$$

where $\mathcal{P}_{k_i}$ is the set of all trees of size k_i and r is the degree of root node [6]. For simplicity, we set $T_{n,b} := T_{n,b,r,k_1,...,k_r}$ and $T(z) := T_{r,k_1,...,k_r}(z)$.

Various studies are devoted to a distributional analysis of distances between random nodes in a lot of tree families of interest. For example, Mahmoud and

Neininger [9] for binary search trees, Christophi and Mahmoud [1] for the digital data structure, and Panholzer [13] for simply generated trees. Fewer studies are made to reveal the distribution of distances between specified nodes in labelled tree structures. Dobrow [3] and Dobrow and Smythe [4] have shown a central limit theorem for the distance between the nodes labelled by j and n, respectively, in a random recursive tree of size n and Devroye and Neininger [2] have shown a central limit theorem for the distance between the nodes labelled by j_1 and j_2 in a random binary search tree of size n. Panholzer and Prodinger have studied the level of nodes in increasing trees [14]. Kuba and Panholzer have studied the distribution of distances between specified nodes in increasing trees [7]. Also Moon studied the distance between nodes in recursive trees [12].

If we denote by $D_{n,n}$ the random variable which measures the depth of node containing label n in the our tree model of size n, then it was shown in [6] that $D_{n,n}$ satisfies a central limit law with mean and variance $b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\log n$. More precisely,

$$\begin{aligned} P(D_{n,n} = m) &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\frac{S(n-1,m)}{(n-1)!}, \\ \mathbb{E}(D_{n,n}) = \mathbb{V}ar(D_{n,n}) &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\log n + \mathcal{O}(1), \end{aligned} \tag{4}$$

where $S(m_1, m_2)$ are the signless Stirling numbers of first kind. We study the random variable level of label j, i.e., the number of edges from the root node to the bucket containing label j denoted by $D_{n,j}$ in tree T of size $n \geq j$. In this paper we extend the above results for $D_{n,n}$ to $D_{n,j}$. In passing, we study the random variable $H_{n,j}$, which counts the distance, measured by the number of edges lying on the connecting path, between bucket containing label j and bucket containing label n. Finally, we extend our results to the random variable $H_{n,i,j}$ which counts the distance between the bucket containing label i and bucket containing label j in our random tree of size n.

2 The depth of label j

We can to sketch a combinatorial approach to obtain the differential equation on the trivariate generating function related to the level of an arbitrary label j. It is better to think of specifically tricolored trees, where the coloring is as follows: one bucket is colored white (containing label j), all buckets with smaller labels than the all labels in white bucket are colored black, and all buckets with larger labels than the white bucket are colored red. We are interested in the level of the white bucket. Assume that the out-degree of the root node is $r \geq 1$ and the white bucket of T is not the root node (the case that the white bucket is the root of the tree corresponds to the initial condition, but does not appear explicitly in the differential equation itself). Then the white bucket is located in one of the r subtrees of the root of T; let us assume that it is in the first subtree. After order preserving relabellings, each subtree $T_1, ..., T_r$ is a bucket recursive tree with variable capacities of buckets by itself. The first subtree is again a tricolored increasing tree with one white, j_1 black and k_1 red buckets, whereas the remaining $r-1$ subtrees are only bicolored. For a proper description of this combinatorial decomposition we use generating functions which are exponential in both variables z and u, where z marks the black buckets and u marks the red buckets, i.e.,

$$\sum_{j\geq 0}\sum_{k\geq 0} f_{j,k}\frac{z^j}{j!}\frac{u^k}{k!}$$

for sequences $f_{j,k}$ and

$$\sum_{j\geq 0}\sum_{k\geq 0}\sum_{m\geq 0} f_{j,k,m}\frac{z^j}{j!}\frac{u^k}{k!}v^m$$

for sequences $f_{j,k,m}$, where v marks the level of the white bucket. Set $f_{j,k} = T_{k+j,b}$ and $f_{j,k,m} = P(D_{k+j+1,j+1} = m)T_{k+j+1,b}$. Thus the $r-1$ bicolored trees and the tricolored tree lead to

$$\alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|} T_{n,b}(z+u)^{r-1}L(z,u,v), \tag{5}$$

just similar to [6] where

$$L(z,u,v) = \sum_{k\geq 0}\sum_{j\geq 0}\sum_{m\geq 0} P(D_{k+j+1,j+1} = m)T_{k+j+1,b}\frac{z^j}{j!}\frac{u^k}{k!}v^m.$$

We recall that the total weights of the r subtrees is

$$\alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|} T_{k_1,b}\cdots T_{k_r,b}.$$

The level of the white bucket in the tree is one more than the level of the white bucket in the subtree. This fact leads to a factor v. We additionally get a factor r, since the white bucket can be in the first, second, ..., r-th subtree. Furthermore, the root has out-degree r that leads to a factor α_r. Thus by summing over $r \geq 1$, (5) leads to

$$\alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|} v\varphi'(T_{n,b}(z+u))L(z,u,v).$$

Since the root node labelled by 1 is colored black,

$$\frac{\partial}{\partial z}L(z,u,v) = \alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|} v\varphi'(T_{n,b}(z+u))L(z,u,v). \tag{6}$$

Equation (6) has the general solution

$$L(z,u,v) = c(u,v)\exp\left\{\alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|} v\int_0^z \varphi'(T_{n,b}(t+u))dt\right\},$$

with a function $c(u,v)$. Evaluating at $z=0$ and adapting to the initial condition gives now $c(u,v) = L(0,u,v) = T'_{n,b}(u) = \alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\varphi(T_{n,b}(u))$. Thus

$$\begin{aligned}
L(z,u,v) &= \alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\varphi(T_{n,b}(u))\exp\left\{\alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|} v\int_0^z \varphi'(T_{n,b}(t+u))dt\right\} \\
&= \alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\varphi(T_{n,b}(u)) \\
&\times \exp\left\{\alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|} v\int_0^z \frac{\varphi'(T_{n,b}(t+t))T'_{n,b}(t+u)}{\alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\varphi(T_{n,b}(t+u))}dt\right\} \\
&= \alpha_1^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\varphi(T_{n,b}(u))\left(\frac{\varphi(T_{n,b}(z+u))}{\varphi(T_{n,b}(u))}\right)^v \\
&= T'_{n,b}(u)\left(\frac{T'_{n,b}(z+u)}{T'_{n,b}(u)}\right)^v.
\end{aligned} \tag{7}$$

In the next results we use from the following facts [5]:

$$[z^n]f(qz) = q^n[z^n]f(z), \tag{8}$$

$$\sum_{n\geq 0}\sum_{m=0}^{n} S(n,m)\frac{z^n}{n!}v^m = \frac{1}{(1-z)^v}, \tag{9}$$

$$[z^n]\log\left(\frac{1}{1-z}\right)(1-z)^{-1} = H_n, \tag{10}$$

$$[z^n]\log^2\left(\frac{1}{1-z}\right)(1-z)^{-1} = H_n^2 - H_n^{(2)}, \tag{11}$$

where H_n, the n-th harmonic number and $H_n^{(2)}$ is the n-th harmonic number of order 2. In the following lemma we see that distribution of $D_{n,j}$ is independent of n.

Lemma 1 *The probabilities* $\mathsf{P}(D_{n,j}=m)$ *are given by the following formula:*

$$\mathsf{P}(D_{n,j}=m)=b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\frac{S(j-1,m)}{(j-1)!},\qquad j\le n \tag{12}$$

and

$$\begin{aligned}\mathbb{E}(D_{n,j}) &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_{j-1},\\ \mathbb{V}ar(D_{n,j}) &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_{j-1}^{2}(1-b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|})\\ &+ b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}(H_{j-1}-H_{j-1}^{(2)}).\end{aligned} \tag{13}$$

Proof. By (3), (7) and (10),

$$\begin{aligned}\mathbb{E}(D_{n,j}) &= \frac{(j-1)!(n-j-1)!}{T_{n,b}}[z^{j-1}u^{n-j-1}]\frac{\partial L(z,u,v)}{\partial v}\Big|_{v=1}\\ &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_{j-1}\end{aligned}$$

and by (11),

$$\begin{aligned}\mathbb{E}(D_{n,j}(D_{n,j}-1)) &= \frac{(j-1)!(n-j-1)!}{T_{n,b}}[z^{j-1}u^{n-j-1}]\frac{\partial^2 L(z,u,v)}{\partial v^2}\Big|_{v=1}\\ &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}(H_{j-1}^{2}-H_{j-1}^{(2)}).\end{aligned}$$

Proof of (13) is completed, since $\mathbb{V}ar(D_{n,j})=\mathbb{E}(D_{n,j}(D_{n,j}-1))+\mathbb{E}(D_{n,j})-\mathbb{E}^2(D_{n,j})$. By (8), the probability generating function

$$\begin{aligned}p(v) &= \sum_{m\ge 0}\mathsf{P}(D_{k+j,j}=m)v^m=\frac{(j-1)!k!}{T_{k+j,b}}[z^{j-1}u^{k}]L(z,u,v)\\ &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\left\{\frac{(j-1)!k!}{(k+j-1)!}\binom{v+j-2}{j-1}\binom{k+j-1}{k}\right\}\\ &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\binom{v+j-2}{j-1}.\end{aligned}$$

Thus one gets (12). Therefore the probability generating function and thus the distribution of $D_{k+j,j}$ is independent of k. □

As an example, in the a tree family with 100 individuals, the probability that the distance from the ancestor to the 45th individual is equal to 10 is calculated from relation (12) under specific conditions. Also, if the ancestor has 5 children, then the average distance to the 45th individual is

$$\mathbb{E}(D_{100,45})=H_{44}=\frac{5884182435213075787}{1345655451257488800}.$$

Corollary 1 *For* $j = n$,

$$\begin{aligned}\mathbb{E}(D_{n,n}) &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_{n-1},\\ \mathbb{V}ar(D_{n,n}) &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_{n-1}^2(1 - b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|})\\ &+ b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}(H_{n-1} - H_{n-1}^{(2)}).\end{aligned}$$

Theorem 1 *As* $j \to \infty$,

$$\mathbb{E}(D_{n,j}) = \mathbb{V}ar(D_{n,j}) = b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\log j + \mathcal{O}(1), \qquad j \le n$$

and

$$\sup_{x\in R}\left|P\left\{\frac{D_{n,j} - b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\log j}{\sqrt{b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\log j}} \le x\right\} - \Phi(x)\right| = \mathcal{O}\left(\frac{1}{\sqrt{\log j}}\right).$$

Proof. This is a direct application of the quasi power theorem for $v = \exp(s)$ in probability generating function $p(v)$ [5]. ☐

3 Distances

In this section we study the random variable $H_{n,j}$, which counts the distance, measured by the number of edges lying on the connecting path, between bucket containing label j and bucket containing label n in a random bucket recursive tree T with variable capacities of buckets of size n. Let

$$W(z,u,v) = \sum_{k\ge1}\sum_{j\ge1}\sum_{m\ge0} P(H_{k+j,j} = m)T_{k+j,b}\frac{z^{j-1}}{(j-1)!}\frac{u^{k-1}}{(k-1)!}v^m.$$

Again we apply a combinatorial description involving the counting of 4-colored bucket recursive tree with variable capacities of buckets. Since the arguments are very similar to [7] we just sketch the derivation. The combinatorial objects considered are all possible 4-colored trees of size ≥ 2 with a coloring as specified next. In each tree T the bucket containing the largest label (i.e., n) is colored green. From the remaining buckets exactly one bucket is colored red (bucket containing label j), all buckets with smaller labels than the red bucket are colored black, and all remaining buckets containing labels larger than the red bucket are colored white. We are interested in the distance between the red bucket and the green bucket. Finally

$$\frac{\partial W(z,u,v)}{\partial z} = b!^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\varphi'(T_{n,b}(z+u))W(z,u,v)$$

$$+ \quad b!^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\frac{\nu^2\varphi''(T_{n,b}(z+u))(T'_{n,b}(z+u))^{2\nu}}{(T'_{n,b}(u)\alpha_0)^{\nu-1}} \qquad (14)$$

with initial condition

$$\begin{aligned} W(0,u,\nu) &= b!^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\frac{\partial}{\partial u}L(u,0,\nu) \\ &= b!^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\nu T''_{n,b}(u)\left(\frac{T'_{n,b}(u)}{\alpha_0}\right)^{\nu-1}, \end{aligned}$$

where z counts the black nodes, u the white nodes, and ν the distance between the red and the green label.

Lemma 2 *The probabilities* $P(H_{n,j}=h)$ *are given by the following formula:*

$$\begin{aligned} P(H_{n,j}=h) &= \frac{b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}}{(n-1)\binom{n-2}{j-1}}\left\{\sum_{\ell=0}^{n-j-1}\binom{n-\ell-2}{j-1}\frac{1}{\ell!}S(\ell,h-1)\right. \\ &+ \left.\sum_{k=0}^{n-j-1}\binom{n-k-2}{j-1}\sum_{\ell=0}^{h-2}\frac{2^\ell}{k!}S(k,h-\ell-2)\right\}, \ 1\le j<n. \end{aligned} \qquad (15)$$

Proof. The equation (14) has the following consequence:

$$\begin{aligned} W(z,u,\nu) &= b!^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\frac{\nu T''_{n,b}(u)T'_{n,b}(z+u)}{T'_{n,b}(u)}\left(\frac{T'_{n,b}(u)}{\alpha_0}\right)^{\nu-1} \\ &+ b!^{-\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\frac{\nu^2 T'_{n,b}(z+u)}{(T'_{n,b}(u)\alpha_0)^{\nu-1}} \\ &\times \int_0^z \varphi''(T_{n,b}(t+u)(T'_{n,b}(t+u))^{2\nu-1}dt. \end{aligned} \qquad (16)$$

By (3),

$$\begin{aligned} W(z,u,\nu) &= \frac{\nu}{b}.\frac{b!^{1+\nu+m\nu}}{(1-b!^{1-m}u)^\nu(1-b!^{1-m}(z+u))} \\ &+ \frac{\nu^2}{b^\nu}\frac{b!^{1-m+\nu+m\nu}(b-1)!^{1-\nu}}{(2\nu-1)(1-b!^{1-m}u)^{1-\nu}(1-b!^{1-m}(z+u))^{2\nu}} \\ &- \frac{\nu^2}{b^\nu}\frac{b!^{1-m+\nu+m\nu}(b-1)!^{1-\nu}}{(2\nu-1)(1-b!^{1-m}u)^{\nu}(1-b!^{1-m}(z+u))}, \end{aligned}$$

where $m=\sum_{i=1}^{r}|\mathcal{P}_{k_i}|$. Thus

$$P(H_{n,j}=h) = \frac{(j-1)!(n-j-1)!}{T_{n,b}}[z^{j-1}u^{n-j-1}\nu^h]W(z,u,\nu),$$

and proof is completed (since these computations are essentially straightforward, but quite lengthy computations, they are omitted here. Similar considerations are done in [7] where the somewhat simpler recurrences appearing there are treated analogously). □

Theorem 2 *For* $1 \leq j < n$,

$$\mathbb{E}(H_{n,j}) = b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\left(H_{n-1} + H_j + \frac{1}{j} - 2\right),$$

$$\begin{aligned}\mathbb{V}ar(H_{n,j}) &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_{n-1}\left(\frac{1}{j} - 1 - \frac{b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}}{j} + 2b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\right)\\ &- b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_j\left(\frac{3}{j} + 1 + \frac{b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}}{j} - 2b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\right)\\ &+ \frac{b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}}{j}(4b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|} - 1) + 4b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}(2 - b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|})\\ &- b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_{n-1}^{(2)} - 3b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_j^{(2)} + b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_{n-1}^2(1 - b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|})\\ &+ b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_j^2(1 - b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}) + 2b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_jH_{n-1}(1 - b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|})\\ &- \frac{b!^{2\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}}{j^2}.\end{aligned}$$

Proof. By (10),

$$\begin{aligned}\mathbb{E}(H_{n,j}) &= \frac{(j-1)!(n-j-1)!}{T_{n,b}}[z^{j-1}u^{n-j-1}]\frac{\partial W(z,u,v)}{\partial v}\Big|_{v=1}\\ &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\left(H_{n-1} + H_j + \frac{1}{j} - 2\right),\end{aligned}$$

and by (11),

$$\begin{aligned}\mathbb{E}(D_{n,j}(D_{n,j}-1)) &= \frac{(j-1)!(n-j-1)!}{T_{n,b}}[z^{j-1}u^{n-j-1}]\frac{\partial^2 W(z,u,v)}{\partial v^2}\Big|_{v=1}\\ &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_{n-1}\left(\frac{1}{j} - 2\right) - b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_j\left(\frac{3}{j} + 2\right)\\ &- 2\frac{b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}}{j} + 10b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|} - b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_{n-1}^{(2)}\\ &- 3b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_j^{(2)} + b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_{n-1}^2 + b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_j^2\\ &+ 2b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}H_jH_{n-1}.\end{aligned}$$

Proof is completed just similar to the Lemma 1. $\square$

Corollary 2 *For* $1 \leq j < n$,

$$\mathbb{E}(H_{n,j}) = \mathbb{V}ar(H_{n,j}) = b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}(\log n + \log j) + \mathcal{O}(1).$$

Theorem 3 *As* $n \to \infty$,

$$Z = \frac{H_{n,j} - b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}(\log n + \log j)}{\sqrt{b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}(\log n + \log j)}} \xrightarrow{d} N(0,1),$$

for arbitrary sequences $(n, j(n))_{n\in\mathbb{N}}$.

Proof. Let $m = \sum_{i=1}^{r}|\mathcal{P}_{k_i}|$ and

$$\psi_{n,j}(v) = \mathbb{E}(v^{H_{n,j}}) = \sum_{h\geq 0} v^h P(H_{n,j} = h),$$

be the probability generating function of $H_{n,j}$. Thus

$$\begin{aligned}
\psi_{n,j}(v) &= \frac{(j-1)!(n-j-1)!}{T_{n,b}}[z^{j-1}u^{n-j-1}]W(z,u,v) \\
&= v\frac{b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\binom{n+v-2}{n-j-1}}{(n-1)\binom{n-2}{j-1}} \\
&+ \frac{v^2}{2v-1}\frac{b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\binom{n+v-2}{n-j-1}}{(n-1)\binom{n-2}{j-1}}\binom{2v+j-2}{j-1} \\
&- \frac{v^2}{2v-1}\frac{b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}\binom{n+v-2}{n-j-1}}{(n-1)\binom{n-2}{j-1}}.
\end{aligned}$$

Let

$$\begin{aligned}
\mu_{n,j} &:= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}(\log n + \log j), \\
\sigma^2_{n,j} &= b!^{\sum_{i=1}^{r}|\mathcal{P}_{k_i}|}(\log n + \log j),
\end{aligned}$$

and

$$M_{n,j}(t) = \mathbb{E}(e^{tZ}) = \sum_{z\geq 0} e^{tz}P(Z = z),$$

be the moment generating function of

$$Z = {\sigma_{n,j}}^{-1}(H_{n,j} - \mu_{n,j}).$$

Then

$$M_{n,j}(t) = e^{-\frac{\mu_{n,j}}{\sigma_{n,j}}} \psi_{n,j}(e^{\frac{t}{\sigma_{n,j}}}).$$

Now we split the region $1 \leq j < n$ into two cases: j big and $j \geq \log n$, and j small and $j \leq \log n$. With the same consideration of [7] proof is completed. □

We get also as a corollary similar results for the random variable $H_{n,i,j}$, which counts the distance between the bucket containing label i and bucket containing label j in our random tree of size n.

Corollary 3 *For* $1 \leq i < j < n$,

$$\mathbb{E}(H_{n,i,j}) = \mathbb{V}ar(H_{n,i,j}) = b!^{\sum_{i=1}^{r} |\mathcal{P}_{k_i}|}(\log i + \log j) + \mathcal{O}(1).$$

If $\mu_{n,i,j} := b!^{\sum_{i=1}^{r} |\mathcal{P}_{k_i}|}(\log i + \log j)$, $\sigma^2_{n,i,j} = b!^{\sum_{i=1}^{r} |\mathcal{P}_{k_i}|}(\log i + \log j)$, *then*

$$Z = \frac{H_{n,i,j} - b!^{\sum_{i=1}^{r} |\mathcal{P}_{k_i}|}(\log i + \log j)}{\sqrt{b!^{\sum_{i=1}^{r} |\mathcal{P}_{k_i}|}(\log i + \log j)}} \xrightarrow{d} N(0,1)$$

for arbitrary sequences $(n, i(n), j(n))_{n \in \mathbb{N}}$.

4 Conclusion

In this paper, we studied the random variable depth of label j in a bucket recursive tree with variable bucket capacities and maximal bucket size b $(n \geq j)$. We obtained a closed formula for the probability distribution, the expectation and the variance. We showed as $j \to \infty$, limiting distributions are Gaussian. In passing, we studied the random variable $H_{n,j}$, which counts the distance, measured by the number of edges lying on the connecting path, between bucket containing label j and bucket containing label n. Finally, we extend our results to the random variable $H_{n,i,j}$ which counts the distance between the bucket containing label i and bucket containing label j in our random tree of size n. We obtained this results by presenting partial differential equations for moment generating functions and solving them.

References

[1] C. Christophi, H. Mahmoud, The oscillatory distribution of distances in random tries, *Ann. Appl. Probab.*, **15** (2005) 1536–1564.

[2] L. Devroye, R. Neininger, Distances and finger search in random binary search trees, *SIAM J. Comput.*, **33** (2004), 647–658.

[3] R. Dobrow, On the distribution of distances in recursive trees, *J. Appl. Probab.*, **33** (1996), 749–757.

[4] R. Dobrow, R. Smythe, Poisson approximations for functionals of random trees, *Random Structures Algorithms*, **9** (1996), 79–92.

[5] P. Flajolet, R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, Cambridge (2008).

[6] R. Kazemi, Depth in bucket recursive trees with variable capacities of buckets, *Act. Math. Sin, English series*, **30(2)** (2014), 305–310.

[7] M. Kuba, A. Panholzer, On the distribution of distances between specified nodes in increasing trees, *Discrete Appl. Math.*, **158 (5)** (2010), 489–506.

[8] M. Kuba, A. Panholzer, Combinatorial approach to the analysis of bucket recursive trees. *Theoret. Comput. Sci.*, **411** (2010), 34–36, 3253–3273.

[9] H. Mahmoud, R. Neininger, Distribution of distances in random binary search trees, *Ann. Appl. Probab.*, **13** (2003), 253–276.

[10] H. Mahmoud, R. Smythe, Probabilistic analysis of bucket recursive trees. *Theor. Comput. Sci.*, **144** (1995), 221–249.

[11] A. Meir, J.W. Moon, Recursive trees with no nodes of out-degree one, *Congressus Numerantium*, **66** (1988), 49–62.

[12] J.W. Moon, The distance between nodes in recursive trees., *London Mathematical Society Lecture Notes.*, **13** (1974), 125–132.

[13] A. Panholzer, The distribution of the size of the ancestor-tree and of the induced spanning subtree for random trees, *Random Structures Algorithms.*, **25** (2004), 179–207.

[14] A. Panholzer, H. Prodinger, Level of nodes in increasing trees revisited, *Random Structures Algorithms.*, **31** (2007), 203–226.

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A new solution to the Rhoades' open problem with an application

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Abstract. We give a new solution to the Rhoades' open problem on the discontinuity at fixed point via the notion of an S-metric. To do this, we develop a new technique by means of the notion of a Zamfirescu mapping. Also, we consider a recent problem called the "fixed-circle problem" and propose a new solution to this problem as an application of our technique.

1 Introduction and preliminaries

Fixed-point theory has been extensively studied by various aspects. One of these is the discontinuity problem at fixed points (see [1, 2, 3, 4, 5, 6, 24, 25, 26, 27] for some examples). Discontinuous functions have been widely appeared in many areas of science such as neural networks (for example, see [7, 12, 13, 14]). In this paper, we give a new solution to the Rhoades' open problem (see [28] for more details) on the discontinuity at fixed point in the setting of an S-metric space which is a recently introduced generalization of a metric space. S-metric spaces were introduced in [29] by Sedgi et al., as follows:

Definition 1 [29] *Let* X *be a nonempty set and* $\mathcal{S} : X \times X \times X \to [0, \infty)$ *a function satisfying the following conditions for all* $x, y, z, a \in X$ *:*

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S1) $\mathcal{S}(x,y,z)=0$ *if and only if* $x=y=z$,
S2) $\mathcal{S}(x,y,z)\leq\mathcal{S}(x,x,a)+\mathcal{S}(y,y,a)+\mathcal{S}(z,z,a)$.
Then $\mathcal{S}$ *is called an* S*-metric on* X *and the pair* $(X,\mathcal{S})$ *is called an* S*-metric space.*

Relationships between a metric and an S-metric were given as follows:

Lemma 1 [9] *Let* (X,d) *be a metric space. Then the following properties are satisfied:*

1. $\mathcal{S}_d(x,y,z)=d(x,z)+d(y,z)$ *for all* $x,y,z\in X$ *is an* S*-metric on* X.
2. $x_n\to x$ *in* (X,d) *if and only if* $x_n\to x$ *in* $(X,\mathcal{S}_d)$.
3. $\{x_n\}$ *is Cauchy in* (X,d) *if and only if* $\{x_n\}$ *is Cauchy in* $(X,\mathcal{S}_d)$.
4. (X,d) *is complete if and only if* $(X,\mathcal{S}_d)$ *is complete.*

The metric $\mathcal{S}_d$ was called as the S-metric generated by d [17]. Some examples of an S-metric which is not generated by any metric are known (see [9, 17] for more details).

Furthermore, Gupta claimed that every S-metric on X defines a metric d_S on X as follows:

$$d_S(x,y)=\mathcal{S}(x,x,y)+\mathcal{S}(y,y,x), \tag{1}$$

for all $x,y\in X$ [8]. However, since the triangle inequality does not satisfied for all elements of X everywhen, the function $d_S(x,y)$ defined in (1) does not always define a metric (see [17]).

In the following, we see an example of an S-metric which is not generated by any metric.

Example 1 [17] *Let* $X=\mathbb{R}$ *and the function* $\mathcal{S}:X\times X\times X\to[0,\infty)$ *be defined as*

$$\mathcal{S}(x,y,z)=|x-z|+|x+z-2y|,$$

for all $x,y,z\in\mathbb{R}$. *Then* $\mathcal{S}$ *is an* S*-metric which is not generated by any metric and the pair* $(X,\mathcal{S})$ *is an* S*-metric space.*

The following lemma will be used in the next sections.

Lemma 2 [29] *Let* $(X,\mathcal{S})$ *be an* S*-metric space. Then we have*

$$\mathcal{S}(x,x,y)=\mathcal{S}(y,y,x).$$

In this paper, our aim is to obtain a new solution to the Rhoades' open problem on the existence of a contractive condition which is strong enough to generate a fixed point but which does not force the map to be continuous at the fixed point. To do this, we inspire of a result of Zamfirescu given in [33].

On the other hand, a recent aspect to the fixed point theory is to consider geometric properties of the set $\mathrm{Fix}(T)$, the fixed point set of the self-mapping T. Fixed-circle problem (resp. fixed-disc problem) have been studied in this context (see [6, 18, 19, 20, 21, 22, 23, 26, 27, 30, 31]). As an application, we present a new solution to these problems. We give necessary examples to support our theoretical results.

2 Main results

From now on, we assume that $(X, \mathcal{S})$ is an S-metric space and $T : X \to X$ is a self-mapping. In this section, we use the numbers defined as

$$M_z(x, y) = \max\left\{ad(x, y), \frac{b}{2}[d(x, Tx) + d(y, Ty)], \frac{c}{2}[d(x, Ty) + d(y, Tx)]\right\}$$

and

$$M_z^S(x, y) = \max\left\{\begin{array}{c} a\mathcal{S}(x, x, y), \frac{b}{2}[\mathcal{S}(x, x, Tx) + \mathcal{S}(y, y, Ty)], \\ \frac{c}{2}[\mathcal{S}(x, x, Ty) + \mathcal{S}(y, y, Tx)] \end{array}\right\},$$

where $a, b \in [0, 1)$ and $c \in \left[0, \frac{1}{2}\right]$.

We give the following theorem as a new solution to the Rhoades' open problem.

Theorem 1 *Let $(X, \mathcal{S})$ be a complete S-metric space and T a self-mapping on X satisfying the conditions*

i) *There exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for each $t > 0$ and*

$$\mathcal{S}(Tx, Tx, Ty) \leq \phi\left(M_z^S(x, y)\right),$$

for all $x, y \in X$,

ii) *There exists a $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < M_z^S(x, y) < \varepsilon + \delta$ implies $\mathcal{S}(Tx, Tx, Ty) \leq \varepsilon$ for a given $\varepsilon > 0$.*

Then T has a unique fixed point $u \in X$. Also, T is discontinuous at u if and only if $\lim\limits_{x \to u} M_z^S(x, u) \neq 0$.

Proof. At first, we define the number

$$\xi = \max\left\{a, \frac{2}{2-b}, \frac{c}{2-2c}\right\}.$$

Clearly, we have $\xi < 1$.

By the condition (i), there exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for each $t > 0$ and

$$\mathcal{S}(Tx, Tx, Ty) \leq \phi\left(M_z^S(x, y)\right),$$

for all $x, y \in X$. Using the properties of ϕ, we obtain

$$\mathcal{S}(Tx, Tx, Ty) < M_z^S(x, y), \tag{2}$$

whenever $M_z^S(x, y) > 0$.

Let us consider any $x_0 \in X$ with $x_0 \neq Tx_0$ and define a sequence $\{x_n\}$ as $x_{n+1} = Tx_n = T^n x_0$ for all $n = 0, 1, 2, 3, \ldots$. Using the condition (i) and the inequality (2), we get

$$\begin{aligned}
\mathcal{S}(x_n, x_n, x_{n+1}) &= \mathcal{S}(Tx_{n-1}, Tx_{n-1}, Tx_n) \leq \phi\left(M_z^S(x_{n-1}, x_n)\right) \\
&< M_z^S(x_{n-1}, x_n) \\
&= \max\left\{\begin{array}{c} a\mathcal{S}(x_{n-1}, x_{n-1}, x_n), \\ \frac{b}{2}\left[\mathcal{S}(x_{n-1}, x_{n-1}, Tx_{n-1}) + \mathcal{S}(x_n, x_n, Tx_n)\right], \\ \frac{c}{2}\left[\mathcal{S}(x_{n-1}, x_{n-1}, Tx_n) + \mathcal{S}(x_n, x_n, Tx_{n-1})\right] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} a\mathcal{S}(x_{n-1}, x_{n-1}, x_n), \\ \frac{b}{2}\left[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(x_n, x_n, x_{n+1})\right], \\ \frac{c}{2}\left[\mathcal{S}(x_{n-1}, x_{n-1}, x_{n+1}) + \mathcal{S}(x_n, x_n, x_n)\right] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} a\mathcal{S}(x_{n-1}, x_{n-1}, x_n), \\ \frac{b}{2}\left[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(x_n, x_n, x_{n+1})\right], \\ \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_{n+1}) \end{array}\right\}.
\end{aligned} \tag{3}$$

Assume that $M_z^S(x_{n-1}, x_n) = a\mathcal{S}(x_{n-1}, x_{n-1}, x_n)$. Then using the inequality (3), we have

$$\mathcal{S}(x_n, x_n, x_{n+1}) < a\mathcal{S}(x_{n-1}, x_{n-1}, x_n) \leq \xi\mathcal{S}(x_{n-1}, x_{n-1}, x_n) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n)$$

and so

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n). \tag{4}$$

Let $M_z^S(x_{n-1}, x_n) = \frac{b}{2}[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(x_n, x_n, x_{n+1})]$. Again using the inequality (3), we get

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \frac{b}{2}[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(x_n, x_n, x_{n+1})],$$

which implies

$$\left(1 - \frac{b}{2}\right)\mathcal{S}(x_n, x_n, x_{n+1}) < \frac{b}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_n)$$

and hence

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \frac{b}{2-b}\mathcal{S}(x_{n-1}, x_{n-1}, x_n) \leq \xi\mathcal{S}(x_{n-1}, x_{n-1}, x_n).$$

This yields

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n). \tag{5}$$

Suppose that $M_z^S(x_{n-1}, x_n) = \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_{n+1})$. Then using the inequality (3), Lemma 2 and the condition (S2), we obtain

$$\begin{aligned}
\mathcal{S}(x_n, x_n, x_{n+1}) &< \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_{n+1}) = \frac{c}{2}\mathcal{S}(x_{n+1}, x_{n+1}, x_{n-1}) \\
&\leq \frac{c}{2}[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + 2\mathcal{S}(x_{n+1}, x_{n+1}, x_n)] \\
&= \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + c\mathcal{S}(x_{n+1}, x_{n+1}, x_n) \\
&= \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + c\mathcal{S}(x_n, x_n, x_{n+1}),
\end{aligned}$$

which implies

$$(1-c)\mathcal{S}(x_n, x_n, x_{n+1}) < \frac{c}{2}\mathcal{S}(x_{n-1}, x_{n-1}, x_n).$$

Considering this, we find

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \frac{c}{2(1-c)}\mathcal{S}(x_{n-1}, x_{n-1}, x_n) \leq \xi\mathcal{S}(x_{n-1}, x_{n-1}, x_n)$$

and so

$$\mathcal{S}(x_n, x_n, x_{n+1}) < \mathcal{S}(x_{n-1}, x_{n-1}, x_n). \tag{6}$$

If we set $\alpha_n = \mathcal{S}(x_n, x_n, x_{n+1})$, then by the inequalities (4), (5) and (6), we find

$$\alpha_n < \alpha_{n-1}, \tag{7}$$

that is, α_n is a strictly decreasing sequence of positive real numbers whence the sequence α_n tends to a limit $\alpha \geq 0$.

