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Maia type fixed point results via C-class function

Arslan Hojat Ansari
Department of Mathematics,
Karaj Branch, Islamic Azad University,
Karaj, Iran
email: mathanalsisamir4@gmail.com

Mohammad Saeed Khan
Department of Mathematics &
Statistics, Sultan Qaboos University,
P. O. Box 36, Al-Khoud 123,Muscat,
Sultanate of Oman
email: mohammad@squ.edu.om

Vladimir Rakočević
Faculty of Sciences and Mathematics,
Višegradska 33, 18000 Niš, Serbia
email: vrakoc@sbb.rs (corresponding author)

Abstract. In 1968, M. G. Maia [16] generalized Banach's fixed point theorem for a set X endowed with two metrics. In 2014, Ansari [2] introduced the concept of C-class functions and generalized many fixed point theorems in the literature. In this paper, we prove some Maia's type fixed point results via C-class function in the setting of two metrics space endowed with a binary relation. Our results, generalized and extended many existing fixed point theorems, for generalized contractive and quasi-contractive mappings, in a metric space endowed with binary relation.

1 Introduction and preliminaries

The classical Banach contraction mapping is one of the most useful in metric fixed point theory. It is very popular tool for solving existence and uniqueness

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problems in many different fields of mathematics. Due to its importance and applications potential, the Banach Contraction Principle has been investigated heavily by many authors. Consequently, a number of generalizations of this celebrated principle have appeared in the literature. For some recent significant book from fixed point theory, we refer to ([1, 7, 11, 14, 17, 24, 26]).

We first recall Maia's fixed point theorem:

Theorem 1 [16] *Let* (X, d, δ) *be a bimetric space and* $T : X \to X$. *Assume that the following conditions are satisfied:*

(i) $d(x, y) \leq \delta(x, y)$ *for all* $x, y \in X$,

(ii) X *is complete with respect to* d,

(iii) T *is continuous with respect to* d,

(iv) *there exists a constant* $\alpha \in [0, 1)$ *such that*

$$\delta(Tx, Ty) \leq \alpha\delta(x, y), \quad \textit{for all} \quad x, y \in X.$$

Then T *has a unique fixed point in* X.

Singh [28] proved that the above theorem is true under much less restricted condition, that is we do not need the continuity of T with respect to d on X, but only the continuity at a point. Many papers deal with fixed point theorems of Maia type and with applications (see eg.,[6, 5, 19, 23, 22, 25, 20, 30]) and references therein). In these direction, in 2019. Petrusel and Rus [20] consider the following: Let X be nonempty set endowed with a metric d, an order relation $\preceq$ and an operator $f : X \to X$, which satisfies two main assumptions:

(1) f is generalized monotone with respect to $\preceq$;

(2) f is a (generalized) contraction with respect to d on a certain subset Y of $X \times X$.

Then, they apply these results to study some problems related to integral and differential equations, and several open questions are discussed. We point out that Turinici [30] have showed that the Ran-Reurings [21] fixed point theorem is but a particular case of Maia's.

In 2012. Samet and Turinici [27] introduced the notion of contractive mapping in a metric space endowed with amorphous binary relation. They showed a theorem subsumes many known results in the literature. For further study about contractive mappings in a metric space endowed with binary relation, we refer the reader to [8] [27] and [31].

In the sequel let (X, d, δ) be a bimetric space and $T : X \to X$ be a mapping. Denote by

$$\mathrm{Fix}(T) = \{x^* \in X : x^* = Tx^*\}.$$

the set of all fixed points of T in X.

Let $\mathcal{R}$ be a binary relation on X and let $\mathcal{S}$ be the *symmetric binary relation* defined by

$$x, y \in X, \quad x\mathcal{S}y \quad \Longleftrightarrow \quad x\mathcal{R}y \quad \text{or} \quad y\mathcal{R}x.$$

For $x_0 \in X$ we define the sequence $\{x_n\}$ by

$$x_n = Tx_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Definition 1 *Let* (X, δ) *be metric space and* $n \in \mathbb{N} \cup \{0\}$*. For* $A \subset X$ *we denote by* $\mathrm{diam}(A) := \sup\{\delta(a, b) : a, b \in A\}$ *the diameter of* A*. For each* $x_0 \in X$ *the orbit sets of* T *at* x_0 *are defined as following*

$$O_n(x_0) = \{x_0, x_1, ..., x_n\} \quad \text{and} \quad O_\infty(x_0) = \{x_0, x_1, , x_2, ...\}.$$

We say that (X, δ) *is* T*-orbitally complete iff every* δ*-Cauchy sequence from* $O_\infty(x)$ *for some* $x \in X$ *converges in* X*.*

Definition 2 [27] *A subset* D *of* X *is called* $\mathcal{R}$*-directed if for every* $x, y \in D$*, there exists* $z \in X$ *such that* $z\mathcal{R}x$ *and* $z\mathcal{R}y$*.*

Definition 3 *A mapping* $T : X \to X$ *is called* $\mathcal{R}$*-preserving mapping if*

$$x, y \in X, \quad x\mathcal{R}y \quad \Longrightarrow \quad Tx\mathcal{R}Ty.$$

Next, we define the set Φ of functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying:

(I) φ is nondecreasing,

(II) $\displaystyle\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for each $t > 0$, where φ^n is the n-th iterate of φ.

Remark 1 *Let* $\varphi \in \Phi$*. We have* $\varphi(t) < t$ *for all* $t > 0$*.*

Remark 2 *Let* $\varphi \in \Phi$*. We have* $\lim_{n\to\infty} \varphi^n(t) = 0$ *for all* $t > 0$*.*

Definition 4 [15] *Assume that for* $T : X \to X$ *there exists* $\varphi \in \Phi$ *such that*

$$\delta(Tx, Ty) \leq \varphi(M_\delta(x, y)) \quad \textit{for all } x, y \in X \quad \textit{with} \quad xSy.$$

A mapping T *is called a generalized contractive with respect to* δ *if*

$$M_\delta(x, y) = \max\left\{\delta(x, y), \frac{\delta(x, Tx) + \delta(y, Ty)}{2}, \frac{\delta(x, Ty) + \delta(Tx, y)}{2}\right\}.$$

A mapping T *is called a generalized quasi-contractive* (*see* [10, 11, 17]) *with respect to* δ *if*

$$M_\delta(x, y) = \max\{\delta(x, y), \delta(x, Tx), \delta(y, Ty), \delta(x, Ty), \delta(Tx, y)\}.$$

Lemma 1 (Lemma 1 of [15]) *Let* (X, δ) *be a metric space, and* $\mathcal{R}$ *a transitive binary relation over* X. *Assume that for* $T : X \to X$, *the following conditions are satisfied:*

(b1) *there exists* $x_0 \in X$ *such that* $x_0 \mathcal{R} Tx_0$,

(b2) T *is* $\mathcal{R}$*-preserving mapping,*

(b3) T *is generalized quasi-contractive with respect to* δ.

Then,

$$\delta(x_i, x_j) \leq \varphi(\operatorname{diam}(O_n(x_0))),$$

for all $i; j \in \{1, ..., n\}$.

In 2014 the concept of C-class functions were introduced by A.H.Ansari [2]. By using this concept we can generalize many fixed point theorems in the literature. C-class functions have been studied by many authors.and some fixed point results with applications (see eg., [3, 12, 18, 4, 13]).

Definition 5 [2] *A mapping* $F : [0, \infty)^2 \to \mathbb{R}$ *is called* C*-class function if it is continuous and satisfies following axioms:*

(1) $F(s, t) \leq s$,
(2) $F(s, t) = s$ *implies that either* $s = 0$ *or* $t = 0$; *for all* $s, t \in [0, \infty)$.

Note for some F we have that $F(0, 0) = 0$.
We denote C-class functions as $\mathcal{C}$.

Example 1 [2] *The following functions* $F : [0,\infty)^2 \to \mathbb{R}$ *are elements of* $\mathcal{C}$, *for all* $s, t \in [0,\infty)$*:*

(1) $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$;

(2) $F(s,t) = ms$, $0<m<1$, $F(s,t) = s \Rightarrow s = 0$;

(3) $F(s,t) = \frac{s}{(1+t)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;

(4) $F(s,t) = \log(t + a^s)/(1+t)$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;

(5) $F(s,t) = \ln(1 + a^s)/2$, $a > e$, $F(s,1) = s \Rightarrow s = 0$;

(6) $F(s,t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$;

(7) $F(s,t) = s \log_{t+a} a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;

(8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t) = s \Rightarrow t = 0$;

(9) $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to [0,1)$, *and is continuous,* $F(s,t) = s \Rightarrow s = 0$;

(10) $F(s,t) = s - \frac{t}{k+t}$, $F(s,t) = s \Rightarrow t = 0$;

(11) $F(s,t) = s - \varphi(s), F(s,t) = s \Rightarrow s = 0$, *here* $\varphi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\varphi(t) = 0 \Leftrightarrow t = 0$;

(12) $F(s,t) = sh(s,t), F(s,t) = s \Rightarrow s = 0$, *here* $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is a continuous function such that* $h(t,s) < 1$ *for all* $t, s > 0$;

(13) $F(s,t) = s - (\frac{2+t}{1+t})t$, $F(s,t) = s \Rightarrow t = 0$;

(14) $F(s,t) = \sqrt[n]{\ln(1 + s^n)}$, $F(s,t) = s \Rightarrow s = 0$;

(15) $F(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0$, *here* $\phi : [0,\infty) \to [0,\infty)$ *is a upper semicontinuous function such that* $\phi(0) = 0$, *and* $\phi(t) < t$ *for* $t > 0$;

(16) $F(s,t) = \frac{s}{(1+s)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$;

(17) $F(s,t) = \vartheta(s)$, $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is a generalized Mizoguchi-Takahashi type function,* $F(s,t) = s \Rightarrow s = 0$;

(18) $F(s,t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+t}\, dx$, *where* Γ *is the Euler Gamma function.*

Denote by Ψ the family of continuous and monotone nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(t) = 0$ if and only if $t = 0$ and by Φ_u the family of continuous functions $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$ for all $t > 0$.

Definition 6 *Assume that for* $T : X \to X$ *there exists* $\varphi \in \Phi_u, \psi \in \Psi, F \in \mathcal{C}$ *such that*

$$\psi(\delta(Tx, Ty)) \le F(\psi(M_\delta(x,y)), \varphi(M_\delta(x,y))), \quad \textit{for all } x, y \in X \quad \textit{with } x\mathcal{S}y. \quad (1)$$

A mapping T *is called a generalized* $F\psi\varphi$*-contractive with respect to* δ*, if*

$$M_\delta(x,y) = \max\left\{\delta(x,y), \frac{\delta(x,Tx)+\delta(y,Ty)}{2}, \frac{\delta(x,Ty)+\delta(Tx,y)}{2}\right\}.$$

A mapping T *is called a generalized quasi-*$F\psi\varphi$*-contractive with respect to* δ*, if*

$$M_\delta(x,y) = \max\{\delta(x,y), \delta(x,Tx), \delta(y,Ty), \delta(x,Ty), \delta(Tx,y)\}.$$

Lemma 2 [9] *Suppose* (X, δ) *be a metric space. Let* $\{x_n\}$ *be a sequence in* X *such that* $\delta(x_n, x_{n+1}) \to 0$ *as* $n \to \infty$*. If* $\{x_n\}$ *is not a Cauchy sequence then there exist an* $\varepsilon > 0$ *and sequences of positive integers* $\{m(k)\}$ *and* $\{n(k)\}$ *with* $m(k) > n(k) > k$ *such that* $\delta(x_{m(k)}, x_{n(k)}) \ge \varepsilon$*,* $\delta(x_{m(k)-1}, x_{n(k)}) < \varepsilon$ *and*

(i) $\lim_{k\to\infty} \delta(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$,

(ii) $\lim_{k\to\infty} \delta(x_{m(k)}, x_{n(k)}) = \varepsilon$,

(iii) $\lim_{k\to\infty} \delta(x_{m(k)-1}, x_{n(k)}) = \varepsilon$,

(iv) $\lim_{k\to\infty} \delta(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon$,

(v) $\lim_{k\to\infty} \delta(x_{m(k)}, x_{n(k)-1}) = \varepsilon$.

Definition 7 *We say* (ψ, ϕ, F) *is monotone if* $x \le y \Longrightarrow F(\psi(x), \phi(x)) \le F(\psi(y), \phi(y))$.

Example 2 *Let* $F(s,t) = s - t, \phi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } \ 0 \le x \le 1, \\ x^2, & \text{if } \ x > 1. \end{cases}$$

Then (ψ, ϕ, F) *is monotone.*

Example 3 *Let* $F(s,t) = s - t, \phi(x) = x^2$

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } \ 0 \le x \le 1, \\ x^2, & \text{if } \ x > 1. \end{cases}$$

Then (ψ, ϕ, F) *is not monotone.*

In this paper we prove some Maia type fixed point results via C-class function in the setting of two metrics space endowed with a binary relation. Our results generalized and extended many existing fixed point theorems for generalized contractive and quasi-contractive mappings in a metric space endowed with binary relation.

2 Main results

Our first main result is the following theorem.

Theorem 2 *Let* (X, d, δ) *be a bimetric space and* $T : X \to X$*. Assume that the following conditions are satisfied:*

(A1) $d(x, y) \leq \delta(x, y)$ *for all* $x, y \in X$*,*

(A2) (X, d) *is* T*-orbitally complete,*

(A3) T *is continuous with respect to* d*,*

(A4) T *is* $\mathcal{S}$*-preserving,*

(A5) *there exists* $x_0 \in X$ *with* $x_0 \mathcal{S} T x_0$*,*

(A6) T *is a generalized* $F\psi\varphi$*-contractive with respect to* δ*.*

Then T *has a fixed point* x^* *in* X*. Moreover, if in addition* $\mathrm{Fix}(T)$ *is* $\mathcal{S}$*-directed then* x^* *is the unique fixed point of* T *in* X*.*

Proof. From (A5), there exists $x_0 \in X$ with $x_0 \mathcal{S} T x_0$ and from (A4) T is $\mathcal{S}$-preserving, we get

$$x_n \mathcal{S} T x_n \quad \text{for all } n \in \mathbb{N}. \tag{2}$$

If $x_n = Tx_n$ then x_n is a fixed point of T. Suppose that $x_n \neq Tx_n$ for all n. Since (2) is satisfied for all $n \geq 1$, by applying the contraction condition (A6), and note that ψ is nondecreasing, we have

$$\begin{aligned}
&\psi(\delta(x_n, x_{n+1}) = \psi(\delta(x_n, Tx_n)) \leq F(\psi(M_\delta(x_{n-1}, x_n)), \varphi(M_\delta(x_{n-1}, x_n)) \\
&< \psi(M_\delta(x_{n-1}, x_n)) \\
&\leq \psi(\max\{\delta(x_{n-1}, x_n), \delta(x_n, x_{n+1})\}).
\end{aligned}$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence in (X, δ). If for some $n \geq 1$ we have $\delta(x_{n-1}, x_n) \leq \delta(x_n, x_{n+1})$, then we get

$$\psi(\delta(x_n, x_{n+1})) = F(\psi(\delta(x_n, x_{n+1})), \varphi(\delta(x_n, x_{n+1}))).$$

Thus, $\psi(\delta(x_n, x_{n+1})) = 0$ or $\varphi(\delta(x_n, x_{n+1})) = 0$, and therefore $\delta(x_n, x_{n+1}) = 0$ which is contradiction. We get $\delta(x_{n-1}, x_n) > \delta(x_n, x_{n+1})$ and

$$\psi(\delta(x_n, x_{n+1})) = F(\psi(\delta(x_{n-1}, x_n)), \varphi(\delta(x_{n-1}, x_n))). \tag{3}$$

Hence $\{\delta(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers. Thus there exist $L \geq 0$ such that

$$\lim_{n\to\infty} \delta(x_n, x_{n+1}) = L. \tag{4}$$

Taking the limit in equation (3) as $n \to \infty$ and using (4) and the properties of F and φ, we have

$$\psi(L) \leq F(\psi(L), \varphi(L)).$$

Thus $\psi(L) = 0$ or $\varphi(L) = 0$, and so $L = 0$. Therefore

$$\lim_{n\to\infty} \delta(x_n, x_{n+1}) = 0. \tag{5}$$

Let us show that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{x_n\}$ is not a Cauchy sequence.

By Lemma 2 there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$\delta(x_{m(k)}, x_{n(k)+1}), \delta(x_{m(k)}, x_{n(k)}), \delta(x_{m(k)-1}, x_{n(k)+1}), \delta(x_{m(k)-1}, x_{n(k)}) \to \varepsilon.$$

Now from (1) we have

$$\begin{aligned} \psi(\delta(x_{m(k)}, x_{n(k)})) &= \psi(\delta(Tx_{m(k)-1}, Tx_{n(k)-1})) \\ &\leq F(\psi(M_\delta(x_{m(k)-1}, x_{n(k)-1})), \varphi(M_\delta(x_{m(k)-1}, x_{n(k)-1}))) \end{aligned} \tag{6}$$

where

$$\begin{aligned} M_\delta(x_{m(k)-1}, x_{n(k)-1}) = \max\Big\{ & \delta(x_{m(k)-1}, x_{n(k)-1}), \\ & \frac{\delta(x_{m(k)-1}, Tx_{m(k)-1}) + \delta(x_{n(k)-1}, Tx_{n(k)-1})}{2}, \\ & \frac{\delta(x_{m(k)-1}, Tx_{n(k)-1}) + \delta(Tx_{m(k)-1}, x_{n(k)-1})}{2} \Big\}. \end{aligned}$$

From above and (6), as $k \to \infty$ we have

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)).$$

Thus $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0 and$ therefore $\varepsilon = 0$ which is contradiction. Consequently the sequence $\{x_n\}$ is δ-Cauchy, so by (A1), $\{x_n\}$ is d-Cauchy too. Since from (A2), we have that the metric space (X, d) is T-orbitally complete, then there exists $x^* \in X$ such that

$$\lim_{n\to\infty} \delta(x_n, x^*) = 0. \tag{7}$$

From (A3), we have that T is continuous with respect to d, and, so it follows that $x^* = \lim\limits_{n\to\infty} Tx_n = T(\lim\limits_{n\to\infty} x_n) = Tx^*$, that is, x^* is a fixed point of T.

Next suppose that Fix(T) is $\mathcal{S}$-directed, and we will show that x^* is the unique fixed point of T in X. Suppose that $y^* \in$ Fix(T) is another fixed point of T. Then, there exists $z_0 \in X$ such that $z_0 \mathcal{S} x^*$ and $z_0 \mathcal{S} y^*$. Define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all $n \geq 0$. Since T is $\mathcal{S}$-preserving, for all $n \geq 0$ we have $z_n \mathcal{S} x^*$ and $z_n \mathcal{S} y^*$. Applying (A6), for all $n \geq 0$ and note $\delta(z_n, z_{n+1}) \leq \delta(z_n, x^*) + \delta(z_{n+1}, x^*)$ we get

$$\begin{aligned}
&\psi(\delta(z_{n+1}, x^*)) = \psi(\delta(Tz_n, Tx^*)) = F(\psi(M_\delta(z_n, x^*)), \varphi(M_\delta(z_n, x^*)))\\
&\leq F\Bigg(\psi\Bigg(\max\Bigg\{\delta(z_n, x^*), \frac{\delta(z_n, Tz_n) + \delta(x^*, Tx^*)}{2}, \frac{\delta(z_n, Tx^*) + \delta(x^*, Tz_n)}{2}\Bigg\}\Bigg),\\
&\quad \varphi\Bigg(\max\Bigg\{\delta(z_n, x^*), \frac{\delta(z_n, Tz_n) + \delta(x^*, Tx^*)}{2}, \frac{\delta(z_n, Tx^*) + \delta(x^*, Tz_n)}{2}\Bigg\}\Bigg)\Bigg)\\
&= F\Bigg(\psi\Bigg(\max\Bigg\{\delta(z_n, x^*), \frac{\delta(z_n, z_{n+1}) + \delta(x^*, x^*)}{2}, \frac{\delta(z_n, x^*) + \delta(x^*, z_{n+1})}{2}\Bigg\}\Bigg),\\
&\quad \varphi\Bigg(\max\Bigg\{\delta(z_n, x^*), \frac{\delta(z_n, z_{n+1}) + \delta(x^*, x^*)}{2}, \frac{\delta(z_n, x^*) + \delta(x^*, z_{n+1})}{2}\Bigg\}\Bigg)\Bigg)\\
&\leq F(\psi(\max\{\delta(z_n, x^*), \delta(z_{n+1}, x^*)\}), \varphi\,(\max\{\delta(z_n, x^*), \delta(z_{n+1}, x^*)\})).
\end{aligned}$$

Now we will show that $\lim\limits_{n\to\infty} \delta(z_n, x^*) = 0$. Without the loss of generality suppose that $\delta(z_n, x^*) > 0$ for all n. Assume that for some n we have $\delta(z_n, x^*) \leq \delta(x^*, z_{n+1})$. Hence we get

$$\psi(\delta(z_{n+1}, x^*)) = F(\psi(\delta(z_{n+1}, x^*)), \varphi(\delta(z_{n+1}, x^*))).$$

Hence $\psi(\delta(z_{n+1}, x^*)) = 0$ or $\varphi(\delta(z_{n+1}, x^*)) = 0$ and therefore $\delta(z_{n+1}, x^*) = 0$, which is a contradiction. Then, for all $n \geq 0$ we have $\delta(z_n, x^*) > \delta(x^*, z_{n+1})$. Consequently, for all n we obtain

$$\psi(\delta(z_{n+1}, x^*)) = F(\psi(\delta(z_n, x^*)), \varphi(\delta(z_n, x^*))) \leq \psi(\delta(z_n, x^*)) \tag{8}$$

that is, $\{\delta(z_n, x^*)\}$ is a non-increasing sequence of positive real numbers. Thus there exist $L \geq 0$ such that

$$\lim_{n\to\infty} \delta(z_n, x^*) = L. \tag{9}$$

Taking the limit in equation (8) as $n \to \infty$ and using (9) and the properties of F and φ we have

$$\psi(L) \leq F(\psi(L), \varphi(L)).$$

Thus $\psi(L) = 0$ or $\varphi(L) = 0$ and therefore $L = 0$. Thus

$$\lim_{n\to\infty} \delta(z_n, x^*) = 0.$$

Similarly we can prove that $\lim_{n\to\infty} \delta(z_n, y^*) = 0$. Hence $x^* = y^*$. $\square$

To prove our next main result we need the following lemmas which will be used in the sequel.

Lemma 3 *Let $n \in \mathbb{N}$, (X, δ) be a metric space, and $\mathcal{R}$ a transitive binary relation over X Assume that for $T : X \to X$ the following conditions are satisfied:*

(a1) *there exists $x_0 \in X$ such that $x_0 \mathcal{R} T x_0$,*

(a2) *T is $\mathcal{R}$-preserving mapping,*

(a3) *T is generalized quasi-$F\varphi$-contractive with respect to δ and (ψ, φ, F) is monotone.*

Then

$$\psi(\delta(x_i, x_j)) \leq F(\psi(\mathrm{diam}(O_n(x_0))), \varphi(\mathrm{diam}(O_n(x_0)))), \tag{10}$$

for all $i, j \in \{1, \ldots, n\}$.

Proof. From (a1) there exists $x_0 \in X$ such that $x_0 \mathcal{R} x_1$. Hence by (a2) we get $x_k \mathcal{R} x_{k+1}$ for all k. Since $\mathcal{R}$ is transitive, then

$$x_{i-1} \mathcal{R} x_{j-1} \quad \text{for all} \quad 1 \leq i < j \leq n. \tag{11}$$

We note that x_{i-1}, x_i, x_{j-1}, $x_j \in O_n(x_0)$. Now using (a3) and (11) we get

$$\begin{aligned}\psi(\delta(Tx_{i-1}, Tx_{j-1})) &\leq F(\psi(M_\delta(x_{i-1}, x_{j-1})), \varphi(M_\delta(x_{i-1}, x_{j-1}))) \\ &= F(\psi(\max\{\delta(x_{i-1}, x_{j-1}), \delta(x_{i-1}, Tx_{i-1}),\end{aligned}$$

$$\delta(x_{j-1}, Tx_{j-1}), \delta(x_{i-1}, Tx_{j-1}), \delta(Tx_{i-1}, x_{j-1})\}),$$
$$\varphi(\max\{\delta(x_{i-1}, x_{j-1}), \delta(x_{i-1}, Tx_{i-1}),$$
$$\delta(x_{j-1}, Tx_{j-1}), \delta(x_{i-1}, Tx_{j-1}), \delta(Tx_{i-1}, x_{j-1})\})),$$

which implies (10). □

Now we are ready to state our second main result.

Theorem 3 *Let* (X, d, δ) *be a bimetric space,* $\mathcal{R}$ *a transitive binary relation over* X *and* $T : X \to X$*. Assume that the following conditions are satisfied:*

(B1) $d(x, y) \leq \delta(x, y)$ *for all* $x, y \in X$*,*

(B2) (X, d) *is* T*-orbitally complete,*

(B3) T *is continuous with respect to* d*,*

(B4) T *is* $\mathcal{R}$*-preserving,*

(B5) *there exists* $x_0 \in X$ *with* $x_0 \mathcal{R} T x_0$*,*

(B6) T *is a generalized quasi-*$F\psi\varphi$*-contractive with respect to* δ *and* (ψ, φ, F) *is monotone.*

Then T *has a fixed point* x^* *in* X*. Moreover if in addition* $\mathcal{R}$ *is symmetric and Fix*(T) *is* $\mathcal{R}$*-directed then* x^* *is the unique fixed point of* T *in* X*.*

Proof. Let $x_0 \in X$ and $x_0 R T x_0$. Define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \geq 0$. Since T is an $\mathcal{R}$-preserving, then $x_n \mathcal{R} x_{n+1}$ for all n. Let n and m, $n < m$ be any positive integers. From (B6) and Lemma 3 it follows

$$\psi(\delta(T^n x_0, T^m x_0)) = \psi(\delta(TT^{n-1}x_0, T^{m-n+1}T^{n-1}x_0))$$
$$F(\psi(\mathrm{diam}(O_{m-n+1}(T^{n-1}x_0))), \varphi(\mathrm{diam}(O_{m-n+1}(T^{n-1}x_0)))).$$

From Remark 1 there exists an integer k_1, $1 \leq k_1 \leq m - n + 1$ such that

$$\mathrm{diam}(O_{m-n+1}(T^{n-1}x_0))) = \delta(T^{n-1}x_0, T^k T^{n-1}x_0).$$

Using Lemma 3 again we get combining the above inequalities

$$\psi(\delta(T^n x_0, T^m x_0)) = \psi(\delta(TT^{n-1}x_0, T^{m-n+1}T^{n-1}x_0))$$
$$F(\psi(\mathrm{diam}(O_{m-n+1}(T^{n-1}x_0))), \varphi(\mathrm{diam}(O_{m-n+1}(T^{n-1}x_0))))$$
$$\psi(\mathrm{diam}(O_{m-n+1}(T^{n-1}x_0))) = \psi(\delta(T^{n-1}x_0, T^k T^{n-1}x_0))$$

$$\begin{aligned}
&\leq F(\psi(\operatorname{diam}(O_{k_1+1}(T^{n-2}x_0))), \varphi(\operatorname{diam}(O_{k_1+1}(T^{n-2}x_0))))\\
&\leq F(\psi(\operatorname{diam}(O_{m-n+2}(T^{n-2}x_0))), \varphi(\operatorname{diam}(O_{m-n+2}(T^{n-2}x_0)))).
\end{aligned}$$

Continue this process we obtain

$$\begin{aligned}
\psi(\delta(T^n x_0, T^m x_0)) &= \psi(\delta(TT^{n-1}x_0, T^{m-n+1}T^{n-1}x_0))\\
&\quad F(\psi(\operatorname{diam}(O_{m-n+1}(T^{n-1}x_0))), \varphi(\operatorname{diam}(O_{m-n+1}(T^{n-1}x_0))))\\
&\quad \psi(\operatorname{diam}(O_{m-n+1}(T^{n-1}x_0))) = \psi(\delta(T^{n-1}x_0, T^k T^{n-1}x_0))\\
&\leq F(\psi(\operatorname{diam}(O_{k_1+1}(T^{n-2}x_0))), \varphi(\operatorname{diam}(O_{k_1+1}(T^{n-2}x_0))))\\
&\leq F(\psi(\operatorname{diam}(O_{m-n+2}(T^{n-2}x_0))), \varphi(\operatorname{diam}(O_{m-n+2}(T^{n-2}x_0))))\\
&\;\vdots\\
&\leq F(\psi(\operatorname{diam}(O_m(x_0))), \varphi(\operatorname{diam}(O_m(x_0))))\\
&\leq F(\psi(\delta(T^{n-1}x_0, T^{m-1}x_0)), \varphi(\delta(T^{n-1}x_0, T^{m-1}x_0))).
\end{aligned}$$

Hence

$$\psi(\varepsilon) \leq F(\psi(\varepsilon), \varphi(\varepsilon)).$$

Thus $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$, that is $\varepsilon = 0$. It follows that the sequence $\{T^n x_0\}$ is a δ-Cauchy sequence. Therefore by (B1) the sequence $\{T^n x_0\}$ is a d-Cauchy sequence too. Since the metric space (X, d) is T-orbitally complete we deduce that the sequence $\{T^n x_0\}$ converges to some x^* in X. From (B3) T is continuous with respect to d, so $x^* = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tx^*$ and x^* is a fixed point of T.

Now suppose that $\mathrm{Fix}(T)$ is $\mathcal{R}$-directed. We claim that the fixed point is unique. Let x^* and y^* be two fixed points of T. Suppose that $x^* \neq y^*$. Since $\mathrm{Fix}(T)$ is $\mathcal{R}$-directed, then there exists $z \in X$ such that $z\mathcal{R}x^*$ and $z\mathcal{R}y^*$. By the transitivity of $\mathcal{R}$ we have $x^*\mathcal{R}y^*$. Then we apply the contraction condition (B6) and get

$$\begin{aligned}
\psi(\delta(x^*, y^*)) &= \psi(\delta(Tx^*, Ty^*))\\
&\leq F(\psi(\max\{\delta(x^*, y^*), \delta(x^*, Tx^*), \delta(y^*, Ty^*), \delta(x^*, Ty^*), \delta(Tx^*, y^*)\}),\\
&\quad \varphi(\max\{\delta(x^*, y^*), \delta(x^*, Tx^*), \delta(y^*, Ty^*), \delta(x^*, Ty^*), \delta(Tx^*, y^*)\})).
\end{aligned}$$

Hence

$$\psi(\delta(x^*, y^*)) \leq F(\psi(\delta(x^*, y^*)), \varphi(\delta(x^*, y^*)))$$

and $\psi(\delta(x^*, y^*)) = 0$ or $\varphi(\delta(x^*, y^*)) = 0$. Therefore $\delta(x^*, y^*) = 0$ and $x^* = y^*$.
□

The following results are an immediate consequences of Theorems 2 and 3.

Corollary 1 *Theorem 1 is a particular case of Theorem 2.*

Corollary 2 *Let* $(X, \preceq)$ *be a partially ordered set and* (X, d, δ) *be a bimetric space and* $T : X \to X$*. Assume that the following conditions are satisfied:*

(C1) $d(x, y) \leq \delta(x, y)$, *for all* $x, y \in X$,

(C2) X *is complete with respect to* d,

(C3) T *is continuous with respect to* d,

(C4) T *is monotone nondecreasing mapping,*

(C5) *there exists* $x_0 \in X$ *with* $x_0 \preceq Tx_0$,

(C6) *there exists* $\varphi \in \Phi$ *such that*

$$\psi(\delta(Tx, Ty)) \leq \psi(M_\delta(x, y)) - \varphi(M_\delta(x, y))), \quad \text{for all } x, y \text{ in } X.$$

Then T *has a unique fixed point in* X.

Corollary 3 *Let* $(X, \preceq)$ *be a partially ordered set and* (X, d) *be a complete metric space. Assume that for* $T : X \to X$*, the following conditions are satisfied:*

(D1) T *is continuous;*

(D2) T *is monotone nondecreasing mapping;*

(D3) *there exists* $x_0 \in X$ *with* $x_0 \preceq Tx_0$;

(D4) *there exists a constant* $\alpha \in [0, 1)$ *such that*

$$\psi(\delta(Tx, Ty)) \leq \alpha\psi(M_\delta(x, y)), \quad \text{for all } x, y \text{ in } X.$$

Then T *has a unique fixed point in* X.

Corollary 4 *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be continuous mapping. Suppose there exists* $\varphi \in \Phi$ *such that*

$$\psi(\delta(Tx, Ty)) \leq \frac{\psi(M_\delta(x, y))}{1 + \varphi(M_\delta(x, y))}, \quad \text{for all } x, y \text{ in } X.$$

Then T *has a unique fixed point in* X.

Remark 3

(1) *If in Theorem 2 we put* $F(s,t) = mt,\ 0 \leq m < 1,\ \psi(t) = \phi(t) = t$*, then we obtain Maia's Theorem 1.*

(2) If we use the same notations as in (1), and if we define relation $\mathcal{S}$ by $x\mathcal{S}y$ if and only if $\alpha d(xTx) \leq d(x,y)$ implies $d(Tx,Ty) \leq \beta d(x,y)$, where $\alpha \in (0, 1/2), \beta \in (0,1)$, then when T is continuous Theorem 2 implies Theorem 2.2 in [29].

(3) Our results, when we put $F(s,t) = mt,\ 0 \leq m < 1$, imply results from [15].

(4) Using Theorem of Singh [28] we note that our results are true under much less restricted condition, that is we do not need the continuity of T with respect to d on X, but only the continuity at a point.

3 Application to Cauchy problem

In this section, we study the Cauchy problem for a class of nonlinear differential equations, using the results obtained in the previous section. We just state the application part and we point out that the proof is on the lines of M.S. Khan at all [15]. So we omit it.

Example 4 *Consider the nonlinear differential equation*

$$\psi(x) = \begin{cases} x'(t) = f(t, x(t)) & t \in [a,b], \\ x(t_0) = x_0 \end{cases} \tag{12}$$

where $a, b, t_0 \in \mathbb{R}$ *and* $f : [a,b] \times \mathbb{R} \to \mathbb{R}$*. Let* $X = C([a,b], \mathbb{R})$ *denotes the space of all continuous* $\mathbb{R}$*-valued functions on* $[a,b]$ *with the metric* d *given by*

$$d(u,v) = \sup_{t \in [a,b]} |f(u(t), v(t))|, \text{ for all } u, v \in X$$

It is well known that (X, d) *is a complete metric space. We define an order relation* $\preceq$ *on* X *by*

$$u \preceq v \Leftrightarrow u(t) \leq v(t), \text{ for all } t \in [a,b]$$

Consider the mapping $T : C([a,b], \mathbb{R}) \to C([a,b], \mathbb{R})$ *defined by*

$$Tx(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds\ ; \qquad t \in [a,b]$$

for all $x \in C([a, b], \mathbb{R})$*. Clearly,* $x^* \in C([a, b], \mathbb{R})$ *is a solution of* (12) *if and only if* x^* *is a fixed point of* T*.*
Furthermore, we consider the following assumptions:

(H1) $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ *is continuous;*

(H2) $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ *is nondecreasing with respect to the second variable;*

(H3) $|f(t, x(t)) - f(t, y(t))| \leq L|x(t) - y(t)|$ *for all* $x(t) \leq y(t)$ *and* $t \in [a, b]$*.*

It is worth noting that condition (H3) *is weaker compared to those used by Maia for studying Cauchy problem in* [16]*, that is,* f *is* L*-Lipschitzien function on the whole space.*

By the proof of Theorem 6 in [15] we have

$$\delta(x, y) \leq \exp(L(b - a))d(x, y)$$

and for $\lambda = \sqrt{1 - \exp(L(a - b))}$, then we have

$$d(T^n x, T^n y) \leq \exp(L(b - a))d(x, y) \qquad \text{for all } \; x \preceq y. \tag{13}$$

We deduce by using Corollary 3 (see also [15]) that T has a unique fixed point $x^* \in C([a, b], \mathbb{R})$.

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On the convergence difference sequences and the related operator norms

P. Baliarsingh
Department of Mathematics, Gangadhar Meher University, Sambalpur-768004, Odisha, India
email: pb.math10@gmail.com

L. Nayak*
Department of Mathematics, School of Applied Sciences, Kalinga Institute of Industrial Technology, Bhubaneswar-751024, India
email: laxmipriyamath@gmail.com

S. Samantaray
Department of Mathematics, School of Applied Sciences, Kalinga Institute of Industrial Technology, Bhubaneswar-751024, India
Kalinga Institute of Industrial Techonology, Bhubaneswar-751024, India
email: samantaraysnigdha1995@gmail.com

Abstract. In this note, we discuss the definitions of the difference sequences defined earlier by Kızmaz (1981), Et and Çolak (1995), Malkowsky et al. (2007), Başar(2012), Baliarsingh (2013, 2015) and many others. Several authors have defined the difference sequence spaces and studied their various properties. It is quite natural to analyze the convergence of the corresponding sequences. As a part of this work, a convergence analysis of difference sequence of fractional order defined earlier is presented. It is demonstrated that the convergence of the fractional difference sequence is dynamic in nature and some of the results involved are also inconsistent. We provide certain stronger conditions on the primary sequence and the results due to earlier authors are substantially modified. Some illustrative examples are provided for each point of the modifications. Results on certain operator norms related to the difference operator of fractional order are also determined.

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*corresponding author

1 Introduction

Recently, one of the most interesting areas of research in Mathematics is the study of difference operators and related sequence spaces which has been attracted in different areas of Mathematical sciences especially in applied and computational fields involving calculus, matrix and approximation theory. The idea of difference sequence spaces plays a key role in most of the scientific problems involving the spectral properties of bounded linear operators(see [2, 7, 11, 15, 16, 28, 29, 30]), related topological structures (see [3, 4, 19, 20, 22, 26, 27]), matrix transformations(see [5, 12, 18, 19, 21, 23]), compact operators (see [1, 14, 24, 25]), fractional calculus [8, 9, 10], etc.

In fact, the study of all the ideas discussed earlier is only feasible and even possible if the related sequences are convergent.

Let $x = (x_k)$ be any sequence in w, the family of all real valued sequences. Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A sequence $x = (x_k)$ is said to be of order k^α, i.e., $x_k = \mathcal{O}(k^\alpha)$ if for a positive constant $\mathcal{C}$, we can write

$$|x_k| \leq \mathcal{C}k^\alpha, \ k = 0, 1, 2, 3, \ldots.$$

By ℓ_∞, c and c_0, we denote the spaces of all bounded, convergent and null sequences, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|.$$

We use the notation ℓ_p, $(1 \leq p < \infty)$ for the space of all $p-$summable sequence with the norm

$$\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p}.$$

The 1st order difference sequence space $X(\Delta)$ for $X \in \{\ell_\infty, c, c_0\}$ was introduced by Kızmaz [20] using forward difference operator Δ, where

$$\Delta x_k = x_k - x_{k+1}, \ (k \in \mathbb{N}_0). \tag{1}$$

Later on, this idea has been generalized to the case of difference sequence spaces of integer order m by Et and Çolak [17] using the operator Δ^m and

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}, \ (k \in \mathbb{N}_0). \tag{2}$$

Using Euler gamma function for a proper fraction α, the fractional difference sequence $\Delta^{\alpha}x$ of order α was defined by Baliarsingh [4](see also [5, 6]) as

$$\Delta^{\alpha}x_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}x_{k+i}, \ (k \in \mathbb{N}_0). \tag{3}$$

By taking inverse transform $\Delta^{-\alpha}$ on the sequence $x = (x_k)$, we write the Eqn. (3) as

$$\Delta^{-\alpha}x_k = x_k + \alpha x_{k+1} + \frac{\alpha(\alpha+1)}{2!}x_{k+2} + \frac{\alpha(\alpha+1)(\alpha+1)}{3!}x_{k+3} + \dots \tag{4}$$

An infinite series has no meaning unless it converges. It is important to mention that in the previous papers, the convergence of the fractional difference sequence defined by (3) and (4) have been presumed without taking any further investigations. Now, in particular, we illustrate the following examples regarding the convergence of these series:

Example 1 *Let α be a proper fraction and $x = (x_k)$ be the convergent sequence defined by $x_k = \frac{1}{3^k}$ for all $k \in \mathbb{N}_0$. Then, we can easily calculate*

$$\Delta^{\alpha}x_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}\frac{1}{3^{k+i}} = \frac{1}{3^k}\left(\frac{2}{3}\right)^{\alpha} = \frac{2^{\alpha}}{3^{k+\alpha}} \to 0 \text{ as } k \to \infty,$$

and

$$\Delta^{-\alpha}x_k = \sum_{i=0}^{\infty}\frac{\alpha(\alpha+1)\dots(\alpha+i-1)}{\Gamma(i+1)}\frac{1}{3^{k+i}} = \frac{1}{3^k}\left(\frac{2}{3}\right)^{-\alpha} = \frac{3^{\alpha-k}}{2^{\alpha}} \to 0 \text{ as } k \to \infty.$$

Example 2 *Let $x = (x_k)$ be the constant sequence with $x_k = 1$ for all $k \in \mathbb{N}_0$. Although the sequence $x = (x_k)$ is convergent, but for a proper fraction α, $\Delta^{\alpha}x_k \to 0$ as $k \to \infty$ whereas, $\Delta^{-\alpha}x_k \to \infty$ as $k \to \infty$.*

Example 3 *Let $x = (x_k)$ be the oscillating sequence, defined by $x_k = (-1)^k$ for all $k \in \mathbb{N}_0$. Clearly, the sequence $x = (x_k)$ is divergent and for a proper fraction α, we have*

$$\Delta^{\alpha}x_k = \begin{cases} 2^{\alpha}, & (k \text{ is even}) \\ -2^{\alpha}, & (k \text{ is odd}), \end{cases}$$

and

$$\Delta^{-\alpha}x_k = \begin{cases} 2^{-\alpha}, & (k \text{ is even}) \\ -2^{-\alpha}, & (k \text{ is odd}) \end{cases}$$

are also divergent.

Example 4 *Let $x = (x_k)$ be the divergent sequence defined by $x_k = k$ for all $k \in \mathbb{N}_0$. Although the sequence $x = (x_k)$ is divergent, but for an integer $\alpha > 1$, $\Delta^{\alpha} x_k \to 0$ as $k \to \infty$ whereas, $\Delta^{-\alpha} x_k \to \infty$ as $k \to \infty$. For a proper fraction α, both of the difference sequences go to ∞ as $k \to \infty$.*

It is remarked that the infinite series defined in (3) and (4) need not be convergent for any arbitrary sequence $x = (x_k)$ and any proper fraction α. Therefore, it is quite difficult to study and analyze the behaviors of the related sequence spaces for fractional cases. As the convergence of the difference sequence $\Delta^{\alpha} x$ depends on the nature and behavior of the sequence x and the value α, it has been observed that the properties such as linearity and exponent rules of the difference operator Δ^{α} are violating in certain particular cases. As a consequence of these violations, it is concluded that Theorems 1, 2 and 3 due to [4, 5] are not stable and need certain additional conditions in order to provide their substantial modifications.

The primary objective of this note is to study the convergence of the fractional difference sequences, the dynamic nature of the fractional difference operator Δ^{α} in detail and apply the same to modify Theorems 1, 2 and 3 of [4, 5]. Now, we analyze the convergence of the difference sequence $\Delta^{\alpha} x$ for different choice of α in detail, (i.e., $\alpha > 0, \alpha < 0$ and $\alpha \in \mathbb{N}$) by using the following theorems.

2 Main results

Theorem 1 *The series defined in (3) is convergent for any $\alpha = n \in \mathbb{N}$ if the sequence $x = (x_k)$ is convergent. The converse of the statement may not hold in general.*

Proof. Let $x = (x_k)$ be a convergent sequence. Then for given $\varepsilon > 0$, there exists a natural number N and real or complex number l such that, for every $k \geq N$, we have $|x_k - l| < \varepsilon$. Now, we have

$$\begin{aligned}
|\Delta^n x_k| &= \left| \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_{k+i} \right| \\
&= \left| \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_{k+i} - \sum_{i=0}^{n} (-1)^i \binom{n}{i} l + \sum_{i=0}^{n} (-1)^i \binom{n}{i} l \right| \\
&= \left| \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x_{k+i} - l) + l \sum_{i=0}^{n} (-1)^i \binom{n}{i} \right|
\end{aligned}$$

$$\leq \sum_{i=0}^{n}(-1)^i\binom{n}{i}|(x_{k+i}-l)| + |l|\left|\sum_{i=0}^{n}(-1)^i\binom{n}{i}\right|$$
$$\leq \varepsilon\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0, \text{ for every } k \geq N.$$

Therefore, $|\Delta^n x_k| \to 0$ as $k \to \infty$. For the converse part we take the following counter example:

For a natural number m, consider the sequence $x = (x_k)$, defined by $x_k = k^m$ for all $k \in \mathbb{N}_0$. Clearly, $x = (x_k)$ is divergent, but its associated difference sequence is

$$\begin{aligned}
\Delta^n x_k &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}(k+i)^m \\
&= k^m - \binom{n}{1}\left[k^m + \binom{m}{1}k^{m-1} + \binom{m}{2} + \cdots + \binom{m}{m}\right] \\
&\quad + \binom{n}{2}\left[k^m + 2\binom{m}{1}k^{m-1} + 2^2\binom{m}{2} + \cdots + \binom{m}{m}2^m\right] + \ldots \\
&\quad + (-1)^n\binom{n}{n}\left[k^m + n\binom{m}{1}k^{m-1} + n^2\binom{m}{2} + \cdots + \binom{m}{m}n^m\right] \\
&= k^m\left[1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\right] \\
&\quad + k^{m-1}\binom{m}{1}\left[-\binom{n}{1} + 2\binom{n}{2} - 3\binom{n}{3} + \cdots + n(-1)^n\right] \\
&\quad + k^{m-2}\binom{m}{2}\left[-\binom{n}{1} + 2^2\binom{n}{2} - 3^2\binom{n}{3} + \cdots + n^2(-1)^n\right] + \ldots \\
&\quad + k^{m-m}\binom{m}{m}\left[-\binom{n}{1} + 2^m\binom{n}{2} - 3^m\binom{n}{3} + \cdots + n^m(-1)^n\right] \\
&= \begin{cases} 0, & (n > m) \\ n!, & (n = m) \\ \infty, & (n < m) \end{cases}.
\end{aligned}$$

Therefore, we conclude that for $n \geq m$ the difference sequence $(\Delta^n(k^m))_k$ is convergent while the primary sequence $x = (k^m)$ is divergent. $\square$

Theorem 2 *The series defined in (3) is convergent for any proper fraction $\alpha > 0$ if the sequence $x = (x_k)$ is convergent. The converse of the statement is true if the sequence involving infinite series*

$$\sum_{i=k}^{\infty} \binom{i-k+\alpha-1}{i-k} \Delta^{\alpha}(x_i) \text{ converges.} \tag{5}$$

Proof. The proof of the sufficient part is similar to that of Theorem (1), hence omitted.

For the necessary part we assume that the difference sequence $\Delta^{\alpha} x_k$ and the infinite series $\sum_{i=k}^{\infty} \binom{i-k+\alpha-1}{i-k} \Delta^{\alpha}(x_i)$ converge for all $k \in \mathbb{N}_0$. Let α be a proper fraction, i.e., $0 < \alpha < 1$. On simplifying (5), we obtain that

$$\begin{aligned}
&\sum_{i=k}^{\infty} \binom{i-k+\alpha-1}{i-k} \Delta^{\alpha}(x_i) \\
&\quad = \binom{\alpha-1}{0} \Delta^{\alpha}(x_k) + \binom{\alpha}{1} \Delta^{\alpha}(x_{k+1}) + \binom{\alpha+1}{2} \Delta^{\alpha}(x_{k+2}) + \dots \\
&\quad = x_k - \binom{\alpha}{1} x_{k+1} + \binom{\alpha}{2} x_{k+2} - \binom{\alpha}{3} x_{k+3} + \dots \\
&\qquad + \binom{\alpha}{1} \left[x_{k+1} - \binom{\alpha}{1} x_{k+2} + \binom{\alpha}{2} x_{k+3} - \binom{\alpha}{3} x_{k+4} + \dots \right] \\
&\qquad + \binom{\alpha+1}{2} \left[x_{k+2} - \binom{\alpha}{1} x_{k+3} + \binom{\alpha}{2} x_{k+4} - \binom{\alpha}{3} x_{k+5} + \dots \right] + \dots \\
&\quad = x_k.
\end{aligned}$$

Thus, from the hypothesis, the sequence (x_k) is convergent. However, from Example 5, it is noticed that for a unbounded sequence $x = (x_k)$ with $x_k = k$ for all $k \in \mathbb{N}_0$, for a proper fraction α, corresponding difference sequence $\Delta^{\alpha} x_k \to \infty$, as $k \to \infty$. This completes the proof. $\square$

Theorem 3 *The series defined in (4) is convergent for any proper $\alpha > 0$ or $\alpha = n \in \mathbb{N}_0$ if the sequence $x = (x_k)$ is convergent with $x_k = \mathcal{O}(k^{-\alpha-1})$. The converse of the statement is true if the sequence involving infinite series*

$$\sum_{i=k}^{\infty} (-1)^{i-k} \binom{\alpha}{i-k} \Delta^{-\alpha}(x_i) \text{ converges.} \tag{6}$$

Proof. We know that the infinite series in (4) represents the inverse fractional difference sequence of the sequence (x_k), thus it always suggests the idea analog to integration or summation. Since the equation is a sum of infinite terms with all positive coefficients of x_k, most of the cases it gives ∞ even if the primary sequence is convergent. As a result, we need to consider strictly the order of the convergence of the primary sequence (x_k) in such a way that the final sum of the series (4) will be dominated.

Let us consider the convergent sequence $x = (x_k)$ with $x_k = \mathcal{O}(k^{-\alpha-1})$ and $\alpha > 0$. Then, there exists a constant M such that

$$\sup_k |x_k| \leq \frac{M}{k^{\alpha+1}}.$$

In fact, the above sequence is a null sequence and the corresponding inverse difference sequence is given below:

$$\begin{aligned}
\Delta^{-\alpha} x_k &= \sum_{i=0}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+i-1)}{\Gamma(i+1)} x_{k+i} \\
&= x_k + \alpha x_{k+1} + \frac{\alpha(\alpha+1)}{2!} x_{k+2} + \frac{\alpha(\alpha+1)(\alpha+1)}{3!} x_{k+3} + \dots \\
&\leq \frac{M}{k^{\alpha+1}} \left(1 + \alpha + \frac{\alpha(\alpha+1)}{2!} + \frac{\alpha(\alpha+1)(\alpha+1)}{3!} + \dots \right).
\end{aligned}$$

The right hand side of the above equation is tending to 0 as $k \to \infty$. The equation contains two terms out of which the term $\frac{M}{k^{\alpha+1}}$ is dominating since it contains $(\alpha+1)$ as power of $1/k$ whereas other term contains α, only, which is a constant. It is rapidly tending to 0 as comparison to the rate at which the other term goes to ∞. The converse part of this theorem is similar to that of Theorem 5. $\square$

Theorem 4 *Let $\alpha > 0$ be either a fraction or a natural number and $\Delta^\alpha : w \to w$ is a linear operator provided the series in (3) is convergent.*

Theorems (1), (2) and (3) can be verified in the light of the above theorem, it can be shown that most of the results are not satisfied in general.

