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An improvement for a mathematical model for distributed vulnerability assessment

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Abstract. Hadarics et. al. gave a Mathematical Model for Distributed Vulnerability Assessment. In this model the extent of vulnerability of a specific company IT infrastructure is measured by the probability of at least one successful malware attack when the users behaviour is also incorporated into the model. The different attacks are taken as independent random experiments and the probability is calculated accordingly. The model uses some input probabilities related to the characteristics of the different threats, protections and user behaviours which are estimated by the corresponding relative frequencies. In this paper this model is further detailed, improved and a numerical example is also presented.

1 Introduction

In recent decades information and infocommunication devices have become widely used. Besides their advantages previously unknown threats and malicious codes [8], [9] appeared. Traditionally measuring cyber risk usually consist of testing malicious activity [3] and penetration testing [10], [1]. Information can be obtained from the traffic of the network hence interactive metrics can

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be evolved [5],[2], [7]. The behaviour of the users is usually regarded as a factor of secondary importance which can result in a model not adequately representing real life situations.

In an adequate model for assessing vulnerability of a specific business all three factors should be considered:

1. Malicious activity from the outher world threatening the IT network of the business.

2. Not properly protected elements of the IT network at the business.

3. Dangerous behaviours of users inside the business.

2 The model

Most of the notation of [4] will be used. For completeness these notations are to be reviewed.

Let $L\{l_1, \dots, l_\tau\}$ be the set of all available threat landscapes. In what follows a specific landscape will be used denoted by l. Let T_{all} be the set of all possible malware. Let $T = \{t_1, \dots, t_k\}$ be the set of all possible malware inside l. Let $U = \{u_1, \dots, u_r\}$ be the set of all users. Let $D = \{d_1, \dots, d_m\}$ be the set of all possible devices inside l. Let $P = \{p_1, \dots, p_n\}$ be the set of all available protections inside l. Let $UT = \{ut_1, \dots, ut_i\}$ be the set of all possible user tricks used by any malware inside l.

An integrated measure of vulnerability accounting for all three sources (attacker ingenuity, infrastructure weakness and adverse user behaviour) can be constructed.

For any given threat or class of threats for which the requisite IT infrastructure vulnerability and user facilitation is known, we can obtain a best estimate of:

1. The probability that an attacker will use a particular threat or class of threats against the enterprise (p_{prev}).
The probability p_{prev} is estimated by

$$p_{prev}(t, l) = \frac{\text{number of computers infected by } t \text{ inside } l}{\text{number of all computers inside } l}$$

for $t \in T$. Note, that p_{prev} can be based on a measurement or estimation and must be related to a time interval. Let

$$P_{prev} = \begin{array}{c|cccc} & t_1 & t_2 & \dots & t_k \\ \hline p & p_{prev}(t_1) & p_{prev}(t_2) & \dots & p_{prev}(t_k) \end{array}$$

be a vector. This means if we examine a particular attack, then the probability that this attack is in the form of the threat t_1 is $p_{prev}(t_1)$, etc.

2. The probability that the enterprise's IT infrastructure will allow the attack to be carried out successfully (p_{device}).

To elaborate the estimation of p_{device} first some auxiliary probabilities are defined and estimated.

The probability $p_{prot}(t,p)$ is introduced

$$p_{prot}(t,p) = \frac{\text{number of successful attempts of } t \text{ through the protection } p}{\text{number of all attempts of } t \text{ through the protection } p}$$

for any $t \in T$ and $p \in P$. Let

$$P_{prot} = \begin{array}{c|cccc} & p_1 & p_2 & \dots & p_n \\ \hline t_1 & p_{prot}(t_1,p_1) & p_{prot}(t_1,p_2) & \dots & p_{prot}(t_1,p_n) \\ t_2 & p_{prot}(t_2,p_1) & p_{prot}(t_2,p_2) & \dots & p_{prot}(t_2,p_n) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ t_k & p_{prot}(t_k,p_1) & p_{prot}(t_k,p_2) & \dots & p_{prot}(t_k,p_n) \end{array}$$

be a $k \times n$ matrix. This means that the probability of a successful attempt of t_1 through the protection p_1 is $p_{prot}(t_1,p_1)$, etc.

The value $z_{device-elements}(d,t)$ is introduced

$$z_{device-elements}(d,t) = \begin{cases} 1 & \text{if } t \text{ can work on } d \\ 0 & \text{if } t \text{ can not work on } d \end{cases}$$

(or shortly $z_{dev-elem}(d,t)$) for any $t \in T$ and $d \in D$. Let

$$Z_{device-elements}$$

$$= \begin{array}{c|cccc} & t_1 & t_2 & \dots & t_k \\ \hline d_1 & z_{dev-elem}(d_1,t_1) & z_{dev-elem}(d_1,t_2) & \dots & z_{dev-elem}(d_1,t_k) \\ d_2 & z_{dev-elem}(d_2,t_1) & z_{dev-elem}(d_2,t_2) & \dots & z_{dev-elem}(d_2,t_k) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ d_m & z_{dev-elem}(d_m,t_1) & z_{dev-elem}(d_m,t_2) & \dots & z_{dev-elem}(d_m,t_k) \end{array}$$

be an $m \times k$ matrix.

The value $z_{device-prot-install}(d,p)$ is introduced

$$z_{device-prot-install}(d,p) = \begin{cases} 1 & \text{if } d \text{ does not have the protection } p \\ 0 & \text{if } d \text{ has the protection } p \end{cases}$$

(or shortly z_{d-p-i}) for any $d \in D$ and $p \in P$. Let

$$Z_{device-prot-install}$$

$$= \begin{array}{c|cccc} & p_1 & p_2 & \dots & p_n \\ \hline d_1 & z_{d-p-i}(d_1, p_1) & z_{d-p-i}(d_1, p_2) & \dots & z_{d-p-i}(d_1, p_n) \\ d_2 & z_{d-p-i}(d_2, p_1) & z_{d-p-i}(d_2, p_2) & \dots & z_{d-p-i}(d_2, p_n) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ d_m & z_{d-p-i}(d_m, p_1) & z_{d-p-i}(d_m, p_2) & \dots & z_{d-p-i}(d_m, p_n) \end{array}$$

be an $m \times n$ matrix. Let

$$P_{device-prot-install-d_j}$$

$$= \begin{array}{c|cccc} & p_1 & p_2 & \dots & p_n \\ \hline t_1 & p_{d-p-i-d_j}(t_1, p_1) & p_{d-p-i-d_j}(t_1, p_2) & \dots & p_{d-p-i-d_j}(t_1, p_n) \\ t_2 & p_{d-p-i-d_j}(t_2, p_1) & p_{d-p-i-d_j}(t_2, p_2) & \dots & p_{d-p-i-d_j}(t_2, p_n) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ t_k & p_{d-p-i-d_j}(t_k, p_1) & p_{d-p-i-d_j}(t_k, p_2) & \dots & p_{d-p-i-d_j}(t_k, p_n) \end{array}$$

be a $k \times n$ matrix where

$$p_{d-p-i-d_j}(t_x, p_y) = \max\{p_{prot}(t_x, p_y), z_{d-p-i}(d_j, p_y)\}$$

for any $j \in \{1, \dots, m\}$, $x \in \{1, \dots, k\}$ and $y \in \{1, \dots, n\}$. This means that if the threat t_1 can work on d_j, then the probability of a successful attempts of the threat t_1 through the protection p_1 on the device d_j is $p_{d-p-i-d_j}(t_1, p_1)$, etc. The probability $p_{device-prot-d_j}(t)$ is introduced

$$p_{device-prot-d_j}(t) = \min_{\text{for all } p \text{ protecting } d_j} p_{prot}(t, p)$$

for any $t \in T$. Let

$$P_{device-prot-d_j} = \begin{array}{c|c} & p \\ \hline t_1 & p_{device-prot-d_j}(t_1) \\ t_2 & p_{device-prot-d_j}(t_2) \\ \vdots & \vdots \\ t_k & p_{device-prot-d_j}(t_k) \end{array}$$

be the column vector where

$$p_{device-prot-d_j}(t_x)$$

$$= \min\{p_{d-p-i-d_j}(t_x, p_1), p_{d-p-i-d_j}(t_x, p_2), \ldots, p_{d-p-i-d_j}(t_x, p_n)\}$$

for any $j \in \{1, \ldots, m\}$ and $x \in \{1, \ldots, k\}$. This means that if the threat t_1 can work on d_j, then the probability of a successful attempts of the threat t_1 through any protection protecting the device d_j is $p_{device-prot-d_j}(t_1)$, etc.
The probability $p_{device-prot}(d, t)$ is introduced

$$p_{device-prot}(d, t) = \min_{\text{for all } p \text{ protecting } d} p_{prot}(t, p)$$

for any $t \in T$ and $d \in D$. Let

$$P_{device-prot} = \begin{array}{c|cccc} & t_1 & t_2 & \ldots & t_k \\ \hline d_1 & p_{device-prot}(d_1, t_1) & p_{device-prot}(d_1, t_2) & \ldots & p_{device-prot}(d_1, t_k) \\ d_2 & p_{device-prot}(d_2, t_1) & p_{device-prot}(d_2, t_2) & \ldots & p_{device-prot}(d_2, t_k) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ d_m & p_{device-prot}(d_m, t_1) & p_{device-prot}(d_m, t_2) & \ldots & p_{device-prot}(d_m, t_k) \end{array}$$

be an $m \times k$ matrix where

$$p_{device-prot}(d_x, t_y) = p_{device-prot-d_x}(t_y)$$

for any $x \in \{1, \ldots, m\}$ and $y \in \{1, \ldots, k\}$.
The probability $p_{device}(d, t)$ is introduced

$$p_{device}(d, t) = z_{decive-elements}(d, t) \cdot p_{device-prot}(d, t)$$

for any $t \in T$ and $d \in D$. Let

$$P_{device} = \begin{array}{c|cccc} & t_1 & t_2 & \ldots & t_k \\ \hline d_1 & p_{device}(d_1, t_1) & p_{device}(d_1, t_2) & \ldots & p_{device}(d_1, t_k) \\ d_2 & p_{device}(d_2, t_1) & p_{device}(d_2, t_2) & \ldots & p_{device}(d_2, t_k) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ d_m & p_{device}(d_m, t_1) & p_{device}(d_m, t_2) & \ldots & p_{device}(d_m, t_k) \end{array}$$

be an $m \times k$ matrix where

$$p_{device}(d_x, t_y) = z_{dev-elem}(d_x, t_y) \cdot p_{device-prot}(d_x, t_y)$$

for any $x \in \{1, \ldots, m\}$ and $y \in \{1, \ldots, k\}$. This means that the probability of a successful attempts of the threat t_1 through any protection protecting the device d_1 is $p_{device}(d_1, t_1)$, etc.

3. The probability that users of the enterprise's IT infrastructure will provide sufficient facilitation for the attack to succeed (p_{user}).
The $p_{usertrick}(t, ut)$ probability is introduced

$$p_{usertrick}(t, ut) = \frac{\text{number of attempts of } t \text{ where } t \text{ used } ut}{\text{number of all attempts of } t}$$

for any $t \in T$ and $ut \in UT$. Let

$$P_{usertrick} = \begin{array}{c|cccc} & ut_1 & ut_2 & \dots & ut_i \\ \hline t_1 & p_{usertrick}(t_1, ut_1) & p_{usertrick}(t_1, ut_2) & \dots & p_{usertrick}(t_1, ut_i) \\ t_2 & p_{usertrick}(t_2, ut_1) & p_{usertrick}(t_2, ut_2) & \dots & p_{usertrick}(t_2, ut_i) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ t_k & p_{usertrick}(t_k, ut_1) & p_{usertrick}(t_k, ut_2) & \dots & p_{usertrick}(t_k, ut_i) \end{array}$$

be a $k \times i$ matrix. This means that the probability that the threat t_1 uses usertrick ut_1 is $p_{usertrick}(t_1, ut_1)$, etc.
The $p_{user-usertrick}(u, ut)$ probability is introduced

$$p_{user-usertrick}(u, ut) = \frac{\text{number of successful attempts of } ut \text{ on } u}{\text{number of all attempts of } ut \text{ on } u}$$

(or shortly $p_{u-utrick}(u, ut)$) for any $u \in U$ and $ut \in UT$. Let

$$P_{user-usertrick} = \begin{array}{c|cccc} & ut_1 & ut_2 & \dots & ut_i \\ \hline u_1 & p_{u-utrick}(u_1, ut_1) & p_{u-utrick}(u_1, ut_2) & \dots & p_{u-utrick}(u_1, ut_i) \\ u_2 & p_{u-utrick}(u_2, ut_1) & p_{u-utrick}(u_2, ut_2) & \dots & p_{u-utrick}(u_2, ut_i) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ u_r & p_{u-utrick}(u_r, ut_1) & p_{u-utrick}(u_r, ut_2) & \dots & p_{u-utrick}(u_r, ut_i) \end{array}$$

be an $r \times i$ matrix. This means that the probability that the user u_1 uses usertrick ut_1 is $p_{u-utrick}(u_1, ut_1)$, etc.
From the probabilities $p_{usertrick}$ and $p_{user-usertrick}$ we can calculate the probability $p_{user}(u, t)$ which is the probability that the threat t infects using at least one usertrick through the user u. This is

$$p_{user}(u, t)$$

$$= 1 - \prod_{\text{for all } ut \text{ used by } t} (1 - p_{usertrick}(t, ut) \cdot p_{user-usertrick}(u, ut))$$

for any $u \in U$, $t \in T$ and $ut \in UT$. Let

$$P_{user} = \begin{array}{c|cccc} & t_1 & t_2 & \dots & t_k \\ \hline u_1 & p_{user}(u_1, t_1) & p_{user}(u_1, t_2) & \dots & p_{user}(u_1, t_k) \\ u_2 & p_{user}(u_2, t_1) & p_{user}(u_2, t_2) & \dots & p_{user}(u_2, t_k) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ u_r & p_{user}(u_r, t_1) & p_{user}(u_r, t_2) & \dots & p_{user}(u_r, t_k) \end{array}$$

be an $r \times k$ matrix where

$$\begin{aligned} & p_{user}(u_1, t_1) \\ & = 1 - (1 - p_{usertrick}(t_1, ut_1) \cdot p_{u-utrick}(u_1, ut_1)) \\ & \cdot (1 - p_{usertrick}(t_1, ut_2) \cdot p_{u-utrick}(u_1, ut_2)) \cdot \dots \\ & \cdot (1 - p_{usertrick}(t_1, ut_i) \cdot p_{u-utrick}(u_1, ut_i)), \end{aligned}$$

etc. This means that the probability that the threat t_1 infects using at least one usertrick through the user u_1 is $p_{user}(u_1, t_1)$, etc.

2.1 The probability of infection

These three probabilities (p_{prev}, p_{device}, p_{user}) can be combined to obtain an overall probability of malicious success, (provided each relevant combination of attack, user, and component of IT infrastructure is accounted for) [6]. The (p_{prev}, p_{device}, p_{user}) values are related to a given threat, a given user and a given device. The aggregated vulnerability would be an index of the whole organization related to all of the users, all of the devices and all of the possible threats. The probability of the infection is p_s which is the probability that the investigated landscape will be infected by at least one malware. This can be calculated in the following form

$$p_s = 1 - \prod_{\text{for all } t,u \text{ and } d} (1 - p_{user}(t, u) \cdot p_{device}(t, d) \cdot p_{prev}(t, l))$$

for any $u \in U$, $t \in T$ and $d \in D$.

The followings were assumed:

1. the attacker usage of the given threat, the IT infrastructure allowance and the user acceptance are different from each other,

2. all of the attack attempts are independent from each other,
3. the computer usage behaviours of all users are the same and equal to the average usage in the organization.

Observe the calculated p_s value is related to the same time interval as the original p_{prev} was related to.

3 A numerical example

Let $T = \{t_1, \ldots, t_4\}$ be the set of malware. Let $U = \{u_1, \ldots, u_7\}$ be the set of all users. Let $D = \{d_1, d_2, d_3\}$ be the set of all devices. Let $P = \{p_1, \ldots, p_5\}$ be the set of all protections. Let $UT = \{ut_1, \ldots, ut_6\}$ be the set of all user tricks used by any malware in T. Let

$$P_{prev} = \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \\ \hline 0.25 & 0.25 & 0.25 & 0.25 \end{array}$$

and

$$P_{prot} = \begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \hline t_1 & 0.01 & 0.02 & 0.03 & 0.04 & 0.02 \\ t_2 & 0.11 & 0.12 & 0.13 & 0.14 & 0.15 \\ t_3 & 0.21 & 0.22 & 0.23 & 0.24 & 0.25 \\ t_4 & 0.31 & 0.32 & 0.33 & 0.34 & 0.35 \end{array}.$$

This means that the probability of a successful attempt of t_1 through the protection p_1 is 0.01, etc.
Let

$$Z_{device\_elements} = \begin{array}{c|cccc} & t_1 & t_2 & t_3 & t_4 \\ \hline d_1 & 1 & 0 & 0 & 0 \\ d_2 & 0 & 1 & 1 & 0 \\ d_3 & 0 & 1 & 0 & 1 \end{array}.$$

This means that t_1 can work on d_1, t_2 can not work on d_1, etc.
Let

$$Z_{device\_prot\_install} = \begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \hline d_1 & 1 & 0 & 1 & 0 & 1 \\ d_2 & 0 & 1 & 1 & 0 & 1 \\ d_3 & 1 & 0 & 0 & 1 & 1 \end{array}.$$

This means that d_1 does not have the protection p_1, d_1 has the protection p_2, etc.

Thus

$$P_{\text{prot\_install\_d}_1} = \begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \hline t_1 & 1 & 0.02 & 1 & 0.04 & 1 \\ t_2 & 1 & 0.12 & 1 & 0.14 & 1 \\ t_3 & 1 & 0.22 & 1 & 0.24 & 1 \\ t_4 & 1 & 0.32 & 1 & 0.34 & 1 \end{array} .$$

Observe

$$\begin{aligned} p_{d-p-i-d_1}(t_1, p_1) &= \max\{p_{prot}(t_1, p_1), z_{d-p-i}(d_1, p_1)\} = \max\{0.01, 1\} = 1, \\ p_{d-p-i-d_1}(t_1, p_2) &= \max\{p_{prot}(t_1, p_2), z_{d-p-i}(d_1, p_2)\} = \max\{0.02, 0\} = 0.02, \end{aligned}$$

etc. This means that the probability of a successful attempts of the threat t_1 through the protection p_1 on the device d_1 is $p_{d-p-i-d_1}(t_1, p_1)$, etc. Similarly

$$P_{\text{prot\_intall\_d}_2} = \begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \hline t_1 & 0.01 & 1 & 1 & 0.04 & 1 \\ t_2 & 0.11 & 1 & 1 & 0.14 & 1 \\ t_3 & 0.21 & 1 & 1 & 0.24 & 1 \\ t_4 & 0.31 & 1 & 1 & 0.34 & 1 \end{array} ,$$

$$P_{\text{prot\_intall\_d}_3} = \begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \hline t_1 & 1 & 0.02 & 0.03 & 1 & 1 \\ t_2 & 1 & 0.12 & 0.13 & 1 & 1 \\ t_3 & 1 & 0.22 & 0.23 & 1 & 1 \\ t_4 & 1 & 0.32 & 0.33 & 1 & 1 \end{array} .$$

Furthermore

$$P_{\text{device\_prot\_D}_1} = \begin{array}{c|c} & P \\ \hline t_1 & 0.02 \\ t_2 & 0.12 \\ t_3 & 0.22 \\ t_4 & 0.32 \end{array} .$$

Observe

$$\begin{aligned} & p_{device-prot-d_1}(t_1) \\ & = \min\{p_{d-p-i-d_1}(t_1, p_1), p_{d-p-i-d_1}(t_1, p_2), \ldots, p_{d-p-i-d_1}(t_1, p_n)\} \\ & \quad \min\{1, 0.02, 0.03, 1, 1\} = 0.02. \end{aligned}$$

This means that if the threat t_1 can work on d_1, then the probability of a successful attempts of the threat t_1 through any protection protecting the device d_1 is 0.02, etc. Similarly

$$P_{device-prot-d_2} = \begin{array}{c|c} & P \\ \hline t_1 & 0.01 \\ t_2 & 0.11 \\ t_3 & 0.21 \\ t_4 & 0.31 \end{array},$$

$$P_{device-prot-d_3} = \begin{array}{c|c} & P \\ \hline t_1 & 0.02 \\ t_2 & 0.12 \\ t_3 & 0.22 \\ t_4 & 0.32 \end{array}.$$

Thus

$$P_{device-prot} = \begin{array}{c|cccc} & t_1 & t_2 & t_3 & t_4 \\ \hline d_1 & 0.02 & 0.12 & 0.22 & 0.32 \\ d_2 & 0.01 & 0.11 & 0.21 & 0.31 \\ d_3 & 0.02 & 0.12 & 0.22 & 0.32 \end{array}.$$

Observe

$$p_{device-prot}(d_1, t_1) = p_{device-prot-d_1}(t_1),$$
$$p_{device-prot}(d_1, t_2) = p_{device-prot-d_1}(t_2),$$

etc. This means that if the threat t_1 can work on d_1, then the probability of a successful attempts of the threat t_1 through any protection protecting the device d_1 is 0.02, etc. Furthermore

$$P_{device} = \begin{array}{c|cccc} & t_1 & t_2 & t_3 & t_4 \\ \hline d_1 & 0.02 & 0 & 0 & 0 \\ d_2 & 0 & 0.11 & 0.21 & 0 \\ d_3 & 0 & 0.12 & 0 & 0.32 \end{array}.$$

Observe

$$p_{device}(d_1, t_1) = z_{dev-elem}(d_1, t_1) \cdot p_{device-prot}(d_1, t_1) = 0.02 \cdot 1 = 0.02,$$
$$p_{device}(d_1, t_2) = z_{dev-elem}(d_1, t_2) \cdot p_{device-prot}(d_1, t_2) = 0.12 \cdot 0 = 0,$$

etc. This means that the probability of a successful attempts of the threat t_1 through any protection protecting the device d_1 is 0.02. Since t_2 can not work on d_1, the probability of a successful attempts of the threat t_2 through any protection protecting the device d_1 is 0, etc. Let

$$P_{usertrick} = \begin{array}{c|cccccc} & ut_1 & ut_2 & ut_3 & ut_4 & ut_5 & ut_6 \\ \hline t_1 & 0.141 & 0.142 & 0.143 & 0.144 & 0.145 & 0.146 \\ t_2 & 0.151 & 0.152 & 0.153 & 0.154 & 0.155 & 0.156 \\ t_3 & 0.161 & 0.162 & 0.163 & 0.164 & 0.165 & 0.166 \\ t_4 & 0.171 & 0.172 & 0.173 & 0.174 & 0.175 & 0.176 \end{array}.$$

This means that the probability that the threat t_1 uses usertrick ut_1 is 0.141, etc. Observe the sum of the probabilities in any row is not greater than 1. Let

$$P_{user\_usertrick} = \begin{array}{c|cccccc} & ut_1 & ut_2 & ut_3 & ut_4 & ut_5 & ut_6 \\ \hline u_1 & 0.031 & 0.032 & 0.033 & 0.034 & 0.035 & 0.036 \\ u_2 & 0.041 & 0.042 & 0.043 & 0.044 & 0.045 & 0.046 \\ u_3 & 0.051 & 0.052 & 0.053 & 0.054 & 0.055 & 0.056 \\ u_4 & 0.061 & 0.062 & 0.063 & 0.064 & 0.065 & 0.066 \\ u_5 & 0.071 & 0.072 & 0.073 & 0.074 & 0.075 & 0.076 \\ u_6 & 0.081 & 0.082 & 0.083 & 0.084 & 0.085 & 0.086 \\ u_7 & 0.091 & 0.092 & 0.093 & 0.094 & 0.095 & 0.096 \end{array}.$$

This means that the probability that the user u_1 uses usertrick ut_1 is 0.031, etc. Thus

$$P_{user} = \begin{array}{c|cccc} & t_1 & t_2 & t_3 & t_4 \\ \hline u_1 & 0.028516 & 0.030477 & 0.032434 & 0.034388 \\ u_2 & 0.036891 & 0.039418 & 0.041939 & 0.044455 \\ u_3 & 0.045206 & 0.048290 & 0.051366 & 0.054434 \\ u_4 & 0.053460 & 0.057094 & 0.060716 & 0.064326 \\ u_5 & 0.061655 & 0.065830 & 0.069989 & 0.074132 \\ u_6 & 0.069791 & 0.074498 & 0.079185 & 0.083852 \\ u_7 & 0.077868 & 0.083099 & 0.088305 & 0.093487 \end{array}.$$

Observe

$$\begin{aligned} p_{user}(u_1, t_1) &= 1 - (1 - p_{usertrick}(t_1, ut_1) \cdot p_{u-utrick}(u_1, ut_1)) \\ &\quad \cdot (1 - p_{usertrick}(t_1, ut_2) \cdot p_{u-utrick}(u_1, ut_2)) \\ &\quad \cdot \ldots \cdot (1 - p_{usertrick}(t_1, ut_i) \cdot p_{u-utrick}(u_1, ut_i)) \\ &= 1 - (1 - 0.141 \cdot 0.031) \cdot (1 - 0.142 \cdot 0.032) \cdot \ldots \cdot (1 - 0.146 \cdot 0.036) \\ &= 0.028516, \end{aligned}$$

etc. Therefore,

$$\begin{aligned} p_s = 1 &- (1 - p_{user}(t_1, u_1) \cdot p_{device}(t_1, d_1) \cdot p_{prev}(t_1)) \\ &\cdot (1 - p_{user}(t_1, u_2) \cdot p_{device}(t_1, d_1) \cdot p_{prev}(t_1)) \\ &\cdot \ldots \cdot (1 - p_{user}(t_4, u_7) \cdot p_{device}(t_4, d_3) \cdot p_{prev}(t_4)) \\ = 1 &- (1 - 0.028516 \cdot 0.02 \cdot 0.25) \cdot (1 - 0.036891 \cdot 0.02 \cdot 0.25) \\ &\cdot \ldots \cdot (1 - 0.093487 \cdot 0.32 \cdot 0.25) = 0.079774. \end{aligned}$$

This means that the probability of the infection of the investigated company with users $u_1, \ldots, u_7$, devices d_1, d_2, d_3, protections $p_1, \ldots, p_5$ and matrices as above is 0.079774. Thus we get that the probability of an infection by at least one malware is 0.079774.

4 Simulations

In this section results of simulation studies are presented. Businesses with different sizes (different number of devices and users) are modelled and the p_s probabilities are calculated when certain number of threats are present. The results are summarized in Table 1 and Table 2.

The *Micro* (*Small*, *Medium*, *Big*, resp.) business is a company (or department) with 10 (50, 100, 1000, resp.) devices and 10 (50, 100, 1000, resp.) users. In real life the probabilities p_{prev}, p_{prot}, $p_{usertrick}$ and $p_{user-usertrick}$ can be estimated by relative frequencies but in the simulations these were estimated by random uniform probabilities. In the Table 1 the probabilities p_{prev} (p_{prot}, $p_{usertrick}$, $p_{user-usertrick}$, resp.) are in the interval $[0.9, 1]$ ($[0, 0.1]$, $[0, 0.1]$, $[0, 0.1]$, resp.). The results in the Table 1 correspond to the case when the number of protections is 5 and the number of usertrick is 5.

The probability 0.25 in the cell of the third row of the second column in Table 1 means that the approximate probability of p_s is 0.25 if there are 10 devices, 10 users in the company, the number of threats is 10, the number of protections is 5, the number of usertricks is 5, the random elements of the vector P_{prev} lie on the interval $[0.9, 1]$, the random elements of the matrix P_{prot} lie on the interval $[0, 0.1]$, the random elements of the matrix $P_{usertrick}$ lie on the interval $[0, 0.1]$ and the random elements of the matrix $P_{user-usertrick}$ lie on the interval $[0, 0.1]$. Of course the matrices $Z_{device-elements}$ and $Z_{device-prot-install}$ are random matrices with elements 0 or 1.

Observe that if the number of the devices (or users) or the number of the threats is large, then the probability is close to 1.

Table 1: The values of p_s probabilities in case of different business sizes

	Micro	Small	Medium	Big
threats	devices=10 users=10	devices=50 users=50	devices=100 users=100	devices=1000 users=1000
10	0.25	0.999935 91999547	1	1
50	0.75	0.999999 99996973	1	1
100	0.85	1	1	1
1000	0.999999 99715744	1	1	1

The probabilities in Table 1 can be regarded as overestimates of the real p_s probabilities since the sum of the elements in the random vector P_{prev} is greater than 1.

In the Table 2 the probabilities p_{prev} (p_{prot}, $p_{usertrick}$, $p_{user-usertrick}$, resp.) are in the interval $[0, 0.1]$ ($[0, 0.1]$, $[0, 0.1]$, $[0, 0.1]$, resp.). The results in the Table 2 correspond to the case when the number of protections is 5 and the number of usertrick is 5.

Table 2: The values of p_s probabilities in case of different business sizes

	Micro	Small	Medium	Big
threats	devices=10 users=10	devices=50 users=50	devices=100 users=100	devices=1000 users=1000
10	0.02	0.25	75	1
50	0.07	0.85	0.9986016 7849174	1
100	0.15	0.996973 10258718	0.999999 99963790	1
1000	0.7	0.999999 99998364	1	1

The difference between the Table 1 and Table 2 is the input random data p_{prev}.

5 Conclusions

From the simulation studies it can be seen that the model presented can be used for defining an index number reflecting the state of vulnerability of a certain company against cyber attacks. However these simulations also show that this model has constraints of applicability because if the size of the company is big enough, then the probability p_s is very close to 1 and no distinction can be made between the vulnerability of different companies. To overcome these constrains of the applicability it can be used either only to a smaller part of a large network or to a randomly selected smaller sample of users and devices.

This index can be a good measuring tool of comparing the vulnerability of different parts of a company or comparing the state of vulnerability of a company at different time instances.

Comparing different user behaviours can give valuable pieces of information for the company managements about the needs of improving employees awareness against cyber attacks.

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Performance and economic analysis of Markovian Bernoulli feedback queueing system with vacations, waiting server and impatient customers

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Abstract. This paper concerns the analysis of a Markovian queueing system with Bernoulli feedback, single vacation, waiting server and impatient customers. We suppose that whenever the system is empty the sever waits for a random amount of time before he leaves for a vacation. Moreover, the customer's impatience timer depends on the states of the server. If the customer's service has not been completed before the impatience timer expires, the customer leaves the system, and via certain mechanism, impatient customer may be retained in the system. We obtain explicit expressions for the steady-state probabilities of the queueing model, using the probability generating function (PGF). Further, we obtain some important performance measures of the system and formulate a cost model. Finally, an extensive numerical study is illustrated.

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1 Introduction

Queueing models with vacations have a great impact in many real life situations, such models occur naturally in different fields such as computer and communication systems, flexible manufacturing systems, telephone services, production line systems, machine operating systems, post offices, etc. Over the past few decades, vacation queueing systems have paid attention of many researchers, excellent surveys on queueing systems with vacations can be found in Doshi [9] and Takagi [16] and in the monographs of Tian [17] and Ke [11]. In recent years, there has been gowning interest in the study of queueing systems with impatient customers (balking and reneging). For related literature, interested readers may refer to Shin and Choo [15], El-Paoumy and Nabwey [10], Kumar et al. [12], Kumar and Sharma [13], Bouchentouf et al. [7], Baek et al. [6], Bouchentouf and Messabihi [8] and references therein.

The studies of queueing models with impatient customers were ranked depending on the causes of the impatience behavior. In queueing literature, models where customers may be impatient because of server vacations have been extensively analyzed. Yue [20] presented the optimal performance analysis of an $M/M/1/N$ queueing system with balking, reneging and server vacation. Altman and Yechiali [2] gave the analysis of some queueing models such as $M/M/1$, $M/G/1$ and $M/M/c$ queues with server vacations and customer impatience, both single and multiple vacation cases were studied. Further, Altman and Yechiali [3] investigated the infinite server queue with vacations and impatient customers. They obtained the probability generating function of the number of customers in the model and derived the performance measures of the system. Queueing systems with vacations and synchronized reneging have been done by Adan et al. [1]. Wu and Ke [19] presented computational algorithm and parameter optimization for a multi-server system with unreliable servers and impatient customers. Later, the model given in Altman and Yechiali [2] were extended by Yue et al. [21] by considering a variant of the multiple vacation policy which includes both single vacation and multiple vacations. In Padmavathy et al. [14], authors studied the steady state behavior of the vacation queues with impatient customers and a waiting server. Further, the transient solution of a $M/M/1$ multiple vacation queueing model with impatient customers has been investigated by Ammar [4]. Then, a study of single server Markovian queueing system with vacations and impatience timers which depend of the state of the server was presented in Yue et al. [22]. Recently, in Ammar [5], author established the transient solution of an $M/M/1$ vacation queue with a waiting server and impatient customers.

The main objective of this article is to study an M/M/1 vacation queueing system with Bernoulli feedback, waiting server, reneging, and retention of reneged customers. It is supposed that whenever the busy period ended the server waits a random duration of time before beginning on a vacation. Moreover, we assume that the impatience timers of customers depend on the server's states. We obtain the steady-state solution of the queueing model, using the probability generating function (PGF). Further, we give explicit expressions of useful measures of effectiveness and formulate a cost model. Then, we present a sensitive numerical experiments to illuminate the interests of our theoretical results and to show the impact of the diverse parameters on the behavior of the system. Finally, an appropriate economic analysis is carried out numerically.

The model analyzed in this paper has a number of applications in practice. In most studies cited earlier, authors considered that the server leaves the system once the system is empty, but in many practical life situations the server waits a certain period of time before he leaves the system even if there is no customers, especially when we deal with a human behavior, examples can be found in post offices, banks, hospitals, etc.

Further, our study has another great scope, in most studies mentioned in the above literature, the basis of the research is the supposition that customers may be impatient because of server vacations. However, there are many situations where the customer can become impatient due to the long wait in the queue even if the server is present in the system, another example when the customer may leave the system during busy period is when he cannot see the server state, these situations can be found in telecommunication systems, call centers and production inventory systems.

The rest of the paper is organized in the following manner. In Section 2, we describe the model. In Section 3, we present the stationary analysis for the queueing model. In Section 4, we obtain different performance measures and formulate a cost model. Section 5 presents numerical results in the form of Tables and Figures. Finally, in Section 6 we conclude the paper.

2 System model

Consider a M/M/1 vacation queueing model with Bernoulli feedback, waiting server, reneging and retention of reneged customers. The model studied in this paper is based on following assumptions:

$*$ Customers arrive into the system according to a Poisson process with arrival rate λ, the service time is assumed to be exponentially distributed with parameter μ. The service discipline is FCFS and there is infinite space for customers to wait.

∗ When the busy period is finished the server waits a random duration of time before beginning on a vacation. This waiting duration is exponentially distributed with parameter η.

∗ If the server comes back from a vacation to an empty system he waits passively the first arrival, then he begins service. Otherwise, if there are customers waiting in the queue at the end of a vacation, the server starts immediately a busy period. That is single vacation policy. The period of vacation has an exponential distribution with parameter γ.

∗ Whenever a customer arrives at the system and finds the server on vacation (respectively. busy), he activates an impatience timer T_0 (respectively. T_1), which is exponentially distributed with parameter ξ_0 (respectively. ξ_1). If the customer's service has not been completed before the impatience timer expires, the customer may abandon the queue. We suppose that the customers timers are independent and identically distributed random variables and independent of the number of waiting customers.

∗ Each reneged customer may leave the system without getting service with probability α and may be retained in the system with probability $\alpha' = (1-\alpha)$.

∗ After completion of each service, the customer can either leave the system definitively with probability β or return to the system and join the end of the queue with probability β', where $\beta + \beta' = 1$.

3 Stationary analysis

In this section, we use the probability generating function (PGF) to obtain the steady-state solution of the queueing system.

Let $L(t)$ be the number of customers in the system at time t, and $J(t)$ denotes the state of the server at time t such that

$$J(t) = \begin{cases} 1, & \text{when the server is in a busy period;} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the process $\{(L(t); J(t)); t \geq 0\}$ is a continuous-time Markov process with state space

$$\Omega = \{(j, n) : j = 0, 1, \ \ n = 0, 1, ...\}.$$

Let $P_{j,n} = \lim_{t\to\infty} P\{J(t) = j, L(t) = n\}, \ \ j = 0, 1, n = 0, 1, ..., \ \ (j, n) \in \Omega$, denote the system state probabilities.

Then, the steady-state balance equations of our model are given as follows:

$$(\lambda+\gamma)P_{0,0}=\alpha\xi_0 P_{0,1}+\eta P_{1,0}, \tag{1}$$

$$(\lambda+\gamma+n\alpha\xi_0)P_{0,n}=\lambda P_{0,n-1}+(n+1)\alpha\xi_0 P_{0,n+1},\ \ n\geq 1, \tag{2}$$

$$(\lambda+\eta)P_{1,0}=\gamma P_{0,0}+(\beta\mu+\alpha\xi_1)P_{1,1}, \tag{3}$$

$$(\lambda+\beta\mu+n\alpha\xi_1)\,P_{1,n}=\lambda P_{1,n-1}+\gamma P_{0,n}+(\beta\mu+(n+1)\alpha\xi_1)P_{1,n+1},\quad n\geq 1, \tag{4}$$

Theorem 1 *If we have a single server Bernoulli feedback queueing system with single vacation, waiting server, server's states-dependent reneging and retention of reneged customers, then*

1. *The steady-state probability* $P_{0,\cdot}$ *is given by*

$$P_{0,\cdot}=\left(\frac{\gamma\alpha\xi_0+\delta_1 K_0(1)(1-\gamma)}{\gamma K_0(1)}\right)P_{0,0}. \tag{5}$$

2. *The steady-state probability* $P_{1,\cdot}$ *is given by*

$$\begin{aligned}P_{1,\cdot}=e^{\frac{\lambda}{\alpha\xi_1}}\Bigg(&\frac{\gamma}{\lambda+\eta}\left(\frac{\beta\mu}{\alpha\xi_1}K_1(1)+\frac{\eta}{\alpha\xi_1}K_2(1)\right)-\frac{\gamma}{\alpha\xi_1}K_3(1)\\&+\frac{\beta\mu+\alpha\xi_1}{\lambda+\eta}\left(\frac{\beta\mu}{\alpha\xi_1}K_1(1)+\frac{\eta}{\alpha\xi_1}K_2(1)\right)\left(\frac{\alpha\xi_0-\delta_1K_0(1)}{\delta_2K_0(1)}\right)\Bigg)P_{0,0},\end{aligned} \tag{6}$$

where

$$\begin{aligned}P_{0,0}=\Bigg\{&\frac{\delta_1\delta_2K_0(1)+\delta_2(\alpha\xi_0-\delta_1K_0(1))}{\gamma\delta_2K_0(1)}+e^{\frac{\lambda}{\alpha\xi_1}}\Bigg[\left(\frac{\beta\mu}{\alpha\xi_1}K_1(1)+\frac{\eta}{\alpha\xi_1}K_2(1)\right)\\&\left(\frac{\gamma}{\lambda+\eta}+\left(\frac{\beta\mu+\alpha\xi_1}{\lambda+\eta}\left(\frac{\alpha\xi_0-\delta_1K_0(1)}{\delta_2K_0(1)}\right)\right)\right)-\frac{\gamma}{\alpha\xi_1}K_3(1)\Bigg]\Bigg\}^{-1},\end{aligned} \tag{7}$$

$$K_0(z)=\int_0^z(1-s)^{\frac{\gamma}{\alpha\xi_0}-1}e^{-\frac{\lambda}{\alpha\xi_0}s}ds,$$

$$K_1(z) = \int_0^z s^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds, \quad K_2(z) = \int_0^z (1-s)^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds,$$

and

$$K_3(z) = \int_0^z \left(1 - \frac{K_0(s)}{K_0(1)}\right) s^{\frac{\beta\mu}{\alpha\xi_1}} (1-s)^{-\left(\frac{\gamma}{\alpha\xi_0}+1\right)} e^{\left(\frac{\lambda}{\alpha\xi_0}-\frac{\lambda}{\alpha\xi_1}\right)s} ds.$$

Proof. Let

$$G_j(z) = \sum_{n=0}^{\infty} P_{j,n} z^n, \quad j = 0, 1.$$

Then, multiplying Equation (2) by z^n, using Equations (1) and (3) and summing all possible values of n, we get

$$\alpha\xi_0(1-z)G_0'(z) - (\lambda(1-z)+\gamma)G_0(z) = -\{\delta_1 P_{00} + \delta_2 P_{11}\}, \tag{8}$$

with

$$\delta_1 = \left(\frac{\gamma\eta}{\lambda+\eta}\right) \text{ and } \delta_2 = \left(\frac{\eta(\beta\mu+\alpha\xi_1)}{\lambda+\eta}\right),$$

where $G_0'(z) = \frac{d}{dz}G_0(z)$.

In the same manner, from Equations (3) and (4) we obtain

$$\alpha\xi_1 z(1-z)G_1'(z) - (\lambda z - \beta\mu)(1-z)G_1(z) = -\gamma z G_0(z) + (\beta\mu(1-z)+\eta z)P_{1,0}. \tag{9}$$

Next, let $\Gamma = \delta_1 P_{00} + \delta_2 P_{11}$. Then, for $z \neq 1$, Equation (8) can be rewritten as follows

$$G_0'(z) - \left(\frac{\lambda}{\alpha\xi_0} + \frac{\gamma}{\alpha\xi_0(1-z)}\right)G_0(z) = -\frac{\Gamma}{\alpha\xi_0(1-z)}. \tag{10}$$

Multiplying both sides of Equation (10) by $e^{\frac{-\lambda}{\alpha\xi_0}}(1-z)^{\frac{\gamma}{\alpha\xi_0}}$, then integrating from 0 to z, we obtain

$$G_0(z) = e^{\frac{\lambda}{\alpha\xi_0}z}(1-z)^{-\frac{\gamma}{\alpha\xi_0}} \left\{G_0(0) - \frac{\Gamma}{\alpha\xi_0}K_0(z)\right\}, \tag{11}$$

with

$$K_0(z) = \int_0^z (1-s)^{\frac{\gamma}{\alpha\xi_0}-1} e^{-\frac{\lambda}{\alpha\xi_0}s} ds. \tag{12}$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{0,n} > 0$ and $z = 1$ is the root of the denominator of the right hand side of Equation (11), so $z = 1$ must be the root of the numerator of the right hand side of Equation (11).

Thus, we get

$$P_{0,0} = G_0(0) = \frac{\Gamma}{\alpha\xi_0} K_0(1). \tag{13}$$

This implies

$$P_{0,0} = \frac{\delta_2 K_0(1)}{\alpha\xi_0 - \delta_1 K_0(1)} P_{1,1}. \tag{14}$$

Consequently

$$P_{1,1} = \frac{\alpha\xi_0 - \delta_1 K_0(1)}{\delta_2 K_0(1)} P_{0,0}. \tag{15}$$

Next, substituting Equation (13) into (11), we obtain

$$G_0(z) = e^{\frac{\lambda}{\alpha\xi_0} z} (1-z)^{-\frac{\gamma}{\alpha\xi_0}} \left\{ 1 - \frac{K_0(z)}{K_0(1)} \right\} P_{0,0}. \tag{16}$$

For $z \neq 1$ and $z \neq 0$, Equation (9) can be rewritten as follows

$$\begin{aligned} G_1'(z) &- \left(\frac{\lambda}{\alpha\xi_1} - \frac{\beta\mu}{\alpha\xi_1 z} \right) G_1(z) \\ &= \left(\frac{\beta\mu}{\alpha\xi_1 z} + \frac{\eta}{\alpha\xi_1(1-z)} \right) P_{1,0} - \frac{\gamma}{\alpha\xi_1(1-z)} G_0(z). \end{aligned} \tag{17}$$

Then, we multiply both sides of Equation (17) by $e^{-\frac{\lambda}{\alpha\xi_1} z} z^{\frac{\beta\mu}{\alpha\xi_1}}$, we get

$$\begin{aligned} &\frac{d}{dz} \left(e^{-\frac{\lambda}{\alpha\xi_1} z} z^{\frac{\beta\mu}{\alpha\xi_1}} G_1(z) \right) \\ &= \left\{ \left(\frac{\beta\mu}{\alpha\xi_1 z} + \frac{\eta}{\alpha\xi_1(1-z)} \right) P_{1,0} - \frac{\gamma}{\alpha\xi_1(1-z)} G_0(z) \right\} e^{-\frac{\lambda}{\alpha\xi_1} z} z^{\frac{\beta\mu}{\alpha\xi_1}}. \end{aligned} \tag{18}$$

Integrating from 0 to z, we have

$$\begin{aligned} G_1(z) = e^{\frac{\lambda}{\alpha\xi_1} z} z^{-\frac{\beta\mu}{\alpha\xi_1}} &\left\{ \left(\frac{\beta\mu}{\alpha\xi_1} K_1(z) + \frac{\eta}{\alpha\xi_1} K_2(z) \right) P_{1,0} \right. \\ &\left. - \frac{\gamma}{\alpha\xi_1} \int_0^z (1-s)^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} G_0(s) ds \right\}, \end{aligned} \tag{19}$$

where

$$K_1(z) = \int_0^z s^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds, \quad K_2(z) = \int_0^z (1-s)^{-1} s^{\frac{\beta\mu}{\alpha\xi_1}} e^{-\frac{\lambda s}{\alpha\xi_1}} ds. \tag{20}$$

Using Equation (14) and substituting Equation (16) into (19), we get

$$G_1(z) = e^{\frac{\lambda z}{\alpha\xi_1}} z^{-\frac{\beta\mu}{\alpha\xi_1}} \left\{ \left(\frac{\beta\mu}{\alpha\xi_1} K_1(z) + \frac{\eta}{\alpha\xi_1} K_2(z) \right) P_{1,0} - \frac{\gamma}{\alpha\xi_1} K_3(z) P_{0,0} \right\}, \tag{21}$$

with

$$K_3(z) = \int_0^z \left(1 - \frac{K_0(s)}{K_0(1)} \right) s^{\frac{\beta\mu}{\alpha\xi_1}} (1-s)^{-\left(\frac{\gamma}{\alpha\xi_0}+1\right)} e^{\left(\frac{\lambda}{\alpha\xi_0} - \frac{\lambda}{\alpha\xi_1}\right)s} ds. \tag{22}$$

Next, putting $z = 1$ in Equation (8), we get the probability that the server is on vacation, $\left(P_{0,.} = G_0(1) = \sum_{n=0}^{\infty} P_{0,n} \right)$,

$$P_{0,.} = \left(\frac{\delta_1 P_{0,0} + \delta_2 P_{1,1}}{\gamma} \right). \tag{23}$$

And, putting $z = 1$ in Equation (21), we find the probability that the server is in busy period, $\left(P_{1,.} = G_1(1) = \sum_{n=0}^{\infty} P_{1,n} \right)$,

$$P_{1,.} = e^{\frac{\lambda}{\alpha\xi_1}} \left\{ \left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) P_{1,0} - \frac{\gamma}{\alpha\xi_1} K_3(1) P_{0,0} \right\}. \tag{24}$$

From Equation (3), it yields

$$P_{1,0} = \left(\frac{\gamma}{\lambda+\eta} \right) P_{0,0} + \left(\frac{\beta\mu + \alpha\xi_1}{\lambda+\eta} \right) P_{1,1}. \tag{25}$$

Substituting Equation (25) into (24), we have

$$\begin{aligned} P_{1,.} = e^{\frac{\lambda}{\alpha\xi_1}} \Bigg\{ & \left(\frac{\gamma}{\lambda+\eta} \left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) - \frac{\gamma}{\alpha\xi_1} K_3(1) \right) P_{0,0} \\ & + \left(\frac{\beta\mu}{\alpha\xi_1} K_1(1) + \frac{\eta}{\alpha\xi_1} K_2(1) \right) \left(\frac{\beta\mu + \alpha\xi_1}{\lambda+\eta} \right) P_{1,1} \Bigg\}. \end{aligned} \tag{26}$$

Next, substituting Equation (15) into (23), we get (5). Then, substituting Equation (15) into (26), we obtain (6).

Finally, using the normalizing condition

$$\sum_{n=0}^{\infty} P_{0,n} + \sum_{n=0}^{\infty} P_{1,n} = 1,$$

which is equivalent to

$$P_{0,.} + P_{1,.} = 1. \tag{27}$$

And substituting Equations (15), (23) and (26) into (27), we find (7) □

4 Performance measures and cost model

4.1 Performance measures

In this subpart useful performance measures are presented.

∗ The probability that the server is in a busy period (P_B).

$$\mathbb{P}(\text{Busy period}) = P_B = P_{1,.}.$$

∗ The probability that the server is on vacation (P_V).

$$\mathbb{P}(\text{Vacation period}) = P_V = 1 - \mathbb{P}(\text{Busy period}).$$

∗ The probability that the server is idle during busy period (P_I).

$$P_I = P_{1,0}.$$

∗ The average number of customers in the system when the server is taking vacation ($\mathbb{E}(L_0)$).

From Equation (8), using L'Hopital rule, we have

$$\mathbb{E}(L_0) = \lim_{z \to 1} G_0'(z) = \frac{-\lambda P_{0,.} + \gamma \mathbb{E}(L_0)}{-\alpha \xi_0}.$$

This implies

$$\mathbb{E}(L_0) = \left(\frac{\lambda}{\gamma + \alpha \xi_0} \right) P_{0,.}.$$

∗ The average number of customers in the system when the server is in busy period ($\mathbb{E}(L_1)$).

From Equation (9), using L'Hopital rule, we get

$$\mathbb{E}(L_1) = \lim_{z\to 1} G_1'(z) = \left(\frac{\lambda-\beta\mu}{\alpha\xi_1}\right) P_{1,.} + \frac{\gamma}{\alpha\xi_1}\mathbb{E}(L_0) + \frac{\beta\mu}{\alpha\xi_1(\lambda+\eta)}\left(\gamma + \frac{(\beta\mu+\alpha\xi_1)(\alpha\xi_0-\delta_1 K_0(1))}{\delta_2 K_0(1)}\right) P_{0,0}.$$

∗ The average number of customers in the system ($\mathbb{E}(L)$).

$$\mathbb{E}(L) = \mathbb{E}(L_0) + \mathbb{E}(L_1).$$

∗ The average number of customers in the queue ($\mathbb{E}(L_q)$).

$$\mathbb{E}(L_q) = \sum_{n=0}^{+\infty} nP_{0n} + \sum_{n=1}^{+\infty}(n-1)P_{1n}$$

$$= \mathbb{E}(L) - (P_{1,.} - P_{1,0}).$$

∗ The mean waiting time of a customer in the system (W_s).

$$W_s = \frac{\mathbb{E}(L_0)+\mathbb{E}(L_1)}{\lambda} = \frac{\mathbb{E}(L)}{\lambda}.$$

∗ The expected number of customers served per unit of time (E_{cs}).

$$E_{cs} = \beta\mu(P_{1,.} - P_{1,0}).$$

∗ The average rates of reneging and retention of impatient customers during vacation period.

$$R_{ren_0} = \alpha\xi_0\mathbb{E}(L_0), \;\; R_{ret_0} = (1-\alpha)\xi_0\mathbb{E}(L_0).$$

∗ The average rates of reneging and retention of impatient customers during busy period.

$$R_{ren_1} = \alpha\xi_1\mathbb{E}(L_1), \;\; R_{ret_1} = (1-\alpha)\xi_1\mathbb{E}(L_1).$$

Thus,

∗ The average rate of abandonment of a customer due to impatience (R_{ren}).

$$R_{ren} = R_{ren_0} + R_{ren_1}.$$

∗ The average rate of retention of impatient customers (R_{ret}).

$$R_{ret} = R_{ret_0} + R_{ret_1}.$$

4.2 Cost model

This subpart is devoted to develop a model for the costs incurred in the queueing system using the following symbols:

- C_1 : Cost per unit time when the server is working during busy period.
- C_2 : Cost per unit time when the server is idle during busy period.
- C_3 : Cost per unit time when the server is on vacation.
- C_4 : Cost per unit time when a customer joins the queue and waits for service.
- C_5 : Cost per service per unit time.
- C_6 : Cost per unit time when a customer reneges.
- C_7 : Cost per unit time when a customer is retained.
- C_8 : Cost per unit time when a customer returns to the system as a feedback customer.

Let
* R be the revenue earned by providing service to a customer.
* Γ be the total expected cost per unit time of the system.

$$\Gamma = C_1 P_B + C_2 P_I + C_3 P_V + C_4 \mathbb{E}(L_q) + C_6 R_{ren} + C_7 R_{ret} + \mu(C_5 + \beta' C_8).$$

* Δ be the total expected revenue per unit time of the system.

$$\Delta = R\mu(1 - P_V - P_{1,0}).$$

* Θ be the total expected profit per unit time of the system.

$$\Theta = \Delta - \Gamma.$$

5 Numerical analysis

5.1 Impact of system parameters on performance measures

Different performance measures of interest computed under different scenarios are given. These measures are obtained by using a MATLAB program coded by the authors. To illustrate the system numerically, the values for default parameters are considered using the following cases

- Table 1: $\lambda = 1.00 : 0.05 : 1.45$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 2: $\lambda = 1.50$, $\mu = 2.00 : 0.40 : 5.60$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 3: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50 : 0.05 : 0.95$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 4: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85 : 0.05 : 1.30$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 5: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10 : 0.05 : 0.55$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 6: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10 : 0.05 : 0.55$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 7: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.10 : 0.10 : 1.00$, and $\alpha = 0.50$.
- Table 8: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.10 : 0.10 : 1.00$.

5.2 General comments

∗ From Table 1 it is clearly seen that with the increases of the arrival rate λ, $P_{0,0}$ and P_V decrease, while P_B increases. Thus, the mean number of customers in the system during the busy period $\mathbb{E}(L_1)$ increases significatively, which leads to an increase in the number of customers served E_{cs}. Moreover, $\mathbb{E}(L_0)$ is not monotone with λ, while W_s increases as the arrival rate increases, this implies an increases in the average reneging and retention rates R_{ren} and R_{ret}.

∗ According to Table 2 we see that along the increases of the service rate μ, $P_{0,0}$, P_V, $\mathbb{E}(L_0)$ and E_{cs} increase, whereas P_B and $\mathbb{E}(L_1)$ both decrease, as it should be expected. Moreover, with the increase in μ, the mean waiting time of a customer in the system W_s deceases, this leads to a decrease in R_{ren} and R_{ret}. Obviously, the higher the service rate, the smaller the average rate of abandonment and the larger the number of customers served.

∗ From Table 3 we remark that when the reneging rate during vacation period ξ_0 increases, P_B, W_s, $\mathbb{E}(L_0)$ and $\mathbb{E}(L_1)$ decrease, while $P_{0,0}$, P_V, R_{ren} and R_{ret} increase. Consequently, E_{cs} decreases. As intuitively expected, the bigger the rate of reneging, the smaller the number of customers served.