Assume that $\alpha > 0$. There exists a positive integer $k \in \mathbb{N}$ such that $n \geq k$ implies

$$\alpha < \alpha_n < \alpha + \delta(\alpha). \tag{8}$$

Using the condition (ii) and the inequality (7), we get

$$\mathcal{S}(Tx_{n-1}, Tx_{n-1}, Tx_n) = \mathcal{S}(x_n, x_n, x_{n+1}) = \alpha_n < \alpha, \tag{9}$$

for $n \geq k$. Then the inequality (9) contradicts to the inequality (8). Therefore, it should be $\alpha = 0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence. Let us fix an $\varepsilon > 0$. Without loss of generality, we suppose that $\delta(\varepsilon) < \varepsilon$. There exists $k \in \mathbb{N}$ such that

$$\mathcal{S}(x_n, x_n, x_{n+1}) = \alpha_n < \frac{\delta}{4},$$

for $n \geq k$ since $\alpha_n \to 0$. Using the mathematical induction and the Jachymski's technique (see [10, 11] for more details), we show

$$\mathcal{S}(x_k, x_k, x_{k+n}) < \varepsilon + \frac{\delta}{2}, \tag{10}$$

for any $n \in \mathbb{N}$. At first, the inequality (10) holds for $n = 1$ since

$$\mathcal{S}(x_k, x_k, x_{k+1}) = \alpha_k < \frac{\delta}{4} < \varepsilon + \frac{\delta}{2}.$$

Assume that the inequality (10) holds for some n. We show that the inequality (10) holds for $n+1$. By the condition (S2), we get

$$\mathcal{S}(x_k, x_k, x_{k+n+1}) \leq 2\mathcal{S}(x_k, x_k, x_{k+1}) + \mathcal{S}(x_{k+n+1}, x_{k+n+1}, x_{k+1}).$$

From Lemma 2, we have

$$\mathcal{S}(x_{k+n+1}, x_{k+n+1}, x_{k+1}) = \mathcal{S}(x_{k+1}, x_{k+1}, x_{k+n+1})$$

and so it suffices to prove

$$\mathcal{S}(x_{k+1}, x_{k+1}, x_{k+n+1}) \leq \varepsilon.$$

To do this, we show

$$M_z^S(x_k, x_{k+n}) \leq \varepsilon + \delta.$$

Then we find

$$a\mathcal{S}(x_k,x_k,x_{k+n}) < \mathcal{S}(x_k,x_k,x_{k+n}) < \varepsilon + \frac{\delta}{2},$$

$$\begin{aligned}
&\frac{b}{2}\left[\mathcal{S}(x_k,x_k,x_{k+1}) + \mathcal{S}(x_{k+n},x_{k+n},x_{k+n+1})\right] \\
<&\mathcal{S}(x_k,x_k,x_{k+1}) + \mathcal{S}(x_{k+n},x_{k+n},x_{k+n+1}) \\
<&\frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{c}{2}\left[\mathcal{S}(x_k,x_k,x_{k+n+1}) + \mathcal{S}(x_{k+n},x_{k+n},x_{k+1})\right] \\
&\le \frac{c}{2}\left[4\mathcal{S}(x_k,x_k,x_{k+1}) + \mathcal{S}(x_{k+1},x_{k+1},x_{k+1+n}) + \mathcal{S}(x_k,x_k,x_{k+n})\right] \\
&= c\left[2\mathcal{S}(x_k,x_k,x_{k+1}) + \frac{\mathcal{S}(x_{k+1},x_{k+1},x_{k+1+n})}{2} + \frac{\mathcal{S}(x_k,x_k,x_{k+n})}{2}\right] \\
&< c\left[\frac{\delta}{2} + \varepsilon + \frac{\delta}{2}\right] < \varepsilon + \delta.
\end{aligned} \tag{11}$$

Using the definition of $M_z^S(x_k,x_{k+n})$, the condition (ii) and the inequalities (10) and (11), we obtain

$$M_z^S(x_k,x_{k+n}) \le \varepsilon + \delta$$

and so

$$\mathcal{S}\left(x_{k+1},x_{k+1},x_{k+n+1}\right) \le \varepsilon.$$

Hence we get

$$\mathcal{S}(x_k,x_k,x_{k+n+1}) < \varepsilon + \frac{\delta}{2},$$

whence $\{x_n\}$ is Cauchy. From the completeness hypothesis, there exists a point $u \in X$ such that $x_n \to u$ for $n \to \infty$. Also we get

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u.$$

Now we prove that u is a fixed point of T. On the contrary, assume that u is not a fixed point of T. Then using the condition (i) and the property of ϕ, we obtain

$$\mathcal{S}(Tu,Tu,Tx_n) \quad \le \quad \phi(M_z^S(u,x_n)) < M_z^S(u,x_n)$$

$$= \max\left\{\begin{array}{c} a\mathcal{S}(u,u,x_n), \frac{b}{2}\left[\mathcal{S}(u,u,Tu)+\mathcal{S}(x_n,x_n,Tx_n)\right], \\ \frac{c}{2}\left[\mathcal{S}(u,u,Tx_n)+\mathcal{S}(x_n,x_n,Tu)\right] \end{array}\right\}.$$

Using Lemma 2 and taking limit for $n \to \infty$, we find

$$\mathcal{S}(Tu,Tu,u) < \max\left\{\frac{b}{2}\mathcal{S}(u,u,Tu), \frac{c}{2}\mathcal{S}(u,u,Tu)\right\} < \mathcal{S}(Tu,Tu,u),$$

a contradiction. It should be $Tu = u$. We show that u is the unique fixed point of T. Let v be another fixed point of T such that $u \neq v$. From the condition (i) and Lemma 2, we have

$$\begin{aligned} \mathcal{S}(Tu,Tu,Tv) &= \mathcal{S}(u,u,v) \leq \phi(M_z^S(u,v)) < M_z^S(u,v) \\ &= \max\left\{\begin{array}{c} a\mathcal{S}(u,u,v), \frac{b}{2}\left[\mathcal{S}(u,u,Tu)+\mathcal{S}(v,v,Tv)\right], \\ \frac{c}{2}\left[\mathcal{S}(u,u,Tv)+\mathcal{S}(v,v,Tu)\right] \end{array}\right\} \\ &= \max\{a\mathcal{S}(u,u,v), c\mathcal{S}(u,u,v)\} < \mathcal{S}(u,u,v), \end{aligned}$$

a contradiction. So it should be $u = v$. Therefore, T has a unique fixed point $u \in X$.

Finally, we prove that T is discontinuous at u if and only if $\lim\limits_{x \to u} M_z^S(x,u) \neq 0$. To do this, we can easily show that T is continuous at u if and only if $\lim\limits_{x \to u} M_z^S(x,u) = 0$. Suppose that T is continuous at the fixed point u and $x_n \to u$. Hence we get $Tx_n \to Tu = u$ and using the condition (S2), we find

$$\mathcal{S}(x_n,x_n,Tx_n) \leq 2\mathcal{S}(x_n,x_n,u) + \mathcal{S}(Tx_n,Tx_n,u) \to 0,$$

as $x_n \to u$. So we get $\lim\limits_{x_n \to u} M_z^S(x_n,u) = 0$. On the other hand, assume $\lim\limits_{x_n \to u} M_z^S(x_n,u) = 0$. Then we obtain $\mathcal{S}(x_n,x_n,Tx_n) \to 0$ as $x_n \to u$, which implies $Tx_n \to Tu = u$. Consequently, T is continuous at u. □

We give an example.

Example 2 *Let* $X = \{0,2,4,8\}$ *and* $(X,\mathcal{S})$ *be the* S*-metric space defined as in Example 1. Let us define the self-mapping* $T : X \to X$ *as*

$$Tx = \left\{\begin{array}{ccc} 4 & ; & x \leq 4 \\ 2 & ; & x > 4 \end{array}\right.,$$

for all $x \in \{0,2,4,8\}$*. Then* T *satisfies the conditions of Theorem 1 with* $a = \frac{3}{4}, b = c = 0$ *and has a unique fixed point* $x = 4$*. Indeed, we get the following table :*

$$\begin{array}{lcl} \mathcal{S}(Tx,Tx,Ty) = 0 & \textit{and} & 3 \leq M_z^S(x,y) \leq 6 \textit{ when } x,y \leq 4 \\ \mathcal{S}(Tx,Tx,Ty) = 4 & \textit{and} & 6 \leq M_z^S(x,y) \leq 12 \textit{ when } x \leq 4, y > 4 \\ \mathcal{S}(Tx,Tx,Ty) = 4 & \textit{and} & 6 \leq M_z^S(x,y) \leq 12 \textit{ when } x > 4, y \leq 4 \end{array}.$$

Hence T *satisfies the conditions of Theorem 1 with*

$$\phi(t)=\left\{\begin{array}{lll} 5 & ; & t\geq 6 \\ \frac{t}{2} & ; & t<6 \end{array}\right.$$

and

$$\delta\left(\varepsilon\right)=\left\{\begin{array}{lll} 6 & ; & \varepsilon\geq 3 \\ 6-\varepsilon & ; & \varepsilon<3 \end{array}\right. .$$

Now we give the following results as the consequences of Theorem 1.

Corollary 1 *Let* $(X,\mathcal{S})$ *be a complete* S*-metric space and* T *a self-mapping on* X *satisfying the conditions*

i) $\mathcal{S}\left(Tx,Tx,Ty\right)<M_z^S\left(x,y\right)$ *for any* $x,y\in X$ *with* $M_z^S\left(x,y\right)>0$,

ii) *There exists a* $\delta=\delta\left(\varepsilon\right)>0$ *such that* $\varepsilon<M_z^S\left(x,y\right)<\varepsilon+\delta$ *implies* $\mathcal{S}\left(Tx,Tx,Ty\right)\leq\varepsilon$ *for a given* $\varepsilon>0$.

Then T *has a unique fixed point* $u\in X$. *Also,* T *is discontinuous at* u *if and only if* $\lim\limits_{x\to u}M_z^S\left(x,u\right)\neq 0$.

Corollary 2 *Let* $(X,\mathcal{S})$ *be a complete* S*-metric space and* T *a self-mapping on* X *satisfying the conditions*

i) *There exists a function* $\phi:\mathbb{R}^+\to\mathbb{R}^+$ *such that* $\phi(\mathcal{S}(x,x,y))<\mathcal{S}(x,x,y)$ *and* $\mathcal{S}(Tx,Tx,Ty)\leq\phi(\mathcal{S}(x,x,y))$,

ii) *There exists a* $\delta=\delta\left(\varepsilon\right)>0$ *such that* $\varepsilon<t<\varepsilon+\delta$ *implies* $\phi(t)\leq\varepsilon$ *for any* $t>0$ *and a given* $\varepsilon>0$.

Then T *has a unique fixed point* $u\in X$.

The following theorem shows that the power contraction of the type $M_z^S\left(x,y\right)$ allows also the possibility of discontinuity at the fixed point.

Theorem 2 *Let* $(X,\mathcal{S})$ *be a complete* S*-metric space and* T *a self-mapping on* X *satisfying the conditions*

i) *There exists a function* $\phi:\mathbb{R}^+\to\mathbb{R}^+$ *such that* $\phi(t)<t$ *for each* $t>0$ *and*

$$\mathcal{S}\left(T^mx,T^mx,T^my\right)\leq\phi\left(M_z^{S^*}\left(x,y\right)\right),$$

where

$$M_z^{S^*}\left(x,y\right)=\max\left\{\begin{array}{c} a\mathcal{S}\left(x,x,y\right),\frac{b}{2}\left[\mathcal{S}\left(x,x,T^mx\right)+\mathcal{S}\left(y,y,T^my\right)\right], \\ \frac{c}{2}\left[\mathcal{S}\left(x,x,T^my\right)+\mathcal{S}\left(y,y,T^mx\right)\right] \end{array}\right\}$$

for all $x,y\in X$,

ii) *There exists a* $\delta = \delta(\varepsilon) > 0$ *such that* $\varepsilon < M_z^{S^*}(x,y) < \varepsilon + \delta$ *implies* $\mathcal{S}(T^m x, T^m x, T^m y) \leq \varepsilon$ *for a given* $\varepsilon > 0$.

Then T *has a unique fixed point* $u \in X$. *Also,* T *is discontinuous at* u *if and only if* $\lim\limits_{x \to u} M_z^{S^*}(x,u) \neq 0$.

Proof. By Theorem 1, the function T^m has a unique fixed point u. Hence we have

$$Tu = TT^m u = T^m Tu$$

and so Tu is another fixed point of T^m. From the uniqueness of the fixed point, we obtain $Tu = u$, that is, T has a unique fixed point u. □

We note that if the S-metric $\mathcal{S}$ generates a metric d then we consider Theorem 1 on the corresponding metric space as follows:

Theorem 3 *Let* (X, d) *be a complete metric space and* T *a self-mapping on* X *satisfying the conditions*

i) *There exists a function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\phi(t) < t$ *for each* $t > 0$ *and*

$$d(Tx, Ty) \leq \phi(M_z(x,y)),$$

for all $x, y \in X$,

ii) *There exists a* $\delta = \delta(\varepsilon) > 0$ *such that* $\varepsilon < M_z(x,y) < \varepsilon + \delta$ *implies* $d(Tx, Ty) \leq \varepsilon$ *for a given* $\varepsilon > 0$.

Then T *has a unique fixed point* $u \in X$. *Also,* T *is discontinuous at* u *if and only if* $\lim\limits_{x \to u} M_z(x,u) \neq 0$.

Proof. By the similar arguments used in the proof of Theorem 1, the proof can be easily obtained. □

3 An application to the fixed-circle problem

In this section, we investigate new solutions to the fixed-circle problem raised by Özgür and Taş in [19] related to the geometric properties of the set $Fix(T)$ for a self mapping T on an S-metric space $(X, \mathcal{S})$. Some fixed-circle or fixed-disc results, as the direct solutions of this problem, have been studied using various methods on a metric space or some generalized metric spaces (see [15, 16, 20, 21, 22, 23, 26, 27, 30, 31, 32]).

Now we recall the notions of a circle and a disc on an S-metric space as follows:

$$C^S_{x_0,r} = \{x \in X : \mathcal{S}(x,x,x_0) = r\}$$

and

$$D^{S}_{x_0,r} = \{x \in X : \mathcal{S}(x,x,x_0) \leq r\},$$

where $r \in [0,\infty)$ [20, 29].

If $Tx = x$ for all $x \in C^{S}_{x_0,r}$ (resp. $x \in D^{S}_{x_0,r}$) then the circle $C^{S}_{x_0,r}$ (resp. the disc $D^{S}_{x_0,r}$) is called as the fixed circle (resp. fixed disc) of T (for more details see [15, 20]).

We begin with the following definition.

Definition 2 *A self-mapping T is called an $\mathcal{S}$-Zamfirescu type x_0-mapping if there exist $x_0 \in X$ and $a, b \in [0,1)$ such that*

$$\mathcal{S}(Tx,Tx,x) > 0 \Longrightarrow \mathcal{S}(Tx,Tx,x) \leq \max\left\{ \begin{array}{c} a\mathcal{S}(x,x,x_0), \\ \frac{b}{2}\left[\mathcal{S}(Tx_0,Tx_0,x) + \mathcal{S}(Tx,Tx,x_0)\right] \end{array} \right\},$$

for all $x \in X$.

We define the following number:

$$\rho := \inf\{\mathcal{S}(Tx,Tx,x) : Tx \neq x, x \in X\}. \tag{12}$$

Now we prove that the set $\mathrm{Fix}(T)$ contains a circle (resp. a disc) by means of the number ρ.

Theorem 4 *If T is an $\mathcal{S}$-Zamfirescu type x_0-mapping with $x_0 \in X$ and the condition*

$$\mathcal{S}(Tx,Tx,x_0) \leq \rho$$

holds for each $x \in C^{S}_{x_0,\rho}$ then $C^{S}_{x_0,\rho}$ is a fixed circle of T, that is, $C^{S}_{x_0,\rho} \subset \mathrm{Fix}(T)$.

Proof. At first, we show that x_0 is a fixed point of T. On the contrary, let $Tx_0 \neq x_0$. Then we have $\mathcal{S}(Tx_0,Tx_0,x_0) > 0$. By the definition of an $\mathcal{S}$-Zamfirescu type x_0-mapping and the condition $(S1)$, we obtain

$$\begin{aligned} \mathcal{S}(Tx_0,Tx_0,x_0) &\leq \max\left\{a\mathcal{S}(x_0,x_0,x_0), \frac{b}{2}\left[\mathcal{S}(Tx_0,Tx_0,x_0) + \mathcal{S}(Tx_0,Tx_0,x_0)\right]\right\} \\ &= b\mathcal{S}(Tx_0,Tx_0,x_0), \end{aligned}$$

a contradiction because of $b \in [0,1)$. This shows that $Tx_0 = x_0$.

We have two cases:

Case 1: If $\rho = 0$, then we get $C^{S}_{x_0,\rho} = \{x_0\}$ and clearly this is a fixed circle of T.

Case 2: Let $\rho > 0$ and $x \in C^S_{x_0,\rho}$ be any point such that $Tx \neq x$. Then we have

$$\mathcal{S}(Tx, Tx, x) > 0$$

and using the hypothesis we obtain,

$$\begin{aligned}\mathcal{S}(Tx, Tx, x) &\leq \max\left\{a\mathcal{S}(x, x, x_0), \frac{b}{2}\left[\mathcal{S}(Tx_0, Tx_0, x) + \mathcal{S}(Tx, Tx, x_0)\right]\right\} \\ &\leq \max\{a\rho, b\rho\} < \rho,\end{aligned}$$

which is a contradiction with the definition of ρ. Hence it should be $Tx = x$ whence $C^S_{x_0,\rho}$ is a fixed circle of T. □

Corollary 3 *If T is an $\mathcal{S}$-Zamfirescu type x_0-mapping with $x_0 \in X$ and the condition*

$$\mathcal{S}(Tx, Tx, x_0) \leq \rho$$

holds for each $x \in D^S_{x_0,\rho}$ then $D^S_{x_0,\rho}$ is a fixed disc of T, that is, $D^S_{x_0,\rho} \subset Fix(T)$.

Now we give an illustrative example to show the effectiveness of our results.

Example 3 *Let $X = \mathbb{R}$ and $(X, \mathcal{S})$ be the S-metric space defined as in Example 1. Let us define the self-mapping $T : X \to X$ as*

$$Tx = \begin{cases} x & ; \quad x \in [-3, 3] \\ x + 1 & ; \quad x \notin [-3, 3] \end{cases},$$

for all $x \in \mathbb{R}$. Then T is an $\mathcal{S}$-Zamfirescu type x_0-mapping with $x_0 = 0$, $a = \frac{1}{2}$ and $b = 0$. Indeed, we get

$$\mathcal{S}(Tx, Tx, x) = 2\left|Tx - x\right| = 2 > 0,$$

for all $x \in (-\infty, -3) \cup (3, \infty)$. So we obtain

$$\begin{aligned}\mathcal{S}(Tx, Tx, x) &= 2 \leq \max\left\{aS\left(x, x, 0\right), \frac{b}{2}\left[\mathcal{S}(0, 0, x) + \mathcal{S}(x + 1, x + 1, 0)\right]\right\} \\ &= \frac{1}{2}.2\left|x\right|.\end{aligned}$$

Also we have

$$\rho = \inf\left\{\mathcal{S}(Tx, Tx, x) : Tx \neq x, x \in X\right\} = 2$$

and

$$\mathcal{S}(Tx, Tx, 0) = \mathcal{S}(x, x, 0) \leq 2,$$

for all $x \in C^S_{0,2} = \{x : \mathcal{S}(x, x, 0) = 2\} = \{x : 2\left|x\right| = 2\} = \{x : |x| = 1\}$. Consequently, T fixes the circle $C^S_{0,2}$ and the disc $D^S_{0,2}$.

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References

[1] R. K. Bisht and R. P. Pant, A remark on discontinuity at fixed point, *J. Math. Anal. Appl.*, **445** (2017), 1239–1242.

[2] R. K. Bisht and R. P. Pant, Contractive definitions and discontinuity at fixed point, *Appl. Gen. Topol.*, **18** (1) (2017), 173–182.

[3] R. K. Bisht and N. Hussain, A note on convex contraction mappings and discontinuity at fixed point, *J. Math. Anal.*, **8** (4) (2017), 90–96.

[4] R. K. Bisht and V. Rakočević, Generalized Meir-Keeler type contractions and discontinuity at fixed point, *Fixed Point Theory*, **19** (1) (2018), 57–64.

[5] R. K. Bisht, $(\varepsilon - \delta)$ conditions and fixed point theorems, *Tbil. Math. J.* **12** (3) (2019), 39–49.

[6] R. K. Bisht and N. Özgür, Geometric properties of discontinuous fixed point set of $(\epsilon - \delta)$ contractions and applications to neural networks, *Aequationes Math.*, **94** (5) (2020), 847–863. https://doi.org/10.1007/s00010-019-00680-7

[7] M. Forti and P. Nistri, Global convergence of neural networks with discontinuous neuron activations, *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, **50** (11) (2003), 1421–1435.

[8] A. Gupta, Cyclic contraction on S-metric space, *Int. J. Anal. Appl.*, **3** (2) (2013), 119–130.

[9] N. T. Hieu, N. T. Ly and N. V. Dung, A generalization of Ciric quasi-contractions for maps on S-metric spaces, *Thai J. Math.*, **13** (2) (2015), 369–380.

[10] J. Jachymski, Common fixed point theorems for some families of maps, *Indian J. Pure Appl. Math.*, **25** (1994), 925–937.

[11] J. Jachymski, Equivalent conditions and Meir–Keeler type theorems, *J. Math. Anal. Appl.*, **194** (1995), 293–303.

[12] X. Liu, T. Chen, J. Cao and W. Lu, Dissipativity and quasi-synchronization for neural networks with discontinuous activations and parameter mismatches, *Neural Networks*, **24** (10) (2011), 1013–1021.

[13] W. Lu and T. Chen, Dynamical behaviors of Cohen–Grossberg neural networks with discontinuous activation functions, *Neural Networks*, **18** (3) (2005), 231–242.

[14] W. Lu and T. Chen, Dynamical behaviors of delayed neural network systems with discontinuous activation functions, *Neural Computation*, **18** (3) (2006), 683–708.

[15] N. Mlaiki, U. Çelik, N. Taş, N. Y. Özgür and A. Mukheimer, Wardowski type contractions and the fixed-circle problem on S-metric spaces, *J. Math.*, 2018, Article ID 9127486.

[16] N. Mlaiki, N. Taş and N. Y. Özgür, On the fixed-circle problem and Khan type contractions, *Axioms*, **7** (2018), 80.

[17] N. Y. Özgür and N. Taş, Some new contractive mappings on S-metric spaces and their relationships with the mapping $(S25)$, *Math. Sci.* (Springer), **11** (1) (2017), 7–16.

[18] N. Y. Özgür, N. Taş, *Generalizations of metric spaces: from the fixed-point theory to the fixed-circle theory*, In: Rassias T. (eds) Applications of Nonlinear Analysis. Springer Optimization and Its Applications, vol 134. Springer, Cham, 2018.

[19] N. Y. Özgür and N. Taş, Some fixed-circle theorems on metric spaces, *Bull. Malays. Math. Sci. Soc.*, **42** (4) (2019), 1433–1449.

[20] N. Y. Özgür and N. Taş, Fixed-circle problem on S-metric spaces with a geometric viewpoint, *Facta Universitatis. Series: Mathematics and Informatics*, **34** (3) (2019), 459–472.

[21] N. Y. Özgür, N. Taş and U. Çelik, New fixed-circle results on S-metric spaces, *Bull. Math. Anal. Appl.*, **9** (2) (2017), 10–23.

[22] N. Y. Özgür and N. Taş, Some fixed-circle theorems and discontinuity at fixed circle, *AIP Conference Proceedings*, **1926**, 020048 (2018).

[23] N. Y. Özgür, Fixed-disc results via simulation functions, *Turkish J. Math.*, **43** (6) (2019), 2794–2805.

[24] A. Pant and R. P. Pant, Fixed points and continuity of contractive maps, *Filomat*, **31** (11) (2017), 3501–3506.

[25] R. P. Pant, Discontinuity and fixed points, *J. Math. Anal. Appl.*, **240** (1999), 284–289.

[26] R. P. Pant, N.Y. Özgür and N. Taş, On discontinuity problem at fixed point, *Bull. Malays. Math. Sci. Soc.*, **43** (1) (2020), 499–517.

[27] R. P. Pant, N. Y. Özgür and N. Taş, Discontinuity at fixed points with applications, *Bulletin of the Belgian Mathematical Society-Simon Stevin*, **26** (4) (2019), 571–589.

[28] B. E. Rhoades, Contractive definitions and continuity, Fixed point theory and its applications (Berkeley, CA, 1986), 233–245, *Contemp. Math.*, **72**, Amer. Math. Soc., Providence, RI, 1988.

[29] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, *Mat. Vesnik*, **64** (3) (2012), 258–266.

[30] N. Taş, N. Y. Özgür and N. Mlaiki, New types of F_C-contractions and the fixed-circle problem, *Mathematics*, **6** (10) (2018), 188.

[31] N. Taş, Suzuki-Berinde type fixed-point and fixed-circle results on S-metric spaces, *J. Linear Topol. Algebra*, **7** (3) (2018), 233–244.

[32] N. Taş, Various types of fixed-point theorems on S-metric spaces, *J. BAUN Inst. Sci. Technol.*, **20** (2) (2018), 211–223.

[33] T. Zamfirescu, Fix point theorems in metric spaces, *Arch. Math.*, **23** (1972), 292–298.

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A determinantal expression and a recursive relation of the Delannoy numbers

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To Professor Sen-Lin Xu, my PhD supervisor, on his 81th birthday anniversary

Abstract. In the paper, by a general and fundamental, but non-extensively circulated, formula for derivatives of a ratio of two differentiable functions and by a recursive relation of the Hessenberg determinant, the author finds a new determinantal expression and a new recursive relation of the Delannoy numbers. Consequently, the author derives a recursive relation for computing central Delannoy numbers in terms of related Delannoy numbers.

1 Motivations

The Delannoy numbers, denoted by $D(p, q)$ for $p, q \geq 0$, form an array of positive integers which are related to lattice paths enumeration and other problems in combinatorics. For more information on their history and status in combinatorics, please refer to [1] and closely related references therein.

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In [1, Section 2] and [7], the explicit formulas

$$D(p,q)=\sum_{i=0}^{p}\binom{p}{i}\binom{q}{i}2^i \quad \text{and} \quad D(p,q)=\sum_{i=0}^{q}\binom{q}{i}\binom{p+q-i}{q}$$

were given. It is well known [7] that the Delannoy numbers $D(p,q)$ satisfy a simple recurrence

$$D(p,q)=D(p-1,q)+D(p-1,q-1)+D(p,q-1)$$

and can be generated by

$$\frac{1}{1-x-y-xy}=\sum_{p,q=0}^{\infty}D(p,q)x^py^q.$$

When taking $n=p=q$, the numbers $D(n)=D(n,n)$ are known [7] as central Delannoy numbers which have the generating function

$$\frac{1}{\sqrt{1-6x+x^2}}=\sum_{n=0}^{\infty}D(n)x^k=1+3x+13x^2+63x^3+\cdots. \tag{1}$$

In [6, Theorems 1.1 and 1.3], considering the generating function (1), among other things, the authors expressed central Delannoy numbers $D(n)$ by an integral

$$D(n)=\frac{1}{\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}}\frac{1}{\sqrt{\left(t-3+2\sqrt{2}\right)\left(3+2\sqrt{2}-t\right)}}\frac{1}{t^{n+1}}\,\mathrm{d}t \tag{2}$$

and by a determinant

$$D(n)=(-1)^n\begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 & 0\\ a_2 & a_1 & 1 & \cdots & 0 & 0 & 0\\ a_3 & a_2 & a_1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_1 & 1 & 0\\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_2 & a_1 & 1\\ a_n & a_{n-1} & a_{n-2} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}$$

for $n\in\mathbb{N}$, where

$$a_n=\frac{(-1)^{n+1}}{6^n}\sum_{\ell=1}^{n}(-1)^\ell 6^{2\ell}\frac{(2\ell-3)!!}{(2\ell)!!}\binom{\ell}{n-\ell}.$$

Making use of the integral expression 2, the authors derived in [5, 6] some new analytic properties, including some product inequalities and determinantal inequalities, of central Delannoy numbers $D(n)$.

In this paper, by a general and fundamental, but non-extensively circulated, formula for derivatives of a ratio of two differentiable functions in [2, p. 40] and by a recursive relation of the Hessenberg determinant in [3, p. 222, Theorem], we find a new determinantal expression and a new recursive relation of the Delannoy numbers $D(p, q)$. Consequently, we derive a recursive relation for computing central Delannoy numbers $D(n)$ in terms of related Delannoy numbers $D(p, q)$.

2 A determinantal expression of the Delannoy numbers

In this section, by virtue of a general and fundamental, but non-extensively circulated, formula for derivatives of a ratio of two differentiable functions in [2, p. 40], we find a new determinantal expression of the Delannoy numbers $D(p, q)$.

Theorem 1 *For $p, q \geq 0$, the Delannoy numbers $D(p, q)$ can be determinantally expressed by*

$$D(p, q) = \frac{(-1)^q}{q!} \left| L_{(q+1)\times 1}(p) \quad M_{(q+1)\times q}(p) \right|_{(q+1)\times(q+1)}, \tag{3}$$

where

$$L_{(q+1)\times 1}(p) = \left(\langle p\rangle_0, \langle p\rangle_1, \dots, \langle p\rangle_q\right)^T,$$

$$M_{(q+1)\times q}(p) = \left((-1)^{i-j}\binom{i-1}{j-1}\langle p+1\rangle_{i-j}\right)_{\substack{1\leq i\leq q+1\\ 1\leq j\leq q}},$$

$$\langle z\rangle_n = \begin{cases} z(z-1)\cdots(z-n+1), & n \geq 1;\\ 1, & n = 0\end{cases}$$

is known as the n-th falling factorial of the number $z \in \mathbb{C}$, and T denotes the transpose of a matrix. Consequently, central Delannoy numbers $D(n)$ for $n \geq 0$ can be determinantally expressed as

$$D(n) = \frac{(-1)^n}{n!} \left| L_{(n+1)\times 1}(n) \quad M_{(n+1)\times n}(n) \right|_{(n+1)\times(n+1)}. \tag{4}$$

Proof. We recall a general and fundamental, but non-extensively circulated, formula for derivatives of a ratio of two differentiable functions. Let $u(t)$ and $v(t) \neq 0$ be two n-th differentiable functions for $n \in \mathbb{N}$. Exercise 5) in [2, p. 40] reads that the n-th derivative of the ratio $\frac{u(t)}{v(t)}$ can be computed by

$$\frac{d^n}{dx^n}\left[\frac{u(t)}{v(t)}\right] = (-1)^n \frac{\left|W_{(n+1)\times(n+1)}(t)\right|}{v^{n+1}(t)}, \tag{5}$$

where $U_{(n+1)\times 1}(t)$ is an $(n+1)\times 1$ matrix whose elements satisfy $u_{k,1}(t) = u^{(k-1)}(t)$ for $1 \le k \le n+1$, $V_{(n+1)\times n}(t)$ is an $(n+1)\times n$ matrix whose elements meet $v_{i,j}(t) = \binom{i-1}{j-1} v^{(i-j)}(t)$ for $1 \le i \le n+1$ and $1 \le j \le n$, and $|W_{(n+1)\times(n+1)}(t)|$ is the determinant of the $(n+1)\times(n+1)$ matrix

$$W_{(n+1)\times(n+1)}(t) = \begin{pmatrix} U_{(n+1)\times 1}(t) & V_{(n+1)\times n}(t) \end{pmatrix}_{(n+1)\times(n+1)}.$$

It is easy to see that

$$\frac{\partial^p}{\partial x^p}\left(\frac{1}{1-x-y-xy}\right) = \frac{p!(1+y)^p}{[1-x-(1+x)y]^{p+1}}.$$

Making use of the formula (5) gives

$$\begin{aligned}
\frac{\partial^{p+q}}{\partial y^q \partial x^p}\left(\frac{1}{1-x-y-xy}\right) &= p!\frac{\partial^q}{\partial y^q}\frac{(1+y)^p}{[1-x-(1+x)y]^{p+1}} \\
&= p!\frac{(-1)^q}{[1-x-(1+x)y]^{(p+1)(q+1)}}
\end{aligned}$$

$$\times \begin{vmatrix}
(1+y)^p & [1-x-(1+x)y]^{p+1} \\
\langle p\rangle_1(1+y)^{p-1} & (-1)^1\langle p+1\rangle_1(1+x)^1[1-x-(1+x)y]^p \\
\langle p\rangle_2(1+y)^{p-2} & (-1)^2\langle p+1\rangle_2(1+x)^2[1-x-(1+x)y]^{p-1} \\
\langle p\rangle_3(1+y)^{p-3} & (-1)^3\langle p+1\rangle_3(1+x)^3[1-x-(1+x)y]^{p-2} \\
\vdots & \vdots \\
\langle p\rangle_{q-2}(1+y)^{p-q+2} & (-1)^{q-2}\langle p+1\rangle_{q-2}(1+x)^{q-2}[1-x-(1+x)y]^{p-q+3} \\
\langle p\rangle_{q-1}(1+y)^{p-q+1} & (-1)^{q-1}\langle p+1\rangle_{q-1}(1+x)^{q-1}[1-x-(1+x)y]^{p-q+2} \\
\langle p\rangle_q(1+y)^{p-q} & (-1)^q\langle p+1\rangle_q(1+x)^q[1-x-(1+x)y]^{p-q+1}
\end{vmatrix}$$

$$
\begin{array}{c}
0\\
\binom{1}{1}[1-x-(1+x)y]^{p+1}\\
\binom{2}{1}(-1)^1\langle p+1\rangle_1(1+x)^1[1-x-(1+x)y]^{p}\\
\binom{3}{1}(-1)^2\langle p+1\rangle_2(1+x)^2[1-x-(1+x)y]^{p-1}\\
\vdots\\
\binom{q-2}{1}(-1)^{q-3}\langle p+1\rangle_{q-3}(1+x)^{q-3}[1-x-(1+x)y]^{p-q+4}\\
\binom{q-1}{1}(-1)^{q-2}\langle p+1\rangle_{q-2}(1+x)^{q-2}[1-x-(1+x)y]^{p-q+3}\\
\binom{q}{1}(-1)^{q-1}\langle p+1\rangle_{q-1}(1+x)^{q-1}[1-x-(1+x)y]^{p-q+2}
\end{array}
$$

$$
\begin{array}{cc}
0 & \cdots\\
0 & \cdots\\
\binom{2}{2}[1-x-(1+x)y]^{p+1} & \cdots\\
\binom{3}{2}(-1)^1\langle p+1\rangle_1(1+x)^1[1-x-(1+x)y]^{p} & \cdots\\
\vdots & \ddots\\
\binom{q-2}{2}(-1)^{q-4}\langle p+1\rangle_{q-4}(1+x)^{q-4}[1-x-(1+x)y]^{p-q+5} & \cdots\\
\binom{q-1}{2}(-1)^{q-3}\langle p+1\rangle_{q-3}(1+x)^{q-3}[1-x-(1+x)y]^{p-q+4} & \cdots\\
\binom{q}{2}(-1)^{q-2}\langle p+1\rangle_{q-2}(1+x)^{q-2}[1-x-(1+x)y]^{p-q+3} & \cdots
\end{array}
$$

$$
\begin{array}{c}
0\\
0\\
0\\
0\\
\vdots\\
\binom{q-2}{q-2}[1-x-(1+x)y]^{p+1}\\
\binom{q-1}{q-2}(-1)\langle p+1\rangle_1(1+x)[1-x-(1+x)y]^{p}\\
\binom{q}{q-2}\langle p+1\rangle_2(1+x)^2[1-x-(1+x)y]^{p-1}
\end{array}
$$

$$
\left.
\begin{array}{c}
0\\
0\\
0\\
0\\
\vdots\\
0\\
\binom{q-1}{q-1}[1-x-(1+x)y]^{p+1}\\
\binom{q}{q-1}(-1)\langle p+1\rangle_1(1+x)[1-x-(1+x)y]^{p}
\end{array}
\right|
$$

$$
\to (-1)^q p! \begin{vmatrix}
\langle p\rangle_0 & (-1)^0\langle p+1\rangle_0 & 0 & 0 & \cdots & 0 & 0 \\
\langle p\rangle_1 & (-1)^1\langle p+1\rangle_1 & \binom{1}{1}(-1)^0\langle p+1\rangle_0 & 0 & \cdots & 0 & 0 \\
\langle p\rangle_2 & (-1)^2\langle p+1\rangle_2 & \binom{2}{1}(-1)^1\langle p+1\rangle_1 & \binom{2}{2}(-1)^0\langle p+1\rangle_0 & \cdots & 0 & 0 \\
\langle p\rangle_3 & (-1)^3\langle p+1\rangle_3 & \binom{3}{1}(-1)^2\langle p+1\rangle_2 & \binom{3}{2}(-1)^1\langle p+1\rangle_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\langle p\rangle_{q-2} & (-1)^{q-2}\langle p+1\rangle_{q-2} & \binom{q-2}{1}(-1)^{q-3}\langle p+1\rangle_{q-3} & \binom{q-2}{2}(-1)^{q-4}\langle p+1\rangle_{q-4} & \cdots & \binom{q-2}{q-2}(-1)^0\langle p+1\rangle_0 & 0 \\
\langle p\rangle_{q-1} & (-1)^{q-1}\langle p+1\rangle_{q-1} & \binom{q-1}{1}(-1)^{q-2}\langle p+1\rangle_{q-2} & \binom{q-1}{2}(-1)^{q-3}\langle p+1\rangle_{q-3} & \cdots & \binom{q-1}{q-2}(-1)^1\langle p+1\rangle_1 & \binom{q-1}{q-1}(-1)^0\langle p+1\rangle_0 \\
\langle p\rangle_{q} & (-1)^{q}\langle p+1\rangle_{q} & \binom{q}{1}(-1)^{q-1}\langle p+1\rangle_{q-1} & \binom{q}{2}(-1)^{q-2}\langle p+1\rangle_{q-2} & \cdots & \binom{q}{q-2}(-1)^2\langle p+1\rangle_2 & \binom{q}{q-1}(-1)^1\langle p+1\rangle_1
\end{vmatrix}
$$

as $x, y \to 0$. Consequently, we have

$$
\begin{aligned}
D(p,q) &= \frac{1}{p!q!}\frac{\partial^{p+q}}{\partial y^q \partial x^p}\left(\frac{1}{1-x-y-xy}\right) \\
&= \frac{(-1)^q}{q!}\left| \left(\langle p\rangle_{ij}\right)_{\substack{0\le i\le q\\ j=1}} \quad \left((-1)^{i-j}\binom{i-1}{j-1}\langle p+1\rangle_{i-j}\right)_{\substack{1\le i\le q+1\\ 1\le j\le q}} \right|_{(q+1)\times(q+1)}.
\end{aligned}
$$

The determinantal expression (3) is thus proved.