Theorem 5 *For any proper fractions α, α_1 and α_2, in general we have*

(i) $\Delta^{\alpha_1}(\Delta^{\alpha_2} x_k) \neq \Delta^{\alpha_1+\alpha_2}(x_k)$ *and* $\Delta^{\alpha_2}(\Delta^{\alpha_1} x_k) \neq \Delta^{\alpha_1+\alpha_2}(x_k)$,

(ii) $\Delta^{\alpha}(\Delta^{-\alpha} x_k) \neq x_k$ *and* $\Delta^{-\alpha}(\Delta^{\alpha} x_k) \neq x_k$,

Proof. We prove theorem by using suitable counter examples.

Example 5 *Consider the sequence* $x = (x_k)$*, defined by* $x_k = k$ *for all* $k \in \mathbb{N}_0$*. Clearly it is a divergent sequence. Let us take* $\alpha_1 = 1/2 = \alpha_2$ *and therefore,* $\alpha_1 + \alpha_2 = 1$*. Then, we can calculate*

$$\begin{aligned}\Delta^{\alpha_2} x_k = (\Delta^{1/2} k)_k &= k - \binom{1/2}{1}(k+1) + \binom{1/2}{2}(k+2) - \binom{1/2}{3}(k+3) + \dots \\ &= k\left[1 - \binom{1/2}{1} + \binom{1/2}{2} - \binom{1/2}{3} + \dots\right] \\ &\quad - \frac{1}{2}\left[1 - \binom{-1/2}{1} + \binom{-1/2}{2} - \binom{-1/2}{3} + \dots\right] \\ &= \infty.\end{aligned}$$

Now, $\Delta^{\alpha_1}(\Delta^{\alpha_2}(x_k)) = \Delta^{1/2}(\Delta^{1/2}(k)) = \Delta^{1/2}(\infty) = \infty$*, but* $\Delta^{\alpha_1+\alpha_2}(x_k) = \Delta^{1/2+1/2}(k) = \Delta(k) = k - (k+1) = -1$*. Interchanging* α_1 *and* α_2 *in above expression we can prove the second condition. This completes the proof of Part* (i) *of Theorem 5.*

Example 6 *Let us consider the sequence* $x = (x_k)$*, defined by* $x_k = r$ *for all* $k \in \mathbb{N}_0$ *and* $r \in \mathbb{R}$*, the set of all real numbers. Clearly,* $x = (x_k)$ *is a convergent sequence. Taking* $\alpha = 1/2$*, we have*

$$\begin{aligned}\Delta^{-\alpha} x_k = (\Delta^{-1/2} r)_k &= r\left[1 - \binom{-1/2}{1} + \binom{-1/2}{2} - \binom{-1/2}{3} + \dots\right] \\ &= \infty.\end{aligned}$$

Thus, the left hand side of 1st equation of Part (ii) *is* $\Delta^{\alpha}(\Delta^{-\alpha}(x_k)) = \Delta^{1/2}(\Delta^{-1/2}(r)) = \Delta^{1/2}(\infty) = \infty$*, whereas the right hand side is* $x_k = r$*. Again by interchanging the positions of* α *and* $-\alpha$*, it is also noticed that*

$$\begin{aligned}\Delta^{\alpha} x_k = (\Delta^{1/2} r)_k &= r\left[1 - \binom{1/2}{1} + \binom{1/2}{2} - \binom{1/2}{3} + \dots\right] \\ &= 0.\end{aligned}$$

Now, the left hand side of the second equation of Part (ii) can be found as $\Delta^{-\alpha}(\Delta^{\alpha}(x_k)) = \Delta^{-1/2}(\Delta^{1/2}(r)) = \Delta^{-1/2}(0) = 0$ *which is not equal to the right hand side i.e.,* $x_k = r$*. This completes the proof of Part* (ii) *of Theorem 5.*

□

Above examples conclude that linearity and exponent rules involving the fractional difference operator Δ^{α} for any sequence in w are not uniformly posed. Eventually, these rules are deviating due to lack of convergence of related infinite series. In fact, the convergence of the related infinite series is completely depending on the nature of the primary sequence (x_k) and the choice of the values of α. It is understood that if the primary sequence (x_k) and the value α are suitably chosen then obviously, this deviation can be restricted to a given domain. This idea suggests that Theorems 1, 2 and 3 of [4] need relevant modifications and the modified results are as follows.

Theorem 6 *For any positive proper fractions α, α_1 and α_2, we have*

(i) *Let the sequence $x = (x_k)$ be convergent, then*

$$\Delta^{\alpha_1}(\Delta^{\alpha_2}(x_k)) = \Delta^{\alpha_1+\alpha_2}(x_k) = \Delta^{\alpha_2}(\Delta^{\alpha_1}(x_k)),$$

(ii) *Let the sequence $(\Delta^{-\alpha}x_k)$ be convergent, then*

$$\Delta^{\alpha}(\Delta^{-\alpha}x_k) = x_k,$$

(iii) *Let the sequence $(\Delta^{\alpha}x_k)$ be of $\mathcal{O}(k^{-\alpha-1})$, then*

$$\Delta^{-\alpha}(\Delta^{\alpha}x_k) = x_k.$$

Combining all points, Theorem 6 can be restated as follows:

Remark 1 *Let $\alpha > 0$ and β be a real such that $\alpha + \beta > 0$ and the sequence (x_k) be of $\mathcal{O}(k^{-m-1})$, where $m = \min(|\alpha|, |\beta|)$, then*

$$\Delta^{\alpha}(\Delta^{\beta}(x_k)) = \Delta^{\alpha+\beta}(x_k) = \Delta^{\beta}(\Delta^{\alpha}(x_k)).$$

Corollary 1 *For any $n \in \mathbb{N}$, let Δ^{-n} be the negative integral difference operator, then*

(i) $\Delta^{-1}(x_k) = \sum_{i=1}^{\infty} x_{k+i}$, *if the sequence (x_k) is convergent with $x_k = \mathcal{O}(k^{-2})$,*

(ii) $\Delta^{-2}(x_k) = \sum_{i=1}^{\infty}(i+1)x_{k+i}$, *if the sequence (x_k) is convergent with $x_k = \mathcal{O}(k^{-3})$,*

(iii) $\Delta^{-3}(x_k) = \sum_{i=1}^{\infty}(s_i)x_{k+i}$, *where $s_i = \sum_{j=1}^{i} j$, if the sequence (x_k) is convergent with $x_k = \mathcal{O}(k^{-4})$.*

To next, we discuss some operator norms involving the difference operator of fractional order.

Let $A = (a_{nk})$ be an infinite matrix with $a_{nk} \geq 0$ for all $n, k \in \mathbb{N}_0$. Then we have the following theorems on operator norms via the infinite matrix A:

Theorem 7 *Let $X \in \{c_0, c, \ell_\infty\}$. Then the infinite matrix A is a bounded operator from X to $X(\Delta^\alpha)$ if*

$$\mathcal{M} = \sup_n \left\{ \sum_{k=0}^{\infty} \left| \sum_{i=0}^{\infty} \frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)} a_{n+i,k} \right| \right\} < \infty$$

and

$$\|A\|_{(\infty,\Delta^\alpha)} = \mathcal{M}.$$

Proof. Suppose $X = \ell_\infty$ and $x \in X$. Then, we have

$$\begin{aligned}
\|Ax\|_{(\infty,\Delta^\alpha)} &= \sup_n \left| \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)} a_{n+i,k} x_k \right| \\
&\leq \sup_n \left\{ \sum_{k=0}^{\infty} \left| \sum_{i=0}^{\infty} \frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)} a_{n+i,k} x_k \right| \right\} \\
&\leq \mathcal{M} \|x\|_\infty.
\end{aligned}$$

Also, for $x = e = (1, 1, 1, \dots)$, we have

$$\begin{aligned}
\|Ae\|_{(\infty,\Delta^\alpha)} &= \sup_n \left| \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)} a_{n+i,k} \right| \\
&= \sup_n \left\{ \sum_{k=0}^{\infty} \left| \sum_{i=0}^{\infty} \frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)} \right| a_{n+i,k} \right\} \\
&= \mathcal{M}.
\end{aligned}$$

This proves the result. □

Theorem 8 *The infinite matrix A is a bounded operator from ℓ_1 to $\ell_1(\Delta^\alpha)$ if*

$$\overline{\mathcal{M}} = \sup_k \left\{ \sum_{n=0}^{\infty} \left| \sum_{i=0}^{\infty} \frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)} a_{n+i,k} \right| \right\} < \infty,$$

and

$$\|A\|_{(1,\Delta^\alpha)} = \overline{\mathcal{M}}.$$

Proof. Suppose that $x \in \ell_1$ and A be an infinite matrix, then

$$\begin{aligned}
\|Ax\|_{(1,\Delta^\alpha)} &= \sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)}a_{n+i,k}x_k\right| \\
&\le \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left|\sum_{i=0}^{\infty}\frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)}a_{n+i,k}x_k\right| \\
&\le \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left|\sum_{i=0}^{\infty}\frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)}a_{n+i,k}\right||x_k| \\
&\le \overline{\mathcal{M}}\|x\|_1.
\end{aligned}$$

Now, for the sequence $x = e^{(m)}$(having 1 at m-th place and 0 otherwise), one can get

$$\begin{aligned}
\|Ae^{(m)}\|_{(1,\Delta^\alpha)} &= \sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)}a_{n+i,k}x_k\right| \\
&= \sum_{n=0}^{\infty}\left|\sum_{i=0}^{\infty}\frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)}a_{n+i,m}\right| \\
&= \overline{\mathcal{M}}.
\end{aligned}$$

This concludes the proof. □

Theorem 9 *The infinite matrix* A *is a bounded operator from* $\ell_p, (1 \le p < \infty)$ *to* $\ell_p(\Delta^\alpha)$ *if*

$$\overline{\mathcal{M}}_p = \sup_k\left\{\sum_{n=0}^{\infty}\left|\sum_{i=0}^{\infty}\frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)}a_{n+i,k}\right|^p\right\} < \infty,$$

and

$$\|A\|^p_{(p,\Delta^\alpha)} = \overline{\mathcal{M}}_p.$$

Proof. This follows from the proof of Theorem 8. □

Theorem 10 *The identity matrix* I *is a bounded operator from* X *to* $X(\Delta^\alpha)$ *for* $X \in \{c, c_0, \ell_\infty, \ell_1\}$ *and*

$$\|I\|_{(\infty,\Delta^\alpha)} = \|I\|_{(1,\Delta^\alpha)} = 2^\alpha.$$

Proof. Suppose the infinite matrix $A = I$, then from Theorem 7, we can write

$$\begin{aligned}\mathcal{M}_n &= \sum_{k=0}^{\infty}\left|\sum_{i=0}^{\infty}\frac{(-\alpha)(-\alpha+1)\dots(-\alpha+i-1)}{\Gamma(i+1)}a_{n+i,k}\right| \\ &= \sum_{k=n}^{\infty}\left|\frac{(-\alpha)(-\alpha+1)\dots(-\alpha+k-n-1)}{\Gamma(k-n+1)}\right|.\end{aligned}$$

Therefore, we have

$$\|I\|_{(\infty,\Delta^{\alpha})} = \sup_n \mathcal{M}_n = 2^{\alpha}.$$

Similarly, using Theorem 8, one can prove $\|I\|_{(1,\Delta^{\alpha})} = 2^{\alpha}$. □

Conclusion

We have investigated some idea on the convergence of difference sequence for fractional-order which may be very similar to that of integer orders but most of the cases they are nonuniform and dynamic in nature. As an application of this idea, some existing results in the literature have been modified. Certain operator norms involving the difference operator of fractional order is determined.

In the next study, we will extend this idea to the case of the statistical convergence of difference sequence and study the variations in the cases of integer and fractional orders.

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On some properties of split Horadam quaternions

Dorota Bród
Rzeszow University of Technology,
Faculty of Mathematics and Applied Physics,
al. Powstańców Warszawy 12,
35-959 Rzeszów, Poland
email: dorotab@prz.edu.pl

Abstract. In this paper we introduce and study the split Horadam quaternions. We give some identities, among others Binet's formula, Catalan's, Cassini's and d'Ocagne's identities for these numbers.

1 Introduction

Let $\mathbb{C}$ be the field of complex numbers. A quaternion x is a hyper-complex number represented by

$$\mathbb{H} = \{x = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} : a_s \in \mathbb{R},\ s = 0, 1, 2, 3\},$$

where $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal basis in $\mathbb{R}^4$, which satisfies the quaternion multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji},\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj},\ \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

The quaternions were introduced by W. R. Hamilton in 1843.

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Another extension of the complex numbers is the algebra of split quaternions. The split quaternions were introduced by J. Cockle in 1849 [2]. The set of split (or coquaternions) can be represented as

$$\hat{\mathbb{H}} = \{y = b_0 + b_1 i + b_2 j + b_3 k \colon b_s \in \mathbb{R},\ s = 0, 1, 2, 3\},$$

where $\{1, i, j, k\}$ is the basis of $\hat{\mathbb{H}}$ satisfying the following equalities

$$i^2 = -j^2 = -k^2 = -1, \tag{1}$$

$$ij = k = -ji,\ jk = -i = -kj,\ ki = j = -ik. \tag{2}$$

The split quaternion can be rewritten as

$$y = (b_0 + b_1 i) + (b_2 + b_3 i)j = z_1 + z_2 j,\ z_1, z_2 \in \mathbb{C}.$$

The split quaternions contain nontrivial zero divisors, nilpotent elements and idempotents. The conjugate of a split quaternion $y = b_0 + b_1 i + b_2 j + b_3 k$, denoted by $\overline{y}$, is given by $\overline{y} = b_0 - b_1 i - b_2 j - b_3 k$. The norm of y is defined as

$$N(y) = y\overline{y} = b_0^2 + b_1^2 - b_2^2 - b_3^2. \tag{3}$$

Let $y_1, y_2 \in \hat{\mathbb{H}}$, $y_1 = a_1 + b_1 i + c_1 j + d_1 k, y_2 = a_2 + b_2 i + c_2 j + d_2 k$. Then addition and subtraction of the split quaternions is defined as follows

$$y_1 \pm y_2 = (a_1 \pm a_2) + (b_1 \pm b_2)i + (c_1 \pm c_2)j + (d_1 \pm d_2)k.$$

Multiplication of the split quaternions is defined by

$$\begin{aligned} y_1 \cdot y_2 = {} & a_1 a_2 - b_1 b_2 + c_1 c_2 + d_1 d_2 + (a_1 b_2 + b_1 a_2 - c_1 d_2 + d_1 c_2)i \\ & + (a_1 c_2 + c_1 a_2 - b_1 d_2 + d_1 b_2)j + (a_1 d_2 + d_1 a_2 + b_1 c_2 - c_1 b_2)k. \end{aligned} \tag{4}$$

For the basics on split quaternions theory, see [5].

2 The Horadam numbers

In [3] Horadam introduced a sequence $\{W_n\}$ defined by the following relation

$$W_0 = a, W_1 = b,\ W_n = pW_{n-1} + qW_{n-2} \text{ for } n \geq 2 \tag{5}$$

for arbitrary $a, b, p, q \in \mathbb{Z}$. This sequence is a certain generalization of famous sequences such as Fibonacci sequence $\{F_n\}$ $(a = 0, b = 1,\ p = q = 1)$, Lucas sequence $\{L_n\}$ ($a = 2, b = 1$, $p = q = 1$), Jacobsthal sequence $\{J_n\}$ ($a = 0, b = 1$, $p = 1, q = 2$), Pell sequence $\{P_n\}$ ($a = 0, b = 1$, $p = 2, q = 1$), Pell-Lucas sequence $\{PL_n\}$ ($a = b = 1$, $p = 2, q = 1$). The sequences defined by (5) are called sequences of the Fibonacci type.

The characteristic equation associated with the recurrence (5) is

$$r^2 - pr - q = 0.$$

Assuming that $p^2 + 4q > 0$, the equation has the following roots

$$r_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad r_2 = \frac{p - \sqrt{p^2 + 4q}}{2}. \tag{6}$$

Note that

$$r_1 + r_2 = p, \tag{7}$$

$$r_1 - r_2 = \sqrt{p^2 + 4q}, \tag{8}$$

$$r_1 r_2 = -q. \tag{9}$$

The Binet's formula for the sequence $\{W_n\}$ has the following form

$$W_n = \frac{(b - ar_2)r_1^n - (b - ar_1)r_2^n}{r_1 - r_2}.$$

Let

$$\alpha = \frac{b - ar_2}{r_1 - r_2}, \quad \beta = \frac{b - ar_1}{r_1 - r_2}. \tag{10}$$

Then

$$W_n = \alpha r_1^n - \beta r_2^n. \tag{11}$$

In the next section we will use the following result.

Theorem 1 *Let* n, p, q *be integers such that* $n \geq 0$, $p^2 + 4q > 0$. *Then*

$$\sum_{l=0}^{n-1} W_l = \frac{W_n + qW_{n-1} + a(p-1) - b}{p + q - 1}. \tag{12}$$

Proof. Using formula (11), (7) and (9), we get

$$\sum_{l=0}^{n-1} W_l = \sum_{l=0}^{n-1} (\alpha r_1^l - \beta r_2^l) = \alpha \frac{1 - r_1^n}{1 - r_1} - \beta \frac{1 - r_2^n}{1 - r_2}$$

$$= \frac{\alpha - \beta - (\alpha r_2 - \beta r_1) - (\alpha r_1^n - \beta r_2^n) + r_1 r_2(\alpha r_1^{n-1} - \beta r_2^{n-1})}{1 - (r_1 + r_2) + r_1 r_2}$$
$$= \frac{\alpha - \beta - (\alpha r_2 - \beta r_1) - W_n - qW_{n-1}}{1 - p - q}.$$

By simple calculations we have $\alpha - \beta = a$, $\alpha r_2 - \beta r_1 = ap - b$. Hence

$$\sum_{l=0}^{n-1} W_l = \frac{W_n + qW_{n-1} + a(p-1) - b}{p + q - 1}.$$

□

Numbers of the Fibonacci type appear in many subjects of mathematics. In [4] Horadam defined the Fibonacci and Lucas quaternions. In [1] the split Fibonacci quaternions Q_n and split Lucas quaternions T_n were introduced by the following relations

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$
$$T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3},$$

where F_n, L_n is nth Fibonacci and Lucas number, resp. and $\{i, j, k\}$ is the standard basis of split quaternions. In the literature there are many generalizations of the Fibonacci and Lucas sequences, among others k-Fibonacci sequence $\{F_{k,n}\}$, k-Lucas sequence $\{L_{k,n}\}$, defined for $k \in \mathbb{N}$ in the following way

$$F_{k,0} = 0, F_{k,1} = 1, \;\; F_{k,n} = kF_{k,n-1} + F_{k,n-2} \text{ for } n \geq 2,$$
$$L_{k,0} = 2, L_{k,1} = k, \;\; L_{k,n} = kL_{k,n-1} + L_{k,n-2} \text{ for } n \geq 2.$$

Some interesting results for the split k-Fibonacci and split k-Lucas quaternions can be found in [6]. In [7] the authors studied split Pell quaternions SP_n and split Pell-Lucas quaternions SPL_n defined by

$$SP_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3},$$
$$SPL_n = PL_n + iPL_{n+1} + jPL_{n+2} + kPL_{n+3},$$

where P_n and PL_n is nth Pell and Pell-Lucas number, resp.

We will focus on split Horadam quaternions. We will present some identities for the split Horadam quaternions, which generalize the results for the split Fibonacci quaternions, the split Lucas quaternions, the split Pell quaternions and the split Pell-Lucas quaternions.

3 The split Horadam quaternions

For $n \geq 0$ define the split Horadam quaternion H_n by

$$H_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}, \tag{13}$$

where W_n is the nth Horadam number and i, j, k are split quaternionic units which satisfy the multiplication rules given by (1) and (2).

By (5) and (13) we obtain

$$\begin{aligned} H_0 &= a + bi + j(pb + qa) + k(p^2b + pqa + qb) \\ H_1 &= b + i(pb + qa) + j(p^2b + pqa + qb) + k(p^3b + p^2qa + 2pqb + q^2a) \\ H_2 &= pb + qa + i(p^2b + pqa + qb) + j(p^3b + p^2qa + 2pqb + q^2a) \\ &\quad + k(p^4b + p^3qa + 2pq(pb + qa) + p^2qb + q^2b). \end{aligned} \tag{14}$$

For any $n \geq 0$ we obtain the norm of H_n.

Proposition 1 *Let n, p, q be integers such that $n \geq 0$, $p^2 + 4q > 0$. Then*

$$\begin{aligned} N(H_n) = &(1 - q^2 - p^2q^2)W_n^2 + (1 - p^2 - (p^2 + q^2)^2)W_{n+1}^2 \\ &- 2pq(1 + p^2 + q)W_nW_{n+1}. \end{aligned}$$

Proof. Using formula (3) and (13), we get

$$\begin{aligned} N(H_n) &= W_n^2 + W_{n+1}^2 - W_{n+2}^2 - W_{n+3}^2 \\ &= W_n^2 + W_{n+1}^2 - (pW_{n+1} + qW_n)^2 - \Big((p^2 + q)W_{n+1} + pqW_n\Big)^2 \\ &= W_n^2 + W_{n+1}^2 - (p^2W_{n+1}^2 + 2pqW_nW_{n+1} + q^2W_n^2) \\ &\quad - ((p^2 + q)^2W_{n+1}^2 + 2pq(p^2 + q)W_nW_{n+1} + p^2q^2W_n^2). \end{aligned}$$

By simple calculations we get the result. □

By (13) we get a recurrence relation for the split Horadam quaternions.

Proposition 2 *Let n, p, q be integers such that $n \geq 2$, $p^2 + 4q > 0$. Then*

$$H_n = pH_{n-1} + qH_{n-2},$$

where H_0, H_1 are given by (14).

Proof. By formula (13) and (5) we get

$$\begin{aligned}
pH_{n-1} + qH_{n-2} &= p(W_{n-1} + iW_n + jW_{n+1} + kW_{n+2}) \\
&\quad + q(W_{n-2} + iW_{n-1} + jW_n + kW_{n+1}) \\
&= pW_{n-1} + qW_{n-2} + i(pW_n + qW_{n-1}) \\
&\quad + j(pW_{n+1} + qW_n) + k(pW_{n+2} + qW_{n+1}) \\
&= W_n + iW_{n+1} + jW_{n+2} + kW_{n+3} = H_n,
\end{aligned}$$

which ends the proof. □

Theorem 2 *Let* n, p, q *be integers such that* $n \geq 0$, $p^2 + 4q > 0$. *Then*

(i) $H_n + \overline{H_n} = 2W_n$,

(ii) $N(H_n) = 2W_n H_n - H_n^2$.

Proof. (i) Using the definition of the conjugate of a split quaternion we obtain the result.
(ii) By formula (13) we have

$$\begin{aligned}
H_n^2 &= W_n^2 - W_{n+1}^2 + W_{n+2}^2 + W_{n+3}^2 \\
&\quad + 2iW_nW_{n+1} + 2jW_nW_{n+2} + 2kW_nW_{n+3} \\
&= -W_n^2 - W_{n+1}^2 + W_{n+2}^2 + W_{n+3}^2 \\
&\quad + 2(W_n^2 + iW_nW_{n+1} + jW_nW_{n+2} + kW_nW_{n+3}) \\
&= 2W_n(W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}) \\
&\quad - W_n^2 - W_{n+1}^2 + W_{n+2}^2 + W_{n+3}^2 \\
&= 2W_nH_n - N(H_n).
\end{aligned}$$

Hence we get the result. □

The next theorem presents the Binet's formula for the split Horadam quaternions.

Theorem 3 (*Binet's formula*) *Let* n, p, q *be integers such that* $n \geq 0$, $p^2 + 4q > 0$. *Then*

$$H_n = \alpha \hat{r}_1 r_1^n - \beta \hat{r}_2 r_2^n, \tag{15}$$

where r_1, r_2, α, β *are given by* (6), (10), *resp. and* $\hat{r}_1 = 1 + ir_1 + jr_1^2 + kr_1^3$, $\hat{r}_2 = 1 + ir_2 + jr_2^2 + kr_2^3$.

Proof. By (11) we have

$$\begin{aligned}
H_n &= W_n + iW_{n+1} + jW_{n+2} + kW_{n+3} \\
&= \alpha r_1^n - \beta r_2^n + i(\alpha r_1^{n+1} - \beta r_2^{n+1}) + j(\alpha r_1^{n+2} - \beta r_2^{n+2}) \\
&\quad + k(\alpha r_1^{n+3} - \beta r^{n+3}) \\
&= \alpha r_1^n \left(1 + ir_1 + jr_1^2 + kr_1^3\right) - \beta r_2^n \left(1 + ir_2 + jr_2^2 + kr_2^3\right) \\
&= \alpha \hat{r}_1 r_1^n - \beta \hat{r}_2 r_2^n.
\end{aligned}$$

□

Using the Binet's formula (15), we can obtain some new identities for the split Horadam quaternions. We will use the following lemma.

Lemma 1 *Let* $\hat{r}_1 = 1 + ir_1 + jr_1^2 + kr_1^3$, $\hat{r}_2 = 1 + ir_2 + jr_2^2 + kr_2^3$, *where* r_1, r_2 *are given by* (6). *Then*

$$\begin{aligned}
\hat{r}_1\hat{r}_2 =& 1 + q + q^2 - q^3 + i(p + q^2\sqrt{p^2+4q}) \\
&+ j(p^2 + 2q - pq\sqrt{p^2+4q}) + k(p^3 + 3pq + q\sqrt{p^2+4q}), \qquad (16)\\
\hat{r}_2\hat{r}_1 =& 1 + q + q^2 - q^3 + i(p - q^2\sqrt{p^2+4q}) \\
&+ j(p^2 + 2q + pq\sqrt{p^2+4q}) + k(p^3 + 3pq - q\sqrt{p^2+4q}). \qquad (17)
\end{aligned}$$

Proof. Using formula (4), we have

$$\begin{aligned}
\hat{r}_1\hat{r}_2 &= 1 - r_1r_2 + (r_1r_2)^2 + (r_1r_2)^3 + i(r_1 + r_2 + (r_1r_2)^2(r_1 - r_2)) \\
&\quad + j(r_1^2 + r_2^2 + r_1r_2(r_1^2 - r_2^2)) + k(r_1^3 + r_2^3 - r_1r_2(r_1 - r_2)), \\
\hat{r}_2\hat{r}_1 &= 1 - r_1r_2 + (r_1r_2)^2 + (r_1r_2)^3 + i(r_1 + r_2 - (r_1r_2)^2(r_1 - r_2)) \\
&\quad + j(r_1^2 + r_2^2 - r_1r_2(r_1^2 - r_2^2)) + k(r_1^3 + r_2^3 + r_1r_2(r_1 - r_2)).
\end{aligned}$$

By (7) and (9) we get

$$\begin{aligned}
r_1^2 + r_2^2 &= (r_1 + r_2)^2 - 2r_1r_2 = p^2 + 2q, \\
r_1^3 + r_2^3 &= (r_1 + r_2)^3 - 3r_1r_2(r_1 + r_2) = p^3 + 3pq.
\end{aligned}$$

Hence

$$\begin{aligned}
\hat{r}_1\hat{r}_2 &= 1 + q + q^2 - q^3 + i(p + q^2\sqrt{p^2+4q}) \\
&\quad + j(p^2 + 2q - pq\sqrt{p^2+4q}) + k(p^3 + 3pq + q\sqrt{p^2+4q}),
\end{aligned}$$

$$\hat{r}_2\hat{r}_1 = 1 + q + q^2 - q^3 + i(p - q^2\sqrt{p^2+4q}) \\ + j(p^2 + 2q + pq\sqrt{p^2+4q}) + k(p^3 + 3pq - q\sqrt{p^2+4q}).$$

□

Corollary 1

$$\hat{r}_1\hat{r}_2 + \hat{r}_2\hat{r}_1 = 2(1 + q + q^2 - q^3 + pi + j(p^2 + 2q) + k(p^3 + 3pq)). \tag{18}$$

Theorem 4 (*Catalan's identity*) *Let* n, m, p, q *be integers such that* $n \geq m$, $p^2 + 4q > 0$. *Then*

$$H_{n-m}H_{n+m} - H_n^2 = \alpha\beta(-q)^{n-m}[(-q)^m(\hat{r}_1\hat{r}_2 + \hat{r}_2\hat{r}_1) - r_2^{2m}\hat{r}_1\hat{r}_2 - r_1^{2m}\hat{r}_2\hat{r}_1],$$

where α, β, $\hat{r}_1\hat{r}_2 + \hat{r}_2\hat{r}_1$, $\hat{r}_1\hat{r}_2$, $\hat{r}_2\hat{r}_1$ *are given by* (10), (18), (16), (17), *resp.*

Proof. By (15) we get

$$\begin{aligned} H_{n-m}H_{n+m} - H_n^2 =& (\alpha\hat{r}_1 r_1^{n-m} - \beta\hat{r}_2 r_2^{n-m})(\alpha\hat{r}_1 r_1^{n+m} - \beta\hat{r}_2 r_2^{n+m}) \\ & - (\alpha\hat{r}_1 r_1^n - \beta\hat{r}_2 r_2^n)(\alpha\hat{r}_1 r_1^n - \beta\hat{r}_2 r_2^n) \\ =& \alpha\beta(r_1 r_2)^{n-m}[(r_1 r_2)^m(\hat{r}_1\hat{r}_2 + \hat{r}_2\hat{r}_1) \\ & - r_2^{2m}\hat{r}_1\hat{r}_2 - r_1^{2m}\hat{r}_2\hat{r}_1]. \end{aligned}$$

Using formula (9), we obtain

$$H_{n-m}H_{n+m} - H_n^2 = \alpha\beta(-q)^{n-m}\left((-q)^m(\hat{r}_1\hat{r}_2 + \hat{r}_2\hat{r}_1) - r_2^{2m}\hat{r}_1\hat{r}_2 - r_1^{2m}\hat{r}_2\hat{r}_1\right).$$

□

Corollary 2 (*Cassini's identity*) *Let* n, p, q *be integers such that* $n \geq 0$, $p^2 + 4q > 0$. *Then*

$$H_{n-1}H_{n+1} - H_n^2 = -\alpha\beta(-q)^{n-1}\left(q(\hat{r}_1\hat{r}_2 + \hat{r}_2\hat{r}_1) + r_2^2\hat{r}_1\hat{r}_2 + r_1^2\hat{r}_2\hat{r}_1\right).$$

Note that for $p = q = 1$ we get the Cassini's identity for the split Fibonacci quaternions Q_n and the split Lucas quaternions T_n ([1]).

Corollary 3 *Let* $n \geq 1$ *be an integer. Then*

(i) $Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n(2Q_1 - 2i - 3k)$,

(ii) $T_{n-1}T_{n+1} - T_n^2 = 5(-1)^{n+1}(2Q_1 - 2i - 3k)$.

Proof. (i) Using Lemma 1, for $p = q = 1$ we get

$$\begin{aligned}
\widehat{r_1}\widehat{r_2} &= 2 + (1+\sqrt{5})i + (3-\sqrt{5})j + (4+\sqrt{5})k,\\
\widehat{r_2}\widehat{r_1} &= 2 + (1-\sqrt{5})i + (3+\sqrt{5})j + (4-\sqrt{5})k,\\
\widehat{r_1}\widehat{r_2} + \widehat{r_2}\widehat{r_1} &= 4 + 2i + 6j + 8k.
\end{aligned}$$

Hence and by Corollary 2 we have

$$\begin{aligned}
Q_{n-1}Q_{n+1} - Q_n^2 = & -\frac{1}{5}(-1)^{n-1}[4 + 2i + 6j + 8k\\
& + \frac{3-\sqrt{5}}{2}(2 + (1+\sqrt{5})i + (3-\sqrt{5})j + (4+\sqrt{5})k)\\
& + \frac{3+\sqrt{5}}{2}(2 + (1-\sqrt{5})i + (3+\sqrt{5})j + (4-\sqrt{5})k)]\\
= & \ (-1)^n(2 + 4j + 3k) = (-1)^n(2Q_1 - 2i - 3k).
\end{aligned}$$

We omit the proof of (ii). □

Proposition 3 *Let n, p, q be integers such that $n \geq 0$, $p^2 + 4q > 0$. Then*

$$H_{n+1}H_{n-1} - H_n^2 = -\alpha\beta(-q)^{n-1}\left(q(\widehat{r_1}\widehat{r_2} + \widehat{r_2}\widehat{r_1}) + r_1^2\widehat{r_1}\widehat{r_2} + r_2^2\widehat{r_2}\widehat{r_1}\right).$$

For $p = 2$ and $q = 1$ we get the Cassini's identity for the split Pell quaternions SP_n and the split Pell-Lucas quaternions SPL_n ([7]).

Corollary 4 *Let $n \geq 1$ be an integer. Then*

$$\begin{aligned}
SP_{n+1}SP_{n-1} - SP_n^2 &= (-1)^n(2 + 4i + 2j + 16k),\\
SPL_{n+1}SPL_{n-1} - SPL_n^2 &= (-1)^{n-1}(4 + 8i + 4j + 32k).
\end{aligned}$$

Theorem 5 (*d'Ocagne's identity*) *Let m, n, p, q be integers such that $n \geq 0$, $p^2 + 4q > 0$. Then*

$$H_nH_{m+1} - H_{n+1}H_m = \frac{(-q)^m(b - ar_2)(b - ar_1)}{r_1 - r_2}\left(r_1^{n-m}\widehat{r_1}\widehat{r_2} - r_2^{n-m}\widehat{r_2}\widehat{r_1}\right),$$

where $\widehat{r_1}\widehat{r_2}$, $\widehat{r_2}\widehat{r_1}$ are given by (16), (17), resp.

Proof. By (15) we get

$$\begin{aligned}
H_nH_{m+1} - H_{n+1}H_m &= (\alpha\hat{r}_1 r_1^n - \beta\hat{r}_2 r_2^n)(\alpha\hat{r}_1 r_1^{m+1} - \beta\hat{r}_2 r_2^{m+1}) \\
&\quad - (\alpha\hat{r}_1 r_1^{n+1} - \beta\hat{r}_2 r_2^{n+1})(\alpha\hat{r}_1 r_1^m - \beta\hat{r}_2 r_2^m) \\
&= \alpha\beta(r_1 - r_2)\,(r_1^n r_2^m \hat{r}_1\hat{r}_2 - r_1^m r_2^n \hat{r}_2\hat{r}_1) \\
&= \alpha\beta(r_1 - r_2)(r_1 r_2)^m\left(r_1^{n-m}\hat{r}_1\hat{r}_2 - r_2^{n-m}\hat{r}_2\hat{r}_1\right) \\
&= \frac{(b - ar_2)(b - ar_1)(-q)^m}{r_1 - r_2}\left(r_1^{n-m}\hat{r}_1\hat{r}_2 - r_2^{n-m}\hat{r}_2\hat{r}_1\right).
\end{aligned}$$

□

In the next theorem we give a summation formula for the split Horadam quaternions.

Theorem 6 *Let* n, p, q *be integers such that* $n \geq 0$, $p^2 + 4q > 0$. *Then*

$$\begin{aligned}
\sum_{l=0}^{n} H_l \quad &= \frac{H_{n+1} + qH_n + (ap - a - b)(1 + i + j + k)}{p + q - 1} \\
&\quad -ia - j(a + b) - k(a + b + pb + qa).
\end{aligned}$$

Proof. By formula (12) we get

$$\sum_{l=0}^{n} H_l = \sum_{l=0}^{n} H_l + i\sum_{l=0}^{n} H_{l+1} + j\sum_{l=0}^{n} H_{l+2} + k\sum_{l=0}^{n} H_{l+3}$$

$$\begin{aligned}
= \frac{1}{p + q - 1}&[W_{n+1} + qW_n + a(p - 1) - b + i(W_{n+2} + qW_{n+1} + a(p - 1) - b) \\
&+ j(W_{n+3} + qW_{n+2} + a(p - 1) - b) + k(W_{n+4} + qW_{n+3} + a(p - 1) - b)] \\
&- iW_0 - j(W_0 + W_1) - k(W_0 + W_1 + W_2).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\sum_{l=0}^{n} H_l = \frac{1}{p + q - 1}&[W_{n+1} + iW_{n+2} + jW_{n+3} + kW_{n+4} \\
&+ q(W_n + iW_{n+1} + jW_{n+2} + kW_{n+3} + (ap - a - b)(1 + i + j + k)] \\
&- ia - j(a + b) - k(a + b + pb + qa) \\
= &\frac{H_{n+1} + qH_n + (ap - a - b)(1 + i + j + k)}{p + q - 1} \\
&- ia - j(a + b) - k(a + b + pb + qa).
\end{aligned}$$

□

For $p = q = 1$ and $a = 0, b = 1$ we get the result for the split Fibonacci quaternions Q_n ([1]).

Corollary 5 $\sum_{l=1}^{n} Q_l = Q_{n+2} - Q_2.$

Now we will give the generating function of the split Horadam quaternions.

Theorem 7 *The generating function of the split Horadam quaternions is*

$$f(x) = \frac{H_0 + (H_1 - pH_0)x}{1 - px - qx^2}.$$

Proof. Let $f(x) = H_0 + H_1x + H_2x^2 + \ldots + H_nx^n + \ldots$. Then

$$pxf(x) = pH_0x + pH_1x^2 + pH_2x^3 + \ldots + pH_{n-1}x^n + \ldots$$
$$qx^2f(x) = qH_0x^2 + qH_1x^3 + qH_2x^4 + \ldots + qH_{n-2}x^n + \ldots.$$

Hence, by Proposition 2, we get

$$\begin{aligned} &f(x) - pxf(x) - qx^2f(x) \\ &= H_0 + (H_1 - pH_0)x + (H_2 - pH_1 - qH_0)x^2 + \ldots \\ &= H_0 + (H_1 - pH_0)x. \end{aligned}$$

Thus

$$f(x) = \frac{H_0 + (H_1 - pH_0)x}{1 - px - qx^2}.$$

Moreover, by (14) we obtain

$$H_0 = a + ib + j(pb + qa) + k(p^2b + pqa + qb),$$
$$H_1 - pH_0 = b - pa + iqa + jqb + k(pqb + q^2a).$$

□

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On topological properties of the set of maldistributed sequences

József Bukor
Department of Informatics,
J. Selye University,
Komárno, Slovakia
email: bukorj@ujs.sk

János T. Tóth
Department of Mathematics,
J. Selye University,
Komárno, Slovakia
email: tothj@ujs.sk

Abstract. The real sequence (x_n) is maldistributed if for any non-empty interval I, the set $\{n \in \mathbb{N} : x_n \in I\}$ has upper asymptotic density 1. The main result of this note is that the set of all maldistributed real sequences is a residual set in the set of all real sequences (i.e., the maldistribution is a typical property in the sense of Baire categories). We also generalize this result.

1 Introduction

Following the concept of statistical convergence for real sequences, J. A. Fridy [2] introduced the concept of statistical cluster points of a sequence (x_n). A number α is called a statistical cluster point of the sequence (x_n) provided that for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |x_n - \alpha| < \varepsilon\}$ has a positive upper asymptotic density.

G. Myerson [7] calls a sequence (x_n) maldistributed if for any non-empty interval I the set $\{n \in \mathbb{N} : x_n \in I\}$ has upper asymptotic density 1. In [12] the maldistribution property is characterized by one-jump distribution functions. Examples of maldistributed sequences are given in [12] and [3]. Using the idea from [4] (Example VII) for the generalization of the concept of statistical

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convergence, we can extend the maldistribution property of sequences with the help of weighted densities.

The concept of weighted density as a generalization of asymptotic density was introduced in [1] and [10]. Let $f:\mathbb{N}\to(0,\infty)$ be a weight function with the properties

$$\sum_{n=1}^{\infty} f(n)=\infty, \qquad \lim_{n\to\infty}\frac{f(n)}{\sum\limits_{a\le n} f(a)}=0. \tag{1}$$

For $A\subset\mathbb{N}$ define by

$$\underline{d}_f(A)=\liminf_{n\to\infty}\frac{\sum\limits_{a\le n,\ a\in A} f(a)}{\sum\limits_{a\le n} f(a)} \quad\text{and}\quad \overline{d}_f(A)=\limsup_{n\to\infty}\frac{\sum\limits_{a\le n,\ a\in A} f(a)}{\sum\limits_{a\le n} f(a)}$$

the lower and upper f-densities of A, respectively. Note that the asymptotic densities correspond to $f(n)=1$ and the logarithmic densities to $f(n)=\frac{1}{n}$. It is well-known that each set which has asymptotic density also has the logarithmic one but a set may have a logarithmic density without having an asymptotic one.

The main tool to compare weighted densities is the classical result of C. T. Rajagopal (cf. [9], Theorem 3) which, in terms of weighted densities, says the following.
Let $f,g:\mathbb{N}\to(0,\infty)$ be weight functions with properties (1). If $\frac{f(n)}{g(n)}$ is decreasing, then for any $A\subset\mathbb{N}$ we have

$$\underline{d}_g(A)\le\underline{d}_f(A)\le\overline{d}_f(A)\le\overline{d}_g(A). \tag{2}$$

Now we give a generalization of maldistributed sequences.

Definition 1 *Let $f:\mathbb{N}\to(0,\infty)$ be a weight function with properties (1). The sequence (x_n) is said to be f-maldistributed, if for any non-empty interval I the set $\{n\in\mathbb{N}:x_n\in I\}$ has upper f-density 1.*

Comparing to asymptotic density, logarithmic density is less sensitive to certain perturbations. For example, if a sequence is maldistributed, then it is not necessary f-maldistributed for $f(n)=\frac{1}{n}$ (which defines the logarithmic density).

Let us denote by $\mathcal{M}_f$ the set of all f-maldistributed sequences. The purpose of this note is to show that for any weight function f satisfying (1) the set $\mathcal{M}_f$ is residual in the Fréchet metric space of all real sequences.

Let $\mathbf{s}$ be the Fréchet metric space of all sequences of real numbers with the metric

$$\rho(\mathbf{x},\mathbf{y}) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|},$$

where $\mathbf{x} = (x_k)$, $\mathbf{y} = (y_k)$. It is known that $(\mathbf{s}, \rho)$ is a complete metric space.

In [5] it was proved that the set of all uniformly distributed sequences is a dense subset of the first Baire category in $\mathbf{s}$. The same is true for the set of all statistically convergent sequences of real numbers (cf. [11]).

2 Main results

The main result of this paper is as follows.

Theorem 1 *Let* $f : \mathbb{N} \to (0, \infty)$ *be a weight function with properties* (1)*. Then the set of all* f*-maildistributed sequences* $\mathcal{M}_f$ *is residual in the the Fréchet metric space of all sequences of real numbers* $\mathbf{s}$*.*

For the proof of the theorem we shall use the following lemma.

Lemma 1 *For the interval* $I = [a, b]$ *denote by* $\mathcal{A}(I, \alpha)$ *the set of all* $\mathbf{x} = (x_k) \in \mathbf{s}$ *for which*

$$\overline{d}_f\big(\{n \in \mathbb{N} : x_n \in I\}\big) \leq \alpha\,,$$

where $\alpha \in (0, 1)$*. Then* $\mathcal{A}(I, \alpha)$ *is a set of the first Baire category in* $\mathbf{s}$*.*

Proof of Lemma 1. We define a continuous function $h : \mathbb{R} \to [0, 1]$ by

$$h(x) = \begin{cases} \frac{2x-2a}{b-a} & \text{for} \quad x \in \left[a, \frac{a+b}{2}\right] \\ \frac{2b-2x}{b-a} & \text{for} \quad x \in \left[\frac{a+b}{2}, b\right] \\ 0 & \text{for} \quad x \in \mathbb{R} \setminus [a, b] \end{cases}$$

We choose an arbitrary real number $\beta \in (\alpha, 1)$. Using the function h we define for $\mathbf{x} = (x_k) \in \mathbf{s}$ and fixed n the function $g_n : \mathbf{s} \to [0, 1]$ in the following way:

$$g_n(\mathbf{x}) = \max\left\{ \beta, \ \frac{\sum\limits_{k=1}^{n} h(x_k).f(k)}{\sum\limits_{k=1}^{n} f(k)} \right\}.$$

Denote $\mathcal{A}^*(I,\alpha)$ the set of all $\mathbf{x} = (x_k) \in \mathbf{s}$ for which there exists the limit $\lim\limits_{n\to\infty} g_n(\mathbf{x})$.
One can easily check that for each $\mathbf{x} = (x_k) \in \mathbf{s}$ and natural number n we have

$$\frac{\sum\limits_{k=1}^{n} h(x_k).f(k)}{\sum\limits_{k=1}^{n} f(k)} \le \frac{\sum\limits_{k\le n,\ x_k\in I} f(k)}{\sum\limits_{k\le n} f(k)}. \tag{3}$$

For any $\mathbf{x} \in \mathcal{A}(I,\alpha)$, the right hand side of (3) does not exceed α if n is large enough. Therefore $\lim\limits_{n\to\infty} g_n(\mathbf{x}) = \beta$, and then $\mathcal{A}(I,\alpha) \subset \mathcal{A}^*(I,\alpha)$.
Put $g(\mathbf{x}) = \lim\limits_{n\to\infty} g_n(\mathbf{x})$ for $\mathbf{x} \in \mathcal{A}^*(I,\alpha)$. We shall prove that
(a) the function g_n $(n = 1,2,\dots)$ is a continuous function on $\mathbf{s}$,
(b) g is discontinuous at each point of $\mathcal{A}^*(I,\alpha)$.

(a) Let $\mathbf{x}^0 = (x_k^0)_{k=1}^\infty$, $\mathbf{x}^{(j)} = (x_k^{(j)})_{k=1}^\infty \in \mathbf{s}$ $(j = 1,2,\dots)$ and $\mathbf{x}^{(j)} \to \mathbf{x}^0$ (for $j \to \infty$).
Then from the convergence in the space $\mathbf{s}$ for each fixed k we have $\lim\limits_{j\to\infty} x_k^{(j)} = x_k^0$. The continuity of function h implies $\lim\limits_{j\to\infty} g_n(\mathbf{x}^{(j)}) = g_n(\mathbf{x}^0)$. Thus g_n $(n = 1,2,\dots)$ is continuous on $\mathbf{s}$.

(b) Let $\mathbf{y} = (y_k) \in \mathcal{A}^*(I,\alpha)$. We have the following two possibilities.
(1) $g(\mathbf{y}) < 1$,
(2) $g(\mathbf{y}) = 1$.

In case (1) we choose a positive ε such that $\varepsilon < 1 - g(\mathbf{y})$. It is suffice to prove that in each ball $K(\mathbf{y},\delta) = \{\mathbf{x} \in \mathcal{A}^*(I,\alpha),\ \rho(\mathbf{x},\mathbf{y}) < \delta\}$ $(\delta > 0)$ of the subspace $\mathcal{A}^*(I,\alpha)$ of $\mathbf{s}$ there exists an element $\mathbf{x} = (x_k) \in \mathbf{s}$ with $g(\mathbf{x}) - g(\mathbf{y}) > \varepsilon$.

Let $\delta > 0$. Choose a positive integer m such that $\sum\limits_{k=m+1}^{\infty} 2^{-k} < \delta$, and define the sequence $\mathbf{x} = (x_k)$ in the following way:

$$x_k = \begin{cases} y_k, & \text{if } k \le m, \\ \frac{a+b}{2}, & \text{if } k > m. \end{cases}$$

Hence $\rho(\mathbf{x},\mathbf{y}) < \delta$, further $h(x_k) = 1$ for $k > m$. Then

$$\frac{\sum\limits_{k=1}^{n} h(x_k).f(k)}{\sum\limits_{k=1}^{n} f(k)} \ge \frac{\sum\limits_{k=m+1}^{n} f(k)}{\sum\limits_{k=1}^{n} f(k)} = 1 - \frac{\sum\limits_{k=1}^{m} f(k)}{\sum\limits_{k=1}^{n} f(k)} \to 1 \quad \text{for} \quad n \to \infty,$$

and therefore $g(\mathbf{x}) = \lim_{n\to\infty} g_n(\mathbf{x}) = 1$. Then immediately follows

$$g(\mathbf{x}) - g(\mathbf{y}) = 1 - g(\mathbf{y}) > \varepsilon.$$

In case (2) we have $g(\mathbf{y}) = 1$. Let $\delta, m, \mathbf{x}$ have the previous meaning. Put

$$x_k = \begin{cases} y_k, & \text{if } k \le m, \\ a, & \text{if } k > m. \end{cases}$$

Then, clearly $\rho(\mathbf{x}, \mathbf{y}) < \delta$, and $h(x_k) = 0$ for $k > m$. Then

$$\frac{\sum_{k=1}^{n} h(x_k).f(k)}{\sum_{k=1}^{n} f(k)} \le \frac{\sum_{k=1}^{m} f(k)}{\sum_{k=1}^{n} f(k)} \to 0 \quad \text{for} \quad n \to \infty.$$

So, we have $g(\mathbf{x}) = \lim_{n\to\infty} g_n(\mathbf{x}) = \beta$, and therefore $g(\mathbf{y}) - g(\mathbf{x}) = 1 - \beta > 0$. Hence the discontinuity of g at $\mathbf{y} \in \mathcal{A}^*(I, \alpha)$ has been proved.

The function g is a limit function (on $\mathcal{A}^*(I, \alpha)$) of the sequence of continuous functions $(g_n)_{n=1}^{\infty}$ on $\mathcal{A}^*(I, \alpha)$. Then the function g is a function in the first Baire class on $\mathcal{A}^*(I, \alpha)$. According to the well-known fact that the set of discontinuity points of an arbitrary function of the first Baire class is a set of the first Baire category (cf. [8], p. 32), we see that the set $\mathcal{A}^*(I, \alpha)$ is of the first Baire category in $\mathcal{A}^*(I, \alpha)$ Thus $\mathcal{A}^*(I, \alpha)$ is in $\mathbf{s}$, too. Since $\mathcal{A}(I, \alpha) \subset \mathcal{A}^*(I, \alpha)$, the assertion follows. □

Proof of Theorem 1. Denote by $\mathbb{Q}$ the set of all rational numbers. Denote by $\mathcal{H}$ the set of all $\mathbf{x} = (x_k) \in \mathbf{s}$ for which there exists an interval I with

$$\overline{d}_f\big(\{n \in \mathbb{N} : x_n \in I\}\big) \le \alpha$$

for some $\alpha \in (0, 1)$. Combining Lemma 1 and the fact that for each interval I there exist rational numbers a, b such that $I \subset [a, b]$, we have

$$\mathcal{H} \subset \bigcup_{a,b\in\mathbb{Q},\ a<b} \ \bigcup_{i\in\mathbb{N},\ i\ge 2} A\left([a, b], 1 - \frac{1}{i}\right)$$

from which follows at once that $\mathcal{H}$ is a meager set. But $\mathcal{M}_f = \mathbf{s} \setminus \mathcal{H}$ and therefore the assertion of theorem follows. Hence the property of f-maldistribution is a typical property of real sequences from the topological point of view. □

We now introduce the concept of f-maldistributed integer sequences.