Table 1: Performance measures vs. λ

λ	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
1.00	0.0272	0.7720	0.2280	0.6840	0.7883	1.4022	0.5060	0.5060	0.5440
1.05	0.0248	0.7795	0.2205	0.6931	0.8654	1.4169	0.5411	0.5411	0.5589
1.10	0.0227	0.7869	0.2131	0.7002	0.9439	1.4296	0.5762	0.5762	0.5738
1.15	0.0208	0.7943	0.2057	0.7052	1.0237	1.4407	0.6114	0.6114	0.5886
1.20	0.0191	0.8017	0.1983	0.7083	1.1049	1.4505	0.6466	0.6466	0.6034
1.25	0.0176	0.8090	0.1910	0.7094	1.1874	1.4591	0.6820	0.6820	0.6180
1.30	0.0161	0.8163	0.1837	0.7087	1.2713	1.4667	0.7175	0.7175	0.6325
1.35	0.0148	0.8234	0.1766	0.7063	1.3566	1.4735	0.7531	0.7531	0.6469
1.40	0.0137	0.8305	0.1695	0.7021	1.4434	1.4797	0.7890	0.7890	0.6610
1.45	0.0126	0.8375	0.1625	0.6964	1.5315	1.4853	0.8250	0.8250	0.6750

Table 2: Performance measures vs. μ

μ	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
2.00	0.0144	0.8143	0.1857	0.7959	1.2864	1.3882	0.7457	0.7457	0.7543
2.40	0.0160	0.7938	0.2062	0.8839	1.0741	1.3053	0.6775	0.6775	0.8225
2.80	0.0174	0.7757	0.2243	0.9614	0.8883	1.2331	0.6179	0.6179	0.8821
3.20	0.0186	0.7597	0.2403	1.0300	0.7240	1.1694	0.5652	0.5652	0.9348
3.60	0.0197	0.7455	0.2545	1.0909	0.5775	1.1123	0.5182	0.5182	0.9818
4.00	0.0207	0.7328	0.2672	1.1453	0.4459	1.0607	0.4758	0.4758	1.0242
4.40	0.0216	0.7214	0.2786	1.1941	0.3268	1.0140	0.4374	0.4374	1.0626
4.80	0.0224	0.7111	0.2889	1.2383	0.2187	0.9713	0.4025	0.4025	1.0975
5.20	0.0231	0.7017	0.2983	1.2786	0.1201	0.9325	0.3707	0.3707	1.1293
5.60	0.0238	0.6931	0.3069	1.3154	0.0300	0.8970	0.3416	0.3416	1.1584

∗ According to Table 4, we observe that along the increases of the reneging rate during busy period ξ_1, P_B, $\mathbb{E}(L_1)$ and W_s decrease, this leads to a decrease in E_{cs}. Further, as expected, the increasing of ξ_1 implies an increase in $P_{0,0}$, P_V, $\mathbb{E}(L_0)$, R_{ren} and R_{ret}.

∗ Table 5 illustrates that P_B increases with increasing values of the vacation rate γ, while $P_{0,0}$ is not monotonic with γ. Further, P_V, W_s, $\mathbb{E}(L_0)$ and $\mathbb{E}(L_1)$ decrease with the increase of γ, this implies an increase in E_{cs}. On the other hand, R_{ren} and R_{ret} decrease significantly as the vacation rate increases, which agrees with the intuitive expectation; the higher the rate of vacation, the bigger the probability of busy period and the greater the number of customers served.

Table 3: Performance measures vs. ξ_0

ξ_0	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.50	0.0130	0.8374	0.1626	0.6506	1.5209	1.4477	0.8253	0.8253	0.6747
0.55	0.0134	0.8372	0.1628	0.6106	1.5117	1.4148	0.8256	0.8256	0.6744
0.60	0.0139	0.8370	0.1630	0.5752	1.5036	1.3859	0.8260	0.8260	0.6740
0.65	0.0143	0.8369	0.1631	0.5438	1.4964	1.3601	0.8263	0.8263	0.6737
0.70	0.0148	0.8367	0.1633	0.5157	1.4899	1.3371	0.8266	0.8266	0.6734
0.75	0.0153	0.8365	0.1635	0.4904	1.4842	1.3164	0.8269	0.8269	0.6731
0.80	0.0158	0.8364	0.1636	0.4675	1.4790	1.2976	0.8272	0.8272	0.6728
0.85	0.0163	0.8362	0.1638	0.4466	1.4742	1.2806	0.8275	0.8275	0.6725
0.90	0.0167	0.8361	0.1639	0.4276	1.4699	1.2650	0.8278	0.8278	0.6722
0.95	0.0172	0.8360	0.1640	0.4101	1.4659	1.2507	0.8281	0.8281	0.6719

Table 4: Performance measures vs. ξ_1

ξ_1	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.85	0.0131	0.8310	0.1690	0.7242	1.4598	1.4560	0.8380	0.8380	0.6620
0.90	0.0136	0.8248	0.1752	0.7508	1.3951	1.4306	0.8504	0.8504	0.6496
0.95	0.0140	0.8189	0.1811	0.7763	1.3364	1.4084	0.8623	0.8623	0.6377
1.00	0.0145	0.8132	0.1868	0.8007	1.2828	1.3890	0.8737	0.8737	0.6263
1.05	0.0149	0.8077	0.1923	0.8241	1.2338	1.3719	0.8846	0.8846	0.6154
1.10	0.0153	0.8024	0.1976	0.8467	1.1886	1.3568	0.8951	0.8951	0.6049
1.15	0.0157	0.7974	0.2026	0.8683	1.1469	1.3435	0.9052	0.9052	0.5948
1.20	0.0161	0.7925	0.2075	0.8892	1.1083	1.3317	0.9150	0.9150	0.5850
1.25	0.0164	0.7878	0.2122	0.9093	1.0723	1.3211	0.9244	0.9244	0.5756
1.30	0.0168	0.7833	0.2167	0.9288	1.0389	1.3117	0.9334	0.9334	0.5666

$*$ According to Table 6, it is clearly observed that with the increase in the waiting server rate η, the probability of busy period P_B decreases which leads to a decrease in the mean number of customers served E_{cs}; this is because W_s, P_V and $\mathbb{E}(L_0)$ increase with η, which implies an increase in R_{ren}, R_{ret} and $P_{0,0}$. On the other hand the number of customers in the system during busy period $\mathbb{E}(L_1)$ increases; the reason is that the size of the system during vacation period becomes large with η.

Table 5: Performance measures vs. γ

γ	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0284	0.7420	0.2580	0.9674	2.1932	2.1071	1.1740	1.1740	0.6646
0.15	0.0290	0.7933	0.2067	0.6890	1.9385	1.7517	0.9961	0.9961	0.7106
0.20	0.0290	0.8276	0.1724	0.5172	1.7849	1.5348	0.8879	0.8879	0.7414
0.25	0.0286	0.8522	0.1478	0.4032	1.6856	1.3925	0.8172	0.8172	0.7635
0.30	0.0281	0.8706	0.1294	0.3234	1.6179	1.2942	0.7685	0.7685	0.7801
0.35	0.0276	0.8850	0.1150	0.2654	1.5699	1.2235	0.7336	0.7336	0.7930
0.40	0.0270	0.8965	0.1035	0.2218	1.5348	1.1711	0.7077	0.7077	0.8033
0.45	0.0264	0.9059	0.0941	0.1882	1.5085	1.1311	0.6882	0.6882	0.8118
0.50	0.0258	0.9138	0.0862	0.1617	1.4884	1.1001	0.6730	0.6730	0.8189
0.55	0.0252	0.9204	0.0796	0.1404	1.4728	1.0755	0.6610	0.6610	0.8249

Table 6: Performance measures vs. η

η	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0161	0.7919	0.2081	0.8919	1.5729	1.6432	0.8914	0.8914	0.6532
0.15	0.0187	0.7579	0.2421	1.0375	1.6647	1.8015	0.9669	0.9669	0.6369
0.20	0.0208	0.7316	0.2684	1.1502	1.7899	1.9601	1.0483	1.0483	0.6243
0.25	0.0224	0.7107	0.2893	1.2400	1.9383	2.1189	1.1338	1.1338	0.6142
0.30	0.0237	0.6936	0.3064	1.3132	2.1034	2.2778	1.2223	1.2223	0.6060
0.35	0.0248	0.6794	0.3206	1.3741	2.2811	2.4368	1.3130	1.3130	0.5992
0.40	0.0258	0.6674	0.3326	1.4254	2.4684	2.5959	1.4054	1.4054	0.5935
0.45	0.0266	0.6571	0.3429	1.4694	2.6632	2.7550	1.4992	1.4992	0.5886
0.50	0.0272	0.6483	0.3517	1.5074	2.8639	2.9142	1.5940	1.5940	0.5843
0.55	0.0278	0.6405	0.3595	1.5407	3.0696	3.0735	1.6897	1.6897	0.5806

$*$ The effect of non-feedback probability β is presented in Table 7, we see that P_B and W_s both decrease with increasing values of β. Further, as expected, $P_{0,0}$, P_V and $\mathbb{E}(L_0)$ increase as β increases, whereas $\mathbb{E}(L_1)$ decreases with increasing values of β; this is because the mean system size during vacation period increases with β. Further, it is well shown that R_{ren} and R_{ret} both decrease along the increasing of non-feedback probability β, which results in the increase of E_{cs}.

Table 7: Performance measures vs. β

β	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0020	0.9741	0.0259	0.1109	4.3719	2.9885	1.1207	1.1207	0.3793
0.20	0.0038	0.9503	0.0497	0.2128	3.6255	2.5589	0.9596	0.9596	0.5404
0.30	0.0060	0.9221	0.0779	0.3336	2.9646	2.1988	0.8246	0.8246	0.6754
0.40	0.0083	0.8932	0.1068	0.4578	2.3968	1.9031	0.7137	0.7137	0.7863
0.50	0.0104	0.8658	0.1342	0.5752	1.9133	1.6590	0.6221	0.6221	0.8779
0.60	0.0123	0.8410	0.1590	0.6815	1.4995	1.4541	0.5453	0.5453	0.9547
0.70	0.0140	0.8189	0.1811	0.7760	1.1416	1.2784	0.4794	0.4794	1.0206
0.80	0.0155	0.7995	0.2005	0.8594	0.8282	1.1251	0.4219	0.4219	1.0781
0.90	0.0169	0.7822	0.2178	0.9333	0.5510	0.9895	0.3711	0.3711	1.1289
1.00	0.0181	0.7669	0.2331	0.9991	0.3038	0.8686	0.3257	0.3257	1.1743

Table 8: Performance measures vs. α

α	$P_{0,0}$	P_B	P_V	$\mathbb{E}(L_0)$	$\mathbb{E}(L_1)$	W_s	R_{ren}	R_{ret}	E_{cs}
0.10	0.0019	0.9710	0.0290	0.2900	6.3941	4.4560	0.5580	5.0220	0.9420
0.20	0.0049	0.9273	0.0727	0.5454	3.4759	2.6808	0.6454	2.5817	0.8546
0.30	0.0076	0.8916	0.1084	0.6502	2.4282	2.0523	0.7167	1.6724	0.7833
0.40	0.0101	0.8623	0.1377	0.6886	1.8756	1.7095	0.7754	1.1631	0.7246
0.50	0.0126	0.8375	0.1625	0.6964	1.5315	1.4853	0.8250	0.8250	0.6750
0.60	0.0151	0.8162	0.1838	0.6892	1.2957	1.3233	0.8676	0.5784	0.6324
0.70	0.0178	0.7976	0.2024	0.6746	1.1238	1.1989	0.9048	0.3878	0.5952
0.80	0.0205	0.7813	0.2187	0.6562	0.9926	1.0992	0.9375	0.2344	0.5625
0.90	0.0231	0.7667	0.2333	0.6362	0.8892	1.0170	0.9666	0.1074	0.5334
1.00	0.0258	0.7537	0.2463	0.6157	0.8056	0.9475	0.9926	0.0000	0.5074

$*$ The impact of non-retention probability α is shown in Table 8. As intuitively expected, along the increase of α, P_B and $\mathbb{E}(L_1)$ decrease, while P_V increases as α increases. Further, $\mathbb{E}(L_0)$ is not monotonic with the probability of non-retention. Moreover, W_s and R_{ret} both decrease with increasing of α whereas R_{ren} increases with the probability α, this leads to a decrease of E_{cs}. This is quite reasonable; the smaller the probability of retaining impatient customers, the larger the average rate of reneged customers and the smaller the number of customers served.

5.3 Economic analysis

In this subpart, a sensitive economic analysis of the model is performed numerically and the results are discussed appropriately.

We present the variation in total expected cost, total expected revenue and total expected profit with the change in different parameters of the system. For the whole numerical study we fix the costs at $C_1 = 5$, $C_2 = 3$, $C_3 = 5$, $C_4 = 3$, $C_5 = 4$, $C_6 = 3$, $C_7 = 2$, $C_8 = 2$, and $R = 50$.

Impact of arrival rate λ

We examine the impact of λ by keeping all other variables fixed, to this end we take $\lambda = 1.00 : 0.05 : 1.45$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$. Results of the analysis are summarized in Table 9 and Figure 1.

Table 9: Γ, Δ and Θ for different values of λ

λ	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45
Γ	22.63	23.04	23.45	23.86	24.26	24.67	25.07	25.48	25.88	26.29
Δ	54.39	55.89	57.38	58.86	60.33	61.80	63.25	64.68	66.10	67.50
Θ	31.76	32.84	33.92	35.00	36.06	37.12	38.17	39.20	40.21	41.20

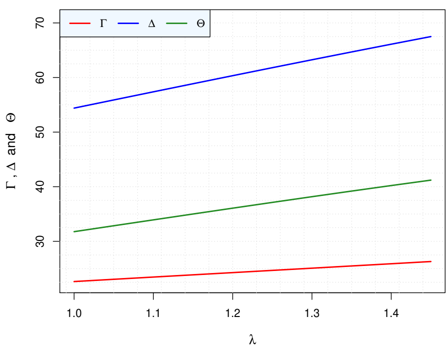

Figure 1: Γ, Δ and Θ vs. λ

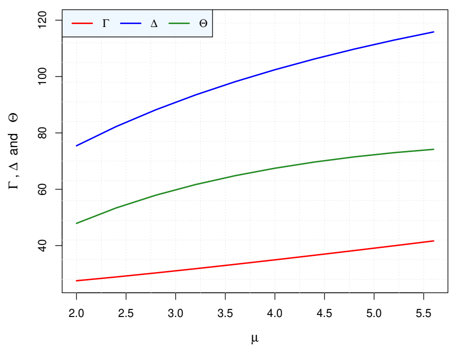

Figure 2: Γ, Δ and Θ vs. μ

Following the obtained results we observe that Γ, Δ, and Θ all increase with the increasing of the arrival rate λ. This result agrees with our intuition; the number of the customers in the system increases with the increasing of

λ, therefore a large number of customers is served. Consequently, the total expected profit increases.

Impact of service rate μ

To check the impact of service rate μ, the values of the parameters are chosen as follows: $\lambda = 1.50$, $\mu = 2.00 : 0.40 : 5.60$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.

Table 10: Γ, Δ and Θ for different values of μ

μ	2.00	2.40	2.80	3.20	3.60	4.00	4.40	4.80	5.20	5.60
Γ	27.53	28.88	30.31	31.80	33.35	34.95	36.58	38.25	39.94	41.66
Δ	75.43	82.25	88.21	93.48	98.18	102.4	106.2	109.7	112.9	115.8
Θ	47.89	53.37	57.90	61.67	64.82	67.46	69.67	71.49	72.98	74.17

According to Table 10 and Figure 2 we see that Γ and Δ increase with increasing values of μ, this generates an increase in Θ. This result makes perfect sense, the higher the service rate, the greater the total expected profit of the system.

Impact of reneging rates ξ_0 and ξ_1

Let's study the effect of reneging rates in vacation and busy periods ξ_0 and ξ_1, to this end we consider the following cases

- Table 11: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 1.00$, $\xi_0 = 2.00 : 0.50 : 6.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.
- Table 12: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85 : 0.05 : 1.30$, $\beta = 0.50$, and $\alpha = 0.50$.

Table 11: Γ, Δ and Θ for different values of ξ_0

ξ_0	2.00	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00	6.50
Γ	23.97	23.99	24.01	24.02	24.03	24.04	24.05	24.05	24.05	24.06
Δ	78.59	78.47	78.37	78.28	78.21	78.15	78.10	78.06	78.03	78.01
Θ	54.62	54.48	54.36	54.26	54.18	54.11	54.06	54.01	53.97	53.95

From Tables 11 and 12 and Figures 3 and 4 we observe that

Table 12: Γ, Δ and Θ for different values of ξ_1

ξ_1	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20	1.25	1.30
Γ	26.24	26.21	26.19	26.17	26.17	26.17	26.18	26.19	26.20	26.22
Δ	66.20	64.96	63.77	62.63	61.54	60.48	59.47	58.50	57.56	56.65
Θ	39.95	38.74	37.58	36.45	35.36	34.31	33.29	32.31	31.36	30.43

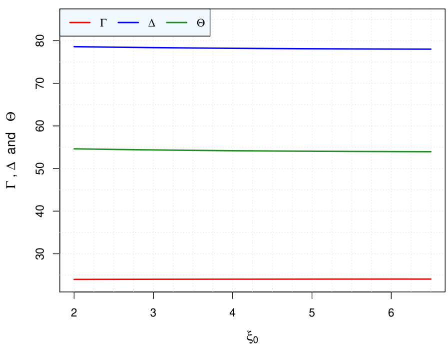

Figure 3: Γ, Δ and Θ vs. ξ_0

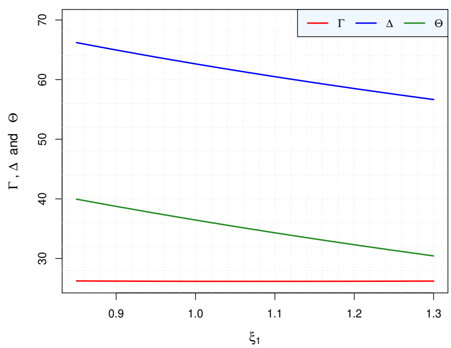

Figure 4: Γ, Δ and Θ vs. ξ_1

$*$ As expected, along the increasing of ξ_0, Γ increases while Θ and Δ decrease with ξ_0, this is because the average rate of reneged customers increases with ξ_0. Therefore the number of customers served decreases, which results in the decrease of the total expected profit.

$*$ With the increase of ξ_1, Δ decreases, while Γ is not monotonic with the parameter ξ_1. Further, Θ decreases with the increasing values of the impatience rate, this is because the number of customers in the system decreases with ξ_1, this implies a decrease in P_B which results in the decrease of E_{cs}.

Impact of vacation rate γ

To examine the impact of the vacation rate γ on the total expected profit, we take $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10 : 0.05 : 0.55$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$.

From Table 13 and Figure 5 it is easily seen that the increases of the vacation rate γ implies a decrease in Γ and a considerable increase in Δ and Θ. This is quite explicable; as γ increases the vacation duration decreases and the server switches to busy period during which customers are served. This leads to a significant increase in the total expected profit.

Table 13: Γ, Δ and Θ for different values of γ

γ	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
Γ	30.58	28.11	26.61	25.61	24.93	24.45	24.09	23.81	23.60	23.43
Δ	66.46	71.06	74.13	76.34	78.00	79.29	80.33	81.18	81.89	82.49
Θ	35.87	42.94	47.53	50.72	53.06	54.85	56.24	57.37	58.29	59.06

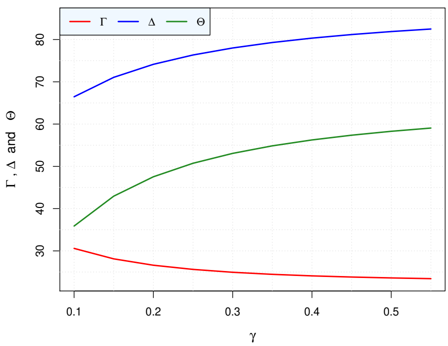

Figure 5: Γ, Δ and Θ vs. γ

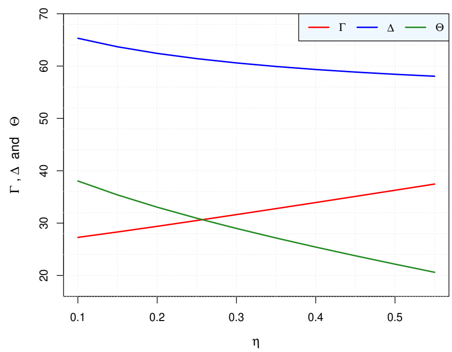

Figure 6: Γ, Δ and Θ vs. η

Impact of waiting rate of a server η

Here, we examine the sensitivity of the total expected profit versus the waiting server rate η. For this case, we put $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10 : 0.05 : 0.55$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.50$. The numerical results are presented in Table 14 and Figure 6.

From the obtained results we remark that with the increase in η, total expected cost Γ increases, while Δ and Θ monotonically decease with the parameter η. This is due to the fact that the probability of busy period during which service is provided decreases with the parameter η. Therefore, the total expected profit decreases considerably.

Table 14: Γ, Δ and Θ for different values of η

η	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
Γ	27.26	28.30	29.38	30.49	31.62	32.77	33.93	35.09	36.27	37.45
Δ	65.31	63.68	62.42	61.42	60.60	59.92	59.34	58.85	58.43	58.06
Θ	38.04	35.38	33.04	30.92	28.98	27.15	25.42	23.75	22.15	20.60

Impact of non-retention probability α

To study the impact of α on the total expect profit, we choose the parameters values as follows: $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.50$, and $\alpha = 0.10 : 0.10 : 1.00$.

Table 15: Γ, Δ and Θ for different values of α

α	0.10	0.20	0.3	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Γ	46.85	34.38	30.05	27.75	26.29	25.26	24.49	23.88	23.39	22.98
Δ	94.20	85.45	78.32	72.45	67.50	63.24	59.52	56.25	53.34	50.74
Θ	47.34	51.07	48.27	44.69	41.20	37.97	35.03	32.36	29.95	27.76

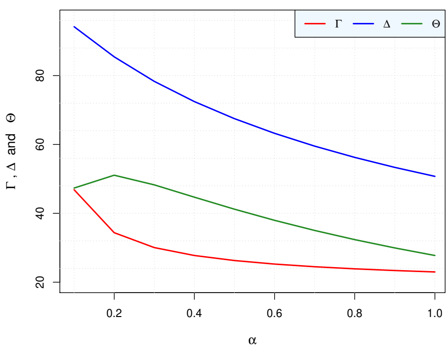

Figure 7: Γ, Δ and Θ vs. α

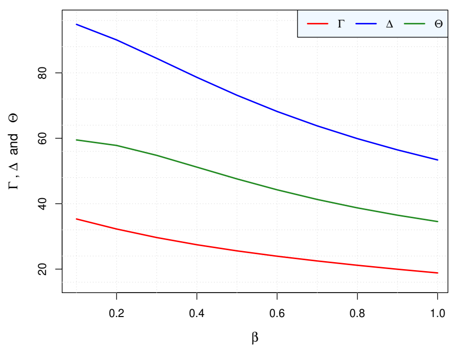

Figure 8: Γ, Δ and Θ vs. β

According to Table 15 and Figure 7 we observe that the increases of non-retention probability α implies a decrease in Γ, Δ and Θ. A slight increase is observed in Θ when the parameter α is below a certain value, ($\alpha = 0.2$). Therefore, we can see that the probability of retaining reneged customers α' has a noticeable effect on the total expected profit of the system. This is because the number of customers served increases with the parameter α'. Thus, it is quite clear that the probability of retention has a positive impact in the economy.

Impact of non-feedback probability β

Here, we put $\lambda = 1.50$, $\mu = 2.00$, $\eta = 0.10$, $\gamma = 0.10$, $\xi_0 = 0.50$, $\xi_1 = 0.85$, $\beta = 0.10 : 0.10 : 1.00$, and $\alpha = 0.50$. The numerical results obtained for this situation are given in Table 16 and Figure 8.

Table 16: Γ, Δ and Θ for different values of β

β	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
Γ	35.32	32.26	29.65	27.45	25.57	23.94	22.49	21.17	19.96	18.83
Δ	94.81	90.07	84.42	78.63	73.15	68.19	63.78	59.89	56.44	53.37
Θ	59.49	57.80	54.77	51.18	47.57	44.24	41.29	38.72	36.48	34.54

From the obtained results, it is clearly shown that Γ, Δ and Θ monotonically decrease as non-feedback probability β increases. The reason is that the number of the customers in the system decreases with the increasing of β, which leads to a decrease in the total expected profit.

6 Conclusion

In this paper we studied an $M/M/1$ Bernoulli feedback queueing system with single exponential vacation, waiting server, reneging and retention of reneged customers, wherein the impatience timers of customers depend on the states of the server. The explicit expressions of the steady-state probabilities are obtained, using probability generating functions (PGFs).

Useful measures of effectiveness of the queueing system are presented and a cost model is developed. Finally, an extensive numerical study is presented. Our system can be considered as a generalized version of the existing queueing models given by Yue et al.[22] and Ammar [5] associated with several practical situations.

The model considered in this paper can be extended to multiserver queueing system with delayed state-dependent service times, breakdowns and repairs.

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On real valued ω-continuous functions

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Abstract. The aim of this paper is to introduce and study upper and lower ω-continuous functions. Some characterizations and several properties concerning upper (resp. lower) ω-continuous functions are obtained.

1 Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologist worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [4]. Several characterizations and properties of ω-closed sets were provided in [1, 2, 3, 4, 5]. Various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them

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have been extended to the setting of multifunction. The purpose of this paper is to define upper and lower ω-continuous functions. Also, some characterizations and several properties concerning upper (lower) ω-continuous functions are obtained.

2 Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is said to be ω-closed [4] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The intersection (resp. union) of all ω-closed (resp. ω-open) set containing (resp. contained in) $A \subset X$ is called the ω-closure (resp. ω-interior) of A and is denoted by $\omega\,\mathrm{Cl}(A)$ (resp. $\omega\,\mathrm{Int}(A)$). The family of all ω-open, ω-closed sets of (X, τ) is, respectively denoted by $\omega O(X)$, $\omega C(X)$. We set $\omega O(X, x) = \{A : A \in \omega O(X) \text{ and } x \in A\}$ and $\omega C(X, x) = \{A : A \in \omega C(X) \text{ and } x \in A\}$. The ω-θ-closure [3] of A, denoted by $\omega\,\mathrm{Cl}_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap \omega\,\mathrm{Cl}(U) \neq \varnothing$ for every $U \in \omega O(X, x)$. A subset A is called ω-θ-closed [3] if and only if $A = \omega\,\mathrm{Cl}_\theta(A)$. The complement of ω-θ-closed set is called ω-θ-open. A subset A is called ω-regular if and only if it is ω-θ-open and ω-θ-closed. The family of all ω-regular sets of (X, τ) is denoted by $\omega R(X)$. We set $\omega R(x) = \{A : A \in \omega R(X)$ and $x \in A\}$. A topological space X is said to be ω-closed if every cover of X by ω-open sets has a finite subcover whose ω-closures cover X. Finally we recall that a function $f : (X, \tau) \to (Y, \sigma)$ is ω-continuous at the point $x \in X$ if for each open set V of Y containing $f(x)$ there exists an ω-open set U in X containing x such that $f(U) \subset V$. If f has the property at each point $x \in X$, then it is said to be ω-continuous [5].

3 On upper and lower ω-continuous functions

Definition 1 *A function* $f : X \to \mathbb{R}$ *is said to be:*

(i) *lower* (*resp. upper*) ω*-continuous at* x_1 *if to each* $\alpha > 0$*, there exists an*

ω-open set U_{x_1} such that $f(x) > f(x_1) - \alpha$ (resp. $f(x) < f(x_1) + \alpha$) for all $x \in U_{x_1}$;

(ii) *lower (resp. upper) ω-continuous if it is respectively so at each point of X.*

Example 1 *Consider $X = \mathbb{R}$ with topology $\tau = \{\emptyset, \mathbb{R}\}$, then $\tau_\omega = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\} \cup \{(\mathbb{R} \setminus \mathbb{Q}) \cup A :$ where A is a subset of $\mathbb{Q}\}$. Define $f : X \to \mathbb{R}$ as follows: $f = \chi_{\mathbb{R}\setminus\mathbb{Q}}$. f is lower ω-continuous but is not upper ω-continuous. In the same form if we define $g : X \to \mathbb{R}$ as follows: $g = \chi_{\mathbb{Q}}$, g is upper ω-continuous but is not lower ω-continuous.*

Theorem 1 *A function $f : X \to \mathbb{R}$ is lower ω-continuous if and only if for each $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) \leq \alpha\}$ is ω-closed.*

Proof. Since the family of sets $T = \{\mathbb{R}, \emptyset\} \cup \{(\alpha, \infty) : \alpha \in \mathbb{R}\}$ forms a topology on $\mathbb{R}$, f is lower ω-continuous if and only if f is ω-continuous from X into the topological space $(\mathbb{R}, T)$. But $(-\infty, \alpha]$ is a closed set in $(\mathbb{R}, T)$ and hence $f^{-1}((-\infty, \alpha])$ is ω-closed in X. But $f^{-1}((-\infty, \alpha]) = \{x \in X : f(x) \leq \alpha\}$. Therefore, $\{x \in X : f(x) \leq \alpha\}$ is ω-closed. □

Corollary 1 *A subset A of X is ω-open if and only if the characteristic function χ_A is lower ω-continuous.*

Similarly for upper ω-continuity, we have the following characterization.

Theorem 2 *A function $f : X \to \mathbb{R}$ is upper ω-continuous if and only if for each $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) \geq \alpha\}$ is ω-closed.*

Corollary 2 *A subset A of X is ω-closed if and only if the characteristic function χ_A is upper ω-continuous.*

Theorem 3 *Let $\{f_\alpha : \alpha \in \Lambda\}$ be a family of lower ω-continuous functions from X into $\mathbb{R}$, then the function $M(x) = \sup_{\alpha\in\Lambda} f_\alpha(x)$ (if it exists) is lower ω-continuous.*

Proof. Let $\lambda \in \mathbb{R}$ and $M(x) < \lambda$. Then $f_\alpha(x) < \lambda$, for all $\alpha \in \Lambda$. Now $\{x \in X : M(x) \leq \lambda\} = \bigcap_{\alpha\in\Lambda} \{x \in X : f_\alpha(x) \leq \lambda\}$. But each f_α being lower ω-continuous, by Theorem 1, each set $\{x \in X : f_\alpha(x) \leq \lambda\}$ is ω-closed in X. Since any intersection of ω-closed sets is ω-closed, M is lower ω-continuous. □

Theorem 4 *Let Λ be a finite index set and $\{f_\alpha : \alpha \in \Lambda\}$ be a family of lower ω-continuous functions from X into $\mathbb{R}$, then the function $m(x) = \min_{\alpha\in\Lambda}\{f_\alpha(x)\}$ (if it exists) is lower ω-continuous.*

Proof. It is enough to prove the case, when $m(x) = \min\{f_1(x), f_2(x)\}$. Let $\lambda \in \mathbb{R}$ and $x_0 \in X$, since f_1, f_2 are lower ω-continuous from X into $\mathbb{R}$, there exists ω-open sets $U_1(x_0)$ (resp. $U_2(x_0)$) such that $f_1(x) > f_1(x_0) + \lambda$ for all $x \in U_1(x_0)$ (resp. $f_2(x) > f_2(x_0) + \lambda$ for all $x \in U_2(x_0)$). It follows that for all $x \in U_1(x_0) \cap U_2(x_0)$, we obtain that $m(x) > m(x_0) + \lambda$ for all $x \in U_1(x_0) \cap U_2(x_0)$. In consequence, the result follows. □

Remark 1 *If Λ be an infinite index set and $\{f_\alpha : \alpha \in \Lambda\}$ be a family of lower ω-continuous functions from X into $\mathbb{R}$. Then the function $m(x) = \inf_{\alpha\in\Lambda}$ $\{f_\alpha(x)\}$ (if it exists) may not be lower ω-continuous.*

Example 2 *For each natural number n, define $f_n = \chi_{(-\frac{1}{n},\frac{1}{n})}$ then $m(x) = \chi_{\{0\}}$, is not lower ω-continuous.*

Theorem 5 *Let $\{f_\alpha : \alpha \in \Lambda\}$ be a family of upper ω-continuous function from X into $\mathbb{R}$, then the function $g(x) = \inf_{\alpha\in\Lambda}\{f_\alpha(x)\}$ (if it exists) is upper ω-continuous.*

Proof. Similar to the proof of Theorem 3. □

Theorem 6 *Let Λ be a finite index set and $\{f_\alpha : \alpha \in \Lambda\}$ be a family of upper ω-continuous functions from X into $\mathbb{R}$, then the function $M(x) = \max_{\alpha\in\Lambda}\{f_\alpha(x)\}$ (if it exists) is upper ω-continuous.*

Proof. Similar to the proof of Theorem 4. □

Remark 2 *If Λ be an infinite index set and $\{f_\alpha : \alpha \in \Lambda\}$ be a family of upper ω-continuous functions from X into $\mathbb{R}$. Then the function $m(x) = \sup_{\alpha\in\Lambda}$ $\{f_\alpha(x)\}$ (if it exists) may not be upper ω-continuous.*

Example 3 *Similar to Example 2.*

Theorem 7 *Let X be an ω-closed space and let $f : X \to \mathbb{R}$ be a lower ω-continuous function. Then f assumes the value $m = \inf_{x\in X}\{f(x)\}$.*

Proof. Let $\alpha \in \mathbb{R}$ be such that $\alpha > m$. Then f being the lower ω-continuous, the set $T_\alpha = \{x \in X : f(x) \leq \alpha\}$ is a nonempty (by the property of infimum) ω-closed set in X. The family $\{T_\alpha : \alpha \in \mathbb{R}$ and $\alpha > m\}$ is a collection of nonempty ω-closed sets with finite intersection property in the ω-closed space X; hence it has nonempty intersection. Let $x^* \in \bigcap_{\alpha > m} T_\alpha$. Then $f(x^*) = m$. □

Theorem 8 *If* X *is* ω*-closed, then any upper* ω*-continuous function* $f : X \to \mathbb{R}$ *attains the value* $M = \sup_{x\in X}\{f(x)\}$.

Proof. Similar to Theorem 7. □

Remark 3 *If a real valued function* $f : X \to \mathbb{R}$ *from an* ω*-closed space is lower* ω*-continuous as well as upper* ω*-continuous, then it is bounded and attains its bounds.*

Definition 2 *Let* $f : X \to Y$ *be a function, where* X *is a topological space and* Y *is a poset. Then*

(i) f *is said to be lower (resp. upper)* ω*-continuous if* $f^{-1}(\{y \in Y : y \leq y_0\})$ *(resp.* $f^{-1}(\{y \in Y : y \geq y_0\})$*) is* ω*-closed in* X *for each* $y_0 \in Y$.

(ii) *a partial order relation* $\leq$ *on a topological space* X *is said to be lower (resp. upper) compatible if the set* $\{x \in X : x \leq x_0\}$ *(resp.* $\{x \in X : x \geq x_0\}$*) is* ω*-closed for each* $x_0 \in X$.

Theorem 9 *A topological space* X *is* ω*-closed if and only if* X *has a maximal element with respect to each upper compatible partial order on* X.

Proof. Suppose that X is not ω-closed. Then there exists a net $\{x_\lambda : \lambda \in \Lambda\}$ which has no ω-accumulation point, where Λ is a well-ordered index set. We define the set $A_\alpha = X \backslash \omega\,\mathrm{Cl}_\theta(\{x_\beta : \beta > \alpha\})$. We claim that for each $x \in X$, $x \in A_\alpha$ for some α. In fact, x is contained in some ω-regular set R such that $R \cap \{x_\beta : \beta \geq \lambda\} = \emptyset$ for some β. Consider $\mathcal{R} = \{R \in \omega R(X) : R \cap \{x_\beta : \beta \geq \lambda\} = \emptyset$ for some $\beta\}$. Let λ_R be the smallest index such that $R \cap \{x_\beta : \beta \geq \lambda_R\} = \emptyset$; let λ_x be the smallest element of $M = \{\lambda_R : R \in \mathcal{R}\}$. We define the relation $\leq$ on X as follows: $x \leq y$ if and only if $A_{\lambda x} \subset A_{\lambda y}$, that is, if and only if $X \backslash \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_x\}) \subset X \backslash \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_y\})$, that is, if and only if $\omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_y\}) \subset \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_x\})$, that is, if and only if $\lambda_x \leq \lambda_y$. Clearly $\leq$ is a partial order relation on X. We claim that λ_x is the first element of M for which $x \in A_{\lambda x}$. In fact if $\alpha < \lambda_x$ and $x \in A_\alpha$, then $x \notin \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \alpha\})$. Then there exists $R \in \omega R(X)$ such that $R \cap \{x_\beta : \beta \geq \alpha\} = \emptyset$, a contradiction. It is obvious that for the corresponding λ_x there exists an $R_{\lambda x} \in \mathcal{R}$ such that $R_{\lambda x} \cap \{x_\beta : \beta \geq \lambda_x\} = \emptyset$ and for any $\alpha < \lambda_x$, $R_{\lambda x} \cap \{x_\beta : \beta \geq \lambda_x\} \neq \emptyset$. Also, $R_{\lambda x} \cap \{x_\beta : \beta \geq \lambda_x\} = \emptyset$. Then $R_{\lambda x} \cap \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_x\}) = \emptyset$, that is $R_{\lambda_x} \subset X \backslash \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_x\}) = A_{\lambda x}$ and this happens for every $x \in X$. To show $\leq$ is upper compatible, it is sufficient to show that $\{x \in X : x \geq x_0\}$ is ω-closed for every $x_0 \in X$. If possible, for some $x_0 \in X$, $\{x \in X : x \geq x_0\}$ is not ω-closed, that is, there exists $y \in \omega\,\mathrm{Cl}(\{x \in X : x \geq x_0\})$ such that $y < x_0$, R_{λ_y} is an ω-regular set containing y such that $x \in R_{\lambda_y}$ with $x > y$, that is, $\lambda_x > \lambda_y$, that is $x \in X \backslash \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_y\}) = A_{\lambda y}$. But λ_x is the first index such that $x \in A_{\lambda_x}$ and thus we arrive at a contradiction. Hence, $\leq$ is upper compatible. Further, $(X, \leq)$ has no maximal element; in fact, if there be any, say x_0, then for some fixed ω, $\omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda\}) \subset \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \alpha\})$ for every $\alpha \in M$, that is, $x_\lambda \in \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \alpha\})$, for all $\alpha \in M$, a contradiction. Conversely, let S be a linearly ordered subset of the topological upper compatible poset X. We denote by S_x the set $\{y \in X : y \geq x\}$. As the partial order on X is upper compatible, each S_x is ω-closed. Since S is a linearly ordered subset of X, $\{S_x : x \in X\}$ has finite intersection property. Then $\bigcap_{x \in S} S_x \neq \emptyset$. Let $x^* \in \bigcap_{x \in S} S_x$. Then $x^* \geq x$, for all $x \in S$. Therefore, by Zorn's lemma X has a maximal element. □

Theorem 10 *A topological space X is ω-closed if and only if X has a maximal element with respect to each lower compatible partial order on X.*

Proof. Similar to Theorem 9. □

Theorem 11 *A topological space X is ω-closed if and only if each upper ω-continuous function from X into a poset assumes a maximal value.*

Proof. Suppose that X is not ω-closed, then there exists a net $\{x_\lambda : \lambda \in M\}$ with no ω-accumulation point, where M is a well-ordered set. We assumes that the topology on M is the order topology. Now, for each $\beta \in M$, $A_\beta = \omega\,\mathrm{Cl}(\{x_\lambda : \lambda \geq \beta\})$. We define a function $f : X \to M$ as follows: $f(x) = \beta_x$, where β_x is the first element of the β's for which $x \notin A_\beta$. This is well defined because from the fact that M is well-ordered, obviously, $f(x)$ has no maximal element. We define the relation $\leq$ on X as follows: $x \leq y$ if and only if $f(x) \leq f(y)$. Clearly, $\leq$ is a partial order relation on X. Now, for each $x \in X$, $S_x = f^{-1}(\{z \in Y : z \geq f(x)\}) = \{y \in X : y \geq x\}$. As f is ω-continuous, each S_x is ω-closed and hence $\leq$ is an upper compatible partial order relation on X. Then X being an

ω-closed space, by Theorem 9 it has a maximal element x^*. Therefore, $f(x^*)$ is the maximal element of $f(X)$. □

Theorem 12 *A topological space* X *is* ω*-closed if and only if each lower* ω*-continuous function from* X *into a poset assumes a minimum value.*

Proof. Similar to Theorem 11. □

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Rejection sampling of bipartite graphs with given degree sequence

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Abstract. Let $A = (a_1, a_2, ..., a_n)$ be a degree sequence of a simple bipartite graph. We present an algorithm that takes A as input, and outputs a simple bipartite realization of A, without stalling. The running time of the algorithm is $\ominus(n_1 n_2)$, where n_i is the number of vertices in the part i of the bipartite graph. Then we couple the generation algorithm with a rejection sampling scheme to generate a simple realization of A uniformly at random. The best algorithm we know is the implicit one due to Bayati, Kim and Saberi (2010) that has a running time of $\mathcal{O}(ma_{max})$, where $m = \frac{1}{2}\sum_{i=1}^{n} a_i$ and a_{max} is the maximum of the degrees, but does not sample uniformly. Similarly, the algorithm presented by Chen et al. (2005) does not sample uniformly, but nearly uniformly. The realization of A output by our algorithm may be a start point for the edge-swapping Markov Chains pioneered by Brualdi (1980) and Kannan et al.(1999).

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1 Introduction

A graph $G(V(G), E(G))$ is said to be *bipartite* if its vertex set $V(G)$ can be partitioned into two different sets $V_1(G)$ and $V_2(G)$ with $V(G) = V_1(G) \cup V_2(G)$ such that $uv \in E$ if $u \in V_1$ and $v \in V_2$. The graphs considered can have possible parallel edges and loops unless otherwise stated. The *Degree Sequence Problem* is to find some or all graphs with a given degree sequence [30, 34]. More detailed analysis of the Degree Sequence Problem and its relevance can be found in [29]. It is much researched upon for its relevance in network modelling in Ecology, Social Sciences, chemical compounds and biochemical networks in the cell. Especially, ecological occurrence matrices, such as the Darwin finches tables, are $(0, 1)$ matrices whose rows are indexed by species of animals and columns are islands, and the (i, j) entry is 1 if animal i lives in island j, and is 0 otherwise. Moreover the row sums and columns sums are fixed by field observation of these islands. These occurrence matrices are thus bipartite graphs G with a fixed degree sequence in which $V_1(G)$ is the set of animals and $V_2(G)$ is the set of islands. Researchers in Ecology [8, 9, 15, 31] are highly interested in sampling easily and uniformly ecological occurrence tables, so that by using Monte Carlo methods, they can approximate test statistics to prove or disprove some null hypothesis about competitions amongst animals. Several algorithms are known to sample random realizations of degree sequences, and each one of them has its strengths and limitations. Most of these use Monte Carlo Markov chain methods based on edge-swapping [6, 9, 10, 11, 12, 13, 18, 22, 21, 24]. Since to start a Markov chain still requires to have a realisation of the degree sequence A, many algorithms are proposed that generate such a realisation [1, 3, 5, 2, 36]. Most of these algorithms are based on random matching methods. In particular, algorithms proposed in [1, 3, 8] are based on inserting edges sequentially according to some probability scheme. The basic ideas of the algorithm presented in the present paper can be seen as implementing a "dual sequential method", as it inserts sequentially vertices instead of edges.

In the theory of the Tutte polynomial, there are two operations, deletion and contraction, that are dual to each other, see [7] for more details on this topic. Let G be a graph having n vertices and m edges. In G, the operation of deleting an edge $e = (v_i, v_j)$ means removing the edge e and the graph thus obtained, denoted by $G\backslash e$, is a graph on n vertices and $m-1$ edges where both the degrees of vertices v_i and v_j decrease by 1. The operation of contracting the graph G by $e = (v_i, v_j)$ consists of deleting the edge e and identifying the vertices v_i and v_j. The graph thus obtained, denoted by G/e, is a graph on $n - 1$ vertices and $m - 1$ edges where the new vertex obtained by identifying

ν_i and ν_j has degree $a_i + a_j - 2$. Deletion is said to be the dual of contraction as the incidence matrix of $G \backslash e$ is orthogonal to the incidence matrix of G^*/e, where G^* is the dual of G if G is planar.

If A is a degree sequence having n entries, it can easily be shown that random matching methods used in [1, 2, 3, 5, 36] are equivalent to starting from a known realization G of A, delete all the edges one by one, and keeping track of the degrees of vertices after each deletion, until one reaches the empty graph having n vertices. Then, reconstructing a random realization of A consists of taking the reverse of the deletion. That is, starting from the empty graph on n vertices, re-insert edges one by one by choosing which edge to insert according to the degrees of the vertices and some probability scheme depending on the stage of the algorithm, and subject to not getting double edges if one would like to get simple graphs or not linking two vertices on the same part if one wants to get bipartite graphs. The algorithm presented in this paper is based on the dual operation of contraction that has been slightly modified to suit our purpose. It is equivalent to starting from a known realization G of A, contract all the edges one by one, and keeping track of the vertices after each contraction, until one reaches the graph with one vertex and $\frac{1}{2}\sum_1^n a_i$ loops. Then, reconstructing a random realization of A consists of reversing the process of contraction. That is, starting from a graph with one vertex and $\frac{1}{2}\sum_1^n a_i$ loops, the algorithm re-inserts vertices one by one by choosing the vertices to be joined according to the degrees of the vertices and some probability that depends on the stage of the algorithm. But, to construct a bipartite realization, we force the algorithm to insert first all the vertices in $V_1(G)$ and then all the vertices in $V_2(G)$.

While algorithms that are based on reversing the deletion operation [1, 3] are easy to implement, our algorithm seems more complex as one has to satisfy not only the degree conditions on the vertices, but also some added graphical structures imposed by the contraction. But this is more of a bonus than an inconvenience, as, apart from the fact that the running time is even better, the extra structure allows an easier analysis of the algorithm. Moreover, the internal structure imposed by the contraction operation allows the algorithm to avoid most of the shortcomings of the previous algorithms. In fact, not only the algorithm never restarts, but the algorithm also allows to sample all bipartite realizations with equal probabilities, making their approximate counting much easier than by the importance sampling used in [1, 3]. Better still, this technique can be extended to construct k-partite realizations of a k-partite degree sequence A, for $k \geq 3$, where a k-partite degree sequence is defined in a natural way extending the definition of a bipartite degree sequence.

The present paper uses the following notations and terminology. Two edges e and f in $E(G)$ are said to be *multiple* edges if they have the same end vertices (in Matroid Theory, multiple edges are said to be *parallel*). A *simple* bipartite graph is without multiple edges and contains no loops. The *degree* a_i of a vertex v_i is the number of edges incident to v_i with a loop contributing twice to the degree of v_i. The degree sequence of a graph G is formed by listing the degrees of vertices of G. If $A = (a_1, a_2, ..., a_n)$ is a sequence of integers and G is a bipartite graph that has A as its degree sequence, we say that G is a *realization* of A, and such a sequence of integers is called a *bipartite degree sequence*. Thus entries of A can be partitioned as A_1 and A_2, where A_i denotes the degree sequence of the part $V_i(G)$. We write V_i and $|A_i|$ to denote the set of vertices with degrees in A_i and the sum of entries in A_i respectively . In the sequel, we denote a bipartite degree sequence A as $(A_1 : A_2)$ and the pair $(A_1 : A_2)$ is called a *bipartition* of A.

Remark 1 *If* $A = (A_1 : A_2)$ *is a bipartite degree sequence having* n *entries, and* A_1 *and* A_2 *have respectively* n_1 *and* n_2 *entries, then the following are true.*

1. $n_1 + n_2 = n$
2. $|A_1| = |A_2|$.
3. The maximal entry of A_1 *is less or equal to* n_2 *and vice versa.*

Conversely, any partition of entries of A into two sets B_1 and B_2 satisfying Observation 1 is a bipartition of A.

In the sequel, we make use of Rejection Sampling to sample all realizations of the degree sequence with equal probability. Indeed, let $\mathcal{S} = S_1, ..., S_r$ be a set of structures, where S_i is obtained with probability $\pi(S_i)$ such that $\sum_i \pi(S_i) = 1$. That is, the set of $\pi(S_i)$ is a probability distribution function. Let $\min(\pi)$ be the minimal probability amongst all $\pi(S_i)$. The Rejection Sampling scheme consists of generating S_i, then accept it with probability $\frac{\min(\pi)}{\pi(S_i)}$ or reject it with probability $1 - \frac{\min(\pi)}{\pi(S_i)}$. It is easy to see that every structure would then be sample with the same probability $\min(\pi)$.

This paper is organized as follows. We first define what is called a recursion chain of a degree sequence, then we present routines for constructing all bipartite realizations. The next section presents criteria and routines to generate simple bipartite realizations only. Then these basic routines are coupled with a rejection sampling routine to get a uniform distribution on the set of all simple bipartite realizations.

2 Construction all bipartite realizations of given degrees

2.1 Recursion chain of degree sequences

Let G be a graph with n vertices and m edges. Throughout we assume that the vertices and edges of G are labelled $v_1, v_2, \cdots, v_n$. Let $A = (a_1, \cdots, a_n)$ be the degree sequence of G, where a_i is the degree of the vertex v_i. Define an arithmetic operation on A, called *contraction*, as follows. For an ordered pair (a_i, a_j) of entries a_i and a_j of A with $i \neq j$, the operation of *contraction* by (a_i, a_j) means changing a_i to $a_i + a_j$ and deleting the entry a_j from A. We write $A/(i, j)$ to denote the new sequence thus obtained. We call the sequence $A/(i, j)$ the (i, j)-*minor* or simply a *minor* of A. The following example illustrates this operation for a bipartite degree sequence.

Example 1 *Let* $A = (4, 3, 3 : 3, 3, 2, 2)$, *where* $a_1 = 4$, $a_2 = 3$, $a_3 = 3$ *and* $a_4 = 3, a_5 = 3, a_6 = 2, a_7 = 2$. *We have* $A/(1, 2) = (7, 3, 3, 3, 2, 2)$ *and* $A/(4, 2) = (4, 3, 6, 3, 2, 2)$.

Let A be the sequence of integers. A is said to be *graphic* if there is a graph G, not necessarily bipartite, such that G has A as its degree sequence. Moreover, it is trivial to observe that a sequence of integers is graphic if and only if the sum of its entries is even.

Theorem 1 *A sequence* A *is graphic if and only if all its minors are graphic.*

Proof. Obviously, if A is graphic, then $A/(a_i, a_j)$ is graphic, as the sum of its entries is even, by definition of contraction. Now suppose that $A/(a_i, a_j)$ is graphic and G'' is a realization of $A/(a_i, a_j)$. To prove that A is also graphic, we present an algorithm, much used in the sequel, that constructs a realization of A, denoted by G, from G''.

Algorithm AddVertex()
Step 1. To G'' add an isolated vertex labelled v_j (as in Figure 1).
Step 2 If the degree of v_j is a_j, stop, output G. Else
Step 3. Amongst the a'_i edges incident to v_i, counting loops twice, choose one edge
$e = (v_i, v_k)$ with probability $\pi(e)$ and connect e to v_j so that e becomes (v_j, v_k). Go to Step 2.

Now, in G the degree of v_j is a_j by Step 2 of algorithm AddVertex(). Moreover, by the definition of contraction the degree of v_i is equal to $a_i + a_j$ in

G''. Since AddVertex() takes a_j edges away from v_i, the degree of v_i is a_i in G. Moreover all other vertices are left unchanged by AddVertex(). Thus G is a realization of A. $\square$

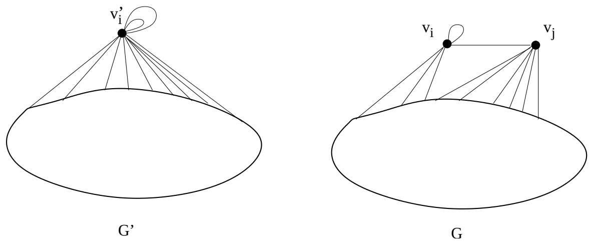

Figure 1: Construction of a graph G from its contract-minor G''

To help the intuition, observe that if G'' is a realization of $A/(a_i, a_j)$ and G is a realization of A constructed by AddVertex(), then G'' is obtained from G by contraction of the edge (v_i, v_j). Now, mimicking the process of recursive contraction of matroid as used in the theory of the Tutte polynomial, we define a process of recursive contraction for a degree sequence. A *recursion chain* of a degree sequence A is a unary tree rooted at A where nodes are integer sequences and every node, except for the root, is a minor of the preceding one. The recursive procedure of contraction is carried on from the root A until a node with a single entry is reached. See Figure 2 for an illustration.

As for the Tutte polynomial, the amazing fact, which is then used to construct all the realizations of A is that the order of contraction is immaterial. Despite this basic fact, we still impose a particular order to ease many proofs in the sequel.

Notes on notations. For the sake of convenience, we denote by $A^{(i)}$ the node of a recursion chain of a degree sequence A, where i is the number of entries in the node. Thus we denote the root A by $A^{(n)}$, the next node by $A^{(n-1)}$, and so on until the last node $A^{(1)}$. Similarly, we denote by $G^{(i)}$ the realization of $A^{(i)}$. The n entries of A are labelled from 1 to n. To keep tract of the vertices, we preserve the labelling of entries of A into its minors so that when a contraction by the pair (a_i, a_j) is performed, the new vertex is labelled a_i, the label a_j is deleted, and all other entries keep the labelling they have before the contraction. In this paper, we consider the recursion chain, called the *accumulating recursion chain*, constructed as follows. Let $A = (A_1 : A_2)$. We order $A = (a_1, a_2, ..., a_n)$ as $(b_1, b_2, ..., b_{n_1} : c_1, c_2, ..., c_{n_2})$, where $A_1 = (b_1, b_2, ..., b_{n_1})$ and $A_2 = (c_1, c_2, ..., c_{n_2})$, such that $b_1 \geq b_2 \geq ... \geq b_{n_1}$ and $c_1 \geq c_2 \geq ... \geq c_{n_2}$ and $n_1 + n_2 = n$. Below is the pseudocode for the recursive construction of the accumulating recursion chain of a bipartite degree sequence.

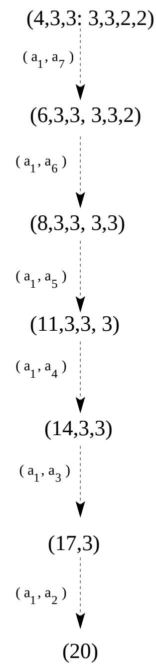

Figure 2: The recursion chain of the bipartition $(4, 3, 3 : 3, 3, 2, 2)$. Nodes of the chain are labelled from $A^{(7)} = A$ to $A^{(1)}$. Notice that we only perform contractions (v_1, v_{last}).

Algorithm ConstructBipartiteRecursionChain()

Given a bipartite degree sequence $A = (a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_{n_1} : c_1, c_2, ..., c_{n_2})$ with $b_1 \geq b_2 \geq ... \geq b_{n_1}$ and $c_1 \geq c_2 \geq ... \geq c_{n_2}$. Let $i = n$.