From (3), we readily see that, when $n = p = q$, central Delannoy numbers $D(n)$ for $n \ge 0$ can be expressed as (4). The proof of Theorem 1 is complete. $\square$

3 A recursive relation of the Delannoy numbers

In this section, by virtue of a recursive relation of the Hessenberg determinant in [3, p. 222, Theorem], we find a recursive relation of the Delannoy numbers $D(p,q)$.

Theorem 2 *For $p, q \ge 0$, the Delannoy numbers $D(p,q)$ satisfy the recursive relation*

$$
D(p,q) = \binom{p}{q} + (-1)^{q-1}\sum_{r=0}^{q-1}(-1)^r\binom{p+1}{q-r}D(p,r). \tag{6}
$$

Consequently, central Delannoy numbers $D(n)$ *for* $n \ge 0$ *satisfy*

$$D(n) = 1 + (-1)^{n+1} \sum_{r=0}^{n-1} (-1)^r \binom{n+1}{r+1} D(n,r). \tag{7}$$

Proof. Let $Q_0 = 1$ and

$$Q_n = \begin{vmatrix} e_{1,1} & e_{1,2} & 0 & \dots & 0 & 0 \\ e_{2,1} & e_{2,2} & e_{2,3} & \dots & 0 & 0 \\ e_{3,1} & e_{3,2} & e_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e_{n-2,1} & e_{n-2,2} & e_{n-2,3} & \dots & e_{n-2,n-1} & 0 \\ e_{n-1,1} & e_{n-1,2} & e_{n-1,3} & \dots & e_{n-1,n-1} & e_{n-1,n} \\ e_{n,1} & e_{n,2} & e_{n,3} & \dots & e_{n,n-1} & e_{n,n} \end{vmatrix}$$

for $n \in \mathbb{N}$. In [3, p. 222, Theorem], it was proved that the sequence Q_n for $n \ge 0$ satisfies $Q_1 = e_{1,1}$ and

$$Q_n = \sum_{r=1}^{n} (-1)^{n-r} e_{n,r} \left(\prod_{j=r}^{n-1} e_{j,j+1} \right) Q_{r-1} \tag{8}$$

for $n \ge 2$, where the empty product is understood to be 1. Replacing the determinant Q_r by $(-1)^{r-1}(r-1)!D(p,r-1)$ in (3) for $1 \le r \le n$ in the recursive relation (8) and simplifying give

$$D(p,n-1) = \frac{\langle p \rangle_{n-1}}{(n-1)!} + (-1)^n \sum_{r=2}^{n} (-1)^r \frac{\langle p+1 \rangle_{n-r+1}}{(n-r+1)!} D(p,r-2)$$

which is equivalent to the recursive relation (6).

When $n = p = q$ in (6), we can see that central Delannoy numbers $D(n)$ satisfy the recursive relation (7). The proof of Theorem 2 is complete. □

Remark 1 *This paper is a shortened version of the electronic preprint* [4].

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References

[1] C. Banderier and S. Schwer, Why Delannoy numbers? *J. Statist. Plann. Inference*, **135** (2005), no. 1, 40–54; available online at https://doi.org/10.1016/j.jspi.2005.02.004.

[2] N. Bourbaki, *Elements of Mathematics: Functions of a Real Variable: Elementary Theory*, Translated from the 1976 French original by Philip Spain. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004; available online at http://dx.doi.org/10.1007/978-3-642-59315-4.

[3] N. D. Cahill and D. A. Narayan, Fibonacci and Lucas numbers as tridiagonal matrix determinants, *Fibonacci Quart.*, **42** (2004), no. 3, 216–221.

[4] F. Qi, *A determinantal expression and a recursive relation of the Delannoy numbers*, arXiv preprint (2020), available online at https://arxiv.org/abs/2003.12572.

[5] F. Qi, V. Čerňanová, and Y. S. Semenov, Some tridiagonal determinants related to central Delannoy numbers, the Chebyshev polynomials, and the Fibonacci polynomials, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, **81** (2019), no. 1, 123–136.

[6] F. Qi, V. Čerňanová, X.-T. Shi, and B.-N. Guo, Some properties of central Delannoy numbers, *J. Comput. Appl. Math.*, **328** (2018), 101–115; available online at https://doi.org/10.1016/j.cam.2017.07.013.

[7] Y. Wang, S.-N. Zheng, and X. Chen, Analytic aspects of Delannoy numbers, *Discrete Math.*, **342** (2019), no. 8, 2270–2277; available online at https://doi.org/10.1016/j.disc.2019.04.003.

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On graphs with minimal distance signless Laplacian energy

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Abstract. For a simple connected graph G of order n having distance signless Laplacian eigenvalues $\rho_1^Q \geq \rho_2^Q \geq \cdots \geq \rho_n^Q$, the distance signless Laplacian energy $DSLE(G)$ is defined as $DSLE(G) = \sum_{i=1}^{n} \left|\rho_i^Q - \frac{2W(G)}{n}\right|$, where $W(G)$ is the Weiner index of G. We show that the complete split graph has the minimum distance signless Laplacian energy among all connected graphs with given independence number. Further, we prove that the graph $K_k \vee (K_t \cup K_{n-k-t})$, $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$ has the minimum distance signless Laplacian energy among all connected graphs with vertex connectivity k.

1 Introduction

A simple and finite graph is denoted by $G(V(G), E(G))$ (or simply by G when there is no confusion), where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is its vertex set and $E(G)$

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is its edge set. The cardinality of $V(G)$ and $E(G)$ are respectively the *order* and *size* of G, and are denoted by n and m. The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to $v \in V(G)$, and its cardinality is the *degree* of v, denoted by $d_G(v)$ (we simply write d_v if it is clear from the context). Throughout this paper, G will be connected. The adjacency matrix $A = [a_{ij}]$ of G is a $(0,1)$-square matrix of order n whose (i,j)-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise. The diagonal matrix of vertex degrees $d_i = d_G(v_i)$, $i = 1, 2, \dots, n$ associated to G is $Deg(G) = diag[d_1, d_2, \dots, d_n]$. The real symmetric and positive semi-definite matrices $L(G) = Deg(G) - A(G)$ and $Q(G) = Deg(G) + A(G)$ are respectively the Laplacian and the signless Laplacian matrices and their spectrum are respectively the Laplacian spectrum and signless Laplacian spectrum of the graph G. Recent work on signless Laplacian spectrum can be seen in [11, 20, 21, 22]. We use standard terminology, K_n denotes a complete graph, $K_{a,n-a}$ is a complete bipartite graph with partite sets of cardinality a and $n-a$. For other undefined notations and terminology, the readers are referred to [5, 13, 15, 16, 23].

In a connected graph G, the *distance* between two vertices $v_1, v_2 \in V(G)$, denoted by $d(v_1, v_2)$, is the length of a shortest path between v_1 and v_2. The *diameter* of G is the maximum distance between any two pair of vertices of G. The *distance matrix* of G, denoted by $D(G)$, is defined as $D(G) = [d(v_i, v_j)]$ where $v_i, v_j \in V(G)$. The *transmission* $Tr_G(v)$ (or simply by $Tr(v)$, when graph under consideration is clear) of a vertex v is defined to be the sum of the distances from v to all other vertices in G, that is, $Tr(v) = \sum\limits_{u \in V(G)} d_{uv}$. The *transmission* number or Wiener index of a graph G, denoted by $W(G)$, is the sum of distances between all unordered pairs of vertices in G. Clearly, $W(G) = \frac{1}{2} \sum\limits_{v \in V(G)} Tr(v)$. For any vertex $v_i \in V(G)$, the transmission $Tr(v_i)$ is called the *transmission degree*, shortly denoted by Tr_i and the sequence $\{Tr_1, Tr_2, \dots, Tr_n\}$ is called the *transmission degree sequence* of the graph G.

If $Tr(G) = diag[Tr_1, Tr_2, \dots, Tr_n]$ is the diagonal matrix of vertex transmissions of G, the matrices $D^L(G) = Tr(G) - D(G)$ and $D^Q(G) = Tr(G) + D(G)$ are respectively called as the *distance Laplacian matrix* and the *distance signless Laplacian matrix* of G [3].

If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the adjacency eigenvalues of a graph G, the *energy* of G [12], denoted by $E(G)$, is defined as $E(G) = \sum\limits_{i=1}^{n} |\lambda_i|$. The reader is referred to the book [15] and for some recent work to [4, 9, 10].

Let $\rho_1^D \geq \rho_2^D \geq \ldots \geq \rho_n^D$ and $\rho_1^Q \geq \rho_2^Q \geq \ldots \geq \rho_n^Q$ be respectively, the distance, and distance signless Laplacian eigenvalues of the graph G. The distance energy [14] of a graph G is the sum of the absolute values of the distance eigenvalues of G, that is, $DE(G) = \sum_{i=1}^{n} |\rho_i^D|$. For some recent works on distance energy, we refer to [2, 6, 8, 18] and the references therein. The distance signless Laplacian energy $DSLE(G)$ [6] of a connected graph G is defined as

$$DSLE(G) = \sum_{i=1}^{n} |\rho_i^Q - \frac{2W(G)}{n}|.$$

Let σ' be the largest positive integer such that $\rho_{\sigma'}^Q \geq \frac{2W(G)}{n}$ and let $B_b^Q(G) = \sum_{i=1}^{b} \rho_i^Q$ be the sum of b largest distance signless Laplacian eigenvalues of G. Then, using $\sum_{i=1}^{n} \rho_i^Q = 2W(G)$, in [6], it is shown that

$$\begin{aligned} DSLE(G) &= 2\left(B_{\sigma'}^Q(G) - \frac{2\sigma' W(G)}{n}\right) = 2 \max_{1\leq j\leq n}\left(\sum_{i=1}^{j} \rho_i^Q(G) - \frac{2jW(G)}{n}\right) \\ &= 2 \max_{1\leq j\leq n}\left(B_j^Q(G) - \frac{2jW(G)}{n}\right). \end{aligned}$$

For some recent works on $DSLE(G)$, see [6, 8, 19].

In the next section, we show that the complete split graph has the minimum distance signless Laplacian energy among all connected graphs with given independence number. Further, we show that among all connected graphs with given vertex connectivity k, the graph $K_k \vee (K_t \cup K_{n-k-t})$, $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$ has the minimum distance signless Laplacian energy.

2 Distance signless Laplacian energy of graphs with given independence number and connectivity

Let $e = v_i v_j$ be an edge of a graph G such that $G - e$ is connected. Then removing the edge e increases the distance by at least one unit. Similarly adding an edge decreases the distance by at least one unit. By Perron-Frobenius theorem, if each entry of the first non negative matrix majorizes the second non negative matrix, then their spectrum is also majorized. This is summarized in next useful result, which can be found in [3].

Lemma 1 *Let* G *be a connected graph of order* n *and size* m*, where* $m \geq n$ *and let* $G' = G - e$ *be a connected graph obtained from* G *by deleting an edge. Let* $\rho_1^Q(G) \geq \rho_2^Q(G) \geq \cdots \geq \rho_n^Q(G)$ *and* $\rho_1^Q(G') \geq \rho_2^Q(G') \geq \cdots \geq \rho_n^Q(G')$ *be respectively, the distance signless Laplacian eigenvalues of* G *and* G'*. Then* $\rho_i^Q(G') \geq \rho_i^Q(G)$ *holds for all* $1 \leq i \leq n$.

Motivated by Lemma 1, we have the following observation, which says that the complete graph has minimum distance signless Laplacian energy among all graphs of order n.

Theorem 1 *Let* G *be a connected graph of order* n*. Then*

$$DSLE(G) \geq 2\left(n + b(n-2) - \frac{2W(G)}{n}\right),$$

equality occurs if and only if $G \cong K_n$.

Proof. By Lemma 1, $\rho_i^Q(G) \geq \rho_i^Q(K_n)$ for each $i = 1, 2, \ldots, n$. So using the definition of $B_b^Q(G)$, we have

$$B_b^Q(G) \geq B_b^Q(K_n) = 2n - 2 + (b-1)(n-2), \tag{1}$$

with equality if and only if $G \cong K_n$. Let σ' be the positive integer such that $\rho_{\sigma'}^Q \geq \frac{2W(G)}{n}$. Then using (1) and the definition of distance signless Laplacian energy, we have

$$\begin{aligned} DSLE(G) &= 2\left(\sum_{i=1}^{\sigma'} \rho_i^Q(G) - \frac{2\sigma' W(G)}{n}\right) = 2 \max_{1 \leq j \leq n}\left(\sum_{i=1}^{j} \rho_i^Q(G) - \frac{2jW(G)}{n}\right) \\ &\geq 2 \max_{1 \leq j \leq n}\left(\sum_{i=1}^{j} \rho_i^Q(K_n) - \frac{2jW(G)}{n}\right) = 2\left(n + b(n-2) - \frac{2(b-1)W(G)}{n}\right). \end{aligned}$$

By Lemma 1 and Inequality (1), equality holds if and only if $G \cong K_n$. □

A graph is complete split, denoted by $CS_{n,\alpha}$, if it can be partitioned into an independent set (a subset of vertices of a graph is said to be an independent set if the subgraph induced by them is an empty graph) on α vertices and a clique on $n - \alpha$ vertices, such that each vertex of the independent set is adjacent to every vertex of the clique.

The following result [17] gives the distance signless Laplacian spectrum of $CS_{n,\alpha}$.

Lemma 2 *Let* $CS_{n,\alpha}$ *be the complete split graph with independence number* α*. Then the distance signless Laplacian spectrum of* $CS_{n,\alpha}$ *is given by* $\left\{\frac{3n+2\alpha-6\pm\sqrt{n^2+12\alpha^2-\alpha(4n+16)+4n+4}}{2}, (n+\alpha-4)^{[\alpha-1]}, (n-2)^{[n-\alpha-1]}\right\}$.

Since independence number of the complete graph K_n is 1 and its distance signless Laplacian energy is discussed in Theorem 1, so we assume $2 \leq \alpha \leq n-2$, and discuss $\alpha = n-1$ separately. The following theorem shows that among all connected graphs with given independence number α, the complete split graph $CS_{n,\alpha}$ has the minimum distance signless Laplacian energy.

Theorem 2 *Let* G *be a connected graph of order* $n \geq 3$ *having independence number* α*, where* $\frac{n+1-\sqrt{n^2+1-10n}}{2} < \alpha < \frac{n+1+\sqrt{n^2+1-10n}}{2}$*. Then*

$$DSLE(G) \geq \begin{cases} 2\Big(2n+\alpha(n-3)+\alpha^2-2-\frac{2(\alpha+1)W(G)}{n}\Big), & \text{if } \alpha \leq \frac{n}{2}, \\ n+\sqrt{\theta}+\alpha(2n-8)+2\alpha^2+2-\frac{4\alpha W(G)}{n}, & \text{if } \alpha > \frac{n}{2}, \end{cases}$$

where $\theta = n^2+12\alpha^2+4n-\alpha(4n+16)+4$*. Equality occurs in each of the inequalities if and only if* $G \cong CS_{n,\alpha}$*.*

Proof. Let G be a connected graph of order $n \geq 3$ having independence number α. Let $CS_{n,\alpha}$ be the complete split graph having independence number α. Clearly, G is a spanning subgraph of $CS_{n,\alpha}$. Therefore, by Lemma 1, we have $\rho_i^Q(G) \geq \rho_i^Q(CS_{n,\alpha})$. Let σ' be the largest positive integer such that $\rho_{\sigma'}^Q(G) \geq \frac{2W(G)}{n}$. With this information, and using the equivalent definition of distance signless Laplacian energy, we have

$$\begin{aligned} DSLE(G) &= 2\left(\sum_{i=1}^{\sigma'} \rho_i^Q(G) - \frac{2\sigma' W(G)}{n}\right) = 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j} \rho_i^Q(G) - \frac{2jW(G)}{n}\right) \\ &\geq 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j} \rho_i^Q(CS_{n,\alpha}) - \frac{2jW(G)}{n}\right). \end{aligned} \tag{2}$$

By using Lemma 2, the trace is $n^2+\alpha^2-n-\alpha$ and the average Wiener index is $\frac{2W(CS_{n,\alpha})}{n} = \frac{n^2+\alpha^2-n-\alpha}{n}$. Therefore, it follows that $\frac{3n+2\alpha-6+\sqrt{n^2+12\alpha^2-\alpha(4n+16)+4n+4}}{2}$ is the distance signless Laplacian spectral radius of $CS_{n,\alpha}$. Next, for the eigenvalue $n+\alpha-4$, we have

$$n+\alpha-4 < \frac{2W(CS_{n,\alpha})}{n} = \frac{n^2+\alpha^2-n-\alpha}{n},$$

provided

$$\alpha^2 - (n+1)\alpha + 3n > 0. \tag{3}$$

Consider the polynomial $f(t) = t^2 - (n+1)t + 3n$, for $1 \leq t \leq n-1$. The zeros of this polynomial are

$$x_1 = \frac{n+1-\sqrt{n^2+1-10n}}{2} \text{ and } x_2 = \frac{n+1+\sqrt{n^2+1-10n}}{2}.$$

This implies that $f(t) > 0$ for all $t < x_1$ and $f(t) > 0$ for all $t > x_2$. From this, for

$$\frac{n+1-\sqrt{n^2+1-10n}}{2} < \alpha < \frac{n+1+\sqrt{n^2+1-10n}}{2},$$

we have $n + \alpha - 4 \geq \frac{2W(CS_{n,\alpha})}{n}$. Similarly, for the second smallest distance signless Laplacian eigenvalue, we have

$$\frac{3n+2\alpha-6-\sqrt{n^2+12\alpha^2-\alpha(4n+16)+4n+4}}{2} < \frac{2W(CS_{n,\alpha})}{n},$$

which after simplification implies that

$$(12-8\alpha)n^3+(-12-4\alpha+12\alpha^2)n^2+(16\alpha-24\alpha^2+8\alpha^3)n+8\alpha^3-4\alpha^2-4\alpha^4 > 0. \tag{4}$$

Inequality (4) is a function of two variables, and putting conditions on the independence number α we have verified that (4) holds true for $\alpha \leq \frac{n}{2}$. Also, the smallest distance signless Laplacian eigenvalue $n-2$ is always less than $\frac{2W(CS_{n,\alpha})}{n}$. Therefore, we have the following cases to consider.

Case (i). If $\alpha \leq \frac{n}{2}$, then $\sigma' = \alpha$. Thus, from (2), it follows that

$$\begin{aligned}
DSLE(G) &\geq 2 \max_{1\leq j\leq n}\left(\sum_{i=1}^{j}\rho_i^Q(CS_{n,\alpha}) - \frac{2jW(G)}{n}\right)\\
&\geq 2\left(\sum_{i=1}^{\alpha}\rho_i^Q(CS_{n,\alpha}) - \frac{2\alpha W(G)}{n}\right)\\
&= 2\left(\frac{3n+2\alpha-6+\sqrt{\theta}}{2} + (\alpha-1)(n+\alpha-4) - \frac{2\alpha W(G)}{n}\right)\\
&= n + \alpha(2n-8) + 2\alpha^2 + 2 + \sqrt{\theta} - \frac{4\alpha W(G)}{n},
\end{aligned}$$

where $\theta = n^2 + 12\alpha^2 + 4n - \alpha(4n+16) + 4$.
Case (ii). If $\alpha > \frac{n}{2}$, then $\sigma' = \alpha + 1$. So, from (2), it follows that

$$\begin{aligned}
DSLE(G) &\geq 2 \max_{1\leq j\leq n} \left(\sum_{i=1}^{j} \rho_i^Q(CS_{n,\alpha}) - \frac{2jW(G)}{n} \right) \\
&\geq 2 \left(\sum_{i=1}^{\alpha+1} \rho_i^Q(CS_{n,\alpha}) - \frac{2(\alpha+1)W(G)}{n} \right) \\
&= 2 \left(3n + 2\alpha - 6 + (\alpha-1)(n+\alpha-4) - \frac{2(\alpha+1)W(G)}{n} \right) \\
&= 2 \left(2n + \alpha(n-3) + \alpha^2 - 2 - \frac{2(\alpha+1)W(G)}{n} \right).
\end{aligned}$$

Equality occurs in all the inequalities above if and only if equality occurs in Inequality (2). It is clear that equality occurs in (2) if and only if $G \cong CS_{n,\alpha}$. This shows that equality occurs in all the inequalities above if and only if $G \cong CS_{n,\alpha}$. This completes the proof. $\square$

When order n of graph G increases, we observe that $\frac{n+1-\sqrt{n^2+1-10n}}{2} \approx 3$ and $\frac{n+1+\sqrt{n^2+1-10n}}{2} \approx n-2$. These remaining cases of independence are discussed as follows.

Proposition 1 *Let G be a graph of order* $n \geq 3$ *with independence number* $\alpha \in \{2,3\}$. *Then*

$$DSLE(G) \geq \begin{cases} 2\left(3n - 2 - \frac{4W(G)}{n}\right), & \text{if } \alpha = 2, \\ 2\left(3n - \frac{4W(G)}{n}\right), & \text{if } \alpha = 3, \end{cases}$$

equality occurs in first and second inequality if and only $G \cong CS_{n,2}$ *and* $CS_{n,3}$ *respectively.*

Proof. By substituting $\alpha = 2$ in Lemma 2, the distance signless Laplacian spectrum of $CS_{n,2}$ is given by $\left\{\frac{1}{2}(3n-2\pm\sqrt{n^2-4n+20}), (n-2)^{[n-2]}\right\}$ and the Wiener index can be calculated to be $\frac{2W(G)}{n} = \frac{n^2-n+2}{n}$. Clearly, $\frac{1}{2}(3n - 2 + \sqrt{n^2-4n+20})$ is the spectral radius and it is always greater or equal to Wiener index. Next $\frac{1}{2}(3n-2-\sqrt{n^2-4n+20}) < \frac{n^2-n+2}{n}$ implies that $n^2(n^2-4n+20) - 16 > 0$ which is true for each $n \geq 1$. Also, the smallest distance signless Laplacian eigenvalue is always strictly less than $\frac{2W(G)}{2}$. Thus, we have $\sigma' = 2$ and the distance signless Laplacian energy is given by

$$\mathsf{DSLE}(G) \geq 2\left(\sum_{i=1}^{2} \rho_i^Q(G) - \frac{4W(G)}{n}\right) = 2\left(3n - 2 - \frac{4W(G)}{2}\right). \tag{5}$$

By using similar arguments, we can easily prove the second inequality. Equality holds as in Theorem 2. ☐

Now, we obtain a lower bound for the distance signless Laplacian energy when independence number is $\alpha = n-2,$ or $n-1$.

Proposition 2 *Let G be a graph of order $n \geq 6$ with independence number $\alpha \in \{n-2, n-1\}$. Then*

$$\mathsf{DSLE}(G) \geq \begin{cases} n(4n-19) + \sqrt{9n^2 - 52n + 84} + 26 - \frac{4W(G)}{n}, & \text{if } \alpha = n-2, \\ 5n + \sqrt{9n^2 - 32n + 32} - 8 - \frac{4W(G)}{n}, & \text{if } \alpha = n-1, \end{cases}$$

equality occurs in first and second inequality if and only $G \cong CS_{n,n-2}$ *and* $CS_{n,n-1}$ *respectively.*

Proof. From Lemma 2, the distance signless Laplacian spectrum of $CS_{n,n-2}$ with independence number $n-2$ is given by

$$\left\{\frac{1}{2}(5n - 10 \pm \sqrt{9n^2 - 52n + 84}), (2n-6)^{[n-3]}, n-2\right\}$$

and Wiener index is $\frac{2W(G)}{n} = \frac{2n^2-6n+6}{n}$. Now, it is clear that $\frac{1}{2}(5n - 10 + \sqrt{9n^2 - 52n + 84})$ is the spectral radius and is always greater or equal to $\frac{2W(G)}{n}$. Also, $2n - 6 < \frac{2W(G)}{n}$ implies that $6 > 0$, which is always true. For the second smallest distance signless Laplacian eigenvalue $\frac{1}{2}(5n-10-\sqrt{9n^2 - 52n + 84})$, we have $\frac{1}{2}(5n - 10 - \sqrt{9n^2 - 52n + 84}) < \frac{n^2-n+2}{n}$, which implies that $n^4 - 7n^3 + 13n^2 + 6n - 18 > 0$, and is true for each $n \geq 2$. Also, the smallest distance signless Laplacian eigenvalue is always strictly less than $\frac{2W(G)}{2}$. Thus, we have $\sigma' = n-2$ and the distance Laplacian energy is given by

$$\begin{aligned} \mathsf{DSLE}(G) &\geq 2\left(\sum_{i=1}^{n-2} \rho_i^Q(G) - \frac{2(n-2)W(G)}{n}\right) \\ &= 2\left(\frac{5n - 10 + \sqrt{9n^2 - 52n + 84}}{2} + (n-3)(2n-6) - \frac{2(n-2)W(G)}{2}\right) \end{aligned}$$

$$=n(4n-19)+26+\sqrt{9n^2-52n+84}-\frac{4(n-2)W(G)}{2}$$

By using similar arguments, we can easily prove the second inequality. By Lemma 1, equality holds as in Theorem 2. □

The vertex connectivity of a graph G, denoted by $\kappa(G)$, is the minimum number of vertices of G whose deletion disconnects G. Let $\mathcal{F}_n$ be the family of simple connected graphs on n vertices and let

$$\mathcal{V}_n^k=\{G\in\mathcal{F}_n:\kappa(G)\leq k\},$$

that is, $\mathcal{V}_n^k$ is the family of graphs with vertex connectivity at most k.

Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be two graphs on disjoint vertex sets V_1 and V_2 with orders n_1 and n_2, respectively. Then their *union* is the graph $G_1\cup G_2=(V_1\cup V_2,E_1\cup E_2)$. The *join* of graphs G_1 and G_2, denoted by $G_1\vee G_2$, is the graph consisting of $G_1\cup G_2$ and all edges joining the vertices in V_1 and the vertices in V_2. In other words, the join of two graphs G_1 and G_2, denoted by $G_1\vee G_2$, is the graph obtained from G_1 and G_2 by joining each vertex of G_1 to every vertex of G_2.

The following result [17] gives the distance signless Laplacian spectrum of the join of a connected graph G_1 with the union of two connected graphs G_2 and G_3, in terms of the adjacency spectrum of the graphs G_1, G_2 and G_3.

Theorem 3 *Let G_i be r_i regular graphs of orders n_i, having adjacency eigenvalues $\lambda_{i,1}=r_i\geq\lambda_{i,2}\geq\cdots\geq\lambda_{i,n_i}$, for $i=1,2,3$. Then the distance signless Laplacian eigenvalues of $G_1\vee(G_2\cup G_3)$ are $(n+n_1-r_1-\lambda_{1,k}-4)^{[n_1-1]}$, $(2n-n_1-r_2-\lambda_{2,k}-4)^{[n_2-1]}$, $(2n-n_1-r_3-\lambda_{3,k}-4)^{[n_3-1]}$, where $k=2,3,\ldots,n_i$, for $i=1,2,3$ and $n=n_1+n_2+n_3$. The remaining three eigenvalues are given by the equitable quotient matrix*

$$\begin{bmatrix} n+3n_1-2r_1-4 & n_2 & n_3\\ n_1 & 2n+2n_2-n_1-2r_2-4 & 2n_3\\ n_1 & 2n_2 & 2n+2n_3-n_1-2r_3-4\end{bmatrix}.$$

Corollary 1 *Let $G=K_k\vee(K_t\cup K_{n-t-k})$, where $\vee$ is the join and $\cup$ is the union, be the connected graph with connectivity k. Then the distance signless Laplacian spectrum of G consists of the eigenvalue $\frac{4n-k-4\pm\sqrt{k^2+16nt-16kt-16t^2}}{2}$, the eigenvalue $(2n-k-t-2)$ with multiplicity $t-1$, the eigenvalue $(n+t-2)$ with multiplicity $n-k-t-1$ and the eigenvalue $(n-2)$ with multiplicity k.*

Proof. Let $G_1 = K_k, G_2 = K_t$ and $G_3 = K_{n-t-k}$. Then substituting $r_1 = k-1, r_2 = t-1$, and $r_3 = n-k-t-1$ and noting that the adjacency spectrum of K_ω is $\{n-1, (-1)^{[\omega]}\}$, the result follows by Theorem 3. $\square$

The following lemma says that for each $G \in \mathcal{V}_n^k$, the graph $K_k \vee (K_t \cup K_{n-t-k})$ has the minimum value of $B_i^Q, 1 \leq i \leq n-1$, that is, the sum of i^{th} largest distance signless Laplacian eigenvalues.

Lemma 3 *Let* G *be a connected graph of order* n *with vertex connectivity* k*,* $1 \leq k \leq n-1$*. Then*

$$B_i^Q(G) \geq B_i^Q(K_k \vee (K_t \cup K_{n-t-k})),$$

with equality if and only if $G \cong K_k \vee (K_t \cup K_{n-t-k})$.

Proof. Let G be a connected graph of order n with vertex connectivity k, $1 \leq k \leq n-1$. We first show that $B_i^Q(G) \geq B_i^Q(K_k \vee (K_t \cup K_{n-t-k}))$, for all $i = 1, 2, \ldots, n$. Suppose that $1 \leq k \leq n-2$. Then G is the connected graph of order n with vertex connectivity k for which the spectral parameter $B_i^Q(G)$ has the minimum possible value. It is clear that $G \in \mathcal{V}_n^k$ and $B_i^Q(G)$ attains the minimum value for G. Let $U \subseteq V(G)$ be such that $G - U$ is disconnected and has r connected components, say $G_1, G_2, \ldots, G_r$. We are required to show that $r = 2$. For if, $r > 2$, then we can construct a new graph $G' = G + e$ by adding an edge between any two components, say G_1 and G_2 of $G - S$, which is such that $G' \in \mathcal{V}_n^k$. By Lemma 1, we have $B_i^Q(G) > B_i^Q(G')$. This is a contradiction to the fact $B_i^Q(G)$ attains the minimum possible value for G. Therefore, we must have $r = 2$. Further, we claim that each of the components G_1, G_2 and the vertex induced subgraph $\langle U \rangle$ are cliques. For if one among them say G_1 is not a clique, then adding an edge between the two non adjacent vertices of G_1 gives a graph $H \in \mathcal{V}_n^k$ and by Lemma 1, we have $B_i^Q(G) > B_i^Q(H)$. This is again a contradiction, as $B_i^Q(G)$ attains minimum possible value for G. Again $|U| \leq k$, and we prove that $|U| = k$. Assume that $|U| < k$. In a similar way, we can form a new graph $G + e = L \in \mathcal{V}_n^k$, where e is adjacent to a vertex of G_1 with a vertex of G_2. Thus, by Lemma 1, $B_i^Q(G) > B_i^Q(H)$, which is not true. Hence G must be of the form $G = K_k \vee (K_t \cup K_{n-k-t})$, $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$. This shows that for all $G \in \mathcal{V}_n^k$, the spectral parameter $B_i^Q(G)$ has the minimum possible value for the graph $K_k \vee (K_t \cup K_{n-k-t})$. $\square$

As $1 \leq k =\leq n-1$ and $t \leq n-k-t$, we have $t \leq \lfloor \frac{n-k}{2} \rfloor$. Also, the distance signless Laplacian energy for $k = n-1$ is given by Theorem 1, so we avoid the case $k = n-1$, and thus $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$ makes sense.