Definition 2 *Let* $f : \mathbb{N} \to (0, \infty)$ *be a weight function with properties* (1). *The sequence* (x_n) *of positive integers is said to be* f*-maldistributed, if for any positive integers* $m \geq 2$ *and* $j \in \{0, 1, \dots, m-1\}$ *the set* $\{n \in \mathbb{N} : x_n \equiv j \pmod m\}$ *has upper* f*-density 1.*

Let $\mathbf{S}$ be the Baire's space of all sequences of positive integers with the metric ρ' defined in the following way.

Let $\mathbf{x} = (x_k) \in \mathbf{S}$, and $\mathbf{y} = (y_k) \in \mathbf{S}$. If $\mathbf{x} = \mathbf{y}$, then $\rho'(\mathbf{x}, \mathbf{y}) = 0$, otherwise

$$\rho'(\mathbf{x}, \mathbf{y}) = \frac{1}{\min\{n : x_n \neq y_n\}}.$$

The space $(\mathbf{S}, \rho')$ is a complete metric space. In [6] the topological properties of the set of all uniformly distributed sequences of positive integers in Baire's space were investigated.

The following auxilary result is similar to Lemma 1.

Lemma 2 *For the positive integers* $m \geq 2$ *and* $j \in \{0, 1, \dots, m-1\}$ *denote by* $\mathcal{A}(j, m, \alpha)$ *the set of all* $\mathbf{x} = (x_k) \in \mathbf{S}$ *for which*

$$\overline{d}_f\big(\{n \in \mathbb{N} : x_n \equiv j \pmod m\}\big) \leq \alpha\,,$$

where $\alpha \in (0, 1)$. *Then* $\mathcal{A}(j, m, \alpha)$ *is a set of the first Baire category in* $\mathbf{S}$.

The proof is analogous to the proof of Lemma 1. The crucial role is played by the function $g_n : \mathbf{S} \to [0, 1]$ given by

$$g_n(\mathbf{x}) = \max\left\{ \sqrt{\alpha},\ \frac{\sum\limits_{\substack{k \leq n \\ x_k \equiv j \pmod m}} f(k)}{\sum\limits_{k=1}^{n} f(k)} \right\}.$$

The following theorem says that the set of all f-maldistributed integer sequences form a residual set in Baire's space.

Theorem 2 *Let* $f : \mathbb{N} \to (0, \infty)$ *be a weight function with properties* (1). *Denote by* $\mathcal{G}$ *the set of all* $\mathbf{x} = (x_k) \in \mathbf{S}$ *for which there exist* $m \geq 2$ *and* $j \in \{0, 1, \dots, m-1\}$ *such that*

$$\overline{d}_f\big(\{n \in \mathbb{N} : x_n \equiv j \pmod m\}\big) \leq \alpha$$

for some $\alpha \in (0, 1)$. *Then* $\mathcal{G}$ *is a set of the first Baire category in* $\mathbf{S}$.

Proof. Combining Lemma 2 with the fact that

$$\mathcal{G} \subset \bigcup_{m=2}^{+\infty} \bigcup_{j=0}^{m-1} \bigcup_{i=2}^{+\infty} A\Big(j, m, 1 - \frac{1}{i}\Big)$$

it immediately follows that $\mathcal{G}$ is a meager set in **S**. □

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On the connection between tridiagonal matrices, Chebyshev polynomials, and Fibonacci numbers

Carlos M. da Fonseca
Kuwait College of Science and Technology,
Doha District, Block 4, P.O. Box 27235,
Safat 13133, Kuwait
email: c.dafonseca@kcst.edu.kw

Abstract. In this note, we recall several connections between the determinant of some tridiagonal matrices and the orthogonal polynomials allowing the relation between Chebyshev polynomials of second kind and Fibonacci numbers. With basic transformations, we are able to recover some recent results on this matter, bringing them into one place.

1 Orthogonal polynomials and tridiagonal matrices

From the elementary orthogonal polynomials theory, we know that any monic sequence orthogonal polynomial sequence $\{P_n(x)\}$ is defined by the recurrence formula

$$P_n(x) = (x - c_n)\, P_{n-1}(x) - \lambda_n\, P_{n-2}(x)\,, \quad \text{for } n = 1, 2, \ldots \tag{1}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, and for complex numbers c_n's and λ_n's, if and only if $\lambda_n \neq 0$ [6, Theorem 4.4]. Moreover, (1) is equivalent

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to

$$P_n(x) = \begin{vmatrix} x-c_1 & 1 & & & \\ \lambda_2 & x-c_2 & 1 & & \\ & \lambda_3 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \lambda_n & x-c_n \end{vmatrix} \tag{2}$$

(cf. [6, Exercise 4.12, p.26]).

Setting $\lambda_n = 1$ and $c_n = 0$, for any positive integer n, in (1) or (2), we obtain $U_n(x) = P_n(2x)$, the Chebyshev polynomial of second kind of degree n (cf. [1, (22.7.5)] or [6, Exercise 4.9, p.25]). The Chebyshev polynomials of second kind satisfy the three-term recurrence relations

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)\,, \quad \text{for all } n = 1, 2, \ldots\,, \tag{3}$$

or, equivalently, we may state

$$U_n(x) = \begin{vmatrix} 2x & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2x \end{vmatrix}_{n\times n} .$$

Among the most important explicit representations for the $U_n(x)$ we have

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}\,, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi), \tag{4}$$

as we can see, for example, in [1, (22.3.16)] or [6, Exercise 1.2, p.5]. From de Moivre's formula we have

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}. \tag{5}$$

Another formula with some relevance is

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k} \tag{6}$$

which can be found for example in [1, (22.3.7)]. There are many other representations and relations for $U_n(x)$. As stated in [8, p.187], many of them are paraphrases of trigonometric identities, derivations from (4). For particular values of $U_n(x)$ the reader is referred to [1, (22.4.5)].

Regarding the generating function for $U_n(x)$, we have

$$\frac{1}{1-2zx+z^2} = \sum_{n=0}^{\infty} U_n(x)\, z^n \tag{7}$$

(cf. [1, (22.9.10)]).

From (3), we can easily deduce that

$$(\sqrt{ab})^n U_n\left(\frac{c}{2\sqrt{ab}}\right) = \begin{vmatrix} c & a & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & a \\ & & b & c \end{vmatrix}_{n\times n} . \tag{8}$$

provided $ab \neq 0$. Perhaps one of the most interesting applications of this identity is

$$F_n = (-\mathrm{i})^{n-1}\, U_{n-1}\left(\frac{\mathrm{i}}{2}\right),$$

where F_n is the nth Fibonacci number (cf. for example [2, 5, 12]), setting $a = c = 1$ and $b = -1$.

Many of these results were recently recast. In this short note, we aim to bring them into attention and put into one place. At the same time, we provide shorter proofs to many of them.

2 Determinants of tridiagonal matrices

In [14], the authors guessed that the determinant $D_n(c)$ in (8), for $a = b = 1$, satisfies

$$D_n(c) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} c^{n-2k}.$$

This is repeatedly proved in [13] and in [16], while indeed it is an immediate consequence of (6). For that propose the authors in [13] proved (7) for $D_n(c)$.

Next, it is proved in [13] that

$$D_n(c) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta},$$

where $\alpha = \frac{1}{\beta} = \frac{c+\sqrt{c^2-4}}{2}$, and the trivial cases are ignored. In [16], three alternative proofs are provided to this fact. Nonetheless, this equality is immediate from (5), if one notices that $\beta = \frac{c-\sqrt{c^2-4}}{2}$. A particular case of (8) is also claimed to be proved by induction when $c = a + b$, using (5).

Another topic discussed in [13] is the inverse of

$$A = \begin{pmatrix} c & a & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & a \\ & & b & c \end{pmatrix}_{n\times n},$$

for $a = b = 1$. We remark that this matrix is occasionally called in [13] "diagonal matrix", but this is obviously not appropriate. It is well-known that if A is nonsingular, i.e., $U_n(d) \neq 0$, with $d = c/(2\sqrt{ab})$, then its inverse is given by

$$(A^{-1})_{ij} = \begin{cases} (-1)^{i+j} \dfrac{a^{j-i}}{\left(\sqrt{ab}\right)^{j-i+1}} \dfrac{U_{i-1}(d)\, U_{n-j}(d)}{U_n(d)} & \text{if } i \leq j \\ (-1)^{i+j} \dfrac{b^{i-j}}{\left(\sqrt{ab}\right)^{i-j+1}} \dfrac{U_{j-1}(d)\, U_{n-i}(d)}{U_n(d)} & \text{if } i > j\,. \end{cases}$$

This can be found in [10, 19, 20, 21] or easily deduced from [11, p.28]. Yet, in [13] we can see this inverse for the particular case of $a = b = 1$. Moreover, in [16] the authors provide a detailed proof for it, while this particular case has been studied for the last 75 years (cf. [9, 17, 18]).

We also want to remark that the eigenpairs of A in [13] are wrong. Indeed, it is a standard result that, for example, the eigenvalues of A are

$$\lambda_k = c + 2\sqrt{ab}\cos\left(\frac{\ell\pi}{n+1}\right), \quad \text{for } \ell = 1, \dots, n\,.$$

On contrary to what is suggested in [13], the real entries of the tridiagonal matrix A are absolutely irrelevant for this formula.

Finally, [13] provides an intricate proof for the equality

$$\begin{vmatrix} -c & 1 & & & \\ 2 & -2c & 1 & & \\ & 6 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & n(n-1) & -nc \end{vmatrix} = (-1)^n n! \begin{vmatrix} c & 1 & & & \\ 1 & c & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & c \end{vmatrix}. \qquad (9)$$

In [16], among others, the authors claim that (9) is "neither trivial nor obvious". Next, using elementary matrix theory, we show that (9) is both trivial and obvious (see also [3]). In fact,

$$\begin{vmatrix} -c & 1 & & & \\ 2 & -2c & 1 & & \\ & 6 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & n(n-1) & -nc \end{vmatrix} = (-1)^n \begin{vmatrix} c & 1 & & & \\ 2 & 2c & 2 & & \\ & 3 & \ddots & \ddots & \\ & & \ddots & \ddots & n-1 \\ & & & n & nc \end{vmatrix}$$
$$= (-1)^n n! \begin{vmatrix} c & 1 & & & \\ 1 & c & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & c \end{vmatrix}.$$

We observe that the same identity can be found and proved using an involved approach in [15].

3 Central Delannoy numbers

Our final comment goes to the last section of the paper [13].

The Legendre polynomials $P_n(x)$ are a particular family of the ultraspherical polynomials [4, p.899]. Their generating function is

$$\frac{1}{\sqrt{1-2tx+t^2}}$$

and they can be defined, for example, by the contour integral

$$P_n(x) = \frac{1}{2\pi i} \oint \frac{1}{\sqrt{1-2xt+t^2}} \frac{1}{t^{n+1}}\, dt$$

where the contour encloses the origin and is traversed in a counterclockwise direction [4, p.416]. The central Delannoy numbers $D(n)$ are defined as ([7, p.81])

$$D(n) = P_n(3)\,.$$

The authors recover a generalization for the central Delannoy numbers, $D_{a,b}(n)$, with generating function

$$\frac{1}{\sqrt{(x+a)(x+b)}} = \sum_{k=0}^{\infty} D_{a,b}(k)\, x^k\,.$$

However, next they claim that squaring both sides of the previous identity one gets

$$\frac{1}{(x+a)(x+b)} = \sum_{k=0}^{\infty} D_{a,b}(k)\, x^k ,$$

which is obviously not true.

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A note on nil-clean rings

Peter V. Danchev
Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
"Acad. G. Bonchev" str., bl. 8, 1113 Sofia, Bulgaria
email: danchev@math.bas.bg;
pvdanchev@yahoo.com

Abstract. We study a special kind of nil-clean rings, namely those nil-clean rings whose nilpotent elements are difference of two "left-right symmetric" idempotents, and prove that in some various cases they are strongly π-regular. We also show that all nil-clean rings having cyclic unit 2-groups are themselves strongly nil-clean of characteristic 2 (and thus they are again strongly π-regular).

1 Introduction and background

Everywhere in the text of the present paper, all our rings R are assumed to be associative, containing the identity element 1, which in general differs from the zero element 0 of R. Our terminology and notations are mainly standard being in agreement with [9]. Exactly, $U(R)$ denotes the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R and $J(R)$ the Jacobson radical of R.

A ring R is called *von Neumann regular* or just *regular* for short if, for any element $r \in R$, there is an element $a \in R$ such that $r = rar$. In the case when $a = 1$, we have that $r = r^2$ and these rings are known to be *boolean*. Generalizing regularity, a ring R is called *π-regular* if, for each $r \in R$, there are $i \in \mathbb{N}$ and $b \in R$ both depending on r such that $r^i = r^i b r^i$. Likewise, a ring R is called *strongly π-regular* if, for every $r \in R$, there exist $j \in \mathbb{N}$ and

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$c \in R$ both depending on r with $r^j = r^{j+1}c$. It is well known that strongly π-regularity implies π-regularity, while the converse is wrong as some critical examples show (see, e.g., [9]).

On the other hand, referring to [7] for a more account, we shall say that a ring is *nil-clean* provided each its element is a sum of a nilpotent and an idempotent. If these two elements commute, the nil-clean ring is said to be *strongly nil-clean*. While nil-clean rings are not completely characterized up to an isomorphism yet, this was successfully done in [4] by proving that a ring R is strongly nil-clean if, and only if, the quotient ring $R/J(R)$ is boolean and $J(R)$ is nil.

That is why, classifying the structure of some special types of nil-clean rings will be of some interest and importance. Our workable purpose here is to examine those nil-clean rings whose nilpotents are differences of two (special) idempotents. Specifically, we shall prove that in Theorem 1 presented below that every nil-clean ring having only nilpotents which are difference of two special (so-called "left-right symmetric") idempotents is strongly π-regular. This contrasts an example due to Šter in [11] who constructed a nil-clean ring of unbounded index of nilpotence which is not strongly π-regular. Note that by an appeal to [6, Corollary 3.12] nil-clean rings of bounded index of nilpotence are always strongly π-regular. We also consider the challenging question of when a nil-clean ring with finite (in particular, cyclic) unit group is strongly nil-clean. It is necessarily such a group to be consisting only of elements of order being a power of 2, and the ring will be of characteristic 2 too.

2 Main results

We separate our chief results into two subsections as follows:

2.1 Nil-clean rings with nilpotents as a sum of two special idempotents

We start our assertions with the next one.

Proposition 1 *If* R *is a nil-clean ring such that each nilpotent is a difference of two commuting idempotents, then* R *is a boolean ring.*

Proof. We first claim that such a ring R is of characteristic 2. Indeed, as $2 \in \mathrm{Nil}(R)$ (see, e.g., [7]), one writes that $2 = e - f$ for some $e, f \in \mathrm{Id}(R)$. Hence, it easily follows that $ef = fe$ even not assuming this a priory and, therefore, $2^3 = (e-f)^3 = e-f = 2$. This means that $6 = 0$, i.e., $2 = 0$ because $3 \in U(R)$ and the claim is sustained.

Moreover, we assert that R has to be abelian, that is, all its idempotents are central. In fact, given an arbitrary $a \in R$ and an arbitrary $e \in Id(R)$, one sees that $ea(1-e) \in Nil(R)$ and thus $ea(1-e) = e_1 + e_2$ for some $e_1, e_2 \in Id(R)$ with $e_1e_2 = e_2e_1$. Squaring this, it follows at once that $0 = e_1 + e_2$ since $2 = 0$ which yields $ea = eae$. Similarly, one derives that $ae = eae$ by looking at the element $(1-e)ae$, which allows us to conclude that $ae = ea$, as asserted.

We next arrive at the fact that R is semi-primitive, which is equivalent to $J(R) = \{0\}$. To verify this, given any element $z \in J(R)$, one may write that $z = e-f$ for some $e, f \in Id(R)$ with $ef = fe$ since $J(R)$ is nil (see, for instance, [7]). Now, taking into account that $2 = 0$, we find that $z^2 = z$ whence $z(z-1) = 0$ ensuring that $z = 0$ because $z-1 \in U(R)$. Thus R is semi-primitive, as claimed.

Furthermore, we may apply either [4] or [7] to get the desired boolean property of R. $\square$

It was established in [8, Proposition 1] that any nilpotent matrix over a field is a difference of two idempotent matrices (for another approach see [10] as well). This major statement allows us to extract the following assertion, independently proved also in [10] and partially in [3].

Lemma 1 *In regular rings all nilpotent elements are difference of two idempotents.*

Proof. Consulting with the main result from [1] which shows that, in an arbitrary ring, a nilpotent with all powers regular can be thought of as locally just a nilpotent matrix in Jordan or Weyr form. With this at hand, the aforementioned matrix result in [8] gives the desired presentation. $\square$

Imitating [3], two idempotents e, f are called *left-right symmetric* if the two equalities $ef = e$ and $fe = f$ hold. It is evident that both e and f are somewhat "left-active" in the sense that they are "preserved on the left multiplication".

So, we have accumulated all the information necessary to establish the following.

Theorem 1 *Every nil-clean ring in which all nilpotents are difference of two left-right symmetric idempotents are strongly π-regular.*

Proof. We foremost assert that for such a ring R it must be that $\mathrm{char}(R) = 2$. To see that, as $2 \in Nil(R)$ holds in view of [7], one writes that $2 = e_1 - e_2$ for two $e_1, e_2 \in Id(R)$. This surely means that e_1 and e_2 do commute, so that

$2^3 = (e_1 - e_2)^3 = e_1 - e_2 = 2$ whence $6 = 0$. Consequently, $2 = 0$ because $3 \in U(R)$, as asserted.

For such a ring R, given an arbitrary $q \in \mathrm{Nil}(R)$, we write that $q = e - f = e + f$ for some two $e, f \in \mathrm{Id}(R)$ with $ef = e$ and $fe = f$. We, therefore, obtain by squaring that $q^2 = 2q = 0$. Thus R is of bounded index of nilpotence and [6, Corollary 3.12] is a guarantor for the validity of our assertion that R is strongly π-regular. □

The given proof allows us to consider whether a more general situation in which we have slightly amended relationships between e and f, that are, $efe = e$ and $fef = f$. Certainly, $ef = e$ forces $efe = e$ as well as $fe = f$ forces $fef = f$. Furthermore, writing $q = e + f$ and squaring this, we deduce that $q^2 - q = ef + fe$. Again squaring the last equality, we derive that $q^4 + q^2 = (q^2 - q)^2 = efef + efe + fef + fefe = ef + e + f + fe = q^2$. Finally, $q^4 = 0$ and hence R is with bounded index of nilpotence, too.

We can now mention some constructions of nil-clean rings having only nilpotent elements which are difference of two idempotents.

Remark 1 *By what we have just previously shown, a crucial example of such a sort of nil-clean rings is any nil-clean ring which is simultaneously regular – in fact, such is, for instance, the ring $\mathbb{M}_n(\mathbb{Z}_2)$ for all $n \geq 1$ by an appeal to [2] and to the well-known fact from [9] that it is a regular ring because so is $\mathbb{Z}_2$. Indeed, this is not always possible as it was recently exhibited in [11] an ingenious example of a nil-clean ring of characteristic 2 which is not strongly π-regular as well as of a nil-clean ring of characteristic 4 which is not π-regular.*

An other interesting example of a nil-clean ring whose nilpotent elements are differences of two idempotents and which ring is not regular (due to the fact that it has a non-zero Jacobson radical) is the upper triangular matrix ring $\mathbb{T}_2(\mathbb{Z}_2)$, which fact we leave to the interested reader for a direct inspection. This ring is, however, strongly π-regular.

Moreover, the indecomposable nil-clean ring $\mathbb{Z}_4$ does not have the indicated above specific property of its nilpotents since $2 \neq 0$ in it.

We end our work in this subsection with the following challenging problem.

Problem 1 *Characterize nil-clean rings whose nilpotent elements are differences of two arbitrary idempotents.*

2.2 Nil-clean rings with cyclic unit group

In [5, p.81] it was asked of whether or not a clean ring with cyclic units is strongly clean. We shall resolve this question in the case of nil-clean rings (note that nil-clean rings are always clean and a *clean* ring is the one whose elements are sums of a unit and an idempotent; if these two elements commute, the clean ring is called *strongly clean*). It was established in [4, Corollary 4.10] that a nil-clean is strongly nil-clean if, and only if, its unit group is a 2-group.

We are now arriving at the following statement.

Theorem 2 *Suppose* R *is a nil-clean ring with cyclic* $U(R)$. *Then* R *is strongly nil-clean of characteristic* 2 *if, and only if,* $U(R)$ *is a* 2*-group.*

Proof. If we assume for a moment that $U(R) = \{1\}$, then $\mathrm{Nil}(R) = \{0\}$ as $1 + \mathrm{Nil}(R) \subseteq U(R)$, so that R must be boolean whence strongly nil-clean. So, we shall assume hereafter that $U(R) \neq \{1\}$.

Firstly, to prove the "right-to-left" implication, assume that $U(R)$ is a cyclic 2-group. Thus, as commented above, it follows immediately from [4, Corollary 4.10] that R is strongly nil-clean. What remain to show is that $2 = 0$ holds in R. Indeed, since $2 \in \mathrm{Nil}(R)$, one observes that the infinite sequence $\{3, 5, 7, \dots, 2k-1, 2k+1, \dots\}$ will invert in R for any $k \in \mathbb{N}$. But as $U(R)$ is finite, there will exist a natural number k with $2k-1 = 2k+1$, so that $2 = 0$ is really fulfilled.

Secondly, the direct application of [4, Corollary 4.10] gives the "left-to-right" part, as desired. □

We finish our work in this subsection with the following useful comments which shed some further light on the explored theme.

Remark 2 *For nil-clean rings with finite unit group the above theorem is not longer true: in fact, as an example we can consider the* 2×2 *matrix ring* $\mathbb{M}_2(\mathbb{Z}_2)$ *which, in accordance with* [2]*, is nil-clean but surely not strongly nil-clean (however, it is strongly* π*-regular being finite). This suggests to extract even the more general claim that nil-clean rings with finite unit group are strongly* π*-regular of characteristic* 2*. In fact, as unipotents (= the sum of* 1 *and a nilpotent) are always units, it readily follows that the set of nilpotents is also finite and so the ring is with bounded index of nilpotence. We, therefore, can apply* [6, Corollary 3.12] *to get the wanted claim. That char*$(R) = 2$ *follows now in the same manner as in the proof of Theorem 2.*

In closing, we pose a few intriguing problems of some interest and importance which immediately arise.

Problem 1. If R is a nil-clean ring with bounded $U(R)$, does it follow that R is (strongly) π-regular?

Problem 2. If R is a nil-clean ring of characteristic 2 and $U(R)$ is a p-group (or, respectively, a $2p$-group) for some prime p, is it true that R is (strongly) π-regular?

For eventual counterexamples in case we have dropped some of the requirements, see Examples 3.1 and 3.2 from [11].

In regard to both sections explored above, one may state the following:

Problem 3. Is any nil-clean ring R such that its nilpotents are differences of two idempotents always π-regular? In particular, if $J(R) = 0$, is then R necessarily von Neumann regular.

In fact, each such nil-clean ring is of characteristic 2. If the above question holds in the affirmative, this will be in sharp contrast to the recent example by Šter from [11] showing that there is a nil-clean ring which is not π-regular.

Letting $QNil(R)$ be the set of all quasi-nilpotent elements of the ring R, we note that both inclusions $Nil(R) \subseteq QNil(R)$ and $J(R) \subseteq QNil(R)$ hold. We thereby come in mind to our next question as follows:

Problem 4. Examine those (nil-clean) rings for which the equality $U(R) = 1 + QNil(R)$ is true.

Notice that the condition $U(R) = 1 + Nil(R) + J(R)$ obviously implies the condition $U(R) = 1 + QNil(R)$, as in the latter situation we shall say that the ring R has *quasi-nilpotent units*.

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Generalized operator for Alexander integral operator

H. Özlem Güney
Dicle University, Faculty of Science,
Department of Mathematics,
21280 Diyarbakır-Turkey
email: ozlemg@dicle.edu.tr

Shigeyoshi Owa
Honorary Professor
"1 Decembrie 1918" University
Alba Iulia, Romania
email: shige21@ican.zaq.ne.jp

Abstract. Let T_n be the class of functions f which are defined by a power series

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots$$

for every z in the closed unit disc $\overline{\mathbb{U}}$. With m different boundary points $z_s, (s = 1, 2, \dots, m)$, we consider $\alpha_m \in e^{i\beta}A_{-j-\lambda}f(\mathbb{U})$, here $A_{-j-\lambda}$ is the generalized Alexander integral operator and $\mathbb{U}$ is the open unit disc. Applying $A_{-j-\lambda}$, a subclass $B_n(\alpha_m, \beta, \rho; j, \lambda)$ of T_n is defined with fractional integral for functions f. The object of present paper is to consider some interesting properties of f to be in $B_n(\alpha_m, \beta, \rho; j, \lambda)$.

1 Introduction

Let T_n be the class of functions

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad n \in \mathbb{N} = \{1, 2, 3, \dots\} \tag{1}$$

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that are analytic in the closed unit disc $\overline{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$. For $f \in T_n$, J.W.Alexander [2] had defined the following the Alexander integral operator $A_{-1}f(z)$ given by

$$A_{-1}f(z) = \int_0^z \frac{f(t)}{t}dt = z + \sum_{k=n+1}^{\infty} \frac{a_k}{k} z^k. \tag{2}$$

The above the Alexander integral operator was applied for some subclasses of analytic functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ by M.Acu [1] and by K. Kugita et al. [4].

For the above the Alexander integral operator $A_{-1}f(z)$, we consider

$$A_{-j}f(z) = A_{-j+1}(A_{-1}f(z)) = z + \sum_{k=n+1}^{\infty} \frac{a_k}{k^j} z^k, \quad j \in \mathbb{N} \tag{3}$$

where $A_0 f(z) = f(z)$.

From the various definitions of fractional calculus of $f \in T_n$ (that is, fractional integrals and fractional derivatives) given in the literature, we would like to recall here the following definitions for fractional calculus which were used by Owa [7] and Owa and Srivastava [8].

Definition 1 *The fractional integral of order* λ *for* $f \in T_n$ *is defined by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \quad (\lambda > 0) \tag{4}$$

where f *is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of* $(z-t)^{\lambda-1}$ *is removed by requiring* $\log(z-t)$ *to be real when* $z-t > 0$ *and* Γ *is the Gamma function.*

With the above definition, we know that

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{k=n+1}^{\infty} \frac{k!}{\Gamma(k+1+\lambda)} a_k z^{k+\lambda} \tag{5}$$

for $\lambda > 0$ and $f \in T_n$. Further applying the fractional integral for $f \in T_n$, we define a new operator $A_{-\lambda}f(z)$ given by

$$A_{-\lambda}f(z) = \frac{\Gamma\left(\frac{3+\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)} z^{\frac{1-\lambda}{2}} D_z^{-\lambda} \left(z^{\frac{-1-\lambda}{2}} f(z) \right), \tag{6}$$

where $0 \leq \lambda \leq 1$. If $\lambda = 0$, then (6) becomes $A_0 f(z) = f(z)$ and if $\lambda = 1$, then (6) leads us that

$$A_{-1} f(z) = D_z^{-1}\left(\frac{f(z)}{z}\right) = \int_0^z \frac{f(t)}{t} dt. \tag{7}$$

With this integral operator, we know

$$A_{-j-\lambda} f(z) = A_{-j}\left(A_{-\lambda} f(z)\right) \tag{8}$$

where $j \in \mathbb{N}$ and $0 \leq \lambda \leq 1$. This operator $A_{-j-\lambda} f(z)$ is the generalization of the Alexander integral operator $A_{-1} f(z)$. Here, we note that

$$A_{-\lambda} f(z) = z + \sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2k+1-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2k+1+\lambda}{2}\right)} a_k z^k \tag{9}$$

and

$$A_{-j-\lambda} f(z) = z + \sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2k+1-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2k+1+\lambda}{2}\right) k^j} a_k z^k \tag{10}$$

where $j \in \mathbb{N}$ and $0 \leq \lambda \leq 1$. From the above, we know that

$$A_{-j-\lambda} f(z) = A_{-j}(A_{-\lambda} f(z)) = A_{-\lambda}(A_{-j} f(z)) \tag{11}$$

for $f \in T_n$. For m different boundary points $z_s (s = 1, 2, 3, ..., m)$ with $|z_s| = 1$, we consider

$$\alpha_m = \frac{1}{m} \sum_{s=1}^{m} \frac{A_{-j-\lambda} f(z_s)}{z_s}, \tag{12}$$

where $\alpha_m \in e^{i\beta} A_{-j-\lambda} f(\mathbb{U}), \alpha_m \neq 1$ and $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$. For such α_m, if $f \in T_n$ satisfies

$$\left| \frac{e^{i\beta} \frac{A_{-j-\lambda} f(z)}{z} - \alpha_m}{e^{i\beta} - \alpha_m} - 1 \right| < \rho, \quad z \in \mathbb{U} \tag{13}$$

for some real $\rho > 0$, we say that the function f belongs to the subclass $B_n(\alpha_m, \beta, \rho; j, \lambda)$ of T_n. With this definition for the class $B_n(\alpha_m, \beta, \rho; j, \lambda)$, we see that the condition (13) is equivalent to

$$\left| \frac{A_{-j-\lambda} f(z)}{z} - 1 \right| < \rho \left| e^{i\beta} - \alpha_m \right|, \quad z \in \mathbb{U}. \tag{14}$$

If we consider the function $f \in T_n$ given by

$$f(z) = z + \frac{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2n+3+\lambda}{2}\right)}{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2n+3-\lambda}{2}\right)}\rho(e^{i\beta} - \alpha_m)(n+1)^j z^{n+1} \tag{15}$$

then f satisfies

$$\left|\frac{A_{-j-\lambda}f(z)}{z} - 1\right| = \rho\left|e^{i\beta} - \alpha_m\right||z|^n < \rho\left|e^{i\beta} - \alpha_m\right|, \quad z \in \mathbb{U}. \tag{16}$$

Therefore, f given by (15) belongs to the class $B_n(\alpha_m, \beta, \rho; j, \lambda)$.

Discussing our problems for $f \in B_n(\alpha_m, \beta, \rho; j, \lambda)$, we have to introduce the following lemma due to S. S. Miller and P. T. Mocanu [5, 6] (also, due to I. S. Jack [3]).

Lemma 1 *Let the function* w *given by*

$$w(z) = a_n z^n + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \quad (n \in \mathbb{N}) \tag{17}$$

be analytic in $\mathbb{U}$ *with* $w(0) = 0$*. If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r$ *at a point* $z_0, (0 < |z_0| < 1)$ *then there exists a real number* $k \geq n$ *such that*

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \tag{18}$$

and

$$\operatorname{Re}\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq k. \tag{19}$$

2 Properties of functions concerning with the class $B_n(\alpha_m, \beta, \rho; j, \lambda)$

Our first property for $f \in T_n$ is as follows.

Theorem 1 *If* $f \in T_n$ *satisfies*

$$\left|\frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)} - 1\right| < \frac{\left|e^{i\beta} - \alpha_m\right| n\rho}{1 + |e^{i\beta} - \alpha_m|\,\rho}, \quad z \in \mathbb{U} \tag{20}$$

for some α_m *defined by* (12) *with* $\alpha_m \neq 1$ *such that* $z_s \in \partial\mathbb{U}\,(s = 1, 2, 3, \dots, m)$, *and for some real* $\rho > 1$*, then*

$$\left|\frac{A_{-j-\lambda}f(z)}{z} - 1\right| < \rho\left|e^{i\beta} - \alpha_m\right|, \quad z \in \mathbb{U} \tag{21}$$

that is, $f \in B_n(\alpha_m, \beta, \rho; j, \lambda)$.

Proof. We introduce the function w by

$$w(z)=\frac{e^{i\beta}\frac{A_{-j-\lambda}f(z)}{z}-\alpha_m}{e^{i\beta}-\alpha_m}-1=\frac{e^{i\beta}}{e^{i\beta}-\alpha_m}\left\{\sum_{k=n+1}^{\infty}\frac{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2k+1-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2k+1+\lambda}{2}\right)k^j}a_kz^{k-1}\right\}. \tag{22}$$

Then, w is analytic in $\mathbb{U}$ with $w(0)=0$ and

$$\frac{A_{-j-\lambda}f(z)}{z}=1+(1-e^{-i\beta}\alpha_m)w(z). \tag{23}$$

It follows from the above that

$$\frac{z(A_{-j-\lambda}f(z))'}{A_{-j-\lambda}f(z)}-1=\frac{(1-e^{-i\beta}\alpha_m)zw'(z)}{1+(1-e^{-i\beta}\alpha_m)w(z)}. \tag{24}$$

Note that

$$z(A_{-j-\lambda}f(z))'=A_{-j-\lambda+1}f(z). \tag{25}$$

So, (24) is the same as

$$\frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)}-1=\frac{(1-e^{-i\beta}\alpha_m)zw'(z)}{1+(1-e^{-i\beta}\alpha_m)w(z)}. \tag{26}$$

Thus, our condition (20) gives that

$$\left|\frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)}-1\right|=\left|\frac{(1-e^{-i\beta}\alpha_m)zw'(z)}{1+(1-e^{-i\beta}\alpha_m)w(z)}\right|<\frac{|e^{i\beta}-\alpha_m|n\rho}{1+|e^{i\beta}-\alpha_m|\rho}. \tag{27}$$

Now, we suppose that there exists a point z_0, $(0<|z_0|<1)$ such that

$$\max\{|w(z)|;|z|\le|z_0|\}=|w(z_0)|=\rho>1. \tag{28}$$

Then, we can write that $w(z_0)=\rho e^{i\theta}$, $(0\le\theta\le 2\pi)$ and $z_0w'(z_0)=kw(z_0)$, $(k\ge n)$ by Lemma 1. For such a point z_0, $(0<|z_0|<1)$ we see that

$$\begin{aligned}\left|\frac{A_{-j-\lambda+1}f(z_0)}{A_{-j-\lambda}f(z_0)}-1\right|&=\left|\frac{(1-e^{-i\beta}\alpha_m)z_0w'(z_0)}{1+(1-e^{-i\beta}\alpha_m)w(z_0)}\right|\\&=\left|\frac{(1-e^{-i\beta}\alpha_m)k\rho}{1+(1-e^{-i\beta}\alpha_m)\rho e^{i\theta}}\right|\\&\ge\frac{|1-e^{-i\beta}\alpha_m|n\rho}{1+|1-e^{-i\beta}\alpha_m|\rho}\\&=\frac{|e^{i\beta}-\alpha_m|n\rho}{1+|e^{i\beta}-\alpha_m|\rho}.\end{aligned} \tag{29}$$

Since (29) contradicts our condition (20), we know that there is no z_0, $(0 < |z_0| < 1)$ such that $|w(z_0)| = \rho > 1$. Therefore, using $|w(z)| < \rho$ for all $z \in \mathbb{U}$, we have that

$$|w(z)| = \left| \frac{e^{i\beta}\left(\frac{A_{-j-\lambda}f(z)}{z} - 1\right)}{e^{i\beta} - \alpha_m} \right| < \rho, \quad z \in \mathbb{U}, \tag{30}$$

that is, that

$$\left| \frac{A_{-j-\lambda}f(z)}{z} - 1 \right| < \rho \left| e^{i\beta} - \alpha_m \right|, \quad z \in \mathbb{U}. \tag{31}$$

This completes the proof of the theorem. □

Example 1 *We consider the function* $f \in T_n$ *given by*

$$f(z) = z + a_{n+1}z^{n+1}, \quad z \in \mathbb{U}. \tag{32}$$

Then, we see that

$$\left| \frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)} - 1 \right| = \left| \frac{P(n,j,\lambda)na_{n+1}z^n}{1 + P(n,j,\lambda)a_{n+1}z^n} \right|. \tag{33}$$

$$< \frac{nP(n,j,\lambda)|a_{n+1}|}{1 - P(n,j,\lambda)|a_{n+1}|}, \quad z \in \mathbb{U},$$

where

$$0 < |a_{n+1}| < \frac{1 - P(n,j,\lambda)}{P(n,j,\lambda)} \tag{34}$$

and

$$P(n,j,\lambda) = \frac{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2n+3-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2n+3+\lambda}{2}\right)(n+1)^j}. \tag{35}$$

Now, we consider five boundary points

$$z_1 = e^{-i\frac{\arg(a_{n+1})}{n}} \tag{36}$$

$$z_2 = e^{i\frac{\pi - 6\arg(a_{n+1})}{6n}} \tag{37}$$

$$z_3 = e^{i\frac{\pi - 4\arg(a_{n+1})}{4n}} \tag{38}$$

$$z_4 = e^{i\frac{\pi - 3\arg(a_{n+1})}{3n}} \tag{39}$$

and

$$z_5 = e^{i\frac{\pi - 2\arg(a_{n+1})}{2n}}. \tag{40}$$

For such $z_s (s = 1, 2, 3, 4, 5)$, *we have that*

$$\frac{A_{-j-\lambda}f(z_1)}{z_1} = 1 + P(n, j, \lambda)|a_{n+1}|, \tag{41}$$

$$\frac{A_{-j-\lambda}f(z_2)}{z_2} = 1 + P(n, j, \lambda)|a_{n+1}|\frac{\sqrt{3} + i}{2}, \tag{42}$$

$$\frac{A_{-j-\lambda}f(z_3)}{z_3} = 1 + P(n, j, \lambda)|a_{n+1}|\frac{\sqrt{2}(1 + i)}{2}, \tag{43}$$

$$\frac{A_{-j-\lambda}f(z_4)}{z_4} = 1 + P(n, j, \lambda)|a_{n+1}|\frac{1 + \sqrt{3}i}{2}, \tag{44}$$

and

$$\frac{A_{-j-\lambda}f(z_5)}{z_5} = 1 + P(n, j, \lambda)|a_{n+1}|i. \tag{45}$$

It follows from the above that

$$\alpha_5 = \frac{1}{5}\sum_{s=1}^{5}\frac{A_{-j-\lambda}f(z_s)}{z_s} = 1 + \frac{(3 + \sqrt{2} + \sqrt{3})P(n, j, \lambda)|a_{n+1}|(1 + i)}{10} \tag{46}$$

and that

$$\left|1 - e^{-i\beta}\alpha_5\right| = \frac{\sqrt{2}(3 + \sqrt{2} + \sqrt{3})P(n, j, \lambda)|a_{n+1}|}{10} \tag{47}$$

with $\beta = 0$. *For such* α_5 *and* β, *we consider* $\rho > 1$ *with*

$$\frac{nP(n, j, \lambda)|a_{n+1}|}{1 - P(n, j, \lambda)|a_{n+1}|} \leq \frac{|e^{i\beta} - \alpha_5|n\rho}{1 + |e^{i\beta} - \alpha_5|\rho}. \tag{48}$$

This gives us that

$$\rho \geq \frac{10}{\sqrt{2}(3 + \sqrt{2} + \sqrt{3})(1 - (1 + |a_{n+1}|)P(n, j, \lambda))} > \frac{10}{\sqrt{2}(3 + \sqrt{2} + \sqrt{3})} > 1. \tag{49}$$

For such α_5 *and* $\rho > 1$, *the function* f *satisfies*

$$\left|\frac{A_{-j-\lambda}f(z)}{z} - 1\right| < P(n, j, \lambda)|a_{n+1}| \leq \rho|e^{i\beta} - \alpha_5|, \quad z \in \mathbb{U}.$$

Next, we derive the following theorem.

Theorem 2 *If* $f \in T_n$ *satisfies*

$$\left|\left(\frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)}-1\right)\left(\frac{A_{-j-\lambda}f(z)}{z}-1\right)\right| < \frac{\left|e^{i\beta}-\alpha_m\right|^2 n\rho^2}{1+|e^{i\beta}-\alpha_m|\,\rho}, \quad z \in \mathbb{U} \tag{50}$$

for some α_m *defined by* (12) *with* $\alpha_m \neq 1$ *and for some real* $\rho > 1$, *then*

$$\left|\frac{A_{-j-\lambda}f(z)}{z}-1\right| < \rho\left|e^{i\beta}-\alpha_m\right|, \quad z \in \mathbb{U} \tag{51}$$

that is, $f \in B_n(\alpha_m, \beta, \rho; j, \lambda)$.

Proof. Define the function w by (22). Applying (25), our condition (50) leads us that

$$\begin{aligned}\left|\left(\frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)}-1\right)\left(\frac{A_{-j-\lambda}f(z)}{z}-1\right)\right| &= \left|\frac{\left(1-e^{-i\beta}\alpha_m\right)^2 zw(z)w'(z)}{1+\left(1-e^{-i\beta}\alpha_m\right)w(z)}\right| \\ &\leq \frac{\left|e^{i\beta}-\alpha_m\right|^2 n\rho^2}{1+|e^{i\beta}-\alpha_m|\,\rho}, \quad z \in \mathbb{U}.\end{aligned} \tag{52}$$

Suppose that there exists a point $z_0, (0 < |z_0| < 1)$ such that

$$\max\{|w(z)|; |z| \leq |z_0|\} = |w(z_0)| = \rho > 1. \tag{53}$$

Then, applying Lemma 1, we write that $w(z_0) = \rho e^{i\theta}, (0 \leq \theta \leq 2\pi)$ and $z_0 w'(z_0) = kw(z_0), (k \geq n)$. This shows us that

$$\begin{aligned}\left|\left(\frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)}-1\right)\left(\frac{A_{-j-\lambda}f(z)}{z}-1\right)\right| &= \left|\frac{\left(1-e^{-i\beta}\alpha_m\right)^2 z_0 w(z_0)w'(z_0)}{1+\left(1-e^{-i\beta}\alpha_m\right)w(z_0)}\right| \\ &= \frac{\left|e^{i\beta}-\alpha_m\right|^2 \rho^2 k}{|1+(1-e^{-i\beta}\alpha_m)\rho e^{i\theta}|} \\ &\geq \frac{\left|e^{i\beta}-\alpha_m\right|^2 n\rho^2}{1+|e^{i\beta}-\alpha_m|\,\rho}\end{aligned} \tag{54}$$

which contradicts our condition (50). Thus there is no $z_0, (0 < |z_0| < 1)$ such that $|w(z_0)| = \rho > 1$. This shows us that

$$\left|\left(\frac{A_{-j-\lambda}f(z)}{z}-1\right)\right| < \rho|e^{i\beta}-\alpha_m|, \quad z \in \mathbb{U}. \tag{55}$$

$\square$

Example 2 *Consider a function* $f \in T_n$ *given by*

$$f(z) = z + a_{n+1}z^{n+1} \;, z \in \mathbb{U} \tag{56}$$

with $0 < |a_{n+1}| < \frac{1}{P(n,j,\lambda)}$, *where* $P(n,j,\lambda)$ *is given by* (35). *It follows that*

$$\begin{aligned} \left| \left(\frac{A_{-j-\lambda+1}f(z)}{A_{-j-\lambda}f(z)} - 1 \right) \left(\frac{A_{-j-\lambda}f(z)}{z} - 1 \right) \right| &= \left| \frac{nP(n,j,\lambda)^2 a_{n+1}^2 z^{2n}}{1 + P(n,j,\lambda)a_{n+1}z^n} \right| \\ &< \frac{nP(n,j,\lambda)^2|a_{n+1}|^2}{1 - P(n,j,\lambda)|a_{n+1}|}, \quad z \in \mathbb{U}. \end{aligned} \tag{57}$$

Considering five boundary points z_1, z_2, z_3, z_4 *and* z_5 *in Example* 1, *we see that*

$$\left| e^{i\beta} - \alpha_5 \right| = \frac{\sqrt{2}(3 + \sqrt{2} + \sqrt{3})P(n,j,\lambda)|a_{n+1}|}{10} \tag{58}$$

with $\beta = 0$. *If we consider* $\rho > 1$ *such that*

$$\frac{nP(n,j,\lambda)^2|a_{n+1}|^2|z|}{1 - P(n,j,\lambda)|a_{n+1}|} \leq \frac{\left|e^{i\beta} - \alpha_5\right|^2 n\rho^2}{1 + |e^{i\beta} - \alpha_5|\,\rho}, \tag{59}$$

then ρ *satisfies*

$$\rho \geq \frac{10}{\sqrt{2}(3 + \sqrt{2} + \sqrt{3})P(n,j,\lambda)|a_{n+1}|} > 1. \tag{60}$$

For such α_5 *and* ρ, f *satisfies*

$$\left| \frac{A_{-j-\lambda}f(z)}{z} - 1 \right| < P(n,j,\lambda)|a_{n+1}| \leq \rho|e^{i\beta} - \alpha_5|, \quad z \in \mathbb{U}. \tag{61}$$

Our next result reads as follows.

Theorem 3 *If* $f \in T_n$ *satisfies*

$$\left| \frac{A_{-j-\lambda+p}f(z)}{z} - 1 \right| < \rho|e^{i\beta} - \alpha_m|(n+1), \quad z \in \mathbb{U}. \tag{62}$$

for some α_m *defined by* (12) *with* $\alpha_m \neq 1$ *and for some real* $\rho > 1$, *then*

$$\left| \frac{A_{-j-\lambda+p-1}f(z)}{z} - 1 \right| < \rho \left| e^{i\beta} - \alpha_m \right|, \quad z \in \mathbb{U} \tag{63}$$

where $p = 0, 1, 2, \ldots, j$.

Proof. We consider the function w defined by

$$
\begin{aligned}
w(z) &= \frac{e^{i\beta}\frac{A_{-j-\lambda+p-1}f(z)}{z} - \alpha_m}{e^{i\beta} - \alpha_m} - 1 \\
&= \frac{e^{i\beta}}{e^{i\beta} - \alpha_m}\left\{\sum_{k=n+1}^{\infty} \frac{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2k+1-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2k+1+\lambda}{2}\right)k^{j-p+1}} a_k z^{k-1}\right\}.
\end{aligned} \tag{64}
$$

Thus w is analytic in $\mathbb{U}$, $w(0) = 0$, and

$$
A_{-j-\lambda+p-1}f(z) = z + (1 - e^{-i\beta}\alpha_m)zw(z). \tag{65}
$$

Noting that

$$
\begin{aligned}
A_{-j-\lambda+p}f(z) &= z\left(A_{-j-\lambda+p-1}f(z)\right)' \\
&= z\left\{1 + (1 - e^{-i\beta}\alpha_m)w(z)\left(1 + \frac{zw'(z)}{w(z)}\right)\right\},
\end{aligned} \tag{66}
$$

we have that

$$
\begin{aligned}
\left|\frac{A_{-j-\lambda+p}f(z)}{z} - 1\right| &= \left|1 - e^{-i\beta}\alpha_m\right| |w(z)| \left|1 + \frac{zw'(z)}{w(z)}\right| \\
&< \rho\left|e^{i\beta} - \alpha_m\right|(n+1), \quad z \in \mathbb{U}
\end{aligned} \tag{67}
$$

by the condition (62). Suppose that there exists a point z_0, $(0 < |z_0| < 1)$ such that

$$
\max\{|w(z)|; |z| \leq |z_0|\} = |w(z_0)| = \rho > 1. \tag{68}
$$

Then, letting $w(z_0) = \rho e^{i\theta}, (0 \leq \theta \leq 2\pi)$ and $z_0 w'(z_0) = k w(z_0), (k \geq n)$ with Lemma 1, we see that

$$
\left|\frac{A_{-j-\lambda+p}f(z_0)}{z_0} - 1\right| = \rho\left|e^{i\beta} - \alpha_m\right|(k+1) \geq \rho\left|e^{i\beta} - \alpha_m\right|(n+1). \tag{69}
$$

This contradicts the inequality (67). Therefore, we don't have any $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho > 1$. This shows us that

$$
|w(z)| = \left|\frac{\alpha_m}{e^{i\beta} - \alpha_m}\left(\frac{A_{-j-\lambda+p-1}f(z)}{z} - 1\right)\right| < \rho, \quad z \in \mathbb{U}, \tag{70}
$$

that is, that

$$
\left|\frac{A_{-j-\lambda+p-1}f(z)}{z} - 1\right| < \rho|e^{i\beta} - \alpha_m|, \quad z \in \mathbb{U}. \tag{71}
$$

This completes the proof of our theorem. ∎

Corollary 1 *If* $f \in T_n$ *satisfies*

$$\left|\frac{A_{-j-\lambda+p}f(z)}{z} - 1\right| < \rho|e^{i\beta} - \alpha_m|(n+1)^p, \quad z \in \mathbb{U} \tag{72}$$

for some α_m *given by* (12) *with* $\alpha_m \neq 1$, *and for some real* $\rho > 1$, *then*

$$\left|\frac{A_{-j-\lambda}f(z)}{z} - 1\right| < \rho|e^{i\beta} - \alpha_m|, \quad z \in \mathbb{U} \tag{73}$$

where $p = 0, 1, 2, ..., j$.

Proof. With Theorem 3, we say that if $f \in T_n$ satisfies

$$\left|\frac{A_{-j-\lambda+p}f(z)}{z} - 1\right| < \rho|e^{i\beta} - \alpha_m|(n+1)^p, \quad z \in \mathbb{U}, \tag{74}$$

then

$$\left|\frac{A_{-j-\lambda+p-1}f(z)}{z} - 1\right| < \rho|e^{i\beta} - \alpha_m|(n+1)^{p-1}, \quad z \in \mathbb{U}. \tag{75}$$

Further, we have that

$$\left|\frac{A_{-j-\lambda+p-2}f(z)}{z} - 1\right| < \rho|e^{i\beta} - \alpha_m|(n+1)^{p-2}, \quad z \in \mathbb{U}, \tag{76}$$

from (75). Finally, we obtain that

$$\left|\frac{A_{-j-\lambda}f(z)}{z} - 1\right| < \rho|e^{i\beta} - \alpha_m|, \quad z \in \mathbb{U}. \tag{77}$$

□

Example 3 *Consider the function* $f \in T_n$ *given by*

$$f(z) = z + a_{n+1}z^{n+1}, z \in \mathbb{U}. \tag{78}$$

Since

$$A_{-j-\lambda+p}f(z) = z + \frac{\Gamma\left(\frac{3+\lambda}{2}\right)\Gamma\left(\frac{2n+3-\lambda}{2}\right)}{\Gamma\left(\frac{3-\lambda}{2}\right)\Gamma\left(\frac{2n+3+\lambda}{2}\right)(n+1)^{j-p+2}} a_{n+1}z^{n+1}, \tag{79}$$

we have

$$\left|\frac{A_{-j-\lambda+p}f(z)}{z} - 1\right| = \left|P(n,j,\lambda)(n+1)^{p-2}a_{n+1}z^n\right| < P(n,j,\lambda)(n+1)^{p-2}|a_{n+1}| \tag{80}$$

where

$$0 < |a_{n+1}| < \frac{1}{P(n,j,\lambda)} \tag{81}$$

and $P(n,j,\lambda)$ *is given by* (35).