Step 1 If $i = 1$, stop, return $\{A^{(1)}, A^{(2)}, ..., A^{(n)}\}$. Else

Step 2 Let $A^{(i-1)} = A^{(i)}/(1, i)$. That is, get the $(i-1)^{th}$ recursive minor of A by contracting the $(i)^{th}$ recursive minor by its first entry and the last entry.

Step 3 Decrement i by 1 and go back to Step 1.

The accumulation recursion chain of A is denoted by $W = (A^{(1)}, A^{(2)}, ..., A^{(n)})$.

The following algorithm generates all the bipartite realizations of A. The graph constructed is not necessarily simple. Loosely speaking, this algorithm consists of reversing the recursive process of contraction as implemented by ConstructBipartiteRecursionChain(). This algorithm starts from $G^{(1)}$ the sole realization of $A^{(1)}$, and by calling AddVertex() recursively it constructs $G^{(2)}$, then $G^{(3)}$, and so on until $G^{(n)}$ that is a realization of $A^{(n)} = A$. The only conditions imposed on the choice of edges is that up to the n_1^{th} iteration, only edges (v_1, v_j), with $j \le n_1$, are constructed. That is, we insert vertices of V_1. After the n_1^{th} iteration, only edges (v_k, v_j), with $k > n_1$ and $j \le n_1$, are constructed. That is, we insert vertices of V_2. We call the graphs $G^{(1)}$, $G^{(2)}$,..., $G^{(n)}$ the *partial realizations* of A.

Algorithm ConstructBipartiteRealization()

Given $W = (A^{(1)}, A^{(2)}, ..., A^{(n)})$, the bipartite accumulating recursion chain of A, do the following.

Step 1. Let $i = 1$ and build the realization of the node $A^{(1)}$, denoted by $G^{(1)}$, which is the graph consisting of one vertex and m loops, where $m = \frac{1}{2}\sum_{i=1}^{n} a_i$.

Step 2. Let $G = G^{(i)}$. If G has n vertices, stop, return G. Else,

Step 3. Using $G^{(i)}$ and $A^{(i+1)}$ as input, Call Algorithm AddVertex() to construct $G^{(i+1)}$ as a realization of $A^{(i+1)}$. If $i \le n_1$, AddVertex() only concedes loops. If $i > n_1$ Addvertex() concedes only edges (v_1, v_j) with $1 \le j \le n_1$. Increment i by 1, go back to Step 2.

See Figure 3 for an illustration of Algorithm ConstructBipartiteRealization().

The following definitions are needed in the sequel. In the process of contraction implemented by the accumulating recursion chain, we observe that the degrees are accumulating on a_1. If we think of recursive contractions of a graph, this is equivalent to saying that the edges are accumulating on v_1 as v_1 seems to swallow the other vertices one by one. Hence when reversing the contraction operation in ConstructBipartiteRealization(), vertex v_1 plays the role of the 'mother that spawns' all the other vertices one by one and concedes some edges to them according to their degrees. Thus AddVertex() can attach an edge e to a new vertex v_s only if e is incident to v_1. This observation prompts the following formal definitions. Let $A = (A_1 : A_2)$ be a bipartite degree sequence, where A_1 and A_2 have respectively n_1 and n_2 entries such that $n_1 + n_2 = n$. Up to the n_1^{th} iteration of ConstructBipartiteRealization(), an edge is *available* if it is a loop incident to v_1. An edge e is *lost* otherwise. From the $(n_1 + 1)^{th}$ iteration of ConstructBipartiteRealization() onwards, an edge is *available* if it is incident to v_1 and a vertex v_j with $1 \le j \le n_1$. An edge e is *lost* otherwise. In the obvious way, we say that a vertex is *available* if it is incident to some available edge. Let V_{av}, E_{av} and E_{v_j} respectively denote the set of all available vertices, the set of all available edges and the set of available edges that are incident to the vertex v_j, for $j \leq n_1$. An edge $e = (v_1, v_j)$ is *conceded* if AddVertex() disconnects it from v_1 so that e becomes $e = (v_j, v_k)$ for some vertex $v_k \neq v_1$. We then say that v_1 (or sometimes E_{v_j} or just v_j) concedes the edge e. A vertex v_s having degree a_s is *fully inserted* if a_s edges are conceded to it. A graph G is said to be *(re)constructed* if it is an output of ConstructBipartiteRealization().

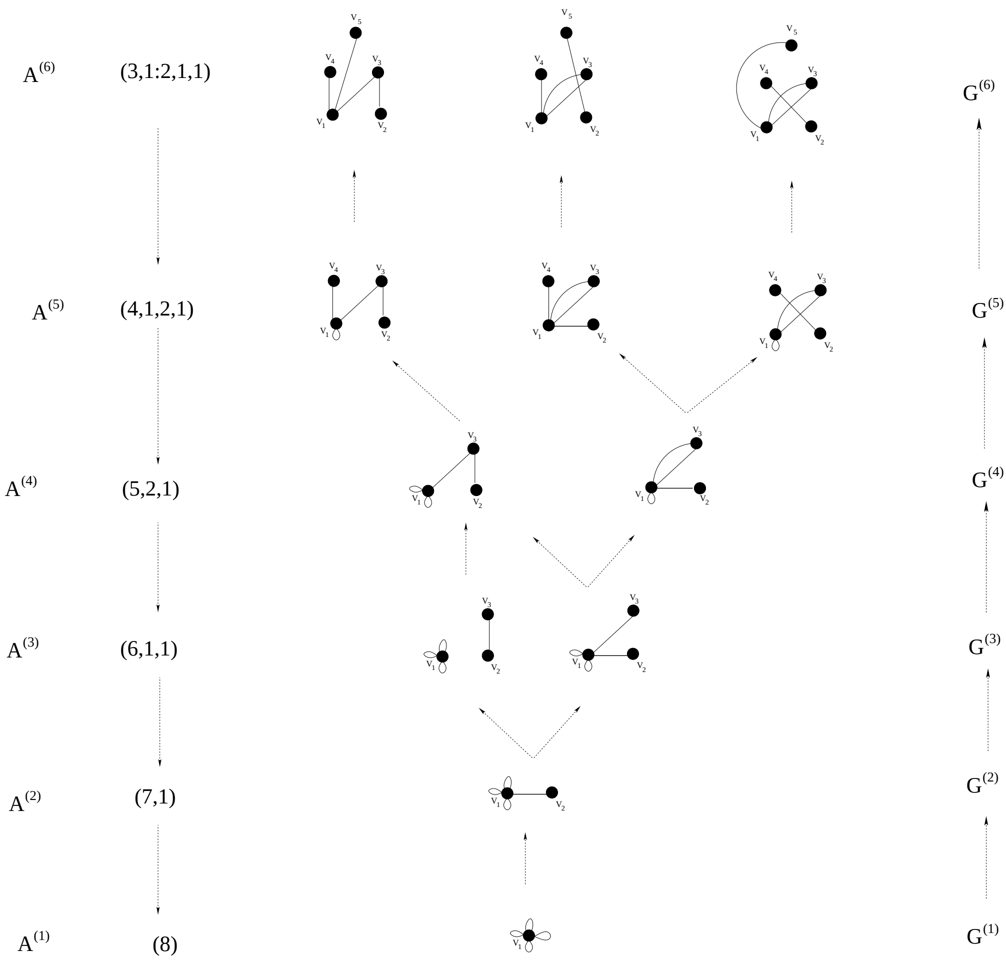

Figure 3: Random reconstruction tree of $(3, 1 : 2, 1, 1)$. Graphs drawn on the same height as the degree sequence $A^{(i)}$ corresponds to all the graphs having $A^{(i)}$ as their degree sequence. Notice that only realizations of $A^{(6)}$ are bipartite.

The next observation is an obvious consequence of the definition of the algorithm ConstructBipartiteRealization(). We single it out for the sake of clarity as it is used in the sequel.

Remark 2 *From the* $(n_1+1)^{th}$ *iteration of ConstructBipartiteRealization(), the number of available edges is equal to the number of edges left to be inserted until ConstructBipartiteRealization() terminates.*

It is because the number of available edges at the end of $(n_1)^{th}$ iteration is equal to half the sum of degrees $a_i \in A_1$, and by the definition of the bipartite degree sequence, this number is equal to half the sum of degrees $a_j \in A_2$.

Theorem 2 *Let* $A = (a_i, a_2, \cdots, a_n) = (A_1 : A_2)$ *be a bipartite degree sequence having* n *entries where* A_1 *and* A_2 *respectively have* n_1 *and* n_2 *entries, such that* $n_1 + n_2 = n$. *Let* W *be the bipartite recursion chain of* A. *Then Algorithm ConstructBipartiteRealization() constructs in time linear on* $m = \frac{a_i+a_2+\cdots+a_n}{2}$ *a bipartite graph* G *having* n *vertices and* m *edges such that* G *is a realization of* A. *Moreover, every bipartite realization of* A *can be constructed in this way.*

Proof. By Algorithm AddVertex(), the graph $G^{(n)}$ output by Algorithm ConstructBipartiteRealization() is assured to be a realization of A. We need only to prove that $G^{(n)}$ is bipartite. Now, since up to the n_1^{th} iteration of ConstructBipartiteRealization(), the routine AddVertex() always chooses loops incident to v_1, vertices inserted from the second iteration up to the n_1^{th} iteration of ConstructBipartiteRealization() (i.e., vertices in V_1) can never be adjacent to each other. Moreover, from the $(n_1+1)^{th}$ to the n^{th} iteration, AddVertex() never chooses an edge (v_1, v_j) with $j > n_1$. Thus all the vertices inserted from the $(n_1+1)^{th}$ iteration onwards (i.e., vertices in V_2) are never adjacent to each other. Thus, we only have to show that (in G^n), v_1 is not adjacent to any vertex inserted before the n_1^{th} iteration of AddVertex(). So, suppose that G^n contains an edge $e = (v_1, v_j)$ with $j \le n_1$. But, at the beginning of the $(n_1+1)^{th}$ iteration, the number of all edges incident to v_1 is equal to the sum of the degrees of the vertices left to insert until the end of the Algorithm. Thus one vertex v_j with $j > n_1$ is not fully inserted. This is a contradiction.

It remains to prove that any bipartite realization G of A can be constructed in this way. So, let G be a realization of A and let $e = (v_i, v_j)$, where $v_i \in V_1$ and $v_j \in V_2$, be any edge of G such that vertex v_i has degree a_i and vertex v_j has degree a_j. Also suppose that vertex v_i and v_j respectively were inserted at the i^{th} and j^{th} iteration of ConstructBipartiteRealization(), with $i \leq n_1$ and $j > n_1$. We need to show that at the j^{th} iteration, there is a positive probability to have an edge e that is incident to v_i and e is available. Assume to the contrary, that is, at the j^{th} iteration all the edges incident to v_i must be lost. Now all the edges incident to v_i are lost before that j^{th} iteration only if at some stage of the running of Algorithm ConstructBipartiteRealization(), there are only the edges that are available and these are exhausted before reaching the j^{th} iteration. Thus, at the j^{th} iteration there are no more available edges. That is, there is no edge incident to v_1. But this means that $a_{n_1+1} + a_{n_1+2} + ... + a_{j-1} \geq m$, contradicting Observation 2.

As for the running time, Algorithm ConstructBipartiteRealization() calls Algorithm AddVertex() once for every new vertex v_k to be inserted. If v_k has degree a_k, Algorithm AddVertex() has to go through a_k iterations to insert the a_k edges of v_k. Hence the total number of iterations to terminate ConstructBipartiteRealization() is $a_1 + a_2 + ... + a_n = 2m$. □

3 Construction of simple bipartite graphs

Till now, ConstructBipartiteRealization() generates any bipartite realization of the bipartite degree sequence A. But, it is easy to modify AddVertex() so that the output of ConstructBipartiteRealization() is always a simple graph. One obvious condition can be stated as follows.

(a) If the Algorithm is inserting the j^{th} edge of vertex v_s (with $j > 1$ and $v_s \in V_2$) and v_k ($v_k \in V_1$) is already adjacent to v_s, then no more available edge incident to v_k should be chosen. This would prevent ConstructBipartite-Realization() from outputting graphs with multiple edges (v_s, v_k). Thus this condition is necessary, but it is not sufficient. Indeed, it is easy to see that the following must also apply.

(b) While inserting vertex v_s and avoiding choosing edges incident to v_k so as not to construct multiple edges (v_s, v_k), ConstructBipartiteRealization() may fall into a stage where there are more edges incident to v_k than there are vertices left to insert, and G, the graph output by ConstructBipartiteRealization() would then have a multiple edge (v_1, v_k).

(c) Let A_1 and A_2 be (separatly) ordered in non decreasing order, where a_1 is the largest entry of A_1 and a_{n_1+1} is the largest entry of A_2. Let M_k be the set of the last k entries of A_1 and let $\max(k) = a_{n_1-k+1}$. Let there be an entry a_s in A_2 satisfying the following.

(f1). $s - n_1 \geq \max(k)$, (i.e., the number of entries of A_2 preceding a_s is greater or equal to the maximal entry in M_k)

(f2). $a_s > n_1 - k$, (i.e., inserting v_s would require more neighbours than there are vertices in $V_1 \backslash M_k$) and,

f(3).

$$\sum_{j=n_1+1}^{s-1} a_j \geq \sum_{i=k}^{n_1} a_i + \sum_{i=1}^{k-1} \max(0, a_i - n + s).$$

(that is, the number of edges required to insert vertices of V_2 prior to v_s exceeds the number of edges available on vertices in M_k plus the minimum number of edges that a vertex v_i (with $v_i \in V_1 \backslash M_k$) has to concede prior to the s^{th} iteration to prevent v_i from having more edges than there are vertices left to be inserted from the s^{th} iteration onwards.)

If $a_s \in A_2$ satisfies (f1), (f2) and (f3), then a_s is said to be *k-fat*. Let F_k denote the set of all the entries that are k-fat. See an illustration in Figure 4.

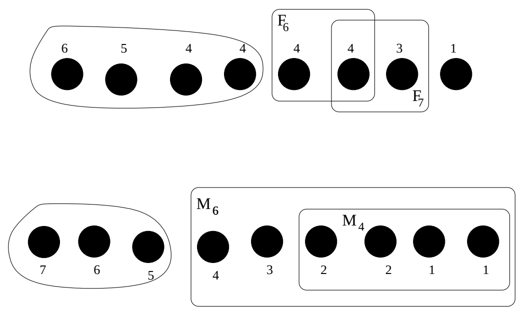

Figure 4: A=(7,6,5,4,3,2,2,1,1 : 6,5,4,4,4,4,3,1), where $A_1 = (7, 6, 5, 4, 3, 2, 2, 1, 1)$ and $A_2 = (6, 5, 4, 4, 4, 4, 3, 1)$. Entries are labelled so that the leftmost entry of A_1 is a_1 and the rightmost entry of A_2 is a_{17}. The entries a_{14} and a_{15} are 6-fat while a_{15} and a_{16} are 7-fat.

Now, if (a) is to be respected and ConstructBipartiteRealization() chose every vertex in M_k to concede an edge to every one of the $s - n_1$ vertices preceding v_s, then ConstructBipartiteRealization() would get stuck at the stage of inserting vertex v_s. This is because by (f1) and (f3), no vertex in M_k would have any edge to concede to v_s and so there would be a maximum of $n_1 - k$ available vertices. But by (f2), vertex v_s needs more adjacent neighbors than the only $n_1 - k$ available vertices. Hence, ConstructBipartiteRealization() must take some precautionary measures by not exhausting all the edges incident to vertices in M_k prior to the insertion of v_s.

Figure 5 illustrates how the Algorithm would get stuck at its s^{th} iteration.

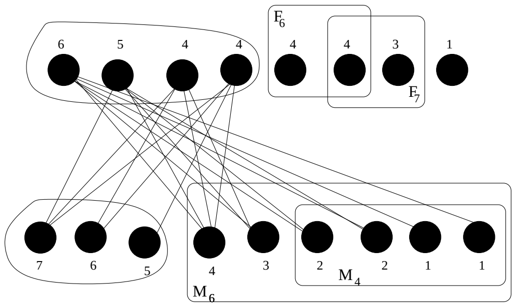

Figure 5: This is a choice of edges that may exhaust all the edges incident to vertices in M_6 prior to the 14^{th} iteration. In this choice, vertex v_1, v_2 and v_3 must concede $3, 2$ and 1 edges respectively lest they would have too many edges after the 13^{th} iteration. Still, vertex v_{14} would not get inserted fully and the Algorithm would stall.

Although (a), (b) and (c) seem to contradict each other, this section defines all these conditions in a formal settings and proves that they can be satisfied simultaneously. Although the analysis seems lengthy, this set of conditions are just inequalities involving the number of edges and vertices already inserted and the number of edges and vertices left to be inserted at each stage of the Algorithm. Moreover, checking these conditions at each iteration of AddVertex() requires checking $\mathcal{O}(n^2)$ inequalities altogether. Thus it does not add to the running time.

Let $A = (A_1 : A_2)$ be a bipartite degree sequence of a simple graph, where A_1 and A_2 have respectively n_1 and n_2 entries such that $n_1 + n_2 = n$. We recall that E_{av} represents the set of available edges. That is, edges that are incident to v_1 and vertices inserted before the n_1^{th} iteration of ConstructBipartiteRealization(), that is, the vertices of V_1. For $v_j \in V_1$, we recall that E_{v_j} is the set of available edges incident to v_j. That is, the set of parallel edges connecting v_1 and v_j. Obviously $E_{v_j} \subseteq E_{av}$ for all j. In particular, E_{v_1} is the set of loops incident to v_1.

Some of the Algorithms given in the literature, such as in [1], have the disadvantage that it has to restart. The algorithm given here allows to choose only edges such that it never has to restart. In order to be able to do that, the choice of edges at every stage must be such that no vertex is incident to too many edges of the 'wrong type'.

If at its s^{th} iteration, Algorithm ConstructBipartiteRealization() is inserting the vertex v_s that has degree a_s, then ConstructBipartiteRealization() has to *call* the routine AddVertex() that has to go through a_s iterations. We recall that the $(s,t)^{th}$ stage of ConstructBipartiteRealization() is the iteration where AddVertex() inserts the t^{th} edge of the s^{th} vertex. Let $X_{s,t}$ and $|X|_{s,t}$ denote respectively a set and its cardinality at the $(s,t)^{th}$ stage of ConstructBipartiteRealization().

To help the reader, we first introduce the motivation for the definitions. At each stage of constructing a simple graph, every vertex v_j, where $v_j \in V_1$, must be connected by at most one edge to any other v_k, where $v_k \in V_2$. So, if some vertex v_j has more available edges than the vertices left to be inserted after its s^{th} iteration, ConstructBipartiteRealization() would never be able to get rid of all these multiple edges, which would then appear in the final graph. This prompts the following definitions. The vertex v_j where $j \leq n_1$ (i.e., $v_j \in V_1$) is *due* if

$$|E_{v_j}|_{st} = n - (s-1), \tag{1}$$

that is, E_{v_j} has as many edges as there are vertices left to be inserted. The vertex v_j is *overdue* if

$$|E_{v_j}|_{st} > n - (s-1), \tag{2}$$

that is, there are too many available edges incident to v_j and whatever are the future choices, the Algorithm would never output a simple graph. The vertex v_j is *undue* if it is neither due nor overdue. Obviously, a stage is *due, undue, overdue* if there is a vertex that is due, undue or overdue,respectively.

Let M_k be the set of the last k entries of A_1. An entry a_s in A_2 is *k-fat* if conditions (f1), (f2) and (f3) are satisfied. We let F_k to denote the set of vertices that are k-fat. A bipartite degree sequence A is *fat* if it contains a k-fat entry for some integer $k > 0$.

Let $r_i = a_i - n_1 + k$, where k is the largest integer such that a_i is k-fat. The $(s,t)^{th}$ stage is *ruined* if there is an entry a_i with $i > s$ (that is, the vertex v_i is not inserted yet) that is fat and the number of vertices in M_k that are available is less than r_i. It is *not ruined* otherwise.

The next lemma indicates that once ConstructBipartiteRealization() has taken a 'wrong path', it is impossible to mend the situation.

Lemma 1 *Suppose ConstructBipartiteRealization() is inserting the vertex* v_s *such that* $s > n_1$, *(i.e., inserting* v_s *into* V_2). *Then the following hold.*

(a) *If the vertex* v_j *is due, it is due or overdue at the next stage. If it is overdue, it is overdue at any future stage.*

(b) *If the* $(s,t)^{th}$ *stage is overdue, then the previous stage (the stage inserting the previous edge) is either due or overdue.*

(c) *If the* $(s,t)^{th}$ *stage is ruined, then the next stage is also ruined.*

Proof.

(a) Suppose v_j is due and Addvertex() does not choose an edge from E_{v_j}. Since no edge of E_{v_j} is chosen, the left side of Equation 1 remains same while the right hand side either goes down by one if ConstructBipartiteRealization() moves to a new vertex v_{s+1} or stays the same if ConstructBipartiteRealization() moves to another edge $t+1$ of the same vertex v_s. Hence the next stage is due or overdue. On the other hand, if Addvertex() chooses an edge from E_{v_j}, the left hand side goes down by 1 and the right one stays the same. But if E_{v_j} concedes only one edge to v_s (as we shall see shortly), E_{v_j} is still due at the insertion of vertex v_{s+1}. Similar arithmetical arguments as above show that if v_j is overdue, it stays overdue.

(b) Suppose v_j is overdue at the $(s,t)^{th}$ stage but is undue at the stage inserting the previous edge. Then at the previous stage, we have

$$|E_{v_j}| < n - (s-1). \tag{3}$$

Now, either the last edge inserted is chosen from E_{v_j} or not. Moreover, in either case, Algorithm ConstructBipartiteRealization() moves to a new vertex or not. If it stays on the same vertex and the chosen edge is not from E_{v_j}, the right and the left hand sides of Equation 3 are both unchanged. Hence v_j is undue at the $(s,t)^{th}$ stage, which is a contradiction. If it stays on the same vertex and the chosen edge is from E_{v_j}, the left hand side of Equation 3 goes down by 1 while the right hand side is unchanged. Hence v_j is also undue at the $(s,t)^{th}$ stage and this is again is a contradiction.

Suppose ConstructBipartiteRealization() moves to a new vertex. If the chosen edge is not from E_{v_j}, the right hand side of Equation 3 goes down by 1 while the right hand side is unchanged. Hence v_j is due at the $(s,t)^{th}$ stage, a contradiction. If the chosen edge is from E_{v_j}, both left hand and right hand sides of Equation 3 go down by 1. Hence v_j is normal at the $(s,t)^{th}$ stage, a contradiction.

(c) Assume that the $(s,t)^{th}$ stage is ruined. That is, there is a fat vertex v_i that is not inserted yet, but the number of vertices in M_k which are available is less than r_i. But, at the next stage, this number can never increase. Thus it would also be ruined. □

While Lemma 1 says that once ConstructBipartiteRealization() takes a wrong path, it is impossible to mend it, the next routine gives preventive measures to avoid getting into that wrong path in the first place.

ChooseCorrectEdge()

Let A be not fat and ConstructBipartiteRealization() is at its $(s,t)^{th}$ stage with $s > n_1$ (that is, inserting vertex v_s into V_2). Then,

(1) for each vertex $v_j \in V_1$, do not choose an edge in E_{v_j} if there is already an edge (v_s, v_j).

(2) if the vertex v_j is due, pick an edge from E_{v_j}. If many vertices are due, pick an edge uniformly at random from the vertices that are due.

Now assume that A is fat and for some integer $k > 0$, F_k is not empty. Then, for every entry $a_i \in F_k$ choose at random $r_i = a_i - n_1 + k$ different entries in M_k. The only condition imposed on the choice is that an entry a_j can be chosen at most once for each fat vertex and at most a_j times for all the fat vertices combined. If a_i is k-fat, let R_i, called the *reserve pool* of a_i, be the set of vertices in M_k chosen for a_i. Let R_{ij}, the *reserve matrix*, be an n_1 by n_2 matrix whose columns are indexed from 1 to n_1 (indices of entries of A_1), and rows are indexed from $n_1 + 1$ to n (indices of entries of A_2), and $R_{ij} = 1$ if the entry $a_j \in R_i$, and zero otherwise. Obviously, the sum of entries in row i is equal to r_i and the sum of entries of column j must be less or equal to a_j. At the $(s,t)^{th}$ stage, a vertex $v_j \in V_1$ is *exhausted* if the sum of row j plus the number of vertices adjacent to v_j equals a_j. (that is, the number of edges already conceded by v_j and the number of edges of v_j in the reserve pools equals a_j).

(3) If ConstructBipartiteRealization() is at its $(s,t)^{th}$ stage with $s > n_1$ and a_s is not fat, then apply (1) and (2) subject to not choosing a vertex v_j if v_j is exhausted. If a_s is fat, first choose all the vertices in R_s, then apply (1) and (2) if necessary.

Complexity Issues

Before proving that the conditions set in routine ChooseCorrectEdge() are necessary and sufficient to sample a simple bipartite graph at random, we observe that, if $A = (a_1, a_2, ..., a_n) = (A_1 : A_2)$ where A_1 and A_2 have respectively n_1 and n_2 entries such that $n_1 + n_2 = n$ and $\sum_{i=1}^{n} a_i = 2m$, ChooseCorrectEdge() runs altogether in $\mathcal{O}(n_1 n_2)$ steps. Indeed, at the s^{th} iteration of ConstructBipartiteRealization(), ChooseCorrectEdge() has to check Equation 1 only once for every vertex $v_j \in V_1$. But there are n_2 iterations and n_1 vertices v_j with $j \le n_1$. This takes $\mathcal{O}(n_1 n_2)$ steps. Constructing the Reserve Matrix R requires $\mathcal{O}(n_1 n_2)$ steps as one has to check Conditions (f1), (f2) and (f3) for each of the n_2 entries of A_2 and writing the $n_1 n_2$ entries of the matrix R.

Theorem 3 *Algorithm ConstructBipartiteRealization() reconstructs a simple graph if and only if AddVertex() calls the routine ChooseCorrectEdge(). In other words, ConstructBipartiteRealization() outputs a simple graph if and only if the choice of edges satisfies Conditions (1), (2) and (3).*

Proof. Assume to the contrary that Conditions (1) and (2) hold but ConstructBipartiteRealization() outputs a bipartite graph G with multiple edges or loops. By Condition (1) there can not be a multiple edge connecting two vertices v_j and v_k such that $j \le n_1$ and $k > n_1$. Moreover, by the definition of the routine ConstructBipartiteRealization(), there can not be a double edge (v_k, v_l) where $k, l > n_1$. Hence if G fails to be a simple graph, it must have either a loop or a multiple edge incident to v_1 and v_j such that $j \le n_1$.

So, in G, let the vertex v_1 is incident to either a loop e or a multiple edge (v_1, v_j) such that $j \le n_1$. But, by the definition of the bipartition, the number of edges incident to v_1 at the end of the n_1^{th} iteration of ConstructBipartite-Realization() equals the number of edges left to be inserted until Construct-BipartiteRealization() terminates. Hence, some vertex v_k such that $k > n_1$ is not fully inserted. This is a contradiction.

Conversely, let the condition (1) or (2) be not satisfied and let G be the realization output by ConstructBipartiteRealization(). If condition (1) is not satisfied at the $(s,t)^{th}$ stage, this would create a double edge (v_j, v_s) with $j \le n_1$ and $s > n_1$. Now, since Algorithm Addvertex() can not concede the double edge (v_j, v_s) anymore as they are lost, the double edge (v_j, v_s) would appear in G. Hence G would not be simple. Assume that the condition (2) is not satisfied. That is, there is a vertex v_j with $j \le n_1$ that is due at the $(s,t)^{th}$ stage, where $s > n_1$, but Algorithm Addvertex() does not pick any of the elements of E_{v_j} for all the remaining edges conceded to v_s. Then v_j is overdue at the insertion of vertex v_{s+1}, and by Lemma 1(b) it remains overdue until the end of Algorihm 2. Hence G is not simple as it must have a multiple edge (v_i, v_j). If condition (3) is not satisfied, Algorithm ConstructBipartite-Realization() may stall. □

Let a *correct edge* and *vertex* be an edge chosen by Algorithm ChooseCorrectEdge and a vertex incident to a correct edge, respectively. So if Construct-BipartiteRealization() terminates, we have shown that it always outputs a simple graph. It remains to show that it always terminates by showing that there is always a correct edge so that conditions (1) and (2) can be satisfied at every stage of ConstructBipartiteRealization().

Theorem 4 *Algorithm ConstructBipartiteRealization() always terminates. That is, Conditions (1) and (2) are always satisfied at every stage of ConstructBipartiteRealization().*

Proof. Suppose A does not contain any fat entry. That is, as long as an edge $e = (v_1, v_j)$ is a correct vertex, it can be chosen. Obviously, Condition (1) can always be forced on AddVertex(). But, while trying hard to satisfy Condition (1), the algorithm may let a vertex v_j of V_1, to become overdue. If at the $(s,t)^{th}$ stage the vertex v_j is due, we prove that it is always possible to concede an edge from E_{v_j} to v_s.

So assume to the contrary that v_j is due but Addvertex() can not pick an edge from E_{v_j}. This is possible only if there are too many vertices that are due. That is, $a_s < n_1' \leq n_1$, where n_1' is the number of vertices that are due at the $(s,t)^{th}$ stage. But we also have $a_s \geq a_{s+1} \geq ... \geq a_n$. Moreover, as all these n_1' vertices are due, each of them is incident to $n - s$ available edges. Hence we have $a_s + a_{s+1} + \cdots + a_n < n_1'(n - s)$. That is, there are more available edges than there are edges left to be inserted until ConstructBipartiteRealization() terminates. This contradicts Observation 2.

Let the entry a_i be k-fat. If all the correct edges $e = (v_1, v_j)$ such that $v_j \in M_k$ are conceded prior to the insertion of the vertex v_i, then by definition of fat entry, Algorithm ConstructBipartiteRealization() would stall as there would not be enough edges to connect to v_i. But, we assume that the Algorithm reserved r_i edges to concede to v_i. Hence v_i can always be inserted. So, we only need to check (c1), whether putting some edges in reserve would prevent some non-fat vertex v_s from being inserted for lack of correct edges and, (c2), whether it is always possible to construct the reserve matrix R_{ij}.

(c1) Assume that $s < i$. That is, v_s precedes v_i. Let all vertices preceding v_s have been inserted but there are not enough correct edges to insert v_s. This is possible if reserving edges for vertices in F_k and inserting vertices preceding v_s exhausts q vertices of V_1 and $a_s > n_1 - q$. Without loss of generality, we may assume that the last q vertices of V_1 are exhausted. So, let the available vertices be vertices $v_1, \ldots, v_{n_1 - q+1}$. If the number of available edges is less than a_s, then $A_1 < A_2$. This is a contradiction. So, let the number of available edges be greater or equal to a_s. Thus the number of available vertices is less than a_s, so that Condition (1) prevents a_s edges from being connected to v_s. Let H

be the graph obtained after the insertion of v_{s-1} by 'fully' connecting all the vertices in $V_2\backslash v_s$, making sure to connect vertices in F_k with edges that are reserved for them in R_{ij}. Then, by the definition of r_i, it is easy to check that every vertex in F_k is adjacent to every vertex in $V_1\backslash M_k$. Also, since all the vertices in M_q are exhausted after the insertion of v_{s-1}, one can check that none of the vertices in $M_q\backslash M_k$ is adjacent to a vertex in $V_2\backslash(F_k \cup V_{\leq s})$, where $V_{\leq s}$ denotes the set of vertices from v_{n_1+1} up to v_s. (i.e., $V_2\backslash(F_k \cup V_{\leq s})$ is the set of vertices between v_s and F_k). Thus, only the vertices in $V_1\backslash M_q$ are adjacent to vertices in $V_2\backslash(F_k \cup V_{\leq s})$. Since all the vertices, except for v_s are properly connected and $|A_1| = |A_2|$, the number of available edges is a_s but the number of available vertices is less than a_s. Therefore, by the pigeonhole principle, there is an available vertex having at least two available edges. Without loss of generality, we may consider v_1 to be the culprit.

Now, since only the vertices in $V_1\backslash M_q$ are adjacent to the vertices in $V_2\backslash(F_k \cup V_{\leq s})$, either v_1 is adjacent to all the vertices in $V_2\backslash(F_k \cup V_{\leq s})$ or it is not. If it is, then v_1 was due during an iteration prior to or during the insertion of v_{s-1} and the algorithm did not select it to concede an edge. This is a contradiction. Suppose that it is not adjacent to some vertex $v_t \in V_2\backslash(F_k \cup V_{\leq s})$. Then $a_t < n_1 - q$, since v_t is fully connected. But, by the non decreasing ordering of A_2, we also have $a_t \geq a_i$. Moreover, since $a_i \in F_k$, we have $a_i \geq n_1 - k$. Hence we have $a_i \geq n_1 - k > n_1 - q > a_t$. This is also a contradiction. Therefore vertex v_s can be fully inserted. See Figure 6 which helps to understand notations in part (c1).

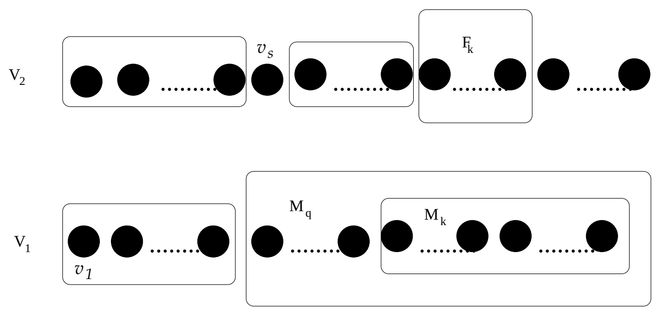

Figure 6:

Finally, let a_i be k-fat, a_s be not k-fat and $s > i$. (that is, v_s is to be inserted after v_i). If there are not enough correct edges to connect to v_s, then $|A_1| < |A_2|$. This is a contradiction.

(c2) Suppose that it is not possible to built the reserve matrix. But, since

$a_i \leq n_1$ for all entries in F_k, this would imply either $\sum_{F_k} a_i > \sum_{V_1 \setminus M_k} a_i + \sum_{M_k} a_i = |A_1|$, or $a_i > n_1$ for some entry $a_i \in F_k$. This is a contradiction. □

It still remains to show that the algorithm constructs all the simple realizations of A.

Lemma 2 *Let* G_{n_1,n_2} *be the* n_1, n_2*- complete bipartite graph. That is, the bipartite graph where one part contains* n_1 *vertices each having degree* n_2 *and the second part contains* n_2 *vertices each of degree* n_1*. Then ConstructBipartiteRealization() satisfying Conditions (1) and (2) can reconstruct* G_{n_1,n_2} *as a realization of* $A = (A_1 : A_2)$ *where* A_1 *has* n_1 *entries* $a_i = n_2$ *and* A_2 *has* n_2 *entries* $a_j = n_1$*.*

Proof. At the beginning of the $(n_1 + 1)^{th}$ iteration, $E_{v_j} = n_1$ for each of the n_1 vertices already inserted. Hence each such vertex is due. Now, the vertex v_{n_1+1} has degree $a_{n_1+1} = n_1$ by the definition of A. Hence, by Condition (2), AddVertex() chooses one edge from each of the n_1 vertices v_j with $j \leq n_1$ and inserts v_{n_1+1} completely. By Lemma 1, each v_j is still due at the $(n_1 + 2)^{th}$ iteration. Again, by Condition (2), AddVertex() chooses one edge from each of the n_1 vertices v_j with $j \leq n_1$ and inserts v_{n_1+2} completely. And so on, until the insertion of vertex v_n, and Algorithm ConstructBipartiteRealization() outputs the graph G_{n_1,n_2}. □

Let G be a graph, a *delete-minor* of $G' = G \backslash e$ is the graph obtained from G by deleting the edge e. If $A = (A_1 : A_2)$ is a bipartite degree sequence, let A' be the degree sequence obtained from A by subtracting 1 from two of its entries a_i and a_j, where $a_i \in A_1$ and $a_j \in A_2$. Thus, if A is the degree sequence of a bipartite graph G, then A' is the degree sequence of some delete-minor of G.

Lemma 3 *If ConstructBipartiteRealization() satisfying Conditions (1) and (2) can reconstruct* G *as a realization of* A*, then it can reconstruct all the delete-minors of* G *that are realizations of* A'*.*

Proof. Let G be a bipartite graph output by Algorithm ConstructBipartiteRealization() and let $G \backslash e$ be a delete-minor of G. In the graph G, let the edge e be incident to vertices v_j and v_k having respectively degrees a_j and a_k, where $j \leq n_1$ and $k > n_1$. Thus in $G \backslash e$, vertices v_j and v_k have degrees $a_j - 1$ and $a_k - 1$. Let f be any edge of $G \backslash e$. Since G is output by ConstructBipartiteRealization(), there is a series of choices of correct edges such that f can be inserted. In that series of choices either e is inserted before or after f. If e is inserted after f, the same series of choices would insert f in $G \backslash e$. If e is inserted before f, the same series of choices, minus the insertion of e, would also lead to the insertion of f in $G \backslash e$, since Algorithm ConstructBipartiteRealization() does not need to insert any edge incident to v_j and v_k as their degrees are down by 1. □

Corollary 1 *Let* G *be a simple bipartite realization of a degree sequence* $A = (A_1 : A_2)$ *where* A_1 *and* A_2 *have* n_1 *and* n_2 *entries respectively. Then there is a positive probability that* G *is output by Algorithm ConstructBipartiteRealization() if Conditions (1) and (2) are satisfied.*

Proof. Every simple bipartite graph having one part of n_1 vertices and another of n_2 vertices can be obtained from G_{n_1,n_2} by a series of deletions. □

3.1 Sampling all bipartite realizations uniformly

Although Theorem 2 shows that the routine ConstructBipartiteRealization() can construct a realization of A in time linear on the number of edges of its realizations, we need the next result to show that it can construct any bipartite realization of A with equal probability, provided we define the probability $\pi(e)$ with which AddVertex() has to insert the edge e. If at its k^{th} iteration ConstructBipartiteRealization() is to insert the vertex v_k that has degree a_k, then ConstructBipartiteRealization() has to *call* AddVertex() that has to go through a_k iterations. Let the $(s,t)^{th}$ stage of ConstructBipartiteRealization() be the iteration where AddVertex() inserts the t^{th} edge of the s^{th} vertex and let $G^{(s,t)}$ denote the graph obtained at that $(s,t)^{th}$ stage. With this notation, let $G^{(s)}$ be the graph $G^{(s,a_s)}$. The *random reconstruction tree*, denoted by $\mathcal{T}$, is a directed rooted tree where the root is the sole realization of the degree sequence $A^{(1)}$, and the $(s,t)^{th}$ level contains all those possible graphs obtainable after inserting the t^{th} edge of the s^{th} vertex, and there is an arc from a graph H at level i to the graph G at level $i+1$ if it is possible to move from H to G by the concession of a single available edge. Realizations of A are thus the leaves of the tree $\mathcal{T}$. With this formalism, sampling a random bipartite realization of the degree sequence A is equivalent to performing a random walk from the root until a leaf is reached, and every step of the random walk consists of walking along a random arc of $\mathcal{T}$. See Figure 7 for an illustration.

Rejection sampling

Let G be a realization of A. That is, G is a leaf of the tree $\mathcal{T}$. Obviously, there are many paths of $\mathcal{T}$ leading to G. Let p be such a path and let $\pi_p(G)$ denote the probability to reach G along the path p. Now $\pi_p(G)$ can easily be computed on the fly since $\pi_p(G) = \prod_{e \in E(G)} \pi(e)$, where $E(G)$ denotes the set of edges of G and $\pi(e)$ is the probability to choose the edge e. Now $\pi(e) = \frac{1}{|V_{cor}}|$, where V_{cor} is the set of all correct vertices at the insertion of e. The only problem is that G can be reached from many paths. The next result proves that all these paths have equal probability.

Lemma 4 *Let* G *be a realization of* A *that can be reached through the paths* p *and* q *of* $\mathcal{T}$*. Then* $\pi_p(G) = \pi_q(G)$.

Proof. Let $E(G)$ denote the set of edges of G. Then, p can be seen as a re-ordering of a subset of edges chosen along q. Now, since the vertices are added in the same order along q as along p, we may only consider the case where p and q differ on a single vertex and edges e and f are interchanged in p and q. Let $V_{cor}(e)$ and $V_{cor}(f)$ denote the sets of correct vertices at the insertion of e and f, respectively. If the Algorithm can choose either the edge e or f, then $V_{cor}(e) = V_{cor}(f)$ and the probability to choose either must be the same. □

Lemma 4 allows to compute $\pi(G)$ on the fly. For any path p leading to G, we have

$$\pi(G) = \prod_{e \in G} \pi(e) = \prod_{e \in G} \frac{1}{|V_{corr}(e)|},$$

where $V_{corr}(e)$ is the set of vertices in V_1 that are incident to some correct edge. Hence, to get $\pi(G)$ on the fly, one set $\pi(G) = \pi(G^{n_1}) = 1$. For every partial realization $G^{(i)}$ from (G^{n_1}) to G multiply $\pi(G)$ by $\frac{1}{|V_{corr}(e)|}$. Finally output $\pi(G)$ with G. Now let $\min(\pi)$ be a lower bound of the probabilities to reach of the realizations of A. This lower bound can be calculated using only parameters of A. Indeed, if $|V_{av}(e)|$ stands for the number of vertices in V_1 that are adjacent to v_1 at the insertion of edge e, then we have the inequality $\frac{1}{|V_{av}(e)|} \leq \frac{1}{|V_{corr}(e)|} \leq \pi(e)$ and, for any realization G, we have

$$\prod_{e \in G} \frac{1}{|V_{av}(e)|} \leq \prod_{e \in G} \pi(e) \leq \pi(G).$$

Finally, since $|V_{v_1}(e)| \leq n_1$ and every realization of A has m edges, we get

$$\frac{1}{n_1^m} \leq \prod_{e \in G} \frac{1}{|V_{av}(e)|} \leq \prod_{e \in G} \pi(e) \leq \pi(G).$$

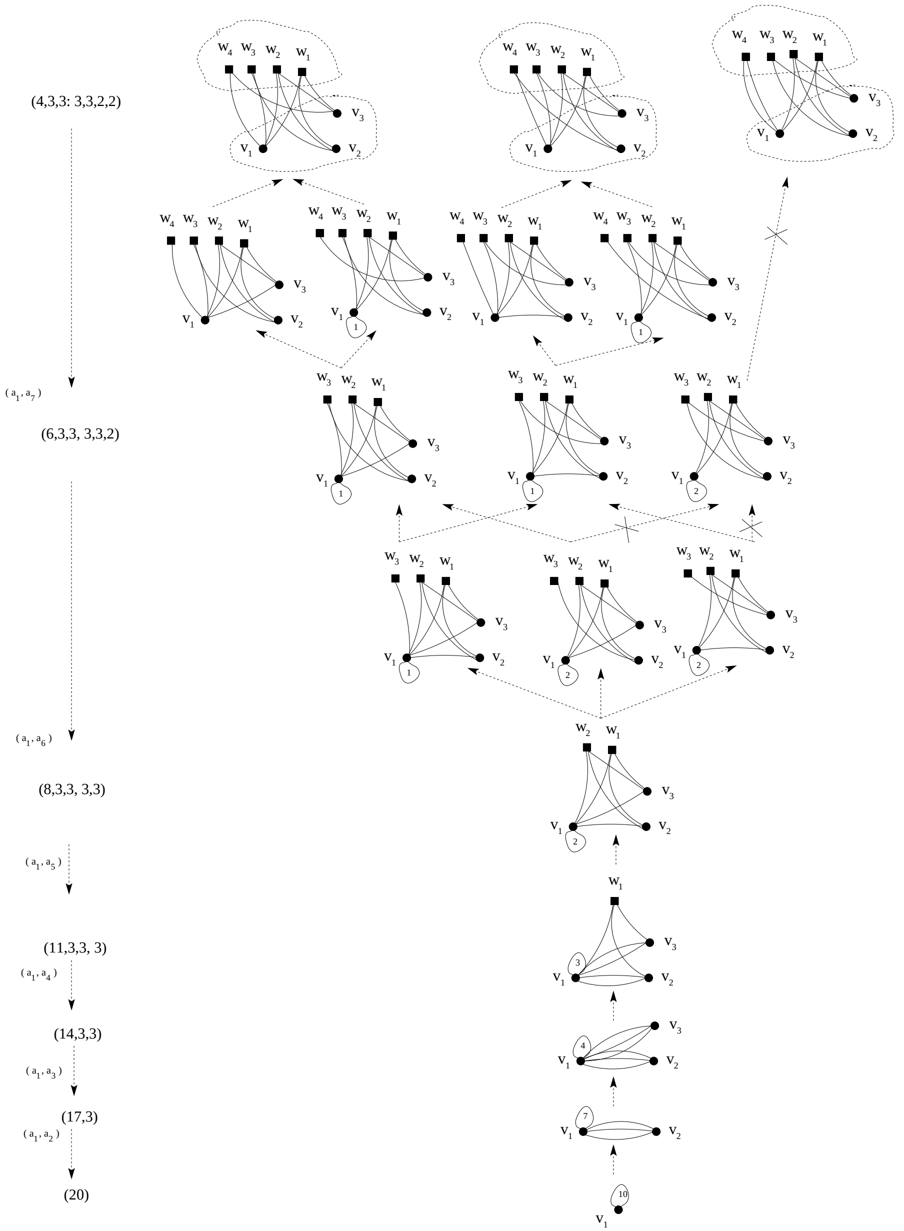

Figure 7: Random reconstruction tree of $(4, 3, 3 : 3, 3, 2, 2)$. The level of $\mathcal{T}$ on the same height as the degree sequence $A^{(i)}$ corresponds to all the graphs having $A^{(i)}$ as their degree sequence. The arrows that are crossed denote the edges that would not lead to a simple realization.

Algorithm RejectionSampling()

Input: Bipartite degree sequence $A = (A_1 : A_2)$, where A_1 and A_2 have n_1 and n_2 entries respectively such that $n_1 + n_2 = n$ and an integers r_1.

Output: A sequence of r_1 bipartite simple realizations of A where every realization has equal probability.

Step 1 Put A_1 and A_2 in non decreasing order.

Step 2 Construct the recursion chain of A by calling the routine ConstructBipartiteRecursionChain().

Step 3 Call ConstructBipartiteRealization() to construct the realization G. Let $\pi(G)$ be the probability computed on the fly and get u, a random number in $(0, 1)$. If $u < \frac{\min(\pi)}{\pi(G)}$, accept G and go back to Step 3 until one gets r_1 realizations. Else, reject G and go back to Step 3 until one gets r_1 realizations.

Obviously, Algorithm RejectionSampling() samples every realization of A with the same probability equal to $\min(\pi)$. Now, it is known that Step 1 takes $\log(n_1) + \log(n_2)$ iterations and, as shown earlier, Step 2 takes $n_1 + n_2$ iterations. In Step 3, ChooseCorrectEdge() does $\mathcal{O}(n_1 n_2)$ inequality checks altogether while AddVertex() needs $2m$ iterations to insert all the vertices. Thus, the overall running time to get the minimum probability is given by

$$\log(n_1) + \log(n_2) + r_1(n_1 n_2 + 2m) = \mathcal{O}(r_1(n_1 n_2 + 2m)) \asymp \mathcal{O}(3r_1 m) \asymp \mathcal{O}(m).$$

Finally, T, the running time of generating a realization of A uniformly, is a geometric random variable with expected running time given by $\frac{1}{(\pi(acc)}$ where $\pi(acc)$ is the acceptance probability for the realization G with the highest probability of being output by ConstructBipartiteRealization(). So

$$\pi(acc) = \frac{\min(\pi)}{\pi(G)} = \frac{\min(\pi)}{\prod_{e \in G} |V_{corr}(e)|}.$$

Now if $n_2 \to \infty$, then $|V_{corr}(e)| \to \frac{n_1}{2}$ on average. Therefore,

$$\pi(acc) \to \frac{\min(\pi)}{(\frac{2}{n_1})^m} = \frac{\frac{1}{n_1^m}}{(\frac{2}{n_1})^m} = \frac{1}{2^m}.$$

Hence $T \to 2^m$. For the typical Darwin tables m is about 40 edges. Thus 2^m is a manageable running time.

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Some sufficient conditions for certain class of meromorphic multivalent functions involving Cho-Kwon-Srivastava operator

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Abstract. Making use of a meromorphic analogue of the Cho-Kwon-Srivastava operator for normalized analytic functions, we introduce below a new class of meromorphic multivalent function in the punctured unit disk and obtain certain sufficient conditions for functions to belong to this class. Some consequences of the main result are also mentioned.

1 Introduction and motivation

Let $\sum_p$ denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, \cdots\}) \tag{1}$$

which are analytic in the punctured unit disk:

$$\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

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having the only pole of order p at origin. In particular for $p = 1$, we write $\sum_1 := \sum$.

For functions $f \in \sum_p$ given by (1) and $g \in \sum_p$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (z \in \mathbb{U}^*), \tag{2}$$

we define $f * g$ by

$$(f * g)(z) := \frac{z^p f(z) * z^p g(z)}{z^p} =: \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z) \quad (z \in \mathbb{U}^*), \tag{3}$$

where $*$ denotes the usual Hadamard product(or convolution) of analytic functions.

Let $\sum_p^*(\alpha)$, $\sum_p^k(\alpha)$ and $\sum_p^c(\alpha)$ be the subclasses of the class $\sum_p$ consists of meromorphic multivalent functions which are respectively starlike, convex and close-to-convex funcions of order α $(0 \leq \alpha < p)$.

Analytically, a function $f \in \sum_p$ is said to be in the class $\sum_p^*(\alpha)$ if and only if

$$\Re\left[-\frac{zf'(z)}{f(z)}\right] > \alpha \quad (z \in \mathbb{U}^*). \tag{4}$$

Similarly, a function $f \in \sum_p$ is said to be in the class $\sum_p^k(\alpha)$ if and only if

$$\Re\left[-1 - \frac{zf''(z)}{f'(z)}\right] > \alpha \quad (z \in \mathbb{U}^*). \tag{5}$$

Furthermore, a function $f \in \sum_p^c(\alpha)$ if and only if f is of the form (1) and satisfies

$$\Re\left[-\frac{f'(z)}{z^{-p-1}}\right] > \alpha \quad (z \in \mathbb{U}^*). \tag{6}$$

We observe that $\sum_1^*(\alpha) := \sum^*(\alpha)$, $\sum_1^k(\alpha) := \sum_k(\alpha)$, $\sum_1^c(\alpha) := \sum_c(\alpha)$ where $\sum^*(\alpha)$, $\sum_k(\alpha)$ and $\sum_c(\alpha)$ are subclasses of $\sum$ consisting of meromorphic univalent functions which are respectively starlike, convex and close-to-convex of order α $(0 \leq \alpha < 1)$. For recent expository work on meromorphic functions see([5, 7, 11, 14, 16]).

For the purpose of defining transform, Liu and Srivastava [7] studies meromorphic analogue of the Carlson-Shaffer operator [1] by introducing the function $\phi_p(a,c;z)$ given by

$$\phi_p(a,c;z) := \frac{{}_2F_1(a,1;c;z)}{z^p} =: \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p}, \quad (z \in \mathbb{U}^*; a \in \mathbb{C}, c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mathbb{Z}_0^- := \{0,1,2,\cdots\}) \tag{7}$$

where ${}_2F_1(a,1;c;z)$ is the Gauss hypergeometric series and $(\lambda)_n$ is the *Pochhammer symbol* (or shifted factorial) given by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)...(\lambda+n-1) & (n \in \mathbb{N}). \end{cases}$$

Recently, Mishra et al. [9] (also see [10]) defined the function $\phi_p^{\dagger}(a,c;z)$, the generalized multiplicative inverse of $\phi_p(a,c;z)$ given by the relation

$$\phi_p(a,c;z) * \phi_p^{\dagger}(a,c;z) = \frac{1}{z^p(1-z)^{\lambda+p}} \quad (a,c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda > -p, z \in \mathbb{U}^*). \tag{8}$$

If $\lambda = -p+1$, then $\phi_p^{\dagger}(a,c;z)$ is the inverse of $\phi_p(a,c;z)$ with respect to the Hadamard product $*$. Using this function $\phi_p(a,c;z)$, they considered an operator $\mathcal{L}_p^{\lambda}(a,c) : \sum_p \longrightarrow \sum_p$ as follows:

$$\begin{aligned} \mathcal{L}_p^{\lambda}(a,c)f(z) :&= \phi_p^{\dagger}(a,c;z) * f(z) = \frac{{}_2F_1(\lambda+p,c;a;z)}{z^p} \\ &= \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{(a)_k(1)_k} a_{k-p} z^{k-p} \quad (z \in \mathbb{U}^*). \end{aligned} \tag{9}$$

The holomorphic version of the function $\phi_p^{\dagger}(a,c;z)$ is given by the relation:

$$z^p{}_2F_1(a,1;c;z) * \phi_p^{\dagger}(a,c;z) := \frac{z^p}{(1-z)^{\lambda+p}} \quad (a,c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda > -p; z \in \mathbb{U}),$$

and the associated transform $\mathcal{L}_p^{\lambda}(a,c)f(z) = \phi_p^{\dagger}(a;c;z) * f(z)$ were studied by Cho et al. [2]. The transform $\mathcal{L}_p^{\lambda}(a,c)$ is popularly known as the Cho-Kwon-Srivastava operator in literature (see, for details [4, 12, 15]).

Recently, Prajapat [13] (also see [3]) introduced a class of analytic and multivalent function $\mathbb{B}(p,n,\mu,\alpha)$ and investigated some sufficient conditions for this class. Furthermore, Goyal and Prajapat [5] introduced the class $\mathcal{T}_p(\lambda,\mu,\alpha)$

by making use of an extended derivative operator of Ruscheweyh type and investigated some sufficient conditions for a certain function to belong to this class.

Motivated by the aforementioned work, in this paper the authors introduce a new class $\mathcal{T}_p^{\lambda,\alpha}(\mu, a, c)$ by making use of a meromorphic analogue of Cho-Kwon-Srivastava operator $\mathcal{L}_p^{\lambda}(a, c)$ for normalized multivalent analytic function as follows:

Definition 1 *A function* $f \in \sum_p$ *is said to be in the class* $\mathcal{T}_p^{\lambda,\alpha}(\mu, a, c)$ *if it satisfies the following condition:*

$$\left|\frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}+p\right|<p-\alpha \qquad (10)$$
$$(z \in \mathbb{U}^*,\ p \in \mathbb{N},\ \lambda > -p,\ \mu \geq 0,\ 0 \leq \alpha < p,\ a,\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$

The condition (10) implies that

$$\Re\left\{-\frac{z^{p+1}(\mathcal{L}_p^{\lambda}(a,c)f(z))'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}\right\} > \alpha \qquad (11)$$

It is clear from the above definition that
$\mathcal{T}_p^{-p+1,\alpha}(2, a, a) = \sum_p^*(\alpha)$ and $\mathcal{T}_p^{-p+1,\alpha}(1, a, a) = \sum_p^c(\alpha)$.