Now, we prove that among all connected graphs with given vertex connectivity k, the graph $K_k \vee (K_t \cup K_{n-k-t})$ has the minimum distance signless Laplacian energy.

Theorem 4 *Let $G \in \mathcal{V}_n^k$ be a connected graph of order $n \geq 4$ with vertex connectivity number k satisfying $a_2 \leq k \leq a_1$. Then*

$$DSLE(G) \geq \begin{cases} \sqrt{D} + 2t(2n-k-t-1) + k - \frac{4tW(G)}{n}, \\ \sqrt{D} + 2n^2 + n(4t-2k-6) - 4kt - 4t^2 + 5k + 4 - \frac{4(n-k-1)W(G)}{n}, \end{cases}$$

according as $k < \frac{n(t+1)}{2t} - t$ or $k \geq \frac{n(t+1)}{2t} - t$, where $a_i = \frac{n^2(10t+1)-n^3-n(10t^2+4t)+8t^3 \pm \sqrt{n^4-n^3(12t+2)+n^2(40t^2+12t+1)+n(8t^3-36t^2)+4t^4}}{4(nt-2t^2)}$ *and* $D = k^2 + 16nt - 16kt - 16t^2$. *Equality occurs in each of these inequalities if and only if $G \cong K_k \vee (K_t \cup K_{n-k-t})$ with $1 \leq t \leq \lfloor \frac{n-k}{2} \rfloor$.*

Proof. Let G be a connected graph of order n with vertex connectivity k, $2 \leq k \leq n-2$. Then, by Lemma 3, for each $G \in \mathcal{V}_n^k$, the spectral parameter $B_i^Q(G)$ has the minimum possible value for the graph $K_k \vee (K_t \cup K_{n-k-t})$. That is, for all $G \in \mathcal{V}_n^k$, we have $B_i^Q(G) \geq B_i^Q(K_k \vee (K_t \cup K_{n-t-k}))$. With this, from the definition of distance signless Laplacian energy, it follows that

$$\begin{aligned} DSLE(G) &= 2\left(B_{\sigma'}(G) - \frac{2\sigma' W(G)}{n}\right) = 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^{j} \rho_i^Q(G) - \frac{2jW(G)}{n}\right) \\ &\geq 2 \max_{1 \leq j \leq n} \left(\sum_{i=1}^{j} \rho_i^Q(K_k \vee (K_t \cup K_{n-t-k})) - \frac{2jW(G)}{n}\right). \end{aligned} \tag{6}$$

By Corollary 1, the distance signless Laplacian spectrum of the graph $K_k \vee (K_t \cup K_{n-t-k})$ is $\left\{\frac{4n-k-4 \pm \sqrt{k^2+16nt-16kt-16t^2}}{2}, (2n-k-t-2)^{[t-1]}, (n+t-2)^{[n-k-t-1]}, (n-2)^k\right\}$. Let σ' be the number of distance signless Laplacian eigenvalues of $K_k \vee (K_t \cup K_{n-t-k})$ which are greater than or equal to that $\frac{2W(K_k \vee (K_t \cup K_{n-t-k}))}{n} = \frac{n^2-n+2nt-2t^2-2kt}{n}$. Clearly, $\frac{4n-k-4+\sqrt{k^2+16nt-16kt-16t^2}}{2}$ is the distance signless Laplacian spectral radius of the graph $K_k \vee (K_t \cup K_{n-t-k})$ and is always greater than $\frac{2W(K_k \vee (K_t \cup K_{n-t-k}))}{n}$. Now, for the eigenvalue $2n-k-t-2$, we have

$$2n-k-t-2 \geq \frac{2W(K_k \vee (K_t \cup K_{n-t-k}))}{n} = \frac{n^2-n+2nt-2t^2-2kt}{n}$$

which implies that

$$2t^2-(3n-2k)t+(n^2-n-kn)\geq 0. \tag{7}$$

The roots of the polynomial $g_1(t)=2t^2-(3n-2k)t+(n^2-n-kn)=0$ are

$$r_1=\frac{3n-2k+\sqrt{(n-2k)^2+8n}}{2} \text{ and } r_2=\frac{3n-2k-\sqrt{(n-2k)^2+8n}}{2}.$$

This shows that $g_1(t)\geq 0$, for all $t\leq r_2$ and $t\geq r_1$. Since,

$$t=\frac{n-k}{2}<\frac{3n-2k-\sqrt{(n-2k)^2+8n}}{2}=r_2$$

gives $k\leq n-2$, which is the maximum value for connectivity. Thus, $g_1(t)\geq 0$, for all $t\leq\frac{n-k}{2}$. For the eigenvalue $n+t-2$, we have

$$n+t-2\geq\frac{2W(K_k\vee(K_t\cup K_{n-t-k}))}{n}=\frac{n^2-n+2nt-2t^2-2kt}{n}$$

which implies that $k\geq\frac{n(t+1)}{2t}-t$. This shows that

$$n+t-2\geq\frac{2W(K_k\vee(K_t\cup K_{n-t-k}))}{n},$$

for all $k\geq\frac{n(t+1)}{2t}-t$ and

$$n+t-2<\frac{2W(K_k\vee(K_t\cup K_{n-t-k}))}{n},$$

for all $k<\frac{n(t+1)}{2t}-t$. For the second smallest distance signless Laplacian eigenvalue

$$\frac{4n-k-4-\sqrt{k^2+16nt-16kt-16t^2}}{2},$$

we have

$$\begin{aligned}\frac{4n-k-4+\sqrt{k^2+16nt-16kt-16t^2}}{2}&\geq\frac{2W(K_k\vee(K_t\cup K_{n-t-k}))}{n}\\&=\frac{n^2-n+2nt-2t^2-2kt}{n}\end{aligned}$$

implying that

$$\begin{aligned}f(k)=&\,k^2(16t^2-8nt)+k(4n^2-4n^3-16nt+40n^2t-40nt^2+32t^3)\\&-8n^3+4n^4+4n^2+16n^2t-32n^3t-16nt^2+48n^2t^2-32nt^3+16t^4\\&\geq 0.\end{aligned}$$

which in turn implies that

$$\frac{4n-k-4-\sqrt{k^2+16nt-16kt-16t^2}}{2}<\frac{2W(K_k\vee(K_t\cup K_{n-t-k}))}{n}$$

for $a_2<k<a_1$, where
$a_i=\frac{n^2(10t+1)-n^3-n(10t^2+4t)+8t^3\pm\sqrt{n^4-n^3(12t+2)+n^2(40t^2+12t+1)+n(8t^3-36t^2)+4t^4}}{4(nt-2t^2)}$,
$i=1,2$, are the zeros of $f(k)$. From these calculations it follows that, if $k<\frac{n(t+1)}{2t}-t$, then $\sigma'=t$, and if $k\geq\frac{n(t+1)}{2t}-t$, then $\sigma'=n-k-1$. Therefore, for $k<\frac{n(t+1)}{2t}-t$, it follows from Inequality (6) that

$$\begin{aligned}
DSLE(G)&\geq 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j}\rho_i^Q(K_k\vee(K_t\cup K_{n-t-k}))-\frac{2jW(G)}{n}\right)\\
&\geq 2\left(\sum_{i=1}^{t}\rho_i^Q(K_k\vee(K_t\cup K_{n-t-k}))-\frac{2tW(G)}{n}\right)\\
&=2\left(\frac{4n-k-4+\sqrt{k^2+16nt-16kt-16t^2}}{2}\right.\\
&\quad\left.+(t-1)(2n-k-t-2)-\frac{2tW(G)}{n}\right)\\
&=\sqrt{k^2+16nt-16kt-16t^2}+2t(2n-k-t-1)+k-\frac{4tW(G)}{n}.
\end{aligned}$$

If $k\geq\frac{n(t+1)}{2t}-t$, from (6), we have

$$\begin{aligned}
&DSLE(G)\geq 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j}\rho_i^Q(K_k\vee(K_t\cup K_{n-t-k}))-\frac{2jW(G)}{n}\right)\\
&\geq 2\left(\sum_{i=1}^{n-k-1}\rho_i^Q(K_k\vee(K_t\cup K_{n-t-k}))-\frac{2(n-k-1)W(G)}{n}\right)\\
&=2\left(\frac{4n-k-4+\sqrt{D}}{2}+(t-1)(2n-k-t-2)+(n-k-t-1)(n+t-2)\right)\\
&-\frac{4(n-k-1)W(G)}{n}=\sqrt{D}+2n^2+n(4t-2k-6)-4kt-4t^2+5k+4\\
&-\frac{4(n-k-1)W(G)}{n},
\end{aligned}$$

where $D=k^2+16nt-16kt-16t^2$. By Lemmas 1 and 3, equality holds if and only if $G\cong K_k\vee(K_t\cup K_{n-t-k})$. This completes the proof. $\square$

The next result is the special case of $G \in \mathcal{V}_n^k$, for $t = 1$.

Proposition 3 *Let* $G \in \mathcal{V}_n^k$ *and* $t = 1$. *Then*

$$\mathrm{DSLE}(G) \geq 2\left(4n - k - 4 - \frac{8W(G)}{n}\right)$$

with equality if and only if $G \cong K_k \vee (K_1 \cup K_{n-1-k})$.

Proof. By letting $t = 1$ in Corollary 1, the distance signless Laplacian spectrum of $K_k \vee (K_1 \cup K_{n-1-k})$ is given by

$$\left\{\frac{4n - k - 4 \pm \sqrt{k^2 - 16k + 16n - 16}}{2}, (n-1)^{[n-k-2]}, (n-2)^{[k]}\right\}.$$

Clearly, the distance signless Laplacian eigenvalue $\frac{4n-k-4+\sqrt{k^2-16k+16n-16}}{2}$ is the distance signless spectral radius and is always greater than

$$\frac{2W(K_k \vee (K_1 \cup K_{n-1-k}))}{n} = \frac{n^2 + n - 2k - 2}{n}.$$

For the eigenvalue $n - 1$, we have

$$n - 1 < \frac{2W(K_k \vee (K_1 \cup K_{n-1-k}))}{n}$$

if $n + k > 1$, which is always true as $n \geq 4$ and $k \geq 2$.
Lastly, for the eigenvalue $\frac{4n-k-4-\sqrt{k^2-16k+16n-16}}{2}$, we see if

$$\frac{4n - k - 4 - \sqrt{k^2 - 16k + 16n - 16}}{2} < \frac{2W(K_k \vee (K_1 \cup K_{n-1-k}))}{n},$$

then after simplification, we have

$$h(k) = k^2(8n-16) - k(44n^3 - 4n^3 - 56n + 32) - 4n^4 + 40n^3 - 68n^2 + 48n - 16 < 0.$$

The zeros of $h(k)$ are $n - 1$ and $\frac{9n^2-n^3-8n+4}{2(n-2)}$. This implies that

$$\frac{4n - k - 4 - \sqrt{k^2 - 16k + 16n - 16}}{2} \geq \frac{2W(K_k \vee (K_1 \cup K_{n-1-k}))}{n}$$

for

$$\frac{9n^2 - n^3 - 8n + 4}{2(n-2)} \leq k \leq n - 1.$$

Thus, from (6), we have

$$\begin{aligned}\mathrm{DSLE}(G) &\geq 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j}\rho_i^Q(K_k\vee(K_1\cup K_{n-1-k}))-\frac{2jW(G)}{n}\right)\\ &\geq 2\left(\sum_{i=1}^{2}\rho_i^Q(K_k\vee(K_1\cup K_{n-1-k}))-\frac{2jW(G)}{n}\right)\\ &= 2\left(4n-k-4-\frac{4W(G)}{n}\right).\end{aligned}$$

Clearly, equality occurs by Lemma 1. □

For $G\in\mathcal{V}_n^k$, with $k=t=1$, we have the following observation.

Corollary 2 *Let* $G\in\mathcal{V}_n^1$. *Then, for* $t=1$, *we have*

$$\mathrm{DSLE}(G)\geq 2\left(4n-k-4-\frac{8W(G)}{n}\right)$$

with equality if and only if $G\cong K_1\vee(K_1\cup K_{n-2})$.

Proof. From Corollary 1, the distance signless Laplacian spectrum of $K_1\vee(K_1\cup K_{n-2})$ is given by

$$\left\{\frac{4n-5\pm\sqrt{16n-31}}{2},(n-1)^{[n-3]},n-2\right\}.$$

It can be easily seen that $\frac{4n-5+\sqrt{16n-31}}{2}$ is the distance signless spectral radius and is always greater than $\frac{2W(K_k\vee(K_1\cup K_{n-t-k}))}{n}=\frac{n^2+n-4}{n}$. For the eigenvalue $n-1$, we have $n-1<\frac{2W(K_k\vee(K_1\cup K_{n-t-k}))}{n}$ if $n>2$, which is always true. Next for the eigenvalue $\frac{4n-5-\sqrt{16n-31}}{2}$, we see that $\frac{4n-5-\sqrt{16n-31}}{2}\geq\frac{n^2+n-4}{n}$, which after simplification gives $n^4-11n^3+28n^2-28n+16\geq 0$, which is true for $n\geq 8$. Thus, from (6), we have

$$\begin{aligned}\mathrm{DSLE}(G) &\geq 2\max_{1\leq j\leq n}\left(\sum_{i=1}^{j}\rho_i^Q(K_1\vee(K_1\cup K_{n-2}))-\frac{2jW(G)}{n}\right)\\ &\geq 2\left(\sum_{i=1}^{2}\rho_i^Q(K_1\vee(K_1\cup K_{n-2}))-\frac{2jW(G)}{n}\right)\\ &= 2\left(4n-5-\frac{4W(G)}{n}\right).\end{aligned}$$

□

Conclusions

We observe that the investigation of the graph invariant $B_b^Q(G) = \sum_{i=1}^{b} \rho_i^Q, 1 \leq b \leq n-1$, that is, the sum of the $b \geq 1$ largest signless Laplacian eigenvalues is an interesting problem. By Lemma 1, Theorem 2, Lemma 3 and Theorem 4, we see that $CS_{n,\alpha}$ and $K_k \vee (K_t \cup K_{n-t-k})$ have minimum value of B_b^Q among the graphs with independence α and connectivity k. In a similar manner, it can be shown that K_n and $K_{a,n-a}$ have minimum value of B_b^Q among all graphs and among all the bipartite graphs. In [1], upper bounds for B_b^Q were discussed for graphs with diameter 1 and 2, split graphs, threshold graphs and a conjecture was also put forward. It will be interesting to find the lower bounds for B_b^Q for an arbitrary graph G and characterization of the extremal graphs. By using Lemma 1 and proceeding as in Theorems 2 and 4, we can show that $K_{a,n-a}$ has the minimum distance signless Laplacian energy among all graphs bipartite graphs. A difficult problem is to investigate the graphs with maximum distance signless Laplacian energy. In particular, it will be interesting to study the graphs with maximum signless Laplacian energy among bipartite graphs, split graphs, graphs with fixed connectivity, perfect matching and other families. The graph invariant σ', that is, the number of distance signless Laplacian eigenvalues which are greater or equal to $\frac{2W(G)}{n}$ is an interesting graph invariant. Several papers exist in the literature in this direction and various open problems were asked in case of Laplacian [7] and signless Laplaian matrices. The same is true for distance signless Laplacian matrix and attractive problems of σ' can be investigated, like characterization of graphs having $\sigma' = 1, 2, \frac{n}{2}$ and $\sigma' = n-1$.

References

[1] A. Alhevaz, M. Baghipur H. A. Ganie and S. Pirzada, Brouwer type conjecture for the eigenvalues of distance signless Laplacian of a graph, *Linear and Multilinear Algebra*, DOI:10.1080/03081087.2019.1679074 (2019).

[2] M. Aouchiche and P. Hansen, Distance spectra of graphs: a survey, *Linear Algebra Appl.*, **458** (2014), 301–386.

[3] M. Aouchiche and P. Hansen, Two Laplacians for the distance matrix of a graph, *Linear Algebra Appl.*, **439** (2013), 21–33.

[4] Mushtaq A. Bhat and S. Pirzada, On unicyclic graphs with given number of pendent vertices and minimal energy, *Linear Algebra Appl.*, 597 (2020), 86–93.

[5] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of graphs. Theory and application,* Pure and Applied Mathematics, 87. Academic Press, Inc. New York, 1980.

[6] K. C. Das, M. Aouchiche and P. Hansen, On (distance) Laplacian energy and (distance) signless Laplacian energy of graphs, *Discrete Appl. Math.*, **243** (2018), 172–185.

[7] K. C. Das and S. A. Mojallal , Open problems on σ-invariant, *Taiwanese. J. Math.*, **23** (2019), 1041–1059.

[8] R. C. Diaz and O. Rojo, Sharp upper bounds on the distance energies of a graph, *Linear Algebra Appl.*, **545** (2018), 55–75.

[9] Hilal A. Ganie, U. Samee, S. Pirzada and Ahmad M. Alghamdi, Bounds for graph energy in terms of vertex covering and clique numbers, *Electronic J. Graph Theory Appl.*, **7, 2** (2019), 315–328.

[10] Hilal A. Ganie, U. Samee and S. Pirzada, On graph energy, maximum degree and vertex cover number, *Le Matematiche*, **74, 1** (2019), 163–172.

[11] Hilal A. Ganie, Bilal A. Chat and S. Pirzada, Signless Laplacian energy of a graph and energy of a line graph, *Linear Algebra Appl.*, **544** (2018), 306–324.

[12] I. Gutman, The energy of a graph, *Ber. Math. Statist. Sekt. Forschungsz. Graz*, **103** (1978), 1–22.

[13] R. Horn and C. Johnson, *Matrix Analysis*, Second Edition Cambridge University press, 2013.

[14] G. Indulal, I. Gutman and A. Vijayakumar, On distance energy of graphs, *MATCH Commun. Math. Comput. Chem.*, **60** (2008), 461–472.

[15] X. Li, Y. Shi and I. Gutman, *Graph Energy*, Springer, New York, 2012.

[16] S. Pirzada, *An Introduction to Graph Theory*, Universities Press, Orient BlackSwan, Hyderabad. 2012.

[17] S. Pirzada, B. A. Rather, M. Aijaz and T. A. Chishti, Distance signless Laplacian spectrum of graphs and spectrum of zero divisor graphs of $\mathbb{Z}_n$, *Linear Multilinear Algebra*, http://doi.org/10.1080/03081087.2020.1838425.

[18] S. Pirzada, Bilal A. Rather and T. A. Chishti, On distance Laplacian spectrum of zero divisor graphs of $\mathbb{Z}_n$, *Carpathian Mathematical Publications* **13(1)** (2021), 48–57.

[19] S. Pirzada, Hilal A. Ganie, A. Alhevaz and M. Baghipur, On sum of the powers of distance signless Laplacian eigenvalues of graphs, *Indian J. Pure and Applied Mathematics*, **51, 3** (2020), to appear.

[20] S. Pirzada, Hilal A. Ganie and Rezwan Ul Shaban, On a conjecture for the sum of the signless Laplacian eigenvalues of a graph, *Italian J. Pure and Applied Mathematics*, to appear.

[21] S. Pirzada, Hilal A. Ganie and Ahmad Alghamdi, On the sum of signless Laplacian spectra of graphs, *Carpathian Math. Publications*, **11, 2** (2019), 407–417.

[22] S. Pirzada and Hilal A. Ganie, On the bounds of signless Laplacian energy of a graph, *Discrete Appl. Math.*, **228** (2017), 3–13.

[23] F. Zhang, *Matrix Theory: Basic Results and Techniques*, Second Edition, Springer-Verlag, New York, 2010.

[24] W. T. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.*, **82** (1956), 99–116.

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Para-Kenmotsu manifolds admitting semi-symmetric structures

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Abstract. The object of the present paper is to study para-Kenmotsu manifolds satisfying different conditions of semi-symmetric type.

1 Introduction

Recently, A. A. Shaikh and H. Kundu [8] studied the equivalency of various geometric structures. They have proved that the conditions

i) $R \cdot R = 0$, $R \cdot \tilde{C} = 0$ and $R \cdot P = 0$ are equivalent and we call such a class G_1;

ii) $C \cdot R = 0$, $C \cdot \tilde{C} = 0$ and $C \cdot P = 0$ are equivalent and we call such a class G_2;

iii) $\tilde{C} \cdot R = 0$, $\tilde{C} \cdot \tilde{C} = 0$ and $\tilde{C} \cdot P = 0$ are equivalent and we call such a class G_3;

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iv) $H \cdot R = 0$, $H \cdot \tilde{C} = 0$ and $H \cdot P = 0$ are equivalent and we call such a class G_4;

v) $R \cdot C = 0$ and $R \cdot H = 0$ are equivalent and we call such a class G_5;

vi) $C \cdot H = 0$ and $C \cdot C = 0$ are equivalent and we call such a class G_6;

vii) $\tilde{C} \cdot H = 0$ and $\tilde{C} \cdot C = 0$ are equivalent and we call such a class G_7;

viii) $H \cdot H = 0$ and $H \cdot C = 0$ are equivalent and we call such a class G_8, where R, C, $\tilde{C}$, H and P are the Riemannian, conformal, concircular, conharmonic and projective curvature tensors, respectively.

In an n-dimensional Riemannian manifold (M^n, g) $(n > 3)$, the conformal curvature tensor C [2], conharmonic curvature tensor H [3], concircular curvature tensor $\tilde{C}$ [13] and projective curvature tensor P [7] are defined respectively by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1}$$

$$\begin{aligned} H(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{2}$$

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{4}$$

where Q, S and r are the Ricci operator, the Ricci curvature tensor and the scalar curvature of M^n. The Ricci operator Q and the $(0,2)$-tensor S^2 are defined as

$$S(X,Y) = g(QX,Y) \text{ and } S^2(X,Y) = S(QX,Y) = g(Q^2X,Y).$$

The present paper is structured as follows. In Section 2, we briefly recall some known results for para-Kenmotsu manifolds. In Section 3, we study para-Kenmotsu manifolds belonging to the class G_i $(i = 1, 2, ..., 8)$ and we prove that a para-Kenmotsu manifold belonging to the class G_1 is Einstein, whereas such a manifold belonging to the class G_5 is η-Einstein.

2 Para-Kenmotsu manifolds

The notion of almost paracontact structure was introduced by I. Sato. According to his definition [9], an *almost paracontact structure* (Φ, ξ, η) on an odd-dimensional manifold M^n consists of a $(1,1)$-tensor field Φ, called the structure endomorphism, a vector field ξ, called the characteristic vector field and a 1-form η, called the contact form, which satisfy the following conditions:

$$\Phi^2 = I - \eta \otimes \xi, \tag{5}$$

$$\eta(\xi) = 1, \tag{6}$$

$$\Phi\xi = 0, \ \eta \circ \Phi = 0, \ \text{rank } \Phi = n-1. \tag{7}$$

Moreover, if g is a pseudo-Riemannian metric satisfying

$$g(\Phi X, \Phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{8}$$

for any vector fields X and Y on M^n, then the manifold M^n [9] is said to admit an *almost paracontact Riemannian structure* (Φ, ξ, η, g). Remark that from the above conditions we get

$$g(X, \xi) = \eta(X), \tag{9}$$

for any vector field X on M^n. Examples of almost paracontact metric structures are given in [4, 1].

An analogue of the Kenmotsu manifold [5] in paracontact geometry will be further considered.

Definition 1 [6] *The almost paracontact metric structure* (Φ, ξ, η, g) *is called para-Kenmotsu if the Levi-Civita connection* ∇ *of* g *satisfies*

$$(\nabla_X \Phi)Y = g(\Phi X, Y)\xi - \eta(Y)\Phi X,$$

for any vector fields X *and* Y *on* M^n*.*

The para-Kenmotsu structure was also considered by J. Welyczko in [12] for 3-dimensional normal almost paracontact metric structures. A similar notion called P-Kenmotsu structure appears in the paper of B. B. Sinha and K. L. Sai Prasad [11]. We shall further give some immediate properties of this structure.

Proposition 1 *If* $(M^n, \Phi, \xi, \eta, g)$ *is a para-Kenmotsu manifold, then* [11]*:*

$$S(X, \xi) = -(n-1)\eta(X), \tag{10}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{11}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{12}$$

where S *is the Ricci curvature tensor and* R *is the Riemannian curvature tensor.*

In view of (12), one can easily bring out the followings:

$$\begin{aligned} g(C(X, Y)Z, \xi) &= \eta(C(X, Y)Z)) \\ &= \frac{1}{n-2}\Bigg[\Bigg(\frac{r}{n-1} + 1)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\Bigg) \\ &- \Bigg(S(Y, Z)\eta(X) - S(X, Z)\eta(Y)\Bigg)\Bigg], \end{aligned} \tag{13}$$

$$\begin{aligned} g(H(X, Y)Z, \xi) &= \eta(H(X, Y)Z)) \\ &= \frac{1}{n-2}[g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \\ &- (S(Y, Z)\eta(X) - S(X, Z)\eta(Y))], \end{aligned} \tag{14}$$

$$\begin{aligned} g(\tilde{C}(X, Y)Z, \xi) &= \eta(\tilde{C}(X, Y)Z)) \\ &= \left(\frac{r}{n(n-1)} + 1\right)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \end{aligned} \tag{15}$$

$$\begin{aligned} g(P(X, Y)Z, \xi) &= \eta(P(X, Y)Z) \\ &= g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \\ &- \frac{1}{n-1}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)]. \end{aligned} \tag{16}$$

Definition 2 [10] *An almost paracontact Riemannian manifold* M^n *is said to be an* η*-Einstein manifold if the Ricci curvature tensor* S *is of the form*

$$S = ag + b\eta \otimes \eta,$$

where a *and* b *are smooth functions on* M^n *and* η *is a* 1*-form.*

In particular, if $b = 0$, then M^n is said to be an *Einstein manifold.*

3 Main results

In this section we consider different types of semi-symmetric para-Kenmotsu manifolds, namely, para-Kenmotsu manifolds belonging to the classes G_i $(i = 1, 2, ..., 8)$.

3.1 Para-Kenmotsu manifolds belonging to the class G_1

We consider para-Kenmotsu manifolds admitting the condition

$$(R(X, Y) \cdot R)(Z, U)V = 0,$$

which implies

$$\begin{aligned} &g(R(\xi, Y)R(Z, U)V, \xi) - g(R(R(\xi, Y)Z, U)V, \xi) \\ &-g(R(Z, R(\xi, Y)U)V, \xi) - g(R(Z, U)R(\xi, Y)V, \xi) = 0. \end{aligned} \tag{17}$$

Putting $Y = Z = e_i$ in (17), where $\{e_1, e_2, e_3, ..., e_{n-1}, e_n = \xi\}$ is an orthonormal basis of the tangent space at each point of the manifold M^n and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(R(\xi, e_i)R(e_i, U)V, \xi) - \sum_{i=1}^{n} g(R(R(\xi, e_i)e_i, U)V, \xi) \\ &- \sum_{i=1}^{n} g(R(e_i, R(\xi, e_i)U)V, \xi) - \sum_{i=1}^{n} g(R(e_i, U)R(\xi, e_i)V, \xi) = 0 \end{aligned} \tag{18}$$

Using (10)-(12) we obtain

$$\sum_{i=1}^{n} g(R(\xi, e_i)R(e_i, U)V, \xi) = -g(U, V) + \eta(U)\eta(V) - S(U, V), \tag{19}$$

$$\sum_{i=1}^{n} g(R(R(\xi, e_i)e_i, U)V, \xi) = -(n-1)[-g(U, V) + \eta(U)\eta(V)], \tag{20}$$

$$\sum_{i=1}^{n} g(R(e_i, R(\xi, e_i)U)V, \xi) = -g(U, V) + \eta(U)\eta(V), \tag{21}$$

$$\sum_{i=1}^{n} g(R(e_i, U)R(\xi, e_i)V, \xi) = (n-1)\eta(U)\eta(V). \tag{22}$$

By virtue of (19), (20), (21) and (22), the equation (18) yields

$$S(U,V) = -(n-1)g(U,V). \tag{23}$$

Thus, we state the following theorem.

Theorem 1 *A para-Kenmotsu manifold belonging to the class* G_1 *is always an Einstein manifold with the Ricci curvature tensor given by* (23).

3.2 Para-Kenmotsu manifolds belonging to the class G_2

We consider para-Kenmotsu manifolds admitting the condition

$$(C(X,Y)\cdot R)(Z,U)V = 0,$$

which implies

$$\begin{aligned} &g(C(\xi,Y)R(Z,U)V,\xi) - g(R(C(\xi,Y)Z,U)V,\xi) \\ &-g(R(Z,C(\xi,Y)U)V,\xi) - g(R(Z,U)C(\xi,Y)V,\xi) = 0. \end{aligned} \tag{24}$$

Putting $Y = Z = e_i$ in (24) and taking the summation over i, $1 \le i \le n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(C(\xi,e_i)R(e_i,U)V,\xi) - \sum_{i=1}^{n} g(R(C(\xi,e_i)e_i,U)V,\xi) \\ &-\sum_{i=1}^{n} g(R(e_i,C(\xi,e_i)U)V,\xi) - \sum_{i=1}^{n} g(R(e_i,U)C(\xi,e_i)V,\xi) = 0. \end{aligned} \tag{25}$$

Using (10)-(12) and (1) we obtain

$$\begin{aligned} \sum_{i=1}^{n} g(C(\xi,e_i)R(e_i,U)V,\xi) &= \frac{1}{n-2}\Bigg[\left(\frac{r}{n-1}+1\right)S(U,V) - S^2(U,V) \\ &- \left(\frac{r}{n-1}+n\right)\eta(R(\xi,U)V)\Bigg], \end{aligned} \tag{26}$$

$$\sum_{i=1}^{n} g(R(C(\xi,e_i)e_i,U)V,\xi) = 0, \tag{27}$$

$$\begin{aligned} \sum_{i=1}^{n} g(R(e_i,C(\xi,e_i)U)V,\xi) &= \frac{1}{n-2}\Bigg[-\left(\frac{r}{n-1}+1\right)\eta(R(\xi,U)V) \\ &- S(U,V) - (n-1)\eta(U)\eta(V)\Bigg], \end{aligned} \tag{28}$$

$$\sum_{i=1}^{n} g(R(e_i, U)C(\xi, e_i)V, \xi) = 0. \tag{29}$$

By virtue of (26), (27), (28), (29) and using (12), the equation (25) yields

$$\left(\frac{r}{n-1} + 2\right) S(U, V) = -(n-1)g(U, V) + S^2(U, V). \tag{30}$$

Thus, we state the following theorem.

Theorem 2 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_2 *satisfies* (30).

3.3 Para-Kenmotsu manifolds belonging to the class G_3

We consider para-Kenmotsu manifolds admitting the condition

$$(\tilde{C}(X, Y) \cdot R)(Z, U)V = 0,$$

which implies

$$\begin{aligned} &g(\tilde{C}(\xi, Y)R(Z, U)V, \xi) - g(R(\tilde{C}(\xi, Y)Z, U)V, \xi) \\ &-g(R(Z, \tilde{C}(\xi, Y)U)V, \xi) - g(R(Z, U)\tilde{C}(\xi, Y)V, \xi) = 0. \end{aligned} \tag{31}$$

Putting $Y = Z = e_i$ in (31) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(\tilde{C}(\xi, e_i)R(e_i, U)V, \xi) - \sum_{i=1}^{n} g(R(\tilde{C}(\xi, e_i)e_i, U)V, \xi) \\ &- \sum_{i=1}^{n} g(R(e_i, \tilde{C}(\xi, e_i)U)V, \xi) - \sum_{i=1}^{n} g(R(e_i, U)\tilde{C}(\xi, e_i)V, \xi) = 0. \end{aligned} \tag{32}$$

Using (10)-(12) and (3) we obtain

$$\sum_{i=1}^{n} g(\tilde{C}(\xi, e_i)R(e_i, U)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)[-g(U, V) + \eta(U)\eta(V) - S(U, V)], \tag{33}$$

$$\sum_{i=1}^{n} g(R(\tilde{C}(\xi, e_i)e_i, U)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)(n-1)[g(U, V) - \eta(U)\eta(V)], \tag{34}$$

$$\sum_{i=1}^{n} g(R(e_i, \tilde{C}(\xi, e_i)U)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)[-g(U, V) + \eta(U)\eta(V)], \quad (35)$$

$$\sum_{i=1}^{n} g(R(e_i, U)\tilde{C}(\xi, e_i)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)(n-1)\eta(U)\eta(V). \quad (36)$$

By virtue of (33), (34), (35) and (36), the equation (32) yields

$$\left(\frac{r}{n(n-1)} + 1\right)[S(U, V) + (n-1)g(U, V)] = 0. \quad (37)$$

Thus, we state the following theorem.

Theorem 3 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_3 *satisfies* (37).

3.4 Para-Kenmotsu manifolds belonging to the class G_4

We consider para-Kenmotsu manifolds admitting the condition

$$(H(X, Y) \cdot R)(Z, U)V = 0,$$

which implies

$$\begin{aligned} &g(H(\xi, Y)R(Z, U)V, \xi) - g(R(H(\xi, Y)Z, U)V, \xi) \\ &-g(R(Z, H(\xi, Y)U)V, \xi) - g(R(Z, U)H(\xi, Y)V, \xi) = 0. \end{aligned} \quad (38)$$

Putting $Y = Z = e_i$ in (38) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(H(\xi, e_i)R(e_i, U)V, \xi) - \sum_{i=1}^{n} g(R(H(\xi, e_i)e_i, U)V, \xi) \\ &-\sum_{i=1}^{n} g(R(e_i, H(\xi, e_i)U)V, \xi) - \sum_{i=1}^{n} g(R(e_i, U)H(\xi, e_i)V, \xi) = 0. \end{aligned} \quad (39)$$

Using (10)-(12) and (2) we obtain

$$\begin{aligned} \sum_{i=1}^{n} g(H(\xi, e_i)R(e_i, U)V, \xi) &= \frac{1}{n-2}[ng(U, V) - n\eta(U)\eta(V) \\ &+ S(U, V) - S^2(U, V)], \end{aligned} \quad (40)$$

$$\sum_{i=1}^{n} g(R(H(\xi, e_i)e_i, U)V, \xi) = \frac{1}{n-2}[rg(U, V) - r\eta(U)\eta(V)], \tag{41}$$

$$\sum_{i=1}^{n} g(R(e_i, H(\xi, e_i)U)V, \xi) = \frac{1}{n-2}[g(U, V) - n\eta(U)\eta(V) - S(U, V)], \tag{42}$$

$$\sum_{i=1}^{n} g(R(e_i, U)H(\xi, e_i)V, \xi) = \frac{1}{n-2} r\eta(U)\eta(V). \tag{43}$$

By virtue of (40), (41), (42) and (43), the equation (39) yields

$$S(U, V) = -\frac{n-r-1}{2} g(U, V) + \frac{1}{2} S^2(U, V). \tag{44}$$

Thus, we state the following theorem.