Consider five boundary points z_1, z_2, z_3, z_4 *and* z_5 *in Example* 1*. Then* α_5 *satisfies* (46) *and* $|1 - e^{-i\beta}\alpha_5|$ *satisfies* (47) *for* $\beta = 0$*. For such* α_5 *and* β*, we consider* $\rho > 1$ *by*

$$\left|\frac{A_{-j-\lambda+p}f(z)}{z} - 1\right| < P(n,j,\lambda)(n+1)^{p-2}|a_{n+1}| \leq \rho\left|e^{i\beta} - \alpha_5\right|(n+1)^{p-2}, \quad z \in \mathbb{U}, \tag{82}$$

Then ρ *satisfies*

$$\rho \geq \frac{P(n,j,\lambda)|a_{n+1}|}{|e^{i\beta} - \alpha_5|} = \frac{10}{\sqrt{2}(3+\sqrt{2}+\sqrt{3})} > 1. \tag{83}$$

With the above α_5 *and* ρ*, we have*

$$\left|\frac{A_{-j-\lambda}f(z)}{z} - 1\right| < P(n,j,\lambda)|a_{n+1}| \leq \rho\left|e^{i\beta} - \alpha_5\right|, \quad z \in \mathbb{U}. \tag{84}$$

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On $\lambda^D - R_0$ and $\lambda^D - R_1$ spaces

Sarhad F. Namiq
Department of Mathematics,
College of Education,
University of Garmian,
Kurdistan-Region, Iraq
email: sarhad1983@gmail.com

Ennis Rosas
Departamento de Ciencias
Naturales y Exactas,
Universidad de la Costa, Barranquilla,
Colombia & Departmento de
Matemáticas, Universidad de Oriente,
Cumaná, Venezuela
email: ennisrafael@gmail.com

Abstract. In this paper we introduce the new types of separation axioms called $\lambda^D - R_0$ and $\lambda^D - R_1$ spaces, by using λ^D-open set. The notion $\lambda^D - R_0$ and $\lambda^D - R_1$ spaces are introduced and some of their properties are investigated.

1 Introduction

In 1943, the notion of R_0 topological space was introduced by Shanin [6]. Later, Davis [3] rediscovered it and studied some properties of this weak separation axiom. In the same paper, Davis also introduced the notion of R_1 topological space which are independent of both T_0 and T_1, but strictly weaker than T_2. The notion of λ−open (λ^*−open) sets was introduced by Alais B. Khalaf and Sarhad F. Namiq [1]. The notion of λ^D-open sets was introduced by Sarhad F. Namiq [5]. In this paper, we continue the study of the above mentioned classes of topological spaces satisfying these axioms by introducing two more notions in terms of λ^D−open sets called $\lambda^D - R_0$ and $\lambda^D - R_1$.

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2 Preliminaries

Throughout, X denote a topological space. Let A be a subset of X, the closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$ respectively. A subset A of a topological space (X, τ) is said to be dense set [7] if $Cl(A) = X$. A subset A of a topological space (X, τ) is said to be semi open [4] if $A \subseteq Cl(Int(A))$. The complement of a semi open set is said to be semi closed [4]. The family of all semi open (resp. semi closed) sets in a topological space (X, τ) is denoted by $SO(X, \tau)$ or $SO(X)$ (resp. $SC(X, \tau)$ or $SC(X)$). We consider λ as a function defined on $SO(X)$ into $\mathcal{P}(X)$ and $\lambda : SO(X) \to \mathcal{P}(X)$ is called an s-operation if $V \subseteq \lambda(V)$ for each non-empty semi open set V. It is assumed that $\lambda(\emptyset) = \emptyset$ and $\lambda(X) = X$ for any s-operation λ.

Definition 1 [1] *Let (X, τ) be a topological space and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation, then a subset A of X is called a λ^*-open set which is equivalent to $\lambda-$open set, if for each $x \in A$, there exists a semi open set U such that $x \in U$ and $\lambda(U) \subseteq A$. The complement of a λ^*-open set is said to be λ^*-closed set which is equivalent to $\lambda-$closed set. The family of all λ^*-open (resp., λ^*-closed) subsets of a topological space (X, τ) is denoted by or $SO_\lambda(X)$ (resp. $SC_\lambda(X, \tau)$ or $SC_\lambda(X)$).*

Definition 2 [5] *Let (X, τ) be a topological space and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation, then a λ^*-open subset A of X is called a λ^D-open set, if for each $x \in A$, there exists a dense set D such that $x \in D \subseteq A$. The complement of a λ^D-open set is said to be λ^D-closed. The family of all λ^D-open (resp., λ^D-closed) subsets of a topological space (X, τ) is denoted by or $SO_{\lambda^D})(X)$ or $SO_{\lambda^D}(X, \tau)$ (resp. $SC_{\lambda^D}(X, \tau)$ or $SC_{\lambda^D}(X)$).*

Example 1 *Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. The $SO(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}\}$. Define $\lambda : SO(X) \to \mathcal{P}(X)$ as:*

$$\lambda(A) = \begin{cases} A & \text{if } a \in A \\ X & \text{if } a \notin A \end{cases}$$

The $SO_{\lambda^D}(X) = \{\emptyset, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$.

Definition 3 [5] *Let (X, τ) be a topological space and let A be a subset of X. Then:*

1. *The λ-closure of A (denoted by $\lambda^D Cl(A)$) is the intersection of all λ^D- closed sets containing A.*

2. *The λ-interior of A (denoted by $\lambda^D Int(A)$) is the union of all λ^D−open sets of X contained in A.*

Proposition 1 [5] *For each point $x \in X$, $x \in \lambda^D Cl(A)$ if and only if $V \cap A \neq \emptyset$, for every $V \in SO_{\lambda^D}(X)$ such that $x \in V$.*

3 On $\lambda^D - R_0$ and $\lambda^D - R_1$ spaces

We introduce the following definitions.

Definition 4 *For any s-operation $\lambda : SO(X) \to \mathcal{P}(X)$ and any subset A of a space (X, τ) the λ^D-kernel of A, denoted by $\lambda^D \operatorname{Ker}(A)$ is defined as:*

$$\lambda^D \operatorname{Ker}(A) = \cap\{G \in SO_{\lambda^D}(X) : A \subseteq G\}.$$

Lemma 1 *Let (X, τ) be a topological space, $A \subseteq X$ and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation. Then $\lambda^D \operatorname{Ker}(A) = \{x \in X : \lambda^D Cl(\{x\}) \cap A \neq \emptyset\}$.*

Proof. Let $x \in \lambda^D \operatorname{Ker}(A)$ such that $\lambda^D Cl(\{x\}) \cap A = \emptyset$. Since $x \notin X \backslash \lambda^D Cl(\{x\})$ which is a λ^D−open set containing A. Thus $x \notin \lambda^D \operatorname{Ker}(A)$ a contradiction.

Conversely, let $x \in X$ be such that $\lambda^D Cl(\{x\}) \cap A \neq \emptyset$. If possible, let $x \notin \lambda^D \operatorname{Ker}(A)$. Then there exist a λ^D−open set G such that $x \notin G$ and $A \subseteq G$. Let $y \in \lambda^D Cl(\{x\}) \cap A$. This implies that $y \in \lambda^D Cl(\{x\})$ and $y \in G$, which gives $x \in G$, a contradiction. □

Theorem 1 *Let (X, τ) be a topological space, A and B be subsets of X. Then:*

(1) *$x \in \lambda^D \operatorname{Ker}(A)$ if and only if $A \cap F \neq \emptyset$; for any λ^D-closed set F containing x.*

(2) *$A \subseteq \lambda^D \operatorname{Ker}(A)$ and $A = \lambda^D \operatorname{Ker}(A)$ if A is λ^D-open.*

(3) *If $A \subseteq B$, then $\lambda^D \operatorname{Ker}(A) \subseteq \lambda^D \operatorname{Ker}(B)$.*

Proof. Obvious. □

Definition 5 *Let $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation, a topological space (X, τ) is called $\lambda^D - R_0$, if $U \in SO_{\lambda^D}(X)$ and $x \in U$ then $\lambda^D Cl(\{x\}) \subseteq U$.*

Example 2 *Let $X = \{a, b, c, d\}$, and $\tau = \mathcal{P}(X)$. We define an s-operation $\lambda : SO(X) \to \mathcal{P}(X)$ as:*
$\lambda(A) = A$, for every subset A of X.
$SO(X) = \mathcal{P}(X) = SO_{\lambda^D}(X) = SC_{\lambda^D}(X)$.

Theorem 2 *For any topological space* X *and any s-operation* $\lambda : SO(X) \to \mathcal{P}(X)$*, the following statements are equivalent:*

(1) X *is* $\lambda^D - R_0$.

(2) $F \in SC_{\lambda^D}(X)$ *and* $x \notin F$ *implies that* $F \subseteq U$ *and* $x \notin U$ *for some* $U \in SO_{\lambda^D}(X)$.

(3) $F \in SC_{\lambda}(X)$ *and* $x \notin F$ *implies that* $F \cap \lambda^D Cl(\{x\}) \neq \emptyset$.

(4) *For any two distinct points* x, y *of* X*, either* $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\})$ *or* $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$.

Proof.
(1) $\Rightarrow$ (2): Let $F \in SC_{\lambda^D}(X)$ and $x \notin F$. This implies that $x \in X \backslash F \in SO_{\lambda^D}(X)$, then $\lambda^D Cl(\{x\}) \subseteq X \backslash F$ (by (1)). Put $U = X \backslash \lambda^D Cl(\{x\})$. Then $x \notin U \in SO_{\lambda^D}(X)$ and $F \subseteq U$.
(2) $\Rightarrow$ (3): $F \in SC_{\lambda^D}(X)$ and $x \notin F$ then there exists $U \in SO_{\lambda^D}(X)$ such that $x \notin U$ and $F \subseteq U$ (by(2)), then $U \cap \lambda^D Cl(\{x\}) = \emptyset$ and $F \cap \lambda^D Cl(\{x\}) = \emptyset$.
(3) $\Rightarrow$ (4): Suppose that for any two distinct points x, y of X, if $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$ Then, without loss of generality, we suppose that there exists some $z \in \lambda^D Cl(\{x\})$ such that $z \notin \lambda^D Cl(\{y\})$. Thus, there exists a λ^D-open set V such that $z \in V$ and $y \notin V$ but $x \in V$. Thus $x \notin \lambda^D Cl(\{y\})$. Hence by (3), $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$.
(4) $\Rightarrow$ (1): Let $U \in SO_{\lambda^D}(X)$ and $x \in U$. Then for each $y \notin U$, $x \notin \lambda^D Cl(\{y\})$. Thus $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$. Hence by (4), $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$, for each $y \in X \backslash U$. So $\lambda^D Cl(\{x\}) \cap [\cup\{\lambda^D Cl(\{y\}) : y \in X \backslash U\}] = \emptyset$. Now, $U \in SO_{\lambda^D}(X)$ and $y \in X \backslash U$ then $\{y\} \subseteq \lambda^D Cl(\{y\}) \subseteq \lambda^D Cl(X \backslash U) = X \backslash U$. Thus $X \backslash U = \cup\{\lambda^D Cl(\{y\}) : y \in X \backslash U\}$. Hence, $\lambda^D Cl(\{y\}) \cap X \backslash U = \emptyset$ then $\lambda^D Cl(\{x\}) \subseteq U$. This showing that (X, τ) is $\lambda^D - R_0$. $\square$

Lemma 2 *Let* $\lambda : SO(X) \to \mathcal{P}(X)$ *be an s-operation. Then* $y \in \lambda^D \operatorname{Ker}(\{x\})$ *if and only if* $x \in \lambda^D Cl(\{y\})$.

Proof. Let $y \notin \lambda^D \operatorname{Ker}(\{x\})$. Then there exists $V \in SO_{\lambda^D}(X)$ containing x such that $y \notin V$. Therefore $x \notin \lambda^D Cl(\{y\})$. The converse part can be proved in a similar way. $\square$

Theorem 3 *Let* $\lambda : SO(X) \to \mathcal{P}(X)$ *be an s-operation. Then for any two points* x, y *in* X*,* $\lambda^D \operatorname{Ker}(\{x\}) \neq \lambda^D \operatorname{Ker}(\{y\})$ *if and only if* $\lambda^D Cl(\{y\}) \neq \lambda^D Cl(\{x\})$.

Proof. Suppose that $\lambda^D \operatorname{Ker}(\{x\}) \neq \lambda^D \operatorname{Ker}(\{y\})$. Then there exists $z \in \lambda^D \operatorname{Ker}(\{x\})$ such that $z \notin \lambda^D \operatorname{Ker}(\{y\})$. Now, $z \in \lambda^D \operatorname{Ker}(x)$ if and only if $x \in \lambda^D \operatorname{Ker}(\{z\})$ by Lemma 2 and $z \notin \lambda^D \operatorname{Ker}(\{y\})$ if and only if $y \in \lambda^D Cl(\{x\})$ by Lemma 2. Hence $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$.

Conversely, suppose that $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$. Then there exists $z \in X$ such that $z \in \lambda^D Cl(\{x\})$ and $z \notin \lambda^D Cl(\{y\})$ so there exists $U \in SO_{\lambda^D}(X)$ such that $z \in U$, $y \notin U$ and $x \in U$. Then $y \notin \lambda^D \operatorname{Ker}(\{x\})$. Thus $\lambda^D \operatorname{Ker}(\{x\}) \neq \lambda^D \operatorname{Ker}(\{y\})$. □

Theorem 4 *Let* $\lambda : SO(X) \to \mathcal{P}(X)$ *be an s-operation. Then* (X, τ) *is* $\lambda^D - R_0$ *if and only if for any two points* $x, y \in X$, $\lambda^D \operatorname{Ker}(\{x\}) \notin \lambda^D \operatorname{Ker}(\{y\})$, *implies that* $\lambda^D \operatorname{Ker}(\{x\}) \cap \lambda^D \operatorname{Ker}(\{y\}) = \emptyset$.

Proof. Let x, y be any two points in a $\lambda^D - R_0$ space X such that $\lambda^D \operatorname{Ker}(\{x\}) \neq \lambda^D \operatorname{Ker}(\{y\})$. Hence by Theorem 3, $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$. We show that $\lambda^D \operatorname{Ker}(\{x\}) \cap \lambda^D \operatorname{Ker}(\{y\}) = \emptyset$. In fact, if $z \in \lambda^D \operatorname{Ker}(\{x\}) \cap \lambda^D \operatorname{Ker}(\{y\})$, then by Lemma 2, we have $x, y \in \lambda^D Cl(z)$ and by Theorem 2, we obtain that $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{z\}) = \lambda^D Cl(\{y\})$ which is impossible.

Conversely, suppose that for any points $x, y \in X$, $\lambda^D \operatorname{Ker}(\{x\}) \neq \lambda^D \operatorname{Ker}(\{y\})$. Thus $\lambda^D \operatorname{Ker}(\{x\}) \cap \lambda^D \operatorname{Ker}(\{y\}) = \emptyset$. Hence we get $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$. In fact $z \in \lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\})$, this implies that $x, y \in \lambda^D \operatorname{Ker}(\{z\})$. Thus $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{z\}) \neq \emptyset$. Hence by hypothesis, we get $\lambda^D \operatorname{Ker}(\{x\}) = \lambda^D \operatorname{Ker}(\{z\})$. By similar way it follows that $\lambda^D \operatorname{Ker}(\{x\}) = \lambda^D \operatorname{Ker}(\{z\})$. Thus $\lambda^D \operatorname{Ker}(\{x\}) \neq \lambda^D \operatorname{Ker}(\{y\})$ which is a contradiction. Hence $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) \neq \emptyset$ and then by Theorem 2, the space X is $\lambda^D - R_0$. □

Theorem 5 *Let* (X, τ) *be a topological space and for any s-operation* $\lambda : SO(X) \to \mathcal{P}(X)$ *the following statements are equivalent:*

(1) X *is a* $\lambda^D - R_0$ *space.*

(2) *For any non-empty set* A *in* X *and any* $G \in SO_{\lambda^D}(X)$ *such that* $A \cap G \neq \emptyset$ *there exists* $F \in SC_{\lambda^D}(X)$ *such that* $A \cap F \neq \emptyset$ *and* $F \subseteq G$.

(3) *For any* $G \in SO_{\lambda^D}(X)$, $G = \cup\{F \in SC_{\lambda^D}(X) : F \subseteq G\}$.

(4) *For any* $F \in SC_{\lambda^D}(X)$, $F = \cap\{G \in SO_{\lambda^D}(X) : F \subseteq G\}$.

(5) *For any* $x \in X$, $\lambda^D Cl(\{x\}) \subseteq \lambda^D \operatorname{Ker}(\{x\})$.

Proof.

(1) $\Rightarrow$ (2): Let A be a non-empty subset of X and $G \in SO_{\lambda^D}(X)$ such that $A \cap G \neq \emptyset$. Let $x \in A \cap G$. Then as $x \in G \in SO_{\lambda^D}(X)$, by (1), we get $\lambda^D Cl(\{x\}) \subseteq G$. Put $F = \lambda^D Cl(\{x\})$. Then $F \in SC_{\lambda^D}(X)$, $F \subseteq G$ and $A \cap F \neq \emptyset$.

(2) $\Rightarrow$ (3): Let $G \in SO_{\lambda^D}(X)$. Then $\bigcup\{F \in SC_{\lambda^D}(X) : F \subseteq G\} \subseteq G$. Let $x \in G$. Then there exists $F \in SC_{\lambda^D}(X)$ such that $x \in F$ and $F \subseteq G$. Thus $x \in F \cup \{K \in SC_{\lambda^D}(X) : K \subseteq G\}$. Hence (3) follows.

(3) $\Rightarrow$ (4): Straight forward.

(4) $\Rightarrow$ (5): Let $x \in X$. Now, $y \notin \lambda^D \operatorname{Ker}(\{x\})$ implies there exists $V \in SO_{\lambda^D}(X)$ such that $x \in V$ and $y \notin V$ then $\lambda^D Cl(\{y\}) \cap V = \emptyset$. This implies by (4) $[\cap\{G \in SO_{\lambda^D}(X) : \lambda^D Cl(\{y\}) \subseteq G\}] \cap V = \emptyset$. Then there exists $G \in SO_{\lambda^D}(X)$ such that $x \in G$ and $\lambda^D Cl(\{y\}) \subseteq G$, so $y \notin \lambda^D Cl(\{x\})$.

(5) $\Rightarrow$ (1): Let $G \in SO_{\lambda^D}(X)$ and $x \in G$. Let $y \in \lambda^D \operatorname{Ker}(\{x\})$. Then $x \in \lambda^D Cl(\{y\})$ and hence $y \in G$. This implies that $\lambda^D \operatorname{Ker}(\{x\}) \subseteq G$. Thus $x \in \lambda^D Cl(\{x\}) \subseteq \lambda^D \operatorname{Ker}(\{x\}) \subseteq G$. Hence X is $\lambda^D - R_0$. $\square$

Corollary 1 *Let $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation. Then X is $\lambda^D - R_0$ if and only if $\lambda^D Cl(\{x\}) = \lambda^D \operatorname{Ker}(\{x\})$, for all $x \in X$.*

Proof. Suppose that X is $\lambda^D - R_0$. By Theorem 5, $\lambda^D Cl(\{x\}) \subseteq \lambda^D \operatorname{Ker}(\{x\})$. For each $x \in X$. Let $y \in \lambda^D \operatorname{Ker}(\{x\})$. Then $x \in \lambda^D Cl(\{y\})$ (by Lemma 2), and hence by Theorem 2, $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\})$. Thus $y \in \lambda^D Cl(\{x\})$ and hence $\lambda^D \operatorname{Ker}(\{x\}) \subseteq \lambda^D Cl(\{x\})$. Thus $\lambda^D Cl(\{x\}) = \lambda^D \operatorname{Ker}(\{x\})$. $\square$

The converse is obvious in view of Theorem 5.

Theorem 6 *Let (X, τ) be a topological space and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation. A space X is $\lambda^D - R_0$ if and only if for any $x, y \in X$, whenever $x \in \lambda^D Cl(\{y\})$ implies $y \in \lambda^D Cl(\{x\})$ and conversely.*

Proof. Suppose that a topological space (X, τ) is $\lambda^D - R_0$. Let $x \in \lambda^D Cl(\{y\})$. Then by Theorem 5, we have $\lambda^D Cl(\{y\}) \subseteq \lambda^D \operatorname{Ker}(\{x\})$. Thus $x \in \lambda^D \operatorname{Ker}(\{y\})$. Hence by Lemma 1, we have $y \in \lambda^D Cl(\{x\})$.

Conversely, let $U \in SO_{\lambda^D}(X)$ and $x \in U$. Let $y \in \lambda^D Cl(\{x\})$ hence by hypothesis, $x \in \lambda^D Cl(\{y\})$. Since $x \in U$, so $y \in U$. Hence $\lambda^D Cl(\{x\}) \subseteq U$. Thus X is $\lambda^D - R_0$. $\square$

Theorem 7 *Let X be a topological space and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation. Then the following statements are equivalent:*

(1) X *is* $\lambda^D - R_0$.

(2) *If* $F \in SC_{\lambda^D}(X)$ *then* $F = \lambda^D \operatorname{Ker}(F)$.

(3) *If* $F \in SC_{\lambda^D}(X)$ *and* $x \in F$, *then* $\lambda^D \operatorname{Ker}(\{x\}) \subseteq F$.

(4) *If* $x \in X$, *then* $\lambda^D \operatorname{Ker}(\{x\}) \subseteq \lambda^D Cl(\{x\})$.

Proof.
(1) $\Rightarrow$ (2): Follows from Theorem 5.
(2) $\Rightarrow$ (3): Follows from the fact that $x \in F$ then $\lambda^D \operatorname{Ker}(\{x\}) \subseteq \lambda^D \operatorname{Ker}(F) = F$ by part 3 of Theorem 1.
(3) $\Rightarrow$ (4): Since $x \in \lambda^D Cl(\{x\}) \in SC_{\lambda^D}(X)$ we have by (3), $\lambda^D \operatorname{Ker}(\{x\}) \subseteq \lambda^D Cl(\{x\})$ and (4) follows.
(4) $\Rightarrow$ (1): Let $U \in SO_{\lambda^D}(X)$ and $x \in U$. To show $\lambda^D Cl(\{x\}) \subseteq U$. If possible, suppose that, there exists $y \in \lambda^D Cl(\{x\})$ such that $y \notin U$. Then $y \in X \backslash U$. This by (4) implies that $\lambda^D \operatorname{Ker}(\{y\}) \subseteq X \backslash U$. Therefore $U \subseteq X \backslash \lambda^D \operatorname{Ker}(\{x\})$. So $x \notin \lambda^D \operatorname{Ker}(\{y\})$. Then, there exists a λ^D-open set G such that $y \in G$ but $x \notin G$. This implies that $y \notin \lambda^D Cl(\{x\})$ which is impossible. Hence $\lambda^D Cl(\{x\}) \sqsubseteq U$. Thus X is a $\lambda^D - R_0$ space. $\square$

Definition 6 *Let* (X, τ) *be a topological space and* $\lambda : SO(X) \to \mathcal{P}(X)$ *be an s-operation. The space* X *is said to be* $\lambda^D - R_1$ *if for* $x, y \in X$ *with* $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(y)$ *there exist disjoint* λ^D*-open sets* U *and* V *such that* $\lambda^D Cl(\{x\}) \subseteq U$ *and* $\lambda^D Cl(\{y\}) \subseteq V$.

Remark 1 *A space* X *in Example 2 is* $\lambda^D - R_1$.

Theorem 8 *Every* $\lambda^D - R_1$ *space is a* $\lambda^D - R_0$ *space.*

Proof. Let $U \in SO_{\lambda^D}(X)$ and $x \in U$. If $y \notin U$ then $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$ (as $x \notin \lambda^D Cl(\{y\})$). Hence there exists $V \in SO_{\lambda^D}(X)$ such that $\lambda^D Cl(\{y\}) \subseteq V$ and $x \notin V$. This gives $y \notin \lambda^D Cl(\{y\})$, proving that $\lambda^D Cl(\{x\}) \subseteq U$. So X is a $\lambda^D - R_0$ space. $\square$

The converse of Theorem 8 is not true, we can show it by the following example:

Example 3 *Let* $X = \{a, b, c, d\}$, *and* $\tau = \mathcal{P}(X)$. *We define an s-operation* $\lambda : SO(X) \to \mathcal{P}(X)$ *as:*

$$\lambda(A) = \begin{cases} X & \text{Otherwise} \\ A & \text{if } A = \emptyset \text{ or} \{b, c\} \text{ or } \{a, c\} \text{ or } \{a, b\}. \end{cases}$$

Now:

$SO(X) = \mathcal{P}(X)$.

$SO_{\lambda^D}(X) = \{\emptyset, \{b, c\}, \{a, c\}, \{a, b\}, X\}$.

$SC_{\lambda^D}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, X\}$.

Clearly X *is* $\lambda^D - R_0$ *but it is not* $\lambda^D - R_1$.

Theorem 9 *Let* (X, τ) *be a topological space and* $\lambda : SO(X) \to \mathcal{P}(X)$ *be an s-operation. Then the following statements are equivalent:*

(1) X *is* $\lambda^D - R_1$.

(2) *For any* $x, y \in X$, *one of the following holds:*

a) For $U \in SO_\lambda(X)$, $x \in U$ *if and only if* $y \in U$;

b) There exist disjoint λ^D*-open sets* U *and* V *such that* $x \in U$, $y \in V$.

(3) *If* $x, y \in X$, *such that* $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$ *then there exist* λ^D*-closed sets* F *and* H *such that* $x \in F$, $y \notin F$, $y \in H$, $x \notin H$ *and* $X = F \cup H$.

Proof.

(1) $\Rightarrow$ (2): Let $x, y \in X$. Then $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\})$ or $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$. If $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\})$ and $U \in SO_{\lambda^D}(X)$, then for any $U \in SO_{\lambda^D}(X)$, $x \in U$ then $y \in \lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\}) \subseteq U$ then (as X is $\lambda^D - R_0$). If $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$, then there exist $U, V \in SO_{\lambda^D}(X)$ such that $x \in \lambda^D Cl(\{x\}) \subseteq U$, $y \in \lambda^D Cl(\{y\}) \subseteq V$ and $U \cap V = \emptyset$.

(2) $\Rightarrow$ (3): Let $x, y \in X$ such that $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$. Then $x \notin \lambda^D Cl(\{y\})$, so that there exists $G \in SO_{\lambda^D}(X)$, such that $x \in G$ and $y \notin G$. Thus by (2), there exist disjoint λ^D-open sets U and V such that $x \in U$, $y \in V$. Put $F = X \backslash V$ and $H = X \backslash U$. Then $F, H \in SO_{\lambda^D}(X)$, $x \in F$, $y \notin F$, $y \in H$, $x \notin H$ and $X = F \cup H$.

(3) $\Rightarrow$ (1): Let $U \in SO_\lambda(X)$ and $x \in U$. Then $\lambda^D Cl(\{x\}) \subseteq U$. In fact, otherwise there exists $y \in \lambda^D Cl(\{x\}) \cap X \backslash U$. Implies that $\lambda^D Cl(\{x\}) \neq \lambda^D Cl(\{y\})$ (as $x \notin \lambda^D Cl(\{y\})$) and so by (3), there exist $F, H \in SO_{\lambda^D}(X)$ such that $x \in F$, $y \notin F$, $y \in H$, $x \notin H$ and $X = F \cup H$. Then $y \in H \backslash F = X \backslash F$ and $x \notin X \backslash F$, where $X \backslash F \in SO_{\lambda^D}(X)$, which is a contradiction to the fact that $y \in \lambda^D Cl(\{x\})$. Hence $\lambda^D Cl(\{x\}) \subseteq U$. Thus X is $\lambda^D - R_0$. To show X to be $\lambda^D - R_1$. Assume that $a, b \in X$ with $\lambda^D Cl(\{a\}) \neq \lambda^D Cl(\{b\})$. Then as above, there exist $K, L \in SC_{\lambda^D}(X)$ such that $a \in K$, $b \notin K$, $b \in L$, $a \notin L$ and $X = K \cup L$. Thus $a \in K \backslash L \in SO_{\lambda^D}(X)$, $b \in L \backslash K \in SO_{\lambda^D}(X)$. So $\lambda^D Cl(\{a\}) \subseteq K \backslash L$, $\lambda^D Cl(\{b\}) \subseteq L \backslash K$. Thus X is $\lambda^D - R_1$. $\square$

Proposition 2 *Let* (X, τ) *be a topological space and* $\lambda : SO(X) \to \mathcal{P}(X)$ *be an s-operation. Then* X *is* $\lambda^D - R_1$*, if and only if for* $x, y \in X$*, with* $\lambda^D \operatorname{Ker}(\{x\}) \neq \lambda^D \operatorname{Ker}(\{y\})$ *there exist disjoint* λ^D*-open sets* U *and* V *such that* $\lambda^D Cl(\{x\}) \subseteq U$ *and* $\lambda^D Cl(\{y\}) \subseteq V$.

Proof. Follows from Theorem 3 and Definition 6. □

4 Conclusion

Introduced by Alais B. Khalaf and Sarhad F. Namiq [1]. The main results are the following:

(1) Let (X, τ) be a topological space, and $\lambda : SO(X) \to \mathcal{P}(X)$ be an s-operation and $A \subseteq X$. Then $\lambda^D \operatorname{Ker}(\{A\}) = \{x \in X : \lambda^D Cl(\{x\}) \cap A \neq \emptyset\}$.

(2) For any topological space X and any s-operation $\lambda : SO(X) \to \mathcal{P}(X)$, the following statements are equivalent:

a) X is $\lambda^D - R_0$.

b) $F \in SC_{\lambda^D}(X)$ and $x \in F$ implies that $F \subseteq U$ and $x \in U$ for some $U \in SO_{\lambda^D}(X)$.

c) $F \in SC_{\lambda^D}(X)$ and $x \notin F$ implies that $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$.

d) For any two distinct points x, y of X, either $\lambda^D Cl(\{x\}) = \lambda^D Cl(\{y\})$ or $\lambda^D Cl(\{x\}) \cap \lambda^D Cl(\{y\}) = \emptyset$.

(3) Every $\lambda^D - R_1$ space is a $\lambda^D - R_0$ space.

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Soft covered ideals in semigroups

Şerif Özlü
Department of Mathematics
Gaziantep University
27300 Gaziantep, Turkey
email: serif.ozlu@hotmail.com

Aslıhan Sezgin
Department of Mathematics
and Science Education
Amasya University
05100 Amasya, Turkey
email: aslihan.sezgin@amasya.edu.tr

Abstract. In this work, soft covered ideals in semigroups are constructed and in this concept, soft covered semigroups, soft covered left (right) ideals, soft covered interior ideals, soft covered (generalized) bi-ideals and soft covered quasi ideals of a semigroup are defined. Various properties of these ideals are introduced and the interrelations of these soft covered ideals and the relations of soft anti covered ideals and soft covered ideals are investigated.

1 Introduction

Soft set theory introduced by Molodstov in 1999 [1] is applied uncertainness complicated problems especially in economy, medical, engineering and over classical Mathematics such as groups [2], semirings [3], rings [4], BCK/BCI-algebras [5, 6, 7], d-algebras [8], BL-algebras [9], BCH-algebras [10] and near-rings [11].

The works over soft set theory progressed rapidly and after then, this theory started to extend over fuzzy set, rough set and so on and a lot of mathematical structures were constructed over these sets. Moreover, soft set theory has

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received abroad attention by several authors in game theory, intelligent system and especially decision making method. After decision making method was applied over uncertainness sets, it could solve complicated problems including medical diagnoses, finding the best alternative solution for a company for example; the best worker, the best job etc., logistic industry and so on. A relation has been developed between soft set and decision making method and it has overcome several problems[12, 13, 14, 15].

A lot of ideals were introduced over fuzzy sets. Mandal [16] introduced fuzzy ideals and definition of fuzzy interior ideals of ordered semirings. Dib and Galhum [17] defined fuzzy grupoid, a fuzzy semigroup and also fuzzy ideals and fuzzy bi-ideals of a semigroup. Moreover, Kazancıand Yamak [18] gave a kind of generalized fuzzy bi-ideals of semigroups. Kavikumar and Khamis [19] studied fuzzy ideals and fuzzy quasi ideals in ternary semirings. Then, Changphas and Summagrap [20] worked semigroups in covered ideals. Sezer and others wrote several papers [21, 22] over soft ideals in semigroups. In one of papers, they introduced soft union semigroup, ideals and bi-ideals [21]. After then, they defined soft intersection quasi ideals, generalized bi- ideals of a semigroup and surveyed regular, weakly regular, intra regular, completely regular, quasi regular semigroups with help of these ideals [22].

Fabrici [23, 24] investigated covered left, right ideal of a semigroup. By inspiring covered ideals, anti-covered (AC)-left (right), interior ideals were defined in [25] and Xie and Yan [25] obtained fuzzy anti-covered (AC)-left (right), bi-ideal, interior ideal of a semigroup and studied their basic properties. In [26], soft anti covered (AC)-ideals of semigroups were defined and studied in detailed.

In this manuscript, we first give the basic definitions of soft sets and covered ideals in semigroups. By using these basic definitions, we introduce covered ideals of semigroups by defining soft covered semigroups, soft covered left (right) ideals, soft covered bi-ideals, soft covered interior ideals, soft covered quasi-ideals, soft covered generalized bi-ideals. We discuss their properties and the interrelations with each others. Also, we study the relationship between soft anti covered ideals defined in [26] and soft covered ideals which is the most important point in this article.

2 Methodology

From now on, U refers to an initial universe, E is a set of parameters, $\mathsf{P}(\mathsf{U})$ is the power set of U and $\mathsf{A}, \mathsf{B}, \mathsf{C} \subseteq \mathsf{E}$.

Definition 1 [1, 13] *A soft set f_A over U is a set defined by*

$$f_A : E \to P(U) \text{ such that} f_A(x) = \emptyset \text{ if } x \notin A.$$

Here f_A is also called an approximate function. A soft set over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}.$$

It is clear to see that a soft set is a parameterized family of subsets of the set U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2 [13] *Let $f_A, f_B \in S(U)$. Then, f_A is called a soft subset of f_B and denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

Definition 3 [13] *Let f_A, $f_B \in S(U)$. Then, union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is defined as $f_A \tilde{\cup} f_B = f_{A \tilde{\cup} B}$, where $f_{A \tilde{\cup} B}(x) = f_A(x) \cup f_B(x)$, intersection of f_A and f_B, denoted by $f_A \tilde{\cap} f_B$, is defined as $f_A \tilde{\cap} f_B = f_{A \tilde{\cap} B}$, where $f_{A \tilde{\cap} B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.*

Definition 4 [13] *Let f_A, $f_B \in S(U)$. Then, $\tilde{\wedge}$-product of f_A and f_B, denoted by $f_A \tilde{\wedge} f_B$, is defined as $f_A \tilde{\wedge} f_B = f_{A \tilde{\wedge} B}$, where $f_{A \tilde{\wedge} B}(x, y) = f_A(x) \cap f_B(y)$ for all $(x, y) \in E \times E$.*

Definition 5 [27] *Let f_A and f_B be soft sets over the common universe U and Ψ be a function from A to B. Then, soft image of f_A under Ψ, denoted by $\Psi(f_A)$, is a soft set over U by*

$$(\Psi(f_A))(b) = \begin{cases} \bigcap\{f_A(a) \mid a \in A \text{ and } \Psi(a) = b\}, & \text{if } \Psi^{-1}(b) \neq \emptyset, \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $b \in B$. And soft pre-image (or soft inverse image) of f_B under Ψ, denoted by $\Psi^{-1}(f_B)$, is a soft set over U by $(\Psi^{-1}(f_B))(a) = f_B(\Psi(a))$ for all $a \in A$.

From now on, S denotes a semigroup.

Definition 6 [21] *Let f_S and g_S be soft sets over the common universe U. Then, soft union product $f_S * g_S$ is defined by*

$$(f_S * g_S)(x) = \begin{cases} \bigcap_{x=yz}\{f_S(y) \cup g_S(z)\}, & \text{if } \exists y, z \in S \text{ such that } x = yz, \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $x \in S$.

Theorem 1 [21] *Let* $f_S, g_S, h_S \in S(U)$*. Then,*

i) $(f_S * g_S) * h_S = f_S * (g_S * h_S)$.

ii) $f_S * g_S \neq g_S * f_S$, *generally.*

iii) $f_S * (g_S \widetilde{\cup} h_S) = (f_S * g_S) \widetilde{\cup} (f_S * h_S)$ *and* $(f_S \widetilde{\cup} g_S) * h_S = (f_S * h_S) \widetilde{\cup} (g_S * h_S)$.

iv) $f_S * (g_S \widetilde{\cap} h_S) = (f_S * g_S) \widetilde{\cap} (f_S * h_S)$ *and* $(f_S \widetilde{\cap} g_S) * h_S = (f_S * h_S) \widetilde{\cap} (g_S * h_S)$.

v) *If* $f_S \tilde{\subseteq} g_S$*, then* $f_S * h_S \tilde{\subseteq} g_S * h_S$ *and* $h_S * f_S \tilde{\subseteq} h_S * g_S$.

vi) *If* $t_S, l_S \in S(U)$ *such that* $t_S \tilde{\subseteq} f_S$ *and* $l_S \tilde{\subseteq} g_S$*, then* $t_S * l_S \tilde{\subseteq} f_S * g_S$.

From now on, if $f_S : S \to P(U)$ is a soft set satisfying $f_S(x) = \emptyset$ for all $x \in S$, then f_S is denoted by θ and if $f_S : S \to P(U)$ is a soft set satisfying $f_S(x) = U$ for all $x \in S$, then f_S is denoted by $\mathbb{S}$.

Lemma 1 *Let* f_S *be any soft set over* U*. Then, we have the followings.*

i) $\theta * \theta \tilde{\supseteq} \theta$.

ii) $(\mathbb{S} - f_S) * \theta \tilde{\supseteq} \theta$ *and* $\theta * (\mathbb{S} - f_S) \tilde{\supseteq} \theta$.

iii) $(\mathbb{S} - f_S) \widetilde{\cup} \theta = (\mathbb{S} - f_S)$ *and* $(\mathbb{S} - f_S) \widetilde{\cap} \theta = \theta$.

Soft anti covered (AC)-ideals of a semigroup were defined in [26] as following:

Definition 7 [26] f_S *is called as soft AC-semigroup over* U*, if*

$$f_S(xy) \supseteq (\mathbb{S} - f_S)(x) \cap (\mathbb{S} - f_S)(y)$$

for all $x, y \in S$.

Definition 8 [26] f_S *is called a soft AC-left ideal over* U*, if* $f_S(xy) \supseteq (\mathbb{S} - f_S)(y)$*, AC-right ideal of* S *over* U*, if* $f_S(xy) \supseteq (\mathbb{S} - f_S)(x)$*, AC-ideal of* S *over* U*, if* $f_S(xy) \supseteq (\mathbb{S} - f_S)(y)$ *and* $f_S(xy) \supseteq (\mathbb{S} - f_S)(x)$ *for all* $x, y \in S$.

Definition 9 [26] *A soft AC-semigroup* f_S *over* U *is called a soft AC-bi-ideal over* U*, if*

$$f_S(xyz) \supseteq (\mathbb{S} - f_S)(x) \cap (\mathbb{S} - f_S)(z)$$

Definition 10 [26] f_S *is called a soft AC-interior ideal over* U*, if*

$$f_S(xyz) \supseteq (\mathbb{S} - f_S)(y)$$

and is called soft AC-generalized bi-ideal of S*, if*

$$f_S(xyz) \supseteq (\mathbb{S} - f_S)(x) \cap (\mathbb{S} - f_S)(y)$$

for all $x, y, z \in S$*.*

From now on, all covered ideals are denoted by C-ideals for the sake of brevity. C-ideals of a semigroup (and fuzzy C-ideals of a semigroup) are defined in [23, 24] as following:

L is called C-left ideal of S, if $L \subseteq S(S-L)$ and R is called C-right ideal of S, if $R \subseteq (S-R)S$. By *C-two-sided ideal*, it is meant a subset of S, which is both C-left and C-right ideal of S. X is called a *C-interior* of S if $X \subseteq (S-X)S(S-X)$. X is called C-bi-ideal of S if $X \subseteq S(S-X)S$ and X is called C-quasi-ideal of S if $X \subseteq (S-X)S \cup S(S-X)$.

3 Results and discussion

3.1 Soft C-semigroup

In this section, we construct soft covered semigroup and study some properties of it.

Definition 11 *Let* S *be semigroup and* f_S *be soft set over* U*.* f_S *is called a soft covered semigroup, if*

$$f_S(xy) \subseteq (\mathbb{S} - f_S)(x) \cup (\mathbb{S} - f_S)(y)$$

for all $x, y \in S$*.*

From now on, soft covered semigroup is denoted by soft C-semigroup for the sake of brevity.

Example 1 *Consider the semigroup* $S = \{a, b, c, d\}$ *constructed by the following table:*

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $U = D_3 = \{< x, y >: x^3 = y^2 = e, xy = yx^2\} = \{e, x, x^2, y, yx, yx^2\}$ *be the universal set and* f_S *be a soft set over* U *such that* $f_S(a) = \emptyset$, $f_S(b) = \{y\}$, $f_S(c) = \{e, yx^2\}$, $f_S(d) = \{e, x^2, yx, yx^2\}$ *and so* $(\mathbb{S} - f_S)(a) = \{e, x, x^2, y, yx, yx^2\}$, $(\mathbb{S} - f_S)(b) = \{e, x, x^2, yx, yx^2\}$, $(\mathbb{S} - f_S)(c) = \{x, x^2, y, yx\}$, $(\mathbb{S} - f_S)(d) = \{x, y\}$. *One can show that* f_S *is a soft C-semigroup. However, if* $f_S(b) = \{x, x^2, yx\}$, *we can easily show that* f_S *is not a soft C-semigroup.*

It is easy to see that if $f_S(x) = \emptyset$ *for* $x \in S$, *then* f_S *is a soft C-semigroup over* U. *We denote such kind of C-semigroup by* θ.

Definition 12 [21] *Let* X *be a subset of* S. *We denote by* $\mathcal{S}_X^c$ *the soft anti characteristic function of* X *and define as*

$$\mathcal{S}_X^c(x) = \begin{cases} \emptyset, & x \in X \\ U, & x \in \mathcal{S} - X \end{cases}$$

It is clear that soft anti characteristic function is a soft set over U *clearly,*

$$\mathcal{S}_X^c : S \to P(U).$$

Theorem 2 *If* X *is a C-semigroup of* S, *then* $\mathcal{S}_X^c$ *is a soft C-semigroup of* S.

Proof. Let X be a C-semigroup and $x = pq \in X$. Since $X \subseteq (S - X)(S - X)$, then it follows that $x = pq \in (S - X)(S - X)$, and so $p, q \in (S - X)$. In this statement, $\mathcal{S}_X^c(pq) \subseteq (\mathbb{S} - \mathcal{S}_X^c)(p) \cup (\mathbb{S} - \mathcal{S}_X^c)(q)$, that is, $\mathcal{S}_X^c$ is a soft C-semigroup of S. In fact,

$$\begin{aligned} \emptyset &= \mathcal{S}_X^c(pq) \\ &\subseteq (\mathbb{S} - \mathcal{S}_X^c)(p) \cup (\mathbb{S} - \mathcal{S}_X^c)(q) \\ &= (U - U) \cup (U - U) \\ &= \emptyset. \end{aligned}$$

□

Theorem 3 *Let* f_S *be a soft set over* U. *Then,* f_S *is a soft AC-semigroup over* U *of* S *if and only if* f_S^c *is a soft C-semigroup over* U *of* S.

Proof. Let f_S be a soft AC-semigroup over U of S. In this statement,

$$\begin{aligned} f_S^c(xy) &= (U - f_S)(xy) \\ &\subseteq U - ((\mathbb{S} - f_S)(x) \cap (\mathbb{S} - f_S)(y)) \\ &= (U - (\mathbb{S} - f_S)(x)) \cup (U - (\mathbb{S} - f_S)(y)) \\ &= (U - (U - f_S(x)) \cup (U - (U - f_S(y)) \\ &= (U - f_S^c(x)) \cup (U - f_S^c(y)) \\ &= (\mathbb{S} - f_S^c)(x) \cup (\mathbb{S} - f_S^c)(y) \end{aligned}$$

for all $x, y \in S$. Conversely, let f_S^c be a soft C-semigroup over U of S. Then,

$$\begin{array}{rcl} f_S(xy) & = & (U - f_S^c)(xy) \\ & \supseteq & U - ((\mathbb{S} - f_S^c)(x) \cup (\mathbb{S} - f_S^c)(y)) \\ & \supseteq & (U - (\mathbb{S} - f_S^c)(x)) \cap (U - (\mathbb{S} - f_S^c)(y)) \\ & = & (U - (U - f_S^c(x)) \cap (U - (U - f_S^c(y)) \\ & = & (U - f_S(x)) \cap (U - f_S(y)) \\ & = & (\mathbb{S} - f_S)(x)) \cap (\mathbb{S} - f_S)(y)) \end{array}$$

for all $x, y \in S$. This completes the proof. $\square$

Theorem 4 *Let f_S be a soft set over U. Then, f_S is a soft C-semigroup over U if and only if we have:*

$$f_S \widetilde{\subseteq} (\mathbb{S} - f_S) * (\mathbb{S} - f_S).$$

Proof. To prove this, we assume that f_S is a soft C-semigroup over U. If $f_S = \emptyset$, then it is trivial since

$$f_S(x) \subseteq ((\mathbb{S} - f_S) * (\mathbb{S} - f_S))(x)$$

thus, $f_S \subseteq (\mathbb{S} - f_S) * (\mathbb{S} - f_S)$. Otherwise, there exist elements $m, n \in S$ such that $x = mn$. Then, since f_S is a soft C-semigroup over U, we have:

$$\begin{array}{rcl} ((\mathbb{S} - f_S) * (\mathbb{S} - f_S))(x) & = & \bigcap_{x=mn} (\mathbb{S} - f_S)(m) \cup (\mathbb{S} - f_S)(n) \\ & \supseteq & \bigcap_{x=mn} f_S(mn) \\ & = & f_S(x) \end{array}$$

thus, $f_S \widetilde{\subseteq} (\mathbb{S} - f_S) * (\mathbb{S} - f_S)$.

Conversely, suppose that $f_S \widetilde{\subseteq} (\mathbb{S} - f_S) * (\mathbb{S} - f_S)$. Let $m, n \in S$ and $x = mn$. Hence, we have:

$$\begin{array}{rcl} f_S(mn) & = & f_S(x) \\ & \subseteq & ((\mathbb{S} - f_S) * (\mathbb{S} - f_S))(x) \\ & = & \bigcap_{x=mn} (\mathbb{S} - f_S)(m) \cup (\mathbb{S} - f_S)(n) \\ & \subseteq & (\mathbb{S} - f_S)(m) \cup (\mathbb{S} - f_S)(n). \end{array}$$

Then, this means that f_S is a soft C-semigroup over U. This completes the proof. $\square$

Proposition 1 *Let f_S and f_T be soft C-semigroups over U. Then, $f_S \widetilde{\wedge} f_T$ is a soft C-semigroup over U.*

Proof. Let $(x_1, y_1), (x_2, y_2)$ be any two elements of $S \times T$. Then,

$$\begin{aligned} f_{S\tilde{\wedge}T}((x_1,y_1),(x_2,y_2)) &= f_{S\wedge T}(x_1x_2, y_1y_2) \\ &= f_S(x_1x_2) \cap f_T(y_1y_2) \\ &\subseteq [(\mathbb{S}-f_S)(x_1) \cup (\mathbb{S}-f_S)(x_2)] \cap [(\mathbb{S}-f_T)(y_1) \\ &\quad \cup(\mathbb{S}-f_T)(y_2)] \\ &= [U-(f_S(x_1)\cap f_S(x_2))] \cap [(U-(f_T(y_1)\cap f_T(y_2))] \\ &= U-[(f_S(x_1)\cap f_S(x_2)) \cup (f_T(y_1)\cap f_T(y_2))] \\ &\subseteq U-[(f_S(x_1)\cap f_S(x_2)) \cap (f_T(y_1)\cap f_T(y_2))] \\ &= U-[(f_S(x_1)\cap f_T(y_1)) \cap (f_S(x_2)\cap f_T(y_2))] \\ &= (\mathbb{S}-f_{S\tilde{\wedge}T})(x_1,y_1) \cup (\mathbb{S}-f_{S\tilde{\wedge}T})(x_2,y_2). \end{aligned}$$

This implies that $f_S \tilde{\wedge} f_T$ is a soft C-semigroup over U. □

Theorem 5 *Let f_S and g_S be two soft C-semigroups over U, then $f_S \tilde{\cap} g_S$ is also so.*

Proof. Let f_S and g_S be two soft C-semigroups over U for $x, y \in S$. Then,

$$\begin{aligned} (f_S\tilde{\cap}g_S)(xy) &= f_S(xy) \cap g_S(xy) \\ &\subseteq [(\mathbb{S}-f_S)(x) \cup (\mathbb{S}-f_S)(y)] \cap [(\mathbb{S}-g_S)(x) \cup (\mathbb{S}-g_S)(y)] \\ &= [U-(f_S(x)\cap f_S(y))] \cap [U-(g_S(x)\cap g_S(y))] \\ &= U-[(f_S(x)\cap f_S(y)) \cup (g_S(x)\cap g_S(y))] \\ &\subseteq U-[(f_S(x)\cap f_S(y) \cap (g_S(x)\cap g_S(y))] \\ &= U-[(f_S(x)\cap g_S(x)) \cap ((f_S(y)\cap g_S(y)] \\ &= U-[(f_S\tilde{\cap}g_S)(x) \cap (f_S\tilde{\cap}g_S)(y)] \\ &= (\mathbb{S}-(f_S\tilde{\cap}g_S))(x) \cup (\mathbb{S}-(f_S\tilde{\cap}g_S))(y). \end{aligned}$$

This completes the proof. □

The union of two C-semigroups needs not to be a soft C-semigroup as shown in following example.