In the present paper, we obtain certain sufficient conditions for functions f to be in the class $\mathcal{T}_p^{\lambda,\alpha}(\mu, a, c)$.

We need the following lemma for our investigation.

Lemma 1 (*see* [6, 8]) *Let the function* $w(z)$ *be non-constant and regular in* $\mathbb{U}$ *such that* $w(0) = 0$*. If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r < 1$ *at a point* $z_0 \in \mathbb{U}$*, then*

$$z_0 w'(z_0) = k w(z_0),$$

where k *is real and* $k \geq 1$*.*

2 Main results

Unless otherwise stated, we mention throughout the sequel that

$$p \in \mathbb{N},\ \mu \geq 0,\ \lambda > -p,\ a,\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ 0 \leq \alpha < p.$$

Theorem 1 *If* $f \in \sum_p$ *given by* (1) *satisfies anyone of the following inequalities:*

$$\left| -\frac{z^{p+1}\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^\lambda(a,c)f(z)\right)^{\mu-1}}\left[1+p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)''}{(\mathcal{L}_p^\lambda(a,c)f(z))'} -(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\mathcal{L}_p^\lambda(a,c)f(z)}\right\}\right]\right| < p-\alpha, \tag{12}$$

$$\left|\frac{1+p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)''}{(\mathcal{L}_p^\lambda(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\mathcal{L}_p^\lambda(a,c)f(z)}\right\}}{-\frac{z^{p+1}\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^\lambda(a,c)f(z)\right)^{\mu-1}}}\right| < \frac{p-\alpha}{(2p-\alpha)^2}, \tag{13}$$

$$\left|\frac{1+p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)''}{(\mathcal{L}_p^\lambda(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\mathcal{L}_p^\lambda(a,c)f(z)}\right\}}{-\left[\frac{z^{p+1}\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^\lambda(a,c)f(z)\right)^{\mu-1}}+p\right]}\right| < \frac{1}{2p-\alpha}, \tag{14}$$

and

$$\Re\left[\frac{z^{p+1}\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^\lambda(a,c)f(z)\right)^{\mu-1}}\left\{\frac{1+p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)''}{(\mathcal{L}_p^\lambda(a,c)f(z))'}-(\mu-1)\left(p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\mathcal{L}_p^\lambda(a,c)f(z)}\right)}{\frac{z^{p+1}\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^\lambda(a,c)f(z)\right)^{\mu-1}}+p}\right\}\right] < 1, \tag{15}$$

then $f \in \mathcal{T}_p^{\lambda,\alpha}(\mu,a,c)$.

Proof. Let $f(z) \in \sum_p$ be given by (1). Define the function $w(z)$ by

$$-\frac{z^{p+1}\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^\lambda(a,c)f(z)\right)^{\mu-1}} = p+(p-\alpha)w(z). \tag{16}$$

Clearly $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=0$. Taking logarithmic differentiation on both sides of (16) with respect to z, we obtain

$$1+p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)''}{(\mathcal{L}_p^\lambda(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\mathcal{L}_p^\lambda(a,c)f(z)}\right\} = \frac{(p-\alpha)zw'(z)}{p+(p-\alpha)w(z)}. \tag{17}$$

From (16) and (17), we have

$$\begin{aligned}\phi_1(z) = -\frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}\Bigg[1+p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)''}{(\mathcal{L}_p^{\lambda}(a,c)f(z))'} \\ -(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\mathcal{L}_p^{\lambda}(a,c)f(z)}\right\}\Bigg] = (p-\alpha)zw'(z),\end{aligned} \tag{18}$$

$$\begin{aligned}\phi_2(z) &= \frac{1+p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)''}{(\mathcal{L}_p^{\lambda}(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\mathcal{L}_p^{\lambda}(a,c)f(z)}\right\}}{-\frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}} \\ &= \frac{(p-\alpha)zw'(z)}{[p+(p-\alpha)w(z)]^2},\end{aligned} \tag{19}$$

$$\begin{aligned}\phi_3(z) &= \frac{1+p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)''}{(\mathcal{L}_p^{\lambda}(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\mathcal{L}_p^{\lambda}(a,c)f(z)}\right\}}{-\left[\frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}+p\right]} \\ &= \frac{zw'(z)}{w(z)[p+(p-\alpha)w(z)]},\end{aligned} \tag{20}$$

and

$$\begin{aligned}\phi_4(z) &= \frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}\,\frac{1+p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)''}{(\mathcal{L}_p^{\lambda}(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\mathcal{L}_p^{\lambda}(a,c)f(z)}\right\}}{\left[\frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}+p\right]} \\ &= \frac{zw'(z)}{w(z)}.\end{aligned} \tag{21}$$

Now we claim that $|w(z)| < 1$ in $\mathbb{U}$. For otherwise there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1. \tag{22}$$

Then from Lemma 1 we find that

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1). \tag{23}$$

Therefore, letting $w(z_0) = e^{i\theta}$ in each of the equation (18) to (21), we obtain

$$|\phi_1(z_0)| = |(p-\alpha)z_0 w'(z_0)| = |(p-\alpha)ke^{i\theta}| \geq (p-\alpha), \tag{24}$$

$$|\phi_2(z_0)| = \left|\frac{(p-\alpha)z_0 w'(z_0)}{[p+(p-\alpha)w(z_0)]^2}\right| = \frac{|(p-\alpha)ke^{i\theta}|}{|p+(p-\alpha)e^{i\theta}|^2} \geq \frac{(p-\alpha)}{(2p-\alpha)^2}, \qquad (25)$$

$$\begin{aligned} |\phi_3(z_0)| &= \left|\frac{z_0 w'(z_0)}{w(z_0)[p+(p-\alpha)w(z_0)]}\right| \\ &= \left|\frac{k}{[p+(p-\alpha)e^{i\theta}]}\right| \geq \frac{1}{2p-\alpha}, \end{aligned} \qquad (26)$$

$$\Re\{\phi_4(z_0)\} = \Re\left\{\frac{z_0 w'(z_0)}{w(z_0)}\right\} = k \geq 1, \qquad (27)$$

which contradicts our assumption (12) to (15), respectively. Therefore, $|w(z)| < 1$ holds true for all $z \in \mathbb{U}$. Then (16) we have

$$\left|\frac{z^{p+1}(\mathcal{L}_p^{\lambda}(a,c)f(z))'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}} + p\right| = |(p-\alpha)w(z)| < (p-\alpha)$$

which implies that

$$f \in \mathcal{T}_p^{\lambda,\alpha}(\mu, a, c).$$

□

3 Consequences of main result

Putting $a = c,\ \lambda = -p+1,\ \mu = 1$ in Theorem 1, we get the following result:

Corollary 1 *Let the function* $f(z)$ *defined by* (1) *belong to the class* $\sum_p$. *If* $f(z)$ *satisfies any one of the following inequalities:*

$$\left|-\frac{f'(z)}{z^{-p-1}}\left(1+p+\frac{zf''(z)}{f'(z)}\right)\right| < p-\alpha,$$

$$\left|\frac{1+p+\frac{zf''(z)}{f'(z)}}{-\frac{f'(z)}{z^{-p-1}}}\right| < \frac{p-\alpha}{(2p-\alpha)^2},$$

$$\left|\frac{1+p+\frac{zf''(z)}{f'(z)}}{-\frac{f'(z)}{z^{-p-1}}-p}\right| < \frac{1}{2p-\alpha},$$

and

$$\Re\left\{\frac{-\frac{f'(z)}{z^{-p-1}}\left(1+p+\frac{zf''(z)}{f'(z)}\right)}{-\frac{f'(z)}{z^{-p-1}}-p}\right\} < 1,$$

then $f(z) \in \sum_p^c(\alpha)$.

Letting $p = 1$ in Corollary 1 we obtain the following result.

Corollary 2 *If* $f(z) \in \sum$ *satisfies any one of the following inequalities:*

$$\left|-\frac{f'(z)}{z^{-2}}\left(2+\frac{zf''(z)}{f'(z)}\right)\right| < 1-\alpha,$$

$$\left|\frac{2+\frac{zf''(z)}{f'(z)}}{-\frac{f'(z)}{z^{-2}}}\right| < \frac{1-\alpha}{(2-\alpha)^2},$$

$$\left|\frac{2+\frac{zf''(z)}{f'(z)}}{-\frac{f'(z)}{z^{-2}}-1}\right| < \frac{1}{2-\alpha},$$

and

$$\Re\left\{-\frac{f'(z)}{z^{-2}}\left(\frac{2+\frac{zf''(z)}{f'(z)}}{-\frac{f'(z)}{z^{-2}}-1}\right)\right\} < 1,$$

then $f(z) \in \sum_c(\alpha)$.

Further in the special case when $\alpha = 0$, Corollary 2 reduces to Corllary 3 stated below:

Corollary 3 *If* $f(z) \in \sum$ *satisfies anyone of the following inequalities:*

$$\left|-z^2f'(z)\left(2+\frac{zf''(z)}{f'(z)}\right)\right| < 1,$$

$$\left|-\frac{2+\frac{zf''(z)}{f'(z)}}{z^2f'(z)}\right| < \frac{1}{4},$$

$$\left|-\frac{2+\frac{zf''(z)}{f'(z)})}{z^2f'(z)+1}\right| < \frac{1}{2},$$

and

$$\Re\left\{\left(2+\frac{zf''(z)}{f'(z)}\right)\frac{z^2f'(z)}{z^2f'(z)+1}\right\} < 1,$$

then $f(z) \in \sum_c \quad (\equiv \sum_c(0))$.

Letting $a = c,\ \lambda = -p+1, \mu = 2$ in Theorem 12, we obtain the following:

Corollary 4 *If* $f \in \sum_p$ *given by (1) satisfies anyone of the following inequalities:*

$$\left|-\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < p-\alpha,$$
$$\left|-\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < \frac{p-\alpha}{(2p-\alpha)^2},$$
$$\left|1+\frac{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{-\frac{zf'(z)}{f(z)}-p}\right| < \frac{1}{2p-\alpha},$$

and

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{\frac{zf'(z)}{f(z)}+p}\right)\right\} < 1,$$

then $f(z) \in \sum_p^*(\alpha)$.

By putting $p = 1$ in Corollary 4, we have

Corollary 5 *If* $f \in \sum$ *satisfies anyone of the following inequalities:*

$$\left|-\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < 1-\alpha,$$
$$\left|-\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < \frac{1-\alpha}{(2-\alpha)^2},$$
$$\left|1+\frac{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{-\frac{zf'(z)}{f(z)}-1}\right| < \frac{1}{2-\alpha},$$

and

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{\frac{zf'(z)}{f(z)}+1}\right)\right\} < 1,$$

then $f(z) \in \sum^*(\alpha)$.

On further setting $\alpha = 0$ in Corollary 5, we get:

Corollary 6 *If* $f(z) \in \sum$ *satisfies any one of the following inequalities:*

$$\left|-\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < 1,$$

$$\left|-\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right|<\frac{1}{4},$$

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{-\frac{zf'(z)}{f(z)}-1}\right|<\frac{1}{2},$$

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{\frac{zf'(z)}{f(z)}+1}\right)\right\}<1,$$

then $f(z) \in \sum^*$.

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Alternative proofs of some formulas for two tridiagonal determinants

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Abstract. In the paper, the authors provide five alternative proofs of two formulas for a tridiagonal determinant, supply a detailed proof of the inverse of the corresponding tridiagonal matrix, and provide a proof for a formula of another tridiagonal determinant. This is a companion of the paper [F. Qi, V. Čerňanová, and Y. S. Semenov, *Some tridiagonal determinants related to central Delannoy numbers, the Chebyshev polynomials, and the Fibonacci polynomials*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **81** (2019), in press.

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1 Introduction

For $c \in \mathbb{C}$ and $k \in \mathbb{N}$, define the $k \times k$ tridiagonal matrix $M_k(c)$ by

$$M_k(c) = \begin{pmatrix} c & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & c & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & c & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & c & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & c \end{pmatrix}_{k\times k}$$

and denote the determinant $|M_k(c)|$ of the $k \times k$ tridiagonal matrix $M_k(c)$ by $D_k(c)$. In [7, Remark 4.4], the explicit expression

$$D_k(-6) = \frac{1}{6^k} \sum_{\ell=0}^{k} (-1)^\ell 6^{2\ell} \binom{\ell}{k-\ell}$$

was derived from some results in [7, Theorem 1.2] for the Cauchy products of central Delannoy numbers, where $\binom{p}{q} = 0$ for $q > p \geq 0$. For information on central Delannoy numbers, please refer to the papers [6, 7] and plenty of references cited therein. In [7, Remar 4.4], the authors guessed that the explicit formula

$$D_k(c) = (-1)^k \sum_{\ell=0}^{k} (-1)^\ell c^{2\ell-k} \binom{\ell}{k-\ell} = \sum_{m=0}^{k} (-1)^m c^{k-2m} \binom{k-m}{m} \qquad (1)$$

should be valid for all $c \in \mathbb{C}$ and $k \in \mathbb{N}$ and claimed that the equality (1) can be verified by induction on $k \in \mathbb{N}$ straightforwardly.

In the paper [6], the authors discovered a generating function of the sequence $D_k(c)$, provided an analytic proof of the explicit formula (1), established a simple formula for computing the tridiagonal determinant $D_k(c)$, found a determinantal expression for $D_k(c)$, presented the inverse of the symmetric tridiagonal matrix $M_k(c)$, connected $D_k(c)$ with the Chebyshev polynomials [6, 9, 11] and the Fibonacci numbers and polynomials [1, 6, 8], reviewed computation of general diagonal determinants, supplied two new formulas for computing general diagonal determinants, generalized central Delannoy numbers [6, 7], and represented the Cauchy product of the generalized central Delannoy numbers [6] in terms of $D_k(c)$.

In this paper, we pay our attention on the following four conclusions.

Theorem 1 ([6, Theorem 2.2]) *For* $k \geq 0$ *and* $c \in \mathbb{C}$*, the formula* (1) *is valid.*

Theorem 2 ([6, Theorem 3.1]) *For* $c \in \mathbb{C}$, $\alpha = \frac{1}{\beta} = \frac{c+\sqrt{c^2-4}}{2}$*, and* $k \geq 0$*, the tridiagonal determinant* $D_k(c)$ *can be computed by*

$$D_k(c) = \begin{cases} \dfrac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}, & c \neq \pm 2; \\ k+1, & c = 2; \\ (-1)^k(k+1), & c = -2. \end{cases} \tag{2}$$

Theorem 3 ([6, Theorem 5.1]) *For* $k \in \mathbb{N}$*, the inverse of the symmetric tridiagonal matrix* $M_k(c)$ *can be computed by* $M_k^{-1}(c) = \left(R_{ij}\right)_{k\times k}$*, where*

$$R_{ij} = \begin{cases} -\dfrac{\left(\lambda^i - \mu^i\right)\left(\lambda^{k-j+1} - \mu^{k-j+1}\right)}{(\lambda - \mu)(\lambda^{k+1} - \mu^{k+1})}, & c \neq \pm 2 \\ (-1)^{i+j}\dfrac{i(k-j+1)}{k+1}, & c = 2 \\ -\dfrac{i(k-j+1)}{k+1}, & c = -2 \end{cases}$$

for $i < j$, $R_{ij} = R_{ji}$ *for* $i > j$*, and* λ *and* μ *are defined by*

$$\lambda = \frac{1}{\mu} = \frac{2}{\sqrt{c^2-4} - c} = -\alpha = -\frac{1}{\beta}.$$

Theorem 4 ([6, Section 8]) *For* $n \in \mathbb{N}$ *and* $a, b, c \in \mathbb{C}$*, we have*

$$D_n = \begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{vmatrix}_{n\times n}$$

$$= \begin{cases} \dfrac{\left(a + \sqrt{a^2-4bc}\,\right)^{n+1} - \left(a - \sqrt{a^2-4bc}\,\right)^{n+1}}{2^{n+1}\sqrt{a^2-4bc}}, & a^2 \neq 4bc; \\ (n+1)\left(\dfrac{a}{2}\right)^n, & a^2 = 4bc. \end{cases} \tag{3}$$

In Section 2 of this paper, we will supply two alternative proofs of Theorem 1. In Section 3, we will provide three alternative proofs of Theorem 2. In Section 4, we will present a detailed proof of Theorem 3. In Section 5, we will provide a proof of Theorem 4. In the last section of this paper, we will list several remarks.

2 Two alternative proofs of Theorem 1

Now we are in a position to supply two alternative proofs of Theorem 1.

Proof. [First alternative proof of Theorem 1] Let $D_0(c) = 1$. Theorem 2.1 in [6] states that the sequence $D_k(c)$ for $k \ge 0$ can be generated by

$$F_c(t) = \frac{1}{t^2 - ct + 1} = \sum_{k=0}^{\infty} D_k(c) t^k. \tag{4}$$

By the formula for the sum of a geometric progression, the generating function $F_c(t)$ can be expanded as

$$F_c(t) = \sum_{\ell=0}^{\infty} (-1)^{\ell} \left(t^2 - ct\right)^{\ell} = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} c^{\ell-m} t^{\ell+m} \tag{5}$$

for $\left|t^2 - ct\right| < 1$. Hence, it follows for $k \ge 0$ that

$$\begin{aligned} [F_c(t)]^{(k)} &= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} c^{\ell-m} \left(t^{\ell+m}\right)^{(k)} \\ &\to \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} c^{\ell-m} \lim_{t \to 0} \left(t^{\ell+m}\right)^{(k)} \\ &= (-1)^k k! \sum_{\ell=0}^{k} (-1)^{\ell} \binom{\ell}{k-\ell} c^{2\ell-k} \end{aligned}$$

for $\left|t^2 - ct\right| < 1$ and as $t \to 0$. The formula (1) is thus proved. □

Proof. [Second alternative proof of Theorem 1] Taking $k = \ell + m$ in (5) leads to

$$F_c(t) = \sum_{k=0}^{\infty} \left[\sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{\ell}{k-\ell} c^{2\ell-k} \right] t^k = \sum_{k=0}^{\infty} D_k(c) t^k$$

for $\left|t^2 - ct\right| < 1$. The formula (1) is proved again. The proof of Theorem 1 is complete. □

3 Three alternative proofs of Theorem 2

We now start out to provide three alternative proofs of Theorem 2.

Proof. [First alternative proof of Theorem 2] It is clear that the generating function $F_c(t)$ in (4) can be rewritten as $F_c(t) = \frac{1}{t-\alpha}\frac{1}{t-\beta}$. By virtue of the Leibniz theorem for the product of two functions, we have

$$[F_c(t)]^{(k)} = \left(\frac{1}{t-\alpha}\frac{1}{t-\beta}\right)^{(k)} = \sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{1}{t-\alpha}\right)^{(\ell)}\left(\frac{1}{t-\beta}\right)^{(k-\ell)}$$
$$= \sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^{\ell}\ell!}{(t-\alpha)^{\ell+1}}\frac{(-1)^{k-\ell}(k-\ell)!}{(t-\beta)^{k-\ell+1}} \to \sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^{\ell}\ell!}{(-\alpha)^{\ell+1}}\frac{(-1)^{k-\ell}(k-\ell)!}{(-\beta)^{k-\ell+1}}$$
$$= k!\sum_{\ell=0}^{k}\frac{1}{\alpha^{\ell+1}}\frac{1}{\beta^{k-\ell+1}} = \frac{k!}{\beta^k}\sum_{\ell=0}^{k}\left(\frac{\beta}{\alpha}\right)^{\ell} = \frac{k!}{\beta^k}\frac{1-(\beta/\alpha)^{k+1}}{1-\beta/\alpha} = k!\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}$$

as $t \to 0$. The formula (2) is thus proved. □

Proof. [Second alternative proof of Theorem 2] The generating function $F_c(t)$ can also be rewritten as

$$F_c(t) = \frac{1}{\alpha-\beta}\left(\frac{1}{t-\alpha} - \frac{1}{t-\beta}\right). \tag{6}$$

Then a straightforward computation reveals

$$[F_c(t)]^{(k)} = \frac{1}{\alpha-\beta}\left[\frac{(-1)^k k!}{(t-\alpha)^{k+1}} - \frac{(-1)^k k!}{(t-\beta)^{k+1}}\right]$$
$$\to -k!\frac{1}{\alpha-\beta}\left(\frac{1}{\alpha^{k+1}} - \frac{1}{\beta^{k+1}}\right) = k!\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}$$

as $t \to 0$. The proof of Theorem 2 is complete. □

Proof. [Third alternative proof of Theorem 2] The formula for the sum of a geometric progression yields

$$\frac{1}{t-\alpha} = -\sum_{k=0}^{\infty}\frac{t^k}{\alpha^{k+1}} \quad \text{and} \quad \frac{1}{t-\beta} = -\sum_{k=0}^{\infty}\frac{t^k}{\beta^{k+1}}$$

for $|t| < \min\{|\alpha|, |\beta|\}$. Thus, in view of $\alpha\beta = 1$ and (6), we obtain

$$F_c(t) = \frac{1}{\alpha-\beta}\sum_{k=0}^{\infty}\left(\frac{1}{\beta^{k+1}} - \frac{1}{\alpha^{k+1}}\right)t^k = \sum_{k=0}^{\infty}\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}t^k = \sum_{k=0}^{\infty}D_k(c)t^k$$

for $|t| < \min\{|\alpha|, |\beta|\}$. The formula (2) is thus proved. The proof of Theorem 2 is complete. □

4 A detailed proof of Theorem 3

We now present a detailed proof of Theorem 3.

In the paper [2], the inverse of the symmetric tridiagonal matrix $M_k(c)$ was discussed. We denote the inverse matrix of $M_k(c)$ by $M_k^{-1}(c) = (R_{ij})_{k\times k}$. Then, basing on discussions in [2, Eq. (9)], one can see without difficulty that the elements R_{ij} can be represented as

$$R_{ij} = (-1)^{i+j}\frac{D_{i-1}(c)D_{k-j}(c)}{D_k(c)}, \quad 1 \le i < j \le k$$

and $R_{ij} = R_{ji}$ for $1 \le j < i \le k$. Making use of the formula (2) yields

$$\begin{aligned}
R_{ij} &= \begin{cases} (-1)^{i+j}\dfrac{\dfrac{\alpha^{i-1+1}-\beta^{i-1+1}}{\alpha-\beta}\,\dfrac{\alpha^{k-j+1}-\beta^{k-j+1}}{\alpha-\beta}}{\dfrac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}}, & c \ne \pm 2\\ (-1)^{i+j}\dfrac{(\pm1)^{i-1}(i-1+1)(\pm1)^{k-j}(k-j+1)}{(\pm1)^k(k+1)}, & c = \pm 2 \end{cases}\\
&= \begin{cases} (-1)^{i+j}\dfrac{\left(\alpha^i-\beta^i\right)\left(\alpha^{k-j+1}-\beta^{k-j+1}\right)}{(\alpha-\beta)(\alpha^{k+1}-\beta^{k+1})}, & c \ne \pm 2\\ (-1)^{i+j}(\pm1)^{i-j-1}\dfrac{i(k-j+1)}{k+1}, & c = \pm 2 \end{cases}\\
&= \begin{cases} -\dfrac{\left[(-\alpha)^i-(-\beta)^i\right]\left[(-\alpha)^{k-j+1}-(-\beta)^{k-j+1}\right]}{[(-\alpha)-(-\beta)][(-\alpha)^{k+1}-(-\beta)^{k+1}]}, & c \ne \pm 2\\ (-1)^{i+j}\dfrac{i(k-j+1)}{k+1}, & c = 2\\ -\dfrac{i(k-j+1)}{k+1}, & c = -2 \end{cases}\\
&= \begin{cases} -\dfrac{\left(\lambda^i-\mu^i\right)\left(\lambda^{k-j+1}-\mu^{k-j+1}\right)}{(\lambda-\mu)(\lambda^{k+1}-\mu^{k+1})}, & c \ne \pm 2\\ (-1)^{i+j}\dfrac{i(k-j+1)}{k+1}, & c = 2\\ -\dfrac{i(k-j+1)}{k+1}, & c = -2 \end{cases}
\end{aligned}$$

for $1 \le i < j \le k$. The proof of Theorem 3 is complete.

5 A proof of Theorem 4

The determinant D_n satisfies the recurrence relation $D_n = aD_{n-1} - bcD_{n-2}$. Solving the equation $x^2 - ax + bc = 0$ reaches to two roots $\alpha = \frac{a+\sqrt{a^2-4bc}}{2}$ and $\beta = \frac{a-\sqrt{a^2-4bc}}{2}$. These two roots satisfy $\alpha+\beta = a$ and $\alpha\beta = bc$. Then by the above recurrence relation one can write

$$\begin{aligned} D_n - \alpha D_{n-1} &= \beta[D_{n-1} - \alpha D_{n-2}] = \beta^2[D_{n-2} - \alpha D_{n-3}] = \cdots \\ &= \beta^{n-2}[D_2 - \alpha D_1] = \beta^{n-2}[(a^2 - bc) - \alpha a] = \beta^n. \end{aligned}$$

Similarly, one can deduce that $D_n - \beta D_{n-1} = \alpha^n$. Accordingly, when $\alpha \neq \beta$, that is, $a^2 \neq 4bc$, one finds $(\alpha - \beta)D_n = \alpha^{n+1} - \beta^{n+1}$, that is,

$$D_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{\left(a + \sqrt{a^2 - 4bc}\,\right)^{n+1} - \left(a - \sqrt{a^2 - 4bc}\,\right)^{n+1}}{2^{n+1}\sqrt{a^2 - 4bc}}.$$

When $\alpha = \beta$, that is, $a^2 = 4bc$, we have

$$\begin{aligned} D_n &= \alpha^n + \alpha D_{n-1} = \alpha^n + \alpha(\alpha^{n-1} + \alpha D_{n-2}) = \cdots = (n-1)\alpha^n + \alpha^{n-1}D_1 \\ &= (n-1)\alpha^n + \alpha^{n-1}(2\alpha) = (n+1)\alpha^n = (n+1)\left(\frac{a}{2}\right)^n. \end{aligned}$$

The formula (3) is thus proved. The proof of Theorem 4 is complete.

6 Several remarks

Finally, we list several remarks on tridiagonal determinants.

Remark 1 *The identities*

$$\mathcal{D}_k(c) \triangleq \begin{vmatrix} -c & 1 & 0 & \cdots & 0 & 0 & 0 \\ 2 & -2c & 1 & \cdots & 0 & 0 & 0 \\ 0 & 6 & -3c & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -(k-2)c & 1 & 0 \\ 0 & 0 & 0 & \cdots & (k-1)(k-2) & -(k-1)c & 1 \\ 0 & 0 & 0 & \cdots & 0 & k(k-1) & -kc \end{vmatrix}$$

$$= (-1)^k k! \begin{vmatrix} c & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & c & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & c & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & c & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & c \end{vmatrix}_{k\times k}$$

$$= \frac{k!}{c^k} \sum_{\ell=0}^{k} (-1)^\ell c^{2\ell} \binom{\ell}{k-\ell} = \begin{cases} k! \dfrac{\lambda^{k+1} - \mu^{k+1}}{\lambda - \mu}, & c \ne \pm 2 \\ (-1)^k (k+1)!, & c = 2 \\ (k+1)!, & c = -2 \end{cases}$$

are neither trivial nor obvious, where $\lambda = \frac{1}{\mu} = \frac{2}{\sqrt{c^2-4}-c} = -\alpha = -\frac{1}{\beta}$. *The determinant* $\mathcal{D}_k(c)$ *satisfies*

$$\mathcal{D}_0(c) = 1, \quad \mathcal{D}_1(c) = -c, \quad \mathcal{D}_2(c) = 2(c^2 - 1),$$

and

$$\mathcal{D}_k(c) = -kc\mathcal{D}_{k-1}(c) - k(k-1)\mathcal{D}_{k-2}(c), \quad k \ge 2. \tag{7}$$

Then, if letting $\mathcal{F}_c(t) = \sum_{k=0}^{\infty} \mathcal{D}_k(c) t^k$, *we have*

$$\sum_{k=2}^{\infty} \mathcal{D}_k(c) t^k = -ct \sum_{k=2}^{\infty} k\mathcal{D}_{k-1}(c) t^{k-1} - t^2 \sum_{k=2}^{\infty} k(k-1)\mathcal{D}_{k-2}(c) t^{k-2},$$

$$\sum_{k=0}^{\infty} \mathcal{D}_k(c) t^k - \mathcal{D}_0(c) - \mathcal{D}_1(c) t = -ct \sum_{k=1}^{\infty} (k+1)\mathcal{D}_k(c) t^k - t^2 \sum_{k=0}^{\infty} (k+2)(k+1)\mathcal{D}_k(c) t^k,$$

$$\mathcal{F}_c(t) - 1 + ct = -ct \frac{d}{dt}\left[\sum_{k=1}^{\infty} \mathcal{D}_k(c) t^{k+1}\right] - t^2 \frac{d^2}{dt^2}\left[\sum_{k=0}^{\infty} \mathcal{D}_k(c) t^{k+2}\right],$$

$$\mathcal{F}_c(t) - 1 + ct = -ct \frac{d}{dt}\left[t \sum_{k=1}^{\infty} \mathcal{D}_k(c) t^{k}\right] - t^2 \frac{d^2}{dt^2}\left[t^2 \sum_{k=0}^{\infty} \mathcal{D}_k(c) t^{k}\right],$$

$$\mathcal{F}_c(t) - 1 + ct = -ct \frac{d}{dt}[t(\mathcal{F}_c(t) - 1)] - t^2 \frac{d^2}{dt^2}\left[t^2 \mathcal{F}_c(t)\right],$$

$$t^4 \mathcal{F}_c''(t) + t^2(4t + c)\mathcal{F}_c'(t) + \left(2t^2 + ct + 1\right)\mathcal{F}_c(t) - 1 = 0.$$

This means that the generating function of the sequence $\mathcal{D}_k(c) = (-1)^k k! D_k(c)$ *is the solution of the second order linear ordinary differential equation*

$$t^4 f''(t) + t^2(4t + c)f'(t) + \big(2t^2 + ct + 1\big)f(t) - 1 = 0$$

with initial values $f(0) = 1$ *and* $f'(0) = -c$*. This differential equation is solvable, but its solution is not elementary.*

Remark 2 *The method used in the proof of* [6, Theorem 3.1] *can not be applied to the sequence* $\mathcal{D}_k(c)$*, since its recurrence relation* (7) *is not a homogeneous linear recurrence relation with constant coefficients.*

Remark 3 *The central Delannoy numbers* $D(k)$ *were generalized in* [10] *as*

$$D_{a,b}(k) = \frac{1}{\pi}\int_a^b \frac{1}{\sqrt{(t-a)(b-t)}}\frac{1}{t^{k+1}}\,\mathrm{d}\,t, \quad k \ge 0, \quad b > a > 0$$

and, by [7, Lemma 2.4]*, we find that* $D_{a,b}(k)$ *can be generated by*

$$\frac{1}{\sqrt{(x+a)(x+b)}} = \sum_{k=0}^{\infty} D_{a,b}(k)x^k.$$

By virtue of conclusions in [4, Section 2.4] *and* [3, Remark 4.1]*, the generalized central Delannoy numbers* $D_{a,b}(k)$ *for* $k \ge 0$ *can be computed by*

$$D_{a,b}(k) = \frac{1}{a^{k+1}}\,{}_2F_1\left(k+1, \frac{1}{2}; 1; 1 - \frac{b}{a}\right), \quad 2a > b > a > 0, \quad k \ge 0,$$

where ${}_2F_1$ *is the classical hypergeometric function which is a special case of the generalized hypergeometric series*

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!}$$

for complex numbers $a_i \in \mathbb{C}$ *and* $b_i \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$*, for positive integers* $p, q \in \mathbb{N}$*, and for*

$$(x)_\ell = \begin{cases} \prod_{k=0}^{\ell-1}(x+k), & \ell \ge 1 \\ 1, & \ell = 0 \end{cases}$$

which is called the rising factorial of $x \in \mathbb{R}$.

Remark 4 *This paper and* [6] *are extracted from different parts of the preprint* [5].

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Computing metric dimension of compressed zero divisor graphs associated to rings

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Abstract. For a commutative ring R with $1 \neq 0$, a compressed zero-divisor graph of a ring R is the undirected graph $\Gamma_E(R)$ with vertex set $Z(R_E) \setminus \{[0]\} = R_E \setminus \{[0],[1]\}$ defined by $R_E = \{[x] : x \in R\}$, where $[x] = \{y \in R : ann(x) = ann(y)\}$ and the two distinct vertices $[x]$ and $[y]$ of $Z(R_E)$ are adjacent if and only if $[x][y] = [xy] = [0]$, that is, if and only if $xy = 0$. In this paper, we study the metric dimension of the compressed zero divisor graph $\Gamma_E(R)$, the relationship of metric dimension between $\Gamma_E(R)$ and $\Gamma(R)$, classify the rings with same or different metric dimension and obtain the bounds for the metric dimension of $\Gamma_E(R)$. We provide a formula for the number of vertices of the family of graphs given by $\Gamma_E(R \times \mathbb{F})$. Further, we discuss the relationship between metric dimension, girth and diameter of $\Gamma_E(R)$.

1 Introduction

Beck [7] first introduced the notion of a zero divisor graph of a ring R and his interest was mainly in coloring of zero divisor graphs. Anderson and Livingston [3] studied zero divisor graph of non-zero zero divisors of a commutative ring

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R. For a commutative ring R with $1 \neq 0$, let $Z^*(R) = Z(R) \setminus \{0\}$ be the set of non-zero zero divisors of R. A *zero divisor graph* $\Gamma(R)$ is the undirected graph with vertex set $Z^*(R)$ and the two vertices x and y are adjacent if and only if $xy = 0$. This zero divisor graph has been studied extensively and even more the idea has been extended to the ideal based zero divisor graphs in [15, 23] and modules in [20]. Inspired by ideas from Mulay [16], we study the zero divisor graph of equivalence classes of zero divisors of a ring R. Anderson and LaGrange [4] studied this under the term compressed zero divisor graph $\Gamma_E(R)$ with vertex set $Z(R_E) \setminus \{[0]\} = R_E \setminus \{[0], [1]\}$, constructed by taking the vertices to be equivalence classes $[x] = \{y \in R \mid ann(x) = ann(y)\}$, for every $x \in R \setminus ([0] \cup [1])$ and each pair of distinct classes $[x]$ and $[y]$ is joined by an edge if and only if $[x][y] = 0$, that is, if and only if $xy = 0$. If x and y are distinct adjacent vertices in $\Gamma(R)$, we note that $[x]$ and $[y]$ are adjacent in $\Gamma_E(R)$ if and only if $[x] \neq [y]$. It is clear that $[0] = \{0\}$ and $[1] = R \setminus Z(R)$ and that $[x] \subseteq Z(R) \setminus \{0\}$, for each $x \in R \setminus ([0] \cup [1])$. Some results on the compressed zero divisor graph can be seen in [5].

For example, consider $R = \mathbb{Z}_{12}$. Here, $Z^*(R) = \{2, 3, 4, 6, 8, 9, 10\}$ is the vertex set of $\Gamma(R)$, see Fig 1(a). For the vertex set of $\Gamma_E(R)$, we have
$ann(2) = \{6\}$, $ann(3) = \{4, 8\}$, $ann(4) = \{3, 6, 9\}$, $ann(6) = \{2, 4, 6, 8, 10\}$, $ann(8) = \{3, 6, 9\}$, $ann(9) = \{4, 8\}$, $ann(10) = \{6\}$.

So, $Z(R_E) = \{[2], [3], [4], [6]\}$ is the vertex set of $\Gamma_E(R)$, see Fig 1(b).

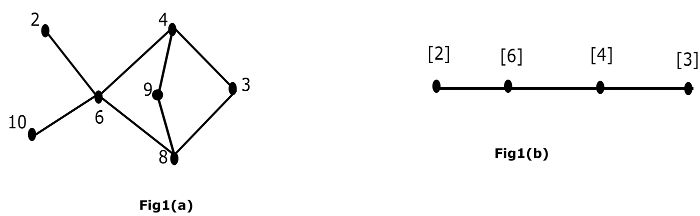

Figure 1: $\Gamma(\mathbb{Z}_{12})$ and $\Gamma_E(\mathbb{Z}_{12})$

We note that the vertices of the graph $\Gamma_E(R)$ correspond to *annihilator* ideals in the ring and hence *prime* ideals if R is a Noetherian ring in which case $Z(R_E)$ is called as the *spectrum* of a ring. Clearly $\Gamma_E(R)$ is connected and $diam(\Gamma_E(R)) \leq 3$. Also $diam(\Gamma_E(R)) \leq diam(\Gamma(R))$. Anderson and LaGrange [5] showed that $gr(\Gamma_E(R)) \leq 3$ if $\Gamma_E(R)$ contains a cycle and determined the structure of $\Gamma_E(R)$ when it is acyclic and the monoids R_E when $\Gamma_E(R)$ is a star graph. In [4], they also show that $\Gamma_E(R) \cong \Gamma_E(S)$ for a Noetherian or finite commutative ring S.

The compressed zero-divisor graph has some advantages over the earlier studied zero divisor graph $\Gamma(R)$ as seen in [1, 2, 3] or subsequent zero divisor graph determined by ideal of R as seen in [15, 23]. For example, Spiroff and Wickham [[27], Proposition 1.10] showed that there are no finite regular graphs $\Gamma_E(R)$ for any ring R with more than two vertices. Further, they showed that R is a local ring (a ring R is said to be a local ring if it has a unique maximal ideal) if $\Gamma_E(R)$ is a star graph with at least four vertices.

Another important aspect of studying graphs of equivalence classes is the connection to associated primes of the ring. In general, all the associated primes of a ring R correspond to distinct vertices in $\Gamma_E(R)$. Through out, R will denote a commutative ring with unity, $U(R)$ its set of units. We will denote a finite field on q elements by $\mathbb{F}_q$, ring of integers modulo n by $\mathbb{Z}_n$ and all graphs are simple graphs in the sense that there are no loops. For basic definitions from graph theory we refer to [11, 17], and for commutative ring theory we refer to [6, 13].

A graph G is *connected* if there exists a path between every pair of vertices in G. The *distance* between two vertices u and v in G, denoted by $d(u, v)$, is the length of the shortest $u - v$ path in G. If such a path does not exist, we define $d(u, v)$ to be infinite. The *diameter* of a graph is the maximum distance between any two vertices of G. The diameter is 0 if the graph consists of a single vertex. Also, the *girth* of a graph G, denoted by $gr(G)$, is the length of a smallest cycle in G. Slater [25] introduced the concept of a *resolving set* for a connected graph G under the term locating set. He referred to a minimum resolving set as a reference set for G and called the cardinality of a minimum resolving set (reference set) the location number of G. Independently, Harary and Melter [12] discovered these concepts as well but used the term metric dimension, rather than location number. The concept of metric dimension has appeared in various applications of graph theory, as diverse as, pharmaceutical chemistry [8, 9], robot navigation [14], combinatorial optimization [24], sonar and coast guard Loran [26]. We adopt the terminology of Harary and Melter.

In this paper, we study the notion of metric dimension of $\Gamma_E(R)$. We explore the relationship between metric dimension of $\Gamma_E(R)$ and $\Gamma_E(R)$. We obtain the metric dimension of $\Gamma_E(R)$ whenever it exists. We also classify the rings having the same or different metric dimension and obtain bounds for the metric dimension of $\Gamma_E(R)$. We also provide relationship between the metric dimension, girth and diameter of $\Gamma_E(R)$.

2 Metric dimension of some graphs $\Gamma_E(R)$

Let G be a connected graph with $n \geq 2$ vertices. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$, we refer to the k-vector as the metric representation (locating code) of v with respect to W as

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

The set W is a resolving set of G if distinct vertices have distinct metric representations (codes) and a resolving set containing the minimum number of vertices is called a *metric basis* for G and the *metric dimension*, denoted by $\dim(G)$, of G is the cardinality of a metric basis. If W is a finite metric basis, we say that $r(v|W)$ are the metric coordinates of vertex v with respect to W. The only vertex of G whose metric coordinate with respect to W has 0 in its i^{th} coordinate of $r(v|W)$ is $\{w_i\}$. So the vertices of W necessarily have distinct metric representations. Since only those vertices of G that are not in W have coordinates all of which are positive, it is only these vertices that need to be examined to determine if their representations are distinct. This implies that the metric dimension of G is at most $n-1$. In fact for every connected graph G of order $n \geq 2$, we have $1 \leq \dim(G) \leq n-1$.

For example, consider the graph G given in Figure 2. Take $W_1 = \{v_1, v_3\}$. So, $r(v_1|W_1) = (0, 1)$, $r(v_2|W_1) = (1, 1)$, $r(v_3|W_1) = (1, 0)$, $r(v_4|W_1) = (1, 1)$, $r(v_5|W_1) = (2, 1)$. Notice, $r(v_2|W_1) = (1, 1) = r(v_4|W_1)$, therefore W_1 is not a resolving set. However, if we take $W_2 = \{v_1, v_2\}$, then $r(v_1|W_2) = (0, 1)$, $r(v_2|W_2) = (1, 0)$, $r(v_3|W_2) = (1, 1)$, $r(v_4|W_2) = (1, 2)$, $r(v_5|W_2) = (2, 1)$. Since distinct vertices have distinct metric representations, W_2 is a minimum resolving set and thus this graph has metric dimension 2.

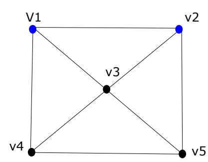

Figure 2: $\dim(G) = 2$

Now, we have the following observation.

Lemma 1 *A connected graph* G *of order* n *has metric dimension* 1 *if and only if* $G \cong P_n$*, where* P_n *denotes a path on* n *vertices of length* $n-1$*.*

Proof. Suppose $G \cong P_n$. Let $x_1 - x_2 - \cdots - x_n$ be a path on n vertices of G. Since $d(x_i, x_1) = i-1$ for $1 \le i \le n$, it follows $\{x_1\}$ is a minimum resolving set and therefore metric basis for $\Gamma_E(R)$. So $dim(P_n) = 1$.

Conversely, let G be not a path. Then either G is a cycle or it contains a vertex v whose degree is at least 3. But, G can not be a cycle as $dim(G) = 2$, see ([18], Lemma 2.3). Let u_1, u_2, $\ldots$, u_k be the vertices adjacent to v. Since $dim(G) = 1$ and if $W = \{w\}$ is a metric basis for G, then the metric representation of every vertex has a single coordinate. If d is the length of the shortest path from v to w, the coordinates of each u_i with respect to W is one of $\{d-1, d, d+1\}$, but $d(u_i, w) = d$ can not occur for all i $(1 \le i \le k)$. Therefore, it follows that at least two adjacent vertices of v have the same metric coordinates, which is a contradiction. Hence G is a path. □

A graph $G(V, E)$ in which each pair of distinct vertices is joined by an edge is called a complete graph. A *complete* graph of n vertices is denoted by K_n. A graph G is said to be *bipartite* if its vertex set V can be partitioned into two sets V_1 and V_2 such that every edge of G has one end in V_1 and another in V_2. A bipartite graph is complete if each vertex of one partite set is joined to every vertex of the other partite set. We denote the complete bipartite graph with partite sets of order m and n by $K_{m,n}$. More generally, a graph is complete r-partite if the vertices can be partitioned into r distinct subsets, but no two elements of the same subset are adjacent. Based on the above definitions, we have the following observations.

Proposition 1 *The metric dimension of the compressed zero divisor graph $\Gamma_E(R)$ is 0 if and only if the zero divisor graph $\Gamma(R)$ of R ($R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$) is a complete graph.*

Proof. If $\Gamma(R) \cong K_n$, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z^*(R)$. Let $v_1, v_2, \ldots, v_n$ be the zero divisors of $\Gamma(R)$, then $[v_1] = [v_2], \cdots = [v_n]$ implies that all the vertices of $\Gamma(R)$ would collapse to a single vertex in $\Gamma_E(R)$ and we know the metric dimension of a single vertex graph is 0.

Conversely, assume that $\Gamma(R)$ is not isomorphic to K_n. Then $\Gamma(R)$ contains at least one vertex not adjacent to all the other vertices. Thus $|\Gamma_E(R)| \ge 2$, so that $dim(\Gamma_E(R)) \ge 1$. □

We can also obtain the converse part by letting $dim(\Gamma_E(R)) = 0$. Then $\Gamma_E(R) = \{[a]\}$ for some $a \in Z^*(R)$, that is, $\Gamma_E(R)$ is a graph on a single vertex, which then implies $\Gamma(R)$ is either isomorphic to a single vertex or a complete graph K_n, for all $n \ge 1$. If G is a connected graph of order $n \ge 2$, we say two distinct vertices u and v are distance similar, if $d(u, a) = d(v, a)$ for all

$a \in V(G) - \{u, v\}$. It can be seen that the distance similar relation $(\sim)$ is an equivalence relation on $V(G)$ and two distinct vertices are distance similar if either $uv \notin E(G)$ and $N(u) = N(v)$, or $uv \in E(G)$ and $N[u] = N[v]$. Further we can find several results on metric dimension for zero divisor graphs of rings in [18, 19, 21].

Proposition 2 *The metric dimension of* $\Gamma_E(R)$ *is* 1 *if* $\Gamma(R)$ *is isomorphic to a complete bipartite graph* $K_{m,n}$*, with* m *or* $n \geq 2$.

Proof. Let $\Gamma(R)$ be isomorphic to a complete bipartite graph $K_{m,n}$ with two distance similar classes V_1 and V_2. Let $V_1 = \{u_1, u_2, \cdots, u_m\}$ and $V_2 = \{v_1, v_2, \cdots, v_n\}$ such that $u_i v_j = 0$ for all $i \neq j$. Clearly, each of V_1 and V_2 is an independent set. We see that $[u_1] = [u_2] = \cdots = [u_m]$ and $[v_1] = [v_2] = \cdots = [v_n]$, so that V_1 and V_2 each represents a single vertex in $\Gamma_E(R)$. Since the graph is connected, $\Gamma_E(R)$ is isomorphic to $K_{1,1}$, a path on two vertices. Therefore by Lemma 1, we have $\dim(\Gamma_E(R)) = 1$. □

Remark 1 *Note that the converse of this result need not be true, the graph illustrated in Fig.1 being a counter example. However, if* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$*, then* $\Gamma_E(R) \cong K_{1,1}$ *with metric dimension* 1 *and* $\Gamma(R) \cong \Gamma_E(R)$.

One of the important differences between $\Gamma(R)$ and $\Gamma_E(R)$ is that the later can not be complete with at least three vertices, as seen in ([27], Proposition 1.5). However, if $\Gamma_E(R)$ is complete r-partite, then $r = 2$ and $\Gamma_E(R) \cong K_{n,1}$, for some $n \geq 1$, see ([27], Proposition 1.7). A second look at the above result allows us to deduce some facts about star graphs. A complete bipartite graph of the form $K_{n,1}, n \in \mathbb{N} \cup \{\infty\}$ is called a *star graph*. If $n = \infty$, we say the graph is an infinite star graph.

Corollary 1 *If* R *is a ring such that* $\Gamma_E(R)$ *is a star graph* $K_{n,1}$ *with* $n \geq 2$*, then* $\dim(\Gamma_E(R)) = n - 1$.

Proof. First we identify a centre vertex of $K_{n,1}$ adjacent to n vertices. Then partition the vertex set V of order $n+1$ into two distance similar classes, with centre vertex in one class V_1 and the remaining n vertices in another class V_2 which is clearly an independent set. Choose a subset of vertices W of V and $u \sim v$. Then $r(u|W) = r(v|W)$ whenever both $u, v \notin W$. Hence the metric basis contains all except at most two vertices one from each class V_i, $1 \leq i \leq 2$. Therefore, $\dim(K_{n,1}) = |V(\Gamma_E(R))| - 2 = n + 1 - 2 = n - 1$. □

For example the metric dimension of $K_{1,3}$ is 2, see Figure 3.

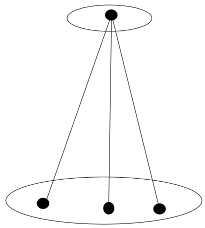

Figure 3: $\dim(K_{1,3} = 2)$

Corollary 2 *If* R *is a commutative ring such that* $\Gamma_E(R)$ *has at least* $n \geq 3$ *vertices, then* $\dim(\Gamma_E(R)) \neq n-1$*.*

Proof. Suppose $\dim(\Gamma_E(R)) = n-1$, $(n \geq 3)$. Then, by [18, Lemma 2.2], $\Gamma_E(R)$ is a complete graph on n vertices which is a contradiction to the argument prior to Corollary 1. Therefore $\dim(\Gamma_E(R)) \neq n-1$. □

Remark 2 *It is not known whether for each positive integer* n*, the star graph* $K_{n,1}$ *can be realized as* $\Gamma_E(R)$ *for some ring* R*. However, there is a ring* $R = \mathbb{Z}_2[x,y,z]/(x^2,y^2)$ *whose* $\Gamma_E(R)$ *is a star graph with infinitely many ends, that is,* $\Gamma_E(R)$ *is an infinite star graph. This ring also shows that the Noetherian condition is not enough to force* $\Gamma_E(R)$ *to be finite, see [27]. For* $n = 3$*, if the local ring* R *is isomorphic to* $\mathbb{Z}_4[x]/(x^2)$ *or* $\mathbb{Z}_2[x,y]/(x^2,y^2)$ *or* $\mathbb{Z}_4[x,y]/(x^2,y^2,xy-2,2x,2y)$*, then* $\Gamma_E(R) \cong K_{1,3}$ *and therefore* $\dim(\Gamma_E(R)) = 2$*. For* $n = 4$*, if the local ring* R *is isomorphic to* $\mathbb{Z}_8[x,y]/(x^2,y^2,4x,4y,2xy)$*, then* $\Gamma_E(R) \cong K_{1,4}$ *and therefore* $\dim(\Gamma_E(R)) = 3$*. For* $n = 5$*, if* $R \cong \mathbb{Z}_2[x,y,z]/(x^2,y^2,z^2,xy)$*, then* $\Gamma_E(R) \cong K_{5,1}$ *and therefore* $\dim(\Gamma_E(R)) = 4$*. This star graph* $K_{1,5}$ *is the smallest star graph that can be realized as* $\Gamma_E(R)$*, but not as a zero divisor graph.*

By definition of the compressed zero divisor graph $\Gamma_E(R)$ of a ring R, it is clear that each vertex in $\Gamma_E(R)$ is a representative of a distinct class of zero divisor activity in R. Thus, $\dim(\Gamma_E(R)) \leq \dim(\Gamma(R))$. However, the strict inequality holds if $\Gamma_E(R)$ has at least 3 vertices.

Example 1 *In the rings* $R = \frac{\mathbb{Z}_2[x,y]}{(x^2,xy,2x)}$*,* $R = \frac{\mathbb{Z}_4[x,]}{(x^2)}$*,* $R = \mathbb{Z}_{16}$*,* $R = \frac{\mathbb{Z}_8[x]}{(2x,x^2)}$*, it is easy to find that* $\dim(\Gamma_E(R)) < \dim(\Gamma(R))$*.*

It will be interesting to see the family of rings in which the equality $\dim(\Gamma_E(R) = \dim(\Gamma(R)$ occurs.

A ring R is called a *Boolean ring* if $a^2 = a$ for every $a \in R$. Clearly a Boolean ring R is commutative with char(R) = 2, where char(R) denotes the characteristic of a ring R. More generally, a commutative ring is *von Neumann regular* ring if for every $a \in R$, there exists $b \in R$ such that $a = a^2 b$, or equivalently, R is a reduced zero dimensional ring, see [13, Theorem 3.1]. A Boolean ring is clearly a von Neumann regular, but not conversely. For example, let $\{F_i\}_{i \in I}$ be a family of fields, then $\prod_{i \in I} F_i$ is always von Neumann regular, but it is Boolean if and only if $F_i \cong \mathbb{Z}_2$ for all $i \in I$. Also the set $B(R) = \{a \in R \mid a^2 = a\}$ of idempotents of a commutative ring R becomes a Boolean ring with multiplication defined in the same way as in R, and addition defined by the mapping $(a, b) \mapsto a + b - 2ab$. In [13, Lemma 3.1], if $r, s \in \Gamma(R)$, the conditions $N(r) = N(s)$ and $[r] = [s]$ are equivalent if R is a reduced ring, and these are equivalent to the condition $rR = sR$ if R is a von Neumann regular ring. Furthermore, if R is a von Neumann regular ring and $B(R)$ is the set of idempotent elements of R, the mapping defined by $e \mapsto [e]$ is isomorphism from the subgraph of $\Gamma(R)$ induced by $B(R) \setminus \{0, 1\}$ onto $\Gamma_E(R)$ [13, Proposition 4.5]. In particular, if R is a Boolean ring (i.e., R = B(R)), then $\Gamma_E(R) \cong \Gamma(R)$. From this discussion, we have the following characterization.

Proposition 3 *Let R be a reduced commutative ring with unity. Then, metric dimension of the zero divisor graph $\Gamma(R)$ equals to metric dimension of its corresponding compressed zero divisor graph if R is a Boolean ring.*

Note that the converse of this result is not true in general. For example, the graphs in Figure 4 being a counter example, where $\dim(\Gamma(\mathbb{Z}_6)) = \dim(\Gamma_E(\mathbb{Z}_6))$, but R is not a Boolean ring.

Figure 4: $\dim(\Gamma(\mathbb{Z}_6)) = \dim(\Gamma_E(\mathbb{Z}_6)) = 1$

Corollary 3 *Let R and S be commutative reduced rings with unity 1. If $\Gamma(R) \cong \Gamma(S)$, then $\dim(\Gamma_E(R)) = \dim(\Gamma_E(S))$.*

Remark 3 *As seen in* [21, Theorem 2], *for the graph $\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)$ of a finite Boolean ring*

$$\dim(\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)) \leq n, \quad \dim(\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)) \leq n - 1$$

for $n = 2, 3, 4$ and $\dim(\Gamma(\Pi_{i=1}^{n}\mathbb{Z}_2)) = n$ for $n = 5$. This is also true for $\Gamma_E(R)$, follows by Proposition 3. The case $n > 5$ is still open.

3 Bounds for the metric dimension of $\Gamma_E(R)$

In this section, we investigate the role of metric dimension in the study of the structure of the graph $\Gamma_E(R)$. We also obtain metric dimension of some special type of rings that exhibit $\Gamma_E(R)$. Pirzada et al [18] characterized those graphs $\Gamma(R)$ for which the metric dimension is finite and for which the metric dimension is undefined [18, Theorem 3.1]. The analogous of this result is as follows.

Theorem 1 *Let* R *be a commutative ring. Then*
(i) $\dim(\Gamma_E(R))$ *is finite if and only if* R *is finite.*
(ii) $\dim(\Gamma_E(R))$ *is undefined if and only if* R *is an integral domain.*

However, $\dim(\Gamma_E(R))$ may be finite if R is infinite. For example, $R = \mathbb{Z}[x, y]/(x^3, xy)$ has $\Gamma_E(R)) \cong K_{1,3} + e$ (or paw graph), see Figure 5, and therefore has $\dim = 2$.