Theorem 4 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_4 *satisfies* (44).

3.5 Para-Kenmotsu manifolds belonging to the class G_5

We consider para-Kenmotsu manifolds admitting the condition

$$(R(X, Y) \cdot C)(Z, U)V = 0,$$

which implies

$$\begin{aligned} &g(R(\xi, Y)C(Z, U)V, \xi) - g(C(R(\xi, Y)Z, U)V, \xi) \\ &-g(C(Z, R(\xi, Y)U)V, \xi) - g(C(Z, U)R(\xi, Y)V, \xi) = 0. \end{aligned} \tag{45}$$

Putting $Y = Z = e_i$ in (45) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(R(\xi, e_i)C(e_i, U)V, \xi) - \sum_{i=1}^{n} g(C(R(\xi, e_i)e_i, U)V, \xi) \\ &- \sum_{i=1}^{n} g(C(e_i, R(\xi, e_i)U)V, \xi) - \sum_{i=1}^{n} g(C(e_i, U)R(\xi, e_i)V, \xi) = 0. \end{aligned} \tag{46}$$

Using (10)-(12) and (1) we obtain

$$\sum_{i=1}^{n} g(R(\xi, e_i)C(e_i, U)V, \xi) = \eta(C(\xi, U)V), \tag{47}$$

$$\sum_{i=1}^{n} g(C(R(\xi, e_i)e_i, U)V, \xi) = -(n-1)\eta(C(\xi, U)V), \tag{48}$$

$$\sum_{i=1}^{n} g(C(e_i, R(\xi, e_i)U)V, \xi) = \eta(C(\xi, U)V), \tag{49}$$

$$\sum_{i=1}^{n} g(C(e_i, U)R(\xi, e_i)V, \xi) = 0. \tag{50}$$

By virtue of (47), (48), (49), (50) and using (13), the equation (46) yields

$$S(U, V) = \left(\frac{r}{n-1} + 1\right) g(U, V) - \left(\frac{r}{n-1} + n\right) \eta(U)\eta(V). \tag{51}$$

Thus, we state the following theorem.

Theorem 5 *A para-Kenmotsu manifold belonging to the class G_5 is always an η-Einstein manifold with the Ricci curvature tensor given by (51).*

3.6 Para-Kenmotsu manifolds belonging to the class G_6

We consider para-Kenmotsu manifolds admitting the condition

$$(C(X, Y) \cdot H)(Z, U)V = 0,$$

which implies

$$\begin{aligned} &g(C(\xi, Y)H(Z, U)V, \xi) - g(H(C(\xi, Y)Z, U)V, \xi) \\ &- g(H(Z, C(\xi, Y)U)V, \xi) - g(H(Z, U)C(\xi, Y)V, \xi) = 0. \end{aligned} \tag{52}$$

Putting $Y = Z = e_i$ in (52) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(C(\xi, e_i)H(e_i, U)V, \xi) - \sum_{i=1}^{n} g(H(C(\xi, e_i)e_i, U)V, \xi) \\ &- \sum_{i=1}^{n} g(H(e_i, C(\xi, e_i)U)V, \xi) - \sum_{i=1}^{n} g(H(e_i, U)C(\xi, e_i)V, \xi) = 0. \end{aligned} \tag{53}$$

Using (10)-(12), (1) and (2) we obtain

$$\sum_{i=1}^{n} g(C(\xi, e_i)H(e_i, U)V, \xi) \tag{54}$$

$$
\begin{aligned}
= & \frac{1}{(n-2)^2}\Big[-r\Big(\frac{r}{n-1}+1\Big)g(U,V)-nS^2(U,V)+rS(U,V)\\
+ & \parallel Q\parallel^2 g(U,V)-(n-2)\Big(\frac{r}{n-1}+n\Big)\eta(H(\xi,U)V)\Big],
\end{aligned}
$$

$$
\sum_{i=1}^{n} g(H(C(\xi,e_i)e_i,U)V,\xi)=0, \tag{55}
$$

$$
\begin{aligned}
& \sum_{i=1}^{n} g(H(e_i,C(\xi,e_i)U)V,\xi) \\
= & \frac{1}{(n-2)^2}\Big[-(n-2)\Big(\frac{r}{n-1}+1\Big)\eta(H(\xi,U)V)-S^2(U,V)\\
+ & S(U,V)+n(n-1)\eta(U)\eta(V)\Big],
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
& \sum_{i=1}^{n} g(H(e_i,U)C(\xi,e_i)V,\xi) \\
= & -\frac{1}{(n-2)^2}\Big[\frac{r^2}{n-1}+2r-\parallel Q\parallel^2+n(n-1)\Big]\eta(U)\eta(V).
\end{aligned} \tag{57}
$$

By virtue of (54), (55), (56), (57) and using (14), the equation (53) yields

$$
\begin{aligned}
& (n+r-2)S(U,V) \\
= & \Big[r\Big(\frac{r}{n-1}+1\Big)+n-1-\parallel Q\parallel^2\Big]g(U,V)\\
+ & \Big[\parallel Q\parallel^2-r\Big(\frac{r}{n-1}+2\Big)-n(n-1)\Big]\eta(U)\eta(V)+(n-1)S^2(U,V).
\end{aligned} \tag{58}
$$

Thus, we state the following theorem.

Theorem 6 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_6 *satisfies* (58).

3.7 Para-Kenmotsu manifolds belonging to the class G_7

We consider para-Kenmotsu manifolds admitting the condition

$$
(\tilde{C}(X,Y)\cdot H)(Z,U)V=0,
$$

which implies

$$\begin{aligned} & g(\tilde{C}(\xi, Y)H(Z, U)V, \xi) - g(H(\tilde{C}(\xi, Y)Z, U)V, \xi) \\ & -g(H(Z, \tilde{C}(\xi, Y)U)V, \xi) - g(H(Z, U)\tilde{C}(\xi, Y)V, \xi) = 0. \end{aligned} \tag{59}$$

Putting $Y = Z = e_i$ in (59) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} & \sum_{i=1}^{n} g(\tilde{C}(\xi, e_i)H(e_i, U)V, \xi) - \sum_{i=1}^{n} g(H(\tilde{C}(\xi, e_i)e_i, U)V, \xi) \\ & - \sum_{i=1}^{n} g(H(e_i, \tilde{C}(\xi, e_i)U)V, \xi) - \sum_{i=1}^{n} g(H(e_i, U)\tilde{C}(\xi, e_i)V, \xi) = 0. \end{aligned} \tag{60}$$

Using (10)-(12), (2) and (3) we obtain

$$\sum_{i=1}^{n} g(\tilde{C}(\xi, e_i)H(e_i, U)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)\left[\eta(H(\xi, U)V) + \frac{r}{n-2} g(U, V)\right], \tag{61}$$

$$\sum_{i=1}^{n} g(H(\tilde{C}(\xi, e_i)e_i, U)V, \xi) = -\left(\frac{r}{n(n-1)} + 1\right)(n-1)\eta(H(\xi, U)V), \tag{62}$$

$$\sum_{i=1}^{n} g(H(e_i, \tilde{C}(\xi, e_i)U)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)\eta(H(\xi, U)V), \tag{63}$$

$$\sum_{i=1}^{n} g(H(e_i, U)\tilde{C}(\xi, e_i)V, \xi) = \left(\frac{r}{n(n-1)} + 1\right)\frac{r}{n-2}\eta(U)\eta(V). \tag{64}$$

By virtue of (61), (62), (63), (64) and using (14), the equation (60) yields

$$\left(\frac{r}{n(n-1)} + 1\right)\left[S(U, V) - \left(\frac{r}{n-1} + 1\right)g(U, V) - \left(\frac{r}{n-1} - n\right)\eta(U)\eta(V)\right] = 0. \tag{65}$$

Thus, we state the following theorem.

Theorem 7 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_7 *satisfies* (65).

3.8 Para-Kenmotsu manifolds belonging to the class G_8

We consider para-Kenmotsu manifolds admitting the condition

$$(H(X,Y)\cdot H)(Z,U)V = 0,$$

which implies

$$\begin{aligned} & g(H(\xi,Y)H(Z,U)V,\xi) - g(H(H(\xi,Y)Z,U)V,\xi) \\ & -g(H(Z,H(\xi,Y)U)V,\xi) - g(H(Z,U)H(\xi,Y)V,\xi) = 0. \end{aligned} \tag{66}$$

Putting $Y = Z = e_i$ in (66) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} & \sum_{i=1}^{n} g(H(\xi,e_i)H(e_i,U)V,\xi) - \sum_{i=1}^{n} g(H(H(\xi,e_i)e_i,U)V,\xi) \\ & -\sum_{i=1}^{n} g(H(e_i,H(\xi,e_i)U)V,\xi) - \sum_{i=1}^{n} g(H(e_i,U)H(\xi,e_i)V,\xi) = 0. \end{aligned} \tag{67}$$

Using (10)-(12) and (2) we obtain

$$\begin{aligned} & \sum_{i=1}^{n} g(H(\xi,e_i)H(e_i,U)V,\xi) \\ = & \frac{1}{(n-2)^2}[-n(n-2)\eta(H(\xi,U)V) - nS^2(U,V) + rS(U,V) \\ + & \parallel Q \parallel^2 g(U,V) - rg(U,V)], \end{aligned} \tag{68}$$

$$\sum_{i=1}^{n} g(H(H(\xi,e_i)e_i,U)V,\xi) = -\frac{r}{n-2}\eta(H(\xi,U)V), \tag{69}$$

$$\begin{aligned} & \sum_{i=1}^{n} g(H(e_i,H(\xi,e_i)U)V,\xi) \\ = & \frac{1}{(n-2)^2}[-(n-2)\eta(H(\xi,U)V) - S^2(U,V) \\ & +S(U,V) + n(n-1)\eta(U)\eta(V)], \end{aligned} \tag{70}$$

$$\sum_{i=1}^{n} g(H(e_i,U)H(\xi,e_i)V,\xi) = -\frac{1}{(n-2)^2}[2r - \parallel Q \parallel^2 + n(n-1)]\eta(U)\eta(V). \tag{71}$$

By virtue of (68), (69), (70), (71) and (14), the equation (67) yields

$$\begin{aligned}(n-2)S(U,V) &= \left[(n-1)-\parallel Q \parallel^2\right]g(U,V) \\ -\left[n(n-1)-(n-2)r\right. &- \left.\parallel Q \parallel^2\right]\eta(U)\eta(V)+(n-1)S^2(U,V).\end{aligned} \tag{72}$$

Thus, we state the following theorem.

Theorem 8 *The Ricci curvature tensor of a para-Kenmotsu manifold belonging to the class* G_8 *satisfies* (72).

We can conclude the followings.

Remark 1 *Let* M^n *be a para-Kenmotsu manifold of dimension* $n > 3$.

i) *If* M^n *belongs to the class* G_1, *then* M^n *is an Einstein manifold of constant negative scalar curvature* $r = -n(n-1)$.

ii) *If* M^n *belongs to the class* G_2, *then the Ricci operator satisfies*

$$\parallel Q \parallel^2 \geq (n-1)^2.$$

iii) *If* M^n *belongs to the class* G_3, *then* M^n *is of constant negative scalar curvature* $r = -n(n-1)$.

iv) *If* M^n *belongs to the class* G_4, *then the scalar curvature satisfies*

$$r = \frac{n(n-1)}{n-2} - \frac{1}{n-2} \parallel Q \parallel^2 \leq \frac{n(n-1)}{n-2}.$$

v) *If* M^n *belongs to the class* G_5, *then* M^n *is an* η*-Einstein manifold.*

vi) *If* M^n *belongs to the class* G_7, *then* M^n *is of vanishing or constant negative scalar curvature* $r = -n(n-1)$.

References

[1] P. Dacko, Z. Olszak, On weakly para-cosymplectic manifolds of dimension 3, *J. Geom. Phys.*, **57** (2007), 561–570.

[2] L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1949.

[3] Y. Ishii, On conharmonic transformations, *Tensor (N.S.)*, **7** (1957), 73–80.

[4] S. Ivanov, D. Vassilev, S. Zamkovoy, Conformal paracontact curvature and the local flatness theorem, *Geom. Dedicata*, **144** (2010), 79–100.

[5] K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tohoku Math. J.*, **24** (1972), 93–103.

[6] Z. Olszak, The Schouten-van Kampen affine connection adapted to an almost (para) contact metric structure, *Publ. Inst. Math. nouv. ser.*, **94** (108) (2013), 31–42.

[7] G. P. Pokhariyal and R. S. Mishra, Curvature tensors and their relativistics significance II, *Yokohama Math. J.*, **19** (1971), 97–103.

[8] A. A. Saikh and H. Kundu, On equivalency of various geometric structures, *J. Geom.*, DOI 10.1007/s00022-013-0200-4

[9] I. Sato, On a structure similar to almost contact structure, *Tensor (N.S.)*, **30** (1976), 219–224.

[10] I. Sato and K. Matsumoto, On P-Sasakian satisfying certain conditions, *Tensor (N.S.)*, **33** (1979), 173–178.

[11] B. B. Sinha, K. L. Sai Prasad, A class of almost para contact metric manifolds, *Bull. Cal. Math. Soc.*, **87** (1995), 307–312.

[12] J. Welyczko, Slant curves in 3-dimensional normal almost paracontact metric manifolds, *Mediterr. J. Math.*, DOI 10.1007/s00009-013-0361-2 (2013).

[13] K. Yano and S. Bochner, Curvature and Betti numbers, *Annals of Math. Studies*, **32**, Princeton University Press, 1953.

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On generalized γ_μ-closed sets and related continuity

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Abstract. In this paper our main interest is to introduce a new type of generalized open sets defined in terms of an operation on a generalized topological space. We have studied some properties of this newly defined sets. As an application, we have introduced some weak separation axioms and discussed some of their properties. Finally, we have studied some preservation theorems in terms of some irresolute functions.

1 Introduction

In 1979, Kasahara [5] introduced the notion of an operation on a topological space and introduced the concept of α-closed graph of a function. After then Janković defined [4] the concept of α-closed sets and investigated some properties of functions with α-closed graphs. On the other hand, in 1991 Ogata [7] introduced the notion of γ-open sets to investigate some new separation axioms on a topological space. The notion of operations on the family of all semi-open sets and pre-open sets are investigated by Krishnan et al. [6] and Van An et al. [11]. Recently, the concept of γ_μ-Lindelöf spaces was studied by Roy and Noiri in [9, 10].

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In this paper our aim is to study an operation based on generalized γ_μ-closed like sets, where the operation is defined on the collection of generalized open sets. The most common properties of different open like sets or weakly open sets are that they are closed under arbitary unions and contain the null set. Observing these, Császár introduced the concept of generalized open sets. We now recall some notions defined in [1]. Let X be a non-empty set. A subcollection $\mu \subseteqq \mathcal{P}(X)$ (where $\mathcal{P}(X)$ denotes the power set of X) is called a generalized topology [1], (briefly, GT) if $\varnothing \in \mu$ and any union of elements of μ belongs to μ. A set X with a GT μ on the set X is called a generalized topological space (briefly, GTS) and is denoted by (X, μ). If for a GTS (X, μ), $X \in \mu$, then (X, μ) is known as a strong GTS. Throughout the paper, we assume that (X, μ) and (Y, λ) are strong GTS's. The elements of μ are called μ-open sets and μ-closed sets are their complements. The μ-closure of a set $A \subseteqq X$ is denoted by $c_\mu(A)$ and defined by the smallest μ-closed set containing A which is equivalent to the intersection of all μ-closed sets containing A. We use the symbol $i_\mu(A)$ to mean the μ-interior of A and it is defined as the union of all μ-open sets contained in A i.e., the largest μ-open set contained in A (see [3, 2, 1]).

2 $\gamma_\mu g$-closed sets and their related properties

Definition 1 [9] *Let (X, μ) be a GTS. An operation γ_μ on a generalized topology μ is a mapping from μ to $\mathcal{P}(X)$ (where $\mathcal{P}(X)$ is the power set of X) with $G \subseteqq \gamma_\mu(G)$, for each $G \in \mu$. This operation is denoted by $\gamma_\mu : \mu \to \mathcal{P}(X)$. Note that $\gamma_\mu(A)$ and A^{γ_μ} are two different notations for the same set.*

Definition 2 [9] *Let (X, μ) be a GTS and γ_μ be an operation on μ. A subset G of (X, μ) is called γ_μ-open if for each point x of G, there exists a μ-open set U containing x such that $\gamma_\mu(U) \subseteqq G$.*

A subset of a GTS (X, μ) is called γ_μ-closed if its complement is γ_μ-open in (X, μ). We shall use the symbol γ_μ to mean the collection of all γ_μ-open sets of the GTS (X, μ).

Definition 3 [9] *Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. It is easy to see that the family of all γ_μ-open sets forms a GT on X. The γ_μ-closure of a set A of X is denoted by $c_{\gamma_\mu}(A)$ and is defined as $c_{\gamma_\mu}(A) = \cap\{F : F$ is a γ_μ-closed set and $A \subseteqq F\}$.*

It is easy to check that for each $x \in X$, $x \in c_{\gamma_\mu}(A)$ if and only if $V \cap A \neq \varnothing$, for any $V \in \gamma_\mu$ with $x \in V$. Note that if $\gamma_\mu = id_\mu$, then $c_{\gamma_\mu}(A) = c_\mu(A)$.

Definition 4 *Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. A subset A of X is said to be $\gamma_\mu g$-closed if $c_{\gamma_\mu}(A) \subseteqq U$, whenever $A \subseteqq U$ and U is a γ_μ-open set in (X, μ).*

Every γ_μ-closed set is $\gamma_\mu g$-closed but the converse is not true as shown in the next example. Also note that if $\gamma_\mu = id_\mu$, then $\gamma_\mu g$-closed set reduces to a μg-closed set [8].

Example 1 *Let $X = \{1, 2, 3\}$ and $\mu = \{\varnothing, \{1\}, \{1, 2\}, \{2, 3\}, X\}$. Then (X, μ) is a GTS. Now $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by*

$$\gamma_\mu(A) = \begin{cases} A, & \text{if } 1 \in A \\ \{2, 3\}, & \text{otherwise} \end{cases}$$

is an operation. It can be easily checked that $\{1, 3\}$ is a $\gamma_\mu g$-closed set but not a γ_μ-closed set.

Theorem 1 *Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. A subset A of X is $\gamma_\mu g$-closed if and only if $c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing$, holds for every $x \in c_{\gamma_\mu}(A)$.*

Proof. First let the given condition hold and let U be a γ_μ-open set with $A \subseteqq U$ and $x \in c_{\gamma_\mu}(A)$. As $c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing$, there exists a $z \in c_{\gamma_\mu}(\{x\})$ such that $z \in A \subseteqq U$. Thus $U \cap \{x\} \neq \varnothing$. Hence $x \in U$. Thus $c_{\gamma_\mu}(A) \subseteqq U$, proving A to be a $\gamma_\mu g$-closed set.

Conversely, let A be a $\gamma_\mu g$-closed subset of X and $x \in c_{\gamma_\mu}(A)$ with $c_{\gamma_\mu}(\{x\}) \cap A = \varnothing$. Then $A \subseteqq X \setminus c_{\gamma_\mu}(\{x\})$ which implies that $c_{\gamma_\mu}(A) \subseteqq X \setminus c_{\gamma_\mu}(\{x\})$ (as A is $\gamma_\mu g$-closed), which is a contradiction to the fact that $x \in c_{\gamma_\mu}(\{x\})$. Thus $c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing$. □

Theorem 2 *Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. If $c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing$ for every $x \in c_{\gamma_\mu}(A)$, then $c_{\gamma_\mu}(A) \setminus A$ does not contain any non-empty γ_μ-closed set.*

Proof. If possible, let there exist a non-empty γ_μ-closed set F such that $F \subseteqq c_{\gamma_\mu}(A) \setminus A$. Let $x \in F$. Then $x \in c_{\gamma_\mu}(A)$. Since $\varnothing \neq c_{\gamma_\mu}(\{x\}) \cap A \subseteqq F \cap A$, we have $F \cap A \neq \varnothing$, which is a contradiction. □

Corollary 1 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. A subset* A *of a GTS* (X, μ) *is* $\gamma_\mu g$*-closed if and only if* $A = F \setminus N$*, where* F *is* γ_μ*-closed and* N *contains no non-empty* γ_μ*-closed subset of* X*.*

Proof. One part of the theorem follows from Theorems 1 and 2 by taking $F = c_{\gamma_\mu}(A)$ and $N = c_{\gamma_\mu}(A) \setminus A$.

Conversely, suppose that $A = F \setminus N$ and $A \subseteqq U$, where U is γ_μ-open. Then $F \cap (X \setminus U)$ is a γ_μ-closed subset of N and hence it must be empty. Thus $c_{\gamma_\mu}(A) \subseteqq F \subseteqq U$. □

Theorem 3 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Let* A *be a* $\gamma_\mu g$*-closed subset of* X *and* $A \subseteqq B \subseteqq c_{\gamma_\mu}(A)$*. Then* B *is also* $\gamma_\mu g$*-closed.*

Proof. Let A be a $\gamma_\mu g$-closed set such that $A \subseteqq B \subseteqq c_{\gamma_\mu}(A)$ and U be a γ_μ-open set with $B \subseteqq U$. Then $A \subseteqq U$ and hence $c_{\gamma_\mu}(A) \subseteqq U$. Thus $c_{\gamma_\mu}(B) \subseteqq U$. Thus B is $\gamma_\mu g$-closed. □

Theorem 4 *Let* (X, μ) *be a strong GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. For each* $x \in X$*, either* $\{x\}$ *is* γ_μ*-closed or* $X \setminus \{x\}$ *is a* $\gamma_\mu g$*-closed set in* (X, μ)*.*

Proof. If $\{x\}$ is γ_μ-closed, then we have nothing to prove. Suppose that $\{x\}$ is not γ_μ-closed. Then $X \setminus \{x\}$ is not γ_μ-open. Let U be any γ_μ-open set such that $X \setminus \{x\} \subseteqq U$. Hence $U = X$. Thus $c_{\gamma_\mu}(X \setminus \{x\}) \subseteqq U$. Thus $X \setminus \{x\}$ is a $\gamma_\mu g$-closed set. □

Definition 5 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is said to be* γ_μ*-*$T_{\frac{1}{2}}$ *if every* $\gamma_\mu g$*-closed set is* γ_μ*-closed.*

Theorem 5 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is* γ_μ*-*$T_{\frac{1}{2}}$ *if and only if* $\{x\}$ *is either* γ_μ*-closed or* γ_μ*-open.*

Proof. Suppose that (X, μ) be a γ_μ-$T_{\frac{1}{2}}$ space and $\{x\}$ is not γ_μ-closed. Then by Theorem 4, $X \setminus \{x\}$ is a $\gamma_\mu g$-closed set and hence a γ_μ-closed set. So $\{x\}$ is a γ_μ-open set.

Conversely, suppose that F be a $\gamma_\mu g$-closed set in (X, μ). Let $x \in c_{\gamma_\mu}(F)$. Then $\{x\}$ is either γ_μ-open or γ_μ-closed. If $\{x\}$ is γ_μ-open, then $\{x\} \cap F \neq \varnothing$. Hence $x \in F$. Thus $c_{\gamma_\mu}(F) \subseteqq F$, which implies that F is γ_μ-closed. If $\{x\}$ is γ_μ-closed, suppose that $x \notin F$. Then $x \in c_{\gamma_\mu}(F) \setminus F$, which is impossible by Theorem 2. Thus $x \in F$. Hence $c_{\gamma_\mu}(F) \subseteqq F$, so that F is γ_μ-closed. □

Definition 6 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is said to be*

(i) γ_μ-T_0 *if for any two distinct points* x *and* y *of* X*, there exists a* γ_μ*-open set* U *containing* x *but not containing* y *or a* γ_μ*-open set* V *containing* y *but not containing* x*.*

(ii) γ_μ-T_1 *if for any two distinct points* x *and* y *of* X*, there exist two* γ_μ*-open sets* U *and* V *such that* $x \in U$*,* $y \notin U$ *and* $y \in V$*,* $x \notin U$*.*

(iii) γ_μ-T_2 *if for any two distinct points* x *and* y *of* X*, there exist two disjoint* γ_μ*-open sets* U *and* V *such that* $x \in U$ *and* $y \in V$*.*

Definition 7 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. A subset* A *of* (X, μ) *is said to be a* γ_μ-D_μ *set if there exist two* γ_μ*-open sets* U *and* V *such that* $U \neq X$ *and* $A = U \setminus V$*.*

It follows from the definition that every γ_μ-open set (other than X) is a γ_μ-D_μ set.

Definition 8 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is said to be a*

(i) γ_μ-D_0 *space if for any pair of distinct points, there exists a* γ_μ-D_μ *set containing* x *but not containing* y *or a* γ_μ-D_μ *set containing* y *but not containing* x*.*

(ii) γ_μ-D_1 *space if for any pair of distinct points* x *and* y *of* X*, there exist a* γ_μ-D_μ *set containing* x *but not* y *and a* γ_μ-D_μ *set containing* y *but not containing* x*.*

(iii) γ_μ-D_2 *space if for any two two distinct points* x *and* y *of* X*, there exist disjoint* γ_μ-D_μ *sets* U *and* V *containing* x *and* y *respectively.*

Remark 1 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then the following hold:*

(i) *If* (X, μ) *is* γ_μ-T_i*, then it is* γ_μ-T_{i-1} *for* $i = 1, 2$*.*

(ii) *If* (X, μ) *is* γ_μ-T_i*, then it is* γ_μ-D_i *for* $i = 0, 1, 2$*.*

(iii) *If* (X, μ) *is* γ_μ-D_i*, then it is* γ_μ-D_{i-1} *for* $i = 1, 2$*.*

Proposition 1 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is* γ_μ-T_1 *if and only if every singleton is* γ_μ*-closed.*

Proof. Obvious. $\square$

Definition 9 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is said to be* γ_μ*-symmetric if for* $x, y \in X$, $x \in c_{\gamma_\mu}(\{y\}) \Rightarrow y \in c_{\gamma_\mu}(\{x\})$.

Proposition 2 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is a* γ_μ*-symmetric space if and only if* $\{x\}$ *is a* $\gamma_\mu g$*-closed set, for each* $x \in X$.

Proof. Assume that $\{x\} \subseteq U$, where U is a γ_μ-open set. If possible, let $c_{\gamma_\mu}(\{x\}) \nsubseteq U$. Then $c_{\gamma_\mu}(\{x\}) \cap (X \setminus U) \neq \varnothing$. Let $y \in c_{\gamma_\mu}(\{x\}) \cap (X \setminus U) \neq \varnothing$. Then by hypothesis, $x \in c_{\gamma_\mu}(\{y\}) \subseteq X \setminus U$ and hence $x \notin U$, which is a contradiction. Therefore $\{x\}$ is a $\gamma_\mu g$-closed set.

Conversely, assume that $x \in c_{\gamma_\mu}(\{y\})$ but $y \notin c_{\gamma_\mu}(\{x\})$. Then $y \in X \setminus c_{\gamma_\mu}(\{x\})$. Thus $c_{\gamma_\mu}(\{y\}) \subseteq X \setminus c_{\gamma_\mu}(\{x\})$. Thus $x \notin c_{\gamma_\mu}(\{x\})$, which is a contradiction. Thus (X, μ) is γ_μ-symmetric. $\square$

Corollary 2 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. If* (X, μ) *is* γ_μ-T_1, *then it is* γ_μ*-symmetric.*

Proof. It follows from Proposition 1 that in a γ_μ-T_1 space every singleton is γ_μ-closed and hence $\gamma_\mu g$-closed. $\square$

Corollary 3 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is* γ_μ-T_1 *if and only if it is* γ_μ-T_0 *and* γ_μ*-symmetric.*

Proof. One part of the theorem follows from Remark 1 and Corollary 2. Let (X, μ) be a γ_μ-T_0 and γ_μ-symmetric space and x, y be any two distinct points of X. We may assume that $x \in U$ but $y \notin U$, for some γ_μ-open set U of X. Thus $x \notin c_{\gamma_\mu}(\{y\})$ and hence $y \notin c_{\gamma_\mu}(\{x\})$. Thus there exists a γ_μ-open set V containing y such that $x \notin V$. Thus (X, μ) is γ_μ-T_1. $\square$

Proposition 3 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then*

(a) *Every* γ_μ-T_1 *space is* γ_μ-$T_{\frac{1}{2}}$ *and every* γ_μ-$T_{\frac{1}{2}}$ *space is* γ_μ-T_0.

(b) *For a* γ_μ*-symmetric space the following are equivalent:*

  (i) (X, μ) *is a* γ_μ-T_0 *space.*

(ii) (X, μ) *is a* γ_μ-T_1 *space.*
(iii) (X, μ) *is a* γ_μ-$T_{\frac{1}{2}}$ *space.*
(iv) (X, μ) *is a* γ_μ-D_1 *space.*

Proof. (a) Follows from Theorem 5 and Proposition 1.
(b) If (X, μ) is γ_μ-symmetric and a γ_μ-T_0 space, then by Corollary 3, (X, μ) is a γ_μ-T_1 space and hence (by (a) above) (X, μ) is γ_μ-$T_{\frac{1}{2}}$ and again by (a) above, (X, μ) is γ_μ-T_0. Thus (i), (ii) and (iii) are equivalent.
Again by Remark 1, (ii) $\Rightarrow$ (iv) is obvious.
(iv)$\Rightarrow$ (i) : Let (X, μ) be a γ_μ-D_1 space. Hence (X, μ) is a γ_μ-D_0 space. Thus for each pair of distinct points $x, y \in X$, at least one of x, y, say x, belongs to a γ_μ-D_μ set S but $y \notin S$. Let $S = U_1 \setminus U_2$, where U_1 and U_2 are γ_μ-sets and $U_1 \neq X$. Then $x \in U_1$. If $y \notin U_1$, then the proof is complete. If $y \in U_1 \cap U_2$, then $y \in U_2$ but $x \notin U_2$. Thus (X, μ) is γ_μ-T_0. □

Theorem 6 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is* γ_μ-D_2 *if and only if it is* γ_μ-D_1.

Proof. One part follows from Remark 1. Conversely, let x and y be two distinct points of X. Then there exist γ_μ-D_μ sets G_1 and G_2 in X such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1 \setminus U_2$ and $G_2 = U_3 \setminus U_4$, where U_1, U_2, U_3 and U_4 are γ_μ-open sets in X and $U_1 \neq X$, $U_3 \neq X$. From $x \notin G_2$, it follows that either $x \in U_3 \cap U_4$ or $x \notin U_3$. We will discuss the two cases separately.
Case - 1: $x \in U_3 \cap U_4$: Then $y \in G_2$ and $x \in U_4$, with $G_2 \cap U_4 = \varnothing$.
Case - 2: $x \notin U_3$: By $y \notin G_1$ the following two cases may arise. If $y \notin U_1$, as $x \in U_1 \setminus U_2$, it follows that $x \in U_1 \setminus (U_2 \cup U_3)$ with $y \in U_3 \setminus (U_1 \cup U_4)$ and $(U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_1 \cup U_4)) = \varnothing$. In the case if $y \in U_1 \cap U_2$, we have $x \in U_1 \setminus U_2$ and $y \in U_2$ such that $(U_1 \setminus U_2) \cap U_2 = \varnothing$. □

Definition 10 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then the* γ_μ*-kernel of a subset* A *of* X *is denoted by* $\ker_{\gamma_\mu}(A) = \cap\{U : A \subsetneqq U$ *and* U *is* γ_μ*-open*$\}$.

Proposition 4 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* $y \in \ker_{\gamma_\mu}(\{x\})$ *if and only if* $x \in c_{\gamma_\mu}(\{y\})$.

Proof. Suppose that $y \notin \ker_{\gamma_\mu}(\{x\})$. Then there exists a γ_μ-open set V containing x such that $y \notin V$. Therefore we have, $x \notin c_{\gamma_\mu}(\{y\})$. The other part can be done in the similar manner. □

Proposition 5 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then for any subset* A *of* X, $\ker_{\gamma_\mu}(A) = \{x : c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing\}$.

Proof. Let $x \in \ker_{\gamma_\mu}(A)$ and $c_{\gamma_\mu}(\{x\}) \cap A = \varnothing$. Then $A \subseteqq X \setminus c_{\gamma_\mu}(\{x\})$, where $X \setminus c_{\gamma_\mu}(\{x\})$ is a γ_μ-open set not containing x.

Conversely, let $x \in X$ and $c_{\gamma_\mu}(\{x\}) \cap A \neq \varnothing$, with $x \notin \ker_{\gamma_\mu}(A)$. Then there exists a γ_μ-open set V containing A such that $x \notin V$. Let $y \in c_{\gamma_\mu}(\{x\}) \cap A$. Then V is a γ_μ-open set containing y (as $A \subseteqq V$), but not containing x. □

Proposition 6 *Let* (X, μ) *be a GTS and* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. If a singleton set* $\{x\}$ *is a* γ_μ-D_μ *set of* X, *then* $\ker_{\gamma_\mu}(\{x\}) \neq X$.

Proof. Let $\{x\}$ be a γ_μ-D_μ set. Then there exist two γ_μ-open sets V_1 and V_2 such that $\{x\} = V_1 \setminus V_2$ and $V_1 \neq X$. Thus $\ker_{\gamma_\mu}(\{x\}) \subseteqq V_1 \neq X$. □

Proposition 7 *Let* (X, μ) *be a* γ_μ-$T_{\frac{1}{2}}$ *GTS having at least two points, where* $\gamma_\mu : \mu \to \mathcal{P}(X)$ *be an operation. Then* (X, μ) *is* γ_μ-D_1 *if and only if* $\ker_{\gamma_\mu}(\{x\}) \neq X$, *for every point* $x \in X$.

Proof. Let $x \in X$ and X be γ_μ-D_1. For any point y other than x, there exists a γ_μ-D_μ set V such that $x \in V$ and $y \notin V$. Then $V = V_1 \setminus V_2$, where V_1 and V_2 are γ_μ-open sets such that $V_1 \neq X$. Thus $\{x\} \subseteqq V_1$ and $V_1 \neq X$, where V_1 is a γ_μ-open set. Hence $\ker_{\gamma_\mu}(\{x\}) \neq X$.