Example 2 *Let f_S and g_S be two soft sets over $U = D_3$ of semigroup $S = \{a, b, c, d\}$. We accept that f_S is the same soft set in Example 1 and define the soft set, g_S as following; $g_S(a) = \emptyset$, $g_S(b) = \{e, x\}$, $g_S(c) = \{x^2, yx, yx^2\}$ and $g_S(d) = \{y, yx, yx^2\}$. Since $(f_S\tilde{\cup}g_S)(dd) = (f_S\tilde{\cup}g_S)(b) \nsubseteq (\mathbb{S}-(f_S\tilde{\cup}g_S))(d) \cup (\mathbb{S}-(f_S\tilde{\cup}g_S))(d)$, $f_S\tilde{\cup}g_S$ is not a soft C-semigroup over U.*

Proposition 2 *Let f_S and f_T be soft C-semigroups over U and Ψ is a semigroup isomorphism from S to T. If f_S is a soft C-semigroup over U, then $\Psi(f_S)$ is a soft C-semigroup.*

Proof. Let $m_1, m_2 \in T$. Since Ψ is surjective, then there exist $k_1, k_2 \in S$ such that $\Psi(k_1) = m_1$ and $\Psi(k_2) = m_2$. Then,

$$\begin{aligned}
(\Psi(f_S))(m_1 m_2) &= \bigcap\{f_S(k) : k \in S, \Psi(k) = m_1 m_2\} \\
&= \bigcap\{f_S(k) : k \in S, k = \Psi^{-1}(m_1 m_2)\} \\
&= \bigcap\{f_S(k) : k \in S, k = \Psi^{-1}(\Psi(k_1 k_2)) = k_1 k_2\} \\
&= \bigcap\{f_S(k_1 k_2) : k_i \in S, \Psi(k_i) = m_i, i = 1, 2\} \\
&\subseteq \bigcap\{(\mathbb{S} - f_S)(k_1) \cup (\mathbb{S} - f_S)(k_2) : m_i \in S, \Psi(k_i) = m_i, i = 1, 2\} \\
&= (\bigcap\{(\mathbb{S} - f_S)(k_1) : k_1 \in S, \Psi(k_1) = m_1\}) \\
&\quad \cup (\bigcap\{(\mathbb{S} - f_S)(k_2) : k_2 \in S, \Psi(k_2) = m_2\}) \\
&\subseteq (\bigcup\{(\mathbb{S} - f_S)(k_1) : k_1 \in S, \Psi(k_1) = m_1\}) \\
&\quad \cup (\bigcup\{(\mathbb{S} - f_S)(k_2) : k_2 \in S, \Psi(k_2) = m_2\}) \\
&= \{U - \bigcap(f_S(k_1)) : k_1 \in S, \Psi(k_1) = m_1\} \\
&\quad \cup \{U - \bigcap(f_S(k_2)) : k_2 \in S, \Psi(k_2) = m_2\} \\
&= (\mathbb{S} - \Psi(f_S))(m_1) \cup (\mathbb{S} - \Psi(f_S))(m_2).
\end{aligned}$$

Consequently, $\Psi(f_S)$ is a soft C-semigroup over U. □

Proposition 3 *Let f_S and f_T be soft C-semigroups over U and Ψ be a semigroup homomorphism from S to T. If f_T is a soft C-semigroup over U, then so $\Psi^{-1}(f_T)$ is.*

Proof. Suppose $k_1, k_2 \in S$. Then,

$$\begin{aligned}
(\Psi^{-1}(f_T))(k_1 k_2) &= f_T(\Psi(k_1 k_2) \\
&= f_T(\Psi(k_1)\Psi(k_2)) \\
&\subseteq (\mathbb{S} - f_T)(\Psi(k_1)) \cup (\mathbb{S} - f_T)(\Psi(k_2)) \\
&= U - (f_T(\Psi(k_1) \cap f_T(\Psi(k_2)) \\
&= U - (\Psi^{-1}(f_T(k_1)) \cap \Psi^{-1}(f_T(k_2))) \\
&= (\mathbb{S} - \Psi^{-1}(f_T))(k_1) \cup (\mathbb{S} - \Psi^{-1}(f_T))(k_2).
\end{aligned}$$

Hence, $\Psi^{-1}(f_T)$ is a soft C-semigroup over U. □

3.2 Soft C-left (right), C-ideals of semigroups

In this subsection, soft covered left (right) ideal are introduced and also we survey some properties of these ideals.

Definition 13 *Let S be semigroup and f_S be soft set over U. f_S is called a soft covered left ideal over U if $f_S(xy) \subseteq (\mathbb{S} - f_S)(y)$; covered right ideal of S over U if $f_S(xy) \subseteq (\mathbb{S} - f_S)(x)$; covered ideal of S over U if $f_S(xy) \subseteq (\mathbb{S} - f_S)(y)$ and $f_S(xy) \subseteq (\mathbb{S} - f_S)(x)$ for all $x, y \in S$.*

From now on, soft covered left ideal is denoted by soft C-left ideal, soft covered right ideal by soft C-right ideal and soft covered ideal by soft C-ideal for the sake of brevity.

Example 3 *Let* $S = \{a, b, c, d\}$ *in Example 1. Define the soft set* f_S *over* $U = D_4$ *as following,* $f_S(a) = \emptyset$, $f_S(b) = \{x, y\}$, $f_S(c) = \{e, x^2, yx^2\}$, $f_S(d) = \{x^2, yx, yx^2\}$ *and so* $(\mathbb{S} - f_S)(a) = \{e, x, x^2, y, yx, yx^2\}$, $(\mathbb{S} - f_S)(b) = \{e, x^2, yx, yx^2\}$, $(\mathbb{S} - f_S)(c) = \{x, y, yx\}$, $(\mathbb{S} - f_S)(d) = \{e, x, y\}$. *Then,* f_S *forms a soft C-left ideal of* S *over* U *but now suppose* $f_S(b) = \{x, x^2, yx, yx^2\}$*, then since* $f_S(b) = f_S(dd) \nsubseteq (\mathbb{S} - f_S)(d)$*,* f_S *is not a soft C-left ideal.*

It is easy to see that if $f_S(x) = \emptyset$ for $x \in S$, then f_S is a soft C-left (right) ideal over U. We denote such kind of C-left (right) by θ.

Theorem 6 *If* X *is a C-left ideal of* S*, then* $\mathcal{S}_X^c$ *is a soft C-left ideal of* S*.*

Proof. Assume that X is a C-left ideal and $x = mn \in X$. Then, since $X \subseteq S(S - X)$, $x = mn \in S(S - X)$, implying that $m \in S$ and $n \in S - X$. Hence, $\mathcal{S}_X^c(mn) \subseteq (\mathbb{S} - \mathcal{S}_X^c)(n)$. In fact,

$$\emptyset = \mathcal{S}_X^c(mn) \subseteq (\mathbb{S} - \mathcal{S}_X^c)(n) = U - U = \emptyset.$$

Thus, $\mathcal{S}_X^c$ is a soft C-left ideal of S. □

Theorem 7 *Let* f_S *be a soft set over* U*. Then,* f_S *is a soft AC-left (right, ideal) over* U *of* S *if and only if* f_S^c *is a soft C-left (right, ideal) ideal over* U *of* S*.*

Proof. We give the proof for soft AC-left ideals. Let f_S be a soft AC-left ideal over U of S. In this statement,

$$\begin{aligned} f_S^c(xy) &= (U - f_S)(xy) \\ &\subseteq U - (\mathbb{S} - f_S)(y) \\ &\subseteq U - (U - f_S(y)) \\ &= (U - f_S^c(y)) \\ &= (\mathbb{S} - f_S^c)(y) \end{aligned}$$

for all $x, y \in S$. Conversely, let f_S^c be a soft C-left ideal over U of S. Then,

$$\begin{aligned} f_S(xy) &= (U - f_S^c)(xy) \\ &\supseteq U - ((\mathbb{S} - f_S^c))(y) \\ &\supseteq U - (U - f_S^c(y)) \\ &= (U - f_S(y)) \\ &= (\mathbb{S} - f_S)(y) \end{aligned}$$

for all $x, y \in S$. This completes the proof. $\square$

Theorem 8 *Let* f_S *be soft set over* U. *Then,* f_S *is a soft C-left ideal over* U *if and only if*

$$f_S \tilde{\subseteq} \theta * (\mathbb{S} - f_S)$$

Proof. Assume that f_S is a soft C-left ideal over U. In this statement, if $f_S = \emptyset$, it is clear that $f_S \tilde{\subseteq} (\theta * (\mathbb{S} - f_S))$. According to assume, since f_S is a soft C-left ideal for $m, n \in S$ over U, we have:

$$\begin{array}{rcl} (\theta * (\mathbb{S} - f_S))(mn) & = & \bigcap_{s=mn} \theta(m) \cup (\mathbb{S} - f_S)(n) \\ & \supseteq & \bigcap_{s=mn} \emptyset \cup (f_S)(mn) \\ & = & \bigcap_{s=mn} (f_S)(mn) \\ & = & f_S(s) \\ & = & f_S(mn) \end{array}$$

hence, $f_S \tilde{\subseteq} \theta * (\mathbb{S} - f_S)$. Conversely, suppose that $f_S \tilde{\subseteq} (\theta * (\mathbb{S} - f_S))$. Let $m, n \in S$ and $s = mn$. Then, we have:

$$\begin{array}{rcl} (f_S)(mn) & = & (f_S)(s) \\ & \subseteq & (\theta * (\mathbb{S} - f_S))(mn) \\ & = & \bigcap_{mn=xy} \theta(x) \cup (\mathbb{S} - f_S)(y) \\ & \subseteq & \emptyset \cup (\mathbb{S} - f_S)(n) \\ & = & (\mathbb{S} - f_S)(n). \end{array}$$

$\square$

Theorem 9 *Let* f_S *be soft set over* U. *Then,* f_S *is a soft C-right ideal over* U *if and only if*

$$f_S \tilde{\subseteq} (\mathbb{S} - f_S) * \theta.$$

Theorem 10 *Let* f_S *be soft set over* U. *Then,* f_S *is a soft C-ideal over* U *if and only if* $f_S \tilde{\subseteq} \theta * (\mathbb{S} - f_S)$ *and* $f_S \tilde{\subseteq} (\mathbb{S} - f_S) * \theta$.

Theorem 11 *Let* f_S *and* g_S *be two soft subsets of* S *such that* $g_S \tilde{\subseteq} f_S$. *If* f_S *is a soft C-left ideal, then* g_S *is also a soft C-left ideal.*

Proof. First assume that f_S is soft C-left ideal and $g_S \tilde{\subseteq} f_S$. Let $p, q \in S$ and

$s = pq$. Then,

$$\begin{array}{rcl} g_S(pq) \subseteq f_S(pq) & \subseteq & (\theta * (\mathbb{S} - f_S))(pq) \\ & = & \bigcap_{s=pq} \theta(p) \cup (\mathbb{S} - f_S)(q) \\ & = & \bigcap_{s=pq} \emptyset \cup (\mathbb{S} - f_S)(q) \\ & = & \bigcap_{s=pq} (\mathbb{S} - f_S)(q) \\ & \subseteq & \bigcap_{s=pq} (\mathbb{S} - g_S)(q) \\ & = & \bigcap_{s=pq} \theta(p) \cup (\mathbb{S} - g_S)(q) \\ & = & (\theta * (\mathbb{S} - g_S))(pq). \end{array}$$

Consequently, $g_S(pq) \subseteq (\theta * (\mathbb{S} - g_S))(pq)$ meaning that g_S is a soft C-left ideal. □

Definition 14 *Soft union left and soft C-left ideal* f_S *of* S *is called a completely soft C-left ideal of* S.

Theorem 12 *Let* f_S *be a soft subset of* S. *Then, the followings are equivalent.*

i) f_S *is a completely soft C-left ideal of* S.

ii) $(\forall x, y \in S) f_S(xy) \subseteq f_S(y) \cap (\mathbb{S} - f_S)(y)$.

iii) $(\forall s \in S^2) f_S(s) \subseteq \bigcap_{s=xy} f_S(y) \cap \bigcap_{s=xy} (\mathbb{S} - f_S)(y)$.

Proof.

i $\Rightarrow$ ii Suppose that f_S is a completely soft C-left ideal of S. Then, since f_S is a soft C-left ideal $f_S(xy) \subseteq (\mathbb{S} - f_S)(y)$. Since f_S is a soft union left ideal of S, $f_S(xy) \subseteq f_S(y)$. This means that $f_S(xy) \subseteq f_S(y) \cap (\mathbb{S} - f_S)(y)$.

ii $\Rightarrow$ iii Accept that $(\forall x, y \in S) f_S(xy) \subseteq f_S(y) \cap (\mathbb{S} - f_S)(y)$. Then, by taking into account the sets, $f_S(xy) \subseteq f_S(y)$ and $f_S(xy) \subseteq (\mathbb{S} - f_S)(y)$. Let $s = xy$ for any $x, y \in S$. This implies that $f_S(s) = f_S(xy) \subseteq \bigcap_{s=xy} f_S(y) \cap \bigcap_{s=xy} (\mathbb{S} - f_S)(y)$.

iii $\Rightarrow$ i Assume that $\forall s = xy \in S^2$, $f_S(s) \subseteq \bigcap_{s=xy} f_S(y) \cap \bigcap_{x=yz} (\mathbb{S} - f_S)(y)$. Thus, $f_S(xy) \subseteq \bigcap_{s=xy} f_S(y)$ and $f_S(xy) \subseteq \bigcap_{s=xy} (\mathbb{S} - f_S)(y)$. Then,

$$\begin{array}{rcl} f_S(s) = f_S(xy) & \subseteq & \bigcap_{s=xy} (\mathbb{S} - f_S)(y) \\ & = & \bigcap_{s=xy} \emptyset \cup (\mathbb{S} - f_S)(y) \\ & = & \bigcap_{s=xy} \theta(x) \cup (\mathbb{S} - f_S)(y) \\ & = & (\theta * (\mathbb{S} - f_S))(s) \end{array}$$

and also,

$$\begin{array}{rcl} f_S(xy) & \subseteq & \bigcap_{s=xy} f_S(y) \\ & \subseteq & f_S(y). \end{array}$$

Clearly, f_S is a completely soft C-left ideal of S. □

Theorem 13 *If* f_S *and* g_S *are two soft C-left ideals of* S *over* U*, then* $f_S\widetilde{\cap}g_S$ *is a soft C-left ideal over* U*.*

Proof. Assume that f_S and g_S are two soft C-left ideals. Then,

$$\begin{aligned}(f_S\widetilde{\cap}g_S)(mn) &= f_S(mn)\cap g_S(mn)\\ &\subseteq (\mathbb{S}-f_S)(n)\cap(\mathbb{S}-g_S)(n)\\ &= U-(f_S(n)\cup g_S(n))\\ &\subseteq U-(f_S(n)\cap g_S(n))\\ &= (\mathbb{S}-(f_S\widetilde{\cap}g_S))(n)\end{aligned}$$

for all $m,n\in S$. This means that $f_S\widetilde{\cap}g_S$ is a soft C-left ideal over U.

Now, we show that if f_S and g_S are two soft C-left ideals of S over U, then $f_S\widetilde{\cup}g_S$ is not a soft C-left ideal of S with the following example. □

Example 4 *Let consider the semigroup* $S=\{a,b,c,d\}$ *over* $U=D_4$ *in Example* 1 *and introduce the soft set* f_S *over* U *as following* $f_S(a)=\emptyset$, $f_S(b)=\{e,x\}$, $f_S(c)=\{x^2,yx,yx^2\}$, $f_S(d)=\{y,x^2,yx^2\}$ *so* $(\mathbb{S}-f_S)(a)=\{e,x,x^2,y,yx,yx^2\}$, $(\mathbb{S}-f_S)(b)=\{y,x^2,yx,yx^2\}$, $(\mathbb{S}-f_S)(c)=\{e,x,y\}$, $(\mathbb{S}-f_S)(d)=\{e,x,yx\}$. *Also let* $g_S(a)=\emptyset$, $g_S(b)=\{e\}$, $g_S(c)=\{y,x^2,yx,yx^2\}$, $g_S(d)=\{x,x^2,yx,yx^2\}$ *so* $(\mathbb{S}-g_S)(a)=\{e,x,x^2,y,yx\ ,yx^2\}$, $(\mathbb{S}-g_S)(b)=\{x,y,x^2,yx^2\}$, $(\mathbb{S}-g_S)(c)=\{e,x\}$, $(\mathbb{S}-g_S)(d)=\{e,y\}$. *This shows that* $(f_S\widetilde{\cup}g_S)(dd)=(f_S\widetilde{\cup}g_S)(b)\nsubseteq(\mathbb{S}-(f_S\widetilde{\cup}g_S))(d)$.

Proposition 4 f_S *is a soft C-ideal over* U *of* S *if and only if*

$$f_S(mn)\subseteq(\mathbb{S}-f_S)(m)\cap(\mathbb{S}-f_S)(n)$$

for all $m,n\in S$*.*

Proof. Let f_S be a soft C-ideal of S over U and $m,n\in S$. Then, $f_S(mn)\subseteq(\mathbb{S}-f_S)(m)$ and $f_S(mn)\subseteq(\mathbb{S}-f_S)(n)$. Then, $f_S(mn)\subseteq(\mathbb{S}-f_S)(m)\cap(\mathbb{S}-f_S)(n)$.

Conversely, suppose that $f_S(mn)\subseteq(\mathbb{S}-f_S)(m)\cap(\mathbb{S}-f_S)(n)$ for all $m,n\in S$. It follows that

$$f_S(mn)\subseteq(\mathbb{S}-f_S)(m)\cap(\mathbb{S}-f_S)(n)\subseteq(\mathbb{S}-f_S)(m)$$

and

$$f_S(mn)\subseteq(\mathbb{S}-f_S)(m)\cap(\mathbb{S}-f_S)(n)\subseteq(\mathbb{S}-f_S)(n)$$

so f_S is a soft C-ideal of S over U. □

Theorem 14 *Let* f_S *be soft set over* U*. If* f_S *is a soft C-left (right) ideal over* U*,* f_S *is a soft C-semigroup over* U*.*

Proof. We give the proof for soft C-left ideals. Similarly, it can be indicated for soft C-right ideals. Let f_S be a soft C-left ideal of S over U. Then, $f_S(pq) \subseteq (\mathbb{S}{-}f_S)(q)$ for all $p, q \in S$. Thus, $f_S(pq) \subseteq (\mathbb{S}{-}f_S)(q) \subseteq (\mathbb{S}{-}f_S)(p) \cup (\mathbb{S}{-}f_S)(q)$, therefore f_S is a soft C-semigroup. □

Proposition 5 *Let* f_S *be soft set over* U*. Then,* $(\mathbb{S}-f_S)\widetilde{\cap}(\theta * (\mathbb{S}-f_S))$ *is a soft C-left ideal over* U *and* $(\mathbb{S}-f_S)\widetilde{\cap}((\mathbb{S}-f_S) * \theta)$ *is a soft C-right ideal of* S *over* U*.*

Proof. Assume that f_S is a soft C-left ideal of S. Then,

$$\begin{aligned}
\theta * [((\mathbb{S}-f_S)\widetilde{\cap}(\theta * (\mathbb{S}-f_S)))] &= [(\theta * (\mathbb{S}-f_S))]\widetilde{\cap}[(\theta * (\theta * (\mathbb{S}-f_S))] \\
&= (\theta * (\mathbb{S}-f_S))\widetilde{\cap}((\theta * \theta) * (\mathbb{S}-f_S)) \\
&\widetilde{\supseteq} (\theta * (\mathbb{S}-f_S))\widetilde{\cap}((\theta * (\mathbb{S}-f_S)) \\
&= (\theta * (\mathbb{S}-f_S)) \\
&\widetilde{\supseteq} (\mathbb{S}-f_S)\widetilde{\cap}(\theta * (\mathbb{S}-f_S)).
\end{aligned}$$

Clearly, $(\mathbb{S}-f_S)\widetilde{\cap}(\theta * (\mathbb{S}-f_S))$ is a soft C-left ideal of S over U. Also,

$$\begin{aligned}
[((\mathbb{S}-f_S)\widetilde{\cap}((\mathbb{S}-f_S) * \theta))] * \theta &= [((\mathbb{S}-f_S) * \theta)]\widetilde{\cap}[(((\mathbb{S}-f_S) * \theta) * \theta)] \\
&= ((\mathbb{S}-f_S) * \theta)\widetilde{\cap}(((\mathbb{S}-f_S) * (\theta * \theta)) \\
&\widetilde{\supseteq} ((\mathbb{S}-f_S) * \theta)\widetilde{\cap}((\mathbb{S}-f_S) * \theta) \\
&= ((\mathbb{S}-f_S) * \theta) \\
&\widetilde{\supseteq} (\mathbb{S}-f_S)\widetilde{\cap}((\mathbb{S}-f_S) * \theta).
\end{aligned}$$

Hence, $(\mathbb{S}-f_S)\widetilde{\cap}((\mathbb{S}-f_S) * \theta)$ is a soft C-right ideal of S over U. □

Theorem 15 *Let* f_S *and* g_S *be a soft C-right ideal and soft C-left ideal of* S *over* U*, respectively. Then,*

$$f_S \widetilde{\cup} g_S \widetilde{\subseteq} (\mathbb{S}-f_S) * (\mathbb{S}-g_S).$$

Proof. We know that f_S is a soft C-R-right ideal of S over U and g_S is a soft C-left ideal of S over U and also $\theta \widetilde{\subseteq} (\mathbb{S}-f_S), \theta \widetilde{\subseteq} (\mathbb{S}-g_S)$. Thus,

$$g_S \widetilde{\subseteq} \theta * (\mathbb{S}-g_S) \widetilde{\subseteq} (\mathbb{S}-f_S) * (\mathbb{S}-g_S)$$

and

$$f_S \widetilde{\subseteq} (\mathbb{S}-f_S) * \theta \widetilde{\subseteq} (\mathbb{S}-f_S) * (\mathbb{S}-g_S)$$

from here

$$f_S \widetilde{\cup} g_S \widetilde{\subseteq} (\mathbb{S} - f_S) * (\mathbb{S} - g_S).$$

Now, we survey that again, let f_S and g_S be a soft C-right ideal a soft C-left ideal of S over U, respectively. In this statement,

$$f_S \widetilde{\cap} g_S \widetilde{\not\supseteq} (\mathbb{S} - f_S) * (\mathbb{S} - g_S).$$

with the following example. □

Example 5 *Think the semigroup* $S = \{a, b, c, d\}$ *over* $U = D_4$ *in Example* 1 *and let the soft set* f_S *and* g_S *be* $f_S(b) = \{x, yx\}$, $(\mathbb{S} - f_S)(c) = \{e, x, yx\}$, $(\mathbb{S} - f_S)(d) = \{x, y, yx\}$ *and* $g_S(b) = \{yx, yx^2\}$, $(\mathbb{S} - g_S)(c) = \{x, yx, yx^2\}$, $(\mathbb{S} - g_S)(d) = \{yx, yx^2\}$. *Then,* f_S *is a soft C-right ideal and* g_S *is a soft C-left ideal.*

$$\begin{aligned}((\mathbb{S}-f_S) * (\mathbb{S}-g_S))(b) &= \{(\mathbb{S}-f_S)(d) \cup (\mathbb{S}-g_S)(d)\} \cap \{(\mathbb{S}-f_S)(c) \cup (\mathbb{S}-g_S)(c)\} \\ &\cap \{(\mathbb{S} - f_S)(d) \cup (\mathbb{S} - g_S)(c)\} \\ &= \{x, yx, yx^2\} \\ &\not\subseteq (f_S \widetilde{\cap} g_S)(b) \\ &= \{yx\}.\end{aligned}$$

Proposition 6 *Let* f_S *and* f_T *be soft C-left (right) ideals of* S *over* U. *Again,* $f_S \widetilde{\wedge} f_T$ *is a soft C-left (right) ideal of* $S \times T$ *over* U.

Proof. We accept that f_S and f_T are soft C-left ideals of S over U and $(x_1, y_1), (x_2, y_2) \in S \times T$,

$$\begin{aligned} f_{S\widetilde{\wedge}T}((x_1, y_1), (x_2, y_2)) &= f_{S\widetilde{\wedge}T}(x_1x_2, y_1y_2) \\ &= f_S(x_1x_2) \cap f_T(y_1y_2) \\ &\subseteq (\mathbb{S} - f_S)(x_2) \cap (\mathbb{S} - f_T)(y_2) \\ &= U - (f_S(x_2) \cup f_T(y_2)) \\ &\subseteq U - (f_S(x_2) \cap f_T(y_2)) \\ &= (\mathbb{S} - f_{S\widetilde{\wedge}T})(x_2, y_2). \end{aligned}$$

Therefore, $f_S \widetilde{\wedge} f_T$ is a soft C-left ideal over U. □

We give following propositions without proof. The proofs are similar to those in section 2.

Proposition 7 *Let* f_S *and* h_S *be soft sets over* U *and* Ψ *is a semigroup isomorphism from* S *to* T. *If* f_S *is a soft C-left (right) ideal of* S *over* U, *then so is* $\Psi(f_S)$ *of* T *over* U.

Proposition 8 *Let* f_S *and* h_S *be soft sets over* U *and* Ψ *is a semigroup homomorphism from* S *to* T*. If* f_T *is a soft C-left (right) ideal of* T *over* U*, then so is* $\Psi^{-1}(f_T)$ *of* S *over* U*.*

3.3 Soft C-bi-ideals of semigroups

In this subsection, we define soft covered bi-ideals and provide their basic properties by using soft set operations and soft intersection products and also support them with examples.

Definition 15 *A soft C-semigroup* f_S *over* U *is called a soft covered bi-ideal over* U *if*

$$f_S(xyz) \subseteq (\mathbb{S} - f_S)(x) \cup (\mathbb{S} - f_S)(z).$$

For the sake of brevity, soft covered bi-ideal is denoted by soft C-bi-ideal.

Example 6 *Define operation over* $S = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ *by the following table:*

.	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

Now let $U = D_2 = \{e, x, y, yx\}$ *be universal set and* f_S *be a soft set over* U *defined by* $f_S(0) = \emptyset$, $f_S(1) = \{x\}$, $f_S(2) = \{x, y\}$, $f_S(3) = \{yx\}$ *and so* $(\mathbb{S} - f_S)(0) = U$, $(\mathbb{S} - f_S)(1) = \{e, y, yx\}$, $(\mathbb{S} - f_S)(2) = \{e, yx\}$, $(\mathbb{S} - f_S)(3) = \{e, x, y\}$*. Then,* f_S *is a soft C-bi-ideal, but if* $f_S(1) = \{e, yx\}$*, then* f_S *is not a soft C-bi-ideal.*

It is easy to see that if $f_S(x) = \emptyset$ for $x \in S$, then f_S is a soft C-bi-ideal over U. We denote such kind of C-bi-ideal by θ.

Theorem 16 *If* X *is a C-bi-ideal of* S*, then* $\mathcal{S}_X$ *is a soft C-bi-ideal of* S*.*

Proof. We accept that X is a C-bi-ideal of S. Let $x = mnp \in X$, then it is clear that $x = mnp \in (S - X)S(S - X)$, implying that $m, p \in S - X$ and $n \in S$. In this statement, $\mathcal{S}_X(mnp) \subseteq (\mathbb{S} - \mathcal{S}_X)(m)) \cup (\mathbb{S} - \mathcal{S}_X)(p)$. In fact,

$$\begin{aligned}
\emptyset &= \mathcal{S}_X^c(mnp) \\
&\subseteq (\mathbb{S} - \mathcal{S}_X^c)(m) \cup (\mathbb{S} - \mathcal{S}_X^c)(p) \\
&= (U - U) \cup (U - U) \\
&= \emptyset.
\end{aligned}$$

□

Theorem 17 *Let f_S be a soft set over U. Then, f_S is a soft AC-bi-ideal over U of S if and only if f_S^c is a soft C-bi-ideal over U of S.*

Proof. Let f_S be a soft AC-bi-ideal over U of S. In this statement,

$$\begin{array}{rcl} f_S^c(xyz) & = & (U - f_S)(xyz) \\ & \subseteq & U - ((\mathbb{S} - f_S)(x) \cap (\mathbb{S} - f_S)(z)) \\ & = & U - ((U - f_S(x)) \cap (U - f_S(z)) \\ & = & (U - (U - f_S(x)) \cup (U - (U - f_S(z)) \\ & = & (U - f_S^c(x)) \cup (U - f_S^c(z)) \\ & = & (\mathbb{S} - f_S^c)(x) \cup (\mathbb{S} - f_S^c)(z) \end{array}$$

for all $x, y, z \in S$. Conversely, let f_S^c be a soft C-bi-ideal over U of S. Then,

$$\begin{array}{rcl} f_S(xyz) & = & (U - f_S^c)(xyz) \\ & \supseteq & U - ((\mathbb{S} - f_S^c)(x) \cup (\mathbb{S} - f_S^c)(z)) \\ & = & (U - (U - f_S^c(x)) \cap (U - (U - f_S^c(z)) \\ & = & (U - f_S(x)) \cap (U - f_S(z)) \\ & = & (\mathbb{S} - f_S)(x) \cap (\mathbb{S} - f_S)(z) \end{array}$$

for all $x, y, z \in S$. This completes the proof. □

Theorem 18 *A soft subset f_S of S is a soft C-bi-ideal if and only if*

$$f_S \widetilde{\subseteq} (\mathbb{S} - f_S) * \theta * (\mathbb{S} - f_S)$$

Proof. Let f_S be a soft C-bi-ideal of S. Assume that $f_S = \emptyset$, then it is clear $f_S \widetilde{\subseteq} ((\mathbb{S} - f_S) * \theta) * (\mathbb{S} - f_S))$. Otherwise, let $a, b, t, p, z \in S$. Since f_S is a soft C-bi-ideal of S over U, then,

$$\begin{array}{rcl} ((\mathbb{S} - f_S) * \theta * (\mathbb{S} - f_S))(s) & = & \bigcap_{s=ab}[((\mathbb{S} - f_S) * \theta)(a) \cup (\mathbb{S} - f_S)(b)] \\ & = & \bigcap_{s=ab}[\bigcap_{a=tp}(\mathbb{S} - f_S)(t) \cup \theta(p) \cup (\mathbb{S} - f_S)(b)] \\ & = & \bigcap_{s=ab}[\bigcap_{a=tp}(\mathbb{S} - f_S)(t) \cup \emptyset \cup (\mathbb{S} - f_S)(b)] \\ & = & \bigcap_{s=ab}\bigcap_{a=tp}(\mathbb{S} - f_S)(t) \cup (\mathbb{S} - f_S)(b) \\ & \supseteq & \bigcap_{s=ab}\bigcap_{a=tp} f_S(tpb) \\ & = & \bigcap_{s=ab} f_S(ab) \\ & = & f_S(s) \end{array}$$

hence, $f_S \widetilde{\subseteq} ((\mathbb{S} - f_S) * \theta * (\mathbb{S} - f_S))$.

Conversely, let us assume that $f_S \widetilde{\subseteq} ((\mathbb{S}-f_S)*\theta)*(\mathbb{S}-f_S))$. Let $s = abz \in S$,

$$\begin{array}{rcl} f_S(abz) & = & f_S(s) \\ & \subseteq & ((\mathbb{S}-f_S)*\theta*(\mathbb{S}-f_S))(s) \\ & = & \bigcap_{s=abz=mn}((\mathbb{S}-f_S)*\theta)(m)\cup(\mathbb{S}-f_S)(n)) \\ & \subseteq & ((\mathbb{S}-f_S)*\theta)(ab)\cup(\mathbb{S}-f_S)(z)) \\ & = & \bigcap_{ab=tp}(\mathbb{S}-f_S)(t)\cup\theta(p)\cup(\mathbb{S}-f_S)(z) \\ & \subseteq & (\mathbb{S}-f_S)(a)\cup\emptyset\cup(\mathbb{S}-f_S)(z) \\ & = & (\mathbb{S}-f_S)(a)\cup(\mathbb{S}-f_S)(z). \end{array}$$

Consequently, f_S is a soft C-bi-ideal of S over U. This completes the proof. □

Theorem 19 *The intersection of two soft C-bi-ideals over* U *is a soft C-bi-ideal over* U.

Proof. Let f_S and g_S be soft C-bi-ideals over U. Then,

$$\begin{array}{rcl} (f_S\widetilde{\cap}g_S)(mnp) & = & f_S(mnp)\cap g_S(mnp) \\ & \subseteq & [(\mathbb{S}-f_S)(m)\cup(\mathbb{S}-f_S)(p)]\cap[(\mathbb{S}-g_S)(m)\cup(\mathbb{S}-g_S)(p)] \\ & = & [U-(f_S(m)\cap f_S(p))]\cap[U-(g_S(m)\cap g_S(p))] \\ & = & U-[(f_S(m)\cap f_S(p))\cup(g_S(m)\cap g_S(p))] \\ & \subseteq & U-[(f_S(m)\cap f_S(p)\cap(g_S(m)\cap g_S(p))] \\ & = & U-((f_S(m)\cap g_S(m))\cap((f_S(p)\cap g_S(p)) \\ & = & U-((f_S\widetilde{\cap}g_S)(m)\cap(f_S\widetilde{\cap}g_S)(p)) \\ & = & (U-(f_S\widetilde{\cap}g_S)(m))\cup(U-(f_S\widetilde{\cap}g_S)(p)) \\ & = & (\mathbb{S}-(f_S\widetilde{\cap}g_S))(m)\cup(\mathbb{S}-(f_S\widetilde{\cap}g_S))(p) \end{array}$$

for $m, n, p \in S$. This completes the proof. Now, we show that if f_S and g_S are two C-bi-ideals of S over U, then $f_S\widetilde{\cup}g_S$ is not a soft C-bi-ideal of S with the following example. □

Example 7 *Consider the semigroup* $S = Z_4 = \{0,1,2,3\}$ *and define the soft set* f_S *and* g_S *over* $U = D_2$ *in Example 6 as following, respectively.* $f_S(0) = \emptyset$, $f_S(1) = \{x\}$, $f_S(2) = \{e\}$, $f_S(3) = \{y, yx\}$ *so* $(\mathbb{S}-f_S)(0) = U$, $(\mathbb{S}-f_S)(1) = \{e, y, yx\}$, $(\mathbb{S}-f_S)(2) = \{x, y, yx\}$, $(\mathbb{S}-f_S)(3) = \{e, x\}$ *and* $g_S(0) = \emptyset$, $g_S(1) = \{e\}$, $g_S(2) = \{e\}$, $g_S(3) = \{x, yx\}$ *so* $(\mathbb{S}-g_S)(0) = U$, $(\mathbb{S}-g_S)(1) = \{x, y, yx\}$, $(\mathbb{S}-g_S)(2) = \{x, y, yx\}$, $(\mathbb{S}-g_S)(3) = \{e, y\}$. *Then,* f_S *and* g_S *are two C-bi-ideals of* S *over* U, *since* $(f_S\widetilde{\cup}g_S)(333) = (f_S\widetilde{\cup}g_S)(1) \nsubseteq (\mathbb{S}-(f_S\widetilde{\cup}g_S)(3))\cup(\mathbb{S}-(f_S\widetilde{\cup}g_S)(3))$, $(f_S\widetilde{\cup}g_S)$ *is not a soft C-bi-ideal.*

Theorem 20 *Every soft C-left (right) ideal of semigroup* S *over* U *is a soft C-bi-ideal of* S *over* U.

Proof. Assume that f_S is a soft C-Left ideal of S over U for $m, p, q \in S$. Then,

$$f_S(mpq) = f_S((mp)q) \subseteq (\mathbb{S} - f_S)(q) \subseteq (\mathbb{S} - f_S)(m) \cup (\mathbb{S} - f_S)(q).$$

Hence, f_S is a soft C-bi-ideal of S. □

Proposition 9 *Let* f_S *and* f_T *be soft C-bi-ideals of* S *over* U. *Then,* $f_S \widetilde{\wedge} f_T$ *is a soft C-bi-ideal of* $S \times T$ *over* U.

Proof. We know that f_S and f_T are soft C-bi-ideals and there exists (x_1, y_1), $(x_2, y_2), (x_3, y_3) \in S \times T$,

$$\begin{aligned}
f_{S\widetilde{\wedge}T}((x_1, y_1), (x_2, y_2), (x_3, y_3)) &= f_{S\wedge T}(x_1x_2x_3, y_1y_2y_3)) \\
&= f_S(x_1x_2x_3) \cap f_T(y_1y_2y_3) \\
&\subseteq [(\mathbb{S} - f_S)(x_1) \cup (\mathbb{S} - f_S)(x_3)] \cap [(\mathbb{S} - f_T)(y_1) \\
&\cup\ (\mathbb{S} - f_T)(y_3)] \\
&= [U-(f_S(x_1)\cap f_S(x_3))]\cap[(U-(f_T(y_1)\cap f_T(y_3))] \\
&= U - [(f_S(x_1) \cap f_S(x_3)) \cup (f_T(y_1) \cap f_T(y_3))] \\
&\subseteq U - [(f_S(x_1) \cap f_S(x_3)) \cap (f_T(y_1) \cap f_T(y_3))] \\
&= U - [(f_S(x_1) \cap f_T(x_3)) \cap (f_S(y_1) \cap f_T(y_3))] \\
&= (\mathbb{S} - f_{S\widetilde{\wedge}T})(x_1, y_1) \cup (\mathbb{S} - f_{S\widetilde{\wedge}T})(x_3, y_3).
\end{aligned}$$

This shows that $f_S \widetilde{\wedge} f_T$ is a soft C-bi-ideal over U. □

Proposition 10 *Let* f_S *and* h_S *be soft sets over* U *and* Ψ *is a semigroup isomorphism from* S *to* T. *If* f_S *is a soft C-bi-ideal of* S *over* U, *then so is* $\Psi(f_S)$ *of* T *over* U.

Proposition 11 *Let* f_S *and* h_S *be soft sets over* U *and* Ψ *is a semigroup homomorphism from* S *to* T. *If* f_T *is a soft C-bi-ideal of* T *over* U, *then so is* $\Psi^{-1}(f_T)$ *of* S *over* U.

3.4 Soft C-interior ideal of semigroups

In this section, we introduce soft covered interior ideals of semigroups, obtain their basic properties with respect to soft operations and soft intersection product.

Definition 16 *Let* f_S *be soft set over* U *and* $x, y, z \in S$*. If*

$$f_S(xyz) \subseteq (\mathbb{S} - f_S)(y)$$

f_S *called a soft covered interior ideal over* U*.*

For the sake of brevity, soft covered interior ideal is denoted by soft C-interior ideal over U.

Example 8 *Let think the semigroup* $\mathbb{Z}_4$ *and the soft set* f_S *over* $U = D_2 = \{e, x, y, yx\}$ *in Example 6. Then, one can easily show that* f_S *is a soft C-interior ideal of* S *over* U*. But we accept that* $f_S(1) = \{e, y\}$*, then* $f_S(333) = f_S(1) \nsubseteq (\mathbb{S} - f_S)(3)$ *which implies* f_S *is not a soft C-interior ideal.*

It is easy to see that if $f_S(x) = \emptyset$ for $x \in S$, then f_S is a soft C-interior-ideal over U. We denote such kind of C-interior-ideal by θ.

Theorem 21 *If* X *is a C-interior ideal of* S*, then* $\mathcal{S}_X^c$ *is a soft C-interior ideal of* S*.*

Proof. Let X be C-interior ideal and $x = mnp \in X$. Since $X \subseteq S(S - X)S$, then $x = mnp \in S(S - X)S$, implying that $m, p \in S$ and $n \in S - X$. In this statement, $\mathcal{S}_X^c(mnp) \subseteq (\mathbb{S} - \mathcal{S}_X^c)(n)$. In fact,

$$\begin{aligned} \emptyset &= \mathcal{S}_X^c(mnp) \\ &\subseteq (\mathbb{S} - \mathcal{S}_X^c)(n) \\ &= U - U \\ &= \emptyset. \end{aligned}$$

□

Theorem 22 *Let* f_S *be a soft set over* U*. Then,* f_S *is a soft AC-interior ideal over* U *of* S *if and only if* f_S^c *is a soft C-interior ideal over* U *of* S*.*

Proof. Let f_S is a soft AC-interior ideal over U of S. In this statement,

$$\begin{aligned} f_S^c(xyz) &= (U - f_S)(xyz) \\ &\subseteq U - (\mathbb{S} - f_S)(y) \\ &= U - (U - f_S(y)) \\ &= U - f_S^c(y) \\ &= (\mathbb{S} - f_S^c)(y) \end{aligned}$$

for all $x, y, z \in S$. Conversely, let f_S^c be a soft C-interior ideal over U of S. Then,

$$\begin{aligned} f_S(xyz) &= (U - f_S^c)(xyz) \\ &\supseteq U - ((\mathbb{S} - f_S^c)(y)) \\ &= U - (U - f_S^c(y)) \\ &= U - f_S(y) \\ &= (\mathbb{S} - f_S)(y) \end{aligned}$$

for all $x, y, z \in S$. This completes the proof. □

Theorem 23 *Let* f_S *be soft set over* U*. Then,* f_S *is a soft C-interior ideal over* U *if and only if*

$$f_S \widetilde{\subseteq} \theta * (\mathbb{S} - f_S) * \theta$$

Proof. Let f_S be a soft C-interior ideal over U and $x \in S$. If $f_S = \emptyset$, it is clear that $f_S(x) \widetilde{\subseteq} (\theta * (\mathbb{S} - f_S) * \theta)(x)$, thus $f_S \widetilde{\subseteq} \theta * (\mathbb{S} - f_S) * \theta$. If there exist elements y, z, u, v of S such that $x = yz$ and $y = mp$, we can write:

$$f_S(x) = f_S(yz) = f_S(mpz) \subseteq (\mathbb{S} - f_S)(p).$$

Then,

$$\begin{aligned} (\theta * (\mathbb{S} - f_S) * \theta)(x) &= ((\theta * (\mathbb{S} - f_S)) * \theta)(x) \\ &= \bigcap_{x=yz} (\theta * (\mathbb{S} - f_S))(y) \cup \theta(z) \\ &= \bigcap_{x=yz} [\bigcap_{y=mp} (\theta(m) \cup (\mathbb{S} - f_S)(p))] \cup \theta(z) \\ &= \bigcap_{x=yz} \bigcap_{y=mp} (\emptyset \cup (\mathbb{S} - f_S)(p)) \cup \emptyset \\ &\supseteq \bigcap_{x=yz} \bigcap_{y=mp} \emptyset \cup f_S(mpz) \cup \emptyset \\ &= f_S(x). \end{aligned}$$

Thus, $f_S \widetilde{\subseteq} \theta * (\mathbb{S} - f_S) * \theta$.

Conversely, accept that $f_S \widetilde{\subseteq} \theta * (\mathbb{S} - f_S) * \theta$ for $x, a, y, m, n, p, q \in S$. Then,

$$\begin{aligned} f_S(xay) &\subseteq (\theta * (\mathbb{S} - f_S) * \theta)(xay) \\ &= \bigcap_{xay=mn} \{(\theta * (\mathbb{S} - f_S))(m) * \theta(n)\} \\ &\subseteq (\theta * (\mathbb{S} - f_S))(xa) \cup \theta(y) \\ &= (\theta * (\mathbb{S} - f_S))(xa) \cup \emptyset \\ &= \bigcap_{xa=pq} \theta(p) \cup (\mathbb{S} - f_S)(q) \\ &\subseteq \theta(x) \cup (\mathbb{S} - f_S)(a) \\ &= (\mathbb{S} - f_S)(a) \end{aligned}$$

hence, f_S is a soft C-interior ideal. This completes the proof. □

Theorem 24 *Every soft union right C-Left ideal of S is a soft C-interior ideal of S.*

Proof. We accept that f_S is a soft union right C-Left ideal. Then,

$$\begin{array}{rcl}(\theta * (\mathbb{S}-f_S) * \theta)(mnp) & = & [(\theta * (\mathbb{S}-f_S)] * \theta)(mnp) \\ & = & \bigcap_{mnp=uv}[(\theta * (\mathbb{S}-f_S)](u) \cup \theta(v) \\ & = & \bigcap_{mnp=uv}(\theta * (\mathbb{S}-f_S))(u) \\ & \supseteq & \bigcap_{mnp=uv} f_S(u) \\ & \supseteq & \bigcap_{mnp=uv} f_S(uv) \\ & = & f_S(mnp)\end{array}$$

f_S is a C-interior ideal of S. □

Theorem 25 *Let f_S and g_S be two soft C-interior ideals of S over U. Then, $f_S \widetilde{\cap} g_S$ is a soft C-interior ideal over U.*

Proof. Let $m, n, p \in S$. Then,

$$\begin{array}{rcl}(f_S \widetilde{\cap} g_S)(mnp) & = & f_S(mnp) \cap g_S(mnp) \\ & \subseteq & (\mathbb{S}-f_S)(n) \cap (\mathbb{S}-g_S)(n) \\ & \subseteq & (\mathbb{S}-f_S)(n) \cup (\mathbb{S}-g_S)(n) \\ & = & (\mathbb{S}-(f_S \widetilde{\cap} g_S))(n)\end{array}$$

thus, $f_S \widetilde{\cap} g_S$ is a soft C-interior ideal over U. Now, we show that if f_S and g_S are two soft C-interior ideals of S over U, $f_S \widetilde{\cup} g_S$ is not a soft C-interior ideal with the following example. □

Example 9 *Let $\mathbb{Z}_4$ be the semigroup over $U = D_2 = \{e, x, y, yx\}$ in Example 6. Let the soft set f_S and g_S over U be as following: $f_S(0) = \emptyset$, $f_S(1) = \{e, x\}$, $f_S(2) = \{x\}$, $f_S(3) = \{y, yx\}$ and $g_S(0) = \emptyset$, $g_S(1) = \{e\}$, $g_S(2) = \{y\}$, $g_S(3) = \{x, y, yx\}$. We see that $(f_S \widetilde{\cup} g_S)(333) = (f_S \widetilde{\cup} g_S)(1) \nsubseteq \mathbb{S} - (f_S \widetilde{\cup} g_S)(3)$.*

Definition 17 *A soft set f_S over U is defined soft semi prime , if for all $a \in S$,*

$$f_S(a) \subseteq f_S(a^2)$$

Proposition 12 *Let f_S be soft semi prime C-interior ideal of a semigroup S. Then, $f_S(a^n) \subseteq (\mathbb{S}-f_S)(a^{n+1})$ for all positive integers n.*

Proof. Let n be any positive integer. Then,

$$f_S(a^n) \subseteq f_S(a^{2n}) \subseteq f_S(a^{4n}) = f_S(a^{3n-2}a^{n+1}a) \subseteq (\mathbb{S} - f_S)(a^{n+1}).$$

□

Proposition 13 *Let f_S and h_S be soft sets over U and Ψ is a semigroup isomorphism from S to T. If f_S is a soft C-interior ideal of S over U, then so is $\Psi(f_S)$ of T over U.*

Proposition 14 *Let f_S and h_S be soft sets over U and Ψ is a semigroup homomorphism from S to T. If f_S is a soft C-interior ideal of T over U, then so is $\Psi^{-1}(f_T)$ of S over U.*

3.5 Soft C- quasi ideals of semigroups

In this subsection, we introduce soft covered quasi-ideals of semigroups, define their basic properties with respect to soft set operations, soft intersection product and certain kinds of soft C-ideals.

Definition 18 *A soft set f_S over U is called a soft covered quasi-ideal of S over U if*

$$f_S \tilde{\subseteq} ((\mathbb{S} - f_S) * \theta) \tilde{\cup} (\theta * (\mathbb{S} - f_S)).$$

For the sake of brevity, soft covered quasi-ideal of S is denoted by soft C-quasi-ideal.

Proposition 15 *Every soft C-quasi-ideal of S is a soft C-semigroup of S.*

Proof. We accept that f_S is a soft C-quasi-ideal of S. Then, since $\theta \subseteq (\mathbb{S} - f_S)$,

$$\theta * (\mathbb{S} - f_S) \tilde{\subseteq} (\mathbb{S} - f_S) * (\mathbb{S} - f_S)$$

and

$$(\mathbb{S} - f_S) * \theta \tilde{\subseteq} (\mathbb{S} - f_S) * (\mathbb{S} - f_S)$$

from here

$$f_S \tilde{\subseteq} (\theta * (\mathbb{S} - f_S)) \tilde{\cup} ((\mathbb{S} - f_S) * \theta) \tilde{\subseteq} (\mathbb{S} - f_S) * (\mathbb{S} - f_S)$$

since f_S is a soft C-quasi-ideal of S. Hence, f_S is a soft C-semigroup of S. □

Theorem 26 *If X is a C-quasi-ideal of S, then $\mathcal{S}_X^c$ is a soft C-quasi-ideal of S.*

Proof. Let X be a C-quasi-ideal and $x = mn \in X$. Since $X \subseteq ((S-X)S) \cup (S(S-X))$, then it is clear that $x = mn \in ((S-X)S) \cup (S(S-X))$, implying that $mn \in (S-X)S$ or $mn \in S(S-X)$ and so $m \in (S-X)$ and $n \in S$ or $m \in S$ and $n \in (S-X)$. Hence since $\mathcal{S}_X^c(x) = \mathcal{S}_X^c(mn) = \emptyset$, in any case

$$\mathcal{S}_X^c \widetilde{\subseteq} ((\mathbb{S}-\mathcal{S}_X^c) * \theta) \widetilde{\cup} (\theta * (\mathbb{S}-\mathcal{S}_X^c)).$$

In fact, if we consider $[(\mathbb{S}-\mathcal{S}_X^c) * \theta) \widetilde{\cup} (\theta * (\mathbb{S}-\mathcal{S}_X^c)](x)$ and if $m \in (S-X)$, then

$$\begin{aligned} ((\mathbb{S}-\mathcal{S}_X^c) * \theta)(x) &= \textstyle\bigcap_{x=mn} (\mathbb{S}-\mathcal{S}_X^c)(m) \cup \theta(n) \\ &= \emptyset \end{aligned}$$

and

$$\begin{aligned} (\theta * (\mathbb{S}-\mathcal{S}_X^c))(x) &= \textstyle\bigcap_{x=mn} \theta(m) \cup (\mathbb{S}-\mathcal{S}_X^c)(n) \\ &= \emptyset \end{aligned}$$

and so $[(\mathbb{S}-\mathcal{S}_X^c) * \theta) \widetilde{\cup} (\theta * (\mathbb{S}-\mathcal{S}_X^c)](x) = \emptyset \cup \emptyset = \emptyset$. Hence,

$$\mathcal{S}_X^c \widetilde{\subseteq} ((\mathbb{S}-\mathcal{S}_X^c) * \theta) \widetilde{\cup} (\theta * (\mathbb{S}-\mathcal{S}_X^c)).$$

Also if $m \in X$, then

$$\begin{aligned} ((\mathbb{S}-\mathcal{S}_X^c) * \theta)(x) &= \textstyle\bigcap_{x=mn} (\mathbb{S}-\mathcal{S}_X^c)(m) \cup \theta(n) \\ &= U \end{aligned}$$

and

$$\begin{aligned} (\theta * (\mathbb{S}-\mathcal{S}_X^c))(x) &= \textstyle\bigcap_{x=mn} \theta(m) \cup (\mathbb{S}-\mathcal{S}_X^c)(n) \\ &= U \end{aligned}$$

and so $[(\mathbb{S}-\mathcal{S}_X^c) * \theta) \widetilde{\cup} (\theta * (\mathbb{S}-\mathcal{S}_X^c)](x) = U \cup U = U$. Hence,

$$\mathcal{S}_X^c \widetilde{\subseteq} ((\mathbb{S}-\mathcal{S}_X^c) * \theta) \widetilde{\cup} (\theta * (\mathbb{S}-\mathcal{S}_X^c)).$$

□

Proposition 16 *Every soft C-left (right) ideal of S is a soft C-quasi-ideal of S.*

Proof. We accept that f_S is a soft C-Left ideal of S over U which is defined $f_S \subseteq (\theta * (\mathbb{S}-f_S))$. In this statement, we have:

$$f_S \widetilde{\subseteq} (\theta * (\mathbb{S}-f_S)) \widetilde{\subseteq} ((\mathbb{S}-f_S) * \theta) \widetilde{\cup} (\theta * (\mathbb{S}-f_S))$$

hence, f_S is a soft C-quasi-ideal. □

The converse of the above proposition does not hold in general as shown in the table.