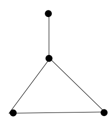

Figure 5:

The following lemma will be used to find the metric dimension of finite local rings.

Lemma 2 *If* R *is a finite local ring, then* $|R| = p^n$*, for some prime* p *and some positive integer* n*.*

Now, we have the following results.

Proposition 4 *If* R *is a local ring with* $|R| = p^2$ *and* $p = 2, 3, 5$*, then* $\dim(\Gamma_E(R))$ *is either* 0 *or undefined.*

Proof. Consider all local rings of order p^2 with p a prime. According to [10, p. 687] local rings of order p^2 are precisely $\mathbb{F}_{p^2}$, $\dfrac{\mathbb{F}_p[x]}{(x^2)}$, and $\mathbb{Z}_{p^2}$. If R is a field of order p^2, i.e., $R \cong \mathbb{F}_{p^2}$, then $\Gamma_E(R)$ is an empty graph, which implies $\dim(\Gamma_E(R))$ is undefined. If R is not a field and $|R| = p^2$, i.e., $R \cong \dfrac{\mathbb{F}_p[x]}{(x^2)}$, or

$\mathbb{Z}_{p^2}$ then $\Gamma_E(R)$ is a single vertex, when $p = 2,\ 3$ or 5 which then immediately gives that $\mathrm{dim}(\Gamma_E(R)) = 0$. □

From the above result, we also observe that $\mathrm{dim}(\Gamma(R)) = \mathrm{dim}(\Gamma_E(R))$, if

$$R \cong \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_4[x]/(2x, x^2 - 2).$$

Proposition 5 *If* R *is a local ring (not a field) of order*

(i) p^3 *with* $p = 2$ *or* 3, *then* $\mathrm{dim}(\Gamma_E(R))$ *is* 0, *and* $\mathrm{dim}(\Gamma_E(R)) = 1$ *only if* $R \cong \mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_8$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$, $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_9[x]/(3x, x^2 - 3)$, $\mathbb{Z}_9[x]/(3x, x^2 - 6)$ *or* $\mathbb{Z}_{27}$

(ii) p^4 *with* $p = 2$, *then* $\mathrm{dim}(\Gamma_E(R))$ *is* 0, 1 *or* 2.

Proof. (i) The following is the list of all the local rings of order p^3.

$$\mathbb{F}_{p^3}, \frac{\mathbb{F}_p[x, y]}{(x, y)^2}, \frac{\mathbb{F}_p[x]}{(x^3)}, \frac{\mathbb{Z}_p{}^2[x]}{(px, x^2)}, \frac{\mathbb{Z}_p{}^2[x]}{(px, x^2 - p)}$$

Case(a). When $p = 2$, the equivalence classes of the zero divisors in the local rings $\mathbb{Z}_2[x, y]/(x, y)^2$ and $\mathbb{Z}_4[x]/(2x, x^2)$ are same and is given by $[a] = \{x, y, x + y\}$ for any zero divisor a of the first ring and $[b] = \{2, x, x + 2\}$ for any zero divisor b of the second ring, that is, they get collapsed to a single vertex. Therefore $\mathrm{dim}(\Gamma_E(R)) = 0$. However, $\Gamma_E(R)$ of the rings $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_8$ and $\mathbb{Z}_4[x]/(2x, x^2 - 2)$ is isomorphic to the graph $K_{1,1}$, which then, by Lemma 1, gives $\mathrm{dim}_E(R) = 1$.

Case(b). When $p = 3$ in the above list of local rings, we find that the compressed zero divisor graph structure of the rings $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_9[x]/(3x, x^2 - 3)$, $\mathbb{Z}_9[x]/(3x, x^2 - 6)$ and $\mathbb{Z}_{27}$ is same and is isomorphic to $K_{1,1}$. Then, by Lemma 1, we have $\mathrm{dim}_E(R) = 1$. Also, in the rings $\dfrac{\mathbb{Z}_3{}^2[x]}{(3x, x^2)}$ and $\dfrac{\mathbb{Z}_3[x, y]}{(x, y)^2}$, the equivalence classes of all the zero divisors is same and is given by $[a] = \{3, 6, x, 2x, x + 3, x + 6, 2x + 3, 2x + 6\}$ for any non-zero zero divisor a of the first ring and $[b] = \{x, 2x, y, 2y, x + y, 2x + y, x + 2y, 2x + 2y\}$ for any non-zero zero divisor b of the later ring. Thus, $\Gamma_E(R)$ for both rings is a graph on a single vertex and follows that $\mathrm{dim}(\Gamma_E(R) = 0$.

(ii) Consider the local rings of order p^4, when $p = 2$. Corbas and Williams [10] conclude that there are 21 non-isomorphic commutative local rings with identity of order 16. The rings with $\mathrm{dim}(\Gamma_E(R)) = 0$ are $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_2[x, y, z]/(x, y, z)^2$ and $\mathbb{Z}_4[x]/(x^2 + x + 1)$. The rings with $\mathrm{dim}(\Gamma_E(R)) = 1$ are $Z_2[x]/(x^4)$, $Z_2[x, y]/(x^3, xy, y^2)$, $Z_4[x]/(2x, x^3 - 2)$, $\mathbb{Z}_4[x]/(x^2 - 2)$, $\mathbb{Z}_8[x]/(2x, x^2)$, $\mathbb{Z}_{16}$, $\mathbb{Z}_4[x]$

$/(x^2-2x-2)$, $\mathbb{Z}_8[x]/(2x, x^2-2)$, $\mathbb{Z}_4[x]/(x^2-2x)$, $\mathbb{Z}_2[x]/(x^4)$ and $\mathbb{Z}_2[x]/(x^4)$. Further the rings with $\mathrm{dim}(\Gamma_E(R)) = 2$ are $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_2[x, y]/(x^2, y^2)$ and $\mathbb{Z}_2[x, y]/(x^2-y^2, xy)$. □

Now, we find the metric dimension of $\Gamma_E(\mathbb{Z}_n)$.

Proposition 6 *Let* p *be a prime number.*
(i) *If* $n = 2p$ *and* $p > 2$*, then* $\mathrm{dim}(\Gamma_E(\mathbb{Z}_n)) = 1$*.*
(ii) *If* $n = p^2$*, then* $\mathrm{dim}(\Gamma_E(\mathbb{Z}_n)) = 0$*.*

Proof. (i) If $p = 2$, since $\Gamma_E(\mathbb{Z}_4)$ is a graph with single vertex. So, $\mathrm{dim}(\Gamma_E(\mathbb{Z}_4)$ $= 0$.

If $p > 2$, the zero divisor set of $\mathbb{Z}_n$ is $\{2, 2.2, 2.3, \ldots, 2.(p-1), p\}$. Since, $\mathrm{char}(\mathbb{Z}_n) = 2p$, it follows that p is adjacent to all other vertices. Thus the equivalence classes of these zero divisors are given by
$[p] = \{2, 2.2, 2.3, \ldots, 2.(p-1)\}$, $[2] = [2.2] = \cdots = [2.(p-1)] = \{p\}$.

So, the vertex set of $\Gamma_E(\mathbb{Z}_n)$ is $Z(R_E) = \{[p], [2x]\}$ for any positive integer $x = 1, 2, \ldots, p-1$. Thus $\Gamma_E(\mathbb{Z}_n)$ is a path P_2 which then, by Lemma 1, gives $\mathrm{dim}(\Gamma_E(\mathbb{Z}_n) = 1$.
(ii) If $n = p^2$ and $p > 2$, the zero divisor set of $\mathbb{Z}_n$ is $\{p, p.2, p.3, \ldots, p(p-1)\}$. Since $\mathrm{char}(\mathbb{Z}_n) = p^2$, it follows that the equivalence class of all these zero divisors is same and is $\{p, p.2, p.3, \ldots, p.(p-1)\}$. Thus, $\Gamma_E(R)$ in this case is a graph on a single vertex and therefore $\mathrm{dim}(\Gamma_E(R) = 0$. □

From the above result, we have the following observations.

Corollary 4 *Let* p *be a prime number*
(i) *If* $n = 2p$ *and* $p > 2$*, then* $|\Gamma_E(\mathbb{Z}_n)| = 2$*.*
(ii) *If* $n = p^2$*, then* $|\Gamma_E(\mathbb{Z}_n)| = 1$*.*
(iii) *If* $n = p^k$*,* $k > 3$ *and* $p > 2$*, then* $|\Gamma_E(\mathbb{Z}_n)| = k-1$*.*

Proof. (i) and (ii) follow from Proposition 6.
(iii) When $n = p^k$, $k > 3$ and $p \geq 2$, the zero divisors of $\mathbb{Z}_n$ are
$Z(\mathbb{Z}_n) = \{up^i | u \in U(\mathbb{Z}_n)\}$, for $i = 1, 2, \ldots, k-1$. Now the equivalence classes of zero divisors are $[up] = \{up^{k-1}\}$, $[up^2] = \{up^{k-1}, up^{k-2}\}$, $\ldots$, $[up^{k-1}] = \{up^{k-1}, up^{k-2}, \ldots, up^2, up\}$.
In this way, we get $k-1$ distinct equivalence classes. Thus, $|\Gamma_E(\mathbb{Z}_n)| = k-1$. □

Corollary 5 $\mathrm{dim}(\Gamma_E(\mathbb{Z}_n)) \leq 2k-2$*, where* $n = p^k$*, for any prime* $p > 2$ *and* $k > 3$*.*

Proof. By [18, Theorem 2.1]. If G is a connected graph with G partitioned into m distance similar classes that consist of a single vertex, then $\dim(G) \leq |V(G)| + m$.

Using part (iii) of Corollary 4, the result follows. □

The following important lemma, which is used later in the proof of several results, provides a combinatorial formula for the number of vertices of the compressed zero divisor graph $\Gamma_E(R \times \mathbb{F}_q)$.

Lemma 3 *Let R be a finite commutative local ring with unity 1 and $|R| = p^k$ and let $\mathbb{F}_q$ be a finite prime field. Then $|Z^*((R \times F_q)_E)| = 2k$ or $2(1 + |Z^*(R_E)|$.*

Proof. Let R be a finite commutative local ring with unity and $|R| = p^k$, $k \geq 1$. We consider the following three cases.

Case 1. $R \cong \mathbb{F}_p$, for some prime p. Then the zero divisor set of $Z^*(\mathbb{F}_p \times \mathbb{F}_q) = \{\{(a, 0)\}, \{(0, x)\}\}$, for every $a \in U(R)$ and $0 \neq x \in \mathbb{F}_q$. Now, to find the equivalence classes of these zero divisors, the set $\{(a, 0)\}$ and $\{(0, x)\}$ respectively correspond to vertices $[(a, 0)]$ and $[(0, x)]$ in $\Gamma_E(R \times \mathbb{F}_q)$, for any $a \in U(R)$ and for any $x \in \mathbb{F}_q$. Therefore, $|Z^*(R \times \mathbb{F}_q)_E| = 2k$, where $k = 1$.

Case 2. $R \cong \mathbb{Z}_p^k$, $(k \geq 2)$. The equivalence class of each element $(a, 0)$, for every $a \in U(R)$ is same, since $[(a, 0)] = \{(0, x)\}$, for all $x \in \mathbb{F}_q$. In this way, we get one vertex of $\Gamma_E(R \times \mathbb{F}_q)$. Also, the equivalence classes of each element $(0, x)$, for every $0 \neq x \in \mathbb{F}_q$ is same, since $[(0, x)] = \{(a, 0)\}$. So, this gives another vertex of $\Gamma_E(R \times \mathbb{F}_q)$. Moreover, for any unit u in R, we get two zero divisor sets of equivalence classes given by

$$Z_1 = \{[(up, 0)], [(up^2, 0)], \ldots, [(up^{k-1}, 0)]\}$$
$$Z_2 = \{[(up, 1)], [(up^2, 1)], \ldots, [(up^{k-1}, 1)]\}.$$

We note that there is no other possible equivalence class. Claim $[(up^{k-1}, 1)] = [(up^{k-1}, x_i]$, for all $1 \leq i \leq q - 2$. If $[(up^{k-1}, 1)] \neq [(up^{k-1}, x_i]$, there exists some zero divisor in $R \times \mathbb{F}_q$, say $(a_1, 0)$ adjacent to $(up^{k-1}, 1)$ but not adjacent to (up^{k-1}, x_i), which is a contradiction.

The total number of zero divisors is $|Z^*((R \times \mathbb{F})_E)| = 2 + |Z_1| + |Z_2| = 2 + k - 1 + k - 1 = 2k$ or $2 + 2|Z^*(R_E)| = 2(1 + |Z^*(R_E)|)$.

Case 3. R is a local ring other than $\mathbb{F}_p$ and $\mathbb{Z}_{p^k}$. So, we consider all local rings R with $|R| = p^k$, especially $k = 2$, 3 or 5 and the rings of order p^2, p^3 or p^4 are mentioned in proof of Proposition 4 and 5. Then the set of zero divisors of equivalence classes include

$$[(a, 0)],\ a \in U(R)$$

$$[(0, x_i)], \text{ for any } i,\ 1 \le i \le q-2$$
$$Z_1 = \{[(a_1, 0)], [(a_2, 0)], \ldots, [(a_r, 0)]\}$$
$$Z_2 = \{[(a_1, 1)], [(a_2, 1)], \ldots, [(a_r, 1)]\}.$$
where $a_1, a_2, \ldots, a_r$ are the non-zero zero divisors of the set $Z(R_E)$.

There is no other possible equivalence class as a zero divisor. Claim $[(a_i, 1)] = [(a_i, x_j)]$, $1 \le i \le r$ and $1 \le j \le q-2$. For if, $[(a_i, 1)] \ne [(a_i, x_j)]$, there exists some zero divisor $(a_k, 0)$ adjacent to one of $[(a_i, 1)]$ or $[(a_i, x_j)]$, but not to the other, which is a contradiction.

Thus, $|Z^*((R \times \mathbb{F}_q)_E)| = 2 + 2|Z^*(R_E)| = 2(1 + |Z^*(R_E)|$. □

Example 2 *Consider the ring* $\mathbb{Z}_8 \times \mathbb{Z}_3$*, Here,* $R = \mathbb{Z}_{2^3}$*,* $k = 3$*, and* $U(R) = \{1, 3, 5, 7\}$*. For the zero divisors of equivalence classes, we have*
$[(1, 0)] = \{(0, 1), (0, 2)\}$, $[(3, 0)] = \{(0, 1), (0, 2)\}$, $[(5, 0)] = \{(0, 1), (0, 2)\}$, $[(7, 0)] = \{(0, 1), (0, 2)\}$.
Also, $[(2, 0)] = \{(0, 1), (0, 2), (4, 0), (4, 1), (4, 2)\}$,
$[(4, 0)] = \{(0, 1), (0, 2), (2, 0), (2, 1), (2, 2), (4, 0), (4, 1), (4, 2), (6, 0), (6, 1), (6, 2)\}$,
$[(6, 0)] = \{(0, 1), (0, 2), (4, 0), (4, 1), (4, 2)\}$.
Moreover, $[(0, 1)] = \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0)\}$,
$[(0, 2)] = \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0)\}$, $[(2, 1)] = \{(4, 0)\}$,
$[(4, 2)] = \{(2, 0), (4, 0), (6, 0)\}$, $[(4, 1)] = \{(2, 0), (4, 0), (6, 0)\}$, $[(2, 2)] = \{(4, 0)\}$,
$[(6, 1)] = \{(4, 0)\}$, $[(6, 2)] = \{(4, 0)\}$
Thus, $|\Gamma_E(\mathbb{Z}_8 \times \mathbb{Z}_3)| = \{[(0, 1)], [(1, 0)], [(2, 0)], [(4, 0)], [(2, 1)], [(4, 1)]\}$.
Using Lemma 3, we can directly have, $|\Gamma_E(\mathbb{Z}_8 \times \mathbb{Z}_3)| = 2 \times 3 = 6$.

Remark 4 *Lemma 3 holds if we replace* $\mathbb{F}_q$ *by any finite field* $\mathbb{F}$*. More generally, let* R *be any finite commutative ring with unity* 1*. We know* $R \cong R_1 \times R_2$*, where each* R_i*,* $1 \le i \le 2$*, is a local ring. If either* R_1 *or* R_2 *is a field, the number of vertices is always given by the formula* $2(1 + |Z^*(R_{1E})|$ *or* $2(1 + |Z^*(R_{2E})|$*, since the equivalence classes of zero divisors of* $\Gamma_E(R_1 \times R_2)$ *are always of the form* $\{[(0, 1)], [(1, 0)], [(a, 0)], [(0, b)], [(a, 1)], [(1, b)], [(a, b)]$*, where* a *and* b *are the non-zero zero divisors and* $Z^*(R_{1E})$*,* $Z^*(R_{2E})$ *denote the number of zero divisor equivalence classes of* R_1 *and* R_2 *respectively. The result holds trivially if both* R_1 *and* R_2 *are fields.*

Theorem 2 *Let* R *be a finite commutative local ring with unity* 1 *and finite field* $\mathbb{F}_q$*. Then,* $\dim(\Gamma_E(R \times \mathbb{F}_q)) = 1$ *or at most* $4k$ *or* $4t$ *where* $k \ge 2$ *and* t *are integers,* $t = 1 + |Z^*(R_E)|$*.*

Proof. Let R be a finite commutative local ring with unity 1. We consider the following three cases.

Case 1. R is a field. Then, by Case 1 of Lemma 3, $\Gamma_E(R \times \mathbb{F}_q)$ is a path on two vertices. Therefore, by Lemma 2.1, $\dim(\Gamma_E(R \times \mathbb{F}_q)) = 1$.
Case 2. $R \cong \mathbb{Z}_{p^k}$, $k \geq 2$. In this case, we partition the vertices into distance similar classes in $\Gamma_E(R)$ given by

$$\begin{aligned} V_1 &= \{[(a,0)]\}, \text{ for any } a \in U(R) \\ V_2 &= \{[(0,x)]\}, \text{ for any } x \in \mathbb{F}_q \\ Z_1 &= \{[(up,0)]\}, Z_2 = \{[(up^2,0)]\}, \dots, Z_{k-1} = \{[(up^{k-1},0)]\} \\ W_1 &= \{[(up,1)]\}, W_2 = \{[(up^2,1)]\}, \dots, W_{k-1} = \{[(up^{k-1},1)]\} \end{aligned}$$

Then, $\dim(\Gamma_E(R\times\mathbb{F}_q) \leq |Z^*((R\times\mathbb{F}_q)_E)+m$ where m is the number of distance similar classes that consist of a single vertex. Hence by case 2 of Lemma 3, we have

$$\dim(\Gamma_E(R \times \mathbb{F}_q) \leq 2k + 2(k-1) + 2 = 4k.$$

Case 3. R is a local ring other than $\mathbb{Z}_p^k$ and $\mathbb{F}_p^k(k \geq 1)$. Then, by Case 3 of Lemma 3, $\dim(\Gamma_E(R\times\mathbb{F}_q)) \leq 2(1+|Z^*(R_E)|+2|Z^*(R_E)|+2 = 4(1+|Z^*(R_E)|) = 4t$ where t is any integer given by $t = 1 + |Z^*(R_E)|$. □

We say that a graph G has a bounded degree if there exists a positive integer M such that the degree of every vertex is at most M. In the next theorems, we obtain an upper bound for the number of zero divisors in a finite commutative ring R with unity 1 with finite metric dimension. The analogous of these results holds in case of $\Gamma_E(R)$.

Proposition 7 *If $\Gamma(R)$ is a zero divisor graph with finite metric dimension k, then $|Z^*(R)| \leq 3^k + k$.*

Proof. Let $\Gamma(R)$ be a zero divisor graph with metric dimension k. We choose two vertices, say w_1 and w_2, from the metric basis W. Since the diameter of $\Gamma(R)$ is at most 3, each coordinate of metric representation is an integer between 0 and 3 and only the vertices of a metric basis have one coordinate 0. The remaining vertices must get a unique code from one of the 3^k possibilities. Therefore, $|Z^*(R)| \leq 3^k + k$. □

Proposition 8 *Let R be a commutative ring and $\Gamma_E(R)$ be a corresponding compressed zero divisor graph with $|Z^*(R)| \geq 2$. Then $\dim(\Gamma_E(R)) \leq |Z^*(R_E)| - d$, where d is the diameter of $\Gamma_E(R)$.*

Proof. By [21, Theorem 5.2], if R is a commutative ring and $\Gamma(R)$ is the corresponding zero divisor graph of R such that $|Z^*(R)| \geq 2$, then $\dim(\Gamma(R)) \leq |Z^*(R)| - d'$ where d' is the diameter of $\Gamma(R)$. Since

$$\dim(\Gamma_E(R)) \leq \dim(\Gamma(R)) \quad \text{and} \quad |Z^*(R_E)| \leq |Z^*(R)|,$$

therefore

$$\dim(\Gamma_E(R)) \leq |Z^*(R_E)| - d,$$

where d is the diameter of $\Gamma_E(R)$. □

Proposition 9 *If $\Gamma(R)$ is a finite graph with metric dimension k, then every vertex of this graph has degree at most $3^k - 1$.*

Proof. Let $W = \{w_1, w_2, \ldots, w_k\}$ be a metric basis of $\Gamma(R)$ with cardinality k. Consider a vertex v with metric representation

$$(d(v, w_1), d(v, w_2), \ldots, d(v, w_k)).$$

If u is adjacent to v, then $r(v|W) \neq r(u|W)$ and $|d(v, w_i) - d(u, w_i)| \leq 1$ for all $w_i \in W$, $1 \leq i \leq k$. If d is distance from v to w_i, then the distance of u from w_i is one of the numbers $\{d, d-1, d+1\}$. Thus, there are three possible numbers for each of the k coordinates of $r(u|W)$, but $d(u, w_i) \neq d(v, w_i)$ for all $1 \leq i \leq k$. This implies that there are at most $3^k - 1$ different possibilities for $r(u|W)$. Since all vertices must have distinct metric coordinates, the degree of v is at most $3^k - 1$. □

A graph G is *realizable* as $\Gamma_E(R)$ if $G \cong \Gamma_E(R)$ for some ring R. There are many results which imply that most graphs are not realizable as $\Gamma_E(R)$, like $\Gamma_E(R)$ is not a cycle graph, nor a complete graph with at least three vertices.

Proposition 10 *The metric dimension of realizable graphs $\Gamma_E(R)$ with 3 vertices is 1.*

Proof. Spiroff et al. proved that the only one realizable graph $\Gamma_E(R)$ with exactly three vertices as a graph of equivalence classes of zero divisors for some ring R is P_3, see Figure 6. Clearly, its metric dimension is 1. □

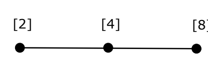

Figure 6: $\mathbb{Z}_{16}$

Proposition 11 *The metric dimension of realizable graphs* $\Gamma_E(R)$ *with* 4 *vertices is either* 1 *or* 2*.*

Proof. All the realizable graphs $\Gamma_E(R)$ on 4 vertices are shown in Figure 7. It is easy to see their metric dimension is either 1 or 2. □

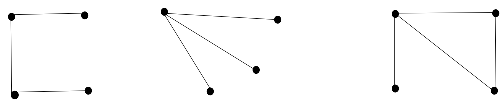

Figure 7: $(\mathbb{Z}_4 \times \mathbb{F}_4)$ $\quad \mathbb{Z}_4[x]/(x^2)$ $\quad \frac{\mathbb{Z}[x,y]}{(x^3,xy)}$

Proposition 12 *The metric dimension of realizable graphs* $\Gamma_E(R)$ *with* 5 *vertices is either* 2 *or* 3*.*

Proof. The only realizable graphs of equivalence classes of zero divisors of a ring R with 5 vertices are shown in Figure 8. It is easy to see the metric dimension of the first three graphs is 2 and for the star graph is 3 (by Corollary 1). □

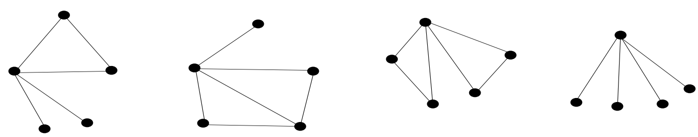

Figure 8: $(\mathbb{Z}_9[x]/(x^2)$, $\mathbb{Z}_{64}$, $Z_3[x,y]/(xy, x^3, y^3, x^2 - y^2)$, $\mathbb{Z}_8[x,y]/(x^2, y^2, 4x, 4y, 2xy)$

4 Relationship between metric dimension, girth and diameter of $\Gamma_E(R)$

In this section, we examine the relationship between girth, diameter and metric dimension of $\Gamma_E(R)$. Since $gr(\Gamma_E(R)) \in \{3, \infty\}$, it is worth to mention that, for a reduced commutative ring R with $1 \neq 0$, $gr(\Gamma_E(R)) = 3$ if and only if $gr(\Gamma(R)) = 3$ and that $gr(\Gamma_E(R)) = \infty$ if and only if $gr(\Gamma(R)) \in \{4, \infty\}$. However, if R is not reduced, then we may have $gr(\Gamma(R)) = 3$ and either $gr(\Gamma_E(R)) = 3$ or ∞. The following result gives the metric dimension of $\Gamma_E(R)$ in terms of the girth of $\Gamma_E(R)$ of a ring R.

Theorem 3 *Let* R *be a finite commutative ring with* $\mathrm{gr}(\Gamma_E(R)) = \infty$*.*

(i) *If* R *is a reduced ring, then* $\dim(\Gamma_E(R)) = 1$*.*

(ii) *If* $R \cong \mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3)$ *or* $\mathbb{Z}_4[x]/(2x, x^2-2)$*, then* $\dim(\Gamma_E(R)) = |Z^*(R_E)| - 1$*.*

(iii) *If* $R \cong \mathbb{Z}_4,\ \mathbb{Z}_9,\ \mathbb{Z}_2[x]/(x^2)$*, then* $\dim(\Gamma_E(R)) = 0$*.*

(iv) $\dim(\Gamma_E(R)) = 0$ *or* 1 *if and only if* $\mathrm{gr}(\Gamma(R)) \in \{4, \infty\}$*.*

Proof. If R is a reduced ring and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then we know $R \cong \mathbb{Z}_2 \times A$ for some finite field A. Therefore, by Remark 4, R has two equivalence classes of zero divisors $[(0,1)]$ and $[(1,0)]$, adjacent to each other. Hence, $\dim(\Gamma_E(R)) = 1$. Also, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then R being a Boolean ring, implies $\Gamma(R) \cong \Gamma_E(R)$. Therefore, by Case 1 of Lemma 3, the result follows. In part (ii), these rings are non reduced and $\Gamma_E(R)$ are isomorphic to $K_{1,1}$. Rings listed in part (iii) represents $\Gamma_E(R)$ on a single vertex, part (iv) follows from the above comments. □

We can also prove the Part (i) by using the fact that if R is reduced and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $R \cong \mathbb{Z}_2 \times A$ for some finite field A. Thus $\Gamma(R)$ is a complete bipartite and the result follows from Proposition 2. Now, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma(R) \cong K_{1,1}$, whose metric dimension is 1. Since, $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a Boolean ring, therefore by Proposition 3, we have $\dim(\Gamma_E(R)) = 1$.

If R is a reduced ring with non-trivial zero divisor graph, then $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_k$ for some integer $k \geq 2$ and for finite fields $\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_k$. If R is not a reduced ring, then either R is local or $R \cong R_1 \times R_2 \times \cdots \times R_t$, for some integer $t \geq 2$ and local rings $R_1, R_2, \ldots, R_t$, where at least one R_i is not a field. Now, we have the following observations for the finite commutative rings whose zero divisor graphs can be seen in [22].

Corollary 6 *If* R *is a finite commutative ring with unity* 1 *and* $\mathrm{gr}(\Gamma_E(R)) = \infty$*, then the compressed zero divisor graph of the reduced rings* $R \times \mathbb{F}$ *where* $\mathbb{F}$ *is a finite field, is isomorphic to the compressed zero divisor graph of the following local rings with metric dimension* 1*,* R *being any local ring.*
$\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x^3, xy, y^2)$, $\mathbb{Z}_8[x]/(2x, x^2)$,
$\mathbb{Z}_4[x]/(x^3, 2x^2, 2x)$, $\mathbb{Z}_9[x]/(3x, x^2-6)$, $\mathbb{Z}_9[x]/(3x, x^2-3)$, $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_{27}$.

Proof. The reduced rings $R \times \mathbb{F}$ with $\mathrm{gr}(\Gamma_E(R)) = \infty$, all have compressed zero divisor graph isomorphic to $K_{1,1}$, by Case 2 of Lemma 3. Also, the local rings listed above have the same compressed zero divisor graph isomorphic to $K_{1,1}$. □

Proposition 13 *Let* R *be a finite commutative ring with* 1 *and* $gr(\Gamma_E(R)) = \infty$*. The following are the non reduced rings with* $dim(\Gamma_E(R)) = 1$
$\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_5 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_9$, $\mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_5 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2,x)^2$.

Proof. If R is not a local ring, we can write $R \cong R_1 \times R_2 \times \cdots \times R_k$, where $k \geq 2$ and each R_i is a local ring. In case of above rings $R \cong R_1 \times R_2$, where either R_1 or R_2 is a field. Therefore, using Remark 4, we have $|\Gamma_E(R)| = 4$ and it is easy to see that $\Gamma_E(R)$ isomorphic to a path on 3 vertices. Thus, $gr(\Gamma_E(R)) = \infty$ and $dim(\Gamma_E(R)) = 1$. □

If $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$, then it is easy to see that the three vertices $[(1,0,0)]$, $[(0,1,0)]$ and $[(0,0,1)]$ are adjacent with ends $[(0,1,1)]$, $[(1,0,1)]$, and $[(1,1,0)]$ respectively and thus $|\Gamma_E(R)| = 6$.

Proposition 14 *Let* R *be a reduced commutative ring and* $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$*. Then,* $gr(\Gamma_E(R)) = 3$ *and* $dim(\Gamma_E(R)) = 2$*.*

We now proceed to study the relationship between diameter and metric dimension of compressed zero divisor graphs. Since $diam(\Gamma_E(R)) \leq 3$, if $\Gamma_E(R)$ contains a cycle. We have the following results.

Theorem 4 *Let* R *be commutative ring and* $\Gamma_E(R)$ *be its corresponding compressed zero divisor graph.*

(i) $dim(\Gamma_E(R)) = 0$ *if and only if* $diam(\Gamma_E(R)) = 0$*.*

(ii) $dim(\Gamma_E(R)) = 0$ *if and only if* $diam(\Gamma(R)) = 0$ *or* 1*,* $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$*.*

(iii) $dim(\Gamma_E(R)) = diam(\Gamma_E(R)) = 1$ *if* $R \cong \mathbb{F}_1 \times \mathbb{F}_2$*, where* $\mathbb{F}_1$ *and* $\mathbb{F}_2$ *are fields.*

(iv) $dim(\Gamma_E(R)) = 1$ *and* $diam(\Gamma_E(R)) = 3$*, if* R *is non reduced ring isomorphic to the rings given in Proposition 13.*

(v) $dim(\Gamma_E(R)) = 0$ *if* $Z(R)^2 = 0$ *and* $|Z(R)| \geq 2$*.*

Proof.

(i) $dim(\Gamma_E(R)) = 0$ if and only if $\Gamma_E(R)$ is a single vertex graph if and only if $diam(\Gamma_E(R)) = 0$.

(ii) Let $\mathrm{dim}(\Gamma_E(R)) = 0$. Then $\Gamma(R)$ is complete and thus $\mathrm{diam}(\Gamma(R)) = 0$ or 1. Conversely, let $\mathrm{diam}(\Gamma(R)) = 0$ or 1, then $\Gamma(R)$ is complete, thus $\mathrm{dim}(\Gamma_E(R)) = 0$ unless $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) Let $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, then by Case 1 of Lemma 3, $|\Gamma_E(R)| = 2$, since the only equivalence classes of zero divisors are $[(0,1)]$ and $[(1,0)]$. So, $\Gamma_E(R) \cong K_{1,1}$. Thus, $\mathrm{dim}(\Gamma_E(R)) = \mathrm{diam}(\Gamma_E(R)) = 1$.

(iv) Rings listed in this case correspond to a path of length 3.

(v) Let $|Z(R)| \geq 2$ and $(Z(R))^2 = 0$. Hence $\mathrm{ann}(a) = \mathrm{ann}(b)$, for each $a, b \in Z(R)^*$, which implies that $\mathrm{diam}(\Gamma_E(R)) = 0$. Therefore, $\mathrm{dim}(\Gamma_E(R)) = 0$.

□

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Θ-modifications on weak spaces

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Abstract. In this article, we want to study and investigate if it is possible to use the notions of weak structures to develop a new theory of θ - modifications in weak spaces and study their properties, finally we study some forms of weak continuity using this modifications.

1 Introduction

In [5], Császar and Makai Jr. introduced and studied the notions of $\delta_{\mu_1\mu_2}$-open sets and $\theta_{\mu_1\mu_2}$-open sets defined by two generalized topologies μ_1 and μ_2 on a nonempty set X and they proved that: $\delta_{\mu_1\mu_2}$ and $\theta_{\mu_1\mu_2}$ are generalized topologies on X and $\theta_{\mu_1\mu_2} \subseteq \delta_{\mu_1\mu_2} \subseteq \mu_1$. The notions of $(\theta_{w_1w_2}, \theta_{\sigma_1\sigma_2})$-continuous was introduced and characterized by W. K. Min in [6], also introduced and characterized the notions of $(\delta_{w_1w_2}, \delta_{\sigma_1\sigma_2})$-continuous on generalized topological

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spaces and $(\delta_{w_1w_2}, \theta_{\nu_1\nu_2})$-continuous. W. K. Min in [7], introduced the notions of mixed weak $(\mu, \nu_1\nu_2)$-continuity between a generalized topology μ and two generalized topologies ν_1, ν_2, also he introduced and characterized continuity in terms of mixed generalized $(\nu_1, \nu_2)'$-semiopen sets, $(\nu_1, \nu_2)'$-preopen sets, (ν_1, ν_2)-preopen sets [4] ,(ν_1, ν_2)'-β'-open sets and $\theta(\nu_1, \nu_2)$'-open sets [5] . Ugur Sengul in [12], using the δ and θ-modifications in bigeneralized topologies, introduced the notion of $(\delta_{\mu_1\mu_2}, \theta_{\sigma_1\sigma_2})$-continuity between two Bi-GTSś. Also he characterized such continuity in terms of mixed generalized open sets: $\delta_{\mu_1\mu_2}$-open sets, $\theta_{\mu_1\mu_2}$-open sets. In this article, we want to study if it is possible, using weak structures to make a new theory related to θ -modifications of weak spaces and study some weak forms of continuity.

2 Preliminaries

Definition 1 [9] *Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a weak structure on X if it satisfies the following:*

1. $\emptyset \in w_X$ *and* $X \in w_X$.
2. *For* $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$

The pair (X, w_X) is called a w-space on X. An element $U \in w_X$ is called w-open set and the complement of a w-open set is a w-closed set

Definition 2 [9] *Let (X, w_X) be a w-space. For a subset A of X,*

1. *The w-closure of A is defined as* $wC(A) = \bigcap\{F : A \subseteq F, X \setminus F \in w_X\}$.
2. *The w-interior of A is defined as* $wI(A) = \bigcup\{U : U \subseteq A, U \in w_X\}$.

Theorem 1 [9] *Let (X, w_X) be a w-space on X and A, B subsets of X. Then the following hold:*

1. *If* $A \subseteq B$, *then* $wI(A) \subseteq wI(B)$ *and* $wC(A) \subseteq wC(B)$.
2. $wI(wI(A)) = wI(A)$ *and* $wC(wC(A)) = wC(A)$.
3. $wC(X \setminus A) = X \setminus wI(A)$ *and* $wI(X \setminus A) = X \setminus wC(A)$.
4. $x \in wC(A)$ *if and only if* $U \cap A \neq \emptyset$, *for all* $U \in w_X$ *with* $x \in U$.
5. $x \in wI(A)$ *if and only if there exists* $U \in w_X$ *with* $x \in U$, *such that* $U \subseteq A$.

6. *If* A *is* w*-closed (resp.* w*-open), then* $wC(A) = A$ *(resp.* $wI(A) = A$*).*

Theorem 2 [11] *Let* (X, w_X) *be a* w*-space on* X *and* A, B *subsets of* X*. Then the following hold:*

1. $wI(A \cap B) = wI(A) \cap wI(B)$.
2. $wC(A \cup B) = wC(A) \cup wC(B)$.

Theorem 3 *Let* (X, w_X) *be a* w*-space on* X *and* A, B *subsets of* X*. Then the following hold:*

1. $wI(A) \cup wI(B) \subseteq wI(A \cup B)$.
2. $wC(A \cap B) \subseteq wC(A) \cap wC(B)$.

3 Modification on weak structures

Throughout this paper if w_1, w_2 are two weak structures on a nonempty set X. Then (X, w_1, w_2) is called a biweak space. Recall that Császar, A. [3], showed that the δ and θ-modifications of topological spaces can be generalized for the case when the topology is replaced by the generalized topologies μ_1, μ_2 in the sense of [1]. W. K. Min [6], gave a characterization for $(\theta_{\mu_1\mu_2}, \theta_{\sigma_1\sigma_2})$-continuity and introduce the concepts of $(\delta_{\mu_1\mu_2}, \delta_{\sigma_1\sigma_2})$-continuity on generalized topological spaces and investigate the relationship between $(\delta_{\mu_1\mu_2}, \theta_{\sigma_1\sigma_2})$-continuity, $(\theta_{\mu_1\mu_2}, \theta_{\sigma_1\sigma_2})$-continuity and $(\delta_{\mu_1\mu_2}, \delta_{\sigma_1\sigma_2})$-continuity. In our case, we want to study what happen when the generalized topologies are replaced by weak structures.

Definition 3 *Let* (X, w_1, w_2) *be a biweak space. A subset* A *of* X *is said to be* $\Upsilon_{w_1w_2}$*-open (resp.* $\Upsilon_{w_1w_2}$*-closed) if* $A = w_1I(w_2C(A))$ *(resp.* $A = w_1C(w_2I(A))$*).*

Example 1 *Let* (X, w_1, w_2) *be a biweak space, where* $X = \{a, b, c\}$*,* $w_1 = \{\emptyset, X, \{a\}, \{b\}\}$ *and* $w_2 = \{\emptyset, X, \{a\}, \{c\}\}$*.*
Observe that the set $A = \{b\}$ *is* $\Upsilon_{w_1w_2}$*-open, the set* $B = \{c\}$ *is* $\Upsilon_{w_2w_1}$*-open and the set* $C = \{a, b\}$ *is not* $\Upsilon_{w_2w_1}$*-open set.*

Definition 4 *Let* (X, w_1, w_2) *be a biweak space.*

1. $A \in \theta_{w_1w_2}$ *if and only if for each* $x \in A$*, there exists an* $U \in w_1$ *such that* $x \in U \subseteq w_2C(U) \subseteq A$*.*

2. $A \in \delta_{w_1w_2}$ *if and only if* $A \subset X$ *and if* $x \in A$*, there exists a* w_2*-closed set* F *such that* $x \in w_1 I(F) \subseteq A$.

Example 2 *In Example 1:*

1. $\theta_{w_1w_2} = \{\emptyset, X, \{b\}, \{a, b\}\}$,
2. $\delta_{w_1w_2} = \{\emptyset, X, \{b\}, \{a, b\}\}$,
3. $\theta_{w_2w_1} = \{\emptyset, X, \{c\}, \{a, c\}\}$,
4. $\delta_{w_2w_1} = \{\emptyset, X, \{a, c\}, \{c\}\}$.

Example 3 *Let* (X, w_1, w_2) *be a biweak space, where* $X = \{a, b, c\}$*,* $w_1 = \{\emptyset, X, \{a\}, \{b\}\}$ *and* $w_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{c\}\}$.
Observe that:

1. $\theta_{w_1w_2} = \{\emptyset, X, \{b\}, \{a, b\}\}$,
2. $\delta_{w_1w_2} = \{\emptyset, X, \{a, b\}, \{b\}\}$,
3. $\theta_{w_2w_1} = \{\emptyset, X, \{c\}, \{a, c\}\}$,
4. $\delta_{w_2w_1} = \{\emptyset, X, \{a, c\}, \{c\}\}$.

Example 4 *Let* (X, w_1, w_2) *be a biweak space, where* $X = \{a, b, c\}$*,* $w_1 = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}$ *and* $w_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{c\}\}$*. Observe that:*

1. $\theta_{w_1w_2} = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$,
2. $\delta_{w_1w_2} = \{\emptyset, X, \{c\}, \{a, b\}, \{b, c\}\}$,
3. $\theta_{w_2w_1} = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$,
4. $\delta_{w_2w_1} = \{\emptyset, X, \{a, b\}, \{c\}\}$.

Example 5 *Let* $X = \{a, b, c\}$ *with weak structures* $w_1 = \{\emptyset, X, \{b\}\}$ *and* $w_2 = \{\emptyset, X, \{a\}\}$*. Observe that:*

1. $\theta_{w_1w_2} = \{\emptyset, X\}$,
2. $\delta_{w_1w_2} = \{\emptyset, X, \{b\}\}$,
3. $\theta_{w_2w_1} = \{\emptyset, X\}$,

4. $\delta_{w_2 w_1} = \{\emptyset, X, \{a\}\}$.

Remark 1 *According with Example 4, $\delta_{w_1 w_2}$ is not necessary a weak structures on X, then first of all, we have an answer. We can not doing similarly modification as [5], if we replace generalized topology by weak structure.*

Theorem 4 *Let (X, w_1, w_2) be a biweak space. The collection $\theta_{w_1 w_2}$ is a strong generalized topology on X.*

Proof. It is easy to see that: $\emptyset$ and X belong to $\theta_{w_1 w_2}$. Now consider $\{U_i : i \in I\}$ a collection of elements of $\theta_{w_1 w_2}$ and $x \in \bigcup_{i \in I} U_i$, then for some $i \in I$, $x \in U_i$ and then there is $V_i \in w_1$, such that $x \in U_i \subseteq w_2C(V_i) \subseteq U_i \subseteq \bigcup_{i \in I} U_i$. It follows that $\bigcup_{i \in I} U_i \in \theta_{w_1 w_2}$. □

Theorem 5 *Let (X, w_1, w_2) be a biweak space. The collection $\theta_{w_1 w_2}$ is a weak structure on X.*

Proof. It is easy to see that: $\emptyset$ and X belong to $\theta_{w_1 w_2}$. Now consider U_1, U_2 two elements of $\theta_{w_1 w_2}$ and $x \in U_1 \cap U_2$, then $x \in U_i$ for $i = 1, 2$. Then there exists $V_i \in w_i$ for $i = 1, 2$, such that $x \in V_i$ and $w_2C(V_i) \subseteq U_i$. It follows that $x \in V_1 \cap V_2$ and $w_2C(V_1) \cap w_2C(V_2) \subseteq U_1 \cap U_2$. But $V_1 \cap V_2 \in w_1$ and $V_1 \cap V_2 \subseteq w_2C(V_1 \cap V_2) \subseteq w_2C(V_1) \cap w_2C(V_2) \subseteq U_1 \cap U_2$. Hence $U_1 \cap U_2 \in \theta_{w_1 w_2}$. □

Remark 2 *Observe that if (X, w_1, w_2) is a biweak space, $\theta_{w_1 w_2}$ is a topology on X.*

Theorem 6 *Let (X, w_1, w_2) be a biweak space. The collection $\delta_{w_1 w_2}$ is a strong generalized topology on X.*

Proof. It is easy to see that: $\emptyset$ and X belong to $\delta_{w_1 w_2}$. Consider $\{V_i : i \in I\}$ a collection of elements of $\delta_{w_1 w_2}$ and $x \in \bigcup_{i \in I} V_i$, then for some $i \in I$, $x \in V_i$ and then there is w_2-closed set F such that $x \in w_1I(F) \subseteq V_i$ and hence, $x \in w_1I(F) \subseteq V_i \subseteq \bigcup_{i \in I} V_i$. In consequence, $\bigcup_{i \in I} V_i \in \delta_{w_1 w_2}$. □

Remark 3 *According with Example 3, $\theta_{w_1 w_2} \subsetneq w_1$ and by Example 4, $\theta_{w_1 w_2} \subsetneq \delta_{w_1 w_2}$ and $\delta_{w_1 w_2} \subsetneq w_1$.*

Remark 4 *Let (X, w_1, w_2) be a biweak space. There are no relation between $\theta_{w_1 w_2}$ and $\delta_{w_1 w_2}$, see Examples 4 and 5.*

Remark 5 *If we start with a biweak space* (X, w_1, w_2). *We obtain that* $\theta_{w_1w_2}$ *is a topology on* X, *see Remark 2.* $\delta_{w_1w_2}$ *is a strong generalized topology on* X, *see Theorem 6 and there are no relation between* $\theta_{w_1w_2}$ *and* $\delta_{w_1w_2}$, *see Examples 4 and 5.*

Definition 5 $A \in \theta_{w_1w_2}$ *is called* $\theta_{w_1w_2}$*-open set and its complement is called* $\theta_{w_1w_2}$*-closed.*

According with Definition 5, we define the $\theta_{w_1w_2}$-closure of a subset A of X, as follows:

Definition 6 *Let* (X, w_1, w_2) *be a biweak space.*

1. *The* $\theta_{w_1w_2}$*-closure of* A *is defined as:*
   $C\theta_{w_1w_2}(A) = \bigcap\{F : A \subseteq F, F \text{ is } \theta_{w_1w_2}\text{-closed set in } X\}$.

2. *The* $\theta_{w_1w_2}$*-interior of* A *is defined as:*
   $I\theta_{w_1w_2}(A) = \bigcup\{U : U \subseteq A, U \text{ is } \theta_{w_1w_2}\text{-open set in } X\}$.

3. $\gamma\theta_{w_1w_2}(A) = \{x \in X : w_2C(U) \cap A \neq \emptyset, \text{ for every } U \in w_1$
   containing $x\}$.

Example 6 *In Example 2. The* $C\theta_{w_1w_2}(\emptyset) = \emptyset$, $C\theta_{w_1w_2}(X) = X$, $C\theta_{w_1w_2}(\{a\}) = \{a, c\}$, $C\theta_{w_1w_2}(\{b\}) = X$, $C\theta_{w_1w_2}(\{c\}) = \{c\}$, $C\theta_{w_1w_2}(\{a, b\}) = X$, $C\theta_{w_1w_2}(\{a, c\}) = \{a, c\}$, $C\theta_{w_1w_2}(\{b, c\}) = X$.

Theorem 7 *Let* (X, w_1, w_2) *and* (X, v_1, v_2) *be two biweak space and* $A \subseteq X$. *If* $w_1 \subseteq v_1$ *and* $w_2 \subseteq v_2$. *Then* $\theta_{w_1w_2} \subseteq \theta_{v_1v_2}$

Proof. Let $A \in \theta_{w_1w_2}$ and $x \in A$, then there exists an $U \in w_1$ such that $x \in U \subseteq w_2C(U) \subseteq A$. Since $w_1 \subseteq v_1$, $U \in v_1$ and $v_2C(U) \subseteq w_2C(U) \subseteq A$. □

Theorem 8 *Let* (X, w_1, w_2) *be a biweak space and* $A \subseteq X$. *The following are true:*

1. $A \subseteq \gamma\theta_{w_1w_2}(A) \subseteq C\theta_{w_1w_2}(A)$.

2. A *is* $\theta_{w_1w_2}$*-closed if and only if* $A = \gamma\theta_{w_1w_2}(A)$.

3. $x \in I\theta_{w_1w_2}(A)$ *if and only if there exists a* w_1*-open set* U *containing* x *such that* $x \in U \subseteq w_2C(U) \subseteq A$.

4. *if* A *is* w_2*-open, then* $w_1C(A) = \gamma\theta_{w_1w_2}(A)$.

Proof. 1. Since $\theta_{w_1w_2}$ is a weak space, the result follows.
2. If A is $\theta_{w_1w_2}$-closed, then $A = C\theta_{w_1w_2}(A)$, Now using 1, the result follows.
3. Is a consequence of Definition 6.
4. Let $x \in w_1C(A)$ and U any w_1-open set containing x, then $U \cap A \neq \emptyset$, follows that $w_2C(U) \cap A \neq \emptyset$ and then $w_1C(A) \subseteq \gamma\theta_{w_1w_2}(A)$. Now consider $x \in \gamma\theta_{w_1w_2}(A)$, then for each w_1-open set U containing x, $w_2C(U) \cap A \neq \emptyset$, and then there exists an element $z \in w_2C(U) \cap A$, since A is w_2-open, $z \in A$, therefore $x \in w_1C(A)$. □

4 Modification on weak continuous functions

Definition 7 *Let (X, w_1, w_2) and (Y, v_1, v_2) be two biweak spaces. A function $f : X \to Y$ is said to be $(\Theta_{w_1w_2}, \Theta_{v_1v_2})$-continuous if for every $\Theta_{v_1v_2}$-open set V, $f^{-1}(V)$ is $\Theta_{w_1w_2}$-open.*

Observe that if (X, w_1, w_2) and (Y, v_1, v_2) are two biweak spaces, $\Theta_{w_1w_2}$ and $\Theta_{v_1v_2}$ are topologies, then the notion of $(\Theta_{w_1w_2}, \Theta_{v_1v_2})$-continuous functions is similar to the well known concept of continuous functions.

Example 7 *In Example 4. Observe that:*

1. $\theta_{w_1w_2} = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$,

2. $\theta_{w_2w_1} = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\{b, c\}\}$,

The identity function $f : X \to X$ is $(\Theta_{w_2w_1}, \Theta_{w_1w_2})$-continuous but is not $(\Theta_{w_1w_2}, \Theta_{w_2w_1})$-continuous.

Theorem 9 *Let (X, w_1, w_2) and (Y, v_1, v_2) be two biweak spaces; let $f : X \to Y$. Then the following are equivalent:*

1. *f is $(\Theta_{w_1w_2}, \Theta_{v_1v_2})$-continuous,*

2. *For each $x \in X$ and each $\Theta_{v_1v_2}$-open set V containing $f(x)$, there exists a $\Theta_{w_1w_2}$-open set U containing $x$such that $f(U) \subseteq V$.*

3. *For each $x \in X$ and each $\Theta_{v_1v_2}$-open set V containing $f(x)$, there exists a w_1-open set U containing x such that $f(w_2C(U) \subseteq V$.*

Proof. The proof follows applying definition. □

Definition 8 *Let (X, w_1) be a weak space and (Y, v_1, v_2) be a biweak space. A function $f : (X, w_1) \to (Y, v_1, v_2)$ is said to be faintly $(w_1, \Theta_{v_1 v_2})$-continuous if for every $\Theta_{v_1 v_2}$-open set U, $f^{-1}(U)$ is w_1-open.*

Example 8 *Let (X, w_1, w_2) and (Y, v_1, v_2) be a biweak spaces, where $X = Y = \{a, b, c\}$, $w_1 = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}$, $w_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{c\}\}$, $v_1 = \{\emptyset, Y, \{a\}, \{b\}\}$ and $v_2 = \{\emptyset, Y, \{a\}, \{c\}\}$.*
Observe that:

1. $\theta_{w_1 w_2} = \{\emptyset, X, \{b\}, \{a, b\}\}$,
2. $\theta_{w_2 w_1} = \{\emptyset, X, \{c\}, \{a, c\}\}$,
3. $\theta_{v_1 v_2} = \{\emptyset, X, \{b\}, \{a, b\}\}$,
4. $\theta_{v_2 v_1} = \{\emptyset, X, \{c\}, \{a, c\}\}$.

Consider a function $f : (X, w_2) \to (Y, v_1, v_2)$ defined as $f(a) = b$, $f(b) = a$, $f(c) = c$. Then f is faintly $(w_2, \Theta_{v_1 v_2})$-continuous but is neither $(\Theta_{w_1 w_2}, \Theta_{v_1 v_2})$-continuous nor $(\Theta_{w_2 w_1}, \Theta_{v_1 v_2})$-continuous

Example 9 *The function defined in Example 4 is not faintly $(w_1, \Theta_{w_1 w_2})$-continuous*

Remark 6 *If (X, w_1, w_2), (Y, v_1, v_2) are two biweak spaces and $f : (X, w_1, w_2) \to (Y, v_1, v_2)$ is a function. The concepts of $(\Theta_{w_1 w_2}, \Theta_{v_1 v_2})$-continuous and faintly $(w_1, \Theta_{v_1 v_2})$-continuous are independent.*

Theorem 10 *Let (X, w_1, w_2) and (Y, v_1, v_2) be two biweak spaces. If $f : (X, w_1, w_2) \to (Y, v_1, v_2)$ is $(\Theta_{w_1 w_2}, \Theta_{v_1 v_2})$-continuous, then for every $\Theta_{v_1 v_2}$-closed set F, $f^{-1}(F)$ is a $\Theta_{w_1 w_2}$-closed set.*

Proof. It follows by duality. □

Definition 9 *Let (X, w_1) be a weak space and (Y, v_1, v_2) be a biweak space. A function $f : (X, w_1) \to (Y, v_1, v_2)$ is said to be mixed weakly $(w_1, v_1 v_2)$-continuous at $x \in X$ if for every v_1-open set V, containing $f(x)$, there exists a w_1-open set U containing x such that $f(U) \subseteq v_2 C(V)$. Then f is mixed weakly $(w_1, v_1 v_2)$-continuous if it is mixed weakly $(w_1, v_1 v_2)$-continuous at every point $x \in X$.*

Example 10 *Let* (X, w_1) *be a weak space and* (Y, ν_1, ν_2) *be a biweak space, where* $X = Y = \{a, b, c\}$ *and weak structures:* $w_1 = \{\emptyset, X, \{a\}, \{b\}\}$, $\nu_1 = \{\emptyset, X, \{b\}\}$ *and* $\nu_2 = \{\emptyset, X, \{a\}\}$. *Consider* $f : (X, w_1) \to (Y, \nu_1, \nu_2)$, *defined as* $f(a) = b$, $f(b) = c$, $f(c) = a$. *Then* f *is mixed weakly* $(w_1, \nu_1\nu_2)$*-continuous.*

Remark 7 *Let* (X, w) *be a weak space and* (Y, ν_1, ν_2) *be a biweak space. If* $\nu_1 = \nu_2$, *then the notion of mixed weakly* $(w, \nu_1\nu_2)$*-continuous function is just the notion of weak weakly* (w, ν_1)*-continuous functions, that is, for any* ν_1*-open set* V, *there exists a* w_1*-open set* U *such that* $f(U) \subseteq \nu_1 C(V)$.

Theorem 11 *Let* $f : X \to Y$ *be a function,* w_1 *a weak structure on a nonempty set* X, *and* ν_1, ν_2 *be two weak structures on a nonempty set* Y. *Then:*

1. *If* f *is mixed weakly* $(w_1, \nu_1\nu_2)$*-continuous, then* $f(w_1C(A)) \subseteq \gamma\theta_{\nu_1\nu_2}(f(A))$ *for every subset* A *of* X.