Conversely, let x and y be two distinct points of X. Using Theorem 5, we have the following cases :

Case -1 : $\{x\}$ and $\{y\}$ both are γ_μ-open : Then the case is obvious, as every γ_μ-open set is a γ_μ-D_μ set.

Case -2: $\{x\}$ and $\{y\}$ both are γ_μ-closed : In this case $x \in X \setminus \{y\}$, $y \in X \setminus \{x\}$, $y \notin X \setminus \{y\}$, $x \notin X \setminus \{x\}$ and $X \setminus \{x\}$, $X \setminus \{y\}$ both are γ_μ-D_μ sets.

Case-3: $\{x\}$ is γ_μ-open and $\{y\}$ is γ_μ-closed : Since $\ker_{\gamma_\mu}(\{y\}) \neq X$, there exists a γ_μ-open set V containing y such that $V \neq X$. Clearly $\{y\} = V \setminus (X \setminus \{y\})$, showing $\{y\}$ to be a γ_μ-D_μ set. Thus $\{x\}$ and $\{y\}$ are the two γ_μ-D_μ sets such that $y \notin \{x\}$ and $x \notin \{y\}$.

Case-4 : $\{x\}$ is γ_μ-closed and $\{y\}$ is γ_μ-open : Can be proved as in case 3. Thus (X, μ) is a γ_μ-D_1 space. □

3 $(\gamma_\mu, \beta_\lambda)$-irresolute functions

Throughout the rest of the paper, (X, μ) and (Y, λ) will denote GTS's and $\gamma_\mu : \mu \to \mathcal{P}(X)$ and $\beta_\lambda : \lambda \to \mathcal{P}(Y)$ will denote operations on μ and λ respectively.

Definition 11 *A function $f : (X, \mu) \to (Y, \lambda)$ is said to be $(\gamma_\mu, \beta_\lambda)$-irresolute if for each $x \in X$ and each β_λ-open set V containing $f(x)$, there is a γ_μ-open set U containing x such that $f(U) \subseteqq V$.*

Theorem 7 *Let $f : (X, \mu) \to (Y, \lambda)$ be a function. Then the following are equivalent:*

(i) *f is $(\gamma_\mu, \beta_\lambda)$-irresolute.*

(ii) *$f(c_{\gamma_\mu}(A)) \subseteqq c_{\beta_\lambda}(f(A))$, holds for every subset A of X.*

(iii) *$f^{-1}(B)$ is γ_μ-closed, for every β_λ-closed set B of Y.*

Proof. Obvious. □

Definition 12 *A mapping $f : (X, \mu) \to (Y, \lambda)$ is said to be $(\gamma_\mu, \beta_\lambda)$-closed if for any γ_μ-closed set A of (X, μ), $f(A)$ is β_λ-closed in Y.*

Theorem 8 *Suppose that $f : (X, \mu) \to (Y, \lambda)$ is $(\gamma_\mu, \beta_\lambda)$-irresolute and $(\gamma_\mu, \beta_\lambda)$-closed.*

(i) *If A is $\gamma_\mu g$-closed, then $f(A)$ is $\beta_\lambda g$-closed.*

(ii) *If B be any $\beta_\lambda g$-closed set of Y, then $f^{-1}(B)$ is a $\gamma_\mu g$-closed set in X.*

Proof. (i) Let V be any β_λ-open set such that $f(A) \subseteqq V$. Then by Theorem 7, $f^{-1}(V)$ is γ_μ-open in X. Since $A \subseteqq f^{-1}(V)$ and A is $\gamma_\mu g$-closed, $c_{\gamma_\mu}(A) \subseteqq f^{-1}(V)$ and hence $f(c_{\gamma_\mu}(A)) \subseteqq V$. Now by the assumption, $f(c_{\gamma_\mu}(A))$ is a β_λ-closed set in Y. Thus $c_{\beta_\lambda}(f(A)) \subseteqq c_{\beta_\lambda}(f(c_{\gamma_\mu}(A))) = f(c_{\gamma_\mu}(A)) \subseteqq V$. Hence $f(A)$ is $\beta_\lambda g$-closed.

(ii) Let U be any γ_μ-open set of X with $f^{-1}(B) \subseteqq U$. Let $F = c_{\gamma_\mu}(f^{-1}(B)) \cap (X \setminus U)$. Then F is a γ_μ-closed set in X. Since f is $(\gamma_\mu, \beta_\lambda)$-closed, $f(F)$ is β_λ-closed. Since $f(F) \subseteqq f(c_{\gamma_\mu}(f^{-1}(B))) \cap f(X \setminus U) \subseteqq c_{\beta_\lambda}(f(f^{-1}(B))) \cap f(X \setminus U) \subseteqq c_{\beta_\lambda}(B) \cap (Y \setminus B) = c_{\beta_\lambda}(B) \setminus B$ by Corollary 1, it then follows that $f(F) = \varnothing$ and hence $F = \varnothing$. □

Theorem 9 *Suppose that $f : (X, \mu) \to (Y, \lambda)$ is $(\gamma_\mu, \beta_\lambda)$-irresolute and $(\gamma_\mu, \beta_\lambda)$-closed.*

(i) *If f is injective and (Y, λ) is β_λ-$T_{\frac{1}{2}}$, then (X, μ) is γ_μ-$T_{\frac{1}{2}}$.*

(ii) *If f is surjective and (X, μ) is γ_μ-$T_{\frac{1}{2}}$, then (Y, λ) is β_λ-$T_{\frac{1}{2}}$.*

Proof. (i) Let A be a $\gamma_\mu g$-closed set of (X, μ). Then by the last theorem, $f(A)$ is $\beta_\lambda g$-closed in Y and hence $f(A)$ is β_λ-closed. Since f is $(\gamma_\mu, \beta_\lambda)$-irresolute, by Theorem 7, $A = f^{-1}(f(A))$ is γ_μ-closed and hence (X, μ) is γ_μ-$T_{\frac{1}{2}}$.

(ii) Let B be any $\beta_\lambda g$-closed set in (Y, λ). It is sufficient to show that B is a β_λ-closed set. By Theorem 8, $f^{-1}(B)$ is a $\gamma_\mu g$-closed set in X. Since (X, μ) is a γ_μ-$T_{\frac{1}{2}}$, $f^{-1}(B)$ is γ_μ-closed. Since f is surjective and $(\gamma_\mu, \beta_\lambda)$-closed, $f(f^{-1}(B)) = B$ is a β_λ-closed set. □

Theorem 10 *Let* $f : (X, \mu) \to (Y, \lambda)$ *be a surjective* $(\gamma_\mu, \beta_\lambda)$*-irresolute function. If* E *be a* β_λ-D_λ *set in* Y*, then* $f^{-1}(E)$ *is a* γ_μ-D_μ *set in* (X, μ)*.*

Proof. Let E be a β_λ-D_λ set in Y. Then there exist two β_λ-open sets U_1 and U_2 in Y such that $E = U_1 \setminus U_2$ and $U_1 \neq Y$. Now by Theorem 7, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are γ_μ-open and $f^{-1}(U_1) \neq X$ (as f is surjective and $U_1 \neq Y$). Thus $f^{-1}(E) = f^{-1}(U_1) \setminus f^{-1}(U_2)$ is a γ_μ-D_μ set. □

Theorem 11 *If* (Y, λ) *is* β_λ-D_1 *and* $f : (X, \mu) \to (Y, \lambda)$ *is a* $(\gamma_\mu, \beta_\lambda)$*-irresolute bijective function, then* (X, μ) *is* γ_μ-D_1*.*

Proof. Suppose that (Y, λ) is a β_λ-D_1 space. Let x and y be any two distinct points of X. Since f is injective and Y is β_λ-D_1, there exist β_λ-D_λ sets G_x and G_y of Y containing $f(x)$ and $f(y)$ respectively such that $f(x) \notin G_y$ and $f(y) \notin G_x$. Thus by Theorem 10, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are γ_μ-D_μ sets containing x and y respectively such that $x \notin f^{-1}(G_y)$ and $y \notin f^{-1}(G_x)$. Thus X a γ_μ-D_1 space. □

Theorem 12 *A GTS* (X, μ) *is* γ_μ-D_1 *if for each distinct points* x *and* y *in* X*, there exists a* $(\gamma_\mu, \beta_\lambda)$*-irresolute surjective function* $f : (X, \mu) \to (Y, \lambda)$*, where* (Y, λ) *is a* β_λ-D_1 *space such that* $f(x)$ *and* $f(y)$ *are distinct.*

Proof. Let x and y be two distinct points of X. By hypothesis, there exists a $(\gamma_\mu, \beta_\lambda)$-irresolute function f on X onto a β_λ-D_1 space Y such that $f(x) \neq f(y)$. Then there exist β_λ-D_λ sets G_x and G_y containing $f(x)$ and $f(y)$ respectively such that $f(x) \notin G_y$ and $f(y) \notin G_x$. As f is surjective and $(\gamma_\mu, \beta_\lambda)$-irresolute, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are γ_μ-D_μ sets in X (by Theorem 10) containing x and y respectively such that $x \notin f^{-1}(G_y)$ and $y \notin f^{-1}(G_x)$. Hence X is a γ_μ-D_1 space. □

Conclusion: If we replace μ by different GT's or γ_μ by different operators, we can obtain various forms of generalized closed sets and related continuous functions.

References

[1] Á. Császár, Generalized topology, generalized continuity, *Acta Math. Hungar.*, **96** (2002), 351–357.

[2] Á. Császár, Generalized open sets in generalized topologies, *Acta Math. Hungar.*, **106** (2005), 53–66.

[3] Á. Császár, δ- and θ-modifications of generalized topologies, *Acta Math. Hungar.*, **120(3)** (2008), 275–279.

[4] D. S. Janković, On functions with γ-closed graphs, *Glasnik Mat.*, **18(38)** (1983), 141–148.

[5] S. Kasahara, Operation-compact spaces, *Math. Japon.*, **24** (1979), 97–105.

[6] G. S. S. Krishnan, M. Ganster and K. Balachandran, Operation approaches on semi open sets and its application, *Kochi J. Math.*, **2** (2007), 21–33.

[7] H. Ogata, Operations on topological spaces and associated topology, *Math. Japon.*, **36** (1991), 175–184.

[8] B. Roy, On a type of generalized open sets, *Applied Gen. Topology.*, **12** (2011), 163–173.

[9] B. Roy, Applications of operations on minimal generalized open sets, *Afrika Mat.*, **29** (2018), 1097–1104.

[10] B. Roy and T. Noiri, γ_μ-Lindelöf generalized topological spaces, *Jour. of Linear and Topol. Algebra*, **8(4)** (2019), 257–263.

[11] T. Van An, D. X. Cuong. and H. Maki, On operation-preopen sets in topological space, *Sci. Math. Japon.*, e-(2008), 241–260.

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Some classes of sequence spaces defined by a Musielak-Orlicz function

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Abstract. In the present paper we introduce the sequence spaces $c_0\{\mathcal{M}, \Lambda, p, q\}$, $c\{\mathcal{M}, \Lambda, p, q\}$ and $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$. We study some topological properties and prove some inclusion relations between these spaces.

1 Introduction and preliminaries

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

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Also, it was shown in [3] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function (see [4], [8]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf \Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let w, l_∞, c and c_0 denote the spaces of all, bounded, convergent and null sequences $x = (x_k)$ with complex terms respectively. The zero sequence (0,0,...) is denoted by θ and $p = (p_k)$ is a sequence of strictly positive real numbers. Further the sequence (p_k^{-1}) will be represented by (t_k).

Mursaleen and Noman [6] introduced the notion of λ-convergent and λ-bounded sequences as follows :
Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \cdots \quad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \longrightarrow L$ as $m \to \infty$, where

$$\lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [6] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) \right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m \left| \frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a) \right| = 0$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [14], Theorem 10.4.2, pp. 183). For more details about sequence spaces (see [1], [2], [5], [7], [9], [10], [11], [12], [13]) and references therein.

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers and let (X, q) be a seminormed space seminormed by q. In the present paper, we define the following sequence spaces:

$$c_0\{\mathcal{M}, \Lambda, p, q\} = \Big\{x = (x_k) \in w : \Big[M_k\Big(\frac{q\big(\Lambda_k(x)\big)}{\rho}\Big)\Big]^{p_k} t_k \to 0, \ \text{as} \ k \to \infty, \\ \text{for some} \ \rho > 0\Big\},$$

$$c\{\mathcal{M}, \Lambda, p, q\} = \Big\{x = (x_k) \in w : \Big[M_k\Big(\frac{q\big(\Lambda_k(x)\big)}{\rho}\Big)\Big]^{p_k} t_k \to 0, \ \text{as} \ k \to \infty, \\ \text{for some} \ L \in X \ \text{and for some} \ \rho > 0\Big\}$$

and

$$l_\infty\{\mathcal{M}, \Lambda, p, q\} = \Big\{x = (x_k) \in w : \sup_k \Big[M_k\Big(\frac{q\big(\Lambda_k(x)\big)}{\rho}\Big)\Big]^{p_k} t_k < \infty, \\ \text{for some} \ \rho > 0\Big\}.$$

If we take $p = (p_k) = 1$, we have

$$c_0\{\mathcal{M}, \Lambda, q\} = \Big\{x = (x_k) \in w : \Big[M_k\Big(\frac{q\big(\Lambda_k(x)\big)}{\rho}\Big)\Big] \to 0, \ \text{as} \ k \to \infty, \\ \text{for some} \ \rho > 0\Big\},$$

$$c\{\mathcal{M}, \Lambda, q\} = \Big\{x = (x_k) \in w : \Big[M_k\Big(\frac{q\big(\Lambda_k(x) - L\big)}{\rho}\Big)\Big] \to 0, \ \text{as} \ k \to \infty, \\ \text{for some} \ L \in X \ \text{and for some} \ \rho > 0\Big\}$$

and

$$l_\infty\{\mathcal{M}, \Lambda, q\} = \Big\{x = (x_k) \in w : \sup_k \Big[M_k\Big(\frac{q\big(\Lambda_k(x)\big)}{\rho}\Big)\Big] < \infty, \ \text{for some} \ \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \le p_k \le \sup p_k = K$, $D = \max(1, 2^{K-1})$ then

$$|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^K)$ for all $a \in \mathbb{C}$.

The main aim of this paper is to study some toplogical properties and prove some inclusion relation between these spaces.

2 Main results

Theorem 1 *If $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers, then the spaces $c_0\{\mathcal{M}, \Lambda, p, q\}$, $c\{\mathcal{M}, \Lambda, p, q\}$ and $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_k),\ y = (y_k) \in c\{\mathcal{M}, \Lambda, p, q\}$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\left[M_k\left(\frac{q\big(\Lambda_k(x) - L\big)}{\rho_1}\right)\right]^{p_k} t_k \to 0, \text{ as } k \to \infty$$

and

$$\left[M_k\left(\frac{q\big(\Lambda_k(y) - L\big)}{\rho_2}\right)\right]^{p_k} t_k \to 0, \text{ as } k \to \infty.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is non-decreasing and convex by using inequality (1.1), we have

$$\begin{aligned}
&\left[M_k\left(\frac{q\big((\alpha\Lambda_k(x) + \beta\Lambda_k(y)) - 2L\big)}{\rho_3}\right)\right]^{p_k} t_k \\
&\leq \left[M_k\left(\frac{q\big(\alpha\Lambda_k(x) - L\big)}{\rho_3} + \frac{q\big(\beta\Lambda_k(y) - L\big)}{\rho_3}\right)\right]^{p_k} t_k \\
&\leq D\frac{1}{2^{p_k}}\left[M_k\left(\frac{q\big(\Lambda_k(x) - L\big)}{\rho_1}\right)\right]^{p_k} t_k + D\frac{1}{2^{p_k}}\left[M_k\left(\frac{q\big(\Lambda_k(y) - L\big)}{\rho_2}\right)\right]^{p_k} t_k \\
&\leq D\left[M_k\left(\frac{q\big(\Lambda_k(x) - L\big)}{\rho_1}\right)\right]^{p_k} t_k + D\left[M_k\left(\frac{q\big(\Lambda_k(y) - L\big)}{\rho_2}\right)\right]^{p_k} t_k \\
&\to 0 \text{ as } k \to \infty.
\end{aligned}$$

Thus, $\alpha x + \beta y \in c\{\mathcal{M}, \Lambda, p, q\}$. Hence $c\{\mathcal{M}, \Lambda, p, q\}$ is a linear space. Similarly, we can prove $c_0\{\mathcal{M}, \Lambda, p, q\}$ and $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ are linear spaces over the field of complex numbers $\mathbb{C}$. □

Theorem 2 *$\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers, then $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ is a paranormed space with the paranorm defined by*

$$g(x) = q(x_1) + \inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)t_k{}^{\frac{1}{p_k}}\right\} \leq 1, \quad \rho > 0 bigg\}, $$

where $H = \max(1, K)$.

Proof. (i) Clearly, $g(x) \geq 0$ for $x = (x_k) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$. Since $M_k(0) = 0$, we get $g(\theta) = 0$.
(ii) $g(-x) = g(x)$
(iii) Let $x = (x_k),\ y = (y_k) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$, then there exist $\rho_1, \rho_2 > 0$ such that

$$\sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_1}\right)t_k{}^{\frac{1}{p_k}}\right\} \leq 1$$

and

$$\sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda(y)\big)}{\rho_2}\right)t_k{}^{\frac{1}{p_k}}\right\} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then by Minkowski's inequality, we have

$$\begin{aligned}\sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x+y)\big)}{\rho}\right)t_k{}^{\frac{1}{p_k}}\right\} &= \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x+y)\big)}{\rho_1+\rho_2}\right)t_k{}^{\frac{1}{p_k}}\right\} \\ &\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{k\geq 1}\left[M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_1}\right)t_k{}^{\frac{1}{p_k}}\right] \\ &+ \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{k\geq 1}\left[M_k\left(\frac{q\big(\Lambda_k(y)\big)}{\rho_2}\right)t_k{}^{\frac{1}{p_k}}\right] \\ &\leq 1\end{aligned}$$

and thus

$$\begin{aligned}g(x+y) &= q(x_1+y_1) \\ &+ \inf\left\{(\rho_1+\rho_2)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x)+\Lambda_k(y)\big)}{\rho}\right)\right\}t_k{}^{\frac{1}{p_k}} \leq 1,\ \rho > 0\right\} \\ &\leq q(x_1) + \inf\left\{(\rho_1)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_1}\right)\right\}t_k{}^{\frac{1}{p_k}} \leq 1,\ \rho > 0\right\} \\ &+ q(y_1) + \inf\left\{(\rho_2)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(y)\big)}{\rho_2}\right)\right\}t_k{}^{\frac{1}{p_k}} \leq 1,\ \rho > 0\right\}\end{aligned}$$

$$\leq g(x) + g(y)$$

(iv) Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition,

$$g(\mu x) = q(\mu x_1) + \inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\mu\Lambda_k(x)\big)}{\rho}\right)\right\}t_k^{\frac{1}{p_k}} \leq 1,\ \ \rho > 0\right\}$$

$$= |\mu| q(x_1) + \inf\left\{(|\lambda| r)^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{r}\right)\right\}t_k^{\frac{1}{p_k}} \leq 1,\ \ r > 0\right\},$$

where $r = \frac{\rho}{|\mu|}$. Hence $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ is a paranormed space. □

Theorem 3 *For any Musielak-Orlicz function $\mathcal{M} = (M_k)$ and $p = (p_k) \in l_\infty$, then the spaces $c_0\{\mathcal{M}, \Lambda, p, q\}$, $c\{\mathcal{M}, \Lambda, p, q\}$ and $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ are complete paranormed spaces paranormed by g.*

Proof. Suppose (x^n) is a Cauchy sequence in $l_\infty\{\mathcal{M}, \Lambda, p, q\}$, where $x^n = (x_k^n)_{k=1}^{\infty}$ for all $n \in \mathbb{N}$. So that $g(x^i - x^j) \to 0$ as $i, j \to \infty$. Suppose $\epsilon > 0$ is given and let s and x_0 be such that $\frac{\epsilon}{sx_0} > 0$ and $M_k\left(\frac{sx_0}{2}\right) \geq \sup_{k\geq 1}(p_k)^{t_k}$. Since $g(x^i - x^j) \to 0$, as $i, j \to \infty$ which means that there exists $n_0 \in \mathbb{N}$ such that

$$g(x^i - x^j) < \frac{\epsilon}{sx_0}, \ \text{ for all } \ i, j \geq n_0.$$

This gives $g(x_1^i - x_1^j) < \frac{\epsilon}{sx_0}$ and

$$\inf\left\{\rho^{\frac{p_k}{H}} : \sup_{k\geq 1}\left\{M_k\left(\frac{q\big(\Lambda_k(x^i - x^j)\big)}{\rho}\right)t_k^{\frac{1}{p_k}}\right\} \leq 1, \rho > 0\right\} < \frac{\epsilon}{sx_0}. \qquad (2)$$

It shows that (x_1^i) is a Cauchy sequence in X. Therefore (x_1^i) is convergent in X because X is complete. Suppose $\lim_{i\to\infty} x_1^i = x_1$ then $\lim_{j\to\infty} g(x_1^i - x_1^j) < \frac{\epsilon}{sx_0}$, we get

$$g(x_1^i - x_1) < \frac{\epsilon}{sx_0}.$$

Thus, we have

$$M_k\left(\frac{q\big(\Lambda_k(x^i - x^j)\big)}{g(x^i - x^j)}\right)t_k^{\frac{1}{p_k}} \leq 1.$$

This implies that

$$M_k\left(\frac{q\big(\Lambda_k(x^i - x^j)\big)}{g(x^i - x^j)}\right) \leq (p_k)^{t_k} \leq M_k\left(\frac{sx_0}{2}\right)$$

and thus

$$q(\Lambda_k(x^i - x^j)) < \frac{sx_0}{2}.\frac{\epsilon}{sx_0} < \frac{\epsilon}{2}$$

which shows that $(\Lambda_k(x^i))$ is a Cauchy sequence in X for all $k \in \mathbb{N}$. Therefore, $(\Lambda_k(x^i))$ converges in X. Suppose $\lim\limits_{i\to\infty} \Lambda_k(x^i) = y$ for all $k \in \mathbb{N}$. Also, we have $\lim\limits_{i\to\infty} \Lambda_k(x_2^i) = y_1 - x_1$. On repeating the same procedure, we obtain $\lim\limits_{i\to\infty} \Lambda_k(x_{k+1}^i) = y_k - x_k$ for all $k \in \mathbb{N}$. Therefore by continuity of (M_k), we get

$$\lim_{j\to\infty} \sup_{k\geq 1} M_k\left(\frac{q\big(\Lambda_k(x^i - x^j)\big)}{\rho}\right) t_k^{\frac{1}{p_k}} \leq 1,$$

so that

$$\sup_{k\geq 1} M_k\left(\frac{q\big(\Lambda_k(x^i - x^j)\big)}{\rho}\right) t_k^{\frac{1}{p_k}} \leq 1.$$

Let $i \geq n_0$ and taking infimum of each ρ's, we have

$$g(x^i - x) < \epsilon.$$

So $(x^i - x) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$. Hence $x = x^i - (x^i - x) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$, since $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ is a linear space. Hence, $l_\infty\{\mathcal{M}, \Lambda, p, q\}$ is a complete paranormed space. Similarly, we can prove the spaces $c_0\{\mathcal{M}, \Lambda, p, q\}$ and $c\{\mathcal{M}, \Lambda, p, q\}$ are complete paranormed spaces. □

Theorem 4 *If* $0 < p_k \leq r_k < \infty$ *for each* k*, then*

$$Z\{\mathcal{M}, \Lambda, p, q\} \subseteq Z\{\mathcal{M}, \Lambda, r, q\}$$

for $Z = c_0$ *and* c*.*

Proof. Let $x = (x_k) \in c\{\mathcal{M}, \Lambda, p, q\}$. Then there exists some $\rho > 0$ and $L \in X$ such that

$$M_k\left(\frac{q\big(\Lambda_k(x) - L\big)}{\rho}\right)^{p_k} t_k \to 0 \ \text{ as } \ k \to \infty.$$

This implies that

$$M_k\left(\frac{q\big(\Lambda_k(x) - L\big)}{\rho}\right) < \epsilon, \ \ (0 < \epsilon < 1)$$

for sufficiently large k. Hence we get

$$M_k\left(\frac{q\big(\Lambda_k(x)-L\big)}{\rho}\right)^{r_k} t_k \;\leq\; M_k\left(\frac{q\big(\Lambda_k(x)-L\big)}{\rho}\right)^{p_k} t_k \to 0 \text{ as } k \to \infty.$$

This implies that $x = (x_k) \in c\{\mathcal{M}, \Lambda, r, q\}$. This completes the proof. Similarly, we can prove for the case $Z = c_0$. □

Theorem 5 *Suppose $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ are Musielak-Orlicz functions satisfying the Δ_2-condition then we have the following results:*

(i) *if $(p_k) \in l_\infty$ then $Z\{\mathcal{M}', \Lambda, p, q\} \subseteq Z\{\mathcal{M}'' \circ \mathcal{M}', \Lambda, p, q\}$ for $Z = c, c_0$ and l_∞.*

(ii) *$Z\{\mathcal{M}', \Lambda, p, q\} \cap Z\{\mathcal{M}'', \Lambda, p, q\} \subseteq Z\{\mathcal{M}' + \mathcal{M}'', \Lambda, p, q\}$ for $Z = c, c_0$ and l_∞.*

Proof. If $x = (x_k) \in c_0\{\mathcal{M}, \Lambda, p, q\}$ then there exists some $\rho > 0$ such that

$$\left\{M_k'\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right\}^{p_k} t_k \to 0 \text{ as } k \to \infty.$$

Suppose

$$y_k = M_k'\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right) \text{ for all } k \in \mathbb{N}.$$

Choose $\delta > 0$ be such that $0 < \delta < 1$, then for $y_k \geq \delta$ we have $y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$. Now (M_k'') satisfies Δ_2-condition so that there exists $J \geq 1$ such that

$$M_k''(y_k) < \frac{Jy_k}{2\delta} M_k''(2) + \frac{Jy_k}{2\delta} M_k''(2) = \frac{Jy_k}{\delta} M_k''(2).$$

We obtain

$$\left[(M_k'' \circ M_k')\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right]^{p_k} t_k = \left[M_k''\left\{M_k'\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right\}\right]^{p_k} t_k = \Big[M_k''(y_k)\Big]^{p_k} t_k$$
$$\leq \max\left\{\sup_k \left([M_k''(1)]^{p_k}\right), \sup_k \left([kM_k''(2)\delta^{-1}]^{p_k}\right)\right\}[y_k]^{p_k} t_k \to 0, \text{ as } k \to \infty.$$

Similarly, we can prove the other cases.

(ii) Suppose $x = (x_k) \in c_0\{M'_k, \Lambda, p, q\} \cap c_0\{M''_k, \Lambda, p, q\}$, then there exist $\rho_1,\ \rho_2 > 0$ such that

$$\left\{\left(M'_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_1}\right)\right)^{p_k} t_k\right\} \to 0, \text{ as } k \to \infty$$

and

$$\left\{\left(M''_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_2}\right)\right)^{p_k} t_k\right\} \to 0, \text{ as } k \to \infty.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The remaining proof follows from the inequality

$$\left\{\left[(M'_k + M''_k)\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right]^{p_k} t_k\right\} \leq D\left\{\left[M'_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_1}\right)\right]^{p_k} t_k + \left[M''_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_2}\right)\right]^{p_k} t_k\right\}.$$

Hence $c_0\{M'_k, \Lambda, p, q\} \cap c_0\{M''_k, \Lambda, p, q\} \subseteq c_0\{M'_k + M''_k, \Lambda, p, q\}$. Similarly we can prove the other cases. □

Theorem 6 (i) *If* $0 < \inf p_k \leq p_k < 1$, *then* $l_\infty\{\mathcal{M}, \Lambda, p, q\} \subset l_\infty\{\mathcal{M}, \Lambda, q\}$.
(ii) *If* $1 \leq p_k \leq \sup p_k < \infty$, *then* $l_\infty\{\mathcal{M}, \Lambda, q\} \subset l_\infty\{\mathcal{M}, \Lambda, p, q\}$.

Proof. (i) Let $x = (x_k) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$. Since $0 < \inf p_k \leq 1$, we have

$$\sup_k \left\{\left[M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_2}\right)\right]\right\} \leq \sup_k \left\{\left[M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho_2}\right)\right]^{p_k} t_k\right\}$$

and hence $x = (x_k) \in l_\infty\{\mathcal{M}, \Lambda, q\}$.
(ii) Let $p_k \geq 1$ for each k and $\sup_k p_k < \infty$. Let $x = (x_k) \in l_\infty\{\mathcal{M}, \Lambda, q\}$, then for each ϵ, $0 < \epsilon < 1$, there exists a positive integer $n_0 \in \mathbb{N}$ such that

$$\sup_k \left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right\} \leq \epsilon < 1.$$

This implies that

$$\sup_k \left\{\left[M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right]^{p_k} t_k\right\} \leq \sup_k \left\{M_k\left(\frac{q\big(\Lambda_k(x)\big)}{\rho}\right)\right\}.$$

Thus $x = (x_k) \in l_\infty\{\mathcal{M}, \Lambda, p, q\}$ and this completes the proof. □

References

[1] T. Bilgin, Some new difference sequences spaces defined by an Orlicz function, *Filomat*, **17** (2003), 1–8.

[2] A. Esi, Some new paranormed sequence spaces defined by Orlicz function, I*nternational Journal of Science, Environment and Technology*, **1** (2012), 49–55.

[3] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379–390.

[4] L. Maligranda, *Orlicz spaces and interpolation*, Seminars in Mathematics **5**, Polish Academy of Science, 1989.

[5] M. Mursaleen, Generalized spaces of difference sequences, *J. Math. Anal. Appl.*, **203** (1996), 738–745.

[6] M. Mursaleen and A. K. Noman, On some new sequence spaces of non absolute type related to the spaces l_p and l_∞ I, *Filomat*, **25** (2011), 33–51.

[7] M. Mursaleen and A. K. Noman, On some new sequence spaces of non absolute type related to the spaces l_p and l_∞ II, *Math. Commun.*, **16** (2011), 383–398.

[8] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, **1034** (1983).

[9] K. Raj, A. K. Sharma and S. K. Sharma, A Sequence space defined by Musielak-Orlicz functions, *Int. J. Pure Appl. Math.*, **67** (2011), 475–484.

[10] K. Raj, S. K. Sharma and A. K. Sharma, Some difference sequence spaces in n-normed spaces defined by Musielak-Orlicz function, *Armenian J. Math.,* **3** (2010), 127–141.

[11] K. Raj and S. K. Sharma, Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function, *Acta. Univ. Saprientia Math.*, **3** (2011), 97–109.

[12] K. Raj and S. K. Sharma, Some multiplier sequence spaces defined by a Musielak-Orlicz function in n-normed spaces, *New Zealand J. Math.*, **42** (2012), 45–56.

[13] B. C. Tripathy and B. Sarma, Some Some classes of difference paranormed sequence spaces defined by Orlicz function, *Thai J. Math.*, **3** (2005), 209–218.

[14] A. Wilansky, *Summability through Functional Analysis*, North- Holland Math. Stud. **85** (1984).

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Fixed point and a Cantilever beam problem in a partial b-metric space

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Abstract. We determine the common fixed point of two maps satisfying Hardy-Roger type contraction in a complete partial b-metric space without exploiting any variant of continuity or commutativity, which is indispensable in analogous results. Towards the end, we give examples and an application to solve a Cantilever beam problem employed in the distortion of an elastic beam in equilibrium to substantiate the utility of these improvements.

1 Introduction and preliminaries

Fixed point theory is a major tool in nonlinear analysis, having applications in many real-world problems, which emerged in 1837 with the article of Liouville [10] on solutions of differential equations. In 1890, Picard [13] developed it further as a process of successive approximations which were conceptualized and extracted by Banach [2] as a fixed point result in a complete normed space in 1922. On the other hand, Shukla [16] familiarized partial b-metric blending

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partial metric (Matthews [11]) and $\mathfrak{b}$-metric (Bakhtin [1] and Czerwik [6]) to establish a fixed point via Banach contraction [2] and Kannan contraction [9].

The aim of the current work is to demonstrate the survival of one and only one common fixed point of two maps satisfying classical Hardy-Rogers type contraction [7] in a complete partial $\mathfrak{b}$-metric space without exploiting any variant of continuity [17] or commutativity [18], which is indispensable in analogous results. We support our theoretical consequences by illustrative examples and conclude the paper by giving an application to solve a Cantilever beam problem employed in the distortion of an elastic beam in equilibrium to substantiate the utility of these improvements.

It is worth mentioning here that in numerous cable-driven docile mechanisms, like a fixed pulley or a cable routing channel in a segmented disk, the need for controlled motion in the flexible frameworks often mandates the actuation cables to pass through a fixed point to compel the force angle on the cable. This situation may be modeled as the large deflection problem of a cantilever beam with two parameters. Recently Zeng et al. [19] emphasized the numerical analysis of the large deflection problem of the cantilever beam subjected to a constraint force pointing at a fixed point which permitted widespread analysis of the impact of diverse factors, including the fixed point position, the force magnitude, and the beam length, on the behaviour of the cantilever beam put to a constraint force pointing at a fixed point. This work permitted mathematical model-based design optimization of docile frameworks in areas such as soft robotics and smart materials.

Definition 1 [16] *A function* $\mathfrak{p}_\mathfrak{b} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *on a nonempty set* $\mathcal{X}$ *is a partial* $\mathfrak{b}$*-metric if* $\forall \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in \mathcal{X}$,

1. $\mathfrak{u} = \mathfrak{v}$ *iff* $\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{u}) = \mathfrak{p}_\mathfrak{b}(\mathfrak{v}, \mathfrak{v})$;
2. $\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{u}) \leq \mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v})$;
3. $\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{p}_\mathfrak{b}(\mathfrak{v}, \mathfrak{u})$;
4. $\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v}) \leq \mathfrak{s}[\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{w}) + \mathfrak{p}_\mathfrak{b}(\mathfrak{w}, \mathfrak{v})] - \mathfrak{p}_\mathfrak{b}(\mathfrak{w}, \mathfrak{w})$.

The pair $(\mathcal{X}, \mathfrak{p}_\mathfrak{b})$ *is a partial* $\mathfrak{b}$*-metric space and* $\mathfrak{s} \geq 1$ *is the coefficient of* $(\mathcal{X}, \mathfrak{p}_\mathfrak{b})$.