Example 10 *Think* $S = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ *defined by the following table:*

.	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	1	1

Let f_S *be a soft set over* $U = D_2 = \{e, x, y, yx\}$ *and is defined as following,* $f_S(0) = \emptyset$, $f_S(1) = \{e\}$, $f_S(2) = \{e\}$, $f_S(3) = \{y, yx\}$ *and so* $(\mathbb{S} - f_S)(0) = U$, $(\mathbb{S} - f_S)(1) = \{x, y, yx\}$, $(\mathbb{S} - f_S)(2) = \{x, y, yx\}$, $(\mathbb{S} - f_S)(3) = \{e, x\}$. *Then, one can show that* $f_S(1) \subseteq ((\mathbb{S} - f_S) * \theta))(1) \cup (\theta * (\mathbb{S} - f_S))(1) = (\mathbb{S} - f_S)(3) \cup ((\mathbb{S} - f_S)(3) \cap (\mathbb{S} - f_S)(2)) = (\mathbb{S} - f_S)(3) = \{e, x\}$ *is a soft C-quasi-ideal but since* $f_S(3.2) = f_S(1) \nsubseteq (\mathbb{S} - f_S)(2)$, f_S *is not a soft C-Left ideal.*

Proposition 17 *Every soft C-quasi-ideal is a soft C-bi-ideal of* S.

Proof. Let f_S be a soft C-quasi-ideal of S. Then,

$$\begin{aligned} f_S &\subseteq ((\mathbb{S} - f_S) * \theta)\widetilde{\cup}(\theta * (\mathbb{S} - f_S)) \\ &\subseteq ((\mathbb{S} - f_S) * (\theta * \theta))\widetilde{\cup}((\theta * \theta) * (\mathbb{S} - f_S)) \\ &\subseteq ((\mathbb{S} - f_S) * \theta * (\mathbb{S} - f_S))\widetilde{\cup}((\mathbb{S} - f_S) * \theta) * (\mathbb{S} - f_S)) \\ &= ((\mathbb{S} - f_S) * \theta * (\mathbb{S} - f_S))\widetilde{\cup}((\mathbb{S} - f_S) * \theta * (\mathbb{S} - f_S)) \\ &= (\mathbb{S} - f_S) * \theta * (\mathbb{S} - f_S) \end{aligned}$$

Hence, f_S is a soft C-bi-ideal of S together with Proposition 15. The converse of this proposition does not hold in general as shown in the following example. □

Example 11 *Let* $S = \{a, b, c, d\}$ *be the semigroup in Example 6 and* $U = D_2 = \{e, x, y, yx\}$. *Let* f_S *be a soft set over* U *defined by* $f_S(0) = \emptyset$, $f_S(1) = \{x\}$, $f_S(2) = \{x, y\}$, $f_S(3) = \{yx\}$ *and so* $(\mathbb{S} - f_S)(0) = U$, $(\mathbb{S} - f_S)(1) = \{e, y, yx\}$, $(\mathbb{S} - f_S)(2) = \{e, yx\}$, $(\mathbb{S} - f_S)(3) = \{e, x, y\}$. *Then,* f_S *is a soft C-bi-ideals. However,* f_S *is not a soft C-quasi-ideal since* $f_S(1) = \{x\} \nsubseteq ((\mathbb{S} - f_S) * \theta))(1) \cup (\theta * (\mathbb{S} - f_S))(1) = (\mathbb{S} - f_S)(2) \cap (\mathbb{S} - f_S)(3) = \{e\}$.

Proposition 18 *Let* f_S *and* g_S *be any soft C-Right ideal and soft C-Left ideal of* S *over* U*, respectively. Then,* $f_S \widetilde{\cap} g_S$ *is a soft C-quasi- ideal.*

Proof. Let f_S be any soft C-Right ideal of S and g_S be any soft C-Left ideal of S. Then,

$$\begin{array}{rcl}(((\mathbb{S}-(f_S\widetilde{\cap}g_S))*\theta)\widetilde{\cup}(\theta*(\mathbb{S}-(f_S\widetilde{\cap}g_S))) & \widetilde{\supseteq} & (((\mathbb{S}-(f_S\widetilde{\cap}g_S))*\theta) \\ & \widetilde{\cap} & (\theta*(\mathbb{S}-(f_S\widetilde{\cap}g_S))) \\ & \widetilde{\supseteq} & ((\mathbb{S}-f_S)*\theta)\widetilde{\cap}(\theta*(\mathbb{S}-g_S)) \\ & \widetilde{\supseteq} & f_S\widetilde{\cap}g_S.\end{array}$$

Thus, $f_S\widetilde{\cap}g_S$ is a soft C-quasi-ideal of S over U. □

Proposition 19 *Let f_S and g_S be any soft C-quasi-ideals of S over U. Then, $f_S\widetilde{\cap}g_S$ is a soft C-quasi-ideal .*

Proof. Let f_S and g_S be any soft C-quasi-ideals of S. Then,

$$\begin{array}{rcl}(((\mathbb{S}-(f_S\widetilde{\cap}g_S))*\theta)\widetilde{\cup}(\theta*(\mathbb{S}-(f_S\widetilde{\cap}g_S))) & \widetilde{\supseteq} & ((\mathbb{S}-f_S)*\theta)\widetilde{\cup}(\theta*(\mathbb{S}-f_S)) \\ & \widetilde{\supseteq} & f_S\end{array}$$

and

$$\begin{array}{rcl}(((\mathbb{S}-(f_S\widetilde{\cap}g_S))*\theta)\widetilde{\cup}(\theta*(\mathbb{S}-(f_S\widetilde{\cap}g_S))) & \widetilde{\supseteq} & ((\mathbb{S}-g_S)*\theta)\widetilde{\cup}(\theta*(\mathbb{S}-g_S)) \\ & \widetilde{\supseteq} & g_S.\end{array}$$

Thus, $(((\mathbb{S}-(f_S\widetilde{\cap}g_S))*\theta)\widetilde{\cup}(\theta*(\mathbb{S}-(f_S\widetilde{\cap}g_S)))\widetilde{\supseteq}(f_S\widetilde{\cap}g_S)$. Then, $f_S\widetilde{\cap}g_S$ is a soft C-quasi-ideal. □

Proposition 20 *Let f_S be any soft C-quasi-ideal of a commutative semigroup S and $a\in A$. Then,*

$$f_S(a^{n+1})\subseteq(\mathbb{S}-f_S)(a^n)$$

for every positive integer n.

Proof. For any positive integer n, we have:

$$\begin{array}{rcl}((\mathbb{S}-f_S)*\theta)(a^{n+1}) & = & \bigcap_{a^{n+1}=xy}(\mathbb{S}-f_S)(x)\cup\theta(y) \\ & \subseteq & (\mathbb{S}-f_S)(a^n)\cup\emptyset \\ & = & (\mathbb{S}-f_S)(a^n)\end{array}$$

from here $((\mathbb{S}-f_S)*\theta)(a^{n+1})\subseteq(\mathbb{S}-f_S)(a^n)$.

Moreover, since f_S is a soft C-quasi-ideal of S, we have:

$$\begin{array}{rcl}f_S(a^{n+1}) & \subseteq & [((\mathbb{S}-f_S)*\theta)\widetilde{\cup}(\theta*(\mathbb{S}-f_S))](a^{n+1}) \\ & = & ((\mathbb{S}-f_S)*\theta)(a^{n+1})\cup(\theta*(\mathbb{S}-f_S))(a^{n+1}) \\ & \subseteq & (\mathbb{S}-f_S)(a^n)\cup(\mathbb{S}-f_S)(a^n) \\ & = & (\mathbb{S}-f_S)(a^n).\end{array}$$

This completes the proof. □

3.6 Soft C-generalized Bi-ideals of semigroups

In this subsection, we study soft covered generalized bi-ideals of semigroups, introduce their basic prosperities as regards soft set operations, soft intersection product and certain kinds of soft C-ideals.

Definition 19 *A soft set over* U *is called a soft covered generalized bi-ideal of* S*, if*

$$f_S(xyz) \subseteq (\mathbb{S} - f_S)(x) \cup (\mathbb{S} - f_S)(z)$$

for all $x, y, z \in S$.

For the sake of brevity, soft covered generalized bi-ideal is denoted by soft C-generalized bi-ideal of S. Clearly, every soft C-bi-ideal of S is a soft C-generalized bi-ideal of S but converse of this statement is not true. This is indicated by following example.

Example 12 *Let think the semigroup* $S = \{0, 1, 2, 3\}$ *in Example 6 and define the soft set* f_S *over* $U = D_4$ *such that* $f_S(0) = \emptyset$, $f_S(1) = \{x, x^2, y\}$, $f_S(2) = \{x, yx\}$, $f_S(3) = \{e, yx, yx^2\}$ *and so* $(\mathbb{S} - f_S)(0) = \{e, x, x^2, y, yx, yx^2\}$, $(\mathbb{S} - f_S)(1) = \{e, yx, yx^2\}$, $(\mathbb{S} - f_S)(2) = \{e, y, x^2, yx^2\}$, $(\mathbb{S} - f_S)(3) = \{x, y, x^2\}$.

Then, one can easily show that f_S *is a soft C-generalized bi-ideal of* S *over* U. *However, since* $f_S(33) = f_S(2) \nsubseteq (\mathbb{S} - f_S)(3) \cup (\mathbb{S} - f_S)(3)$, f_S *is not a soft C-bi-ideal of* S.

Theorem 27 *If* X *is a C-generalized bi-ideal of* S*, then* $\mathcal{S}_X^c$ *is a soft C-generalized bi-ideal of* S.

Theorem 28 *Let* f_S *be a soft set over* U. *Then,* f_S *is a soft C-generalized bi-ideal over* U *of* S *if and only if* f_S^c *is a soft C-generalized bi-ideal over* U *of* S.

Theorem 29 *Let* f_S *be a soft set over* U. *Then,* f_S *is a soft C-generalized bi-ideal of* S *over* U *if and only if*

$$f_S \widetilde{\subseteq} (\mathbb{S} - f_S) * \theta * (\mathbb{S} - f_S).$$

Theorem 30 *Every soft C-left (right) ideal of a semigroup* S *over* U *is a soft C-generalized bi-ideal of* S *over* U.

Proof. Let f_S be a soft C-left (right) ideal of S over U and $x, y, z \in S$. Then,

$$f_S(xyz) = f_S((xy)z) \subseteq (\mathbb{S} - f_S)(z) \subseteq (\mathbb{S} - f_S)(x) \cup (\mathbb{S} - f_S)(z)$$

Thus, f_S is a soft C-generalized bi-ideal of S. □

Proposition 21 *Let* $\mathsf{f_S}$ *and* $\mathsf{f_T}$ *be soft C-generalized bi-ideals of* S *over* U. *Then,* $\mathsf{f_S}\widetilde{\wedge}\mathsf{f_T}$ *is a soft C-generalized bi-ideal of* $\mathsf{S} \times \mathsf{T}$ *over* U.

Proposition 22 *Let* $\mathsf{f_S}$ *and* $\mathsf{h_S}$ *be soft sets over* U *and* Ψ *is a semigroup isomorphism from* S *to* T. *If* $\mathsf{f_S}$ *is a soft C-generalized bi-ideal of* S *over* U, *then so is* $\Psi(\mathsf{f_S})$ *of* T *over* U.

Proposition 23 *Let* $\mathsf{f_S}$ *and* $\mathsf{h_S}$ *be soft sets over* U *and* Ψ *is a semigroup homomorphism from* S *to* T. *If* $\mathsf{f_T}$ *is a soft C-generalized bi-ideal of* T *over* U, *then so is* $\Psi^{-1}(\mathsf{f_T})$ *of* S *over* U.

4 Conclusion

In this manuscript, we have introduced soft C-semigroups, C-left (right) ideals, C-bi-ideals, C-interior ideals, C-quasi-ideals and C-generalized bi-ideals. Moreover, we survey the relation between soft AC-ideals and soft C-ideals. Addition to, we obtain the interrelations of various soft C-ideals as in the following figure.

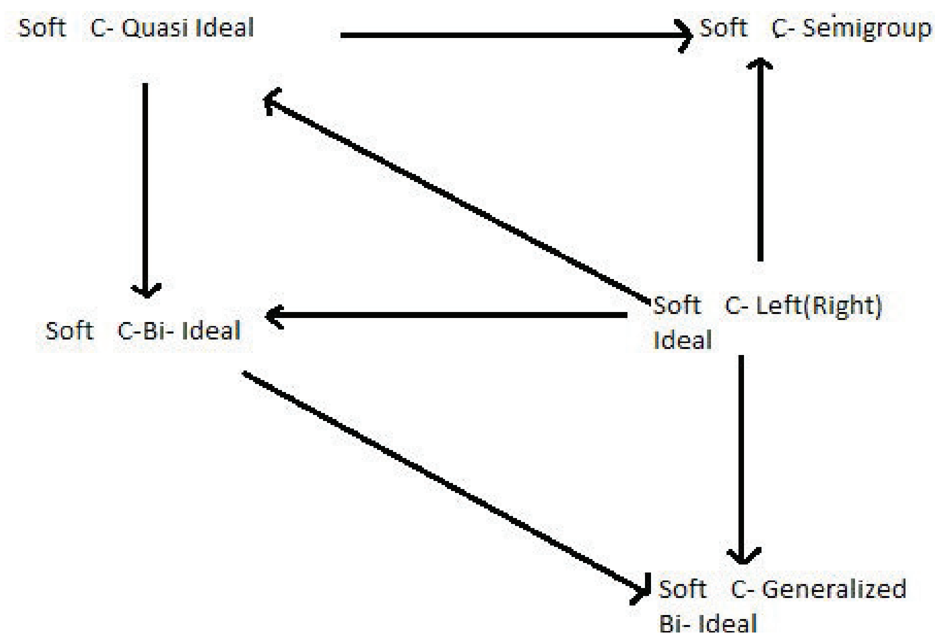

Figure 1

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Some fixed point results on S-metric spaces

Maryam Shahraki
Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
email: m.shahraki@gu.ac.ir

Shaban Sedghi*
Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
email: sedghi.gh@qaemiau.ac.ir

S. M. A. Aleomraninejad
Department of Mathematics, Faculty of Science, Qom University of Technology, Qom, Iran
email: aleomran@qut.ac.ir

Zoran D. Mitrović
University of Banja Luka, Faculty of Electrical Engineering, 78000 Banja Luka, Bosnia and Herzegovina
email: zoran.mitrovic@etf.unibl.org

Abstract. In this paper, a general form of the Suzuki type function is considered on S- metric space, to get a fixed point. Then we show that our results generalize some old results.

1 Introduction and preliminaries

In 1922, Banach [1] proposed a theorem, which is well-known as Banach's Fixed Point Theorem (or Banach's Contraction Principle, BCP for short) to establish the existence of solutions for nonlinear operator equations and integral equations. Since then, because of simplicity and usefulness, it has become a very popular tool in solving a variety of problems such as control theory, economic theory, nonlinear analysis and global analysis. Later, a huge amount

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*Corresponding author

of literature is witnessed on applications, generalizations and extensions of this theorem. They are carried out by several authors in different directions, e.g., by weakening the hypothesis, using different setups.

Many mathematics problems require one to find a distance between tow or more objects which is not easy to measure precisely in general. There exist different approaches to obtaining the appropriate concept of a metric structure. Due to the need to construct a suitable framework to model several distinguished problems of practical nature, the study of metric spaces has attracted and continues to attract the interest of many authors. Over last few decades, a numbers of generalizations of metric spaces have thus appeared in several papers, such as 2-metric spaces, G-metric spaces, D^*-metric spaces, partial metric spaces and cone metric spaces. These generalizations were then used to extend the scope of the study of fixed point theory. For more discussions of such generalizations, we refer to [3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 20, 21, 22, 23]. Sedghi et al [17] have introduced the notion of an S-metric space and proved that this notion is a generalization of a G-metric space and a D^*-metric space. Also, they have proved properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space.

The Banach contraction principle is the most powerful tool in the history of fixed point theory. Boyd and Wong [2] extended the Banach contraction principle to the nonlinear contraction mappings. We begin by briefly recalling some basic definitions and results for S-metric spaces that will be needed in the sequel. For more details please see [1, 14, 18].

Definition 1 [17] *Let* X *be a (nonempty) set, an S-metric on* X *is a function* $S : X^3 \longrightarrow [0, +\infty)$ *that satisfies the following conditions, for each* $x, y, z, a \in X$*,*

(1). $S(x, y, z) \geq 0$,

(2). $S(x, y, z) = 0$ *if and only if* $x = y = z$,

(3). $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$,

for all $x, y, z, a \in X$.
The pair (X, S) *is called an S-metric space.*

Immediate examples of such S-metric spaces are:

Example 1 [15, 18] *Let* $X = \mathbb{R}^n$ *and* $\| . \|$ *a norm on X, then*

$$S(x, y, z) = \| y + z - 2x \| + \| y - z \|$$

is an S-metric on X.
Let X *be a nonempty set, d is ordinary metric on* X*, then*

$$S(x, y, z) = d(x, z) + d(y, z)$$

is an S-metric on X*. This S-metric is called the usual S-metric on* X*.*

Definition 2 [16] *Let* (X, S) *be an S-metric space.*

(i) *A sequence* $\{x_n\} \subset X$ *converges to* $x \in X$ *if* $S(x_n, x_n, x) \to 0$ *as* $n \to +\infty$*. That is, for each* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *we have* $S(x_n, x_n, x) < \varepsilon$*. We write* $x_n \to x$ *for brevity.*

(ii) *A sequence* $\{x_n\} \subset X$ *is a Cauchy sequence if* $S(x_n, x_n, x_m) \to 0$ *as* $n, m \to +\infty$*.*
That is, for each $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n, m \geq n_0$ *we have* $S(x_n, x_n, x_m) < \varepsilon$*.*

(iii) *The S-metric space* (X, S) *is compelet if every Cauchy sequence is a convergent sequence.*

Definition 3 [15] *Let* (X, S) *be an S-metric space. For* $r > 0$ *and* $x \in X$ *we define the open ball* $B_s(x, r)$ *and closed ball* $B_s[x, r]$ *with center* x *and radius* r *as follows respectively:*

$$\begin{aligned} B_s(x, r) &= \{y \in X : S(y, y, x) < r\}, \\ B_s[x, r] &= \{y \in X : S(x, x, y) \leq r\}. \end{aligned}$$

Example 2 [15] *Let* $X = \mathbb{R}$ *and* $S(x, y, z) = |y + z - 2x| + |y - z|$ *for all* $x, y, z \in \mathbb{R}$*. Then*

$$\begin{aligned} B_s(1, 2) &= \{y \in \mathbb{R} : S(y, y, 1) < 2\} = \{y \in \mathbb{R} : |y - 1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} = (0, 2). \end{aligned}$$

Lemma 1 [16] *Let* (X, S) *be an S-metric space. If* $r > 0$ *and* $x \in X$*, then the ball* $B_s(x, r)$ *is open subset of* X*.*

Lemma 2 [15, 16, 18] *In an S-metric space, we have* $S(x, x, y) = S(y, y, x)$*.*

Proof. By third condition of S-metric, we have

$$\begin{aligned} S(x, x, y) &\leq S(x, x, x) + S(x, x, x) + S(y, y, x) \\ &= S(y, y, x) \end{aligned} \tag{1}$$

$$\begin{aligned} S(y,y,x) &\le S(y,y,y)+S(y,y,y)+S(x,x,y) \\ &= S(x,x,y), \end{aligned} \tag{2}$$

hence by (1) and (2), we get $S(x,x,y)=S(y,y,x)$. $\square$

Lemma 3 [18] *Let* (X,S) *be an S-metric space. If sequence* $\{x_n\}$ *in converges to* x*, then* x *is unique.*

Lemma 4 [18] *Let* (X,S) *be an S-metric space. If sequence* $\{x_n\}$ *in* X *is converges to* x*, then* $\{x_n\}$ *is a Cauchy sequence.*

Lemma 5 [15, 16, 18] *Let* (X,S) *be an S-metric space. If there exist sequences* $\{x_n\}$ *and* $\{y_n\}$ *such that* $\lim_{n\to+\infty} x_n = x$ *and* $\lim_{n\to+\infty} y_n = y$*, then* $\lim_{n\to+\infty} S(x_n,x_n,y_n)=S(x,x,y)$.

Definition 4 [15, 19] *Let* X *be a (nonempty) set, a b-metric on* X *is a function* $d: X^2 \longrightarrow [0,+\infty)$ *if there exists a real number* $b\ge 1$ *such that the following conditions hold for all* $x,y,z\in X$,

(1) $d(x,y)=0$ *if and only if* $x=y$,

(2) $d(x,y)=d(y,x)$,

(3) $d(x,z)\le b[d(x,y)+d(y,z)]$.

The pair (X,d) *is called a b-metric space.*

Proposition 1 [16] *Let* (X,S) *be an S-metric space and let*

$$d(x,y)=S(x,x,y),$$

for all $x,y\in X$*. Then we have*

(1) d *is a b-metric on* X;

(2) $x_n\to x$ *in* (X,S) *if and only if* $x_n\to x$ *in* (X,d);

(3) $\{x_n\}$ *is a Cauchy sequence in* (X,S) *if and only if* $\{x_n\}$ *is a Cauchy sequence in* (X,d).

Definition 5 *Let* £ *be the set of all continuous functions* $g:[0,\infty)^4\to[0,+\infty)$*, satisfying the conditions:*

(i) $g(1,1,1,1)<1$,

(ii) g *is subhomogeneous,i.e.,* $g(\alpha x_1,\alpha x_2,\alpha x_3,\alpha x_4)\le \alpha g(x_1,x_2,x_3,x_4)$, *for all* $\alpha\ge 0$,

(iii) *if* $x_i, y_i \in [0, +\infty), x_i \leq y_i$ *for* $i = 1, \dots, 4$ *we have* $g(x_1, x_2, x_3, x_4) \leq g(y_1, y_2, y_3, y_4)$.

Example 3 *The function* $g(x_1, x_2, x_3, x_4) = k \max\{x_i\}_{i=0}^{4}$ *for* $k \in (0, 1)$ *is in class* $\pounds$.

Example 4 *The function* $g(x_1, x_2, x_3, x_4) = k \max\{x_1, x_2, \frac{x_3+x_4}{2}\}$ *for* $k \in (0, 1)$ *is in class* $\pounds$.

Proposition 2 *If* $g \in \pounds$ *and* $u, v \in [0, +\infty]$ *are such that* $u \leq g(v, v, v, u)$, *then* $u \leq hv$, *where* $h = g(1, 1, 1, 1)$.

Proof. If $v < u$, then

$$u \leq g(v, v, v, u) \leq g(u, u, u, u) < ug(1, 1, 1, 1) = hu < u,$$

which is a contradiction. Thus $u \leq v$, which implies

$$u \leq g(v, v, v, u) \leq g(v, v, v, v) < vg(1, 1, 1, 1) = hv.$$

□

Corollary 1 [15] *Let* (X, S) *be a complete* S*-metric space and* $T : X \to X$ *a function such that for, all* $x, y, z, a \in X$,

$$S(Tx, Ty, Tz) \leq LS(x, y, z),$$

where $L \in (0, 1/2)$. *Then there exists a unique point* $u \in X$ *such that* $Tu = u$.

2 Results

Now, we give our main result.

Theorem 1 *Let* (X, S) *be a S- metric space and* $T : X \to X$ *be a function. Suppose that there exist* $g \in \pounds$ *and* $\alpha \in (0, 1)$, *such that* $\alpha(h + 2) \leq 1$ *where* $h = g(1, 1, 1, 1)$. *Suppose also that* $\alpha S(x, x, Tx) \leq S(x, y, z)$ *implies*

$$S(Tx, Ty, Tz) \leq g(S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz)),$$

for all $x, y, z \in X$. *Then* $F(T)$ *is non-empty set.*

Proof. Fix arbitrary $x_0 \in X$ and let $Tx_0 = x_1$. Since

$$\alpha S(x_0, x_0, Tx_0) < S(x_0, x_0, x_1),$$

then by the hypothesis of the theorem and condition (iii) Definition 5, respectively, we have

$$\begin{array}{rcl} S(x_1, x_1, Tx_1) & = & S(Tx_0, Tx_0, Tx_1) \\ & \leq & g(S(x_0, x_0, x_1), S(x_0, x_0, Tx_0), S(x_0, x_0, Tx_0), S(x_1, x_1, Tx_1)) \\ & = & g(S(x_0, x_0, x_1), S(x_0, x_0, x_1), S(x_0, x_0, x_1), S(x_1, x_1, Tx_1)) \end{array}$$

Then, by Proposition 2, we have $S(x_1, x_1, Tx_1) \leq hS(x_0, x_0, x_1)$.
Now let $Tx_1 = x_2$. Since $\alpha S(x_1, x_1, Tx_1) < S(x_1, x_1, x_2)$, by using and the properties of the function g we have

$$\begin{array}{rcl} S(x_2, x_2, Tx_2) & = & S(Tx_1, Tx_1, Tx_2) \\ & \leq & g(S(x_1, x_1, x_2), S(x_1, x_1, Tx_1), S(x_1, x_1, Tx_1), S(x_2, x_2, Tx_2)) \\ & = & g(S(x_1, x_1, x_2), S(x_1, x_1, x_2), S(x_1, x_1, x_2), S(x_2, x_2, Tx_2)). \end{array}$$

Then, by Proposition 2, we have $S(x_2, x_2, Tx_2) \leq hS(x_1, x_1, x_2)$.
In a similar way, we can let $Tx_2 = x_3$. So we have

$$S(x_2, x_2, x_3) < hS(x_1, x_1, x_2) < h^2 S(x_0, x_0, x_1).$$

By continuing this process, we obtain a sequence $\{x_n\}_{n\geq 1}$ in X such that $x_{n+1} = Tx_n$, which satisfies $S(x_n, x_n, Tx_n) \leq hS(x_{n-1}, x_{n-1}, x_n)$ and

$$S(x_n, x_n, x_{n+1}) \leq h^n S(x_0, x_0, x_1).$$

If $x_m = x_{m+1}$ for some $m \geq 1$, then
Then T has a fixed point.
Suppose that $x_n \neq x_{n+1}$, for all $n \geq 1$. Repeated application of the triangle inequality implies

$$\begin{array}{rcl} S(x_n, x_n, x_{n+m}) & \leq & 2S(x_n, x_n, x_{n+1}) + S(x_{n+m}, x_{n+m}, x_{n+1}) \\ & = & 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+m}) \\ & \leq & 2S(x_n, x_n, x_{n+1}) + 2S(x_{n+1}, x_{n+1}, x_{n+2}) \\ & + & S(x_{n+m}, x_{n+m}, x_{n+2}) \\ & \leq & 2[S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}) \\ & + & \cdots + S(x_{n+m-1}, x_{n+m-1}, x_{n+m})] \end{array}$$

$$\leq \quad 2\sum_{k=0}^{k=m-1} h^{k+n}S(x_0,x_0,x_1) \leq \frac{2h^n}{1-h}S(x_0,x_0,x_1).$$

So we get

$$\lim_{n\to+\infty} S(x_n,x_n,x_{n+m}) \to 0$$

and hence $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X,S). Regarding Definition 2, $\{x_n\}_{n\geq 1}$ is also a Cauchy sequence in (X,S).
Since (X,S) is a complete S- metric space, by Definition 2, (X,S) is also complete.
Thus $\{x_n\}_{n\geq 1}$ converges to a limit, say, $x \in X$, that is,

$$\lim_{n\to+\infty} S(x_n,x_n,x) = 0.$$

It is easy to see that $\lim_{n\to\infty} S(x_n,x_{n+1},x) = 0$. Now, we claim that for each $n \geq 1$ one of the relations

$$\alpha S(x_n,x_n,Tx_n) \leq S(x_n,x_n,x)$$

or

$$\alpha S(x_{n+1},x_{n+1},Tx_{n+1}) \leq S(x_n,x_n,x)$$

holds. If for some $n \geq 1$ we have

$$\alpha S(x_n,x_n,Tx_n) > S(x_n,x_n,x) \text{ and } \alpha S(x_{n+1},x_{n+1},Tx_{n+1}) > S(x_{n+1},x_{n+1},x),$$

then

$$\begin{aligned} S(x_n,x_n,x_{n+1}) &\leq 2S(x_n,x_n,x) + S(x_{n+1},x_{n+1},x) \\ &< 2\alpha S(x_n,x_n,Tx_n) + \alpha S(x_{n+1},x_{n+1},Tx_{n+1}) \\ &= 2\alpha S(x_n,x_n,x_{n+1}) + \alpha h S(x_n,x_n,x_{n+1}). \end{aligned}$$

This results in $\alpha(h+2) > 1$, which contradidts the intial assumption. Hence, our claim is proved.
Observe that by the assumption of the theorem, we have either

$$S(Tx_n,Tx_n,Tx) \leq g(S(x_n,x_n,x),S(Tx_n,x_n,x),S(Tx_n,x_n,x),S(Tx,x_n,x_n)),$$

or

$$\begin{aligned} S(Tx_{n+1},Tx_{n+1},Tx) \leq\ & g(S(x_{n+1},x_{n+1},x),S(Tx_{n+1},x_{n+1},x), \\ & S(Tx_{n+1},x_n,x),S(Tx,x_{n+1},x_{n+1})). \end{aligned}$$

Therefore, one of the following cases holds.
Case (i). There exists an infinite subset $I \subseteq N$ such that

$$\begin{aligned} S(x_{n+1}, x_{n+1}, Tx) &= S(Tx_n, Tx_n, Tx) \\ &\leq g(S(x_n, x_n, x), S(Tx_n, x_n, x), S(Tx_n, x_n, x), S(Tx, x_n, x_n)) \\ &= g(S(x_n, x_n, x), S(x_{n+1}, x_n, x), S(x_{n+1}, x_n, x), S(Tx, x_n, x_n)). \end{aligned}$$

for all $n \in I$.
Case (ii). There exists an infinite subset $J \subseteq N$ such that

$$\begin{aligned} S(x_{n+2}, x_{n+2}, Tx) &= S(Tx_{n+1}, Tx_{n+1}, Tx) \\ &\leq g(S(x_{n+1}, x_{n+1}, x), S(Tx_{n+1}, x_{n+1}, x), \\ &\quad S(Tx_{n+1}, x_{n+1}, x), S(Tx, x_{n+1}, x_{n+1})) \\ &= g(S(x_{n+1}, x_{n+1}, x), S(x_{n+2}, x_{n+1}, x), \\ &\quad S(x_{n+2}, x_{n+1}, x), S(Tx, x_{n+1}, x_{n+1})). \end{aligned}$$

for all $n \in I$. In case (i), taking the limit as $n \to +\infty$ we obtain

$$S(x, x, Tx) \leq g(0, 0, 0, S(x, x, Tx))$$

Now by using Definition 5, Proposition 2, we have $S(x, x, Tx) = 0$, and thus $x = Tx$.
In case(ii), taking the limit as $n \to \infty$ we obtain

$$S(x, x, Tx) \leq g(0, 0, 0, S(x, x, Tx))$$

Now by using definition 5, propositions 2, we have $S(x, x, Tx) = 0$, and thus $x = Tx$. This completes the proof. □

Corollary 2 *Let* (X, S) *be a S- metric space and* $T : X \to X$ *be a function. Suppose that there exist* $g \in \pounds$ *and* $\alpha \in (0, 1)$, *such that* $\alpha(h + 2) \leq 1$ *where* $h = g(1, 1, 1, 1)$. *Suppose also that* $\alpha S(y, y, Ty) \leq S(x, y, z)$ *implies*

$$S(Tx, Ty, Tz) \leq g(S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz))$$

for all $x, y, z \in X$. *Then* $F(T)$ *is non-empty.*

Corollary 3 *Let* (X, S) *be a complete S-metric space and* $T : X \to X$ *a function such that for all* $x, y, z \in X$,

$$S(Tx, Ty, Tz) \leq LS(x, y, z),$$

where $L \in (0, 1)$. *Then there exists a unique point* $u \in X$ *such that* $Tu = u$.

Proof. Let $g(x_1, x_2, x_3, x_4) = Lx_1$. $\square$

Corollary 4 *Let* (X, S) *be a complete S-metric space and* $T : X \to X$ *a function such that for all* $x, y, z \in X$*,*

$$S(Tx, Ty, Tz) \leq L \max\{S(x, y, z), S(x, x, Tx), S(y, y, Ty), S(z, z, Tz)\}$$

where $L \in (0, 1)$*. Then there exists a unique point* $u \in X$ *such that* $Tu = u$*.*

Proof. Let $g(x_1, x_2, x_3, x_4) = L \max\{x_1, x_2, x_3, x_4\}$. $\square$

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On the metric dimension of strongly annihilating-ideal graphs of commutative rings

V. Soleymanivarniab
Department of Mathematics,
Science and Research Branch,
Islamic Azad University (IAU),
Tehran, Iran
email: soleymani.vali@yahoo.com

R. Nikandish*
Department of Mathematics,
Jundi-Shapur University of Technology,
Dezful, Iran
email: r.nikandish@ipm.ir

A. Tehranian
Department of Mathematics,
Science and Research Branch,
Islamic Azad University (IAU), Tehran, Iran
email: tehranian@srbiau.ac.ir

Abstract. Let R be a commutative ring with identity and $A(R)$ be the set of ideals with non-zero annihilator. The strongly annihilating-ideal graph of R is defined as the graph $\mathrm{SAG}(R)$ with the vertex set $A(R)^* = A(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $I \cap \mathrm{Ann}(J) \neq (0)$ and $J \cap \mathrm{Ann}(I) \neq (0)$. In this paper, we study the metric dimension of $\mathrm{SAG}(R)$ and some metric dimension formulae for strongly annihilating-ideal graphs are given.

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*Corresponding author

1 Introduction

The problem of finding the metric dimension of a graph was first studied by Harary and Melter [7]. Determining the metric dimension of a graph as an NP-complete problem has attracted many graph theorists and it has appeared in various applications of graph theory, for example pharmaceutical chemistry [5], robot navigation [8], combinatorial optimization [14] and so on. Recently, there was much work done in computing the metric dimension of graphs associated with algebraic structures. Calculating the metric dimension for the commuting graph of a dihedral group was done in [1], for the zero-divisor graphs of commutative rings in [9, 10, 12], for the compressed zero-divisor graphs of commutative rings in [13], for total graphs of finite commutative rings in [6], for some graphs of modules in [11] and for annihilator graphs of commutative rings in [15]. Motivated by these papers, we study the metric dimension of another graph associated with a commutative ring.

Throughout this paper, all rings are assumed to be commutative with identity. The sets of all zero-divisors, nilpotent elements and maximal ideals are denoted by $Z(R)$, $\mathrm{Nil}(R)$ and $\mathrm{Max}(R)$, respectively. For a subset T of a ring R we let $T^* = T \setminus \{0\}$. An ideal with non-zero annihilator is called an *annihilating-ideal*. The set of annihilating-ideals of R is denoted by $A(R)$. For every subset I of R, we denote the *annihilator* of I by $\mathrm{Ann}(I)$. A non-zero ideal I of R is called *essential* if I has a non-zero intersection with every other non-zero ideal of R. The set of essential annihilating-ideal ideals of R is denoted by $\mathrm{Ess}(R)$. The ring R is said to be *reduced* if it has no non-zero nilpotent element. Some more definitions about commutative rings can be find in [2, 4].

We use the standard terminology of graphs following [18]. Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. We recall that a graph is *connected* if there exists a path connecting any two distinct vertices. The *distance* between two distinct vertices x and y, denoted by $d(x, y)$, is the length of the shortest path connecting them (if such a path does not exist, then we set $d(x, y) = \infty$). The *diameter* of a connected graph G, denoted by $\mathrm{diam}(G)$, is the maximum distance between any pair of vertices of G. For a vertex x in G, we denote the set of all vertices adjacent to x by $N(x)$ and $N[x] = N(x) \cup \{x\}$. A k-*partite* graph is one whose vertex set can be partitioned into k subsets so that an edge has both ends in no subset. A *complete* k-*partite* graph is a k-partite graph in which each vertex is adjacent to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with part sizes m and n is denoted by $K_{m,n}$. If $m = 1$, then the bipartite graph is called star. A graph in which each pair of vertices is joined by an edge is called a *complete* graph and use K_n to denote it with n vertices and its complement is denoted by $\overline{K}_n$ (possibly n is zero). Also, a cycle of order n is denoted by C_n. A subset of vertices $S \subseteq V(G)$ resolves a graph G, and S is a *resolving set of* G, if every vertex is uniquely determined by its vector of distances to the vertices of S. In general, for an ordered subset $S = \{v_1, v_2, \ldots, v_k\}$ of vertices in a connected graph G and a vertex $v \in V(G) \setminus S$ of G, the metric representation of v with respect to S is the k-vector $D(v|S) = (d(v, v_1), d(v, v_2), \ldots, d(v, v_k))$. The set S is a resolving set for G if $D(u|S) = D(v|S)$ implies that $u = v$, for all pair of vertices, $v, u \in V(G) \setminus S$. A resolving set S of minimum cardinality is the *metric basis for* G, and the number of elements in the resolving set of minimum cardinality is the *metric dimension of* G. We denote the metric dimension of a graph G by $dim_M(G)$. Let G be a connected graph such that $|V(G)| \geq 2$. Two distinct vertices u and v are *distance similar*, if $d(u, x) = d(v, x)$, for all $x \in V(G) \setminus \{u, v\}$. It can be easily checked that two distinct vertices u and v are distance similar if either $u - v \notin E(G)$ and $N(u) = N(v)$ or $u - v \in E(G)$ and $N[u] = N[v]$.

Let R be a commutative ring with identity and $A(R)$ be the set of ideals with non-zero annihilator. The *strongly annihilating-ideal graph of* R is defined as the graph $\mathrm{SAG}(R)$ with the vertex set $A(R)^* = A(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $I \cap \mathrm{Ann}(J) \neq (0)$ and $J \cap \mathrm{Ann}(I) \neq (0)$. This graph was first introduced and studied in [16, 17]. It is worthy to mention that strongly annihilating-ideal graph is a generalization of annihilating-ideal graph. The *annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, is a graph with the vertex set $A(R)^*$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$ (see [3] for more details). In this paper, we study the metric dimension of $\mathrm{SAG}(R)$ and we provide some metric dimension formulas for $\mathrm{SAG}(R)$.

2 Metric dimension of a strongly annihilating-ideal graph of a reduced ring

Let R be a commutative ring. In this section, we provide a metric dimension formula for a strongly annihilating-ideal graph when R is reduced.

Lemma 1 *Let* R *be a ring which is not an integral domain. Then* $dim_M(\mathrm{SAG}(R))$ *is finite if and only if* R *has only finitely many ideals.*

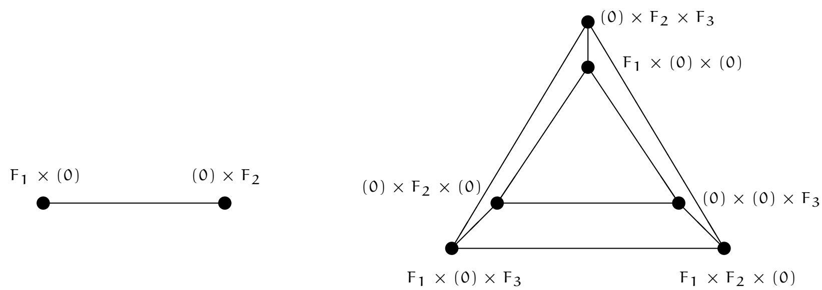
Proof. One side is clear. To prove the other side, suppose that $dim_M(\mathrm{SAG}(R))$ is finite and let $W = \{I_1, I_2, \ldots, I_n\}$ be the metric basis for $\mathrm{SAG}(R)$, where n

is a non-negative. By [16, Theorem 2.1], $\mathrm{diam}(\mathrm{SAG}(R)) \leq 2$ and so for every $I \in A(R)^* \setminus W$, there are $(2+1)^n$ possibilities for $D(I|W)$. Thus $|A(R)^*| \leq 3^n + n$ and hence R has only finitely many ideals. □

If R is a reduced ring with finitely many ideals, then by [2, Theorem 8.7], R is a direct product of finitely many fields. Using this fact, we prove the following result.

Theorem 1 *Let R be a reduced ring which is not an integral domain. If $\dim_M(\mathrm{SAG}(R))$ is finite, then:*
(1) *If $|\mathrm{Max}(R)| \leq 3$, then $\dim_M(\mathrm{SAG}(R)) = |\mathrm{Max}(R)| - 1$.*
(2) *If $|\mathrm{Max}(R)| \geq 4$, then $\dim_M(\mathrm{SAG}(R)) = |\mathrm{Max}(R)|$.*

Proof. (1) Since $\dim_M(\mathrm{SAG}(R))$ is finite, R has only finitely many ideals, by Lemma 1. Also, since R is not an integral domain, $|\mathrm{Max}(R)| \neq 1$. Hence $|\mathrm{Max}(R)| = 2$ or 3. If $|\mathrm{Max}(R)| = 2$, then $R \cong F_1 \times F_2$, where F_i is a field. Thus $\mathrm{SAG}(R) = K_2$ and so $\dim_M(\mathrm{SAG}(R)) = 1$. If $|\mathrm{Max}(R)| = 3$, then $R \cong F_1 \times F_2 \times F_3$, where F_i is a field for every $1 \leq i \leq 3$. Let $W = \{F_1 \times (0) \times F_3, F_1 \times F_2 \times (0)\}$. By the following figure, one may easily get
$D((0) \times F_2 \times (0)|W) = (1, 2)$,
$D(F_1 \times (0) \times (0)|W) = (2, 2)$,
$D((0) \times (0) \times F_3|W) = (2, 1)$,
$D((0) \times F_2 \times F_3|W) = (1, 1)$.
So for every $x, y \in V(\mathrm{SAG}(R)) \setminus W$, $D(x|W) \neq D(y|W)$ and hence $\dim_M(\mathrm{SAG}(R)) = 2$.



$\mathrm{SAG}(F_1 \times F_2)$ $\mathrm{SAG}(F_1 \times F_2 \times F_3)$

(2) Assume that $|\mathrm{Max}(R)| = n \geq 4$. By Lemma 1, $R \cong F_1 \times \cdots \times F_n$, where

F_i is a field for every $1 \leq i \leq n$. We show that $\dim_M(\mathrm{SAG}(R)) = n$. Indeed, we have the following claims:

Claim 1. $\dim_M(\mathrm{SAG}(R)) \geq n$.

Since $R \cong F_1 \times \cdots \times F_n$, by Lemma 1, $\dim_M(\mathrm{SAG}(R))$ is finite. Let $W = \{I_1, I_2, \ldots, I_k\}$ be the metric basis for $\mathrm{SAG}(R)$, where k is a positive integer. On the other hand, by [16, Theorem 2.1], $\mathrm{diam}(\mathrm{SAG}(R)) \in \{1, 2\}$, and so for every $I \in A(R)^* \setminus W$, there are 2^k possibilities for $D(I|W)$. This implies that $|A(R)^*| - k \leq 2^k$. Since $|A(R)^*| = 2^n - 2$, $2^n - 2 - k \leq 2^k$ and hence $2^n \leq 2^k + 2 + k$. Since $n \geq 4$, we deduce that $k \geq n$. Therefore $\dim_M(\mathrm{SAG}(R)) \geq n$.

Claim 2. $\dim_M(\mathrm{SAG}(R)) \leq n$.

For every $1 \leq i \leq n$, let $(F_1, \ldots, F_{i-1}, 0, F_{i+1}, \ldots, F_n) = \mathfrak{m}_i \in A(R)^*$. Put $W = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$ (in fact $W = \mathrm{Max}(R)$). We show that W is the resolving set for $\mathrm{SAG}(R)$. To see this, let $I, J \in V(\mathrm{SAG}(R)) \setminus W$ and $I \neq J$. We need only to show that $D(I|W) \neq D(J|W)$. Let $I = (I_1, I_2, \ldots, I_n)$ and $J = (J_1, J_2, \ldots, J_n)$. Since $I \neq J$, $I_i = 0$ and $J_i = F_i$ or $I_i = F_i$ and $J_i = 0$, for some $1 \leq i \leq n$. Without loss of generality, assume that $I_1 = 0$ and $J_1 = F_1$. It is easy to see that $d(I, \mathfrak{m}_1) = 1$ and $d(J, \mathfrak{m}_1) = 2$. This clearly shows that $D(I|W) \neq D(J|W)$. Therefore $\dim_M(\mathrm{SAG}(R)) \leq n$.

Now, by Claims 1, 2, $\dim_M(\mathrm{SAG}(R)) = n$, for $n \geq 4$. $\square$

3 Metric dimension of a strongly annihilating-ideal graph of a non-reduced ring

In this section, we discuss the metric dimension of strongly annihilating-ideal graphs for non-reduced rings. First we need to recall two lemmas from [16].

Lemma 2 [16, Lemma 2.1] *Let R be a ring and $I, J \in A(R)^*$. Then the following statements hold.*

(1) *If $I - J$ is not an edge of $\mathrm{SAG}(R)$, then $\mathrm{Ann}(IJ) = \mathrm{Ann}(I)$ or $\mathrm{Ann}(IJ) = \mathrm{Ann}(J)$. Moreover, if R is a reduced ring, then the converse is also true.*

(2) *If $I - J$ is an edge of $\mathbb{AG}(R)$, then $I - J$ is an edge of $\mathrm{SAG}(R)$.*

(3) *If $\mathrm{Ann}(I) \nsubseteq \mathrm{Ann}(J)$ and $\mathrm{Ann}(J) \nsubseteq \mathrm{Ann}(I)$, then $I - J$ is an edge of $\mathrm{SAG}(R)$. Moreover if R is a reduced ring, then the converse is also true.*

(4) *Let* $n \geq 1$ *be a positive integer. Suppose that* $R \cong R_1 \times \cdots \times R_n$*, where* R_i *is a ring, for every* $1 \leq i \leq n$*, and* $I = (I_1, \ldots, I_n)$ *and* $J = (J_1, \ldots, J_n)$ *are two vertices of* $\mathrm{SAG}(R)$*. If* $I_i \cap \mathrm{Ann}(J_i) \neq (0)$ *and* $J_j \cap \mathrm{Ann}(I_j) \neq (0)$*, for some* $1 \leq i, j \leq n$*, then* $I - J$ *is an edge of* $\mathrm{SAG}(R)$*. In particular, if* $I_i - J_i$ *is an edge of* $\mathrm{SAG}(R_i)$ *or* $I_i = J_i$ *and* $I_i \cap \mathrm{Ann}(I_i) \neq (0)$*, for some* $1 \leq i \leq n$*, then* $I - J$ *is an edge of* $\mathrm{SAG}(R)$*.*

(5) *If* $I, J \in \mathrm{Ess}(R)$ *or* $\mathrm{Ann}(I), \mathrm{Ann}(J) \in \mathrm{Ess}(R)$*, then* I *is adjacent to* J*.*

(6) *If* $d_{\mathbb{AG}(R)}(I, J) = 3$ *for some distinct* $I, J \in A(R)^*$*, then* $I - J$ *is an edge of* $\mathrm{SAG}(R)$*.*

(7) *If* $I - J$ *is not an edge of* $\mathrm{SAG}(R)$ *for some distinct* $I, J \in A(R)^*$*, then* $d_{\mathbb{AG}(R)}(I, J) = 2$*.*

Lemma 3 [16, Lemma 2.2] *Let* R *be a non-reduced ring and* I *be an ideal of* R *such that* $I^n = (0)$*, for some positive integer* n*. Then* $\mathrm{Ann}(I)$ *is an essential ideal of* R*.*

Remark 1 *Let* G *be a connected graph and* $V_1, V_2, \ldots, V_k$ *be a partition of* $V(G)$ *such that for every* $1 \leq i \leq k$*, if* $x, y \in V_i$*, then* $N(x) = N(y)$*. Then* $\dim_M(G) \geq |V(G)| - k$*.*

Next, we provide some formulas for the metric dimension of strongly annihilating-ideal graphs for non-reduced rings.

Theorem 2 *Suppose that* $R \cong R_1 \times \cdots \times R_n$*, where* R_i *is an Artinian local ring such that for every* $1 \leq i \leq n$*,* $|A(R_i)^*| = 1$*. Then* $\dim_M(\mathrm{SAG}(R)) = 2n$*.*

Proof. Assume that $X = (R_1, 0, \ldots, 0)$ and $Y = (I_1, 0, \ldots, 0)$, where $I_1 \in A(R_1)^*$. By Part 4 of Lemma 2, it is easy to see that $N(X) = N(Y)$. This implies that if W is the metric basis for $\mathrm{SAG}(R)$, then $X \in W$ or $Y \in W$. Without loss of generality, we may assume that $X \in W$. Similarly, we may assume that $W_1 \subseteq W$, where $W_1 = \{(R_1, 0, \ldots, 0), (0, R_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, R_n)\}$.

Now, assume that $X = (0, R_2, \ldots, R_n)$ and $Y = (I_1, R_2, \ldots, R_n)$, where $I_1 \in A(R_1)^*$. It is easy to see that $N(X) = N(Y)$ and so if W is the metric basis for $\mathrm{SAG}(R)$, then $X \in W$ or $Y \in W$. Without loss of generality, we may assume that $X \in W$. Similarly, we may assume that $W_2 \subseteq W$, where

$$W_2 = \{(0, R_2, \ldots, R_n), (R_1, 0, R_3, \ldots, R_n), \ldots, (R_1, \ldots, R_{n-1}, 0)\}.$$

Since $|W_1| = |W_2| = n$ and $W_1 \cup W_2 \subseteq W$, $|W| \geq 2n$. We show that $|W| \leq 2n$. For this, it is enough to show that W is a resolving set and consequently it is the metric basis for the graph $\mathrm{SAG}(R)$. Let $X, Y \notin W$, $X \neq Y$, $X = (I_1, \ldots, I_n)$ and $Y = (J_1, \ldots, J_n)$. We show that $D(X|W) \neq D(Y|W)$. Since $X \neq Y$, for some $1 \leq i \leq n$, we conclude that $I_i \neq J_i$. Without loss of generality, one may assume that $I_1 \supset J_1$. We have the following cases:

Case 1. $I_1 = R_1$.