2. *If* $f(w_1C(A)) \subseteq \gamma\theta_{\nu_1\nu_2}(f(A))$ *for every subset* A *of* X, *then* $w_1C(f^{-1}$ $(\nu_2I(G)) \subseteq f^{-1}(\nu_1C(V))$ *for every* ν_2*-open set* V *of* Y.

Proof. 1. Consider $A \subseteq X$, $x \in w_1C(A)$ and V any ν_1-open set containing $f(x)$. By hypothesis f is mixed weakly $(w_1, \nu_1\nu_2)$-continuous, then there exists a w_1-open set U containing x such that $f(U) \subseteq \nu_2C(V)$. Since $x \in w_1C(A)$ and U is a w_1-open set U containing x, $A \cap U \neq \emptyset$. In consequence, $\emptyset \neq f(A) \cap f(U) \subseteq \nu_2C(V) \cap f(A)$. Follows that $f(x) \in \gamma\theta_{\nu_1\nu_2}(f(A))$ and hence, $f(w_1C(A)) \subseteq \gamma\theta_{\nu_1\nu_2}(f(A))$.
2. Clear. □

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Uniqueness theorems related to weighted sharing of two sets

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Abstract. Using the notion of weighted sharing of sets, we study the uniqueness problem of meromorphic functions sharing two finite sets. Our results are inspired from an article due to J. F. Chen (Open Math., 15 (2017), 1244–1250).

1 Introduction, Definitions and Main results

In this paper, a meromorphic function means a function which is meromorphic in the entire complex plane $\mathbb{C}$. Throughout the paper, we adopt the standard notations of Nevanlinna value distribution theory as explained in [6] and [12]. We denote by $\mathcal{M}(\mathbb{C})$ the class of all meromorphic functions defined in $\mathbb{C}$ and by $\mathcal{M}_1(\mathbb{C})$ the class of meromorphic functions which have finitely many poles in $\mathbb{C}$. For convenience, we denote any set of positive real numbers of finite linear measure by E, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ for $r \to \infty, r \notin E$. The order $\lambda(f)$ of $f \in \mathcal{M}(\mathbb{C})$ is defined as

$$\lambda(f) = \limsup_{r \longrightarrow \infty} \frac{\log T(r,f)}{\log r}.$$

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For a meromorphic function f and a set $S \subset \mathbb{C} \cup \{\infty\}$, we define $E_f(S)$ ($\overline{E}_f(S)$) to be the set of all a-points of f, where $a \in S$, together with their multiplicities (ignoring their multiplicities). We say that two functions f and g share the set S CM (IM) if $E_f(S) = E_g(S)$ ($\overline{E}_f(S) = \overline{E}_g(S)$).

The development of research works related to set sharing problems was broadly initiated due to the following question which was raised by F. Gross [5].

Question 1 *Can one find two finite sets $S_i (i = 1, 2)$ of $\mathbb{C} \cup \{\infty\}$ such that any two nonconstant entire functions f and g satisfying $E_f(S_i) = E_g(S_i)$ for $i = 1, 2$ must be identical?*

In 1994, H. X. Yi [14] proved the following theorem which gives an affirmative answer to Gross's question.

Theorem A *Let $S_1 = \{\omega \mid \omega^n - 1 = 0\}$ and $S_2 = \{a\}$, where $n \geq 5$ is an integer, $a \neq 0$ and $a^{2n} \neq 1$. If f and g are entire functions such that $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.*

In [5], F. Gross also pointed that if the answer of Question 1 is affirmative, then it would be interesting to know how large the sets can be.

In 1998, H. X. Yi [15] proved the following theorem which deals with the above comment.

Theorem B *Let $S_1 = \{0\}$ and $S_2 = \{\omega \mid \omega^2(\omega + a) - b = 0\}$, where a and b are two nonzero constants such that the algebraic equation $\omega^2(\omega + a) - b = 0$ has no multiple roots. If f and g are two entire functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.*

In this direction, a lot of research works have been devoted during the last two decades (see [4], [9], [10], [13]).

We recall the following recent result due to J. F. Chen [2].

Theorem C *Let k be a positive integer and let $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $S_2 = \{\beta_1, \beta_2\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2$ are $k+2$ distinct finite complex numbers satisfying*

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2.$$

If two nonconstant meromorphic functions f and g in $\mathcal{M}_1(\mathbb{C})$ share S_1 CM, S_2 IM, and if the order of f is neither an integer nor infinite, then $f \equiv g$.

In the same paper, the author also proved another result concerning unique range sets. Before stating the result, we present the definition of unique range sets.

Definition 1 *For a family of functions* $\mathcal{G}$, *the subsets* $S_1, S_2, \ldots, S_q$ *of* $\mathbb{C} \cup \{\infty\}$ *such that for any* $f, g \in \mathcal{G}$, f *and* g *share* S_j *CM for* $j = 1, 2, \ldots, q$ *imply* $f \equiv g$, *are called unique range sets (URS, in brief) for the functions in* $\mathcal{G}$.

Theorem D *Let* k *be a positive integer and let* $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $S_2 = \{\beta_1, \beta_2\}$, *where* $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2$ *are* $k+2$ *distinct finite complex numbers satisfying*

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2.$$

If the order of f *is neither an integer nor infinite, then the sets* S_1 *and* S_2 *are the URS of meromorphic functions in* $\mathcal{M}_1(\mathbb{C})$.

The condition $(\beta_1-\alpha_1)^2(\beta_1-\alpha_2)^2 \ldots (\beta_1-\alpha_k)^2 \neq (\beta_2-\alpha_1)^2(\beta_2-\alpha_2)^2 \ldots (\beta_2-\alpha_k)^2$ in Theorems C and D can not be dropped as shown by the following example.

Example 1 [2] *For a positive integer* k, *let* $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{3n}}$, $g(z) = -f(z)$, $S_1 = \{-1, 1, -2, 2, \ldots, -k, k\}$, *and* $S_2 = \{-(k+1), k+1\}$. *Then using the result of* [3, p. 288] *we deduce*

$$\lambda(f) = \frac{1}{\liminf\limits_{n \longrightarrow \infty} \dfrac{\log n^{3n}}{n \log n}} = \limsup_{n \longrightarrow \infty} \frac{n \log n}{\log n^{3n}} = \frac{1}{3}.$$

Clearly $f(z), g(z) \in \mathcal{M}_1(\mathbb{C})$, $f(z)$ *and* $g(z)$ *share* S_1, S_2 *CM. But* $f(z) \not\equiv g(z)$.

The assumption "nonconstant meromorphic functions f and g in $\mathcal{M}_1(\mathbb{C})$" in Theorems C and D cannot be relaxed to "nonconstant meromorphic functions f and g in $\mathcal{M}(\mathbb{C})$" as shown by the following example.

Example 2 [2] *For a positive integer* k, *let* $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{3n}}$, $g(z) = \frac{1}{f(z)}$, $S_1 = \left\{2, \frac{1}{2}, 3, \frac{1}{3}, \ldots, k, \frac{1}{k}\right\}$, $S_2 = \left\{k+1, \frac{1}{k+1}\right\}$. *From Example 1 we note that* $\lambda(f) = \frac{1}{3}$ *and, therefore, using the result of* [3, p. 293] *we see that* $g(z)$ *has infinitely many poles in* $\mathbb{C}$. *Moreover,* $f(z)$ *and* $g(z)$ *share the sets* S_1, S_2 *CM. But* $f(z) \not\equiv g(z)$.

The following example given in [2] shows the necessity of the assumption in Theorems C and D that the order of f is neither an integer nor infinite.

Example 3 *For a positive integer* k, *let* $f(z) = e^z$ (*resp.* $f(z) = e^{e^z}$), $g(z) = \frac{1}{f(z)}$, $S_1 = \left\{2,\ \frac{1}{2},\ 3,\ \frac{1}{3}, \ldots,\ k,\ \frac{1}{k}\right\}$, $S_2 = \left\{k+1,\ \frac{1}{k+1}\right\}$. *Then by Lemma 8 in section 2 we see that* $\lambda(f) = 1$ (*resp.* $\lambda(f) = \infty$). *Though all other conditions of Theorems C and D are satisfied,* $f(z) \not\equiv g(z)$.

However, the research on set sharing problem gained a new dimension when the idea of weighted sharing, introduced by I. Lahiri in 2001 (see [7], [8]), was incorporated. The necessary definitions are as follows:

Definition 2 *Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all* a*-points of* f, *where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and* $k+1$ *times if* $m > k$. *If* $E_k(a; f) = E_k(a; g)$, *we say that* f *and* g *share the value* a *with weight* k.

We write f and g share (a, k) to mean that f and g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p where $0 \leq p < k$. In particular, f and g share a CM (IM) if and only if f and g share (a, ∞) $((a, 0))$.

Definition 3 *Let* S *be a set of distinct elements of* $\mathbb{C} \cup \{\infty\}$ *and* k *be a nonnegative integer or infinity. We denote by* $E_f(S, k)$ *the set* $\cup_{a \in S} E_k(a; f)$. *We say that* f *and* g *share the set* S *with weight* k, *or simply* f *and* g *share* (S, k) *if* $E_f(S, k) = E_g(S, k)$.

Definition 4 *Let* k *be a positive integer and* $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, *where* α_i*'s are nonzero complex constants. Suppose that*

$$P(z) = \frac{z^k - (\sum \alpha_i) z^{k-1} + \ \ldots + (-1)^{k-1} (\sum \alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_{k-1}}) z}{(-1)^{k+1} \alpha_1 \alpha_2 \ldots \alpha_k}, \tag{1}$$

where $\alpha_i \in S_1$ *for* $i = 1, 2, \ldots, k$. *Let* m_1 *be the number of simple zeros of* $P(z)$ *and* m_2 *be the number of multiple zeros of* $P(z)$. *Then we define* $\Gamma_1 := m_1 + m_2$ *and* $\Gamma_2 := m_1 + 2m_2$.

Regarding Theorem C, one may ask the following question:

Question 2 *Is the conclusion of Theorem C still true if* f *and* g *share* $(S_1, 2)$ *and* S_2 *IM instead of sharing* S_1 *CM and* S_2 *IM?*

In this paper, we try to find possible answers to the above question and prove the following theorems:

Theorem 1 *Let* $f,\ g \in \mathcal{M}_1(\mathbb{C})$ *and* $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $S_2 = \{\beta_1, \beta_2\}$, *where* $\alpha_1,\ \alpha_2, \ldots, \alpha_k,\ \beta_1,\ \beta_2$ *are* $k+2$ *distinct nonzero complex constants satisfying* $k > 2\Gamma_2$. *If* $f,\ g$ *share* $(S_1, 2)$ *and* S_2 *IM, then* $f \equiv g$, *provided*

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2$$

and f *is of non-integer finite order.*

Theorem 2 *Let* S_1 *and* S_2 *be stated as in Theorem 1 with* $k > 2\Gamma_2$. *If* $\mathcal{M}_2(\mathbb{C})$ *denote the subclass of meromorphic functions of non-integer finite order in* $\mathcal{M}_1(\mathbb{C})$, *then the sets* S_1 *and* S_2 *are the URS of meromorphic functions in* $\mathcal{M}_2(\mathbb{C})$, *provided*

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2.$$

We now state some more definitions (see [7], [8]).

Definition 5 *For* $a \in \mathbb{C} \cup \{\infty\}$, *we denote by* $\overline{N}(r, a; f| = k)$ *the reduced counting function of the* a*-points of* f *whose multiplicities are exactly* k. *In particular,* $\overline{N}(r, a; f| = 1)$ *or* $N(r, a; f| = 1)$ *is the counting function of the simple* a*-points of* f.

Definition 6 *For a positive integer* m *we denote by* $N(r, a; f| \leq m)$ $(N(r, a; f| \geq m))$ *the counting function of those* a*-points of* f *whose multiplicities are not greater (less) than* m, *where each* a*-point is counted according to its multiplicity.* $\overline{N}(r, a; f| \leq m)$ *and* $\overline{N}(r, a; f| \geq m)$ *are the corresponding reduced counting functions.*

Definition 7 *We denote by* $N_2(r, a; f)$ *the sum* $\overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2)$.

Definition 8 *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share* $(a, 2)$ *for* $a \in \mathbb{C} \cup \{\infty\}$. *Let* z_0 *be an* a*-point of* f *with multiplicity* p *and an* a*-point of* g *with multiplicity* q. *We denote by* $\overline{N}_L(r, a; f)$ $(\overline{N}_L(r, a; g))$ *the reduced counting function of those* a*-points of* f *and* g *where* $p > q \geq 3$ $(q > p \geq 3)$. *Also we denote by* $\overline{N}_E^{(3}(r, a; f)$ *the counting function of those* a*-points of* f *and* g *where* $p = q \geq 3$. *Clearly* $\overline{N}_E^{(3}(r, a; f) = \overline{N}_E^{(3}(r, a; g)$.

Definition 9 *Let* $f,\ g$ *share the value* a *IM. We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g.

Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2 Lemmas

In this section, we present some lemmas which will be needed in the sequel. We denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are two meromorphic functions in $\mathcal{M}_1(\mathbb{C})$.

Lemma 1 [7] *If* F, G *share* $(1,1)$ *and* $H \not\equiv 0$, *then*

$$N(r,1;F|=1) \leq N(r,\infty;H) + S(r,F) + S(r,G).$$

Lemma 2 *Let* $F, G \in \mathcal{M}_1(\mathbb{C})$. *If* F, G *share* $(1,0)$ *and* $H \not\equiv 0$, *then*

$$\begin{aligned} N(r,\infty,H) \leq\ & \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + \overline{N}_*(r,1;F,G) \\ & + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G), \end{aligned}$$

where $\overline{N}_0(r,0;F')$ *is the reduced counting function of those zeros of* F' *which are not the zeros of* $F(F-1)$. $\overline{N}_0(r,0;G')$ *is defined similarly.*

Proof. Noting that $\overline{N}_*(r,\infty;F,G) = S(r,F)+S(r,G)$, this lemma can be proved in a similar manner as in Lemma 4 of [9]. □

Lemma 3 [1] *Let* F *and* G *be two nonconstant meromorphic functions sharing* $(1,2)$. *Then*

$$\begin{aligned} 2\overline{N}_L(r,1;F) + 3\overline{N}_L(r,1;G) + 2\overline{N}_E^{(3}(r,1;F) + \overline{N}(r,1;F| = 2) \\ \leq N(r,1;G) - \overline{N}(r,1;G). \end{aligned}$$

Lemma 4 [11] *Let* f *be a nonconstant meromorphic function and* $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, *where* a_0, a_1, $a_2, \ldots, a_n$ *are constants and* $a_n \neq 0$. *Then* $T(r,P(f)) = nT(r,f) + O(1)$.

Lemma 5 [15] *If* $H \equiv 0$, *then* $T(r,G) = T(r,F) + O(1)$. *If, in addition,*

$$\limsup_{r \to \infty, r \notin E} \frac{\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G)}{T(r)} < 1,$$

where $T(r) = \max\{T(r,F), T(r,G)\}$ *then either* $F \equiv G$ *or* $F.G \equiv 1$.

Remark 1 *We observe that the above lemma holds for* $F, G \in \mathcal{M}(\mathbb{C})$*. As our discussion is restricted in* $\mathcal{M}_1(\mathbb{C})$*, we may drop the terms* $\overline{N}(r,\infty;F)$ *and* $\overline{N}(r,\infty;G)$ *while using this result.*

Lemma 6 *Let* $F, G \in \mathcal{M}_1(\mathbb{C})$*. If* F *and* G *share* $(1,2)$ *and* $H \not\equiv 0$*, then*

(i) $T(r,F) \le N_2(r,0;F) + N_2(r,0;G) - m(r,1;G) - \overline{N}_E^{(3}(r,1;F) - \overline{N}_L(r,1;G) + S(r,F) + S(r,G)$;

(ii) $T(r,G) \le N_2(r,0;G) + N_2(r,0;F) - m(r,1;F) - \overline{N}_E^{(3}(r,1;G) - \overline{N}_L(r,1;F) + S(r,F) + S(r,G)$.

Proof. The proof of this lemma flows in the line of the proof of Lemma 2.13 in [1]. As we are dealing with functions of class $\mathcal{M}_1(\mathbb{C})$, we insist in presenting the proof for the sake of completeness.

From the second fundamental theorem of Nevanlinna , we have

$$T(r,F) \le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) - N_0(r,0;F') + S(r,F);$$

that is,

$$T(r,F) \le \overline{N}(r,0;F) + \overline{N}(r,1;F) - N_0(r,0;F') + S(r,F). \tag{2}$$

Similarly,

$$T(r,G) \le \overline{N}(r,0;G) + \overline{N}(r,1;G) - N_0(r,0;G') + S(r,G). \tag{3}$$

Combining (2) and (3), we obtain

$$\begin{aligned} T(r,F) + T(r,G) \le\ & \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,1;F) + \overline{N}(r,1;G) \\ & - N_0(r,0;F') - N_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{4}$$

We also see that

$$\begin{aligned} \overline{N}(r,1;F) + \overline{N}(r,1:G) \le\ & N(r,1;F|=1) + \overline{N}(r,1;F|=2) + \overline{N}_E^{(3}(r,1;F) \\ & + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;G). \end{aligned} \tag{5}$$

Using Lemma 1 and Lemma 2 in (5), we obtain that

$$\begin{aligned} \overline{N}(r,1;F) + \overline{N}(r,1;G) \le\ & \overline{N}(r,0;F| \ge 2) + \overline{N}(r,0;G| \ge 2) + 2\overline{N}_L(r,1;F) \\ & + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;F|=2) + \overline{N}_E^{(3}(r,1;F) \\ & + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + \overline{N}(r,1;G) \\ & + S(r,F) + S(r,G). \end{aligned}$$

Substituting the value of $\overline{N}(r,1;G)$ from Lemma 3, we obtain

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) &\leq \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+2\overline{N}_L(r,1;F)\\ &\quad +2\overline{N}_L(r,1;G)+\overline{N}(r,1;F|=2)+\overline{N}_E^{(3}(r,1;F)\\ &\quad +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+N(r,1;G)\\ &\quad -2\overline{N}_L(r,1;F)-3\overline{N}_L(r,1;G)-2\overline{N}_E^{(3}(r,1;F)\\ &\quad -\overline{N}(r,1;F|=2)+S(r,F)+S(r,G)\\ &\leq \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)-\overline{N}_L(r,1;G)\\ &\quad -\overline{N}_E^{(3}(r,1;F)+T(r,G)-m(r,1;G)+\overline{N}_0(r,0;F')\\ &\quad +\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \qquad (6)\end{aligned}$$

Noting the fact that $N_2(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f|\geq 2)$, the lemma follows from (4) and (6). □

Lemma 7 *Let* f, $g \in \mathcal{M}_1(\mathbb{C})$. *If* f, g *share the set* $\{\beta_1,\beta_2\}$ *IM, then* $\lambda(f)=\lambda(g)$.

Proof. Proof of this lemma can be extracted from the first part of the proof of Theorem 1.3 in [2] (see p. 1247). □

Lemma 8 (see [12, p. 65]) *Let* h *be an entire function and* $f(z)=e^{h(z)}$. *Then*

(i) *if* $h(z)$ *is a polynomial of* $\deg h$, *then* $\lambda(f)=\deg h$;

(ii) *if* $h(z)$ *is a transcendental entire function, then* $\lambda(f)=\infty$.

Lemma 9 (see [12, p. 115]) *Let* a_1, a_2 *and* a_3 *be three distinct complex numbers in* $\mathbb{C}\cup\{\infty\}$. *If two nonconstant meromorphic functions* f *and* g *share* a_1, a_2 *and* a_3 *CM, and if the order of* f *and* g *is neither an integer nor infinity, then* $f\equiv g$.

3 Proof of the Theorems

Proof. [Proof of Theorem 1] Let $F=P(f)$ and $G=P(g)$ where $P(z)$ is defined as in (1). Clearly F, G share $(1,2)$ as f, g share $(S_1,2)$. From Lemma 4, we obtain

$$T(r,F)=kT(r,f)+S(r,f); \qquad (7)$$

$$T(r,G) = kT(r,g) + S(r,g). \tag{8}$$

Let $H \not\equiv 0$. By Lemma 6, we have

$$\begin{aligned} T(r,F) &\le N_2(r,0;F) + N_2(r,0;G) + S(r,F) + S(r,G) \\ &= N_2(r,0;P(f)) + N_2(r,0;P(g)) + S(r,f) + S(r,g) \\ &\le \Gamma_2\overline{N}(r,0;f) + \Gamma_2\overline{N}(r,0;g) + S(r,f) + S(r,g) \\ &\le \Gamma_2\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned} \tag{9}$$

Similarly,

$$T(r,G) \le \Gamma_2\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \tag{10}$$

From (7)-(10), we obtain

$$k\{T(r,f) + T(r,g)\} \le 2\Gamma_2\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g),$$

which is a contradiction as $k > 2\Gamma_2$. Hence $H \equiv 0$.

Let $T(r) = \max\{T(r,F), T(r,G)\}$. Now,

$$\begin{aligned} \overline{N}(r,0;F) + \overline{N}(r,0;G) &\le \Gamma_1\overline{N}(r,0;f) + \Gamma_1\overline{N}(r,0;g) \\ &\le \Gamma_1\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g) \\ &= \frac{\Gamma_1}{k}\{T(r,F) + T(r,G)\} + S(r,F) + S(r,G) \\ &\le \frac{2\Gamma_1}{k}T(r) + o\{T(r)\}. \end{aligned} \tag{11}$$

As $k > 2\Gamma_2 \ge 2\Gamma_1$, from Lemma 5 and (11), we obtain either $F \equiv G$ or $F.G \equiv 1$.

If possible, let $F.G \equiv 1$. Then $P(f).P(g) \equiv 1$. As $g \in \mathcal{M}_1(\mathbb{C})$, we have $P(g) \in \mathcal{M}_1(\mathbb{C})$. Hence $P(f)$ has at most finitely many zeros. Therefore $P(f) = \mu_1(z)e^{\phi_1(z)}$, where $\mu_1(z)$ is a rational function and $\phi_1(z)$ is an entire function, which is a contradiction by Lemma 8 as the order of f is neither an integer not infinity. Similarly if we consider the case when $P(g)$ has at most finitely many zeros, we arrive at a contradiction as $\lambda(g) = \lambda(f)$, by Lemma 7. Hence the case $F.G \equiv 1$ can not occur.

If $F \equiv G$, we have $P(f) \equiv P(g)$, which gives

$$\frac{(f(z)-\alpha_1)(f(z)-\alpha_2)\ldots(f(z)-\alpha_k)}{(g(z)-\alpha_1)(g(z)-\alpha_2)\ldots(g(z)-\alpha_k)} \equiv 1. \tag{12}$$

From (12) and the assumption

$$(\beta_1-\alpha_1)^2(\beta_1-\alpha_2)^2\cdots(\beta_1-\alpha_k)^2 \neq (\beta_2-\alpha_1)^2(\beta_2-\alpha_2)^2\cdots(\beta_2-\alpha_k)^2,$$

we obtain that $f(z) = \beta_1$ if and only if $g(z) = \beta_1$ since f and g share S_2 IM. Similarly, we see that $f(z) = \beta_2$ if and only if $g(z) = \beta_2$. Consequently, we have f and g share β_1 and β_2 IM. Again, from (12) we see that f and g share β_1, β_2 and ∞ CM. Noting that the order of f is neither an integer nor infinity, the conclusion follows from Lemma 7 and Lemma 9. □

Proof. [Proof of Theorem 2] If f, g share S_1 and S_2 CM, then f, g certainly share $(S_1, 2)$ and S_2 IM, which satisfies the conditions of Theorem 1 and hence the conclusion follows. Here we omit the details. □

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Scaling functions on the spectrum

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Abstract. A generalization of Mallat's classic theory of multiresolution analysis based on the theory of spectral pairs was considered by Gabardo and Nashed [4] for which the translation set $\Lambda = \{0, r/N\} + 2\,\mathbb{Z}$ is no longer a discrete subgroup of $\mathbb{R}$ but a spectrum associated with a certain one-dimensional spectral pair. In this short communication, we characterize the scaling functions associated with such a nonuniform multiresolution analysis by means of some fundamental equations in the Fourier domain.

1 Introduction

Multiresolution analysis (MRA) is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems. The concept of an MRA structure has been extended in various setups in recent years. More precisely, they have been generalized to different dimensionalities, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of MRA to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$ (see [1]). All these concepts were developed on regular lattices, that is the translation set is always a group. Recently, Gabardo and Nashed [3, 4] considered a generalization of Mallat's classical MRA [6] based on the theory of

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spectral pairs, in which the translation set $\Lambda = \{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N-1, r$ is an odd integer relatively prime to N, acting on the scaling function related with an MRA to generate the core subspace V_0 is no longer a group, but a union of two lattices, which is associated with a famous open conjecture of Fuglede on spectral pairs [2]. They call it *nonuniform multiresolution analysis* (NUMRA). By an NUMRA, we mean a sequence of embedded closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of the Hilbert space $L^2(\mathbb{R})$ that satisfies the following conditions:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f(x) \in V_j$ if and only if $f(2Nx) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) there exists a function $\phi \in V_0$ such that $\{\phi(x-\lambda)\}_{\lambda\in\Lambda}$ is an orthonormal basis for V_0.

It is worth noticing that, when $N = 1$, one recovers the standard definition of one dimensional MRA with dyadic dilation 2. When, $N > 1$, the dilation factor of $2N$ ensures that $2N\Lambda \subset \mathbb{Z} \subset \Lambda$.

If ϕ is a scaling function of an NUMRA, then by condition (e) we can express this function in terms of the orthonormal basis $\{\phi(x-\lambda) : \lambda \in \Lambda\}$ as

$$\phi(x) = \sum_{\lambda\in\Lambda} h_\lambda \phi\big(2Nx - \lambda\big). \tag{1}$$

where the convergence is in $L^2(\mathbb{R})$ and $\{h_\lambda\}_{\lambda\in\Lambda} \in l^2$. Refinement equation (1) can be rewritten in the Fourier domain as

$$\hat{\phi}(\xi) = m_0\left(\frac{\xi}{2N}\right)\hat{\phi}\left(\frac{\xi}{2N}\right) \tag{2}$$

where m_0 is the low pass filter associated with the scaling function ϕ and is of the form

$$m_0(\xi) = m_0^1(\xi) + e^{-2\pi i\xi r/N} m_0^2(\xi). \tag{3}$$

One of the fundamental problems in the study of wavelet theory is to find conditions on the scaling functions so that they can generate an MRA for $L^2(\mathbb{R})$. Our main purpose in this short communication is to characterize those functions that are scaling functions for an NUMRA of $L^2(\mathbb{R})$.

To achieve our goal, we need the following technical results obtained in [4, 5, 7] that will be used in sequel.

Theorem 1 [4] *Let* $\{V_j : j \in \mathbb{Z}\}$ *be a sequence of closed subspaces of* $L^2(\mathbb{R})$ *satisfying conditions (a), (d) and (e). Then,* $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

Theorem 2 [5] *Let* $\{V_j : j \in \mathbb{Z}\}$ *be a sequence of closed subspaces of* $L^2(\mathbb{R})$ *satisfying conditions (a), (d) and (e). Assume that the function* ϕ *of condition (e) is such that* $\hat{\phi}$ *is continuous at* $\xi = 0$. *Then the following two conditions are equivalent:*

(i) $\lim_{j\to\infty} \left|\hat{\phi}\big((2N)^{-j}\xi\big)\right| = 1 \quad \text{a.e. } \xi \in \mathbb{R}$;

(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.

Proposition 1 [7] *Let* N *be a positive integer, and* $r \in \{1, 3, \dots, 2N-1\}$ *be an odd integer. Let* $\phi \in L^2(\mathbb{R})$ *with* $\|\phi\|_2 = 1$. *Then,*

(i) *For each fixed odd* r, *the family* $\{\phi(x-\lambda) : \lambda \in \Lambda\}$ *is an orthonormal system in* $L^2(\mathbb{R})$ *if and only if*

$$\sum_{p\in\mathbb{Z}} \left|\hat{\phi}\left(\xi + \frac{p}{2}\right)\right|^2 = 2, \quad \text{for a.e. } \xi \in \mathbb{R} \quad \text{and} \tag{4}$$

$$\sum_{p\in\mathbb{Z}} e^{-i\pi rp/N} \left|\hat{\phi}\left(\xi + \frac{p}{2}\right)\right|^2 = 0, \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{5}$$

(ii) *The collection* $\{\phi(x-\lambda) : \lambda \in \Lambda\}$ *is an orthonormal system for every odd integer* $r \in \{1, 3, \dots, 2N-1\}$ *if and only if*

$$\sum_{\beta\in\Gamma_N} \left|\hat{\phi}(\xi - \beta)\right|^2 = 1, \ \text{for a.e. } \xi \in \mathbb{R}, \tag{6}$$

where $\Gamma_N = \{nN + j/2 : n \in \mathbb{Z},\ j = 0, 1, 2, \dots, N-1\}$.

2 Characterization of scaling functions on the spectrum

In this section we will characterize those functions that are scaling functions for an NUMRA of $L^2(\mathbb{R})$ by means of some basic equations in the Fourier domain.

Before formulating our main result, let us clarify what we mean when we say that a function is a scaling function for an NUMRA. Given a function $\phi \in L^2(\mathbb{R})$, we define the closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ as follows:

$$V_0 = \overline{\text{span}}\{\phi(x-\lambda) : \lambda \in \Lambda\}, \text{ and } V_j = \left\{f : f\big((2N)^{-j}x\big) \in V_0\right\},\ j \in \mathbb{Z}\backslash\{0\}.$$

We say that $\phi \in L^2(\mathbb{R})$ is a scaling function for an NUMRA of $L^2(\mathbb{R})$ if the sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ as defined above forms an NUMRA for $L^2(\mathbb{R})$.

Theorem 3 *A function $\phi \in L^2(\mathbb{R})$ is a scaling function for an NUMRA of $L^2(\mathbb{R})$ if and only if*

$$\sum_{\beta \in \Gamma_N} \left|\hat{\phi}(\xi - \beta)\right|^2 = 1, \ \mathrm{texta.e} \tag{7}$$

$$\lim_{j \to \infty} \left|\hat{\phi}\left((2N)^{-j}\xi\right)\right| = 1 \ \ a.e. \ \xi \in \mathbb{R} \tag{8}$$

and there exists a periodic function m_0 of the form (3) such that

$$\hat{\phi}(\xi) = m_0\left(\frac{\xi}{2N}\right)\hat{\phi}\left(\frac{\xi}{2N}\right), \quad \text{a.e. } \xi \in \mathbb{R}. \tag{9}$$

Proof. Suppose ϕ is a scaling function for an NUMRA. Then, $\{\phi(x-\lambda) : \lambda \in \Lambda\}$ forms an orthonormal system in $L^2(\mathbb{R})$ which is equivalent to equation (7) by Proposition 1. Equality (9) follows from equations (2) and (3). Since $\{V_j : j \in \mathbb{Z}\}$ is an NUMRA for $L^2(\mathbb{R})$, we have $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$. Therefore, from Theorem 2, we infer that

$$\lim_{j\to\infty} \int_{\Gamma_N} \left|\hat{\phi}\left((2N)^{-j}\xi\right)\right|^2 d\xi = 1.$$

Since $m_0(\xi)$ is of the form (3), so it is easy to compute the following two conditions in terms of the 1/2-periodic functions m_0^1, m_0^2 as

$$\sum_{p=0}^{2N-1} \left\{\left|m_0^1\left(\xi + \frac{p}{4N}\right)\right|^2 + \left|m_0^2\left(\xi + \frac{p}{4N}\right)\right|^2\right\} = 1, \quad \text{and} \tag{10}$$

$$\sum_{p=0}^{2N-1} e^{-i\pi r p/N}\left\{\left|m_0^1\left(\xi + \frac{p}{4N}\right)\right|^2 + \left|m_0^2\left(\xi + \frac{p}{4N}\right)\right|^2\right\} = 0. \tag{11}$$

If we take $M_0(\xi) = \left|m_0^1(\xi)\right|^2 + \left|m_0^2(\xi)\right|^2$, then clearly $M_0\left(\xi + \frac{1}{4}\right) = M_0(\xi)$ and

$$M_0(\xi) = \frac{|m_0(\xi + N/2)|^2 + |m_0(\xi)|^2}{2}. \tag{12}$$

Subsequently, Eqs. (10) and (11) takes the form

$$\sum_{p=0}^{2N-1} M_0\left(\xi+\frac{p}{4N}\right)=1, \quad \text{and} \quad \sum_{p=0}^{2N-1} e^{-i\pi rp/N} M_0\left(\xi+\frac{p}{4N}\right)=0.$$

Hence, $M_0(\xi) \leq 1$, a.e. $\xi \in \mathbb{R}$, which together with (12) implies $|m_0(\xi)| \leq 1$ a.e. $\xi \in \mathbb{R}$. This inequality along with equality (9) shows that $\left|\hat{\phi}((2N)^{-j}\xi)\right|$ is non-decreasing for a.e. $\xi \in \mathbb{R}$ as $j \to \infty$. Let

$$\Phi(\xi)=\lim_{j\to\infty}\left|\hat{\phi}((2N)^{-j}\xi)\right|. \tag{13}$$

Since $\left|\hat{\phi}(\xi)\right| \leq 1$ a.e, therefore, Lebesgue's dominated convergence theorem implies that

$$\int_{\Gamma_N} \Phi(\xi)\, d\xi = 1.$$

We now prove the converse. Assume that (7), (8) and (9) are satisfied. The orthonormality of the system $\{\phi(x-\lambda):\lambda\in\Lambda\}$ follows immediately from (7). This fact alongwith the definition of V_0 gives us (e) of the definition of an NUMRA. Moreover, the definition of the subspaces V_j also shows that $f(x) \in V_j$ holds if and only if $f(2Nx) \in V_{j+1}$ which is (d) of the definition of an NUMRA. Thus, we say that if $(2N)^{-j/2} f((2N)^{-j}x) \in V_0$, then there exists a sequence $\{h_\lambda\}_{\lambda\in\Lambda}$ satisfying $\sum_{\lambda\in\Lambda} h_\lambda < \infty$ such that

$$f\left((2N)^{-j}x\right)=(2N)^{j/2}\sum_{\lambda\in\Lambda} h_\lambda \phi(x-\lambda). \tag{14}$$

Taking Fourier transform on both sides of (14), we obtain

$$\hat{f}\left((2N)^{j}\xi\right)=\mu_j(\xi)\hat{\phi}(\xi) \tag{15}$$

where $\mu_j(\xi)=\sum_{\lambda\in\Lambda} h_\lambda e^{-2\pi i\lambda\xi}$. Since $\Lambda=\{0,r/N\}+2\mathbb{Z}$, we can rewrite $\mu_j(\xi)$ as

$$\mu_j(\xi)=\mu_j^1(\xi)+e^{-2\pi i\xi r/N}\mu_j^2(\xi) \tag{16}$$

where μ_j^1 and μ_j^2 are locally L^2, $1/2$-periodic functions. Now, for each $j \in \mathbb{Z}$, we claim that

$$V_j=\left\{f : \hat{f}\left((2N)^{j}\xi\right)=\mu_j(\xi)\hat{\phi}(\xi) \text{ for some periodic function } \mu_j(\xi)\right\}. \tag{17}$$

To prove the inclusion $V_j \subset V_{j+1}$, it is enough to show that $V_0 \subset V_1$. Assume that $f \in V_0$, then by equation (17), it follows that there exists a locally L^2 function say μ_0 such that $\hat{f}(\xi) = \mu_0(\xi)\hat{\varphi}(\xi)$, where $\mu_0(\xi) = \mu_0^1(\xi) + e^{-2\pi i\xi r/N}\mu_0^2(\xi)$. Using (9), we obtain

$$\hat{f}(2N\xi) = \mu_0(2N\xi)\hat{\varphi}(2N\xi) = \mu_0(2N\xi)m_0(\xi)\hat{\varphi}(\xi).$$

Moreover, $\mu_0(2N\xi)m_0(\xi)$ can be further expressed in the form

$$\eta_1(\xi) + e^{-2\pi i\xi r/N}\eta_2(\xi),$$

where

$$\eta_1(\xi) = \left\{\mu_0^1(2N\xi) + e^{-4\pi i\xi r}\mu_0^2(2N\xi)\right\} m_0^1(\xi)$$
$$\eta_2(\xi) = \left\{\mu_0^1(2N\xi) + e^{-4\pi i\xi r}\mu_0^2(2N\xi)\right\} m_0^2(\xi).$$

Using the fact that $|m_0(\xi)| \leq 1$ for a.e. $\xi \in \Gamma_N$, we have

$$\int_{\Gamma_N} \left|\mu_0(2N\xi)\right|^2 \left|m_0(\xi)\right|^2 d\xi \leq \int_{\Gamma_N} \left|\mu_0(2N\xi)\right|^2 d\xi < \infty,$$

which implies that $f \in V_1$. We have already seen that separation property (c) of an NUMRA follows from (a), (d) and (e). Now it remains to prove density property (b) of an NUMRA, that is; $L^2(\mathbb{R}) = \overline{\cup_{j\in\mathbb{Z}} V_j}$. To prove this, we assume that P_j be the orthogonal projection onto the closed subspace V_j of $L^2(\mathbb{R})$, then it suffices to show that

$$\left\|P_j f - f\right\|_2^2 = \|f\|_2^2 - \left\langle P_j(f), f\right\rangle_2 \to 0 \quad \text{as} \quad j \to \infty.$$

Since $\left\{(2N)^{j/2}\varphi\left((2N)^j x - \lambda\right)\right\}_{\lambda\in\Lambda}$ is an orthonormal basis for V_j. Therefore, for any compactly supported function f, we have

$$\left\langle P_j f, f\right\rangle_2 = \int_{\mathbb{R}} \left|\hat{\varphi}\left((2N)^{-j}\xi\right)\right|^2 \left|\hat{f}(\xi)\right|^2 d\xi. \tag{18}$$

Implementing condition (8), it follows that the right hand side of (18) converges to $\|f\|_2^2$ as $j \to \infty$. This completes the proof of Theorem 3. □

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Integrals of polylogarithmic functions with negative argument

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Abstract. The connection between polylogarithmic functions and Euler sums is well known. In this paper we explore the representation and many connections between integrals of products of polylogarithmic functions and Euler sums. We shall consider mainly, polylogarithmic functions with negative arguments, thereby producing new results and extending the work of Freitas. Many examples of integrals of products of polylogarithmic functions in terms of Riemann zeta values and Dirichlet values will be given.

1 Introduction and preliminaries

It is well known that integrals of products of polylogarithmic functions can be associated with Euler sums, see [16]. In this paper we investigate the representations of integrals of the type

$$\int_0^1 x^m \operatorname{Li}_t(-x) \operatorname{Li}_q(-x)\, dx,$$

for $m \geq -2$, and for integers q and t. For $m = -2, -1, 0$ we give explicit representations of the integral in terms of Euler sums and for $m \geq 0$ we give a recurrence relation for the integral in question. We also mention two specific

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integrals with a different argument in the polylogarithm. Some examples are highlighted, almost none of which are amenable to a computer mathematical package. This work extends the results given by [16], who examined a similar integral with positive arguments of the polylogarithm. Devoto and Duke [14] also list many identities of lower order polylogarithmic integrals and their relations to Euler sums. Some other important sources of information on polylogarithm functions are the works of [19] and [20]. In [3] and [12] the authors explore the algorithmic and analytic properties of generalized harmonic Euler sums systematically, in order to compute the massive Feynman integrals which arise in quantum field theories and in certain combinatorial problems. Identities involving harmonic sums can arise from their quasi-shuffle algebra or from other properties, such as relations to the Mellin transform

$$M[f(z)](N) = \int_0^1 dz\, z^N f(z),$$

where the basic functions $f(z)$ typically involve polylogarithms and harmonic sums of lower weight. Applying the latter type of relations, the author in [6], expresses all harmonic sums of the above type with weight $w = 6$, in terms of Mellin transforms and combinations of functions and constants of lower weight. In another interesting and related paper [17], the authors prove several identities containing infinite sums of values of the Roger's dilogarithm function. defined on $x \in [0.1]$, by

$$L_R(x) = \begin{cases} \mathrm{Li}_2(x) + \frac{1}{2}\ln x \ln(1-x); & 0 < x < 1 \\ 0 \quad ; & x = 0 \\ \zeta(2) \quad ; & x = 1 \end{cases}.$$

The Lerch transcendent,

$$\Phi(z,t,a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^t}$$

is defined for $|z| < 1$ and $\Re(a) > 0$ and satisfies the recurrence

$$\Phi(z,t,a) = z\,\Phi(z,t,a+1) + a^{-t}.$$

The Lerch transcendent generalizes the Hurwitz zeta function at $z = 1$,

$$\Phi(1,t,a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^t}$$

and the polylogarithm, or de-Jonquière's function, when $a = 1$,

$$L_{i_t}(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^t}, \ t \in \mathbb{C} \text{ when } |z| < 1; \ \Re(t) > 1 \text{ when } |z| = 1.$$

Let

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \int_0^1 \frac{1-t^n}{1-t}\, dt = \gamma + \psi(n+1) = \sum_{j=1}^{\infty} \frac{n}{j(j+n)}, \qquad H_0 := 0$$

be the nth harmonic number, where γ denotes the Euler-Mascheroni constant, $H_n^{(m)} = \sum_{r=1}^{n} \frac{1}{r^m}$ is the m^{th} order harmonic number and $\psi(z)$ is the digamma (or psi) function defined by

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \text{ and } \psi(1+z) = \psi(z) + \frac{1}{z},$$

moreover,

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right).$$

More generally a non-linear Euler sum may be expressed as,

$$\sum_{n \geq 1} \frac{(\pm 1)^n}{n^p} \left(\prod_{j=1}^{t} \left(H_n^{(\alpha_j)} \right)^{q_j} \prod_{k=1}^{r} \left(J_n^{(\beta_k)} \right)^{m_k} \right)$$

where $p \geq 2$, $t, r, q_j, \alpha_j, m_k, \beta_k$ are positive integers and

$$\left(H_n^{(\alpha)} \right)^q = \left(\sum_{j=1}^{n} \frac{1}{j^\alpha} \right)^q, \ \left(J_n^{(\beta)} \right)^m = \left(\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j^\beta} \right)^m.$$

If, for a positive integer

$$\lambda = \sum_{j=1}^{t} \alpha_j q_j + \sum_{j=1}^{r} \beta_j m_j + p,$$

then we call it a λ-order Euler sum. The polygamma function

$$\psi^{(k)}(z) = \frac{d^k}{dz^k}\{\psi(z)\} = -(-1)^{k+1} k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}}$$

and has the recurrence

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k k!}{z^{k+1}}.$$

The connection of the polygamma function with harmonic numbers is,

$$H_z^{(\alpha+1)} = \zeta(\alpha+1) + \frac{(-1)^\alpha}{\alpha!}\psi^{(\alpha)}(z+1), \; z \neq \{-1,-2,-3,...\}. \tag{1}$$

and the multiplication formula is

$$\psi^{(k)}(pz) = \delta_{m,0} \ln p + \frac{1}{p^{k+1}} \sum_{j=0}^{p-1} \psi^{(k)}(z + \frac{j}{p}) \tag{2}$$

for p a positive integer and $\delta_{p,k}$ is the Kronecker delta. We define the alternating zeta function (or Dirichlet eta function) $\eta(z)$ as

$$\eta(z) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = \left(1 - 2^{1-z}\right) \zeta(z) \tag{3}$$

where $\eta(1) = \ln 2$. If we put

$$S(p,q) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \; H_n^{(p)}}{n^q},$$

in the case where p and q are both positive integers and $p+q$ is an odd integer, Flajolet and Salvy [15] gave the identity:

$$\begin{aligned} 2S(p,q) = (1-(-1)^p)\,\zeta(p)\,\eta(q) + 2(-1)^p \sum_{i+2k=q} \binom{p+i-1}{p-1} \zeta(p+i)\; \eta(2k) \\ + \eta(p+q) - 2 \sum_{j+2k=p} \binom{q+j-1}{q-1} (-1)^j \eta(q+j)\; \eta(2k), \end{aligned} \tag{4}$$

where $\eta(0) = \frac{1}{2}$, $\eta(1) = \ln 2$, $\zeta(1) = 0$, and $\zeta(0) = -\frac{1}{2}$ in accordance with the analytic continuation of the Riemann zeta function. We also know, from the work of [11] that for odd weight $(p+q)$ we have

$$BW(p.q) = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} = (-1)^p \sum_{j=1}^{[\frac{p}{2}]} \binom{p+q-2j-1}{p-1} \zeta(p+q-2j)\, \zeta(2j) \tag{5}$$

$$+\frac{1}{2}\left(1+(-1)^{p+1}\right)\zeta(p)\zeta(q)+(-1)^{p}\sum_{j=1}^{[\frac{p}{2}]}\binom{p+q-2j-1}{q-1}\zeta(p+q-2j)\zeta(2j)$$
$$+\frac{\zeta(p+q)}{2}\left(1+(-1)^{p+1}\binom{p+q-1}{p}+(-1)^{p+1}\binom{p+q-1}{q}\right),$$

where $[z]$ is the integer part of z. It appears that some isolated cases of $BW(p.q)$, for even weight $(p+q)$, can be expressed in zeta terms, but in general, almost certainly, for even weight $(p+q)$, no general closed form expression exits for $BW(p.q)$. (at least at the time of writing this paper). Two examples with even weight are

$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}=\zeta^2(3)-\frac{1}{3}\zeta(6),\quad \sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^4}=\frac{13}{12}\zeta(8).$$

The work in this paper extends the results of [16] and later [25], in which they gave identities of products of polylogarithmic functions with positive argument in terms of zeta functions. Other works including, [1], [4], [8], [10], [13], [18], [22], [23], [24], cite many identities of polylogarithmic integrals and Euler sums, but none of these examine the negative argument case. The following result was obtained by Freitas, [16].

Lemma 1 *For* q *and* t *positive integers*

$$\int_0^1\frac{Li_t(x)\ \ Li_q(x)}{x}dx=\sum_{j=1}^{q-1}(-1)^{j+1}\zeta(t+j)\zeta(q-j+1)+(-1)^{q+1}EU(t+q)$$

where $EU(m)$ *is Euler's identity given in the next lemma.*

The following lemma will be useful in the development of the main theorem.

Lemma 2 *The following identities hold: for* $m\in\mathbb{N}$*. Euler's identity states*

$$EU(m)=\sum_{n=1}^{\infty}\frac{H_n}{n^m}=(m+2)\zeta(m+1)-\sum_{j=1}^{m-2}\zeta(m-j)\zeta(j+1).\qquad(6)$$

For p *a positive even integer,*

$$\begin{aligned} HE(p) = \sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^p} &= \frac{p}{2}\left(\zeta(p+1) + \eta(p+1)\right) - \left(\zeta(p) + \eta(p)\right)\ln 2 \\ &-\frac{1}{2}\sum_{j=1}^{\frac{p}{2}-1}\left(\zeta(p+1-2j) + \eta(p+1-2j)\right)\left(\zeta(2j) + \eta(2j)\right). \end{aligned} \tag{7}$$

For p *a positive odd integer,*

$$\begin{aligned} HO(p) = \sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^p} &= \frac{p}{4}\left(\zeta(p+1) + \eta(p+1)\right) - \left(\zeta(p) + \eta(p)\right)\ln 2 \\ &-\frac{1}{4}\left(\frac{1+(-1)^{\frac{p-1}{2}}}{2}\right)\left(\zeta\left(\frac{p+1}{2}\right) + \eta\left(\frac{p+1}{2}\right)\right) \\ &-\frac{1}{2}\sum_{j=1}^{b}\left(\zeta(p-2j) + \eta(p-2j)\right)\left(\zeta(2j+1) + \eta(2j+1)\right) \end{aligned} \tag{8}$$

where $\eta(z)$ *is the Dirichlet eta function,* $b = \left[\frac{p-1}{4}\right] - \left(\frac{1+(-1)^{\frac{p-1}{2}}}{2}\right)$ *and* $[z]$ *is the greatest integer less than* z*. For* p *and* t *positive integers we have*

$$\begin{aligned} F(p,t) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p (n+1)^t} \\ &= \sum_{r=1}^{p} (-1)^{p-r} \binom{p+t-r-1}{p-r} \eta(r) \\ &+ \sum_{s=1}^{t} (-1)^{p+1} \binom{p+t-s-1}{t-s} (1-\eta(s)), \end{aligned} \tag{9}$$

$$\begin{aligned} G(p,t) = \sum_{n=1}^{\infty} \frac{1}{n^p (n+1)^t} &= (-1)^{p+1} \binom{p+t-1}{p} \\ &+ \sum_{r=2}^{p} (-1)^{p-r} \binom{p+t-r-1}{p-r} \zeta(r) \\ &+ \sum_{s=1}^{t} (-1)^{p} \binom{p+t-s-1}{t-s} \zeta(s), \end{aligned} \tag{10}$$

and

$$HG(p,t)=\sum_{n=1}^{\infty}\frac{H_n}{n^p(n+1)^t}=(-1)^{p+1}\binom{p+t-2}{p-1}\zeta(2)$$
$$+\sum_{r=2}^{p}(-1)^{p-r}\binom{p+t-r-1}{p-r}EU(r) \tag{11}$$
$$+\sum_{s=2}^{t}(-1)^{p}\binom{p+t-s-1}{t-s}(EU(s)-\zeta(s+1)).$$

Proof. The identity (6) is the Euler relation and by manipulation we arrive at (7) and (8). The results (7) and (8) are closely related to those given by Nakamura and Tasaka [21]. For the proof of (9) we notice that

$$\frac{1}{n^p(n+1)^t}=\sum_{r=1}^{p}(-1)^{p-r}\binom{p+t-r-1}{p-r}\frac{1}{n^r}$$
$$+\sum_{s=1}^{t}(-1)^{p}\binom{p+t-s-1}{t-s}\frac{1}{(n+1)^s}$$

therefore, summing over the integers n,

$$F(p,t)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p(n+1)^t}=\sum_{r=1}^{p}(-1)^{p-r}\binom{p+t-r-1}{p-r}\eta(r)$$
$$+\sum_{s=1}^{t}(-1)^{p}\binom{p+t-s-1}{t-s}(1-\eta(s))$$

and hence (9) follows. Consider,

$$\frac{1}{n^p(n+1)^t}=\frac{(-1)^{p+1}}{n(n+1)}\binom{p+t-2}{p-1}+\sum_{r=2}^{p}(-1)^{p-r}\binom{p+t-r-1}{p-r}\frac{1}{n^r}$$
$$+\sum_{s=2}^{t}(-1)^{p}\binom{p+t-s-1}{t-s}\frac{1}{(n+1)^s},$$

and summing over the integers n produces the result (10). The proof of (11) follows by summing $\sum_{n=1}^{\infty}\frac{H_n}{n^p(n+1)^t}$ in partial fraction form. An example, from (8)

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^9}=\frac{9207}{2048}\zeta(10)-\frac{961}{1024}\zeta^2(5)-\frac{889}{512}\zeta(7)\zeta(3)-\frac{511}{256}\zeta(9)\ln 2$$

and from (7),

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^8}=\frac{511}{64}\zeta(9)-\frac{381}{256}\zeta(7)\zeta(2)-\frac{441}{256}\zeta(6)\zeta(3)$$
$$-\frac{465}{256}\zeta(5)\zeta(4)-\frac{255}{128}\zeta(8)\ln 2.$$

□

2 Summation identity

We now prove the following theorems.

Theorem 1 *For positive integers* q *and* t*, the integral of the product of two polylogarithmic functions with negative arguments*

$$I_0(q,t)=\int_0^1 Li_t(-x)\ \ Li_q(-x)\,dx=\int_{-1}^{0} Li_t(x)\,Li_q(x)\,dx$$
$$=\sum_{j=1}^{q-1}(-1)^{j+1}\eta(q-j+1)\,F(t,j)$$
$$+(-1)^q\left(F(t,q+1)-(F(t,q)-G(t,q))\ln 2\right)+(-1)^q\,W_n(q,t) \tag{12}$$

where the sum

$$W_n(q,t)=\sum_{n=1}^{\infty}H_n\left(\frac{1}{(2n)^t(2n+1)^q}-\frac{1}{n^t(n+1)^q}+\frac{1}{(2n+1)^t(2n+2)^q}\right) \tag{13}$$

is obtained from (6)*,* (7)*,* (8) *and the terms* $F(\cdot,\cdot)$*,* $G(\cdot,\cdot)$ *are obtained from* (9) *and* (10) *respectively.*

Proof. By the definition of the polylogarithmic function we have

$$I_0(q,t)=\int_0^1 \mathrm{Li}_t(-x)\ \ \mathrm{Li}_q(-x)\,dx=\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\frac{(-1)^{n+r}}{n^t r^q(n+r+1)}$$
$$=\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\frac{(-1)^{n+r}}{n^t}\left(\frac{(-1)^q}{(n+r+1)(n+1)^q}+\sum_{j=1}^{q}\frac{(-1)^{j+1}}{(n+1)^j\ \ r^{q-j+1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+r}}{n^t} \left(\frac{(-1)^{q+1}}{(n+1)^q} \left(\frac{1}{2} H_{\frac{n+1}{2}} - \frac{1}{2} H_{\frac{n}{2}} \right) + \sum_{j=1}^{q} \frac{(-1)^{j+1} \eta(q-j+1)}{(n+1)^j} \right)$$
$$= \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^t (n+1)^j} + (-1)^q \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^t (n+1)^{q+1}}$$
$$+ (-1)^q \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^t (n+1)^q} \left(\frac{1}{2} H_{\frac{n+1}{2}} - \frac{1}{2} H_{\frac{n}{2}} - \ln 2 \right).$$

Now we utilize the double argument identity (2) together with (9) we obtain

$$I_0(q,t) = \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1) F(t,j) + (-1)^q F(t,q+1)$$
$$+ (-1)^q \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^t (n+1)^q} \left(H_n - H_{\frac{n}{2}} - 2\ln 2 \right),$$

we can use the alternating harmonic number sum identity (4) to simplify the last sum, however we shall simplify further as follows.

$$I_0(q,t) = \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1) F(t,j) + (-1)^q F(t,q+1)$$
$$+ (-1)^q \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^t (n+1)^q} \left((-1)^{n+1} \left(H_{[\frac{n}{2}]} - H_n \right) - (1 + (-1)^n) \ln 2 \right)$$

where $[z]$ is the integer part of z. Now

$$I_0(q,t) = \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1) F(t,j) + (-1)^q F(t,q+1)$$
$$- (-1)^q (F(t,q) - G(t,q)) \ln 2 + (-1)^q W_n(q,t)$$

where

$$W_n(q,t) = \sum_{n=1}^{\infty} H_n \left(\frac{1}{(2n)^t (2n+1)^q} - \frac{1}{n^t (n+1)^q} + \frac{1}{(2n+1)^t (2n+2)^q} \right)$$

and the infinite positive harmonic number sums are easily obtainable from (6), (7), (8), hence the identity (12) is achieved. □

The next theorem investigates the integral of the product of polylogarithmic functions divided by a linear function.