Example 1 *Let* $\mathcal{X} = [0, 10]$ *and* $\mathfrak{p}_\mathfrak{b} : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ *be defined as:* $\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v}) = |\mathfrak{u} - \mathfrak{w}|^2 + 2$. *By routine calculation, one may verify that* $(\mathcal{X}, \mathfrak{p}_\mathfrak{b})$ *is a partial* $\mathfrak{b}$*-metric space for* $\mathfrak{s} = 2$. *However,* $(\mathcal{X}, \mathfrak{p}_\mathfrak{b})$ *is not a partial metric*

space. Since for $\mathfrak{u} = 0$, $\mathfrak{v} = 10$ *and* $\mathfrak{w} = 5$, *we obtain*
$p_{\mathfrak{b}}(0, 10) = |0 - 10|^2 + 2 = 102$,

$$\begin{aligned} p_{\mathfrak{b}}(0,5) + p_{\mathfrak{b}}(5,10) - p_{\mathfrak{b}}(5,5) &= |0-5|^2 + 2 + |5-10|^2 + 2 - 2 \\ &= 25 + 2 + 25 \\ &= 52. \end{aligned}$$

Therefore, $p_{\mathfrak{b}}(0,10) > p_{\mathfrak{b}}(0,5) + p_{\mathfrak{b}}(5,10) - p_{\mathfrak{b}}(5,5)$. *Noticeably,* $(\mathcal{X}, p_{\mathfrak{b}})$ *is also not a* $\mathfrak{b}$*-metric space.*

Definition 2 [12] *A sequence* $\{\mathfrak{u}_n\}$ *in a partial* $\mathfrak{b}$*-metric space* $(\mathcal{X}, p_{\mathfrak{b}})$ *is*

1. *convergent to* $\mathfrak{u} \in \mathcal{X}$ *if* $p_{\mathfrak{b}}(\mathfrak{u}, \mathfrak{u}) = \lim_{n\to\infty} p_{\mathfrak{b}}(\mathfrak{u}, \mathfrak{u}_n)$.

2. *Cauchy sequence if* $\lim_{n\to\infty} p_{\mathfrak{b}}(\mathfrak{u}_n, \mathfrak{u}_m)$ *exists and is finite.*

A partial $\mathfrak{b}$-metric space $(\mathcal{X}, p_{\mathfrak{b}})$ is complete [16] if each $p_{\mathfrak{b}}$-Cauchy sequence in $\mathcal{X}$ converges to $\mathfrak{u} \in \mathcal{X}$, i.e., $p_{\mathfrak{b}}(\mathfrak{u}, \mathfrak{u}) = \lim_{n\to\infty} p_{\mathfrak{b}}(\mathfrak{u}, \mathfrak{u}_n) = \lim_{n,m\to\infty} p_{\mathfrak{b}} (\mathfrak{u}_n, \mathfrak{u}_m)$.
One may notice that the limit of a convergent sequence is not essentially unique in a partial $\mathfrak{b}$-metric space.

2 Main results

Theorem 1 *Let* $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be self maps of a complete partial* $\mathfrak{b}$*-metric space* $(\mathcal{X}, p_{\mathfrak{b}})$ *so that* $\mathcal{T}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$ *and*

$$p_{\mathfrak{b}}(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq \mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) + \mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v}, \mathcal{T}\mathfrak{v}) + \mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u}, \mathcal{T}\mathfrak{v}) + \mathfrak{d}p_{\mathfrak{b}}(\mathfrak{v}, \mathcal{S}\mathfrak{u}) + \mathfrak{e}p_{\mathfrak{b}}(\mathfrak{u}, \mathfrak{v}), \quad (1)$$

$\forall\, \mathfrak{u}, \mathfrak{v} \in \mathcal{X}$ *and* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ *are positive reals satisfying* $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \mathfrak{e} + \mathfrak{d}(2s - 1) \leq 1$ *and* $s > 1$. *Then* $\mathcal{S}$ *and* $\mathcal{T}$ *have a unique common fixed point in* $\mathcal{X}$.

Proof. Assume $\mathfrak{u}_0 \in \mathcal{X}$ and since $\mathcal{T}(\mathcal{X}) \subseteq \mathcal{S}(\mathcal{X})$, so we may inductively define a sequence $\{\mathfrak{u}_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ as

$$\mathfrak{u}_n = \mathcal{T}\mathfrak{u}_{n-1} \text{ and } \mathfrak{u}_{n+1} = \mathcal{S}\mathfrak{u}_n, \quad (2)$$

for $n = 0, 1, 2, \ldots$. If $\mathfrak{u}_n = \mathfrak{u}_{n+1}$, i.e., $\mathfrak{u}_n = \mathcal{S}\mathfrak{u}_n$, i.e., $\mathfrak{u}_n$ is a fixed point of $\mathcal{S}$.

Since, $u_n = u_{n+1} \Longrightarrow u_{n+1} = \mathcal{S}u_n = \mathcal{S}u_{n+1}$. So

$$\begin{aligned}
p_b(u_{n+1}, u_{n+2}) &= p_b(\mathcal{S}u_{n+1}, \mathcal{T}u_n) \\
&\leq \mathfrak{a}p_b(u_{n+1}, \mathcal{S}u_{n+1}) + \mathfrak{b}p_b(u_n, \mathcal{T}u_n) + \mathfrak{c}p_b(u_{n+1}, \mathcal{T}u_n) \\
&\quad + \mathfrak{d}p_b(u_n, \mathcal{S}u_{n+1}) + \mathfrak{e}p_b(u_{n+1}, u_n) \\
&= \mathfrak{a}p_b(u_{n+1}, u_{n+2}) + \mathfrak{b}p_b(u_n, u_{n+1}) + \mathfrak{c}p_b(u_{n+1}, u_{n+1}) \\
&\quad + \mathfrak{d}p_b(u_n, u_{n+2}) + \mathfrak{e}p_b(u_{n+1}, u_n) \\
&\leq \mathfrak{a}p_b(u_{n+1}, u_{n+2}) + \mathfrak{b}p_b(u_n, u_{n+1}) + \mathfrak{c}p_b(u_{n+1}, u_{n+1}) \\
&\quad + \mathfrak{d}s[p_b(u_n, u_{n+1}) + p_b(u_{n+1}, u_{n+2})] - \mathfrak{d}p_b(u_{n+1}, u_{n+1}) \\
&\quad + \mathfrak{e}p_b(u_{n+1}, u_n),
\end{aligned}$$

$$\begin{aligned}
\text{i.e.,}\quad & (1 - \mathfrak{a} - \mathfrak{d}s)p_b(u_{n+1}, u_{n+2}) + \mathfrak{d}p_b(u_{n+1}, u_{n+1}) \leq (\mathfrak{b} + \mathfrak{d}s + \mathfrak{e})p_b(u_n, u_{n+1}) \\
& + \mathfrak{c}p_b(u_{n+1}, u_{n+1}), \\
\text{i.e.,}\quad & (1 - \mathfrak{a} - \mathfrak{d}s)p_b(u_{n+1}, u_{n+2}) + \mathfrak{d}p_b(u_{n+1}, u_{n+2}) \leq (\mathfrak{b} + \mathfrak{d}s + \mathfrak{e})p_b(u_n, u_{n+1}) \\
& + \mathfrak{c}p_b(u_n, u_{n+1}), \\
\text{i.e.,}\quad & (1 + \mathfrak{d} - \mathfrak{a} - \mathfrak{d}s)p_b(u_{n+1}, u_{n+2}) \leq (\mathfrak{b} + \mathfrak{d}s + \mathfrak{e} + \mathfrak{c})p_b(u_n, u_{n+1}), \\
\text{i.e.,}\quad & (1 + \mathfrak{d} - \mathfrak{a} - \mathfrak{d}s)p_b(u_{n+1}, u_{n+2}) \leq (\mathfrak{b} + \mathfrak{d}s + \mathfrak{e} + \mathfrak{c})p_b(u_{n+1}, u_{n+2}), \\
\text{i.e.,}\quad & (1 - \mathfrak{a} - \mathfrak{b} - \mathfrak{c} - \mathfrak{e} - \mathfrak{d}(2s - 1))p_b(u_{n+1}, u_{n+2}) \leq 0, \\
\text{i.e.,}\quad & p_b(u_{n+1}, u_{n+2}) \leq 0 \Longrightarrow p_b(u_{n+1}, u_{n+2}) = 0,
\end{aligned}$$

i.e., $\mathcal{T}u_n = u_{n+1} = u_{n+2}$ and $u_n = u_{n+1} \Longrightarrow \mathcal{T}u_n = u_n$, i.e., u_n is a fixed point of $\mathcal{T}$.
Also, $u_n = u_{n+1} = u_{n+2} = \ldots,$ i.e., u_n is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.
So, presume that for even n, $u_n \neq u_{n+1}$. Then

$$\begin{aligned}
p_b(u_{n+1}, u_n) &= p_b(Su_n, Tu_{n-1}) \\
&\leq \mathfrak{a}p_b(u_n, Su_n) + \mathfrak{b}p_b(u_{n-1}, Tu_{n-1}) + \mathfrak{c}p_b(u_n, Tu_{n-1}) \\
&\qquad + \mathfrak{d}p_b(u_{n-1}, Su_n) + \mathfrak{e}p_b(u_n, u_{n-1}) \\
&\leq \mathfrak{a}p_b(u_n, u_{n+1}) + \mathfrak{b}p_b(u_{n-1}, u_n) + \mathfrak{c}p_b(u_n, u_n) \\
&\qquad + \mathfrak{d}p_b(u_{n-1}, u_{n+1}) + \mathfrak{e}p_b(u_n, u_{n-1}) \\
&\leq \mathfrak{a}p_b(u_n, u_{n+1}) + \mathfrak{b}p_b(u_{n-1}, u_n) + \mathfrak{c}p_b(u_n, u_n) + \mathfrak{d}s[p_b(u_{n-1}, u_n) \\
&\qquad + p_b(u_n, u_{n+1})] - \mathfrak{d}p_b(u_n, u_n) + \mathfrak{e}p_b(u_n, u_{n-1}) \\
&\leq \mathfrak{a}p_b(u_n, u_{n+1}) + \mathfrak{b}p_b(u_{n-1}, u_n) + \mathfrak{c}p_b(u_n, u_n) + \mathfrak{d}sp_b(u_{n-1}, u_n) \\
&\qquad + \mathfrak{d}sp_b(u_n, u_{n+1}) - dp_b(u_n, u_n) + \mathfrak{e}p_b(u_n, u_{n-1}),
\end{aligned}$$

i.e., $(1-\mathfrak{a}-\mathfrak{d}s)p(u_n,u_{n+1}) \leq (\mathfrak{b}+\mathfrak{d}s+\mathfrak{e})p_b(u_{n-1},u_n)+(\mathfrak{c}-\mathfrak{d})p_b(u_n,u_n)$,
i.e., $(1-\mathfrak{a}-\mathfrak{d}s)p(u_n,u_{n+1})+\mathfrak{d}p_b(u_n,u_n) \leq (\mathfrak{b}+\mathfrak{d}s+\mathfrak{e})p_b(u_n,u_{n-1})$
$+\mathfrak{c}p_b(u_n,u_n)$,
i.e., $(1-\mathfrak{a}-\mathfrak{d}s)p(u_n,u_{n+1})+\mathfrak{d}p_b(u_n,u_{n+1}) \leq (\mathfrak{b}+\mathfrak{d}s+\mathfrak{e})p_b(u_nu_{n-1})$
$+\mathfrak{c}p_b(u_n,u_{n-1})$
$(1+\mathfrak{d}-\mathfrak{a}-\mathfrak{d}s)p_bu_n,u_{n+1}) \leq (\mathfrak{b}+\mathfrak{c}+\mathfrak{e}+\mathfrak{d}s)p_b(u_n,u_{n+1})$,
i.e., $p_b(u_n,u_{n+1}) \leq \dfrac{\mathfrak{b}+\mathfrak{c}+\mathfrak{e}+\mathfrak{d}s}{1+\mathfrak{d}-\mathfrak{a}-\mathfrak{d}s}p_b(u_n,u_{n+1})$,

$$\text{i.e.,}\quad p_b(u_n,u_{n+1}) \leq kp_b(u_n,u_{n-1}), \quad \text{where,}\ \ k=\frac{\mathfrak{b}+\mathfrak{c}+\mathfrak{e}+\mathfrak{d}s}{1+\mathfrak{d}-\mathfrak{a}-\mathfrak{d}s} \leq 1. \qquad (3)$$

If n is odd, the same inequality (3) can be obtained analogously.
Continuing this process, we attain
$p_b(u_n,u_{n+1}) \leq k^n p_b(u_0,u_1)$.
We assert that $\{u_n\}$is a Cauchy sequence in $\mathcal{X}$. For $m > n$ and $m,n \in \mathbb{N}$, consider

$$\begin{aligned}
p_b(u_n,u_m) &\leq s[p_b(u_n,u_{n+1})+p_b(u_{n+1},u_m)]-p_b(u_{n+1},u_{n+1})\\
&\leq s[p_b(u_n,u_{n+1})+p_b(u_{n+1},u_m)]\\
&\leq sp_b(u_n,u_{n+1})+s[s\{p_b(u_{n+1},u_{n+2})\\
&\qquad\qquad +p_b(u_{n+2},u_m)\}-p_b(u_{n+2},u_{n+2})]\\
&\leq sp_b(u_n,u_{n+1})+s[s\{p_b(u_{n+1},u_{n+2})+p_b(u_{n+2},u_m)\}]\\
&\leq sk^n p_b(u_0,u_1)+s^2k^{n+1}p_b(u_0,u_1)+....\\
&\leq sk^n p_b(u_0,u_1)[1+sk+(sk)^2+....]\\
&\leq \frac{sk^n}{1-sk}p_b(u_0,u_1) \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty,
\end{aligned}$$

i.e., $\{u_n\}$ is a Cauchy sequence. Using completeness of $\mathcal{X}$, $\{u_n\}$ converges to $u^* \in \mathcal{X}$ and we have $\lim_{n,m\longrightarrow\infty} p_b(u_n,u_m)=\lim_{n\longrightarrow\infty} p_b(u_n,u^*)=p_b(u^*,u^*)=0$. Further, we assert that u^* is a fixed point of $\mathcal{S}$. Let $\{u_{n_i}\}_{i=1}^{\infty}$ be a subsequence of $\{u_n\}$.

So,

$$\begin{aligned}
p_b(u^*,\mathcal{S}u^*) &\leq s[p_b(u^*,u_{n_i})+p_b(u_{n_i},\mathcal{S}u^*)]-p_b(u_{n_i},u_{n_i})\\
&\leq sp_b(u^*,u_{n_i})+sp_b(\mathcal{T}u_{n-1_i},\mathcal{S}u^*)\\
&\leq sp_b(u^*,u_{n_i})+s[p_b(\mathcal{S}u^*,\mathcal{T}u_{n-1_i})]\\
&\leq sp_b(u^*,u_{n_i})+s[\mathfrak{a}p_b(u^*,Su^*)+\mathfrak{b}p_b(u_{n-1_i},\mathcal{T}u_{n-1_i})
\end{aligned}$$

$$
\begin{aligned}
&+ \mathfrak{c}p_{\mathfrak{b}}(u^*, \mathcal{T}u_{n-1_i}) + \mathfrak{d}p_{\mathfrak{b}}(u_{n-1_i}, \mathcal{S}u^*) + \mathfrak{e}p_{\mathfrak{b}}(u^*, u_{n-1_i})] \\
&\leq sp_{\mathfrak{b}}(u^*, u_{n_i}) + s[\mathfrak{a}p_{\mathfrak{b}}(u^*, \mathcal{S}u^*) + \mathfrak{b}p_{\mathfrak{b}}(u_{n-1_i}, u_{n_i}) \qquad (4)\\
&+ \mathfrak{c}p_{\mathfrak{b}}(u^*, u_{n_i}) + \mathfrak{d}p_{\mathfrak{b}}(u_{n-1_i}, \mathcal{S}u^*) + \mathfrak{e}p_{\mathfrak{b}}(u^*, u_{n-1_i})].
\end{aligned}
$$

As $n \to \infty$, $p_{\mathfrak{b}}(u^*, \mathcal{S}u^*) \leq s(\mathfrak{a}+\mathfrak{d})p_{\mathfrak{b}}(u^*, Su^*)$, which gives a contradiction. So, $u^* = \mathcal{S}u^* \Rightarrow u^*$ is fixed point of $\mathcal{S}$.
Furthermore, we assert that u^* is a fixed point of $\mathcal{T}$. Let $\{u_{n+1_i}\}_{i=1}^{\infty}$ be a subsequence of $\{u_n\}$.
So,

$$
\begin{aligned}
p_{\mathfrak{b}}(u^*, \mathcal{T}u^*) &\leq s[p_{\mathfrak{b}}(u^*, u_{n+1_i}) + p_{\mathfrak{b}}(u_{n+1_i}, \mathcal{T}u^*)] - p_{\mathfrak{b}}(u_{n+1_i}, u_{n+1_i}) \\
&\leq sp_{\mathfrak{b}}(u^*, u_{n+1_i}) + sp_{\mathfrak{b}}(\mathcal{S}u_{n_i}, \mathcal{T}u^*) \\
&\leq sp_{\mathfrak{b}}(u^*, u_{n+1_i}) + s[\mathfrak{a}p_{\mathfrak{b}}(u_{n_i}, \mathcal{S}u_{n_i}) + \mathfrak{b}p_{\mathfrak{b}}(u^*, \mathcal{T}u^*) \\
&\qquad + \mathfrak{c}p_{\mathfrak{b}}(u_{n_i}, \mathcal{T}u^*) + \mathfrak{d}p_{\mathfrak{b}}(u^*, \mathcal{S}u_{n_i}) + \mathfrak{e}p_{\mathfrak{b}}(u_{n_i}, u^*)] \\
&\leq sp_{\mathfrak{b}}(u^*, u_{n+1_i}) + s[\mathfrak{a}p_{\mathfrak{b}}(u_{n_i}, u_{n+1_i}) + \mathfrak{b}p_{\mathfrak{b}}(u^*, \mathcal{T}u^*) \\
&\qquad + \mathfrak{c}p_{\mathfrak{b}}(u_{n_i}, \mathcal{T}u^*) + \mathfrak{d}p_{\mathfrak{b}}(u^*, u_{n+1_i}) + \mathfrak{e}p_{\mathfrak{b}}(u_{n_i}, u^*)].
\end{aligned}
$$

As $n \to \infty$, $p_{\mathfrak{b}}(u^*, \mathcal{T}u^*) \leq s(\mathfrak{b}+\mathfrak{c})p_{\mathfrak{b}}(u^*, \mathcal{T}u^*)$, which gives a contradiction.
Therefore, $u^* = \mathcal{T}u^* \Rightarrow u^*$ is a fixed point of $\mathcal{T}$.
If u and u^* are two different common fixed points of $\mathcal{S}$ and $\mathcal{T}$, then we have $\mathcal{S}u = \mathcal{T}u = u$ and $\mathcal{S}u^* = \mathcal{T}u^* = u^*$. Consider

$$
\begin{aligned}
p_{\mathfrak{b}}(u, u^*) &= p_{\mathfrak{b}}(Su, \mathcal{T}u^*) \\
&\leq \mathfrak{a}p_{\mathfrak{b}}(u, \mathcal{S}u) + \mathfrak{b}p_{\mathfrak{b}}(u^*, \mathcal{T}u^*) + \mathfrak{c}p_{\mathfrak{b}}(u, \mathcal{T}u^*) + \mathfrak{d}p_{\mathfrak{b}}(u^*, \mathcal{S}u) + \mathfrak{e}p_{\mathfrak{b}}(u, u^*) \\
&\leq \mathfrak{a}p_{\mathfrak{b}}(u, u) + \mathfrak{b}p_{\mathfrak{b}}(u^*, u^*) + \mathfrak{c}p_{\mathfrak{b}}(u, u^*) + \mathfrak{d}p_{\mathfrak{b}}(u^*, u) + \mathfrak{e}p_{\mathfrak{b}}(u, u^*) \\
&\leq (\mathfrak{c}+\mathfrak{d}+\mathfrak{e})p_{\mathfrak{b}}(u, u^*),
\end{aligned}
$$

a contradiction, i.e., $u = u^* \Rightarrow \mathcal{S}$ and $\mathcal{T}$ has a unique common fixed point in $\mathcal{X}$. □

Next, we provide a non-trivial illustration to exhibit the significance of Theorem 1.

Example 2 *Let* $\mathcal{X} = [-10, 10]$ *and* $p_{\mathfrak{b}} : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ *be defined as:* $p_{\mathfrak{b}}(u, \mathfrak{v}) = (|u| + |\mathfrak{v}| + 2)^2$. *Then* $(\mathcal{X}, p_{\mathfrak{b}})$ *is a complete partial* $\mathfrak{b}$*-metric space and* $s = 2$. *Define* $\mathcal{S}$, $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ *as:* $\mathcal{S}u = \frac{u}{6}$ *and* $\mathcal{T}u = \frac{u}{10}$. *Let* $u \geq \mathfrak{v}$. *Then*

$$
\begin{aligned}
p_{\mathfrak{b}}(\mathcal{S}u, \mathcal{T}\mathfrak{v}) &= p_{\mathfrak{b}}\left(\frac{u}{6}, \frac{\mathfrak{v}}{10}\right) = \left(\frac{|u|}{6} + \frac{|\mathfrak{v}|}{10} + 2\right)^2 \qquad (5)\\
&= \left(\frac{10|u| + 6|\mathfrak{v}| + 120}{60}\right)^2 \quad \textit{and}
\end{aligned}
$$

$$\begin{aligned}
&\mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u},\mathcal{S}\mathfrak{u})+\mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v},\mathcal{T}\mathfrak{v})+\mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u},\mathcal{T}\mathfrak{v})+\mathfrak{d}p_{\mathfrak{b}}(\mathfrak{v},\mathcal{S}\mathfrak{u})+\mathfrak{e}p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v})\\
&=\mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u},\frac{\mathfrak{u}}{6})+\mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v},\frac{\mathfrak{v}}{10})+\mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u},\frac{\mathfrak{v}}{10})+\mathfrak{d}p_{\mathfrak{b}}(\mathfrak{v},\frac{\mathfrak{u}}{6})+\mathfrak{e}p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v})\\
&=\mathfrak{a}\big(|\mathfrak{u}|+|\frac{\mathfrak{u}}{6}|+2\big)^2+\mathfrak{b}\big(|\mathfrak{v}|+|\frac{\mathfrak{v}}{10}|+2\big)^2+\mathfrak{c}\big(\mathfrak{u}+\frac{\mathfrak{v}}{10}+2\big)^2+\mathfrak{d}\big(\mathfrak{v}+\frac{\mathfrak{u}}{6}+2\big)^2\\
&\quad+\mathfrak{e}\big(\mathfrak{u}+\mathfrak{v}+2\big)^2 \qquad (6)\\
&=\mathfrak{a}\left(\frac{7|\mathfrak{u}|+12}{6}\right)^2+\mathfrak{b}\left(\frac{11|\mathfrak{v}|+20}{10}\right)^2+\mathfrak{c}\left(\frac{10|\mathfrak{u}|+|\mathfrak{v}|+20}{10}\right)^2\\
&\quad+\mathfrak{d}\left(\frac{6|\mathfrak{v}|+|\mathfrak{u}|+12}{6}\right)^2+\mathfrak{e}(|\mathfrak{u}|+|\mathfrak{v}|+2)^2.
\end{aligned}$$

From equations (5) and (6) it is clear that for $\mathfrak{a}=\mathfrak{b}=\mathfrak{e}=\frac{1}{6}$, $\mathfrak{c}=\frac{1}{3}$, *and* $\mathfrak{d}=\frac{1}{9}$,

$$p_{\mathfrak{b}}(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{v})\le \mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u},\mathcal{S}\mathfrak{u})+\mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v},\mathcal{T}\mathfrak{v})+\mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u},\mathcal{T}\mathfrak{v})+\mathfrak{d}p_{\mathfrak{b}}(\mathfrak{v},\mathcal{S}\mathfrak{u})+\mathfrak{e}p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v}).$$

Consequently, all postulates of Theorem 1 are verified, and 0 *is the unique common fixed point of* $\mathcal{S}$ *and* $\mathcal{T}$.

Corollary 1 *Inference of Theorem 1 is valid if* $\mathfrak{c}=\mathfrak{d}=0$.

Proof. The proof follows the pattern of Theorem 1. □

Next, we present two examples to understand and support the result proved herein. In one example involved maps are continuous and commutative and in another maps are discontinuous and noncommutative. It is worth mentioning that continuity is difficult to be fulfilled in some daily life applications and is an ideal property.

Example 3 *Let* $\mathcal{X}=\mathbb{R}^+$ *and* $p_{\mathfrak{b}}:\mathcal{X}\times\mathcal{X}\longrightarrow[0,\infty)$ *be defined as:* $p_{\mathfrak{b}}(\mathfrak{u},\mathfrak{v})=\max\{\mathfrak{u},\mathfrak{v}\}^2+|\mathfrak{u}-\mathfrak{v}|^2$. *Then* $(\mathcal{X},p_{\mathfrak{b}})$ *is a complete partial* $\mathfrak{b}$*-metric space and* $s=4$. *Define* $\mathcal{S}$, $\mathcal{T}:\mathcal{X}\longrightarrow\mathcal{X}$ *as:* $\mathcal{S}\mathfrak{u}=\frac{\mathfrak{u}}{4}$ *and* $\mathcal{T}\mathfrak{u}=\frac{\mathfrak{u}}{5}$. *Let* $\mathfrak{u}\ge\mathfrak{v}$. *Then*

$$\begin{aligned}
p_{\mathfrak{b}}(\mathcal{S}\mathfrak{u},\mathcal{T}\mathfrak{v})&=p_{\mathfrak{b}}\left(\frac{\mathfrak{u}}{4},\frac{\mathfrak{v}}{5}\right)\\
&=\max\left\{\frac{\mathfrak{u}}{4},\frac{\mathfrak{v}}{5}\right\}^2+\left|\frac{\mathfrak{u}}{4}-\frac{\mathfrak{v}}{5}\right|^2 \qquad (7)\\
&=\frac{\mathfrak{u}^2}{16}+\left|\frac{5\mathfrak{u}-4\mathfrak{v}}{25}\right|^2 \quad \textit{and}
\end{aligned}$$

$$
\begin{aligned}
&\mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) + \mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v}, \mathcal{T}\mathfrak{v}) + \mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{a}p_{\mathfrak{b}}\left(\mathfrak{u}, \frac{\mathfrak{u}}{4}\right) + \mathfrak{b}p_{\mathfrak{b}}\left(\mathfrak{v}, \frac{\mathfrak{v}}{5}\right) + \mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u}, \mathfrak{v}) \\
&= \mathfrak{a}\left[\max\left\{\mathfrak{u}, \frac{\mathfrak{u}}{4}\right\}^2 + \left|\mathfrak{u} - \frac{\mathfrak{u}}{4}\right|^2\right] + \mathfrak{b}\left[\max\left\{\mathfrak{v}, \frac{\mathfrak{v}}{5}\right\}^2 + \left|\mathfrak{v} - \frac{\mathfrak{v}}{5}\right|^2\right] \\
&\qquad + \mathfrak{c}\left[\max\{\mathfrak{u}, \mathfrak{v}\}^2 + |\mathfrak{u} - \mathfrak{v}|^2\right] \\
&= \mathfrak{a}\left[\mathfrak{u}^2 + \frac{9}{16}\mathfrak{u}^2\right] + \mathfrak{b}\left[\mathfrak{v}^2 + \frac{16}{25}\mathfrak{v}^2\right] + \mathfrak{c}\left[\mathfrak{u}^2 + |\mathfrak{u} - \mathfrak{v}|^2\right] \\
&= \frac{25}{16}\mathfrak{a}\mathfrak{u}^2 + \frac{41}{25}\mathfrak{b}\mathfrak{v}^2 + \mathfrak{c}[\mathfrak{u}^2 + |\mathfrak{u} - \mathfrak{v}|^2].
\end{aligned}
\tag{8}
$$

From Equations (7) *and* (8) *it is clear that for* $\mathfrak{a} = \frac{1}{3}$, $\mathfrak{b} = \mathfrak{c} = \frac{1}{9}$,

$$p_{\mathfrak{b}}(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq \mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) + \mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v}, \mathcal{T}\mathfrak{v}) + \mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u}, \mathfrak{v}).$$

Hence, all postulates of Corollary 1 are verified, and 0 *is the unique common fixed point of* $\mathcal{S}$ *and* $\mathcal{T}$.

Example 4 *Let* $\mathcal{X} = \mathbb{R}^+$ *and* $p_{\mathfrak{b}} : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ *be defined as:* $p_{\mathfrak{b}}(\mathfrak{u}, \mathfrak{v}) = \max\{\mathfrak{u}, \mathfrak{v}\} + |\mathfrak{u} - \mathfrak{v}|^2$. *Then* $(\mathcal{X}, p_{\mathfrak{b}})$ *is a complete partial* $\mathfrak{b}$*-metric space and* $s = 4$. *Define* $\mathcal{S}, \mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ *as:* $\mathcal{S}\mathfrak{u} = \begin{cases} \frac{\mathfrak{u}}{2}, & \mathfrak{u} \in [0,1] \\ 0, & \textit{otherwise} \end{cases}$ *and* $\mathcal{T}\mathfrak{u} = \begin{cases} \frac{\mathfrak{u}^2 - \mathfrak{u}}{2}, & \mathfrak{u} \in [0,1] \\ 0, & \textit{otherwise} \end{cases}$. *Let* $\mathfrak{u}, \mathfrak{v} \in [0, 1]$ *and* $\mathfrak{u} \geq \mathfrak{v}$. *Therefore,*

$$
\begin{aligned}
p_{\mathfrak{b}}(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{v}) &= p_{\mathfrak{b}}\left(\frac{\mathfrak{u}}{2}, \frac{\mathfrak{v}^2 - \mathfrak{v}}{2}\right) \\
&= \max\left\{\frac{\mathfrak{u}}{2}, \frac{\mathfrak{v}^2 - \mathfrak{v}}{2}\right\} + \left|\frac{\mathfrak{u}}{2} - \frac{\mathfrak{v}^2 - \mathfrak{v}}{2}\right|^2 \\
&= \frac{\mathfrak{u}}{2} + \left|\frac{\mathfrak{u} + \mathfrak{v} - \mathfrak{v}^2}{2}\right|^2 \quad \textit{and}
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&\mathfrak{a}p_{\mathfrak{b}}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) + \mathfrak{b}p_{\mathfrak{b}}(\mathfrak{v}, \mathcal{T}\mathfrak{v}) + \mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u}, \mathfrak{v}) = \mathfrak{a}p_{\mathfrak{b}}\left(\mathfrak{u}, \frac{\mathfrak{u}}{2}\right) + \mathfrak{b}p_{\mathfrak{b}}\left(\mathfrak{v}, \frac{\mathfrak{v}^2 - \mathfrak{v}}{2}\right) + \mathfrak{c}p_{\mathfrak{b}}(\mathfrak{u}, \mathfrak{v}) \\
&= \mathfrak{a}\left[\max\left\{\mathfrak{u}, \frac{\mathfrak{u}}{2}\right\} + \left|\mathfrak{u} - \frac{\mathfrak{u}}{2}\right|^2\right] + \mathfrak{b}\left[\max\left\{\mathfrak{v}, \frac{\mathfrak{v}^2 - \mathfrak{v}}{2}\right\} + \left|\mathfrak{v} - \frac{\mathfrak{v}^2 - \mathfrak{v}}{2}\right|^2\right] \\
&\qquad \mathfrak{c}\left[\max\{\mathfrak{u}, \mathfrak{v}\} + \left|\mathfrak{u} - \mathfrak{v}\right|^2\right] \\
&= \mathfrak{a}\left[\mathfrak{u} + \frac{1}{4}\mathfrak{u}^2\right] + \mathfrak{b}\left[\mathfrak{v} + \frac{1}{4}(3\mathfrak{v} - \mathfrak{v}^2)\right] + \mathfrak{c}[\mathfrak{u} + |\mathfrak{u} - \mathfrak{v}|^2\Big].
\end{aligned}
\tag{10}
$$

Next, if $\mathfrak{u} \le \mathfrak{v}$ *and* $\mathfrak{u}, \mathfrak{v} \in [0, 1]$,

$$\mathfrak{p}_\mathfrak{b}(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{v}) = \frac{\mathfrak{u}}{2} + \left|\frac{\mathfrak{u}+\mathfrak{v}-\mathfrak{v}^2}{2}\right|^2 \quad \textit{and} \tag{11}$$

$$\begin{aligned}\mathfrak{a}\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) + \mathfrak{b}\mathfrak{p}_\mathfrak{b}(\mathfrak{v}, \mathcal{T}\mathfrak{v}) + \mathfrak{c}\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v}) &= \mathfrak{a}\left[\mathfrak{u} + \frac{1}{4}\mathfrak{u}^2\right] + \mathfrak{b}\left[\mathfrak{v} + \frac{1}{4}(3\mathfrak{v} - \mathfrak{v}^2)\right] \\ &+ \mathfrak{c}[\mathfrak{v} + |\mathfrak{u} - \mathfrak{v}|^2].\end{aligned} \tag{12}$$

From Equations (9), (10), (11), and (12) it is clear that for $\mathfrak{a} = \frac{1}{3}$, $\mathfrak{b} = \frac{1}{4}$ *and* $\mathfrak{c} = \frac{1}{7}$

$$\mathfrak{p}_\mathfrak{b}(\mathcal{S}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \le \mathfrak{a}\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathcal{S}\mathfrak{u}) + \mathfrak{b}\mathfrak{p}_\mathfrak{b}(\mathfrak{v}, \mathcal{T}\mathfrak{v}) + \mathfrak{e}\mathfrak{p}_\mathfrak{b}(\mathfrak{u}, \mathfrak{v}),\ \ \mathfrak{u}, \mathfrak{v} \in [0, 1]. \tag{13}$$

Hence, all postulates of Corollary 1 are verified, and 0 *is the unique common fixed point of* $\mathcal{S}$ *and* $\mathcal{T}$.

Remark 1

(i) *Above results are also true if* $\mathcal{T}(\mathcal{X})$ *is a complete subspace instead of completeness of* $\mathcal{X}$.

(ii) *Above results become more fascinating if we appraise a better natural postulate of closures of range space, i.e.,* $\overline{\mathcal{T}(\mathcal{X})} \subseteq \mathcal{S}(\mathcal{X})$.