Subcase 1. For some $2 \leq j \leq n$, $J_j \neq 0$. In this case, $Z - Y$ is an edge of $\mathrm{SAG}(R)$ but $Z - X$ is not an edge of $\mathrm{SAG}(R)$, where $Z = (R_1, 0, \ldots, 0)$. Since $Z \in W$, we deduce that $D(X|W) \neq D(Y|W)$.

Subcase 2. For every $2 \leq j \leq n$, $J_j = 0$. Since $I_1 = R_1$ and $(R_1, 0, \ldots, 0) \in W$, for some $2 \leq i \leq n$, $I_i \neq 0$. If $I_i = R_i$, for some $2 \leq i \leq n$, then $Z - Y$ is an edge of $\mathrm{SAG}(R)$ but $Z - X$ is not an edge of $\mathrm{SAG}(R)$, where $Z = (0, \ldots, 0, R_i, 0 \ldots, 0)$. So we can let for every $2 \leq i \leq n$, $I_i \neq R_i$. Now, without loss of generality, we may assume that $I_2 \neq 0$. Obviously, $Z - X$ is an edge of $\mathrm{SAG}(R)$ but $Z - Y$ is not an edge of $\mathrm{SAG}(R)$, where $Z = (R_1, 0, R_3 \ldots, R_n)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$.

Case 2. $I_1 \neq R_1$. Since $I_1 \neq R_1$, $J_1 \neq R_1$. Also, since $X \neq Y$, we may let

$I_1 \in A(R_1)^*$ and $J_1 = 0$. If $I_i \neq R_i$, for some $2 \leq i \leq n$, then $Z - X$ is an edge of $\mathrm{SAG}(R)$ but $Z - Y$ is not an edge of $\mathrm{SAG}(R)$, where $Z = (0, R_2, R_3 \ldots, R_n)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$. So let $X = (I_1, R_2, \ldots, R_n)$. Since $J_1 = 0$ and $Y \notin W$, for some $2 \leq i \leq n$, $J_i \in A(R_1)^*$. Without loss of generality, we may assume that $J_2 \in A(R_2)^*$. If $J_i \neq 0$, for some $3 \leq i \leq n$, then we put $Z = (0, R_2, \ldots, R_{i-1}, 0, R_{i+1}, \ldots, R_n)$. It is not hard to check that $Z - Y$ is an edge of $\mathrm{SAG}(R)$ but $Z - X$ is not an edge of $\mathrm{SAG}(R)$. If for every $3 \leq i \leq n$, $J_i = 0$, then we put $Z = (R_1, R_2, \ldots, 0, \ldots, 0)$. In both cases we have that $D(X|W) \neq D(Y|W)$. Therefore, $|W| \leq 2n$. $\square$

Theorem 3 *Suppose that $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring such that for every $1 \leq i \leq n$, $|A(R_i)^*| \geq 2$. Then $\dim_M(\mathrm{SAG}(R)) = |A(R)^*| - 3^n + 2$.*

Proof. If R is local, then Lemma 3 implies that $\mathrm{SAG}(R)$ is complete and hence $\dim_M(\mathrm{SAG}(R)) = |A(R)^*| - 1$. So let $R \cong R_1 \times \cdots \times R_n$ and $n \geq 2$. Assume that

$X = (I_1, \ldots, I_n)$, $Y = (J_1, \ldots, J_n)$ are vertices of $\mathrm{SAG}(R)$. Define the relation $\sim$ on $V(\mathrm{SAG}(R))$ as follows: $X \sim Y$, whenever, the following two conditions hold.

(1) "$I_i = 0$ if and only if $J_i = 0$" for every $1 \leq i \leq n$.

(2) "$0 \neq I_i \subseteq \mathrm{Nil}(R_i)$ if and only if $0 \neq J_i \subseteq \mathrm{Nil}(R_i)$" for every $1 \leq i \leq n$.

It is easily seen that $\sim$ is an equivalence relation on $V(\mathrm{SAG}(R))$. By [X], we mean the equivalence class of X. Let X_1 and X_2 be two elements of [X]. Since $X_1 \sim X_2$, by Part 4 of Lemma 2, $N(X_1) = N(X_2)$. This, together with the fact that the number of equivalence classes is $3^n - 2$ and Remark 1, implies that

$$\dim_M(\mathrm{SAG}(R)) \geq |A(R)^*| - (3^n - 2) = |A(R)^*| - 3^n + 2.$$

We show that

$$\dim_M(\mathrm{SAG}(R)) \leq |A(R)^*| - 3^n + 2.$$

Let

$A = \{(I_1, \ldots, I_n) \in V(\mathrm{SAG}(R)) \mid I_i \in \{0, \mathrm{Nil}(R_i), \ldots, R_i\}$ for every $1 \leq i \leq n\}$ and $W = A(R)^* \setminus A$.

It is shown that W is a resolving set and consequently it is the metric basis for the graph $\mathrm{SAG}(R)$. To see this, let $X, Y \in A$ and $X \neq Y$. We show that $D(X|W) \neq D(Y|W)$. Let $X = (I_1, \ldots, I_n)$ and $Y = (J_1, \ldots, J_n)$. Since $X \neq Y$, for some $1 \leq i \leq n$, $I_i \neq J_i$. Without loss of generality, we may assume that $I_1 \supset J_1$. We have the following cases:

Case 1. $I_1 = R_1$.

Subcase 1. $J_1 = 0$. In this case $Z - X$ is an edge of $\mathrm{SAG}(R)$ but $Z - Y$ is not an edge of $\mathrm{SAG}(R)$, where $Z = (I_1', R_2, \ldots, R_n)$ and $I_1' \in A(R_1)^* \setminus \{\mathrm{Nil}(R_1)\}$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$. **Subcase 2.** $J_1 = \mathrm{Nil}(R_1)$. In this case $Z - Y$ is an edge of $\mathrm{SAG}(R)$ but $Z - X$ is not an edge of $\mathrm{SAG}(R)$, where $Z = (J_1', 0, \ldots, 0)$, $J_1' \in A(R_1)^*$ and $J_1' \neq \mathrm{Nil}(R_1)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$.

Case 2. $I_1 = \mathrm{Nil}(R_1)$.

Since $I_1 \neq J_1$ and $I_1 \supseteq J_1$, $J_1 = 0$. Hence $Z - X$ is an edge of $\mathrm{SAG}(R)$ but $Z - Y$ is not an edge of $\mathrm{SAG}(R)$, where $Z = (J_1', R_2, \ldots, R_n)$ and $J_1' \in A(R_1)^*$ and $J_1' \neq \mathrm{Nil}(R_1)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$. Therefore,

$$\dim_M(\mathrm{SAG}(R)) \leq |W|.$$

Since $|A| = 3^n - 2$, $|W| = |A(R)^*| - (3^n - 2) = |A(R)^*| - 3^n + 2$. Therefore,

$$\dim_M(\mathrm{SAG}(R)) \leq |A(R)^*| - 3^n + 2.$$

□

Next, we provide some upper and lower bounds for the metric dimension of strongly annihilating-ideal graphs for some other classes of non-reduced rings.

Theorem 4 *Suppose that* $R \cong R_1 \times \cdots \times R_n \times F_{n+1} \times \cdots \times F_{n+m}$*, where* R_i *is an Artinian local ring such that* $|A(R_i)| = 2$ *for every* $1 \leq i \leq n$ *and* F_i *is a field for every* $1 + n \leq i \leq n + m$*. Then* $n + m \leq \dim_M(\mathrm{SAG}(R)) \leq 2^{n+m} - 2$.

Proof. Suppose that $W = \{I_1, I_2, \ldots, I_k\}$ be the metric basis for $\mathrm{SAG}(R)$, for some non-negative integer k. Since $\operatorname{diam}(\mathrm{SAG}(R)) \leq 2$, there are exactly $(2)^k$ possibilities for $D(I|W)$, for every $I \in A(R)^* \setminus W$. On the other hand, since $|A(R)^*| = 3^n 2^m - 2$, we must have $3^n 2^m - 2 - k \leq 2^k$. This implies that $n+m \leq k$. Hence $n+m \leq \dim_M(\mathrm{SAG}(R))$. It is shown that $\dim_M(\mathrm{SAG}(R)) \leq 2^{n+m} - 2$. Let

$W = \{(I_1, \ldots, I_{n+m}) \in V(\mathrm{SAG}(R)) \,|\, I_i \in \{0, R_1, \ldots, R_n, F_1, \ldots, F_m\}$ for every $1 \leq i \leq n + m\}$.

We show that W is a resolving set for $\mathrm{SAG}(R)$. For this, let $X, Y \in A(R)^* \setminus W$ and $X \neq Y$. We show that $D(X|W) \neq D(Y|W)$. Let $X = (I_1, \ldots, I_{n+m})$ and $Y = (J_1, \ldots, J_{n+m})$. Since $X \neq Y$, $I_i \neq J_i$, for some $1 \leq i \leq n + m$.

We have the following cases:

Case 1. For some $n + 1 \leq i \leq n + m$, $I_i \neq J_i$.

Without loss of generality, we may assume that $i = n + m$, $I_{n+m} = F_{n+m}$ and $J_{n+m} = 0$. Now, put $Z = (R_1, \ldots, R_n, F_{n+1} \ldots, F_{n+m-1}, 0)$. Since for some $1 \leq i \leq n$, $I_i \in A(R_i)^*$, one may easily see that $Z - X$ is an edge of $\mathrm{SAG}(R)$ but $Z - Y$ is not an edge of $\mathrm{SAG}(R)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$.

Case 2. For every $n + 1 \leq i \leq n + m$, $I_i = J_i$.

Since $I_i \neq J_i$, for some $1 \leq i \leq n$, one can let $J_1 \subset I_1$. Thus we have the following subcases:

Subcase 1. $J_1 = 0$ and $I_1 \in A(R_1)^*$.

Since $J_1 = 0$, for some $2 \leq i \leq n$, $J_i \in A(R_i)^*$. Hence one can let $J_2 \in A(R_2)^*$. If for some $2 \leq i \leq n$, $I_i \neq R_i$ or for some $1 + m \leq i \leq n + m$, $I_i \neq F_i$, then put $Z = (0, R_2, R_3 \ldots, R_n, F_{n+1}, \ldots, F_{n+m})$. $Z - X$ is an edge of $\mathrm{SAG}(R)$ but $Z - Y$ is not an edge of $\mathrm{SAG}(R)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$. So we let $X = (I_1, R_2 \ldots, R_n, F_{n+1}, \ldots, F_{n+m})$ Similarly, if for some $3 \leq i \leq n$, $J_i \neq R_i$ or for some $1 + m \leq j \leq n + m$, $J_i \neq F_i$, then without loss of generality, we may assume that $J_3 \neq R_3$. Then put $Z = (0, 0, R_3 \ldots, R_n, F_{n+1}, \ldots, F_{n+m})$. Thus $Z - Y$ is an edge of $\mathrm{SAG}(R)$ but $Z - X$ is not an edge of $\mathrm{SAG}(R)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$. Now, let $X = (I_1, R_2 \ldots, R_n, F_{n+1}, \ldots, F_{n+m})$ and $Y = (0, J_2, R_3, \ldots, R_n, F_{n+1}, \ldots, F_{n+m})$. Put $Z = (0, R_2, 0 \ldots, 0, 0, \ldots, 0)$. Therefore, $Z - Y$ is an edge of $\mathrm{SAG}(R)$ but $Z - X$ is not an edge of $\mathrm{SAG}(R)$. Since $Z \in W$, $D(X|W) \neq D(Y|W)$.

Subcase 2. $J_1 = 0$ and $I_1 = R_1$.

Since $J_1 = 0$, for some $2 \le i \le n$, $J_i \in A(R_i)^*$. Hence one may let $J_2 \in A(R_2)^*$. Assume that $Z = (R_1, 0, \dots, 0)$. Thus $Z - Y$ is an edge of SAG(R) but $Z - X$ is not an edge of SAG(R) (note that since $Z \in W$, $Z \neq X$). This implies that $D(X|W) \neq D(Y|W)$.

Subcase 3. $J_1 \in A(R_1)^*$ and $I_1 = R_1$. If $J_i \neq 0$, for some $2 \le i \le n$, then one may assume that $J_2 \neq 0$. Suppose that $Z = (R_1, 0, \dots, 0)$. Then $Z - Y$ is an edge of SAG(R) but $Z - X$ is not an edge of SAG(R). Hence $D(X|W) \neq D(Y|W)$. Let $Y = (J_1, 0, \dots, 0)$. Since $X \notin W$, for some $2 \le i \le n$, $I_i \in A(R_i)^*$. So, we can let $I_2 \in A(R_2)^*$. If $I_i \neq 0$, for some $3 \le i \le n$, then we can assume that $I_3 \neq 0$. If we put $Z = (R_1, R_2, 0, \dots, 0)$, then we easily get $D(X|W) \neq D(Y|W)$. Finally, if $X = (R_1, I_2, 0, \dots, 0)$ and $Y = (J_1, 0, \dots, 0)$, then $D(X|W) \neq D(Y|W)$. Since $Z - X$ is an edge of SAG(R) but $Z - Y$ is not an edge of SAG(R), where $Z = (R_1, 0, R_3, 0 \dots, 0)$. Therefore, $\mathrm{dim}_M(\mathrm{SAG}(R)) \le |W|$. Since $|W| = 2^{n+m} - 2$, $\mathrm{dim}_M(\mathrm{SAG}(R)) \le 2^{n+m} - 2$. □

We end this paper with the following example.

Example 1 (1) *Let* $R = \mathbb{Z}_4 \times \mathbb{Z}_2$. *Then* SAG$(R) = C_4$ *and hence* $\mathrm{dim}_M(\mathrm{SAG}(R)) = 2$. *Also, in Theorem 4,* $n = m = 1$, *and so* $\mathrm{dim}_M(\mathrm{SAG}(R)) = 2$.

(2) *Let* $R = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $\mathrm{dim}_M(\mathrm{SAG}(R)) = k$. *We show that* $3 \le k \le 6$. *Since* $\mathrm{diam}(\mathrm{SAG}(R)) \le 2$ *and* $|A(R)^*| = 10$, $10 - k \le 2^k$. *Thus* $k \ge 3$. *Let* $W = \{((2), \mathbb{Z}_2, \mathbb{Z}_2), ((2), 0, \mathbb{Z}_2), ((2), \mathbb{Z}_2, 0), ((2), 0, 0)\}$. *Then*

$D((\mathbb{Z}_4, 0, 0)|W) = (1, 1, 1, 2)$,
$D((\mathbb{Z}_4, \mathbb{Z}_2, 0))|W) = (1, 1, 2, 2)$,
$D((\mathbb{Z}_4, 0, \mathbb{Z}_2))|W) = (1, 2, 1, 2)$,
$D((0, \mathbb{Z}_2, \mathbb{Z}_2)|W) = (2, 1, 1, 1)$,
$D((0, \mathbb{Z}_2, 0)|W) = (2, 1, 2, 1)$,
$D((0, 0, \mathbb{Z}_2)|W) = (2, 2, 1, 1)$.

Therefore, W *is a resolving set for* SAG(R) *and hence* $k \le 6$.

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On application of differential subordination for Carathéodory functions

V. Agnes Sagaya Judy Lavanya
Dr. MGR Janaki College of Arts and Science
Chennai 600028, TamilNadu, India
email: lavanyaravi06@gmail.com

M. P. Jeyaraman
L. N. Government College
Ponneri, Chennai 601 204,
Tamil Nadu, India
email: jeyaraman_mp@yahoo.co.in

H. Aaisha Farzana
A. M. Jain College
Meenambakkam, Chennai 600 114,
Tamil Nadu, India
email: h.aaisha@gmail.com

Abstract. New sufficient conditions involving the properties of analytic functions to belong to the class of Carathéodory functions are investigated. Certain univalence and starlikeness conditions are deduced as special cases of main results.

1 Introduction

Let $\mathcal{H}$ be the class of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ denote the class of all the functions $f \in \mathcal{H}$ that satisfy the normalization $f(0) = 0$, $f'(0) = 1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. The function $f \in \mathcal{A}$ satisfying the conditions $\mathrm{Re}\{zf'(z)/f(z)\} > 0$, $\mathrm{Re}\{1 + zf''(z)/f'(z)\} > 0$ belong to the familiar classes of starlike and convex functions denoted by $\mathcal{S}^*$ and $\mathcal{C}$ respectively. Let f and g be analytic in $\mathbb{D}$, then we say that f is subordinate to g in $\mathbb{D}$ (written $f \prec g$)

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if there exists a Schwarz function $w(z)$,analytic in $\mathbb{D}$ with $w(0) = 0$ such that $f(z) = g(w(z)), (z \in \mathbb{D})$. In particular, if the function g is univalent in $\mathbb{D}$, then the subordination is equivalent to $f(0) = g(0)$ or $f(\mathbb{D}) \subset g(\mathbb{D})$. Let us denote by $\mathcal{Q}$ the set of functions q that are analytic and injective on $\bar{\mathbb{D}}\backslash E(q)$, where $E(q) = \{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty\}$, and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D}\backslash E(q)$. Further, the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$. Let $\mathcal{P}(\alpha)$ be a class of functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, which are analytic in $\mathbb{D}$, we say that $p(z) \in \mathcal{P}(\alpha)$ if $\mathrm{Re}\{p(z)\} > \alpha$. We note that for $\mathcal{P}(0) := \mathcal{P}$ is the class of Carathéodory functions in $\mathbb{D}$.

The function $q_c(z) = \sqrt{1 + cz}$, maps $\mathbb{D}$ onto a set which is bounded by the lemniscate of Bernoulli. That is, $q_c(\mathbb{D}) = \{w \in \mathbb{C} : |w^2 - 1| < c\}$, and the class $\mathcal{S}^*(q_c)$ given by $\mathcal{S}^*(q_c) = \{f \in \mathcal{A} : |(zf'(z)/f(z))^2 - 1| < c\} \quad (0 < c \leq 1)$, has been briefly discussed in [17]. We consider the class $\mathcal{U}(\lambda)$ of analytic functions satisfying the following condition, $\mathcal{U}(\lambda) := \{f \in \mathcal{A} : |(z/f(z))^2 f'(z) - 1| < \lambda, \quad 0 < \lambda \leq 1\}$. From [16] it is known that the functions in $\mathcal{U}(\lambda)$ are univalent if $0 < \lambda \leq 1$, but not necessarily univalent if $\lambda > 1$.

Various sufficient conditions for Carathéodory functions were studied by authors in [5, 6, 11, 12, 13, 14]. Using differential subordination as a tool, authors in [13] and [14] obtained sufficient conditions for Carathéodory functions.Recently, Kim et al. [5] obtained sufficient conditions involving the argument of the function such that the function is Carathéodory. Motivated by the aforementioned works, in this paper various results involving analytic function to be Carathéodory are obtained and as a consequence, sufficient conditions for functions to belong to the classes $\mathcal{S}^*(q_c)$ and $\mathcal{U}(\lambda)$ are provided. The results thus obtained generalize and extend certain recent results.

2 Main results

To prove the main results we need the following Lemma.

Lemma 1 [4] *Let w be a non constant regular function in $\mathbb{D}$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then*

$$z_0 w'(z_0) = k w(z_0),$$

where $k \geq 1$ is a real number.

Lemma 2 [1, 3] *Let $q \in \mathcal{Q}(a)$, and let $p(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{D}$ with $p(z) \not\equiv a$ and $n \geq 2$, if p is not subordinate to q then there exist points $z_0 = r_0 e^{i\theta_0} \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D}\backslash E(q)$ and an $m \geq n$ for which $p(\mathbb{D}_{r_0}) \subset q(\mathbb{D})$,*

1. $p(z_0) = q(\zeta_0)$
2. $\operatorname{Re}\left\{\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right\} \geq 0$ and $\left|\dfrac{zp'(z)}{q'(\zeta)}\right| \leq m,$
3. $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$
4. $\operatorname{Re}\left\{\dfrac{z_0 p''(z_0)}{p'(z_0)} + 1\right\} \geq m \operatorname{Re}\left\{\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right\}$
5. $\operatorname{Re}\left\{\dfrac{z_0^2 p'''(z_0)}{p'(z_0)}\right\} \geq m^2 \operatorname{Re}\left\{\dfrac{\zeta_0^2 q'''(\zeta_0)}{q'(z_0)}\right\}.$

Theorem 1 *Let $0 \leq \alpha < 1$, $0 < \lambda \leq 1$, β, γ, δ, $\mu \in \mathbb{R}$. For an analytic function p defined in $\mathbb{D}$ with $p(0) = 1$, if*

$$\operatorname{Re}\left\{(p(z)-\alpha)^{\lambda}(\mu + \beta(p(z)-\alpha) + \frac{\gamma}{p(z)-\alpha} + \frac{\delta z p'(z)}{p(z)-\alpha})\right\} > g(\epsilon(\alpha,\lambda),\alpha,\lambda), \tag{1}$$

where

$$g(u,\alpha,\lambda) = -u^{\lambda+1}\left(\beta + \frac{\delta}{2(1-\alpha)}\right)\sin\left(\frac{\lambda\pi}{2}\right) + \mu\cos\left(\frac{\lambda\pi}{2}\right)u^{\lambda} + \left(\gamma - \frac{\delta(1-\alpha)}{2}\right)\sin\left(\frac{\lambda\pi}{2}\right)u^{\lambda-1}$$

and

$$\epsilon(\alpha,\lambda) = \frac{\mu\lambda\cos\left(\frac{\lambda\pi}{2}\right) + \sqrt{\mu^2\lambda^2\cos^2\left(\frac{\lambda\pi}{2}\right) + 4(\lambda^2-1)\left(\beta + \frac{\delta}{2(1-\alpha)}\right)\left(\gamma - \frac{\delta(1-\alpha)}{2}\right)\sin^2\left(\frac{\lambda\pi}{2}\right)}}{2(\lambda+1)\left(\beta + \frac{\delta}{2(1-\alpha)}\right)\sin\left(\frac{\lambda\pi}{2}\right)}$$

then $p \in \mathcal{P}(\alpha)$.

Proof. Define the analytic function $p : \mathbb{D} \to \mathbb{C}$ as

$$p(z) = \frac{1 + (1-2\alpha)w(z)}{1 - w(z)}, \tag{2}$$

where w is an analytic function in $\mathbb{D}$ with $w(0) = 0$. Suppose that there exists a point $z_0 \in \mathbb{D}$, such that

$$\operatorname{Re}\{p(z)\} > \alpha \quad \text{for} \quad |z| < |z_0| \quad \text{and} \quad \operatorname{Re}\{p(z_0)\} = \alpha, \tag{3}$$

then we have

$$|w(z)| < 1 \quad \text{for} \quad |z| < |z_0| \quad \text{and} \quad |w(z_0)| = 1. \tag{4}$$

By Lemma 1 , we have $z_0 w'(z_0) = k w(z_0)$, where k is a real number with $k \geq 1$. Now,

$$\frac{z_0 p'(z_0)}{2k(1-\alpha)} = \frac{w(z_0)}{(1-w(z_0))^2} = \frac{2\{\mathrm{Re}\, w(z_0) - 1\}}{|1-w(z_0)|^4}.$$

Putting $p(z_0) = \alpha + iy,$ we have $w(z_0) = 1 - \dfrac{2(1-\alpha)^2}{(1-\alpha)^2 + y^2} + i\dfrac{2(1-\alpha)y}{(1-\alpha)^2+y^2}$ and

$$z_0 p'(z_0) = -k\frac{(1-\alpha)^2 + y^2}{2(1-\alpha)}, \tag{5}$$

which is a non positive real number. Also we observe that for the case $0 < \lambda < 1$

$$\begin{aligned}
&\mathrm{Re}\left\{(p(z_0)-\alpha)^\lambda\left(\mu + \beta(p(z_0)-\alpha) + \frac{\gamma}{p(z_0)-\alpha} + \delta\frac{z_0 p'(z_0)}{p(z_0)-\alpha}\right)\right\}\\
&= \mathrm{Re}\{\beta(p(z_0)-\alpha)^{\lambda+1} + \mu(p(z_0)-\alpha)^\lambda + \gamma(p(z_0)-\gamma)^{\lambda-1}\\
&\quad + \delta z_0 p'(z_0)(p(z_0)-\alpha)^{\lambda-1}\}\\
&= \mathrm{Re}\left\{\beta(iy)^{\lambda+1} + \mu(iy)^\lambda + \left(\gamma - k\delta\frac{(1-\alpha)^2+y^2}{2(1-\alpha)}\right)(iy)^{\lambda-1}\right\}\\
&= \mathrm{Re}\left\{\left(-\beta\sin\frac{\lambda\pi}{2} + i\beta\cos\frac{\lambda\pi}{2} - k\frac{\delta}{2(1-\alpha)}\left(\sin\frac{\lambda\pi}{2} - i\cos\frac{\lambda\pi}{2}\right)\right)|y|^{\lambda+1}\right.\\
&\quad \left. + \mu\left(\cos\frac{\lambda\pi}{2} + i\sin\frac{\lambda\pi}{2}\right)|y|^\lambda + \left(\gamma - k\delta\frac{1-\alpha}{2}\right)\left(\sin\frac{\lambda\pi}{2} - i\cos\frac{\lambda\pi}{2}\right)|y|^{\lambda-1}\right\}\\
&= \left(-\beta\sin\frac{\lambda\pi}{2} - k\frac{\delta}{2(1-\alpha)}\sin\frac{\lambda\pi}{2}\right)|y|^{\lambda+1} + \mu\left(\cos\frac{\lambda\pi}{2}\right)|y|^\lambda\\
&\quad + \left(\gamma - k\frac{\delta(1-\alpha)}{2}\right)\left(\sin\frac{\lambda\pi}{2}\right)|y|^{\lambda-1}\\
&\leq -\left(\beta + \frac{\delta}{2(1-\alpha)}\right)\sin\frac{\lambda\pi}{2}|y|^{\lambda+1} + \mu\cos\frac{\lambda\pi}{2}|y|^\lambda + \left(\gamma - \frac{\delta(1-\alpha)}{2}\right)\sin\frac{\lambda\pi}{2}|y|^{\lambda-1}\\
&= g(|y|, \alpha, \lambda) \leq \max_{u\in(0,\infty)} g(u, \alpha, \lambda) = g(\epsilon(\alpha,\lambda), \alpha, \lambda),
\end{aligned}$$

which is a contradiction to (1). For the case when $\lambda = 1$,

$$\mathrm{Re}\left\{(p(z_0)-\alpha)\left(\mu + \beta(p(z_0)-\alpha) + \frac{\gamma}{p(z_0)-\alpha} + \delta\frac{z_0 p'(z_0)}{p(z_0)-\alpha}\right)\right\}$$

$$= \mathrm{Re}\left\{(p(z_0)-\alpha)+\left(\mu+\beta(p(z_0)-\alpha)+\frac{\gamma}{p(z_0)-\alpha}-\delta k\left(\frac{(1-\alpha)^2+y^2}{2(1-\alpha)p(z_0)-\alpha}\right)\right)\right\}$$
$$\leq \mathrm{Re}\left\{(iy)\mu+\beta(iy)^2+\gamma-\delta\left(\frac{(1-\alpha)^2+y^2}{2(1-\alpha)}\right)\right\}$$
$$= -\left(\beta+\frac{\delta}{2(1-\alpha)}\right)y^2+\gamma-\delta\frac{(1-\alpha)}{2}$$
$$\leq \gamma-\delta\frac{(1-\alpha)}{2} = g(\epsilon(\alpha,1),\alpha,1).$$

This contradicts (1). Hence the proof. □

Remark 1 *By taking* $\mu = \delta = 1$ *and* $\beta = \gamma = 0$ *in Theorem 1, we get the result obtained in* [6, Theorem 2.29].

By taking $p(z) = zf'(z)/f(z)$ and $\alpha = 0$ in Theorem 1, we have the following result:

Corollary 1 *For a function* $f \in \mathcal{A}$ *and* $0 < \lambda \leq 1$*, if*

$$\mathrm{Re}\left\{(\beta-\delta)\left(\frac{zf'(z)}{f(z)}\right)^{\lambda+1}+\left(\mu+\delta+\delta\frac{zf''(z)}{f'(z)}\right)\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}+\gamma\left(\frac{zf'(z)}{f(z)}\right)^{\lambda-1}\right\} > g(\epsilon(0,\lambda),0,\lambda),$$

then $f \in \mathcal{S}^*$*.*

Theorem 2 *For an analytic function* p *in* $\mathbb{D}$ *with* $p(0) = 1$ *and* $0 \leq \alpha < 1$*, if*

$$\frac{zp'(z)}{(p(z)-\alpha)^{1/\beta}} \neq i\delta \tag{6}$$

for all $\delta \in \mathbb{R}$*, with* $|\delta| \geq 1$ *and* $\beta = \dfrac{1}{2n-1}, n \in \mathbb{N}$,
then $p \in \mathcal{P}(\alpha)$*.*

Proof. Let

$$h(z) = \left(\frac{p(z)-\alpha}{1-\alpha}\right)^{1/\beta}.$$

We note that h is analytic in $\mathbb{D}$, with $h(0) = 1$. Here $p(z) \neq \alpha$ for $z \in \mathbb{D}$, suppose that there exist a point $z_1 \in \mathbb{D}$ such that $p(z_1) = \alpha$, then z_1 is a zero of multiplicity $m \geq 1$ such that

$$h(z) = (z-z_1)^m g(z) \quad (m \in \mathbb{N}),$$

where $g(z)$ is analytic in $\mathbb{D}$ and $g(z_1) \neq 0$. Therefore we have

$$\frac{zp'(z)}{(p(z)-\alpha)} = \beta\Big(\frac{mz}{z-z_1} + \frac{zg'(z)}{g(z)}\Big). \tag{7}$$

But the imaginary part of right-hand side of (7) can take any value when z approaches z_1. This contradicts (6). Therefore $p(z) \neq \alpha$, that is $h(z) \neq 0$. Suppose that there exist a point $z_0 \in \mathbb{D}$ such that

$$\mathrm{Re}\{(h(z))^{\beta}\} > 0 \quad \text{for} \quad |z| < |z_0| \quad \text{and} \quad \mathrm{Re}\{(h(z_0))^{\beta}\} = 0 \quad (h(z_0) \neq 0).$$

Setting

$$\phi(z) = \frac{1-(h(z))^{\beta}}{1+(h(z))^{\beta}},$$

we observe that

$$|\phi(z)| < 1 \;\; \text{for} \;\; |z| < |z_0|, \;\; |\phi(z_0)| = 1 \;\; \text{and} \;\; \phi(0) = 0.$$

Hence the conditions of Lemma 1 are satisfied. By taking

$$(h(z_0))^{\beta} = iy,$$

where y is a non zero positive real number, and by using Lemma 1, we obtain

$$\frac{z_0\phi'(z_0)}{\phi(z_0)} = \frac{-2\beta(h(z_0))^{\beta-1}z_0h'(z_0)}{1-(h(z))^{2\beta}} = k,$$

and

$$-z_0h'(z_0) = \frac{k(1+y^2)}{2\beta(h(z_0))^{-1}(iy)}.$$

Now,

$$\begin{aligned}\frac{z_0p'(z_0)}{(p(z_0)-\alpha)^{1/\beta}} &= \frac{-k(1-\alpha)^{1-(1/\beta)}(1+y^2)}{2((iy)^{1/\beta}} \\ &= \frac{-k(1-\alpha)^{1-(1/\beta)}(1+y^2)}{2y^{1/\beta}}\Big(\cos\frac{\pi}{2\beta} - i\sin\frac{\pi}{2\beta}\Big).\end{aligned}$$

For $\beta = \dfrac{1}{2n-1}$, $n \in \mathbb{N}$

$$\frac{z_0p'(z_0)}{(p(z_0)-\alpha)^{1/\beta}} = \frac{-k(1-\alpha)^{1-(1/\beta)}(1+y^2)}{2y^{1/\beta}}(-1)^{n+1}i = i\delta, \quad \delta \in \mathbb{R},$$

which is a contradiction to (6), where

$$\begin{aligned}|\delta| &= \left|\frac{k(1-\alpha)^{1-(1/\beta)}(1+y^2)(-1)^{n+2}}{2y^{1/\beta}}\right| \\ &\geq \frac{(1-\alpha)^{1-(1/\beta)}(1+y^2)}{2y^{1/\beta}} \geq 1.\end{aligned}$$

Hence the proof. □

For the choice of $p(z) = zf'(z)/f(z)$, $\beta = 1$ and $p(z) = f'(z)$, $\beta = 1$ and $\alpha = 0$, in Theorem 2, we have the following Corollary 2 and Corollary 3 respectively.

Corollary 2 *For $0 \leq \alpha < 1$, if the function $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)\left(1-\frac{zf'(z)}{f(z)}\right)+z^2f''(z)}{zf'(z)-\alpha f(z)} \neq i\delta \qquad (\delta \in \mathbb{R}, |\delta| \geq 1),$$

then $f \in \mathcal{S}^*(\alpha)$.

Corollary 3 *If $f \in \mathcal{A}$ satisfies*

$$\frac{zf''(z)}{f'(z)} \neq i\delta \qquad (\delta \in \mathbb{R}, |\delta| \geq 1),$$

then f is univalent.

Theorem 3 *Let α, β, γ, $\delta \in \mathbb{R}$ with $0 \leq \alpha < 1$, $\gamma < \dfrac{\beta+\delta}{2}$ and let $G(z)$ be a complex valued function defined in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ with $p(0) = 1$ and*

$$\begin{aligned}&\mathrm{Re}\{\gamma z^3p'''(z) + (3\gamma+\beta)z^2p''(z) + (\gamma+2\beta+\delta)zp'(z) + G(z)p(z)\} \\ &\quad > \mu(\alpha,\beta,\gamma,\delta,G(z)),\end{aligned} \tag{8}$$

where

$$\mu(\alpha,\beta,\gamma,\delta,G(z)) = \frac{(1-\alpha)[\mathrm{Im}(G(z))]^2 - [\delta+\beta-2\gamma]^2}{2(\delta+\beta-2\gamma)} + \alpha\,\mathrm{Re}\{G(z)\},$$

then $p \in \mathcal{P}(\alpha)$.

Proof. Let the function p be defined as in (2), suppose that there exists a point z_0 in $\mathbb{D}$ satisfying (3). By defining $h : \mathbb{D} \to \mathbb{C}$ as $h(z) = (1+(1-2\alpha)z)/(1-z)$, we have $p \not\prec h$. By Lemma 2, there exist a $\zeta_0 \in \partial\mathbb{D}$ and $m \geq 1$ such that

$$\begin{aligned} &\operatorname{Re}\left\{1+\frac{z_0 p''(z_0)}{p'(z_0)}\right\} \geq m \operatorname{Re}\left\{1+\frac{\zeta_0 h''(\zeta_0)}{h'(\zeta_0)}\right\} = 0 \\ \text{and} \\ &\operatorname{Re}\left\{\frac{z_0^2 p'''(z_0)}{p'(z_0)}\right\} \geq m^2 \operatorname{Re}\left\{\frac{\zeta_0^2 h'''(\zeta_0)}{h'(\zeta_0)}\right\} > 0. \end{aligned} \tag{9}$$

Using (5) and (9), we obtain

$$\operatorname{Re}\{z_0^2 p''(z_0)\} \leq -z_0 p'(z_0) \quad \text{and} \quad \operatorname{Re}\{z_0^3 p'''(z_0)\} \leq 0. \tag{10}$$

From (5), (10) and by taking $p(z) = \alpha + iy \, (y \in \mathbb{R})$, we have the following inequality

$$\begin{aligned} &\operatorname{Re}\{\gamma z_0^3 p'''(z_0) + (3\gamma+\beta) z_0^2 p''(z_0) + (\gamma + 2\beta + \delta) z_0 p'(z_0) + G(z) p(z_0)\} \\ &\leq (\delta + \beta - 2\gamma) z_0 p'(z_0) + \alpha \operatorname{Re}\{G(z_0)\} - \operatorname{Im}\{G(z_0)\} y \\ &\leq -(\delta+\beta-2\gamma)\left(\frac{(1-\alpha)^2 + y^2}{2(1-\alpha)}\right) + \alpha \operatorname{Re}\{G(z_0)\} - \operatorname{Im}\{G(z_0)\} y \\ &\leq \frac{(1-\alpha)(\operatorname{Im}\{G(z_0)\})^2 - (\delta+\beta-2\gamma)^2}{2(\delta+\beta-2\gamma)} + \alpha \operatorname{Re}\{G(z_0)\} \\ &= \mu(\alpha, \beta, \gamma, \delta, G(z_0)), \end{aligned}$$

which contradicts (8) and completes the proof. □

On taking $\alpha = \beta = \gamma = 0$, $\delta = 1$ and $G(z) \equiv 1$ in Theorem 3, we get the following Corollary that improves the result of Miller [7, p.80].

Corollary 4 *For an analytic function p in $\mathbb{D}$ with $p(0) = 1$, if*

$$\operatorname{Re}\{p(z) + zp'(z)\} > -\frac{1}{2},$$

then $p \in \mathcal{P}$.

By taking $\gamma = \beta = 0$, $G(z) \equiv 1$ and $\gamma = \beta = 0$, $\delta = 1, G(z) \equiv 1$ in Theorem 3, we obtain the Corollary 5 and Corollary 6 respectively, which are due to Kim et al. [6, Theorem 2.6].

Corollary 5 *For an analytic function* p *in* $\mathbb{D}$ *with* $p(0) = 1$, *if*

$$\operatorname{Re}\{\delta z p'(z) + p(z)\} > \alpha - \frac{(1-\alpha)\delta}{2}, \qquad (0 \le \alpha < 1)$$

then $p \in \mathcal{P}(\alpha)$.

Corollary 6 *For an analytic function* p *in* $\mathbb{D}$ *with* $p(0) = 1$, *if*

$$\operatorname{Re}\{z p'(z) + p(z)\} > \frac{(3\alpha - 1)}{2}, \qquad (0 \le \alpha < 1)$$

then $p \in \mathcal{P}(\alpha)$.

Theorem 4 *Let* p *be an analytic function in* $\mathbb{D}$, *with* $p(0) = 1$ *for* $\beta > 0$, *if*

$$\left| \operatorname{Im} \left\{ (p(z))^{1/\beta} + \frac{z p'(z)}{p(z)} \right\} \right| < \frac{1}{2|(p(z))^{1/\beta}|} \Big((2-\beta)|(p(z))^{2/\beta}| - \beta \Big), \qquad (11)$$

then $(p(z))^{1/\beta} \in \mathcal{P}$.

Proof. Define the function $p : \mathbb{D} \to \mathbb{C}$ as

$$p(z) = \left(\frac{1 + w(z)}{1 - w(z)} \right)^{\beta}$$

or equivalently

$$w(z) = \frac{p(z)^{1/\beta} - 1}{p(z)^{1/\beta} + 1},$$

then w is analytic in $\mathbb{D}$ with $w(0) = 0$. Suppose that there exist a point z_0 in $\mathbb{D}$ such that

$$\operatorname{Re}\{(p(z))^{1/\beta}\} > 0 \quad \text{for} \quad |z| < |z_0| \quad \text{and} \quad \operatorname{Re}\{(p(z_0))^{1/\beta}\} = 0,$$

we obtain

$$|w(z)| < 1 \quad \text{for} \quad |z| < |z_0| \quad \text{and} \quad |w(z_0)| = 1.$$

Therefore by using Jack's Lemma, a simple calculation yields

$$\frac{1}{\beta} \frac{z_0 p'(z_0)}{p(z_0)} = \frac{2 z_0 w'(z_0)}{1 - (w(z_0))^2} = \frac{2k w(z_0)}{1 - (w(z_0))^2}\,.$$

Hence

$$\frac{1}{2k\beta} \frac{z_0 p'(z_0)}{p(z_0)} = \frac{w(z_0)}{1 - (w(z_0))^2}\,.$$

On taking $(p(z_0))^{1/\beta} = iy$, where y is a nonzero real number, we obtain

$$w(z_0) = \frac{y^2-1}{1+y^2} + i\frac{2y}{1+y^2}$$

and

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2k\beta w(z_0)}{1-(w(z_0))^2},$$

where k is a real number, with $k \geq 1$. Therefore

$$\begin{aligned}\left|\mathrm{Im}\left\{(p(z_0))^{1/\beta} + \frac{z_0 p'(z_0)}{p(z_0)}\right\}\right| &= \left|\mathrm{Im}\{(p(z_0))^{1/\beta}\} + \mathrm{Im}\left\{\frac{z_0 p'(z_0)}{p(z_0)}\right\}\right| \\ &= \left|y + 2k\beta\frac{(1+y^2)}{4y}\right| \\ &\geq \left|y + \beta\frac{(1+y^2)}{2y}\right| \\ &\geq \left|y - \beta\frac{(1+y^2)}{2y}\right| \geq |y| - \beta\frac{(1+|y|^2)}{2|y|} \\ &= |(p(z_0))^{1/\beta}| - \frac{\beta(1+|(p(z_0))^{2/\beta}|)}{2|(p(z_0))^{1/\beta}|},\end{aligned}$$

which contradicts (11). Hence the proof. □

Theorem 5 *For an analytic function* p *in* $\mathbb{D}$ *with* $p(0) = 1$*, if* p *satisfies*

$$\mathrm{Re}\left\{(p(z))^2 + \frac{zp'(z)}{p(z)}\right\} < 1 - c - \frac{c}{2(1-c)} \quad (0 < c < 1), \tag{12}$$

then $p \in \mathcal{P}$*. Also* $p(z) \prec \sqrt{1+cz}$*.*

Proof. Define a function $p : \mathbb{D} \to \mathbb{C}$ by

$$\begin{aligned} p(z) &= \sqrt{1 + cw(z)}, \quad (z \in \Delta) \\ &= 1 + p_1 z + p_2 z^2 + ..., \end{aligned}$$

or equivalently

$$w(z) = \frac{p^2(z) - 1}{c} = w_1 z + w_2 z^2 + ...,$$

we observe that w is analytic in $\mathbb{D}$ and $w(0) = 0$.

Suppose that there exist a point z_0 in $\mathbb{D}$, such that

$$\mathrm{Re}\{p(z)\} > 0 \quad \text{for} \quad |z| < |z_0| \quad \text{and} \quad \mathrm{Re}\{p(z_0)\} = 0$$

and

$$\max |w(z)| = |w(z_0)| = 1 \quad |z| \le |z_0|.$$

By Lemma 1, there exist a number $k \ge 1$ such that $z_0 w'(z_0) = k w(z_0)$. Without loss of generality we may assume that $w(z_0) = e^{i\theta}$, where $\theta \in [-\pi, \pi]$, for this z_0, we have

$$\begin{aligned}
\mathrm{Re}\left\{(p(z_0))^2 + \frac{z_0 p'(z_0)}{p(z_0)}\right\} &= \mathrm{Re}\{1 + cw(z_0)\} + \mathrm{Re}\left\{\frac{ckw(z_0)}{2(1+cw(z_0)}\right\} \\
&= \mathrm{Re}\{1 + ce^{i\theta}\} + \frac{ck}{2}\mathcal{R}e\left\{\frac{e^{i\theta}}{1+ce^{i\theta}}\right\} \\
&= \mathrm{Re}\{1 + c\cos\theta + i\sin\theta\} + \frac{ck}{2}\mathcal{R}e\left\{\frac{\cos\theta + \sin\theta}{1+ce^{i\theta}}\right\} \\
&\ge 1 + c\cos\theta + \frac{c}{2}\Big(\frac{\cos\theta + c}{1 + c^2 + 2c\cos\theta}\Big) = H(\cos\theta).
\end{aligned}$$

Let $t = \cos\theta$ then

$$H(t) = 1 + ct + \frac{c}{2}\Big(\frac{t+c}{1+c^2+2ct}\Big).$$

Since $H(t)$ is an increasing function,

$$\begin{aligned}
H(t) \ge H(-1) &= 1 - c + \frac{c}{2}\Big(\frac{c-1}{1+c^2-2c}\Big) \\
&= 1 - c - \frac{c}{2}\Big(\frac{1-c}{(1-c)^2}\Big) \\
&= 1 - c - \frac{c}{2(1-c)},
\end{aligned}$$

which is a contradiction to (12) and implies that, $\mathrm{Re}\{p(z)\} > 0$ and $|w(z_0)| < 1$. That is $p(z) \prec \sqrt{1+cz}$ and $p \in \mathcal{P}$, $z \in \mathbb{D}$. $\square$

Following results are obtained as the consequence of Theorem 5.

For the choice of $p(z) = \dfrac{zf'(z)}{f(z)}$ in Theorem 5, we have the following:

Corollary 7 *If* $f \in \mathcal{A}$ *satisfies*

$$\mathrm{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + 1\right\} < (1-c) - \frac{c}{2(1-c)} \qquad (0 < c < 1),$$

then $f \in \mathcal{S}^*(q_c)$.

By taking $p(z) = \dfrac{z\sqrt{f'(z)}}{f(z)}$ in Theorem 5, we have the following:

Corollary 8 *If* $f \in \mathcal{A}$ *satisfies*

$$\mathrm{Re}\left\{\frac{z^2 f'(z)}{(f(z))^2} + \frac{zf''(z)}{2f'(z)} - \frac{zf'(z)}{f(z)} + 1\right\} < (1-\lambda) - \frac{\lambda}{2(1-\lambda)} \qquad (0 < \lambda < 1),$$

then $f \in \mathcal{U}(\lambda)$ *and hence it is univalent.*

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On $\gamma-$countably paracompact sets

Amani Rawshdeh
Department of Mathematics,
Al-Balqa Applied University,
Alsalt 19117, Jordan
email: amanirawshdeh@bau.edu.jo
(Corresponding author)

Heyam H. Al-Jarrah
Department of Mathematics,
Faculty of science Yarmouk University,
Irbid 21163, Jordan
email: hiamaljarah@yahoo.com;
heyam@yu.edu.jo

Khalid Y. Al- Zoubi
Department of Mathematics,
Faculty of science Yarmouk
University,
Irbid 21163, Jordan
email: khalidz@yu.edu.jo

Wasfi A. Shatanawi
Department of Mathematics and General
Courses, Prince Sultan University, Riyadh
11586, Saudi Arabia.
Department of Medical Research, China
Medical University Hospital, China Medical
University, Taichung 40402, Taiwan
email: wshatanawi@yahoo.com

Abstract. In this paper we introduce and study a new class of sets, namely $\gamma-$countably paracompact sets. We characterize $\gamma-$countably paracompact sets and we study some of its basic properties. We obtain that this class of sets is weaker than $\alpha-$countably paracompact sets and stronger than $\beta-$countably paracompact sets.

1 Introduction

In [3] C. E. Aull, presented and studied the concept of $\alpha-$countably paracompact and $\beta-$countably paracompact sets. In connection with the definition of $\alpha-$countably paracompact sets and $\beta-$countably paracompact sets we obtain the definition of $\gamma-$countably paracompact sets. In section 2 of this work, we

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present $\gamma-$countably paracompact sets and then we investigate several characterizations to this types of sets and study some of its basic properties. In section 3 of this work, some of relationships between $\gamma-$countably paracompact sets and other well-known sets are investigated. In particular, we show that this class of sets lies between the classes of $\alpha-$countably paracompact sets and $\beta-$countably paracompact sets. Finally, in section 4, we introduce a class of spaces namely locally $\gamma-$countably paracompact spaces characterized by $\gamma-$countably paracompact sets and study some of their fundamental properties.

Throughout this work a space will always mean a topological space on which no separation axiom is assumed unless explicitly stated. Let (X, τ) be a space and A be a subset of X. The closure of A, interior of A and the relative topology on A in (X, τ) will be denoted by $cl(A)$, $int(A)$ and τ_A, respectively. A space (X, τ) is called countably paracompact [4] if every countable open cover of X has an open locally finite refinement. Now we begin with some known notions and definitions which will be used in this work.

Definition 1 [5] *A subset A of a space (X, τ) is called generalized closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).*

Theorem 1 [4] *If A is dense in X, then for every open $U \subseteq X$ we have $cl(U) = cl(U \cap A)$.*

Definition 2 *Let A, B, C and Y be subsets of a space (X, τ). Then:*

i. *A cover $\mathcal{U}$ of Y is called an $A-$open cover of Y [1] if $Y \subseteq \bigcup_{U \in \mathcal{U}} U$ and U is open in (A, τ_A) for every $U \in \mathcal{U}$.*
ii. *A collection $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ is called $A-$locally finite[1] if $\mathcal{U}$ is locally finite in (A, τ_A).*
iii. *If $\mathcal{U}$ and $\mathcal{V}$ are covers of Y, then $\mathcal{V}$ is called $A-$refinement of $\mathcal{U}$ [1] if for every $V \in \mathcal{V}$, $V \subseteq A$ and there exists $U \in \mathcal{U}$ such that $V \subseteq U$. If for every $V \subseteq \mathcal{V}$, V is open in (A, τ_A) then $\mathcal{V}$ is called an A-open refinement of $\mathcal{U}$.*
iv. *Y is called $\alpha-$countably paracompact of (X, τ) [3] if every countable open cover of Y by members of τ has a locally finite open refinement by members of τ.*
v. *Y is called $\beta-$countably paracompact of (X, τ) [3] if (Y, τ_Y) is countably paracompact as a subspace.*

The proof of the following proposition is obvious.

Proposition 1 *Let* Y *be a subset of a topological space* (X, τ). *If a collection* $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ *is* $X-$*locally finite, then* $\mathcal{U}$ *is* $Y-$*locally finite.*

The following example shows that the converse of the above proposition is not true in general.

Example 1 *Let* $X = \mathbb{R}$ *with the topology* $\tau = \{U : 0 \notin U\} \cup \{\mathbb{R}\}$. *Put* $Y = \mathbb{Q}^* = \mathbb{Q} - \{0\}$. *Then the collection* $\{\{y\} : y \in Y\}$ *is* $Y-$*locally finite but it is not* $X-$*locally finite.*

In the following proposition, we shall show when the converse of the above proposition is true.

Proposition 2 *Let* Y *be a closed subset of a topological space* (X, τ). *If* $\mathcal{U} = \{U_\alpha : \alpha \in I, U_\alpha \subseteq Y\}$ *is a* $Y-$*locally finite collection of subsets of* Y, *then* $\mathcal{U}$ *is* $X-$*locally finite.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be $Y-$locally finite such that $U_\alpha \subseteq Y$ for each $\alpha \in I$. If $x \in X$, then either $x \in Y$ or $x \notin Y$. If $x \in Y$, then there exists an open set W in (Y, τ_Y) such that $x \in W$ and W intersects at most finitely many members of $\mathcal{U}$. Now $W = M \cap Y$ for some $M \in \tau$. As $\mathcal{U}$ is a collection of subsets of Y, so M intersects at most finitely many members of $\mathcal{U}$. Now if $x \notin Y$, then $X - Y$ is open in (X, τ) containing x which intersects no member of $\mathcal{U}$. $\square$

Corollary 1 *Let* Y *be a closed subset of a topological space* (X, τ). *The collection* $\{U_\alpha : \alpha \in I, U_\alpha \subseteq Y\}$ *is* $Y-$*locally finite iff* $\mathcal{U}$ *is* $X-$*locally finite.*

2 $\gamma-$countably paracompact sets

In this section we shall present the concept of $\gamma-$countably paracompact sets.