Theorem 2 *Let* (t,q) *be positive integers, then for* $t+q$ *an odd integer*

$$\begin{aligned} I_1(t,q) &= \int_0^1 \frac{Li_t(-x)\ \ Li_q(-x)}{x}dx = -\int_{-1}^0 \frac{Li_t(x)\ \ Li_q(x)}{x}dx \\ &= \sum_{j=1}^{q-1} (-1)^{j+1}\eta(t+j)\,\eta(q-j+1) \\ &\quad + (-1)^{q+1}(\zeta(t+q)+\eta(t+q))\ln 2 \\ &\quad + (-1)^{q+1}\left(2^{-t-q}-1\right)EU(q+t) + (-1)^{q+1}HO(q+t). \end{aligned} \tag{14}$$

For $t+q$ *an even integer*

$$\begin{aligned} I_1(t,q) = &\sum_{j=1}^{q-1}(-1)^{j+1}\eta(t+j)\eta(q-j+1) + (-1)^{q+1}(\zeta(t+q) \\ &+\eta(t+q))\ln 2 + (-1)^{q+1}\left(2^{-t-q}-1\right)EU(q+t) \\ &+(-1)^{q+1}HE(q+t). \end{aligned} \tag{15}$$

Proof. Consider

$$I_1(t,q) = \int_0^1 \frac{\mathrm{Li}_t(-x)\ \ \mathrm{Li}_q(-x)}{x}dx = \sum_{n\geq 1}\frac{(-1)^n}{n^t}\int_0^1 x^{n-1}\ \mathrm{Li}_q(-x)\,dx,$$

and successively integrating by parts leads to

$$I_1(t,q) = \sum_{n\geq 1}\frac{(-1)^n}{n^{t+j}}\sum_{j=1}^{q-1}\eta(q-j+1) + \sum_{n\geq 1}\frac{(-1)^{n+q+1}}{n^{t+q-1}}\int_0^1 x^{n-1}\ \mathrm{Li}_1(-x)\,dx.$$

Evaluating the inner integral,

$$\int_0^1 x^{n-1}\ \mathrm{Li}_1(-x)\,dx = -\int_0^1 x^{n-1}\ \ln(1+x)\,dx = \frac{1}{n}\left(\frac{1}{2}H_{\frac{n}{2}} - \frac{1}{2}H_{\frac{n-1}{2}} - \ln 2\right),$$

so that

$$\begin{aligned}
I_1(t,q) &= \sum_{n\geq 1} \frac{(-1)^n}{n^{t+j}} \sum_{j=1}^{q-1} (-1)^j \eta(q-j+1) \\
&\quad + \sum_{n\geq 1} \frac{(-1)^{n+q+1}}{n^{t+q}} \left(\frac{1}{2} H_{\frac{n}{2}} - \frac{1}{2} H_{\frac{n-1}{2}} - \ln 2 \right) \\
&= \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1)\, \eta(t+j) \\
&\quad + \sum_{n\geq 1} \frac{(-1)^{n+q+1}}{n^{t+q}} \left(\frac{1}{2} H_{\frac{n}{2}} - \frac{1}{2} H_{\frac{n-1}{2}} - \ln 2 \right).
\end{aligned}$$

If we now utilize the multiplication formula (2) we can write

$$I_1(t,q) = \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1)\, \eta(t+j) + (-1)^{q+1} \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^{t+q}} \left(H_n - H_{\frac{n}{2}} \right).$$

Now consider the harmonic number sum

$$\begin{aligned}
\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^{t+q}} \left(H_n - H_{\frac{n}{2}} \right) &= \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^{t+q}} \begin{pmatrix} (1-(-1)^n)\ln 2 \\ +(-1)^{n+1}\left(H_{[\frac{n}{2}]} - H_n \right) \end{pmatrix} \\
&= \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^{t+q}} \left((1-(-1)^n)\ln 2 + (-1)^{n+1} \sum_{j=1}^{n} \frac{(-1)^j}{j} \right) \\
&= \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^{t+q}} (1-(-1)^n)\ln 2 + \sum_{n\geq 1} \left(\frac{1}{2^{t+q}} - 1 \right) \frac{H_n}{n^{t+q}} + \sum_{n\geq 1} \frac{H_n}{(2n+1)^{t+q}} \\
&= (\zeta(t+q) + \eta(t+q))\ln 2 + \sum_{n\geq 1} \left(\frac{1}{2^{t+q}} - 1 \right) \frac{H_n}{n^{t+q}} + \sum_{n\geq 1} \frac{H_n}{(2n+1)^{t+q}}
\end{aligned}$$

where $[z]$ is the integer part of z. Hence

$$
\begin{aligned}
I_1(t,q) &= \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1)\,\eta(t+j) + (-1)^{q+1}(\zeta(t+q)+\eta(t+q))\ln 2 \\
&\quad + (-1)^{q+1} \sum_{n\geq 1} \left(\frac{1}{2^{t+q}} - 1\right) \frac{H_n}{n^{t+q}} + (-1)^{q+1} \sum_{n\geq 1} \frac{H_n}{(2n+1)^{t+q}} \\
&= \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1)\,\eta(t+j) + (-1)^{q+1}(\zeta(t+q)+\eta(t+q))\ln 2 \\
&\quad + (-1)^{q+1} \left(\frac{1}{2^{t+q}} - 1\right) EU(q+t) \\
&\quad + (-1)^{q+1} \begin{cases} HO(q+t), & \text{for } t+q \text{ odd} \\ HE(q+t), & \text{for } t+q \text{ even} \end{cases},
\end{aligned}
$$

hence (14) and (15) follow. □

Remark 1 *It is interesting to note that, for* $m \in \mathbb{R}$,

$$
\int_0^1 \frac{Li_t(-x^m)\ \ Li_q(-x^m)}{x}\,dx = \frac{1}{m} \int_0^1 \frac{Li_t(-x)\ \ Li_q(-x)}{x}\,dx
$$

The next theorem investigates the integral of the product of polylogarithmic functions divided by a quadratic factor.

Theorem 3 *For positive integers* q *and* t*, the integral of the product of two polylogarithmic functions with negative arguments*

$$
\begin{aligned}
I_2(t,q) &= \int_0^1 \frac{Li_t(-x)\ \ Li_q(-x)}{x^2}\,dx = \int_{-1}^0 \frac{Li_t(x)\ \ Li_q(x)}{x^2}\,dx \\
&= \eta(q+1) + \sum_{j=1}^{q-1} (-1)^j \eta(q-j+1)\,F(j,t) \qquad (16) \\
&\quad + (-1)^q (F(q,t)+G(q,t))\ln 2 + (-1)^q W_n(t,q)
\end{aligned}
$$

where the sum,

$$
W_n(t,q) = \sum_{n=1}^{\infty} H_n \left(\frac{1}{(2n)^q (2n+1)^t} - \frac{1}{n^q (n+1)^t} + \frac{1}{(2n+1)^q (2n+2)^t} \right) \qquad (17)
$$

is obtained from (6), (7), (8) *and the terms* $F(\cdot,\cdot)$, $G(\cdot,\cdot)$ *are obtained from* (9) *and* (10) *respectively.*

Proof. Following the same process as in Theorem 2, we have,

$$I_2(t,q) = \int_0^1 \frac{\mathrm{Li}_t(-x)\ \mathrm{Li}_q(-x)}{x^2} dx = \sum_{n\geq 1} \frac{(-1)^n}{n^t} \int_0^1 x^{n-2}\,\mathrm{Li}_q(-x)\,dx$$
$$= -\int_0^1 x^{-1}\,\mathrm{Li}_q(-x)\,dx + \sum_{n\geq 2} \frac{(-1)^n}{n^t} \int_0^1 x^{n-2}\,\mathrm{Li}_q(-x)\,dx,$$

and re ordering the summation index n, produces

$$I_2(t,q) = \eta(q+1) + \sum_{n\geq 1} \frac{(-1)^{n+1}}{(n+1)^t} \int_0^1 x^{n-1}\,\mathrm{Li}_q(-x)\,dx.$$

Integrating by parts, we have,

$$I_2(t,q) = \eta(q+1) + \sum_{n\geq 1} \frac{(-1)^{n+1}}{(n+1)^t} \left(\begin{array}{c} \sum_{j=1}^{q-1} \frac{(-1)^j \eta(q-j+1)}{n^j} \\ + \frac{(-1)^{q+1}}{n^{q-1}} \int_0^1 x^{n-1}\,\mathrm{Li}_1(-x)\,dx \end{array} \right)$$
$$= \eta(q+1) + \sum_{j=1}^{q-1} (-1)^j \eta(q-j+1) \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^j (n+1)^t}$$
$$+ \sum_{n\geq 1} \frac{(-1)^{n+q}}{n^q (n+1)^t} \left(\frac{1}{2} H_{\frac{n}{2}} - \frac{1}{2} H_{\frac{n-1}{2}} - \ln 2 \right).$$

Using the multiplication Theorem (2) and following the same steps as in Theorem 2, we have

$$I_2(t,q) = \eta(q+1) + \sum_{j=1}^{q-1} (-1)^j \eta(q-j+1)\, F(j,t)$$
$$+ (-1)^q (F(q,t) + G(q,t)) \ln 2 + (-1)^q W_n(t,q),$$

and the proof of Theorem 3 is finalized. □

The following recurrence relation holds for the reduction of the integral of the product of polylogarithmic functions multiplied by the power of its argument.

Lemma 3 *For* $(q,t)\in\mathbb{N}$ *and* $m\geq 0$, *let*

$$J(m,q,t)=\int_0^1 x^m\ Li_t(-x)\ \ Li_q(-x)\,dx=(-1)^m\int_{-1}^0 x^m\ Li_t(x)\ \ Li_q(x)\,dx$$

then

$$(m+1)\,J(m,q,t)=\eta(q)\,\eta(t)-J(m,q,t-1)-J(m,q-1,t)\,.$$

For $q=1$,

$$(m+1)\,J(m,1,t)=\eta(t)+mJ(m-1,1,t)+J(m-1,1,t-1)-J(m,1,t-1)$$
$$-mK(m,t)-K(m,t-1)$$

where

$$K(m,t)=\int_0^1 x^m\ Li_t(-x)\ \ dx.$$

Proof. The proof of the lemma follows in a straight forward manner after integration by parts. □

We list some examples of the results of the integrals in Theorems 1, 2 and 3.

Example 1

$$I_0(3,3)=\int_0^1 (Li_3(-x))^2\ \ dx=\frac{9}{16}\zeta^2(3)+\frac{5}{8}\zeta(4)-\frac{3}{4}\zeta(2)\,\zeta(3)$$
$$+(3\zeta(3)-6\zeta(2)-40)\ln 2+4\zeta(2)+12\ln^2 2+20.$$

$$I_0(3,4)=\int_0^1 Li_3(-x)\ \ Li_4(-x)\,dx=\frac{3}{4}\eta(4)+\zeta(3)\,2\zeta(5)-\frac{49}{64}\zeta(6)-\frac{9}{16}\zeta^2(3)$$
$$+\frac{5}{4}\zeta(3)-\frac{15}{2}\zeta(2)+\left(10\zeta(2)-6\zeta(3)+\frac{7}{4}\zeta(4)+70\right)\ln 2$$
$$-\frac{3}{2}\zeta(4)-\frac{17}{16}\zeta(5)+\frac{3}{2}\zeta(2)\,\zeta(3)-20\ln^2 2-35.$$

$$I_1(2m,2m+1)=\int_0^1 \frac{Li_{2m}(-x)\ \ Li_{2m+1}(-x)}{x}\,dx=\frac{1}{2}\eta^2(2m+1)$$

for $m \in \mathbb{N}$.

$$I_1(4,7) = \int_0^1 \frac{Li_4(-x)\ \ Li_7(-x)}{x} dx = \eta(5)\eta(7) - \frac{1}{2}\eta^2(6).$$

$$\begin{aligned} I_2(3,4) = \int_0^1 \frac{Li_3(-x)\ \ Li_4(-x)}{x^2} dx &= 2\zeta(5) - \frac{49}{64}\zeta(6) + \frac{23}{8}\zeta(4) - \frac{9}{16}\zeta^2(3) \\ &+ 10\zeta(3) - \left(10\zeta(2) + 6\zeta(3) + \frac{7}{4}\zeta(4)\right)\ln 2 \\ &+ 10\zeta(2) - \frac{3}{2}\zeta(2)\zeta(3) - 20\ln^2 2 - \frac{21}{32}\zeta(3)\zeta(4), \end{aligned}$$

$$\begin{aligned} I_2(3,3) = \int_0^1 \left(\frac{Li_3(-x)}{x}\right)^2 dx &= \frac{9}{8}\zeta(4) - \frac{9}{16}\zeta^2(3) + 6\zeta(3) \\ &+ 6\zeta(2) - \frac{3}{4}\zeta(2)\zeta(3) - (6\zeta(2) + 3\zeta(3))\ln 2 - 12\ln^2 2. \end{aligned}$$

These results build on the work of [16] and [25] where they explored integrals of polylogarithmic functions with positive arguments only. Freitas gives many particular examples of identities for $\int_0^1 \frac{Li_q(x)\ Li_t(x)}{x^2} dx$, but no explicit identity of the form (16) is given. Therefore in the interest of presenting a complete record we list the following theorem.

Theorem 4 *For positive integers* q *and* t, *the integral of the product of two polylogarithmic functions with positive arguments,*

$$\begin{aligned} P(q,t) = \int_0^1 \frac{Li_q(x)\ \ Li_t(x)}{x^2} dx &= (-1)^q HG(q,t) \\ &+ \sum_{j=1}^{q-1} (-1)^{j+1}\zeta(t+j)\, G(j,t), \end{aligned}$$

where $G(\cdot,\cdot)$ *and* $HG(\cdot,\cdot)$ *are given by* (10) *and* (11) *respectively.*

Proof. The proof follows the same technique as that used in Theorem 3. □

Example 2

$$P(4,5)=\int_0^1 \frac{Li_4(x)\ \ Li_5(x)}{x^2}dx = 70\zeta(2)-35\zeta(3)-\frac{114}{5}\zeta(4)-10\zeta(5)$$
$$-\zeta(4)\zeta(5)-\frac{31}{4}\zeta(6)-\frac{5}{2}\zeta^2(3)-5\zeta(2)\zeta(3)-3\zeta(3)\zeta(4)$$
$$-\zeta(2)\zeta(5)-\frac{7}{6}\zeta(8)-\zeta(3)\zeta(5),$$

$$\int_0^1 \frac{Li_4(x^3)\ \ Li_4(x^3)}{x}dx=\frac{2}{3}\zeta(4)\zeta(5)+\frac{2}{3}\zeta(2)\zeta(7)-\frac{5}{3}\zeta(9).$$

It is interesting to note the degenerate case, that is when $t=0$, of theorems 1, 2 and 3. The following results are noted.

Remark 2 *For* $t=0,\ Li_0(-x)=-\frac{x}{1+x}$, *hence*

$$I_0(q,0)=\int_0^1 Li_q(-x)\ \ Li_0(-x)\,dx=(-1)^q(1-\eta(q+1))$$
$$+\sum_{j=1}^{q-1}(-1)^{j+1}\eta(q-j+1)(1-\eta(j))-(-1)^q(2-\zeta(q)-\eta(q))\ln 2$$
$$+(-1)^q\left(\frac{1}{2^q}-1\right)(EU(q)-\zeta(q+1))+(-1)^q\begin{cases}HO(q), \text{ for } q \text{ odd}\\ HE(q), \text{ for } q \text{ even}\end{cases}.$$

$$I_1(q,0)=\int_0^1 \frac{Li_q(-x)\ \ Li_0(-x)}{x}dx=\sum_{j=1}^{q-1}(-1)^{j+1}\eta(q-j+1)\eta(j)$$
$$+(-1)^{q+1}\left(\frac{1}{2^q}-1\right)EU(q)+(-1)^{q+1}(\zeta(q)+\eta(q))\ln 2$$
$$+(-1)^{q+1}\begin{cases}HO(q), & \text{for } q \text{ odd}\\ HE(q), & \text{for } q \text{ even}\end{cases}.$$

$$I_2(q,0) = \int_0^1 \frac{Li_q(-x)\ Li_0(-x)}{x^2} dx = \sum_{j=1}^{q-1} (-1)^j \eta(q-j+1)\eta(j)$$
$$+\eta(q+1) + (-1)^q \left(\frac{1}{2^q} - 1\right) EU(q) + (-1)^q (\zeta(q) + \eta(q)) \ln 2$$
$$+(-1)^q \begin{cases} HO(q), & \text{for } q \text{ odd} \\ HE(q), & \text{for } q \text{ even} \end{cases}.$$

Here we notice that

$$I_2(q,0) = \eta(q+1) - I_1(q,0).$$

There are some special cases of polylogarithmic integrals which are worthy of a mention and we list two in the following corollary.

Corollary 1 *Let* $q, t \in \mathbb{N}$ *then,*

$$S1(q,t) = \int_0^1 \frac{Li_q\left(-\frac{1}{x}\right)\ Li_t(-x)}{x} dx = \sum_{j=1}^{q-1} \eta(t+j)\eta(q-j+1)$$
$$+\eta(q+t+1) + (\eta(q+t) + \zeta(q+t)) \ln 2 \tag{18}$$
$$+\left(\frac{1}{2^{q+t}} - 1\right) EU(q+t) + \begin{cases} HE(q+t), & \text{for } q+t \text{ even} \\ HO(q+t), & \text{for } q+t \text{ odd} \end{cases}.$$

$$S2(q) = \int_0^1 \frac{Li_2(1-x)\ Li_q(x)}{x} dx = \zeta(2)\zeta(q+1) - BW(2,q+1), \tag{19}$$

where $BW(2,q+1)$ *is given by* (5).

Proof. If we follow the same procedure as in theorem 2, we obtain

$$S1(q,t) = \int_0^1 \frac{\mathrm{Li}_q\left(-\frac{1}{x}\right)\ \mathrm{Li}_t(-x)}{x} dx = \sum_{j=1}^{q-1} \eta(q-j+1)\eta(t+j) + \eta(q+t+1)$$
$$+\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^{t+q}} \left(H_n - H_{\frac{n}{2}}\right).$$

simplifying as in Theorem 2, we arrive at the identity (18).

From Euler's reflection formula we now that

$$\mathrm{Li}_2(1-x)+\mathrm{Li}_2(x)+\ln x\ln(1-x)=\zeta(2)$$

so that

$$S2(q)=\int_0^1 \frac{(-\mathrm{Li}_2(x)-\ln x\ln(1-x)+\zeta(2))\ \mathrm{Li}_q(x)}{x}dx.$$

Integrating term by term as in theorem 2, we obtain (19) □

Example 3 *Some examples of the corollary follow.*

$$S1(2,5)=\int_0^1 \frac{\mathrm{Li}_2\left(-\frac{1}{x}\right)\ \mathrm{Li}_5(-x)}{x}dx=\frac{2345}{768}\zeta(8)-\eta(3)\eta(5),$$

$$S1(q,q)=\int_0^1 \frac{\mathrm{Li}_q\left(-\frac{1}{x}\right)\ \mathrm{Li}_q(-x)}{x}dx=q\zeta(2q+1),$$

$$S1(9,5)=\int_0^1 \frac{\mathrm{Li}_9\left(-\frac{1}{x}\right)\ \mathrm{Li}_5(-x)}{x}dx=7\zeta(15)-\eta(6)\eta(9)-\eta(7)\eta(8).$$

$$S2(3)=\int_0^1 \frac{\mathrm{Li}_2(1-x)\ \mathrm{Li}_3(x)}{x}dx=\frac{25}{12}\zeta(6)-\zeta^2(3)$$

$$S2(8)=\int_0^1 \frac{\mathrm{Li}_2(1-x)\ \mathrm{Li}_8(x)}{x}dx$$
$$=27\zeta(11)-8\zeta(2)\zeta(9)-6\zeta(4)\zeta(7)-4\zeta(6)\zeta(5)-2\zeta(8)\zeta(3).$$

Summary In this paper we have developed new Euler sum identities (7) and (8) of general weight $p+1$ for $p\in\mathbb{N}$. Moreover, we have developed the new identities (16) and (18). In a series of papers [2], [5], [6], the authors explore linear combinations of associated harmonic polylogarithms and nested harmonic numbers. The multiple zeta value data mine, computed by Blumlein et. al. [7], is an invaluable tool for the evaluation of harmonic numbers. Values with weights of twelve, for alternating sums and weights above twenty for non-alternating sums are presented.

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A note on some relations between certain inequalities and normalized analytic functions

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Abstract. In this note, an extensive result consisting of several relations between certain inequalities and normalized analytic functions is first stated and some consequences of the result together with some examples are next presented. For the proof of the presented result, some of the assertions indicated in [5], [8] and [11] along with the results in [3] and [4] are also considered.

1 Introduction, definitions and motivation

Firstly, *here and throughout this investigation,* let $\mathbb{C}$ be the complex plane, $\mathbb{U}$ be the *unit open* disc, *i.e.,* $\{z \in \mathbb{C} : |z| < 1\}$ and also let $\mathcal{H}$ denote the class of all analytic functions in $\mathbb{U}$. Moreover, a function $f(z) \in \mathcal{H}$ is said to be a *convex function* (in $\mathbb{U}$) *if* $f(\mathbb{U})$ is a *convex* domain. In this respect, let $\mathcal{A}$ be the subclass of all functions $\mathcal{H}$ such that $f(0) = f'(0) - 1 = 0$, that is, $f(z) \in \mathcal{A}$ is of the form $f(z) = z + a_1 z + a_2 z^2 + \cdots$, *where* $z \in \mathbb{U}$ and $a_i \in \mathbb{C}$

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for all $i = 1, 2, 3, \cdots$. In general, the subclass of $\mathcal{A}$ consisting of all univalent functions is denoted by $\mathcal{S}$. At the same time, $f(z) \in \mathcal{A}$ is *convex function* iff $\Re e\{1 + zf''(z)/f'(z)\} > 0$ for all $z \in \mathbb{U}$. Furthermore, $f(z) \in \mathcal{H}$ is said to be *starlike* if $f(z)$ is *univalent* and $f(\mathbb{U})$ is a *starlike* domain (with respect to $z = 0$). It is well-known that $f(z) \in \mathcal{A}$ is *starlike* iff $\Re e\{zf'(z)/f(z)\} > 0$ for all $z \in \mathbb{U}$. The classes $\mathcal{K}$ and $\mathcal{S}^*$ denote the *normalized* functions' class of the functions $f(z)$ in $\mathcal{S}$, *when* $f(\mathbb{U})$ is *convex* and $f(\mathbb{U})$ is *starlike,* respectively. The class $\mathcal{S}^*(\alpha)$ denotes the class of all s*tarlike* functions $f(z)$ of order α $(0 \leq \alpha < 1)$ if $f(z) \in \mathcal{A}$ and $\Re e\{zf'(z)/f(z)\} > \alpha$ for all $z \in \mathbb{U}$. Besides, the class $\mathcal{K}(\alpha)$ denotes the class of all s*convex* functions $f(z)$ of order α $(0 \leq \alpha < 1)$ if $f(z) \in \mathcal{A}$ and $\Re e\{1 + zf'(z)/f(z)\} > \alpha$ for all $z \in \mathbb{U}$. Namely, $\mathcal{K}(\alpha)$ is the class of all *convex* functions $f(z) \in \mathcal{A}$ satisfying the condition $\Re e\{1 + zf''(z)/f'(z)\} > \alpha$ for all $z \in \mathbb{U}$ and for some α $(0 \leq \alpha < 1)$. In addition, let $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$, *which* are the subclasses of *starlike* and *convex* functions with respect to the origin $(z = 0)$ in $\mathbb{U}$, respectively. (See, for the details of the related definitions (and also information), [1], [2], and see also (for novel examples) [3], [4], [6], [7].)

The literature presents us several works including important *or* interesting results between certain inequalities and certain classes of the functions which are *analytic* and *univalent* in the disc $\mathbb{U}$. For those, one may look over the earlier results presented in [3], [8], [9], [10] and [11]. In particularly, in [8], the problem of finding $\lambda > 0$ such that the condition $|f''(z)|$, *where* $f(z) \in \mathcal{A}$ and $z \in \mathbb{U}$, implies $f(z) \in \mathcal{S}^*$, was firstly considered by P. T. Mocanu for $\lambda = 2/3$. Later, in [9], S. Ponnusamy and V. Singh considered the problem for $\lambda = 2/\sqrt{3}$. Afterwards, in [10], M. Obradović focused on the problem for $\lambda = 1$ by proving that his result is sharp. In [11], N. Tuneski also obtained certain results dealing with the same problems, which are also generalizations of the results of M. Obradović in [10].

In this investigation, by using a different technique, developed by S. S. Miller and P. T. Mocanu in [5], certain results determined by the functions $f(z) \in \mathcal{A}$ relating to both condition $|f''(z)| \leq \lambda$ for some values of $\lambda > 0$ and the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ are restated and then their certain consequences which will be important for (analytic and) geometric function theory are given. In addition, only for the proofs of these consequences of our main results derived in the Section 2 of this paper, both the assertion of S. S. Miller and P. T. Mocanu given in [3] and the results of N. Tuneski given in [11] are also used.

The following two assertions (Lemma 1 in [3] and Lemma 2 in [11] below) will be required to prove the main results.

Lemma 1 *Let* $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$ *and* $0 \leq \alpha < 1$. *Then,*

$$(2-\alpha)\left|f''(z)\right| \leq 2(1-\alpha) \;\Rightarrow\; f(z) \in \mathcal{S}^*(\alpha).$$

The result is sharp.

Lemma 2 *Let* $f(z) \in \mathcal{A}$, $z \in \mathbb{U}$ *and* $0 \leq \alpha < 1$. *Then,*

$$(2-\alpha)\left|f''(z)\right| \leq 1-\alpha \;\Rightarrow\; f(z) \in \mathcal{K}(\alpha).$$

The result is sharp.

The following important assertion (see, for its details and also example, [3] (p. 33-34 and $a = 0$)) will be required to prove the main results.

Lemma 3 *Let* $\Omega \subset \mathbb{C}$ *and suppose that the function* $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ *satisfies* $\psi\left(Me^{i\theta}, Ke^{i\theta}; z\right) \notin \Omega$ *for all* $K \geq Mn$, $\theta \in \mathbb{R}$, *and* $z \in \mathbb{U}$. *If the function* $p(z)$ *is in the class:*

$$\mathcal{H}\left[0, n\right] := \left\{p(z) \in \mathcal{H} \;:\; p(z) = a_n z^n + a_{n+1} z^{n+1} + \ldots.\ (z \in \mathbb{U})\right\}$$

and

$$\psi\left(p(z), zp'(z); z\right) \in \Omega\ ,$$

then $|p(z)| < M$, *where for some* $M > 0$ *and for all* $z \in \mathbb{U}$.

2 The main results, implications and examples

By making use of Lemma 3, we shall firstly give and then prove the main result, which is given by

Theorem 1 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < \delta < M$, *and let* $f''(z) \neq 2a_2 - \delta$. *Then,*

$$\Re e\left(\frac{zf'''(z)}{\delta - 2a_2 + f''(z)}\right) < \frac{M(M-\delta)}{\delta^2 + (\delta + M)^2} \quad \Rightarrow \quad \left|f''(z)\right| < M + 2|a_2|\,.$$

Proof. Let us define $p(z)$ by

$$p(z) = f''(z) - 2a_2\ ,$$

where $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$ and $z \in \mathbb{U}$. Clearly, $p(z)$ is in the class $\mathcal{H}[0,1]$ (when, of course, $a_3 \neq 0$). Then, it immediately follows that

$$\frac{zp'(z)}{\delta + p(z)} = \frac{zf'''(z)}{\delta - 2a_2 + f''(z)} \qquad \left(f''(z) \neq 2a_2 - \delta;\ z \in \mathbb{U}\right).$$

Let

$$\psi(r, s; z) := \frac{s}{\delta + r}$$

and

$$\Omega := \left\{ w \ :\ w \in \mathbb{C} \ \text{ and } \ \Re e\{w\} < \frac{M(M-\delta)}{\delta^2 + (\delta + M)^2} \right\}.$$

Then we have

$$\psi\left(p(z), zp'(z); z\right) = \left(\frac{zp'(z)}{\delta + p(z)} =\right) \frac{zf'''(z)}{\delta - 2a_2 + f''(z)} \in \Omega$$

for all z in $\mathbb{U}$. Furthermore, for any $\theta \in \mathbb{R}$, $K \geq nM \geq M$, and $z \in \mathbb{U}$, we obviously obtain that

$$\Re e\left\{\psi\left(Me^{i\theta}, Ke^{i\theta}; z\right)\right\} = \Re e\left(\frac{Ke^{i\theta}}{\delta + Me^{i\theta}}\right) \geq \frac{M(M-\delta)}{\delta^2 + (\delta + M)^2},$$

i.e.,

$$\psi\left(Me^{i\theta}, Ke^{i\theta}; z\right) \notin \Omega.$$

Therefore, in respect of the Lemma 3, the definition of $p(z)$ easily yields that

$$\left|p(z)\right| = \left|f''(z) - 2a_2\right| < M \qquad (M > 0; z \in \mathbb{U}),$$

which completes the desired proof. □

Proposition 1 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < \delta < 1$, *and let* $f''(z) \neq 2a_2 - \delta$. *Then,*

$$\Re e\left(\frac{zf'''(z)}{\delta - 2a_2 + f''(z)}\right) < \Phi(\alpha, \delta, a_2) \quad \Rightarrow \quad f(z) \in \mathcal{S}^*(\alpha),$$

where

$$\Phi(\alpha, \delta, a_2) := \frac{[2(1-\alpha) - 2(2-\alpha)|a_2|]\,[2(1-\alpha) - (2-\alpha)(2|a_2| + \delta)]}{(2-\alpha)^2\delta^2 + [(2-\alpha)(\delta - 2|a_2|) + 2(1-\alpha)]^2}.$$

Proof. If we take

$$M + 2|a_2| := \frac{2(1-\alpha)}{2-\alpha} \quad (0 \leq \alpha < 1)$$

in Theorem 1 and just then use Lemma 1, we easily get the proof. □

By letting $\alpha := 0$ in Proposition 1, we first obtain the following corollary.

Corollary 1 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < \delta < 1$, *and let* $f''(z) \neq 2a_2 - \delta$. *Then,*

$$\Re e\left(\frac{zf'''(z)}{\delta - 2a_2 + f''(z)}\right) < \frac{(1-2|a_2|)(1-2|a_2|-\delta)}{(1-2|a_2|+\delta)^2 + \delta^2} \quad \Rightarrow \quad f(z) \in \mathcal{S}^* .$$

By taking $\delta := 2|a_2|$ in Corollary 1, we next have the following corollary.

Corollary 2 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < 2|a_2| < 1$, *and let* $f''(z) \neq 0$. *Then,*

$$\Re e\left(\frac{zf'''(z)}{f''(z)}\right) < \frac{1-2|a_2|}{1+4|a_2|^2} \quad \Rightarrow \ f(z) \in \mathcal{S}^* .$$

For this result (i.e., for Corollary 2), the following example can be easily given.

Example 1 *Take* $f(z) = z + \frac{1}{4}z^2 + a_3 z^3$ *and let* $|a_3| < \frac{1}{18}$. *Since*

$$\left|f''(z)\right| = \left|\frac{1}{2} + 6a_3 z\right| \geq \frac{1}{2} - 6|a_3| > \frac{1}{2} - \frac{1}{3} = \frac{1}{6} > 0 ,$$

we arrive at $f''(z) \neq 0$. *Besides, it is obvious that* $|a_2| = \frac{1}{4} < \frac{1}{2}$. *At the same time, clearly,*

$$\Re e\left(\frac{zf'''(z)}{f''(z)}\right) = 1 - \Re e\left(\frac{1}{1+12a_3 z}\right) < \frac{1-2|a_2|}{1+4|a_2|^2} = \frac{2}{5} .$$

In that case, as a result of Corollary 2, it is clear that $f(z) \in \mathcal{S}^*$. We also indicate that, since $\left|f''(z)\right| = \left|\frac{1}{2} + 6a_3 z\right| < 1$, Lemma 1 immediately implies that the function $f(z)$ is starlike in $\mathbb{U}$.

Proposition 2 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < \delta < \frac{1}{2}$, *and let* $f''(z) \neq 2a_2 - \delta$. *Then,*

$$\Re\mathrm{e}\left(\frac{zf'''(z)}{\delta - 2a_2 + f''(z)}\right) < \Phi(\alpha, \delta, a_2) \quad \Rightarrow \quad f(z) \in \mathcal{K}(\alpha) ,$$

where

$$\Phi(\alpha, \delta, a_2) := \frac{[(1-\alpha) - 2(2-\alpha)|a_2|]\,[(1-\alpha) - (2-\alpha)(2|a_2| + \delta)]}{(2-\alpha)^2\delta^2 + [(2-\alpha)(\delta - 2|a_2|) + (1-\alpha)]^2} .$$

Proof. If we put

$$M + 2|a_2| := \frac{1-\alpha}{2-\alpha} \quad (0 \le \alpha < 1)$$

in Theorem 1 and just then use Lemma 2, we easily arrive at the desired result in Proposition 2. □

By putting $\alpha = 0$ in Proposition 2, we then get the following result.

Corollary 3 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < \delta < \frac{1}{2}$, *and let* $f''(z) \neq 2a_2 - \delta$. *Then,*

$$\Re\mathrm{e}\left(\frac{zf'''(z)}{\delta - 2a_2 + f''(z)}\right) < \frac{(1-4|a_2|)(1-4|a_2|-2\delta)}{(1-4|a_2|+2\delta)^2 + 4\delta^2} \quad \Rightarrow \quad f(z) \in \mathcal{K} .$$

By setting $\delta := 2|a_2|$ in Corollary 3, we also get the following corollary.

Corollary 4 *Let* $f(z) = z + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $z \in \mathbb{U}$, $0 < 2|a_2| < \frac{1}{2}$, *and let* $f''(z) \neq 0$. *Then,*

$$\Re\mathrm{e}\left(\frac{zf'''(z)}{f''(z)}\right) < \frac{1-4|a_2|}{1+4|a_2|^2} \quad \Rightarrow \quad f(z) \in \mathcal{K} .$$

The following can be also given to exemplify the result given above.

Example 2 *Take* $f(z) = z + \frac{1}{8}z^2 + a_3 z^3$ *and let* $|a_3| < \frac{1}{24\sqrt{2}}$. *Since*

$$|f''(z)| = \left|\frac{1}{4} + 6a_3\right| \ge \frac{1}{4} - 6|a_3| > \frac{1}{4} - \frac{1}{4\sqrt{2}} > 0 ,$$

we obtain $f''(z) \neq 0$. *Furthermore, it is clear that* $|a_2| = \frac{1}{4} < \frac{1}{2}$. *At the same time, obviously,*

$$\Re\mathrm{e}\left(\frac{zf'''(z)}{f''(z)}\right) = 1 - \Re\mathrm{e}\left(\frac{1}{1 + 24a_3 z}\right) < \frac{1-4|a_2|}{1+4|a_2|^2} = \frac{8}{17} .$$

In this case, as a result of Corollary 4, it is clear that $f(z) \in \mathcal{K}$. *Then, since* $|f''(z)| = \left|\frac{1}{4} + 6a_3\right| < 1$, *Lemma2 immediately implies that function* $f(z)$ *is convex in* $\mathbb{U}$.

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Finite groups with a certain number of cyclic subgroups II

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Abstract. In this note we describe the finite groups G having $|G|-2$ cyclic subgroups. This partially solves the open problem in the end of [3].

Let G be a finite group and $C(G)$ be the poset of cyclic subgroups of G. The connections between $|C(G)|$ and $|G|$ lead to characterizations of certain finite groups G. For example, a basic result of group theory states that $|C(G)| = |G|$ if and only if G is an elementary abelian 2-group. Recall also the main theorem of [3], which states that $|C(G)| = |G|-1$ if and only if G is one of the following groups: $\mathbb{Z}_3$, $\mathbb{Z}_4$, S_3 or D_8.

In what follows we shall continue this study by describing the finite groups G for which

$$|C(G)| = |G| - 2. \qquad (*)$$

First, we observe that certain finite groups of small orders, such as $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_{12} and $\mathbb{Z}_2 \times D_8$, have this property. Our main theorem proves that in fact these groups exhaust all finite groups G satisfying $(*)$.

Theorem 1 *Let G be a finite group. Then $|C(G)| = |G| - 2$ if and only if G is one of the following groups: $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_{12} or $\mathbb{Z}_2 \times D_8$.*

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Proof. We will use the same technique as in the proof of Theorem 2 in [3]. Assume that G satisfies $(*)$, let $n = |G|$ and denote by $d_1 = 1, d_2, \ldots, d_k$ the positive divisors of n. If $n_i = |\{H \in C(G) \mid |H| = d_i\}|$, $i = 1, 2, \ldots, k$, then

$$\sum_{i=1}^{k} n_i \phi(d_i) = n.$$

Since $|C(G)| = \sum_{i=1}^{k} n_i = n - 2$, one obtains

$$\sum_{i=1}^{k} n_i(\phi(d_i) - 1) = 2,$$

which implies that we have the following possibilities:

Case 1. There exists $i_0 \in \{1, 2, \ldots, k\}$ such that $n_{i_0}(\phi(d_{i_0}) - 1) = 2$ and $n_i(\phi(d_i) - 1) = 0, \forall\, i \neq i_0$.

Since the image of the Euler's totient function does not contain odd integers > 1, we infer that $n_{i_0} = 2$ and $\phi(d_{i_0}) = 2$, i.e. $d_{i_0} \in \{3, 4, 6\}$. We remark that d_{i_0} cannot be equal to 6 because in this case G would also have a cyclic subgroup of order 3, a contradiction. Also, we cannot have $d_{i_0} = 3$ because in this case G would contain two cyclic subgroups of order 3, contradicting the fact that the number of subgroups of a prime order p in G is $\equiv 1 \pmod p$ (see e.g. the note after Problem 1C.8 in [1]). Therefore $d_{i_0} = 4$, i.e. G is a 2-group containing exactly two cyclic subgroups of order 4. Let $n = 2^m$ with $m \geq 3$. If $m = 3$ we can easily check that the unique group G satisfying $(*)$ is $\mathbb{Z}_2 \times \mathbb{Z}_4$. If $m \geq 4$ by Proposition 1.4 and Theorems 5.1 and 5.2 of [2] we infer that G is isomorphic to one of the following groups:

- M_{2^m};
- $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-1}}$;
- $\langle a, b \mid a^{2^{m-2}} = b^8 = 1,\ a^b = a^{-1},\ a^{2^{m-3}} = b^4 \rangle$, where $m \geq 5$;
- $\mathbb{Z}_2 \times D_{2^{m-1}}$;
- $\langle a, b \mid a^{2^{m-2}} = b^2 = 1,\ a^b = a^{-1+2^{m-4}}c,\ c^2 = [c, b] = 1,\ a^c = a^{1+2^{m-3}} \rangle$, where $m \geq 5$.

All these groups have cyclic subgroups of order 8 for $m \geq 5$ and thus they do not satisfy $(*)$. Consequently, $m = 4$ and the unique group with the desired property is $\mathbb{Z}_2 \times D_8$.

Case 2. There exist $i_1, i_2 \in \{1, 2, \ldots, k\}$, $i_1 \neq i_2$, such that $n_{i_1}(\phi(d_{i_1}) - 1) = n_{i_2}(\phi(d_{i_2}) - 1) = 1$ and $n_i(\phi(d_i) - 1) = 0, \forall\, i \neq i_1, i_2$.

Then $n_{i_1} = n_{i_2} = 1$ and $\phi(d_{i_1}) = \phi(d_{i_2}) = 2$, i.e. $d_{i_1}, d_{i_2} \in \{3, 4, 6\}$. Assume that $d_{i_1} < d_{i_2}$. If $d_{i_2} = 4$, then $d_{i_1} = 3$, that is G contains normal cyclic subgroups of orders 3 and 4. We infer that G also contains a cyclic subgroup of order 12, a contradiction. If $d_{i_2} = 6$, then we necessarily must have $d_{i_1} = 3$. Since G has a unique subgroup of order 3, it follows that a Sylow 3-subgroup of G must be cyclic and therefore of order 3. Let $n = 3 \cdot 2^m$, where $m \geq 1$. Denote by n_2 the number of Sylow 2-subgroups of G and let H be such a subgroup. Then H is elementary abelian because G does not have cyclic subgroups of order 2^i with $i \geq 2$. By Sylow's Theorems,

$$n_2|3 \text{ and } n_2 \equiv 1 \pmod 2,$$

implying that either $n_2 = 1$ or $n_2 = 3$. If $n_2 = 1$, then $G \cong \mathbb{Z}_2^m \times \mathbb{Z}_3$, a group that satisfies $(*)$ if and only if $m = 1$, i.e. $G \cong \mathbb{Z}_6$. If $n_2 = 3$, then $|\mathrm{Core}_G(H)| = 2^{m-1}$ because $G/\mathrm{Core}_G(H)$ can be embedded in S_3. It follows that G contains a subgroup isomorphic with $\mathbb{Z}_2^{m-1} \times \mathbb{Z}_3$. If $m \geq 3$ this has more than one cyclic subgroup of order 6, contradicting our assumption. Hence either $m = 1$ or $m = 2$. For $m = 1$ one obtains $G \cong S_3$, a group that does not have cyclic subgroups of order 6, a contradiction, while for $m = 2$ one obtains $G \cong D_{12}$, a group that satisfies $(*)$. This completes the proof. □

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New classes of local almost contractions

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Abstract. Contractions represents the foundation stone of nonlinear analysis. That is the reason why we propose to unify two different type of contractions: almost contractions, introduced by V. Berinde in [2] and local contractions (Martins da Rocha and Filipe Vailakis in [7]). These two types of contractions operate in different space settings: in metric spaces (almost contractions) and semimetric spaces (for local contractions). That new type of contraction was built up in a new space setting, which is the pseudometric space. The main results of this paper represent the extension for various type of operators on pseudometric spaces, such as: generalized ALC, Ćirić-type ALC, quasi ALC, Ćirić-Reich-Rus type ALC. We propose to study the existence and uniqueness of their fixed points, and also the continuity in their fixed points, with a large number of examples for ALC-s.

1 Introduction

First, we present the concept of almost contraction, following V. Berinde in [2].

Definition 1 (see [2]) *Let* (X, d) *be a metric space.* $T : X \to X$ *is called almost contraction or* (δ, L)*- contraction if there exist a constant* $\delta \in (0, 1)$ *and some* $L \geq 0$ *such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(y, Tx), \forall x, y \in X. \quad (1)$$

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Remark 1 *The term of almost contraction is equivalent to weak contraction, and it was first introduced by V. Berinde in* [2].

Because of the simmetry of the distance, the almost contraction condition (1) *includes the following dual one:*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + L \cdot d(x, Ty), \forall x, y \in X, \tag{2}$$

obtained from (1) *by replacing* $d(Tx, Ty)$ *by* $d(Ty, Tx)$ *and* $d(x, y)$ *by* $d(y, x)$.

Obviously, to prove the almost contactiveness of T*, it is necessary to check both* (1) *and* (2).

A strict contraction satisfies (1)*, with* $\delta = a$ *and* $L = 0$*, therefore it is an almost contraction with a unique fixed point.*

Many examples of almost contractions are given in [1]-[3]. Weak contractions represent a generous concept, due to various mappings satisfying the condition (1). Such examples of weak contraction was given by V. Berinde in [2].

Definition 2 [5] *Let* (X, d) *be a metric space. Any mapping* $T : X \to X$ *is called Ćirić-Reich-Rus contraction if it is satisfied the condition:*

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot [d(x, Tx) + d(y, Ty)], \forall x, y \in X, \tag{3}$$

where $\alpha, \beta \in \mathcal{R}_+$ *and* $\alpha + 2\beta < 1$.

Proposition 1 (see [8]) *Let* (X, d) *be a metric space. Any Ćirić-Reich-Rus contraction,i.e., any mapping* $T : X \to X$ *satisfying the condition (3), represent an almost contraction.*

Theorem 1 *A mapping satisfying the contractive condition: there exists* $0 \leq h < \frac{1}{2}$ *such that*

$$d(Tx, Ty) \leq h \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{4}$$

for all $x, y \in X$*, is a weak contraction.*

An operator satisfying (4) *with* $0 < h < 1$ *is called quasi-contraction.*

Remark 2 *Theorem 1 prove that quasi-contractions with* $0 < h < \frac{1}{2}$ *are always weak contractions. However, there exists quasi-contractions with* $h \geq \frac{1}{2}$*, presented in Example 1 by V. Berinde in* [2]*, as it follows:*

Example 1 *Let* $T : [0,1] \to [0,1]$ *a mapping given by* $Tx = \frac{2}{3}$ *for* $x \in [0,1)$*, and* $T1 = 0$*. Then* T *has the following properties:*
1) T *satisfies* (4) *with* $h \in [\frac{2}{3}, 1)$*, i.e.,* T *is quasi-contraction;*
2) T *satisfies* (1)*, with* $\delta \geq \frac{2}{3}$ *and* $L \geq \delta$*, i.e.,* T *is also weak contraction;*
3) T *has a unique fixed point,* $x^* = \frac{2}{3}$*.*

Since we were familiarized with the class of almost contractions, we introduce the concept of local contractions, another interesting type of operators with unexpected applications. The concept of local contraction was presented by Martins da Rocha and Filipe Vailakis in [7].

Definition 3 (see [7]) *Let* F *be a set and let* $\mathcal{D} = (d_j)_{j \in J}$ *be a family of semidistances defined on* F*. We let* σ *be the weak topology on* F *defined by the family* $\mathcal{D}$*. A sequence* $(f_n)_{n \in \mathbb{N}^*}$ *is said to be* $\sigma - \mathsf{Cauchy}$ *if it is* d_j*-Cauchy,* $\forall j \in J$*. A subset* A *of* F *is said to be sequencially* σ*-complete if every* σ*-Cauchy sequence in* A *converges in* A *for the* σ*-topology. A subset* $A \subset F$ *is said to be* σ*-bounded if* $\mathrm{diam}_j(A) \equiv \sup\{d_j(f,g) : f, g \in A\}$ *is finite for every* $j \in J$*.*
Let r *be a function from* J *to* J*. An operator* $T : F \to F$ *is called local contraction with respect* $(\mathcal{D}, r)$ *if, for every* j*, there exists* $\beta_j \in [0,1)$ *such that*

$$\forall f, g \in F, \quad d_j(Tf, Tg) \leq \beta_j d_{r(j)}(f, g).$$

Definition 4 *The mapping* $d(x,y) : X \times X \to \mathbb{R}_+$ *is said to be a pseudometric if:*

1. $d(x,y) = d(y,x)$*;*
2. $d(x,y) \leq d(x,z) + d(z,y)$*;*
3. $x = y$ *implies* $d(x,y) = 0$
   (instead of $x = y \Leftrightarrow d(x,y) = 0$ *in the metric case).*

Definition 5 (see [11])
Let r *be a function from* J *to* J*. An operator* $T : F \to F$ *is an almost local contraction (ALC) with respect* $(\mathcal{D}, r)$ *or* (δ, L)*- contraction, if there exist a constant* $\delta \in (0,1)$ *and some* $L \geq 0$ *such that*

$$d_j(Tf, Tg) \leq \delta \cdot d_j(f,g) + L \cdot d_{r(j)}(g, Tf), \forall f, g \in F. \tag{5}$$

Theorem 2 [11] *Assume that the space* F *is* σ*- Hausdorff, which means: for each pair* $f, g \in F$*,* $f \neq g$*, there exists* $j \in J$ *such that* $d_j(f,g) > 0$*.*

If A *is a nonempty subset of* F*, then for each* h *in* F*, we let*
$d_j(h, A) \equiv \{d_j(h, g) : g \in A\}$.
Consider a function $r : J \to J$ *and let* $T : F \to F$ *be an almost local contraction with respect to* $(\mathcal{D}, r)$*. Consider a nonempty,* σ*- bounded, sequentially* σ*- complete, and* T*- invariant subset* $A \subset F$*.*
(E) *If the condition*

$$\forall j \in J, \quad \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} \mathrm{diam}_{r^{n+1}(j)}(A) = 0 \tag{6}$$

is satisfied, then the operator T *admits a fixed point* f^* *in* A*.*
(S) *Moreover, if* $h \in F$ *satisfies*

$$\forall j \in J, \quad \lim_{n\to\infty} \beta_j \beta_{r(j)} \cdots \beta_{r^n(j)} d_{r^{n+1}(j)}(h, A) = 0, \tag{7}$$

then the sequence $(T^n h)_{n\in\mathbb{N}}$ *is* σ*- convergent to* f^**.*

Example 2 *Let* $X = [0, n] \times [0, n] \subset \mathbb{R}^2, n \in \mathbb{N}^*, \quad T : X \to X,$

$$T(x, y) = \begin{cases} (\frac{x}{2}, \frac{y}{2}) & \text{if } (x, y) \neq (1, 0) \\ (0, 0) & \text{if } (x, y) = (1, 0) \end{cases}$$

The diameter of the subset $X = [0, n] \times [0, n] \subset \mathbb{R}^2$ *is given by the diagonal line of the square whose four sides have length* n *.*
We shall use the pseudometric:

$$d_j\big((x_1, y_1), (x_2, y_2)\big) = |x_1 - x_2| \cdot e^{-j}, \forall j \in J, \tag{8}$$

where J *is a subset of* $\mathbb{N}$*. This is a pseudometric, but not a metric, take for example:*
$d_j((1, 4), (1, 3)) = |1 - 1| \cdot e^{-j} = 0$*, however* $(1, 4) \neq (1, 3)$
In this case, we shall use the function $r(j) = \frac{j}{2}$*. By applying the inequality* (5) *to our mapping* T*, we get for all* $x = (x_1, y_1), y = (x_2, y_2) \in X$

$$\Big|\frac{x_1}{2} - \frac{x_2}{2}\Big| \cdot e^{-j} \leq \theta \cdot |x_1 - x_2| \cdot e^{\frac{-j}{2}} + L \cdot \Big|x_2 - \frac{x_1}{2}\Big| \cdot e^{\frac{-j}{2}},$$

for all $j \in J$*, which can be write as the equivalent form*

$$|x_1 - x_2| \cdot e^{\frac{-j}{2}} \leq 2\theta \cdot |x_1 - x_2| + L \cdot |2x_2 - x_1|,$$

The last inequality became true if we take $\theta = \frac{1}{2} \in (0, 1), L = 4 \geq 0$*. Hence* T *is an almost local contraction, with the unique fixed point* $(0, 0)$*.*
T *is continuous in the fixed point, at* $(0, 0) \in \mathrm{Fix}(T)$*, but is not continuous at* $(1, 0) \notin \mathrm{Fix}(T)$*.*

Example 3 *With the assumptions from the previous example and the pseudometric defined by (8) where* $j \in J$, *and* $r(j) = \frac{j}{2}$, *we get another example for almost local contractions. Considering* $T : X \to X$,

$$T(x, y) = \begin{cases} (x, -y) & \text{if } (x, y) \neq (1, 1) \\ (0, 0) & \text{if } (x, y) = (1, 1) \end{cases}$$

T *is not a contraction because the contractive condition:*

$$d_j(Tx, Ty) \leq \theta \cdot d_j(x, y), \tag{9}$$

is not valid $\forall\, x, y \in X$, *and for any* $\theta \in (0, 1)$. *Indeed,* (9) *is equivalent with:*

$$|\, x_1 - x_2| \cdot e^{-j} \leq \theta \cdot |x_1 - x_2| \cdot e^{-j}, \forall\, j \in J.$$

The last inequality leads us to $1 \leq \theta$, *which is obviously false, considering* $\theta \in [0, 1)$. *However,* T *becomes an almost local contraction if:*

$$|x_1 - x_2| \cdot e^{-j} \leq \theta \cdot |x_1 - x_2| \cdot e^{\frac{-j}{2}} + L \cdot |x_2 - x_1| \cdot e^{\frac{-j}{2}}$$

which is equivalent to : $e^{\frac{-j}{2}} \leq \theta + L$. *For* $\theta = \frac{1}{3} \in [0, 1)$, $L = 2 \geq 0$ *and* $j \in J$, *the last inequality becomes true, i.e.* T *is an almost local contraction with many fixed points:*

$$\text{Fix}T = \{(x, 0) : x \in \mathbb{R}\}.$$

In this case, we have:

$$\forall\, j \in J, \quad \lim_{n \to \infty} \theta^{n+1} \text{diam}_{r^{n+1}(j)}(A) = \lim_{n \to \infty} \left(\frac{1}{3}\right)^{n+1} \cdot (n-1)^2 = 0$$

This way, the existence of the fixed point is assured, according to condition (E) from Theorem 2. The continuity of T *in* $(0, 0) \in \text{Fix}(T)$ *is valid, but we have discontinuity in* $(1, 1)$, *which is not a fixed point of* T.

Example 4 *Let* X *be the set of positive functions:*

$$X = \{f | f : [0, \infty) \to [0, \infty)\},$$

which is a subset of the real functions $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$.
Let $d_j(f, g) = |f(0) - g(0)| \cdot e^{-j}$, $\forall\, f, g \in X, r(j) = \frac{j}{2}$, $\forall\, j \in J$. *Indeed,* d_j *is a pseudometric, but not a metric, take for example* $d_j(x, x^2) = 0$, *but* $x \neq x^2$.

Considering the mapping $Tf = |f|, \forall f \in X$*, and using the inequality* (1) *from the definition of almost local contractions:*

$$|f(0) - g(0)| \cdot e^{-j} \leq \theta \cdot |f(0) - g(0)| \cdot e^{\frac{-j}{2}} + L \cdot |g(0) - f(0)| \cdot e^{\frac{-j}{2}}$$

which is equivalent to: $e^{-j/2} \leq \theta + L$*. This inequality becames true if* $j > 0$, $\theta = \frac{1}{4} \in (0, 1)$, $\quad L = 3 > 0$*. Hence,* T *is an ALC. However,* T *is not a contraction, because the contractive condition* (9) *leads us again to the false assumption:* $1 \leq \theta$*. The mapping* T *has infinite number of fixed points:* $\mathrm{Fix}T = \{f \in X\} = X$*, by taking:*

$$|f(x)| = f(x), \forall f \in X, x \in [0, \infty)$$

2 Main results

The main results of this paper represent the extension for various type of operators on pseudometric spaces, such as: generalized ALC, Ćirić-type ALC, quasi ALC, Ćirić-Reich-Rus type ALC.

a) **Generalized ALC**

Definition 6 *Let* r *be a function from* J *to* J*. Let* $A \subset F$ *be a* τ*-bounded sequencially* τ*-complete and* T*- invariant subset of* F*. A mapping* $T : A \to A$ *is called generalized almost local contraction if there exist a constant* $\theta \in (0, 1)$ *and some* $L \geq 0$ *such that* $\forall x, y \in X, \forall j \in J$ *we have:*

$$\begin{aligned} d_j(Tx, Ty) \leq \theta \cdot d_{r(j)}(x, y) \\ + L \cdot \min\{d_{r(j)}(x, Tx), d_{r(j)}(y, Ty), d_{r(j)}(x, Ty), d_{r(j)}(y, Tx)\} \end{aligned} \tag{10}$$

Remark 3 *It is obvious that any generalized almost local contraction is an almost contraction, i.e., it does satisfy inequality* (1)*.*

Theorem 3 *Let* $T : A \to A$ *be a generalized almost local contraction, i.e., a mapping satisfying* (10)*, and also verifying the condition* (7) *for the unicity of fixed point. Let* $\mathrm{Fix}(T) = \{f\}$*. Then* T *is continuous at* f*.*

Proof. Since T is a generalized almost local contraction, there exist a constant $\theta \in (0, 1)$ and some $L \geq 0$ such that (10) is satisfied. We know by Theorem 7 that T has a unique fixed point, say f.
Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X converging to f. Then by taking

$$y := y_n, \quad x := f$$

in the generalized almost local contraction condition (10), we get

$$d_j(Tf, Ty_n) \leq \theta \cdot d_{r(j)}(f, y_n), n = 0, 1, 2, \cdots \tag{11}$$

since f is a fixed point for T, we have

$$\min\{d_{r(j)}(x, Tx), d_{r(j)}(y, Ty), d_{r(j)}(x, Ty), d_{r(j)}(y, Tx)\} = d_{r(j)}(f, Tf) = 0.$$

Now, by letting $n \to \infty$ in (11), we get $Ty_n \to Tf$, which shows that T is continuous at f. □

b) **Ćirić-type almost local contraction**

Definition 7 (see Berinde, [4]) *Let* (X, d) *be a complete metric space. The mapping* $T : X \to X$ *is called Ćirić almost contraction if there exist a constant* $\alpha \in [0, 1)$ *and some* $L \geq 0$ *such that*

$$d(Tx, Ty) \leq \alpha \cdot M(x, y) + L \cdot d(y, Tx), \textit{for all } x,y \in X, \tag{12}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

From the above definition the following question arises: it is possible to expand it to the case of almost local contractions? The answer is affirmative and is given by the next definition. But first we need to remind the Lemma of Ćirić ([6]), which will be essential in proving our main results.