(iii) *Suitably choosing the values of constants* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d},$ *and* $\mathfrak{e}$, *we get the extensions, improvements, generalizations of Bakhtin [1], Banach [2], Chatterjea [3],* Kannan [9], *Reich [14], and so on to a partial* $\mathfrak{b}$*-metric space for a noncommutative discontinuous pair of maps.*

(iv) *In Theorem 1* and *Corollary 1 (see, Example 4), a unique common fixed point exists for a pair of discontinuous self maps which does not satisfy even commutativity ([8], [15], [17]) and thereby extend, generalize and improve the comparable theorems present in the literature (for instance, Banach [2], Chatterjea [3], Ćirić [4], Czerwik [6], Hardy-Rogers [7], Kannan [9], Reich [14], and references therein).*

(v) *Following arguments of Theorem 1, we may relax continuity, commutativity, and completeness of numerous celebrated and contemporary results existing in different spaces.*

3 Solution of Cantilever beam problem

Motivated by the fact that the Cantilever structure permits overhanging constructions deprived of peripheral bracing, we solve a system of fourth-order differential equations arising in the two-point boundary value problem of bending of an elastic beam as an application of Corollary 1. Suppose $\mathcal{X} = C[I, \mathbb{R}]$ denotes the set of all continuous functions on $I = [0, 1]$. Define a partial $\mathfrak{b}$-metric $p_{\mathfrak{b}} : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}^+$ as:
$p_{\mathfrak{b}}(u(t), \mathfrak{v}(t)) = \max_{t\in[0,1]} \left(\frac{|u(t)|+|\mathfrak{v}(t)|}{2}\right)^2$ with $s = 3$.

Theorem 2 *The equations of deformations of an elastic beam, one of whose end-point is free while the other is fixed, in its equilibrium state is:*

$$\begin{aligned} &\frac{d^4u}{dt^4} = \psi(t, u(t), u'(t), u''(t), u'''(t)), \\ &u(0) = u'(0) = u''(1) = u'''(1) = 0, \quad t \in [0, 1], \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\frac{d^4\mathfrak{v}}{dt^4} = \phi(t, \mathfrak{v}(t), \mathfrak{v}'(t), \mathfrak{v}''(t), \mathfrak{v}'''(t)), \\ &\mathfrak{v}(0) = \mathfrak{v}'(0) = \mathfrak{v}''(1) = \mathfrak{v}'''(1) = 0, \quad t \in [0, 1], \end{aligned} \tag{15}$$

where, $\psi,\ \phi : [0, 1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ *are continuous functions satisfying:*
$\max_{t\in[0,1]}(|\psi(t, u(t), u'(t), u''(t))|+|\phi(t, \mathfrak{v}(t), \mathfrak{v}'(t), \mathfrak{v}''(t))|)^2 \leq \exp^{-\alpha} \max_{t\in[0,1]} |u(t) + \mathfrak{v}(t)|^2 + \exp^{-\beta} \max_{t\in[0,1]} |u(t)|^2 + \exp^{-\gamma} \max_{t\in[0,1]} |\mathfrak{v}(t)|^2,\ u, \mathfrak{v} \in \mathcal{X},\ \lambda \in [1, \infty),\ t \in [0, 1).$
Then, the Cantilever beam problem (14-15) has a solution in $\mathcal{X}$*.*

Proof. The Cantilever beam problem (14-15) is identical to solving the system of integral equations

$$u(t) = \int_0^1 \mathcal{G}(s, t)\psi(s, u(s), u'(s), u''(s))ds \tag{16}$$

and

$$\mathfrak{v}(t) = \int_0^1 \mathcal{G}(s, t)\phi(s, \mathfrak{v}(s), \mathfrak{v}'(s), \mathfrak{v}''(s))ds,\ t \in [0, 1],\ u \in \mathcal{X}. \tag{17}$$

Here,

$$\mathcal{G}(s, t) = \begin{cases} \frac{1}{6}s^2(3t - s) & ,0 \leq t \leq s \leq 1 \\ \frac{1}{6}t^2(3s - t) & ,0 \leq s \leq t \leq 1 \end{cases}, \tag{18}$$

is a continuous Green function on $[0,1]$. Define maps $\mathcal{S}:\mathcal{X}\longrightarrow\mathcal{X}$ and $\mathcal{T}:\mathcal{X}\longrightarrow\mathcal{X}$ as:

$$\mathcal{S}u(t)=\int_{-0}^{1}G(s,t)\psi(s,u(s),u'(s),u''(s))ds$$

and

$$\mathcal{T}u(t)=\int_{0}^{1}G(s,t)\phi(s,v(s),v'(s),v''(s))ds.$$

Then u is a solution of (14-15) iff u is a single common fixed point of $\mathcal{S}$ and $\mathcal{T}$ respectively.
Clearly, $\mathcal{S},\ \mathcal{T}:\mathcal{X}\longrightarrow\mathcal{X}$are well defined, so

$$\begin{aligned}
&p_b(\mathcal{S}u(t),\mathcal{T}v(t))=\left(\frac{|\mathcal{S}u(t)|+|\mathcal{T}v(t)|}{2}\right)^2\\
&=\left(\frac{\left|\int_0^1\mathcal{G}(s,t)\psi(s,u(s),u'(s),u''(s))ds\right|+\left|\int_0^1\mathcal{G}(s,t)\phi(s,v(s),v'(s),v''(s))ds\right|}{2}\right)^2\\
&\leq\left(\frac{\int_0^1\mathcal{G}(s,t)\left|\psi(s,u(s),u'(s),u''(s))\right|ds+\int_0^1\mathcal{G}(s,t)\left|\phi(s,v(s),v'(s),v''(s))\right|ds}{2}\right)^2\\
&=\frac{1}{4}\left(\int_0^1\mathcal{G}(t,s)\Big(\left|\psi(s,u(s),u'(s),u''(s))\right|+\left|\phi(s,v(s),v'(s),v''(s))\right|\Big)ds\right)^2\\
&\leq\frac{1}{4}\max(\left|\psi(s,u(s),u'(s),u''(s))\right|+\left|\phi(s,v(s),v'(s),v''(s))\right|)^2\left(\int_{-1}^1\mathcal{G}(t,s)ds\right)^2\\
&\leq\frac{1}{4}[\exp^{-\alpha}\max_{t\in[0,1]}|u(t)+v(t)|^2+\exp^{-\beta}\max_{t\in[0,1]}|u(t)|^2+\exp^{-\gamma}\max_{t\in[0,1]}|v(t)|^2]\\
&\qquad\left(\int_{-1}^1\mathcal{G}(t,s)ds\right)^2\\
&\leq\frac{1}{4}\left[\exp^{-\alpha}\max_{t\in[0,1]}|u(t)+v(t)|^2+\exp^{-\beta}\max_{t\in[0,1]}|u(t)|^2+\exp^{-\gamma}\max_{t\in[0,1]}|v(t)|^2\right]\frac{5}{12}\\
&\leq\exp^{-\alpha}p_b(u(t),v(t))+\exp^{-\beta}p_b(u(t),\mathcal{S}u(t))+\exp^{-\gamma}p_b(v(t),\mathcal{T}v(t)).
\end{aligned}\tag{19}$$

Hence all the postulates of Corolarry 1 are verified for $a=\exp^{-\alpha}$, $b=\exp^{-\beta}$, $f=\exp^{-\gamma}$ and the Cantilever beam problem has one and only one solution. □

4 Conclusion

We have established a common fixed point of non-continuous maps exploiting partial b-metric and without exploiting commutativity or its weaker form ([17]), which is indispensable for the survival of one and only one common fixed point in analogous theorems present in the literature. Consequently, our theorems are sharpened versions of the well-known results, wherein any variant of continuity [18] or commutativity is not essentially required for the survival of a single common fixed point. Examples and applications to solve a Cantilever beam problem employed in the distortion of an elastic beam in equilibrium substantiate the utility of these improvements and extensions.

References

[1] I. A. Bakhtin, The contraction mapping principle in quasimetric spaces, *Funct. Anal. Unianowsk Gos Ped. Inst.*, **30** (1989), 26–37.

[2] S. Banach, Sur les operations dans les ensembles abstrait et leurapplication aux equations, *Integrals. Fundam. Math.*, **3** (1922), 133–181.

[3] S. K. Chatterjea, Fixed point theorems, *C. R. Acad. Bulgare Sci.*, **25** (1972), 727–730.

[4] Lj. B. Ćirić, Generalised contractions and fixed-point theorems, *Publ. Inst. Math.*, **12** (26) (1971), 9–26.

[5] M. Cosentino, P. Salimi, P. Vetro, Fixed point results on metric-type spaces, *Acta Math. Sci.*, **34** (4) (2014), 1237–1253.

[6] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Univ. Ostrav.*, **1** (1993), 5–11.

[7] G. E. Hardy, T. D., Rogers, A generalisation of fixed point theorem of Reich, *Canad. Math. Bull.*, **16** (1973), 201–206.

[8] G. Jungck, Commuting mappings and fixed points, *Amer. Math. Monthly*, **83** (4) (1976), 261–263.

[9] R. Kannan, Some Results on Fixed Points—II., *Math. Notes*, **76** (4) (1969), 405–408.

[10] J. Liouville, Second mémoire sur le développement des fonctions ou parties de fonctions en series dont divers termes sont assujettis à satisfaire a une meme equation différentielle du second ordre contenant unparamétre variable, *J. Math. Pure et Appi.*, **2** (1837), 16–35.

[11] S. G. Matthews, Partial metric topology, *Ann. New York Acad. Sci.*, (1) **728** (1994), 183–197.

[12] Z. Mustafa, J. Roshan, V. Parvaneh, Z. Kadelburg, Some common fixed point results in ordered partial b-metric spaces, *J. Inequal. Appl.*, (1) **562** (2013).

[13] Ć. E. Picard, Memoire sur la theorie des equations aux derives partielles et la methode des Approximations successives, *J. Math. Pureset Appl.*, **6** (1890).

[14] S. Reich, Some remarks concerning contraction mappings, *Can. Math. Bull.*, **14** (1971), 121–124.

[15] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math. (Beograd) (N.S.)*, **32**(46) (1982), 149–153.

[16] S. Shukla, Partial b-metric spaces and fixed point theorems, *Mediter. J. Math.*, (2) **11** (2014), 703–711.

[17] S. L. Singh, A. Tomar, Weaker forms of commuting maps and existence of fixed points, *J. Korea. Soc. Math. Educ. ser. B: Pure Appl. Math.*, (03) **10** (2003), 145–161.

[18] A. Tomar, E. Karapinar, On variants of continuity and existence of fixed point via meir-keeler contractions in MC-spaces, *J. Adv. Math. Stud.*, (2) **9** (2016), 348–359.

[19] W. Zeng, J. Yan, Y. Hong, S. S. Cheng, Numerical analysis of large deflection of the cantilever beam subjected to a force pointing at a fixed point, *Appl. Math. Model.*, **92** (2021), 719–730.

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The L_p-mixed quermassintegrals for $0 < p < 1$*

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Abstract. In the paper, L_p-harmonic addition, p-harmonic Blaschke addition and L_p-dual mixed volume are improved. A new p-harmonic Blaschke mixed quermassintegral is introduced. The relationship between p-harmonic Blaschke mixed volume and L_p-dual mixed volume is shown.

1 Notation and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the subset of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. We write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin and contains the origin, its radial function is $\rho(K,\cdot): S^{n-1} \to [0,\infty)$, defined by (see e. g. [1] and [2])

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

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If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies in $\mathbb{R}^n$. We write $S(K)$ for the surface area of star body K. If $k>0$, then for all $u\in\mathbb{R}^n\backslash\{0\}$

$$\rho(kK,u)=k\rho(K,u). \tag{1}$$

Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows, if $K,L\in\mathcal{S}^n$, then (see e. g. [1])

$$\tilde{\delta}(K,L)=|\rho(K,u)-\rho(L,u)|_\infty.$$

1.1 Dual mixed volume

The radial Minkowski linear combination, $\lambda_1K_1\widetilde{+}\cdots\widetilde{+}\lambda_rK_r$, defined by (see [3])

$$\lambda_1K_1\widetilde{+}\cdots\widetilde{+}\lambda_rK_r=\{\lambda_1x_1\widetilde{+}\cdots\widetilde{+}\lambda_rx_r : x_i\in K_i,\ i=1,\ldots,r\},$$

for $K_1,\ldots,K_r\in\mathcal{S}^n$ and $\lambda_1,\ldots,\lambda_r\in\mathbb{R}$. It has the following important property:

$$\rho(\lambda K\widetilde{+}\mu L,\cdot)=\lambda\rho(K,\cdot)+\mu\rho(L,\cdot),$$

for $K,L\in\mathcal{S}^n$ and $\lambda,\mu\geq 0$.

If $K_i\in\mathcal{S}^n$ $(i=1,2,\ldots,r)$ and λ_i $(i=1,2,\ldots,r)$ are nonnegative real numbers, then of fundamental importance is the fact that the dual volume of $\lambda_1K_1\widetilde{+}\cdots\widetilde{+}\lambda_rK_r$ is a homogeneous polynomial in the λ_i given by (see e. g. [3])

$$V(\lambda_1K_1\widetilde{+}\cdots\widetilde{+}\lambda_rK_r)=\sum_{i_1,\ldots,i_n}\lambda_{i_1}\ldots\lambda_{i_n}\widetilde{V}_{i_1\ldots i_n}, \tag{2}$$

where the sum is taken over all n-tuples $(i_1,\ldots,i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1\ldots i_n}$ depends only on the bodies $K_{i_1},\ldots,K_{i_n}$ and is uniquely determined by (2), it is called the dual mixed volume of $K_{i_1},\ldots,K_{i_n}$, and is written as $\widetilde{V}(K_{i_1},\ldots,K_{i_n})$. Let $K_1=\ldots=K_{n-i}=K$ and $K_{n-i+1}=\ldots=K_n=L$, then the mixed volume $\widetilde{V}(K_1\ldots K_n)$ is written as $\widetilde{V}_i(K,L)$. If $K_1=\cdots=K_{n-i}=K$, $K_{n-i+1}=\cdots=K_n=B$, then the mixed volumes $V_i(K,B)$ is written as $\widetilde{W}_i(K)$ and is called the dual quermassintegral of star body K and $(n-i)\widetilde{W}_{i+1}$ is written as $S_i(K)$ and called the mixed surface area of K. The dual quermassintegral of star body K, defined as an integral by (see [4])

$$\widetilde{W}_i(K)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i}dS(u). \tag{3}$$

It is convenient to write relation (2) in the form (see [5, p.137])

$$\widetilde{V}(\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_s K_s)$$
$$= \sum_{p_1+\cdots+p_r=n} \sum_{1\le i_1<\cdots<i_r\le s} \frac{n!}{p_1!\cdots p_r!} \lambda_{i_1}^{p_1}\cdots\lambda_{i_r}^{p_r} \widetilde{V}(\underbrace{K_{i_1},\ldots,K_{i_1}}_{p_1},\ldots,\underbrace{K_{i_r},\ldots,K_{i_r}}_{p_r}). \tag{4}$$

Let $s = 2, \lambda_1 = 1, K_1 = K, K_2 = B$, we have

$$V(K\widetilde{+}\lambda B) = \sum_{i=0}^{n} \binom{n}{i} \lambda^i \widetilde{W}_i(K),$$

known as formula "Steiner decomposition". Moreover, for star bodies K and L, (4) can show the following special case:

$$\widetilde{W}_i(K\widetilde{+}\lambda L) = \sum_{j=0}^{n-i} \binom{n-i}{j} \lambda^j \widetilde{V}(\underbrace{K,\ldots,K}_{n-i-j},\underbrace{B,\ldots,B}_{i},\underbrace{L,\ldots,L}_{j}). \tag{5}$$

1.2 The p-radial addition and p-dual mixed volume

For any $p \neq 0$, the p-radial addition $K\widetilde{+}_p L$ defined by (see [6] and [7])

$$\rho(K\widetilde{+}_p L, u)^p = \rho(K,u)^p + \rho(L,u)^p, \tag{6}$$

for $u \in S^{n-1}$ and $K, L \in \mathcal{S}^n$. When $p = \infty$ or $-\infty$, the p-radial addition is interpreted as $\rho(K\widetilde{+}_\infty L, u) = K \cup L$ or $\rho(K\widetilde{+}_{-\infty} L, u) = K \cap L$ (see e. g. [8]).

The following result follows immediately from (6).

$$\frac{p}{n} \lim_{\varepsilon\to 0^+} \frac{V(K\widetilde{+}_p \varepsilon\cdot L) - V(L)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}} \rho(K.u)^{n-p}\rho(L.u)^p dS(u).$$

Let $K, L \in \mathcal{S}^n$ and $p \neq 0$, the p-dual mixed volume of star K and L, $\widetilde{V}_p(K,L)$, defined by

$$\widetilde{V}_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K.u)^{n-p}\rho(L.u)^p dS(u). \tag{7}$$

The Minkowski inequality for the p-radial addition stated that: If $K, L \in \mathcal{S}^n$ and $0 < p \le n$, then (see [7])

$$\widetilde{V}_p(K,L)^n \le V(K)^{n-p} V(L)^p, \tag{8}$$

with equality if and only if K and L are dilates.

The inequality is reversed for $p > n$ or $p < 0$

2 The L_p-dual mixed volume for $0 < p < 1$

For $p \geq 1$, Lutwak defined the L_p-harmonic addition of star bodies K and L, $K\check{+}_p\varepsilon \diamond L$, defined by (see [9])

$$\rho(K\check{+}_p\varepsilon \diamond L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon\rho(L, \cdot)^{-p}. \tag{9}$$

As defined in (9), $K\check{+}_p\varepsilon \diamond L$ has a constant coefficient p restricted to $p \geq 1$. We now extend the definition so that $K\check{+}_pL$ is defined for $0 < p < 1$.

Definition 1 *(The L_p-harmonic addition for $0 < p < 1$) If $K, L \in \mathcal{S}^n$ and $0 < p < 1$, the L_p-harmonic addition of star bodies K and L, $K\check{+}_p\varepsilon \diamond L$, defined by*

$$\rho(K\check{+}_p\varepsilon \diamond L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon\rho(L, \cdot)^{-p}. \tag{10}$$

From (10), it is easy that for $0 < p < 1$ (and $p \geq 1$)

$$-\frac{p}{n}\lim_{\varepsilon \to 0^+} \frac{V(K\check{+}_p\varepsilon \diamond L) - V(K)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n+p}\rho(L, u)^{-p}dS(u).$$

Definition 2 *If $K, L \in \mathcal{S}^n$ and $0 < p < 1$, the L_p-dual mixed quermassintegral of K and L, $\widetilde{V}_{-p}(K, L)$, defined by*

$$\widetilde{V}_{-p}(K, L) := \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n+p}\rho(L, u)^{-p}dS(u). \tag{11}$$

Theorem 1 (L_p-*Minkowski inequality*) *If $K, L \in \mathcal{S}^n$ and $0 < p < 1$, then*

$$\widetilde{V}_{-p}(K, L)^n \geq V(K)^{n+p}V(L)^{-p}, \tag{12}$$

with equality if and only if K and L are dilates.

Proof. This integral representation (11) and together with Hölder integral inequality, this yields (12). □

The case $p \geq 1$, please see literatures [10] and [11].

Theorem 2 (L_p-*Brunn-Minkowski inequality*) *If $K, L \in \mathcal{S}^n$ and $0 < p < 1$, then*

$$\widetilde{V}(K\check{+}_p\varepsilon \diamond L)^{-p/n} \geq V(K)^{-p/n} + V(L)^{-p/n}, \tag{13}$$

with equality if and only if K and L are dilates.

Proof. This follows immediately from (10) and (12). □

3 The p-harmonic Blaschke addition for $0 < p < 1$

Let us recall the concept, the harmonic Blaschke addition, defined by Lutwak [12]. Suppose K and L are star bodies in $\mathbb{R}^n$, the harmonic Blaschke linear addition, $K\widehat{+}L$, by

$$\frac{\rho(K\widehat{+}L,\cdot)^{n+1}}{V(K\widehat{+}L)} = \frac{\rho(K,\cdot)^{n+1}}{V(K)} + \frac{\rho(L,\cdot)^{n+1}}{V(L)}. \tag{14}$$

Lutwak's Brunn-Minkowski inequality for the harmonic Blaschke addition was established (see [12]). If $K, L \in \mathcal{S}^n$, then

$$V(K\widehat{+}L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{15}$$

with equality if and only if K and L are dilates. More generally, for any $p \geq 1$, the p-harmonic Blaschke addition $K\widehat{+}_pL$ defined by (see [13] and [14]).

$$\frac{\rho(K\widehat{+}_pL,\cdot)^{n+p}}{V(K\widehat{+}_pL)} = \frac{\rho(K,\cdot)^{n+p}}{V(K)} + \frac{\rho(L,\cdot)^{n+p}}{V(L)}. \tag{16}$$

The L_p Brunn-Minkowski inequality for the p-harmonic Blaschke addition was established (see [13]). If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$V(K\widehat{+}_pL)^{p/n} \geq V(K)^{p/n} + V(L)^{p/n}, \tag{17}$$

with equality if and only if K and L are dilates.

As defined in (16), $K\widehat{+}_pL$ has a constant coefficient p restricted to $p \geq 1$. We now extend the definition so that $K\widehat{+}_pL$ is defined for $0 < p < 1$.

Definition 3 (*The p-harmonic Blaschke addition for $0 < p < 1$*) *If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $0 < p < 1$, the p-harmonic Blaschke addition of K and L, $K\widehat{+}_pL$, defined by*

$$\frac{\rho(K\widehat{+}_pL,\cdot)^{n-i+p}}{\widetilde{W}_i(K\widehat{+}_pL)} = \frac{\rho(K,\cdot)^{n-i+p}}{\widetilde{W}_i(K)} + \frac{\rho(L,\cdot)^{n-i+p}}{\widetilde{W}_i(L)}. \tag{18}$$

Obviously, the case $i = 0$ and $p \geq 1$, is just (16), and the case of $p = 1$ and $i = 0$, is just (14).

Definition 4 *Let $K, L \in \mathcal{S}^n$, $0 \leq i < n$, $0 < p < 1$, and $\alpha, \beta \geq 0$ (not both zero), the p-harmonic Blaschke liner combination of K and L, $\alpha\blacklozenge K\widehat{+}_p\beta\blacklozenge L$, defined by*

$$\frac{\rho(\alpha\blacklozenge K\widehat{+}_p\beta\blacklozenge L,u)^{n-i+p}}{\widetilde{W}_i(\alpha\blacklozenge K\widehat{+}_p\beta\blacklozenge L)} = \alpha\frac{\rho(K,u)^{n-i+p}}{\widetilde{W}_i(K)} + \beta\frac{\rho(L,u)^{n-i+p}}{\widetilde{W}_i(L)}. \tag{19}$$

From (19) *with* $\beta = 0$ *and* (1), *it is easy that*

$$\frac{\rho(\alpha\blacklozenge K, u)^{n-i+p}}{\widetilde{W}_i(\alpha\blacklozenge K)} = \alpha\frac{\rho(K, u)^{n-i+p}}{\widetilde{W}_i(K)} = \frac{\rho(\alpha^{1/p}K, u)^{n-i+p}}{\widetilde{W}_i(\alpha^{1/p}K)}.$$

Hence

$$\alpha\blacklozenge K = \alpha^{1/p}K. \tag{20}$$

4 Inequalities for p-harmonic Blaschke mixed quermassintegral for $0 < p < 1$

In order to define the p-harmonic Blaschke mixed quermassintegral for $0 < p < 1$ with respect to p-harmonic Blaschke addition, we need the following lemmas.

Lemma 1 ([15] *and* [16, p.51]) *If* $a, b \geq 0$ *and* $\lambda \geq 1$, *then*

$$a^\lambda + b^\lambda \leq (a+b)^\lambda \leq 2^{\lambda-1}(a^\lambda + b^\lambda). \tag{21}$$

Lemma 2 *Let* $0 < p < 1$, $0 \leq i < n$ *and* $\varepsilon > 0$. *If* $K, L \in \mathcal{S}^n$, *then*

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\frac{\rho(K\widehat{+}_p\varepsilon\blacklozenge L, u)^{n-i} - \rho(K, u)^{n-i}}{\varepsilon}\\ &\geq \frac{n-i}{n-i+p}\left(\frac{S_i(K)}{\widetilde{W}_i(K)}\rho(K, u)^{n-i} + \frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\rho(K, u)^{-p}\rho(L, u)^{n-i+p}\right).\end{aligned} \tag{22}$$

Proof. From (19) and in view of the L'Hôpital's rule, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\frac{\rho(K\widehat{+}_p\varepsilon\blacklozenge L, u)^{n-i} - \rho(K, u)^{n-i}}{\varepsilon}\\ &= \lim_{\varepsilon\to 0^+}\frac{\left(\left(\frac{\rho(K,u)^{n-i+p}}{\widetilde{W}_i(K)} + \varepsilon\frac{\rho(L,u)^{n-i+p}}{\widetilde{W}_i(L)}\right)\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L)\right)^{n-i/(n-i+p)} - \rho(K, u)^{n-i}}{\varepsilon}\\ &= \lim_{\varepsilon\to 0^+}\frac{n-i}{n-i+p}\left(\left(\frac{\rho(K, u)^{n-i+p}}{\widetilde{W}_i(K)} + \varepsilon\frac{\rho(L, u)^{n-i+p}}{\widetilde{W}_i(L)}\right)\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L)\right)^{-p/(n-i+p)}\\ &\times\left(\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L)'\left(\frac{\rho(K, u)^{n-i+p}}{\widetilde{W}_i(K)} + \varepsilon\frac{\rho(L, u)^{n-i+p}}{\widetilde{W}_i(L)}\right) + \widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L)\frac{\rho(L, u)^{n-i+p}}{\widetilde{W}_i(L)}\right).\end{aligned} \tag{23}$$

In the following, we estimate the value of the derivative $\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L)'$. Let $f_i(t)=\widetilde{W}_i(K\widehat{+}_pt\blacklozenge L)$ and from (5), (20) and (21), we obtain

$$\begin{aligned} f_i(t+\varepsilon) &= \widetilde{W}_i(K\widehat{+}_p(t+\varepsilon)\blacklozenge B)\\ &= \widetilde{W}_i(K\widehat{+}_p(t+\varepsilon)^{1/p}B)\\ &\geq \widetilde{W}_i(K\widehat{+}_p(t^{1/p}+\varepsilon^{1/p})B)\\ &\geq \widetilde{W}_i((K\widehat{+}_pt\blacklozenge B)+\varepsilon B)\\ &= \sum_{j=0}^{n-i}\binom{n-i}{j}\varepsilon^j\tilde{W}_{i+j}(K\widehat{+}_pt\blacklozenge B)\\ &= f_i(t)+\varepsilon(n-i)\tilde{W}_{i+1}(K\widehat{+}_pt\blacklozenge B)+o(\varepsilon^2). \end{aligned}$$

Further

$$V(K\widehat{+}_pt\blacklozenge L)'=\lim_{\varepsilon\to 0^+}\frac{f(t+\varepsilon)-f(t)}{\varepsilon}\geq(n-i)\tilde{W}_{i+1}(K\widehat{+}_pt\blacklozenge B). \tag{24}$$

From (23) and (24) and in view of $(n-i)\tilde{W}_{i+1}(K)=S_i(K)$, we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\frac{\rho(K\widehat{+}_p\varepsilon\blacklozenge L,u)^{n-i}-\rho(K,u)^{n-i}}{\varepsilon}\\ &\qquad\geq\frac{n-i}{n-i+p}\left(\frac{S_i(K)}{\widetilde{W}_i(K)}\rho(K,u)^{n-i}+\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\rho(K,u)^{-p}\rho(L,u)^{n-i+p}\right). \end{aligned}$$

□

Theorem 3 *Let* $0<p<1$, $0\leq i<n$ *and* $\varepsilon>0$. *If* $K,L\in\mathcal{S}^n$, *then*

$$\begin{aligned} &\frac{n-i+p}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L,u)-\widetilde{W}_i(K)}{\varepsilon}\\ &\geq\left(S_i(K)+\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(L)}\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{-p}\rho(L,u)^{n-i+p}dS(u)\right). \end{aligned} \tag{25}$$

Proof. This follows immediately from Lemma 2 and (3). □

Definition 5 *Let* $K,L\in\mathcal{S}^n, 0\leq i<n$ *and* $0<p<1$, *we define the* p*-ith harmonic Blaschke mixed quermassintegral of star bodies* K *and* L, *denoted by* $\widehat{W}_{p,i}(K,L)$, *defined by*

$$\widehat{W}_{p,i}(K,L)=\frac{n-i+p}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widehat{+}_p\varepsilon\blacklozenge L,u)-\widetilde{W}_i(K)}{\varepsilon}. \tag{26}$$

When $i = 0$, *the* p*-harmonic Blaschke mixed quermassintegral* $\widehat{W}_{p,i}(K, L)$ *becomes the* p*-harmonic Blaschke mixed volume* $\widehat{V}_p(K, L)$ *and*

$$\widehat{V}_p(K, L) = \frac{n+p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K \widehat{+}_p \varepsilon \blacklozenge L, u)^n - V(K)^n}{\varepsilon}. \tag{27}$$

Theorem 4 (L_p*-Minkowski type inequality*) *If* $K, L \in \mathcal{S}^n$, $0 \le i < n$ *and* $0 < p < 1$, *then*

$$(\widehat{W}_{p,i}(K, L) - S_i(K))^{n-i} \ge \widetilde{W}_i(K)^{n-i-p} \widetilde{W}_i(L)^p \tag{28}$$

Proof. This follows immediately from Theorem 3, (27) and Hölder integral inequality. □

Corollary 1 *If* $K, L \in \mathcal{S}^n$ *and* $0 < p < 1$, *then*

$$(\widehat{V}_p(K, L) - S(K))^n \ge V(K)^{n-p} V(L)^p. \tag{29}$$

Proof. This follows immediately from Theorem 4 with $i = 0$. □

5 The relationship between the two mixed volumes

In the following, we give a relationship between the p-harmonic Blaschke mixed volume $\widehat{V}_p(K, L)$ and the L_p-dual mixed volume $\widetilde{V}_{-p}(K, L)$.

Theorem 5 *If* $K, L \in \mathcal{S}^n$ *and* $0 < p < 1$, *then*

$$\frac{\widehat{V}_p(K, L)}{V(K)} \ge \frac{\widetilde{V}_{-p}(L, K)}{V(L)}. \tag{30}$$

Proof. This follows immediately from (11), (27) and Theorem 3 with $i = 0$. □

We give also a relationship between the p-harmonic Blaschke mixed volume $\widehat{V}_p(K, L)$ and the p-dual mixed volume $\widetilde{V}_p(K, L)$.

Theorem 6 *If* $K, L \in \mathcal{S}^n$ *and* $0 < p < 1$, *then*

$$\widehat{V}_p(K, L) \ge \widetilde{V}_p(K, L). \tag{31}$$

Proof. From (11), (12), (8), (25) and (27), we obtain

$$\begin{aligned}\widehat{V}_p(K,L) &\geq \frac{V(K)}{V(L)}\frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n+p}\rho(K,u)^{-p}dS(u)\\ &= \frac{V(K)}{V(L)}\tilde{V}_{-p}(L,K)\\ &\geq \frac{V(K)}{V(L)}V(L)^{(n+p)/n}V(K)^{-p/n}\\ &= V(K)^{(n-p)/n}V(L)^{p/n}\\ &\geq \tilde{V}_p(K,L).\end{aligned}$$

□

Finally, we establish the Brunn-Minkowski inequality for the p-ith harmonic Blaschke addition.

Theorem 7 *If* $K, L \in \mathcal{S}^n$*,* $0 \leq i < n$*,* $0 < p < 1$ *and* $\lambda, \mu \geq 0$*, then*

$$\widetilde{W}_i(\lambda\blacklozenge K\widehat{+}_p\mu\blacklozenge L)^{p/(n-i)} \geq \lambda\widetilde{W}_i(K)^{p/(n-i)} + \mu\widetilde{W}_i(L)^{p/(n-i)}, \tag{32}$$

with equality if and only if K *and* L *are dilates.*

Proof. This follows immediately from (3), (19) and Minkowski integral inequality. □

This case of $\lambda = \mu = 1$, $p \geq 1$ and $i = 0$ is just (17). This case of $p = 1$, $\lambda = \mu = 1$ and $i = 0$ is just (15).

References

[1] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge Univ. Press, 1993.

[2] R. J. Gardner, *Geometric Tomography*, Cambridge Univ. Press, New York, 1996.

[3] E. Lutwak, Intersection bodies and dual mixed volumes, *Adv. Math.*, **71** (1988), 232–261.

[4] E. Lutwak, Dual mixed volumes, *Pacific J. Math.*, **58** (1975), 531–538.

[5] Y. D. Burago, V. A. Zalgaller, *Geometric Inequalities*, Springer-Verlag, Berlin-New york, 1988.

[6] R. J. Gardner, The Brunn-Minkowski inequality, *Bull. Amer. Math. Soc.,* **39** (2002), 355–405.

[7] E. Grinberg, G. Zhang, Convolutions, transforms, and convex bodies, *Proc. London Math. Soc,.* **78** (1999), 77–115.

[8] R. J. Gardner, D. Hug, W. Weil, Operations between sets in geometry, *J. Eur. Math. Soc.* (JEMS), in press.

[9] E. Lutwak, The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas. *Adv. Math.,* **118** (1996), 244–294.

[10] E. Lutwak, D. Yang, G. Zhang, $\mathrm{L_p}$ affine isoperimetric inequalities, *J. Differential Geom.,* **56** (2000),111–132.

[11] E. Lutwak, D. Yang, G. Zhang, Sharp affine $\mathrm{L_p}$ Sobolev inequalities, *J. Differential Geom.,* **62** (2002), 17–38.

[12] E. Lutwak, Centroid bodies and dual mixed volumes, *Proc. London Math. Soc.*, **60**(1990), 365–391.

[13] Y. B. Feng, W. D. Wang, Shephard type probems for $\mathrm{L_p}$-centroid bodies, *Math. Inqu. Appl.,* **17**(3) (2014), 865–877.

[14] J. Yuan, L. Z. Zhao, G. S. Leng, Inequalities for $\mathrm{L_p}$-centroid body, *Taiwanese. J. Math.,* **11**(5) (2007), 1315–1325.

[15] P. Chunaev, Hölder and Minkowski type inequalityies with alternating signs, *J. Math. Ineq.*, **9**(1)(2015), 61–71.

[16] R. P. Agarwal, P. Y. H. Pang, Opial inequalities with applications in differential and difference equations, Kluwer Academic Publishers, Dordrecht, Boston, London, 1995.

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