Definition 3 *Let* A, B, C *and* Y *be subsets of a space* (X, τ). *Then* Y *is called* $ABC-$*countably paracompact set of* (X, τ), *if every countable* $A-$*open cover of* Y *has a* $B-$*locally finite* $C-$*open refinement.*

Note that a subset Y of a space (X, τ) is $\alpha-$countably paracompact iff it is $XXX-$countably paracompact and it is $\beta-$countably paracompact iff it is $YYY-$countably paracompact.

Definition 4 *Let* Y *be a subset of a topological space* (X, τ)*. Then* Y *is called* $\gamma-$*countably paracompact if it is* $XXY-$*countably paracompact.*

Example 2 *Let* $X = \mathbb{R}$ *with the topology* $\tau = \{U : \mathbb{R} - \mathbb{Q} \subseteq U\} \cup \{\phi\}$*. Then* $Y = \mathbb{Q}$ *is* $\gamma-$*countably paracompact.*

Proposition 3 *Let* Y *be a subset of a topological space* (X, τ)*. Then* Y *is* $\gamma-$*countably paracompact iff for every countable* $X-$*open cover* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *of* Y *there exists an* $X-$*locally finite* $Y-$*open cover* $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ *of* Y *such that* $V_n \subseteq U_n$ *for* $n = 1, 2, ...$

Proof. Let Y be a $\gamma-$countably paracompact set. If $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a countable $X-$open cover of Y, then there exists an $X-$locally finite $Y-$open refinement of $\mathcal{U}$, say $\mathcal{W}$. So for every $W \in \mathcal{W}$ choose a natural number $n(W)$ such that $W \subseteq U_{n(W)}$. Then define $V_n = \underset{n\in\mathbb{N}}{\cup}\{W : n(W) = n\}$. Hence $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ is open in Y and it is $X-$locally finite such that $V_n \subseteq U_n$ for $n = 1, 2, ...$ □

Proposition 4 *Let* Y *be a subset of a topological space* (X, τ)*. If* Y *is* $\gamma-$*countably paracompact, then for every increasing countable* $X-$*open cover* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *of* Y *there exists a* $Y-$*open cover* $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ *of* Y *such that* $cl_Y(V_n) \subseteq U_n$ *for* $n = 1, 2, ...$

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an increasing countable $X-$open cover of Y. Then, by Proposition 3, there exists an $X-$locally finite $Y-$open cover $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ of Y such that $V_n \subseteq U_n$. To show that $cl_Y(V_n) \subseteq U_n$, set $F_n = Y - \underset{m>n}{\cup} V_m$. Then F_n is closed in Y such that $V_n \subseteq F_n \subseteq \underset{m\leq n}{\cup} V_m \subseteq U_n$ and so $cl_Y(V_n) \subseteq U_n$. □

Proposition 5 *Let* Y *be a subset of a topological space* (X, τ) *and suppose that for every countable* $X-$*open cover* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *of* Y *there exists an* $X-$*locally finite* $Y-$*open cover* $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ *of* Y *such that* $V_n \subseteq U_n$ *for* $n = 1, 2, ...$ *If* $W_1 \subseteq W_2 \subseteq ...$ *is an increasing sequence of open sets in* X *such that* $\underset{n\in\mathbb{N}}{\cup} W_n = Y$*, then there exists a sequence* $F_1 \subseteq F_2 \subseteq ...$ *of closed subsets of* Y *such that* $F_n \subseteq W_n$ *for* $n = 1, 2, ...$ *and* $\underset{n\in\mathbb{N}}{\cup} int_Y(F_n) = Y$ *.*

Proof. Let $W_1 \subseteq W_2 \subseteq ...$ be an increasing sequence of $X-$open sets such that $\underset{n\in\mathbb{N}}{\cup} W_n = Y$. Then, there exists an $X-$locally finite $Y-$open cover $\mathcal{V} = \{V_n :$

$n \in \mathbb{N}\}$ of Y such that $V_n \subseteq W_n$ for all n. Now, define $F_n = Y - \underset{j>n}{\cup} V_j$ which is closed in Y and for $n \in \mathbb{N}$ we have $F_n \subseteq \underset{j\leq n}{\cup} V_j \subseteq \underset{j\leq n}{\cup} W_j = W_n$. To show that $\underset{n\in\mathbb{N}}{\cup} int_Y(F_n) = Y$, it is enough to show that $Y \subseteq \underset{n\in\mathbb{N}}{\cup} int_Y(F_n)$. Let $y \in Y$. Then there exists an open O in (X, τ) such that $y \in O$ and $(O \cap Y) \cap \underset{m>n}{\cup} V_m = \phi$ for some $n \in \mathbb{N}$, so we have $y \in O \cap Y \subseteq Y - \underset{m>n}{\cup} V_m = F_n$. Hence $y \in \underset{n\in\mathbb{N}}{\cup} int_Y(F_n)$. □

Proposition 6 *Let Y be a closed subset of a topological space (X, τ). Then every countable $X-$open cover $\mathcal{U}=\{U_n : n \in \mathbb{N}\}$ of Y has an $X-$locally finite $Y-$open cover $\mathcal{V}=\{V_n : n \in \mathbb{N}\}$ such that $V_n \subseteq U_n$ for all n iff for every increasing sequence $W_1 \subseteq W_2 \subseteq ...$ of open sets in X such that $Y = \underset{n\in\mathbb{N}}{\cup} W_n$ there exists a sequence $F_1 \subseteq F_2 \subseteq ...$ of closed subsets of Y such that $F_n \subseteq W_n$ for $n = 1, 2, ...$, moreover $\underset{n\in\mathbb{N}}{\cup} int_Y(F_n) = Y$.*

Proof. We show the sufficiency part. Let $\mathcal{U}=\{U_n : n \in \mathbb{N}\}$ be a countable $X-$open cover of Y. Set $W_n = \underset{j\leq n}{\cup} U_j$. Then $W_1 \subseteq W_2 \subseteq ...$, such that $\underset{n\in\mathbb{N}}{\cup} W_n = Y$. So there exists $F_1 \subseteq F_2 \subseteq ...$ of closed subsets of Y such that $F_n \subseteq W_n$ for $n = 1, 2, ...$ and $\underset{n\in\mathbb{N}}{\cup} int_Y(F_n) = Y$. Define $V_n = (U_n \cap Y) - \underset{j<n}{\cup} F_j$. Then V_n is open in Y and $V_n \subseteq U_n$ for $n = 1, 2, ...$. To show that $\underset{n\in\mathbb{N}}{\cup} V_n = Y$, let $y \in Y$ and j be the first index such that $y \in (U_j \cap Y)$. Therefore, $y \in V_j$. To complete the proof we show that $\mathcal{V}=\{V_n : n \in \mathbb{N}\}$ is $Y-$locally finite. Let $y \in Y$. Then there exists j such that $x \in int_Y(F_j)$ and $int_Y(F_j) \cap V_n = \phi$ for $n > j$. Therefore, $\mathcal{V}$ is $Y-$locally finite and so by Proposition 2, $\mathcal{V}=\{V_n : n \in \mathbb{N}\}$ is $X-$locally finite. □

From above discussion we can get the following Theorem.

Theorem 2 *Let Y be a closed subset of a topological space (X, τ). Then the following are equivalent:*

i. Y *is $\gamma-$countably paracompact.*

ii. *For every countable $X-$open cover $\mathcal{U}=\{U_n : n \in \mathbb{N}\}$ of Y, there exists an $X-$locally finite $Y-$open cover $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ of Y such $V_n \subseteq U_n$ for all n.*

iii. *For every increasing sequence* $W_1 \subseteq W_2 \subseteq \ldots$ *of open sets in* X *such that* $\bigcup_{n\in\mathbb{N}} W_n = Y$*, there exists* $F_1 \subseteq F_2, \ldots$ *of closed subsets of* Y *such that* $F_n \subseteq W_n$ *for* $n = 1, 2, \ldots$*, moreover* $\bigcup_{n\in\mathbb{N}} int_Y(F_n) = Y$.

iv. *For every increasing countable* $X-$*open cover* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *of* Y*, there exists a* $Y-$*open cover* $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ *of* Y *such that* $cl_Y(V_n) \subseteq U_n$ *for* $n = 1, 2, \ldots$.

v. *For every decreasing* $X-$*closed collection* $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ *such that* $(\bigcap_{n\in\mathbb{N}} F_n) \cap Y = \phi$*, there exists a* $Y-$*open cover* $\mathcal{O} = \{O_n : n \in \mathbb{N}\}$ *of* Y *such that* $cl_Y(O_n) \cap F_n = \phi$ *for* $n = 1, 2, \ldots$.

Proof. Only we prove $(iv \to i)$. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable $X-$open cover of Y. Define $W_n = \bigcup_{j\leq n} U_j$. Then the collection $\{W_n : n \in \mathbb{N}\}$ is an increasing countable $X-$open cover of Y, by (iv), there exists a $Y-$open cover $\{V_n : n \in \mathbb{N}\}$ of Y such that $cl_Y(V_n) \subseteq W_n$. Define $O_n = (U_n \cap Y) - \bigcup_{j<n} cl_Y(V_j)$. Then $\{O_n : n \in \mathbb{N}\}$ is an $X-$locally finite $Y-$open refinement of $\mathcal{U}$. □

To identify more characterization of $\gamma-$countably paracompact we need the following theorem.

Theorem 3 *Let* Y *be a* $\gamma-$*countably paracompact set in a space* (X, τ)*. If* F *is a generalized closed subset of* (X, τ) *such that* $F \subseteq Y$*, then* F *is* $\gamma-$*countably paracompact set in* (X, τ)*.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable $X-$open cover of F. Then $F \subseteq \bigcup_{n\in N} \mathcal{U} = U$. Since F is a generalized closed subset in (X, τ) and U is open in X, then $cl(F) \subseteq U$. Therefore, the collection $(X - cl(F)) \cup \{U_n : n \in \mathbb{N}\}$ is an $X-$open cover of the $\gamma-$countably paracompact set Y and so it has an $X-$locally finite open refinement, say $\mathcal{V}^*$. Put $\mathcal{V} = \{V \in \mathcal{V}^* : \exists\ U_V \in \mathcal{U}$ such that $V \subseteq U_V\}$. Finally, define $\mathcal{W} = \{V \cap F : V \in \mathcal{V}\}$. Then, it is clear that $\mathcal{W}$ is $X-$locally finite and it is $F-$open refinement of $\mathcal{U}$ since for each $V \in \mathcal{V}$ there exists an open O_V in (X, τ) such that $V = O_V \cap Y$ and so $V \cap F = O_V \cap Y \cap F = O_V \cap F$, which is open in F. □

Corollary 2 *If* $F \subseteq Y \subseteq X$ *such that* Y *is a* $\gamma-$*countably paracompact set and* F *is a closed set in* (X, τ)*. Then* F *is* $\gamma-$*countably paracompact set in* (X, τ)*.*

Corollary 3 *A closed subset of a countably paracompact space is* $\gamma-$*countably paracompact set.*

Let $\{(X_\alpha,\tau_\alpha) : \alpha \in I\}$ be a collection of topological spaces such that $X_\alpha \cap X_\beta = \phi$ for each $\alpha \neq \beta$. Let $X = \underset{\alpha\in I}{\cup} X_\alpha$ be topologized by $\tau_s = \{G \subseteq X : G \cap X_\alpha \in \tau_\alpha$ for each $\alpha \in I\}$. Then (X,τ_s) is called the sum of the spaces $\{(X_\alpha,\tau_\alpha) : \alpha \in \Delta\}$ and we write $X = \underset{\alpha\in I}{\oplus} X_\alpha$.

Theorem 4 *Let* $A_\alpha \subseteq X$ *for all* $\alpha \in I$ *and* $A = \bigcup\limits_{\alpha\in I} A_\alpha$*. Then* A *is* $\gamma-$*countable paracompact set in* X *iff* A_α *is* $\gamma-$*countable paracompact set in* X_α *for all* $\alpha \in I$.

Proof. Let $\alpha \in I$ and $\mathcal{U}$ be a countable $X_\alpha-$open cover of A_α. Then the collection $\{U : U \in \mathcal{U}\} \cup (\underset{\beta\neq\alpha}{\cup} X_\beta)$ is a countable $X-$open cover of the $\gamma-$countable paracompact set A and so it has an $X-$locally finite $A-$open refinement, say $\mathcal{V}$. Put $\mathcal{V}_{\mathcal{U}} = \{V \cap A_\alpha : V \in \mathcal{V}$ and $V \subseteq U$ for some $U \in \mathcal{U}\}$. It is clear that $\mathcal{V}_{\mathcal{U}}$ is $X_\alpha-$ locally finite $A_\alpha-$open collection such that $\mathcal{V}_{\mathcal{U}} < \mathcal{U}$. To show that $\mathcal{V}_{\mathcal{U}}$ is a cover for A_α. Let $x_\alpha \in A_\alpha$, then there exists $V \in \mathcal{V}$ such that $x_\alpha \in V$. Since $x_\alpha \notin X_\beta$ for all $\beta \neq \alpha$, then $V \subseteq U$ for some $U \in \mathcal{U}$ and $x_\alpha \in V \cap A_\alpha$. Conversely, Let $\mathcal{U}$ be a countable $X-$open cover of A. For all $\alpha \in I$, the collection $\mathcal{U}_\alpha = \{U \cap X_\alpha : U \in \mathcal{U}\}$ is a countable $X_\alpha-$open cover of the $\gamma-$countable paracompact set A_α in X_α, so it has an $X_\alpha-$locally finite $A_\alpha-$open refinement, say $\mathcal{W}_\alpha$. For all $W \in \mathcal{W}_\alpha$, there exists an open set $H_{\alpha(W)}$ in X_α such that $W = A_\alpha \cap H_{\alpha(W)} = A \cap H_{\alpha(W)}$. Put $\mathcal{H} = \{W : W \in \mathcal{W}_\alpha, \alpha \in I\}$. Then, it is clear that $\mathcal{H}$ is an $A-$open refinement of $\mathcal{U}$. To show that $\mathcal{H}$ is $X-$locally finite, let $x \in X$. Then there exists $\alpha_\circ \in I$ such that $x \in X_{\alpha_\circ}$ and $x \notin X_\beta$ for all $\beta \neq \alpha$. Since $\mathcal{W}_{\alpha_\circ}$is $X_{\alpha_\circ}-$locally finite, then there exists an open set K in $X_{\alpha_\circ}$(and so in X) such that K is intersect at most finitely many numbers of $\mathcal{W}_{\alpha_\circ}$ and $K \cap W = \phi$ for all $W \in \mathcal{W}_\alpha, \alpha \neq \alpha_\circ$. Therefore, $\mathcal{H}$ is $X-$locally finite and so A is $\gamma-$countable paracompact set in X. □

Theorem 5 *Let* $f : X \to Y$ *be a perfect onto function and let* B *be a* $\gamma-$*countably paracompact set in the space* (Y,σ)*. Then* $f^{-1}(B)$ *is* $\gamma-$*countably paracompact set in* (X,τ).

Proof. Let $\mathcal{U}$ =$\{U_n : n \in \mathbb{N}\}$ be a countable $X-$open cover of $f^{-1}(B)$. For each $y \in B$, $\mathcal{U}$ is an $X-$open cover of the compact set $f^{-1}(y)$, so there exists a finite subset $\{U_1, U_2, ... U_n\}$ of $\mathcal{U}$ such that $f^{-1}(y) \subseteq \overset{n}{\underset{i=1}{\cup}} U_i = U_y$ and U_y is open in (X,τ). Put $V_y = Y - f(X - U_y)$. Since f is closed then the collection $\mathcal{V} = \{V_y : y \in B\}$ is a countable $Y-$open cover of B, and so it has a $Y-$locally finite $B-$open refinement, say $\mathcal{W} = \{W_j : j = 1,2,...\}$. Since f is continuous, the family $f^{-1}(\mathcal{W}) = \{f^{-1}(W_j) : j = 1, 2, ...\}$ is an X−locally finite $f^{-1}(B)$−open cover of $f^{-1}(B)$ such that for each $j = 1, 2, ...$ $f^{-1}(W_j) \subseteq U_{y_j}$ for some $y_j \in B$. Finally, the collection $\{f^{-1}(W_j) \cap U_i : j = 1, 2, ... , i \in i_{y_j}\}$ is an X−locally finite $f^{-1}(B)$−open refinement of $\mathcal{U}$. Therefore, $f^{-1}(B)$ is γ−countably paracompact. □

An E_1 space [2] is a topological space such that every point is the intersection of a countable number of closed neighborhoods. Note that in [2] show that every E_1 space is T_2.

Theorem 6 *Every γ−countable paracompact subset of E_1 space is closed.*

Proof. Let Y be a γ−countably paracompact subset of an E_1 space (X, τ) and let $x \notin Y$. Let $\{C_n : n \in \mathbb{N}\}$ be a countable family of closed neighborhoods of x such that $\{x\} = \cap C_n$. Now, $\{X - C_n : n \in \mathbb{N}\}$ is a countable X−open cover of Y and $x \notin cl(X - C_n)$ for any n. Hence there is an X−locally finite Y−open refinement of $\{X - C_n : n \in \mathbb{N}\}$, say $\mathcal{W}$. Put $H = \cup\{W : W \in \mathcal{W}\}$, then $cl(H) = \cup\{cl(W) : W \in \mathcal{W}\}$. Finally, put $H^* = X - cl(H)$. So H^* is open in (X, τ) such that $x \in H^*$ and $H^* \cap Y = \phi$. Therefore, $x \notin cl(Y)$ and Y is closed. □

3 The relationship between α−countably paracompact, β−countably paracompact and γ−countably paracompact sets

In this section we study the relationship between α−countably paracompact, β−countably paracompact and γ−countably paracompact sets.

It follows from the definition that every α−countably paracompact set is γ−countably paracompact and every γ−countably paracompact set is β−countably paracompact. The following two examples show that the converse are not true in general.

Example 3 *Let $X = \mathbb{R}$ with the topology $\tau = \{U : \mathbb{R} - \mathbb{Q} \subseteq U\} \cup \{\phi\}$. Put $Y = \mathbb{Q}$. Then Y is γ−countably paracompact, note that if $\mathcal{U}$ is a countably X−open cover of Y, then the collection $\{\{y\} : y \in Y\}$ is an X−locally finite Y−open refinement of $\mathcal{U}$. Now, to show Y is not α−countably paracompact, let $\mathcal{U} = \{(\mathbb{R} - \mathbb{Q}) \cup \{y\} : y \in Y\}$. Then $\mathcal{U}$ is a countable X−open cover of Y. If $\mathcal{V}$ is an X−locally finite X−open refinement of $\mathcal{U}$, then for every $y \in Y$ there exists $y \in V \in \mathcal{V}$ such that $y \in V \subseteq (\mathbb{R} - \mathbb{Q}) \cup \{y\}$. Thus, $V = (\mathbb{R} - \mathbb{Q}) \cup \{y\}$ which means $\mathcal{V}$ is not X−locally finite.*

Example 4 *Let* $X = \mathbb{R}$ *with the topology* $\tau = \{U : 0 \notin U\} \cup \{\mathbb{R}\}$. *Then* $Y = \mathbb{Q}^* = \mathbb{Q} - \{0\}$ *is* $\beta-$*countably paracompact, since* $\tau_Y = \tau_{dis}$. *On the other hand,* Y *is not* $\gamma-$*countably paracompact, since* $\mathcal{U} = \{\{y\} : y \in Y\}$ *is a countable* $X-$*open cover of* Y *by members of* τ *and it is not* $X-$*locally finite.*

So what are the additional conditions that make the reversal of previous relationships true? This is what will be shown in the following Theorem.

Theorem 7 [3] *Let* Y *be a closed* $\beta-$*countably paracompact set in a normal space. Then* Y *is* $\alpha-$*countably paracompact*

Theorem 8 *Let* Y *be a* $\gamma-$*countably paracompact set in a space* (X, τ). *Then* Y *is* $\alpha-$*countably paracompact if one of the following holds:*

i. Y *is closed in the normal space* (X, τ).

ii. Y *is open set in the space* (X, τ).

Proof. The proof of (ii) is clear. The proof of (i) follows by Theorem 7 and from the fact that every $\gamma-$countably paracompact set is $\beta-$countably paracompact. $\square$

Theorem 9 *Let* Y *be a closed* $\beta-$*countably paracompact set in a space* (X, τ). *Then* Y *is* $\gamma-$*countably paracompact.*

Proof. Let Y be a closed $\beta-$countably paracompact subset of (X, τ) and let $\mathcal{U}$ be a countable $X-$open cover of Y. Then the collection $\mathcal{W} = \{U \cap Y : U \in \mathcal{U}\}$ is a countable $Y-$open cover of Y and so it has a $Y-$locally finite $Y-$open refinement say $\mathcal{V}$. Since Y is closed set, by Proposition 2, $\mathcal{V}$ is $X-$locally finite. Also as for every $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U \cap Y \subseteq U \in \mathcal{U}$, so $\mathcal{V}$ is $X-$locally finite $Y-$open refinement of $\mathcal{U}$. Hence Y is $\gamma-$countably paracompact. $\square$

Corollary 4 *Let* Y *be closed in a normal space* (X, τ). *The following are equivalent:*

i. Y *is* $\gamma-$*countably paracompact.*

ii. Y *is* $\alpha-$*countably paracompact.*

iii. Y *is* $\beta-$*countably paracompact.*

4 Locally $\gamma-$countably paracompact spaces

In this section we introduce locally $\gamma-$countably paracompact spaces and we study their properties.

Definition 5 *A space (X, τ) is called locally $\gamma-$ countably paracompact if each point $x \in X$ has an open neighborhood U in (X, τ) such that $cl(U)$ is $\gamma-$countably paracompact in (X, τ).*

The following result follow immediately from Theorem 9.

Proposition 7 *Let (X, τ) be a space. Then (X, τ) is locally $\gamma-$ countably paracompact iff for all $x \in X$ there exists an open neighborhood U in (X, τ) such that $cl(U)$ is $\beta-$countably paracompact.*

Theorem 10 *Every closed subspace of a locally $\gamma-$countably paracompact space is locally $\gamma-$countably paracompact.*

Proof. Let F be a closed subspace of a locally $\gamma-$countably paracompact space (X, τ). For every $x \in F$, there exists an open neighborhood U of the point x in the space (X, τ) such that $cl(U)$ is $\gamma-$countably paracompact space. The intersection $F \cap U$ is an open neighborhood of the point x in the subspace F and, by Corollary 3, $cl_F(F \cap U) = cl(F \cap U) \cap F = cl(F \cap U)$ is $\gamma-$countably paracompact, being a closed subset of the $\gamma-$countably paracompact set $cl(U)$, by Theorem 3. □

Theorem 11 *Every locally $\gamma-$countably paracompact E_1 space is T_3.*

Proof. Let F be a closed subset of a locally $\gamma-$countably paracompact space (X, τ) and $x \notin F$. Let $cl(P_x)$ be the $\gamma-$countably paracompact such that P_x is neighborhood of x and let $\{C_n : n \in N\}$ be a countable family of closed neighborhood of x such that $\{x\} = \cap C_n$. Put $H = cl(P_x) \cap F$. Then, by Theorem 3, H is $\gamma-$countably paracompact set such that $x \notin H$.Thus the collection $\{X - C_n : n \in N\}$ is a countable $X-$open cover of H and so it has an $X-$locally finite $H-$open refinement, say $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$. Since H is closed in X, then $V = (\cup U_\alpha) \cup (X - cl(P_x))$ is an open set containing F such that $x \notin cl(V)$. Hence (X, τ) is regular. □

Example 5 *Let the Hausdroff neighborhoods of a point p in the Euclidean plane consist of open circles with p at the center excluding the points on the* *vertical diameters except* p *itself. Since the resulting topology is a strengthening of the usual topology of the Euclidean plane it is an* E_1 *topology* ([2], Example 2)*. Since this is a* T_2*space which is not* T_3*, it can not be locally* $\gamma-$*countably paracompact.*

Lemma 1 *Let* Y *be an* $\alpha-$*countably paracompact Lindelöf subset of a regular locally* $\gamma-$*countably paracomact space* X*. If* W *is an open set in* (X, τ) *such that* $Y \subseteq W$*, then there is an* $X-$*locally finite collection* $\{F_n : n \in \mathbb{N}\}$ *of closed* $\gamma-$*countably paracompact sets such that* $Y \subseteq \bigcup_{n\in\mathbb{N}} int(F_n) \subseteq \bigcup_{n\in\mathbb{N}} F_n \subseteq W$.

Proof. By the regularity of the space X, then for every $x \in Y$, there exists an open set U_x in X such that $x \in U_x \subseteq cl(U_x) \subseteq W$. On the other hand, X is locally $\gamma-$countably paracompact space and so there exists an open set H_x in X such that $cl(H_x)$ is $\gamma-$countably paracompact set. Put $V_x = cl(H_x) \cap cl(U_x)$. Then, by Theorem 3, V_x is a closed $\gamma-$countably paracompact set such that $x \in int(V_x) \subseteq W$. Therefore, the collection $\mathcal{V} = \{int(V_x) : x \in Y\}$ is an $X-$open cover of the Lindelöf set Y, so it has a countable subcover, say $\mathcal{V}^*$. Since Y is $\gamma-$countably paracompact set, then $\mathcal{V}^*$ has an $X-$locally finite $X-$open refinement $\mathcal{H}$ which cover Y. Now, for every $H \in \mathcal{H}$, $cl(H)$ is a closed set in X such that $cl(H) \subseteq V_x$ for some $x \in Y$ and so $cl(H)$, by Theorem 3, is $\gamma-$countably paracompact set. Thus, the collection $\{cl(H) : H \in \mathcal{H}\}$ is the required collection. □

Theorem 12 *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a perfect function from a space* (X, τ) *onto a locally* $\gamma-$ *countably paracompact space* (Y, σ)*. Then* (X, τ) *is locally* $\gamma-$*countably paracompact.*

Proof. Let $x \in X$. Then there exists an open set V in (Y, σ) such that $f(x) \in V$ and $cl(V)$ is $\gamma-$countably paracompact in (Y, σ). Now, by Theorem 5, $f^{-1}(cl(V))$ is $\gamma-$countably paracompact subset of X. Since $cl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$, then by Theorem 3, $cl(f^{-1}(V))$ is $\gamma-$countably paracompact subset of X. □

Corollary 5 *The product of a compact space* (X, τ) *and a locally* $\gamma-$*paracompact space* (Y, σ) *is locally* $\gamma-$*countably paracompact.*

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On some Hall polynomials over a quiver of type $\tilde{D}_4$

Csaba Szántó
Babeş-Bolyai University, Cluj-Napoca
Faculty of Mathematics and Computer Science
Str. Mihail Kogalniceanu nr. 1
R0-400084 Cluj-Napoca
Romania
email: szanto.cs@gmail.com

István Szöllősi
Babeş-Bolyai University, Cluj-Napoca
Faculty of Mathematics and Computer Science
Str. Mihail Kogalniceanu nr. 1
R0-400084 Cluj-Napoca
Romania
email: szollosi@gmail.com

Abstract. Let k be an arbitrary field and Q a tame quiver of type $\tilde{D}_4$. Consider the path algebra kQ and the category of finite dimensional right modules mod-kQ. We determine the Hall polynomials F^z_{xy} associated to indecomposable modules of defect $\partial z = -2$, $\partial x = \partial y = -1$ or dually $\partial z = 2$, $\partial x = \partial y = 1$.

1 Introduction

Classical Hall algebras associated with discrete valuation rings were introduced by Steinitz and Hall to provide an algebraic approach to the classical combinatorics of partitions. The multiplication is given by Hall polynomials which play an important role in the representation theory of the symmetric groups and the general linear groups. In 1990 Ringel defined Hall algebras for a large class of rings, namely finitary rings, including in particular path algebras of quivers over finite fields. Far reaching analogues of the classical ones, these Ringel-Hall algebras provided a new approach to the study of quantum groups using the representation theory of finite dimensional algebras. They

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can also be used successfully in the theory of cluster algebras or to investigate the structure of the module category.

In case of Ringel-Hall algebras corresponding to Dynkin quivers and tame quivers we know due to Ringel and Hubery, that the structure constants of the multiplication are again polynomials in the number of elements of the base field. These are the generalized Hall polynomials. If we are looking at Hall polynomials associated to indecomposable modules, the classical ones are just 0 or 1, the generalized ones in the Dynkin case are also known and have degree up to 5, however we do not have too much information about the generalized ones in the tame case. The first lists of particular tame Hall polynomials were given by the authors in [6] and in [7]. In [6] we presented all the tame Hall polynomials associated to indecomposable modules of defect -1, 0, 1. In [7] we listed the tame Hall polynomials corresponding to exact sequences of the form $0 \to P \to R \to I \to 0$, where P is a preprojective, I a preinjective indecomposable and R is a homogeneous module of dimension δ (the minimal radical vector of the tame quiver).

In this paper we restrict ourselves to the tame quiver of type $\tilde{D}_4$ and determine all the tame Hall polynomials F^z_{xy} associated to indecomposable modules of defect $\partial z = -2$, $\partial x = \partial y = -1$ or dually $\partial z = 2$, $\partial x = \partial y = 1$.

2 Preliminaries

We begin with some facts related to representations of tame quivers. For a detailed description we refer to [1, 2, 3].

Let $Q = (Q_0, Q_1)$ be a tame quiver without oriented cycles. Suppose that the vertex set Q_0 has n elements and for an arrow $\alpha \in Q_1$ we denote by $t(\alpha), h(\alpha) \in Q_0$ the tail and head of α. The Euler form of Q is a bilinear form on $\mathbb{Z}Q_0 \cong \mathbb{Z}^n$ given by $\langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{t(\alpha)} y_{h(\alpha)}$. Its quadratic form q_Q (called Tits form) is independent from the orientation of Q and in the tame case it is positive semidefinite with radical $\mathbb{Z}\delta$, where δ is a minimal positive imaginary root of the corresponding Kac-Moody root system (which is also the minimal radical vector of the Tits form). The defect of $x \in \mathbb{Z}Q_0$ is then $\partial x = \langle \delta, x \rangle$.

Let k be a field. The category mod-kQ will be identified with the category rep-kQ of the finite dimensional k-representations of the quiver. We will denote by $[M]$ the isomorphism class of the module M, by a_M the number of its automorphisms, by $\dim M \in \mathbb{Q}_0 \cong \mathbb{Z}^n$ its dimension vector and by $\partial M =$

$\partial(\dim M)$ its defect. Using the Euler form one has for $X, Y \in$ mod-kQ

$$\langle \underline{\dim} X, \underline{\dim} Y\rangle = \dim_k \mathrm{Hom}(X, Y) - \dim_k \mathrm{Ext}^1(X, Y).$$

For $\dim_k \mathrm{Hom}(X, Y)$ we will use the notation (X, Y).

The indecomposable modules in mod-kQ are of three types: preprojectives (having negative defect), preinjectives (having positive defect) and regulars (having zero defect).

For P preprojective (i.e. with all its indecomposable components preprojective), I preinjective and R regular module we have $\mathrm{Hom}(R, P) = \mathrm{Hom}(I, P) = \mathrm{Hom}(I, R) = \mathrm{Ext}^1(P, R) = \mathrm{Ext}^1(P, I) = \mathrm{Ext}^1(R, I) = 0$. It follows that the submodules of a preprojective module are always preprojective, preinjectives can project only on preinjectives, a submodule of a regular module cannot have preinjective components and a regular cannot project on preprojectives. Preprojective and preinjective indecomposables are exceptional (i.e. their endomorphism space is one dimensional and they have no self extensions) and are uniquely determined up to isomorphism by their dimension vector, which is a positive real root of the root system of Q. Note also that the possible defects of a preprojective indecomposable are -1 in the $\tilde{A}_n$ case, -1, -2 in the $\tilde{D}_n$ case, -1, -2, -3 in the $\tilde{E}_6$ case, -1, -2, -3, -4 in the $\tilde{E}_7$ case and -1, -2, -3, -4, -5, -6 in the $\tilde{E}_8$ case.

The category of regular modules is an abelian, exact subcategory which decomposes into a direct sum of serial categories with Auslander-Reiten quiver of the form $Z\mathbb{A}_\infty/m$, called tubes of rank m. These tubes are indexed by the points of the projective line $\mathbb{P}^1_k$, the degree of a point $a \in \mathbb{P}^1_k$ being denoted by $\deg a$. A tube of rank 1 is called homogeneous, otherwise it is called non-homogeneous. We have at most 3 non-homogeneous tubes indexed by points a of degree $\deg a = 1$. All the other tubes are homogeneous. We assume that the non-homogeneous tubes are labelled by some subset of $\{0, 1, \infty\}$, whereas the homogeneous tubes are labelled by the closed points of the scheme $\mathbb{H}_k = \mathbb{H}_\mathbb{Z} \otimes k$ for some open integral subscheme $\mathbb{H}_\mathbb{Z} \subset \mathbb{P}^1_\mathbb{Z}$. Let $\mathbb{X}_k \subseteq \mathbb{H}_k$ be the set of points of degree 1. The indecomposables on a homogeneous tube labelled by $a \in \mathbb{H}_k$ are denoted by $R^k(1, a) \subset R^k(2, a) \subset \dots$. For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ let $R^k(\lambda, a) = R^k(\lambda_1, a) \oplus \dots \oplus R^k(\lambda_n, a)$. Note that the homogeneous modules of dimension δ are up to isomorphism $R^k(1, a)$, with $a \in \mathbb{X}_k$. For simplicity we will denote them by $R^k(a)$. Note that $\dim_k \mathrm{End}(R^k(a)) = 1$.

A module without homogeneous regular components can be described combinatorially, field independently, using a system of positive real roots together with the dimension of quasi-socles for the non-homogeneous regular components of dimension $t\delta$. We denote this system by μ and let $M(\mu, k)$ be the corresponding (up to isomorphism) unique module in mod-kQ. A Segre symbol is a multiset $\sigma = \{(\lambda^1, d_1), \dots, (\lambda^r, d_r)\}$, where λ^i are partitions and $d_i \in \mathbb{N}^*$. It will describe the homogeneous regular components of the module. Using the definitions above, a decomposition symbol is pair $\alpha = (\mu, \sigma)$. Given a decomposition symbol $\alpha = (\mu, \sigma)$ and a field k, we define the decomposition class $S(\alpha, k)$ to be the set of isomorphism classes of modules of the form $M(\mu, k) \oplus R$, where $R = R^k(\lambda^1, a_1) \oplus \cdots \oplus R^k(\lambda^r, a_r)$ for some distinct points $a_1, \dots a_r \in \mathbb{H}_k$ such that $\deg a_i = d_i$. We also mention that for a decomposition symbol α the polynomial $n_\alpha(q) = |S(\alpha, k)|$ is strictly increasing in $q > 1$.

Note that for k finite with q elements $|\mathbb{X}_k| = q+1$, q or $q-1$ in the $\tilde{A}_n$ case and $q-2$ for other tame quivers. So if k has 2 elements and the quiver is not of $\tilde{A}_n$ type there are no homogeneous modules of dimension δ.

For simplicity denote by x the decomposition symbol corresponding to a preprojective (preinjective) indecomposable given by the root x. Also denote by δ the symbol corresponding to homogeneous modules of dimension δ.

We mention next some needed facts about Ringel-Hall algebras. Suppose that k is finite. We consider the rational Ringel-Hall algebra $\mathcal{H}(kQ)$ of the algebra kQ. Its $\mathbb{Q}$-basis is formed by the isomorphism classes $[M]$ from mod-kQ and the multiplication is defined by $[N_1][N_2] = \sum_{[M]} F^M_{N_1N_2}[M]$. The structure constants $F^M_{N_1N_2} = |\{U \subseteq M |\ U \cong N_2,\ M/U \cong N_1\}|$ are called Ringel-Hall numbers. The associativity of the Ringel-Hall algebra follows from the equality $\sum_{[N]} F^M_{N_1N} F^N_{N_2N_3} = \sum_{[N]} F^N_{N_1N_2} F^M_{NN_3}$.

Hubery proved the existence of generalized Hall polynomials in tame cases with respect to the decomposition classes.

Theorem 1 ([4]) *Given decomposition symbols α, β and γ, there exists a rational polynomial $F^\gamma_{\alpha\beta}$ such that for any finite field k with $|k| = q$,*

$$F^\gamma_{\alpha\beta}(q) = \sum_{\substack{A \in S(\alpha,k)\\ B \in S(\beta,k)}} F^C_{AB} \qquad \text{for all } C \in S(\gamma, k)$$

and moreover

$$n_\gamma(q) F^\gamma_{\alpha\beta}(q) = n_\alpha(q) \sum_{\substack{B \in S(\beta,k)\\ C \in S(\gamma,k)}} F^C_{AB} \qquad \text{for all } A \in S(\alpha, k),$$

$$n_\gamma(q) F^\gamma_{\alpha\beta}(q) = n_\beta(q) \sum_{\substack{A \in S(\alpha,k)\\ C \in S(\gamma,k)}} F^C_{AB} \qquad \text{for all } B \in S(\beta, k).$$

Remark 1 *The polynomials* F^z_{rx} *or* F^z_{yr} *where* r *is the symbol of a homogeneous regular will denote in our article Hubery's polynomial divided by* $n_r(q)$, *which is again a polynomial.*

We list now the known tame Hall polynomials associated to indecomposables (see the introduction).

Proposition 1 ([6, 7]) *We have the following:*

a) *Suppose we limit ourselves to defects in* $\{-1, 0, 1\}$. *For two roots* x, y *with* $\partial x = \partial y = -1$ *and* $\langle x, y\rangle > 0$ *we have that* $F^y_{rx} = 1$ *for any symbol* r *corresponding to regular indecomposables of dimension* $y - x$. *This dualizes for roots with defect 1. For roots* x, y *with* $\partial x = -1$, $\partial y = 1$ *and* $\langle x, y\rangle \neq 0$ *we have that* $F^r_{yx} = \frac{1}{q-1}\alpha_r$ *for any symbol* r *corresponding to regular indecomposables of dimension* $y - x$ *(where* α_r *is the number of automorphisms). For three symols corresponding to regular indecomposables the Hall polynomial is classical so it is 0 or 1. In all the other cases the Hall polynomial is 0.*

b) *Let* x *be a positive real root with* $\partial x < 0$. *Then* $F^\delta_{\delta - x x} = h_{-\partial x}$, *where*

$$h_1 = 1,$$

$$h_2 = q - 3,$$
$$h_3 = q^2 - 5q + 7,$$
$$h_4 = q^3 - 6q^2 + 15q - 14,$$
$$h_5 = q^4 - 7q^3 + 22q^2 - 37q + 26,$$
$$h_6 = q^5 - 7q^4 + 22q^3 - 45q^2 + 62q - 39.$$

We end this section with a well known lemma:

Lemma 1 *Let* P *and* P' *be preprojective indecomposables with* $\partial P = -1$. *Then every nonzero morphism* $f : P \to P'$ *is a monomorphism.*

3 Reductions

From now on we suppose that Q is of $\tilde{D}_4$ type.

Our aim is to determine the tame Hall polynomials F^z_{xy} associated to indecomposable modules of defect $\partial z = -2$, $\partial x = \partial y = -1$ or dually $\partial z = 2$, $\partial x = \partial y = 1$.

Using reflection functors (and the fact that Hall numbers are preserved via these functors) one can see that we only need to consider a particularly oriented quiver of $\tilde{D}_4$ type (see for example [6] for all the details).

By the arguments above we will consider the quiver Q' of $\tilde{D}_4$ type with all arrows pointing to a non-central vertex (say vertex 1, the central vertex being 5). Thus the unique sink in Q' is 1 (one of the marginal vertexes):

$$Q' : \quad \begin{tikzcd} 2 \arrow[dr] & & 3 \arrow[dl] \\ & 5 \arrow[dl] & \\ 1 & & 4 \arrow[ul] \end{tikzcd}$$

We end this section with the main tool, which will provide us the recursions permitting to compute the Ringel-Hall numbers above.

Proposition 2 [5] *Let* $X, Y, Z \in \text{mod-}kQ$ *where* Q *is an arbitrary quiver and* k *is finite. Denote by* s_X^Y *the number of submodules of* Y *isomorphic to* X*, by* f_X^Y *the number of submodules of* Y *with factor isomorphic to* X*, by* e_X^Y *the number of epimorphisms from* Y *to* X*, by* α_X *the number of automorphisms of* X *and by* h_{XY} *the number of morphisms from* X *to* Y*. Then we have the following formula:*

$$e_X^Y = h_{YX} - \sum_{\substack{Z \in \text{mod-}kQ \\ \underline{\dim} Z < \underline{\dim} X}} f_Z^Y \alpha_Z s_Z^X.$$

Moreover $e_X^Y = \alpha_X f_X^Y$*.*

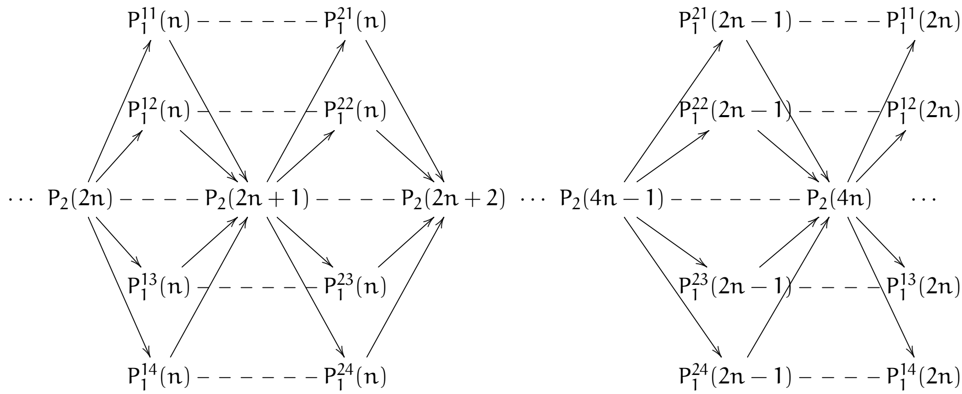
4 Recursions and Hall polynomials

Consider the quiver Q' and the indecomposable preprojectives P_0, P', P with $\underline{\dim} P = \underline{\dim} P_0 + \underline{\dim} P'$ and $\partial P_0 = \partial P' = -1$, $\partial P = -2$. Let S_1 be the projective simple corresponding to the unique sink 1. The rest of the indecomposable preprojectives are:

- $P_2(n)$ the indecomposable preprojective (of defect -2) with dimension vector $(1, 0, 0, 0, 1) + n\delta$;

- $P_1^{1i}(n)$ (for $i = \overline{1,4}$) the indecomposable preprojectives (of defect -1) of dimensions $(0,0,0,0,1) + n\delta$, $(1,1,0,0,1) + n\delta$, $(1,0,1,0,1) + n\delta$, $(1,0,0,1,1) + n\delta$;
- $P_1^{2i}(n)$ (for $i = \overline{1,4}$) the indecomposable preprojectives (of defect -1) of dimensions $(2,1,1,1,2) + n\delta$, $(1,0,1,1,2) + n\delta$, $(1,1,0,1,2) + n\delta$, $(1,1,1,0,2) + n\delta$.

The segment of the preprojective component of the Auslander-Reiten quiver which we will use is the following:



Proposition 3 $F^{P}_{P'P_0} = g_{n-1}(q)$, *where* $n = \langle \underline{\dim} P_0, \underline{\dim} P\rangle = \langle \underline{\dim} P, \underline{\dim} P'\rangle$ *and*

$$g_n = X^n - 3X^{n-1} + \cdots + (-1)^{n-1}(2n-1)X + (-1)^n(n+1)$$

(with $g_0 = 1$ *and* $g_{-1} = 0$*).*

Proof. First of all note that $n = \langle \underline{\dim} P_0, \underline{\dim} P\rangle = \langle \underline{\dim} P_0, \underline{\dim} P_0 + \underline{\dim} P'\rangle = 1 + \langle \underline{\dim} P_0, \underline{\dim} P'\rangle = \langle \underline{\dim} P_0 + \underline{\dim} P', \underline{\dim} P'\rangle = \langle \underline{\dim} P, \underline{\dim} P'\rangle$. Also if $n = \langle \underline{\dim} P_0, \underline{\dim} P\rangle = (P_0, P) = 0$, then $F^{P}_{P'P_0} = 0 = g_{-1}$.

We will use induction on $n \geq 1$. For $n = 1$, the assertion is trivial since $n = 1 = \langle \underline{\dim} P_0, \underline{\dim} P\rangle = (P_0, P)$. Using successive Auslander-Reiten translations, the fact that the modules are indecomposable preprojectives and $\underline{\dim} P = \underline{\dim} P_0 + \underline{\dim} P'$, one can see that $F^{P}_{P'P_0} = F^{P_2}_{P_1 S_1}$, where $n = 1 = \langle \underline{\dim} P_0, \underline{\dim} P\rangle = \langle \underline{\dim} S_1, \underline{\dim} P_2\rangle = (\underline{\dim} P_2)_1$. This means (looking at the dimensions) that $P_2 = P_2(0)$, and $P_1 = P_1^{11}(0)$, so $F^{P}_{P'P_0} = F^{P_2}_{P_1 S_1} = 1$.

Suppose the assertion is true for values under n and prove it for n.

Using again successive Auslander-Reiten translations, one can see (as above) that $F^{P}_{P'P_0} = F^{P_2}_{P_1 S_1}$, where $n = \langle \underline{\dim} P_0, \underline{\dim} P\rangle = \langle \underline{\dim} S_1, \underline{\dim} P_2\rangle = (\underline{\dim} P_2)_1$.

By Proposition 2 we have that

$$e_{P_1}^{P_2} = h_{P_2P_1} - \sum_{\substack{Z\in \text{mod-}kQ' \\ \underline{\dim} Z < \underline{\dim} P_1}} f_Z^{P_2} \alpha_Z s_Z^{P_1}.$$

Note that $F_{P_1S_1}^{P_2} = f_{P_1}^{P_2} = \frac{e_{P_1}^{P_2}}{\alpha_{P_1}} = \frac{e_{P_1}^{P_2}}{q-1}$ and $h_{P_2P_1} = q^{(P_2,P_1)}$. Also if there is a monomorphism $Z \to P_1$ and an epimorphism $P_2 \to Z$ it follows that $Z = 0$ or Z is a indecomposable preprojective of defect -1 such that $(Z, P_1) \neq 0$ and $(P_2, Z) \neq 0$ (here we use the fact that submodules of preprojectives are preprojective and a preprojective of defect -2 can't project on a different preprojective of defect -2). Using the fact that the indecomposable preprojectives are directing, one can see that in the Auslander-Reiten quiver Z follows after P_2 and precedes P_1.

Suppose $n = 2m$. Denote by $g'_{2m} = f_{P_1^{11}(2m)}^{P_2(2m)}$.

Using the previous formula and observations and the Auslander-Reiten segment presented above, performing all the calculations we obtain:

$$\begin{aligned} g'_{2m} &= f_{P_1^{11}(2m)}^{P_2(2m)} \\ &= \frac{q^{2m+1}-1}{q-1} - \sum_{\substack{i=\overline{1,4} \\ j=\overline{0,m-1}}} f_{P_1^{1i}(m+j)}^{P_2(2m)} s_{P_1^{1i}(m+j)}^{P_1^{11}(2m)} - \sum_{\substack{i=\overline{1,4} \\ j=\overline{0,m-1}}} f_{P_1^{2i}(m+j)}^{P_2(2m)} s_{P_1^{2i}(m+j)}^{P_1^{11}(2m)} \end{aligned} \tag{1}$$

By Lemma 1 we have that

$$s_{P_1^{1i}(m+j)}^{P_1^{11}(2m)} = \frac{q^{(P_1^{1i}(m+j), P_1^{11}(2m))} - 1}{q-1}$$

where $(P_1^{11}(m+j), P_1^{11}(2m)) = m - j + 1$ and $(P_1^{1i}(m+j), P_1^{11}(2m)) = m - j$ for $i = \overline{2,4}$. Also

$$s_{P_1^{2i}(m+j)}^{P_1^{11}(2m)} = \frac{q^{(P_1^{2i}(m+j), P_1^{11}(2m))} - 1}{q-1}$$

where $(P_1^{21}(m+j), P_1^{11}(2m)) = m - j - 1$ and $(P_1^{2i}(m+j), P_1^{11}(2m)) = m - j$ for $i = \overline{2,4}$.

The kernel of an epimorphism $P_2(2m) \to P_1^{li}(m+j)$ is preprojective and of defect -1, so it is indecomposable and unique. Denote it by X. This implies that $f_{P_1^{li}(m+j)}^{P_2(2m)} = F_{P_1^{li}(m+j)X}^{P_2(2m)}$. Using the induction hypothesis one can deduce that $F_{P_1^{1i}(m+j)X}^{P_2(2m)} = g_{2j}(q)$ and $F_{P_1^{2i}(m+j)X}^{P_2(2m)} = g_{2j+1}(q)$, since $\langle P_2(2m), P_1^{1i}(m+j)\rangle = 2j+1$ and $\langle P_2(2m), P_1^{2i}(m+j)\rangle = 2j+2$.

Substituting everything in (1) we obtain:

$$\begin{aligned}
g'_{2m} &= \frac{q^{2m+1}-1}{q-1} - \frac{q^{m+1}-1}{q-1} g_0(q) \\
&\quad - \sum_{j=\overline{1,m-1}} \frac{q^{m-j+1}-1}{q-1} \left(g_{2j}(q) + 3g_{2j-1}(q) + 3g_{2j-2}(q) + g_{2j-3}(q)\right) \\
&\quad - 3g_{2m-1}(q) - 3g_{2m-2}(q) - g_{2m-3}(q).
\end{aligned}$$

In case $n = 2m+1$ a similar recursion can be obtained for g'_{2m+1}. More precisely we get:

$$\begin{aligned}
g'_{2m+1} &= \frac{q^{2m+2}-1}{q-1} - \frac{q^{m+1}-1}{q-1} \left(g_1(q) + 3g_0(q)\right) \\
&\quad - \sum_{j=\overline{1,m-1}} \frac{q^{m-j+1}-1}{q-1} \left(g_{2j+1}(q) + 3g_{2j}(q) + 3g_{2j-1}(q) + g_{2j-2}(q)\right) \\
&\quad - 3g_{2m}(q) - 3g_{2m-1}(q) - g_{2m-2}(q).
\end{aligned}$$

By direct calculation we get that $g'_{2m} = g_{2m}(q)$ and $g'_{2m+1} = g_{2m+1}(q)$ that is, $g'_n = g_n(q)$ for all n. $\square$

Remark 2 *Based on calculations done with a computer we conjecture that the polynomials above are irreducible (as integer polynomials).*

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