Lemma 1 *Let* T *be a quasi-contraction on* X *and let* n *be any positive integer. Then, for each* $x \in X$*, and all positive integers* i, j*, where* $i, j \in \{1, 2, \cdots n\}$ *implies*

$$d(T^i x, T^j x) \leq h \cdot \delta[O(x, n)],$$

where we denoted $\delta(A) = \sup\{d(a, b) : a, b \in A\}$ *for a subset* $A \subset X$.

Remark 4 *Observe that, by means of Lemma 1, for each* n*, there exist* $k \leq n$ *such that*

$$d(x, T^k x) = \delta[O(x, n)].$$

Lemma 2 (see [6]) *Let* T *be a quasi-contraction on* X*. Then the inequality*

$$\delta[O(x, n)] \leq \frac{1}{1 - h} d(x, T^k x)$$

holds for all $x \in X$.

Definition 8 *Under the assumptions of definition 5, the operator* $T : A \to A$ *is called Ćirić-type almost local contraction with respect* $(\mathcal{D}, r)$ *if, for every* $j \in J$*, there exist the constants* $\theta \in [0, 1)$ *and* $L \geq 0$ *such that*

$$d_j(Tf, Tg) \leq \theta \cdot M_{r(j)}(f, g) + L \cdot d_{r(j)}(g, Tf), \text{ for all } f,g \in A, \tag{13}$$

where

$$M_{r(j)}(f, g) = \max\left\{d_{r(j)}(f, g), d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\right\}.$$

Remark 5 *Although this class is more wide than the one of almost local contractions, similar conclusions can be stated as in the case of almost local contractions, as it follows:*

Theorem 4 *Consider a function* $r : J \to J$*, let a nonempty,* τ*- bounded, sequentially* τ*- complete, and* T*- invariant subset* $A \subset X$ *and let* $T : A \to A$ *be Ćirić- type almost local contraction with respect to* $(\mathcal{D}, r)$*. Then*

1. T *has a fixed point,i.e.,* $\text{Fix}(T) = \{x \in X : Tx = x\} \neq \phi$;
2. *For any* $x_0 = x \in A$*, the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *converges to* $x^* \in \text{Fix}(T)$;
3. *The following a priori estimate is available:*

$$d_j(x_n, x^*) \leq \frac{\theta^n}{(1-\theta)^2} d_j(x, Tx), \quad n = 1, 2... \tag{14}$$

Proof. For the conclusion of the Theorem, we have to prove that T has at least a fixed point in the subset $A \subset X$. To this end, let $x \in A$ be arbitrary, and let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration defined by $x_{n+1} = Tx_n, \quad n \in \mathbb{N}$ with $x_0 = x$.
Take $x := x_{n-1}, y := x_n$ in (13) to obtain

$$d_j(x_n, x_{n+1}) = d_j(Tx_{n-1}, Tx_n) \leq \theta \cdot M_{r(j)}(x_{n-1}, x_n),$$

since $d_j(x_n, Tx_{n-1}) = d_j(Tx_{n-1}, Tx_{n-1}) = 0$. Continuing in this manner, for $n \geq 1$, by Lemma 1 we have

$$d_j(T^n x, T^{n+1} x) = d_j(TT^{n-1} x, T^2 T^{n-1} x) \leq \theta \cdot \delta[O(T^{n-1} x, 2)].$$

By using Remark 4, we can easily conclude: there exist a positive integer $k_1 \in \{1, 2\}$ such that

$$\delta[O(T^{n-1} x, 2)] = d_j(T^{n-1} x, T^{k_1} T^{n-1} x)$$

and therefore

$$d_j(x_n, x_{n+1}) \leq \theta \cdot d_j(T^{n-1}x, T^{k_1}T^{n-1}x).$$

By using once again Lemma 1, we obtain, for $n \geq 2$,

$$d_j(T^{n-1}x, T^{k_1}T^{n-1}x) = d_j(TT^{n-2}x, T^{k_1+1}T^{n-2}x) \leq$$
$$\leq \theta \cdot \delta[O(T^{n-2}x, k_1 + 1)] \leq \theta \cdot \delta[O(T^{n-2}x, 3)].$$

Continuing in this manner, we get

$$d_j(T^n x, T^{n+1}x) \leq \theta \cdot \delta[O(T^{n-1}x, 2)] \leq \theta^2 \cdot \delta[O(T^{n-2}x, 3)].$$

By applying repeatedly the last inequality, we get

$$d_j(T^n x, T^{n+1}x) \leq \theta \cdot \delta[O(T^{n-1}x, 2)] \leq \cdots \leq \theta^n \cdot \delta[O(x, n+1)]. \tag{15}$$

At this point, by Lemma 2, we obtain

$$\delta[O(x, n+1)] \leq \delta[O(x, \infty)] \leq \frac{1}{1-\theta} d_j(x, Tx),$$

which by (15) yields

$$d_j(T^n x, T^{n+1}x) \leq \frac{\theta^n}{1-\theta} d_j(x, Tx). \tag{16}$$

The last inequality and the triangle inequality can be merged to obtain the following estimate:

$$d_j(T^n x, T^{n+p}x) \leq \frac{\theta^n}{1-\theta} \cdot \frac{1-\theta^p}{1-\theta} d_j(x, Tx). \tag{17}$$

Let us remind the fact that $0 \leq \theta \leq 1$, then, by using (17), we can conclude that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. The subset A is assumed to be sequentially τ-complete, there exists x^* in A such that $\{x_n\}$ is τ- convergent to x^*. After simple computations involving the triangular inequality and the Definition (13), we get

$$\begin{aligned}
d_j(x^*, Tx^*) &\leq d_j(x^*, x_{n+1}) + d_j(x_{n+1}, Tx^*)\\
&= d_j(T^{n+1}x, x^*) + d_j(T^n x, Tx^*)\\
&\leq d_j(T^{n+1}x, x^*) + \theta \max\{d_j(T^n x, u), d_j(T^n x, T^{n+1}x), d_j(x^*, Tx^*),\\
&\qquad d_j(T^n x, Tx^*), d_j(T^{n+1}x, x^*)\} + + L \cdot d_j(x^*, Tx_n)
\end{aligned}$$

Continuing in this manner, we obtain

$$d_j(x^*, Tx^*) \le d_j(T^{n+1}x, x^*) + \theta \cdot [d_j(T^n x, u) + d_j(T^n x, T^{n+1}x) + d_j(x^*, Tx^*) + d_j(T^{n+1}x, x^*)] + L \cdot d_j(x^*, Tx_n).$$

These relations leads us to the following inequalities:

$$d_j(x^*, Tx^*) \le \frac{1}{1-\theta}[(1+\theta)d_j(T^{n+1}x, x^*) + (\theta + L)d_j(x^*, Tx_n) + \theta d_j(T^n x, T^{n+1}x)]. \tag{18}$$

Letting $n \to \infty$ in (18) we obtain

$$d_j(x^*, Tx^*) = 0,$$

which means that x^* is a fixed point of T. The estimate (14) can be obtained from (16) by letting $p \to \infty$.
This completes the proof. □

Remark 6 1) *Theorem 4 represent a very important extension of Banach's fixed point theorem, Kannan's fixed point theorem, Chatterjea's fixed point theorem, Zamfirescu's fixed point theorem, as well as of many other related results obtained on the base of similar contractive conditions. These fixed point theorems mentioned before ensures the uniqueness of the fixed point, but the Ćirić type almost local contraction need not have a unique fixed point.*
2) *Let us remind* (see Rus [9], [10]) *that an operator $T : X \to X$ is said to be a weakly Picard operator (WPO) if the sequence $\{T^n x_0\}_{n=0}^{\infty}$ converges for all $x_0 \in X$ and the limits are fixed point of T. The main merit of Theorem 4 is the very large class of Weakly Picard operators assured by using it.*

The uniqueness of the fixed point of a Ćirić type almost local contraction can be assured by imposing an additional contractive condition, quite similar to (13), according to the next theorem.

Theorem 5 *With the assumptions of Theorem 4, let $T : A \to A$ be a Ćirić type almost local contraction with the additional inequality, which actually means the monotonicity of the pseudometric:*

$$d_{r(j)}(f, g) \le d_j(f, g), \forall f, g \in A, \forall j \in J. \tag{19}$$

If the mapping T satisfies the supplementary condition: there exist the constants $\theta \in [0, 1)$ and some $L_1 \ge 0$ such that

$$d_j(Tf, Tg) \le \theta \cdot d_{r(j)}(f, g) + L_1 \cdot d_{r(j)}(f, Tf), \text{ for all } f,g \in A, \forall j \in J, \tag{20}$$

then

1) T *has a unique fixed point, i.e.,* $\mathsf{Fix}(T) = \{f^*\}$*;*

2) *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* $x_{n+1} = Tx_n, \quad n \in \mathbb{N}$ *converges to* f^**, for any* $x_0 \in A$*;*

3) *The a priori error estimate* (14) *holds;*

4) *The rate of the convergence of the Picard iteration is given by*

$$d_j(x_n, f^*) \leq \theta \cdot d_{r(j)}(x_{n-1}, f^*), \quad n = 1, 2, \dots, \forall j \in J \tag{21}$$

Proof. 1) Suppose, by contradiction, there are two distinct fixed points f^* and g^* of T. Then, by using (20), and condition (19) for every fixed $j \in J$ with $f := f^*, g := g^*$ we get:

$$d_j(f^*, g^*) \leq \theta \cdot d_{r(j)}(f^*, g^*) \leq \theta \cdot d_j(f^*, g^*) \Leftrightarrow (1 - \theta) \cdot d_j(f^*, g^*) \leq 0,$$

which is obviously a contradiction with $d_j(f^*, g^*) > 0$. So, we prove the uniqueness of the fixed point.
The proof for 2) and 3) is quite similar to the proof from the Theorem 4.
4) At this point, letting $g := x_n, f := f^*$ in (20), it results the rate of convergence given by (21). The proof is complete. □

The contractive conditions (13) and (20) can be merged to maintain the unicity of the fixed point, stated by the next theorem.

Theorem 6 *Under the assumptions of definition 8, let* $T : A \to A$ *be a mapping for which there exist the constants* $\theta \in [0, 1)$ *and some* $L \geq 0$ *such that for all* $f, g \in A$ *and* $\forall j \in J$

$$\begin{aligned} d_j(Tf, Tg) \leq \theta \cdot M_{r(j)}(f, g) \\ + L \cdot \min\{d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\}, \end{aligned} \tag{22}$$

where

$$M_{r(j)}(f, g) = \max\{d_{r(j)}(f, g), d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\}.$$

Then

1. T *has a unique fixed point,i.e.,* $\mathsf{Fix}(T) = \{f^*\}$*;*

2. *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* $x_{n+1} = Tx_n, \quad n \in \mathbb{N}$ *converges to* f^**, for any* $x_0 \in A$*;*

3. *The a priori error estimate* (14) *holds.*

Particular case

1. The famous Ćirić' s fixed point theorem for single valued mappings given in [6] can be obtain from Theorems 4, 6, 5 by taking $L = L_1 = 0$ and considering r the identity mapping: $r(j) = j$. The Ćirić' s contractive condition represent one of the most general metrical condition that provide a unique fixed point by means of Picard iteration. Despite this observation, the contractive condition given for Ćirić-type almost local contraction (in (13)) possess a very high level of generalisation. Note that the fixed point could be approximated by means of Picard iteration, just like in the case of Ćirić' s fixed point theorem, although the uniqueness of the fixed point is not ensured by using (13).

2. If the maximum from Theorem 6 becomes:

$$\max\left\{d_{r(j)}(f,g), d_{r(j)}(f,Tf), d_{r(j)}(g,Tg), d_{r(j)}(f,Tg), d_{r(j)}(g,Tf)\right\} = d_j(f,g),$$

   for all $f, g \in A$, then we can easily obtain Theorem 2 (E) from Theorem 4. Also, by Theorem 5 we obtain Theorem 2 (U) (see Zakany,[11]).

In the light of the above informations about the Ćirić-type ALC-s, it is natural to extend it to the Ćirić-type strict almost local contractions.

Definition 9 *Let* X *be a set and let* $\mathcal{D} = (d_j)_{j\in J}$ *be a family of pseudometrics defined on* X. *In order to underline the local character of these type of contractions, we let* $A \subset X$ *a subset of* X. *We let* τ *be the weak topology on* X *defined by the family* $\mathcal{D}$. *Let* r *be a function from* J *to* J. *The operator* $T : A \to A$ *is called Ćirić-type strict almost local contraction with respect* $(\mathcal{D}, r)$ *if it simultaneously satisfies conditions* $(Ci - ALC)$ *and* $(ALC - U)$, *with some real constants* $\theta_C \in [0,1)$, $L_C \geq 0$ *and* $\theta_u \in [0,1)$, $L_u \geq 0$, *respectively.*

$$(Ci - ALC) \quad d_j(Tf,Tg) \leq \theta_C \cdot M_{r(j)}(f,g) + L_C \cdot d_{r(j)}(g,Tf), \text{ for all } f,g \in A,$$

for every $j \in J$, *where*

$$M_{r(j)}(f,g) = \max\left\{d_{r(j)}(f,g), d_{r(j)}(f,Tf), d_{r(j)}(g,Tg), d_{r(j)}(f,Tg), d_{r(j)}(g,Tf)\right\}.$$

$$(ALC - U) \quad d_j(Tf,Tg) \leq \theta_u \cdot d_{r(j)}(f,g) + L_u \cdot d_{r(j)}(f,Tf), \text{ for all } f,g \in A, \forall j \in J,$$

We end with a few examples that have an illustrative role. They presents Ćirić' type almost local contractions, without having unique fixed point.

Example 5 *Let* A *be the set of positive functions* $A = \{f|f : [0,\infty) \to [0,\infty)\}$, *which is the subset of all real functions* $X = \{f : \mathbb{R} \to \mathbb{R}\}, A \subset X$.

We shall use the pseudometric:

$$d_j(f,g) = |f(0) - g(0)| \cdot j, \;\; \forall j \in J; J \subset \mathbb{N}, \;\; \forall f,g \in A.$$

Indeed, d_j *is a pseudometric, but not a metric, take for example* $d_j(x^3,x^2) = 0$, *but* $x^3 \neq x^2$. *Considering the mapping* $Tf = |f|, \;\; \forall f \in A, \; r(j) = j+1$. *Note that the restrictive condition* (19) *is also verified. By using condition* (5) *for almost local contractions:*

$$|f(0) - g(0)| \cdot j \leq \theta \cdot |f(0) - g(0)| \cdot (j+1) + L \cdot |g(0) - f(0)| \cdot (j+1)$$

which is equivalent to: $j \leq (\theta + L)(j+1)$. *This inequality becames true if* $j > 1, \;\; \theta = \frac{1}{5} \in (0,1), \;\; L = 3 > 0$, *and* $\frac{j}{j-1} \in (1,2)$. *Hence,* T *is an almost local contraction. However,* T *is not a contraction, because the contractive condition*

$$d(Tx,Ty) \leq \theta \cdot d(x,y)$$

leads us to the false assumption: $1 \leq \theta$.
The map T *is Ćirić-type almost local contraction, because*

$$M_{r(j)}(f,g) = |f(0) - g(0)| \cdot (j-1),$$

and from (13) *we have the equivalent form*

$$|f(0) - g(0)| \cdot j \leq \theta \cdot |f(0) - g(0)| \cdot (j-1) + L \cdot |f(0) - f(0)| \cdot (j-1).$$

Again, we get the inequality $j \leq (\theta + L)(j-1)$. *The mapping* T *has infinite number of fixed points:* $\mathrm{Fix}T = \{f \in A\} = A$, *by taking:*

$$|f(x)| = f(x), \forall f \in A, \quad x \in [0,\infty).$$

In fact, the uniqueness condition (20) *is not valid, having in view the equivalent form:*

$$|f(0) - g(0)| \cdot j \leq \theta \cdot |f(0) - g(0)| \cdot (j-1) + L_1 \cdot |f(0) - f(0)| \cdot (j-1),$$

which leads us to the contradiction $j \leq \theta(j-1)$, *i.e. the mapping* T *not satisfy the uniqueness condition* (20).
In fact, not even (22) *is satisfied, by computing* $M_{r(j)}(f,g) = |f(0)-g(0)| \cdot (j-1)$ *and* $\min\{d_{r(j)}(f,Tf), d_{r(j)}(g,Tg), d_{r(j)}(f,Tg), d_{r(j)}(g,Tf)\} = |f(0) - g(0)| \cdot (j-1)$ (*since* $j > 1$). *By replacing these values in* (22), *we get*

$$|f(0) - g(0)| \cdot j \leq \theta \cdot |f(0) - g(0)| \cdot (j-1) + L \cdot |f(0) - f(0)| \cdot (j-1),$$

which also lead to the previous contradiction.

Example 6 *By taking the mapping from Example 4, with a small modification, which is: let* X *be the set of positive functions*

$$X = \{f| \quad f : [0, \infty) \to [0, \infty)\},$$

which is a subset of the real functions $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$.
Let $d_j(f, g) = |f(x_0) - g(x_0)| \cdot e^j, \quad \forall f, g \in X, r(j) = \frac{j}{2}, \forall j \in \mathbb{Z}$.

We can conclude in the same manner that T *is also a Ćirić type almost local contraction, i.e., it satisfy the contractive condition* (13).
Indeed, we have $M_{r(j)}(f, g) = |f(x_0) - g(x_0)| \cdot e^{\frac{j}{2}}$. *This way, the condition* (13) *became the contractive condition for almost local contractions* (5).

By considering $L = 0$ in the definition 8 of Ćirić-type almost local contraction, we get a new type of ALC, that is the quasi-almost local contraction.

c) **Quasi-almost local contractions**

Definition 10 *Under the assumptions of definition 5, the operator* $T : A \to A$ *is called quasi-almost local contraction with respect* $(\mathcal{D}, r)$ *if, for every* $j \in J$*, there exist the constant* $\theta \in [0, 1)$ *such that*

$$d_j(Tf, Tg) \le \theta \cdot M_{r(j)}(f, g), \text{ for all } f,g \in A, \tag{23}$$

where

$$M_{r(j)}(f, g) = \max\{d_{r(j)}(f, g), d_{r(j)}(f, Tf), d_{r(j)}(g, Tg), d_{r(j)}(f, Tg), d_{r(j)}(g, Tf)\}.$$

Theorem 7 *Consider a function* $r : J \to J$*, let a nonempty,* τ*- bounded, sequentially* τ*- complete, and* T*- invariant subset* $A \subset X$ *and let* $T : A \to A$ *be quasi-almost local contraction with respect to* $(\mathcal{D}, r)$.
Then

1. T *has a fixed point,i.e.,* $\mathrm{Fix}(T) = \{x \in X : Tx = x\} \neq \phi$*;*

2. *For any* $x_0 = x \in A$*, the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *converges to* $x^* \in \mathrm{Fix}(T)$*;*

3. *The following a priori estimate is available:*

$$d_j(x_n, x^*) \le \frac{\theta^n}{(1-\theta)^2} d_j(x, Tx), \quad n = 1, 2, \dots \tag{24}$$

Proof. Obviously, we have to follow the steps from the proof of Theorem 4, with the only difference that the constant $L = 0$, as in the case of quasi ALC-s. □

The uniqueness of the fixed point is also assured by imposing an additional condition, just like in the class of Ćirić-type almost local contraction, as it follows.

Theorem 8 *With the assumptions of Theorem 4, let* $T : A \to A$ *be a quasi-almost local contraction with the additional inequality:*

$$d_{r(j)}(f, g) \le d_j(f, g), \forall f, g \in A, \forall j \in J. \tag{25}$$

If the mapping T *satisfies the supplementary condition: there exist the constants* $\theta \in [0, 1)$ *such that*

$$d_j(Tf, Tg) \le \theta \cdot d_{r(j)}(f, g) + L_1 \cdot d_{r(j)}(f, Tf), \text{ for all } f,g \in A, \forall j \in J, \tag{26}$$

then

1. T *has a unique fixed point,i.e.,* $Fix(T) = \{f^*\}$*;*
2. *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* $x_{n+1} = Tx_n, \quad n \in \mathbb{N}$ *converges to* f^**, for any* $x_0 \in A$*;*
3. *The a priori error estimate* (14) *holds;*
4. *The rate of the convergence of the Picard iteration is given by*

$$d_j(x_n, f^*) \le \theta \cdot d_{r(j)}(x_{n-1}, f^*), \quad n = 1, 2, ..., \forall j \in J \tag{27}$$

d) **Ćirić-Reich-Rus type almost local contraction**

Definition 11 *Under the assumptions of definition 5, the operator* $T : A \to A$ *is called Ćirić-Reich-Rus type almost local contraction with respect* $(\mathcal{D}, r)$ *if the mapping* $T : A \to A$ *satisfying the condition*

$$d_j(Tf, Tg) \le \delta \cdot d_{r(j)}(f, g) + L \cdot [d_{r(j)}(f, Tf) + d_{r(j)}(g, Tg)], \tag{28}$$

for all f, g *in* A*, where* $\delta, L \in \mathcal{R}_+$ *and* $\delta + 2L < 1$

Theorem 9 *If the pseudometric* d *satisfy the condition:* $d_{r(j)}(f, g) < d_j(f, g), \forall j \in J, \quad \forall f, g \in A$*, then any Ćirić- Reich- Rus type almost local contraction, i.e. any mapping* $T : A \to A$ *satisfying the condition* (28) *with* $L \neq 1$ *is an almost local contraction.*

Proof. Using condition (28) and the triangle rule, we get

$$\begin{aligned} d_j(Tf, Tg) &\leq \delta \cdot d_{r(j)}(f, g) + L \cdot [d_{r(j)}(f, Tf) + d_{r(j)}(g, Tg)] \\ &\leq \delta \cdot d_{r(j)}(f, g) + L \cdot [d_{r(j)}(g, Tf) \\ &\quad + d_{r(j)}(Tf, Tg) + d_{r(j)}(f, g) + d_{r(j)}(g, Tf)] \end{aligned}$$

The condition for the pseudometric leads us to:

$$\begin{aligned} d_j(f, g) &> d_{r(j)}(f, g), \\ d_j(Tf, Tg) &> d_{r(j)}(Tf, Tg), \\ d_j(g, Tf) &> d_{r(j)}(g, Tf) \end{aligned}$$

From this point, we get after simple computations:

$$(1 - L) \cdot d_j(Tf, Tg) \leq (\delta + L) \cdot d_j(f, g) + 2L \cdot d_{r(j)}(g, Tf) \quad (29)$$

and which implies

$$d_j(Tf, Tg) \leq \frac{\delta + L}{1 - L} \cdot d_j(f, g) + \frac{2L}{1 - L} \cdot d_{r(j)}(g, Tf), \forall f, g \in A \quad (30)$$

Considering $\delta, L \in \mathcal{R}_+$ and $\delta + 2L < 1$, the inequality (28) holds, with $\frac{\delta+L}{1-L} \in (0, 1)$ and $\frac{2L}{1-L} \geq 0$. Therefore, any Ćirić-Reich-Rus type almost local contraction with the condition for the pseudometric, is an almost local contraction. □

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Slant helices of (k, m)-type in $\mathbb{E}^4$

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Abstract. In the present work, we define new type slant helices called (k,m)-type and we conclude that there are no (1,k) type $(1 \leq k \leq 4)$ slant helices. Also we obtain conditions for different type slant helices.

1 Introduction

The curve theory has been one of the most studied subject because of having many application area from geometry to the various branch of science. Especially the characterizations on the curvature and torsion play important role to define special curve types such as so-called helices. The curves of this type have drawn great attention from mathematics to natural sciences and engineering. Helices appear naturally in structures of DNA, nanosprings. They are also widely used in engineering and architecture.The concept of slant helix defined by Izumiya and Takeuchi [6] based on the property that the principal normal lines of an α curve(with non-vanishing curvature) make constant angle with a fixed direction of the ambient space. After this lightening work many researcher have characterized this type of curves in various spaces. For instance in [1] authors extended slant helix concept to $\mathbb{E}^n$ and conclude that there are no slant helices with non-zero constant curvatures in the space $\mathbb{E}^4$. The slant helix subject are also considered in 3-, 4-, and n- dimensional Eucliedan spaces, respectively in [7, 10, 12] different dimensions. Moreover different properties

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of helices are also discussed in [4, 8, 11, 13] . On the other hand in A.T Ali, R. Lopez and M. Turgut extended this study to the k-type slant helix in $\mathbb{E}_1^4$. In this study they called α curve as k-type slant helix if there exists on (non-zero) constant vector field $U \in \mathbb{E}_1^4$ such that $\langle V_{k+1}, U \rangle = const$, for $0 \leq k \leq 3$. Here V_{k+1} shows the Frenet vectors of this curve [2].

One may easily conclude that 0-type slant helices are general helices and 1-type slant helices correspond just slant helices. They consider k-type slant helices for partially null and pseudo null curves, and in hyperbolic space.

In accordance with above studies, in this work we define (k, m)-type slant helices in $\mathbb{E}^4$ and we show that there are no $(1, m)$ type slant helices in $\mathbb{E}^4$.

2 Preliminaries

In this section we will present on brief basic tools for the space curves in $\mathbb{E}^4$. A detailed information can be found in [5].

Let $\alpha : I \subseteq \mathbb{R} \to \mathbb{E}^4$ be an arbitrary curve in Euclidean space $\mathbb{E}^4$. The standard scalar product in $\mathbb{E}^4$ given by

$$\langle x, y \rangle = \sum_{i=1}^{4} x_i y_i$$

where $x, y \in \mathbb{E}^4\, (1 \leq i \leq 4)$, Then the curve α is said to be of unit speed (or parametrized by arclength) if it satisfies $\left\langle \alpha'(s), \alpha'(s) \right\rangle = 1$. In addition the norm of an arbitrary vector x in $\mathbb{E}^4$ is given by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Let $\{T, N, B_1, B_2\}$ be the moving frame along the unit speed curve α, where T, N, B_1 and B_2 denote, the tangent, the principal normal, binormal and tri-normal vector fields, respectively. Then the Frenet formulas are given by [3]

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \tag{1}$$

Hence k_1, k_2 and $k_3$are called, the first, the second and the third curvature of α. If $k_3 \neq 0$ for each $s \in I \subseteq \mathbb{R}$, the curve lies fully in $\mathbb{E}^4$.

3 (k, m)-type slant helices in $\mathbb{E}^4$

In this section, we will define (k, m) type slant helices in $\mathbb{E}^4$.

Definition 1 *Let* α *be a regular unit speed curve in* $\mathbb{E}^4$ *with Frenet frame* $\{V_1, V_2, V_3, V_4\}$. *We call* α *is a* (k, m) *type slant helix if there exists a non-zero constant vector field* $U \in \mathbb{E}^4$ *satisfies* $\langle V_k, U\rangle = m$ (m *constant*) *for* $1 \leq k \leq 4$ $k \neq m$. *The constant vector* U *is on axis of* (k, m)*-type slant helix. We decompose* U *with respect to Frenet frame* $\{T, N, B_1, B_2\}$ *as* $U = u_1 T + u_2 N + u_3 B_1 + u_4 B_2$, *where* $u_i = u_i(s)$ *are differentiable functions of* s. *Here we denote* $V_1 = T, V_2 = N, V_3 = B_1, V_4 = B_2$. *From now on, in sake of easinesss, we will use these notations and assume that* $k_i \neq 0$, $(1 \leq i \leq 3)$.

Theorem 1 *There are no* $(1, 2)$ *type slant helices in* $\mathbb{E}^4$.

Proof. Assume that α is a $(1, 2)$ type slant helix. Then for a constant vector field U. $\langle T, U\rangle = a$ is const and $\langle N, U\rangle = b$ is constant. Differentiating this equation and using Frenet equations, we obtain $k_1 \langle N, U\rangle = 0$ means that U is orthogonal to N. □

Theorem 2 *There are no* $(1, 3)$ *type slant helices in* $\mathbb{E}^4$.

Proof. Assume that α is a $(1, 3)$ type slant helix. Then we may write $\langle T, U\rangle = const = a$ $\langle B, U\rangle = const = b$. Also taking account Theorem 1 we decompose U as follows

$$U = aT + bB_1 + u_2 B_2 \tag{2}$$

Differentiating constant vector U, we get

$$a(k_1 N) + b(-k_2 N + k_3 B_2) + u_2' B_2 + u_2(-k_3 B_1) = 0$$

$$ak_1 N - bk_2 = 0 \tag{3}$$

$$-u_2 k_3 = 0 \tag{4}$$

$$u_2' + bk_3 = 0. \tag{5}$$

From (4) we get $u_2 = 0$ and hence $b = 0$, which means that there are no $(1, 3)$ type slant helices in $\mathbb{E}^4$. □

Theorem 3 *There are no* $(1, 4)$ *type slant helix in* $\mathbb{E}^4$.

Proof. Assume that α is a $(1,4)$ type slant helix. Then we may write

$$U = aT + u_1 B_1 + bB_2$$

We know that U is constant then we get

$$a(k_1 N) + u_1^{'} B_1 + u_1(-k_2 N + k_3 B_2) + b(-k_3 B_1) = 0$$
$$ak_1 - u_1 k_2 = 0 \quad (6)$$
$$u_1^{'} - bk_3 = 0 \quad (7)$$
$$u_1 k_3 = 0 \quad (8)$$

means that $u_1 = 0$ and from (7) we get $b = 0$, hence there are no $(1,4)$ type slant helix in $\mathbb{E}^4$. □

Corollary 1 *There are no* $(1,k)$ *type slant helix in* $\mathbb{E}^4$.

Theorem 4 *If* α *is a* $(2,3)$ *type slant helix in* $\mathbb{E}^4 \iff$ *there exist a constant such that*

$$\frac{k_2(t)}{k_1(t)} - \int_0^s k_1(t)\,dt = 0.$$

Proof. Assume that α is a $(2,3)$ type slant helix in $\mathbb{E}^4$. Then we may write

$$U = u_1 T + aN + bB_1 + u_2 B_2.$$

Differentiating constant vector U, one may get

$$u_1(k_1 N) + u_1^{'} T + a(-k_1 T + k_2 B_1)$$
$$+b(-k_2 N + k_3 B_2) + u_2^{'} B_2 + u_2(-k_3 B_1) = 0$$
$$u_1^{'} - ak_1 = 0 \quad (9)$$
$$u_1 k_1 - bk_2 = 0 \quad (10)$$
$$ak_2 - k_3 = 0 \quad (11)$$
$$bk_3 + u_2^{'} = 0 \quad (12)$$

Using (9) and (12)

$$u_1 = a\int_0^s k_1(t)\,dt \quad (13)$$

and

$$u_2 = -b \int_0^s k_3(t)\,dt \tag{14}$$

From (10) we get $u_1 = b\frac{k_2}{k_1}$. Taking into account this result in (13) we get

$$c \int_0^s k_1(t)\,dt = b\frac{k_2}{k_1} = \frac{a}{b}\int_0^s k_1(t)\,dt. \tag{15}$$

We assume a and b are constants so we can denote $\frac{a}{b} = \mathrm{const}$.

Using this fact in (15) we conclude that

$$\frac{k_2(t)}{k_1(t)} - \int_0^s k_1(t)\,dt = 0.$$

This completes the proof. □

Theorem 5 *There are no* $(2, 4)$ *type slant helix in* $\mathbb{E}^4$.

Proof. Assume that α is a $(2, 4)$ type slant helix in $\mathbb{E}^4$. Then we may write

$$U = u_1T + aN + u_2B_1 + bB_2.$$

Recall that U is a constant vector, we obtain

$$u_1'T + u_1(k_1N) + a(-k_1T + k_2B_1)$$
$$+u_2'B_1 + u_2(-k_2N + k_3B_2) + b(-k_3B_1) = 0$$

$$u_1' - ak_1 = 0 \tag{16}$$

$$u_1k_1 - u_2k_2 = 0 \tag{17}$$

$$ak_2 + u_2' - bk_3 = 0 \tag{18}$$

$$u_2k_3 = 0 \tag{19}$$

From (19) we get $u_2 = 0$. Using this in (17) we get $u_1 = 0$ and finally we get $a = b = 0$ which means that there are no $(2, 4)$ type slant helix in $\mathbb{E}^4$. □

Theorem 6 *There are no* $(3, 4)$ *type slant helix in* $\mathbb{E}^4$.

Proof. Assume that α is a $(3,4)$ type slant helix in $\mathbb{E}^4$. Then we may write

$$U = u_1 T + u_2 N + a B_1 + b B_2$$

Taking into account of the constant vector U we get

$$\begin{aligned} \left(u_1' - u_2 k_1\right) T + \left(u_1 k_1 + u_2' - a k_2\right) N & \\ + \left(u_2 k_2 - b k_3\right) B_1 + \left(a k_3\right) B_2 &= 0 \\ u_1' - u_2 k_1 &= 0 \\ u_1 k_1 + u_2' - a k_2 &= 0 \\ u_2 k_2 - b k_3 &= 0 \\ a k_3 &= 0 \end{aligned}$$

$\Rightarrow a = 0$ so there is no $(3,4)$ type slant helix in $\mathbb{E}^4$. □

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On extensions of Baer and quasi-Baer modules

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Abstract. Let R be a ring, M_R a module, S a monoid, $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism and $R * S$ a skew monoid ring. Then $M[S] = \{m_1 g_1 + \cdots + m_n g_n \,|\, n \geq 1,\ m_i \in M \text{ and } g_i \in S \text{ for each } 1 \leq i \leq n\}$ is a module over $R * S$. A module M_R is *Baer* (resp. *quasi-Baer*) if the annihilator of every subset (resp. submodule) of M is generated by an idempotent of R. In this paper we impose S-compatibility assumption on the module M_R and prove: (1) M_R is quasi-Baer if and only if $M[s]_{R*S}$ is quasi-Baer, (2) M_R is Baer (resp. p.p) if and only if $M[S]_{R*S}$ is Baer (resp. p.p), where M_R is S-skew Armendariz, (3) M_R satisfies the ascending chain condition on annihilator of submodules if and only if so does $M[S]_{R*S}$, where M_R is S-skew quasi-Armendariz.

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1 Introduction and preliminaries

Throughout this paper R denotes an associative ring with identity and M_R is a right R-module. According to [16] a ring R is *Baer* if the right annihilator of every nonempty subset of R is generated by an idempotent. Quasi-Baer rings were initially introduced by Clark [10]. A ring R is *quasi-Baer* if the right annihilator of every right ideal of R generated by an idempotent. Another generalization of Baer rings is p.p.-rings. Recall that a ring R is called *right* (resp. *left*) *p.p* if right (left) annihilator of every element of R is generated by an idempotent. Birkenmeier et al. in [7] introduced principally quasi-Baer rings. A ring R is called *right principally quasi-Baer* (or p.q.-Baer for short) if the right annihilator of a principal right ideal of R is generated by an idempotent.

In [1] Armendariz studied the behaver of a polynomial ring over Baer ring. He proved for a reduced ring R, $R[x]$ is Baer if and only if R is Baer [1, Theorem B]. Also, he provid an example to show that the "Armendariz" condition is not superfluous. Birkenmeier and park [9] extended this result to monoid ring.

We now introduce the definitions and notions used in this paper. If A and B are non-empty subsets of a monoid S, then an element $s_0 \in AB = \{ab : a \in A, b \in B\}$ is said to be a *unique product element* (u.p. element for short) in the product of AB if it is uniquely presented in the form of $s = ab$ where $a \in A$ and $b \in B$.

Recall that a monoid S is called *unique product monoid* (*u.p. monoid* for short) if for any two non-empty finite subsets $A, B \subseteq S$ there exist $a \in A$ and $b \in B$ such that ab is u.p. element in the product of AB. The class of u.p. monoids are quite large. For example this class includes the right or left ordered monoid and torsion free nilpotent groups. Every u.p. monoid S is cancellative [9, Lemma 1.1] and has no non-unit element of finite order.

Assume that R is a ring, S a monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. For each $g \in S$ we denote the image of g by ω_g (i.e., $\omega(g) = \omega_g$). Then all finite formal combinations $\sum_{i=1}^{n} a_i g_i$, with point-wise addition and multiplication induced by $(ag)(bh) = (a\omega_g(b))gh$ form a ring that is called *skew monoid ring* and it is denoted by $R * S$. The construction of skew monoid ring generalizes some classical ring construction such as skew polynomial rings, skew Laurent polynomial rings and monoid rings. Hence any result on skew monoid ring has its counterpart in each of the subclasses.

As a generalization of monoid rings, we introduce the notion of modules over skew monoid rings. For a module M_R, let $M[S] = \{m_1g_1 + \cdots + m_ng_n \,|\, n \geq 1,\ m_i \in M \text{ and } g_i \in S \text{ for each } 1 \leq i \leq n\}$. Then $M[S]$ is a right module over $R * S$ under the following scaler product operation: for $m(s) = m_1g_1 + \cdots + m_ng_n \in$

$M[S]$ and $f(s) = a_1h_1 + \cdots + a_mh_m \in R * S$, $m(s)f(s) := \sum_{i,j} m_i\omega_{g_i}(a_j)g_ih_j$.

For a nonempty subset X of M_R, let $ann_R(X) = \{r \in R \,|\, Xr = 0\}$.

The notion of reduced, Armendariz, Baer, p.p and quasi-Baer module introduced in [18] by Lee and Zhou. A module M_R is called *reduced* if for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. A module M_R is called *Baer* if, for any nonempty subset X of M, $ann_R(X) = eR$ where $e^2 = e \in R$. A module M_R is called *p.p* if for any element $m \in M$, $ann_R(m) = eR$ where $e^2 = e \in R$. A module M_R is called *quasi-Baer* if, for any right R-submodule X of M, $ann_R(X) = eR$ where $e^2 = e \in R$. Clearly, R is reduced (resp. Baer, right p.p, quasi-Baer) if and only if R_R is reduced (resp. Baer, right p.p, quasi-Baer). Lee and Zhou [18] proved that M_R is reduced if and only if $M[x]_{R[x]}$ is reduced. Various results of reduced rings were extended to modules in [18, 2].

Recall that from [6] an idempotent $e \in R$ is *left (resp. right) semicentral* in R if $exe = xe$ (resp. $exe = ex$) for all $x \in R$. Equivalently, $e = e^2 \in R$ is left (resp. right) semicentral if eR (resp. Re) is an ideal of R. Since the right annihilator of a right R-module is an ideal, then the right annihilator of a right R-module is generated by a left semicemtral idempotent in a quasi-Baer module. We denote the set of all left (resp. right) semiccentral idempotents of R with $\mathcal{S}_\ell(R)$ (resp. $\mathcal{S}_r(R)$).

A module M_R is called *principally quasi-Baer* (or p.q.-Baer for short) if, for any $m \in M$, $ann_R(mR) = eR$ where $e^2 = e \in R$. Clearly R is a right p.q.-Baer if and only if R_R is p.q.-Baer module.

In this paper we introduce and study the concept of S-skew Armendariz modules as a generalization of S-Armendariz rings [19]. For a u.p. monoid S and monoid homomorphism $\omega : S \longrightarrow End(R)$ we show that reduced module M_R is S-skew Armendariz. We investigate the quasi-Baer and related conditions on right $R * S$-module $M[S]$ for a u.p. monoid S and monoid homomorphism $\omega : S \longrightarrow Aut(R)$. We impose S-compatibility assumption on the module M_R and prove: (1) M_R is quasi-Baer if and only if $M[s]_{R*S}$ is quasi-Baer, (2) M_R is Baer (resp. p.p) if and only if $M[S]_{R*S}$ is Baer (resp. p.p), when M_R is S-skew Armendariz, (3) M_R satisfies the ascending chain condition on annihilator of submodules if and only if so does $M[S]_{R*S}$, when M_R is S-skew quasi-Armendariz. Our results extend Armendariz [1, Theorem B], Groenewald [11, Theorem 2], Birkenmeier, Kim and Park [8, Theorem 1.2], Birkenmeier and Park [9, Theorem 1.2, Corollary 1.3].

2 S-skew Armendariz modules

Let R be a ring with an endomorphism σ. According to [4] for a module M_R and an endomorphism $\sigma : R \longrightarrow R$, we say that M_R is σ-*compatible* if for each $m \in M$ and $r \in R$, we have $mr = 0$ if and only if $m\sigma(r) = 0$. For more details on σ-compatible rings refer to [13, 14].

Definition 1 *Let R be a ring, S a monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. We say that a module M_R is S-compatible if M_R is ω_g-compatible for each $g \in S$.*

Notic that R is S-compatible if and only if R_R is S-compatible. Now we give some examples of S-compatible modules.

Example 1 [4, Example 4.4] *Let R_0 be a domain of characteristic zero, and $R := R_0[t]$. Define $\sigma|_{R_0} = id_{R_0}$ and $\sigma(t) = -t$. Let $M_R := R_0 \oplus R_0 \oplus R_0 \oplus \cdots$, where $t \in R$ acts on M_R as follows: for $(m_0, m_1, m_2, \ldots) \in M$, we set $(m_0, m_1, m_2, \ldots).t := (0, m_0k_0, m_1k_1, m_2k_2, \ldots)$ where the $k_i (i \in \mathbb{N})$ are fixed nonzero integers. We show that M is σ-compatible. For this, it suffices to show that $ann(m) = 0$ whenever $0 \neq m \in M$. Suppose that $(a_0, a_1, a_2, \cdots)(b_rt^r + b_{r+1}t^{r+1} +$ "higher terms"$) = 0$, where $a_i, b_i \in R_0$ for every $i \in \mathbb{N}$ and $b_r \neq 0$. First applying t^r to $(a_0, a_1, a_2, \ldots)$ gives*

$$(0, 0, \cdots, 0, a_0k_0k_1 \cdots k_{r-1}, a_1k_1k_2 \cdots k_r, \ldots)(b_r + b_{r+1}t + \text{“ higher terms”}) = 0.$$

Upon computing this expression, we deduce that $a_0k_0k_1 \ldots k_{r-1}b_r = 0$. Since the characteristic is zero, R is a domain, and $k_0k_1 \ldots k_{r-1}b_r \neq 0$, we deduce that $a_0 = 0$. Now, we may proceed inductively to show that all $a_i = 0$. From this calculation, we deduce that M_R is σ-compatible.

Example 2 [14, Example 1.1] *Let R_1 be a ring, D a domain and $R = T_n(R_1) \oplus D[y]$, where $T_n(R_1)$ is upper $n \times n$ triangular matrix ring over R_1. Let $\alpha : D[y] \longrightarrow D[y]$ be a monomorphism which is not surjective. We define an endomorphism $\overline{\alpha} : R \longrightarrow R$ of R by $\overline{\alpha}(A \oplus f(y)) = A \oplus \alpha(f(y))$ for each $A \in T_n(R_1)$ and $f(y) \in D[y]$. In* [14, Example 1.1] *it is shown that R is an $\overline{\alpha}$-compatible.*

Example 3 *Let R be a ring and σ_i an endomorphism of R such that R be a σ_i-compatible for each $1 \leq i \leq n$. Let S be a monoid generated by $\{x_1, x_2, \ldots, x_n\}$ and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism such that $\omega_{x_i^j} = \sigma_i^j$. One can show that R is S-compatible and $R * S \cong R[x_1, x_2, \ldots, x_n; \sigma_1, \sigma_2, \ldots, \sigma_n]$.*

According to Lee and Zhou [18] a module M_R is *Armendariz* if, for elements $m(x) = m_0 + m_1x + \cdots + m_nx^n \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x]$, $m(x)f(x) = 0$ implies $m_ia_j = 0$ for each $1 \leq i \leq n$, $1 \leq j \leq m$. In [21] Zhang and Chen, introduced the concept of a σ-skew Armendariz module and studied its properties. A module M_R is called σ-*skew Armendariz module*, if, whenever $m(x)f(x) = 0$ where $m(x) = m_0 + m_1x + \cdots + m_nx^n \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x; \sigma]$, we have $m_i\sigma^i(b_j) = 0$ for each $0 \leq i \leq n$, $0 \leq j \leq m$. In [19], Liu introduced the concept of a S-Armendariz ring and studied its properties. In the following we introduce the concept of S-skew Armendariz module as a generalization of S-Armendariz rings.

Definition 2 *Let* R *be a ring,* S *a monoid and* $\omega : S \longrightarrow \mathrm{End}(R)$ *a monoid homomorphism. We say that a module* M_R *is* S*-skew Armendariz module if, for elements* $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ *and* $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$, $m(s)f(s) = 0$ *implies* $m_i\omega_{g_i}(a_j) = 0$ *for each* $1 \leq i \leq n, 1 \leq j \leq t$. *In the case of* ω *is identity homomorphism, we say* M_R *is* S*-Armendariz module.*

Notice that for a ring R and monid S with monoid homomorphism $\omega : S \longrightarrow \mathrm{End}(R)$, R is S-skew Armendariz (resp. S-Armendariz) if and only if R_R is S-skew Armendariz (resp. S-Armendariz).

Theorem 1 *Let* R *be a ring,* S *a monoid and* $\omega : S \longrightarrow \mathrm{End}(R)$ *a monoid homomorphism. Then* M_R *is* S*-skew Armendariz if and only if for every elements* $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ *and* $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$, $m(s)f(s) = 0$ *implies* $m_{i_1}\omega_{g_{i_1}}(a_j) = 0$ *for each* $1 \leq j \leq t$ *and some* $1 \leq i_1 \leq t$.

Proof. The forward direction is clear. For the converse, suppose that $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ and $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$ with $m(s)f(s) = 0$. Then there exists $1 \leq i_1 \leq n$ such that $m_{i_1}\omega_{g_{i_1}}(a_j) = 0$ for each $1 \leq j \leq t$. Without loss of generality we can assume that $i_1 = 1$. Thus $0 = m(s)f(s) = (m_2g_2 + \cdots + m_ng_n)f(s)$. Then by induction on n we can conclude that $m_i\omega_{g_i}(a_j) = 0$ for each $1 \leq i \leq n$ and $1 \leq j \leq t$. Hence M_R is S-skew Armendariz. □

If S is a monoid generated by $\{x\}$ and $\omega : S \longrightarrow \mathrm{End}(R)$ such that $\omega_{x^i} = \sigma^i$ for an endomorphism σ of R, then the skew monoid ring $R * S$ is isomorphic to skew polynomial ring $R[x; \sigma]$ and $M[S]$ is isomorphic to $M[x]$. Thus we have the following equivalent condition for a module to be σ-skew Armendariz.

Corollary 1 *Let* M_R *be a module and* σ *an endomorphism of* R. *Then* M_R *is* σ*-skew Armendariz if and only if for every polynomials* $m(x) = m_0 + m_1x + \cdots + m_nx^n \in M[x]$ *and* $f(x) = a_0 + a_1x + \cdots + a_tx^t \in R[x; \sigma]$, $m(x)f(x) = 0$ *implies* $m_{i_1}\sigma^{i_1}(a_j) = 0$ *for each* $0 \leq j \leq t$ *and some* $0 \leq i_1 \leq n$.

Corollary 2 *Let* R *be a ring and* σ *an endomorphism of* R. *Then* R *is* σ*-skew Armendariz if and only if for every polynomials* $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\sigma]$, $f(x)g(x) = 0$ *implies* $a_{i_0}\sigma^{i_0}(b_j) = 0$ *for each* $0 \leq j \leq m$ *and some* $0 \leq i_0 \leq n$.

Recall that a module M_R is reduced if, for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$.

Lemma 1 *The following are equivalent for a module* M_R.

(i) M_R *is reduced and* S*-compatible.*

(ii) *The following conditions hold for any* $m \in M, a \in R$ *and* $g \in S$,

- (a) $ma = 0$ *implies* $mRa = 0$.
- (b) $ma = 0$ *if and only if* $m\omega_g(a) = 0$.
- (c) $ma^2 = 0$ *implies* $ma = 0$.

Proof. The proof is straightforward. □

For an element $f(s) = a_1g_1 + \cdots + a_ng_n \in R * S$ with $a_i \neq 0$ for each i, we say that length $(f(s)) = n$ and denote it by $\ell(f(s))$. Similarly, we can define $\ell(m(s)) = t$ for an element $m(s) = m_1h_1 + \cdots + m_th_t \in M[S]$.

Proposition 1 *Let* R *be a ring,* S *a u.p. monoid and* $\omega : S \longrightarrow \text{End}(R)$ *a monoid homomorphism. Then* S*-compatible reduced module* M_R *is* S*-skew Armendariz.*

Proof. Assume that $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ and $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$ with $m(s)f(s) = 0$. We proceed by induction on $\ell(m(s)) + \ell(f(s)) = n + t$. If $\ell(m(s)) = 1$ or $\ell(f(s)) = 1$, then the result is clear Since u.p. monoids are cancellative by [6, Lemma 1.1]. From $m(s)f(s) = 0$ there exist $1 \leq i \leq n, 1 \leq j \leq t$ such that g_ih_j is u.p. element in the product of two subsets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_t\}$ of S. Without loss of generality we can assume that $i = j = 1$. Thus $m_1\omega_{g_1}(a_1) = 0$ and so $m_1a_1 = 0$ since M_R is S-compatible. Therefore $0 = m(s)f(s)a_1 = (m_1g_1 + \cdots + m_ng_n)(a_1\omega_{h_1}(a_1)h_1 + \cdots + a_t\omega_{h_t}(a_1)h_t)$. By using of Lemma 1, from $m_1a_1 = 0$ we have $m_1\omega_{g_1}(a_j\omega_{h_j}(a_1)) = 0$ for each $1 \leq j \leq t$ since M_R is reduced and S-Compatible. Thus $0 = m(s)f(s)a_1 = (m_2g_2 + \cdots + m_ng_n)f(s)a_1 = m'(s)(f(s)a_1)$. Since $\ell(m'(s)) + \ell(f(s)a_1) < n+t$ satisfying $m'(s)f(s)a_1 = 0$, by induction hypothesise $m_i\omega_{g_i}(a_j\omega_{h_j}(a_1)) = 0$ which implies that $m_ia_ja_1 = 0$ for each $2 \leq i \leq n, 1 \leq j \leq t$, since M_R is S-compatible. Thus $m_ia_1^2 = 0$

and so $m_i a_1 = 0$ for each $2 \le i \le n$, by Lemma 1. Hence $0 = m(s)f(s) = m(s)(a_2h_2 + \cdots + a_th_t)$. Then by induction $m_i\omega_{g_i}(a_j) = 0$ for each $1 \le i \le n$ and $1 \le j \le t$. Therefore M_R is S-skew Armendariz. □

If ω is identity homomorphism (i.e. $\omega_g = id_R$ the identity homomorphism of R for each $g \in S$) we deduce the following corollary.

Corollary 3 *Let M_R be a reduced and S a u.p. monoid. Then M_R is S-Armendariz.*

Corollary 4 [2, Theorem 2.19] *Every reduced module is Armendariz.*

Corollary 5 *Let R be a reduced ring, S a u.p. monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. Then R is S-skew Armendariz.*

Proposition 2 *Let S be a monoid and M_R a S-skew Armendariz module. If $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ and $f_i(s) = a_1^ih_1^i + \cdots + a_{t_i}^ih_{t_i}^i \in R * S$ for $1 \le i \le k$ are such that $m(s)f_1(s)\cdots f_k(s) = 0$, then*

$$m_j\omega_{g_j}(a_{i_1}^1)\omega_{g_j}\omega_{h_{i_1}^1}(a_{i_2}^2)\cdots\omega_{g_j}\omega_{h_{i_1}^1}\ldots\omega_{h_{i_{k-1}}^{k-1}}(a_{i_k}^k) = 0$$

for each $1 \le j \le n$ and $1 \le i_r \le t_i, 1 \le r \le k$.

Proof. Suppose $m(s)f_1(s)\cdots f_k(s) = 0$. Then from $m(s)(f_1(s)\cdots f_k(s)) = 0$ we have $m_j\omega_{g_j}(a) = 0$ for each $1 \le j \le n$ and each coefficient a of $f_1(s)f_2(s)\cdots f_k(s)$, since M_R is S-skew Armendariz and S-compatible. Thus $(m_jg_jf_1(s))f_2(s)\cdots f_k(s) = 0$ for each $1 \le j \le n$. Thus $m_j\omega_{g_j}(a_{i_1}^1)\omega_{g_j}\omega_{h_{i_1}^1}(a') = 0$ for each $1 \le j \le n$, $1 \le i_1 \le t_1$ and each coefficient a' of $f_3(s)\cdots f_k(s)$. By continuing this manner, we see that $m_j\omega_{g_j}(a_{i_1}^1)\omega_{g_j}\omega_{h_{i_1}^1}(a_{i_2}^2)\cdots\omega_{g_j}\omega_{h_{i_1}^1}\ldots\omega_{h_{i_{k-1}}^{k-1}}(a_{i_k}^k) = 0$ for each $1 \le j \le n$ and $1 \le i_r \le t_i, 1 \le r \le k$. □

As a consequence of Propositions 1 and 2 we have the following result.

Corollary 6 *Let R be a ring, S a u.p. monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. Let M_R be a S-compatible reduced module. If $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ and $f_i(s) = a_1^ih_1^i + \cdots + a_{t_i}^i \in R * S$ for $1 \le i \le k$ are such that $m(s)f_1(s)\cdots f_k(s) = 0$, then*

$$m_j\omega_{g_j}(a_{i_1}^1)\omega_{g_j}\omega_{h_{i_1}^1}(a_{i_2}^2)\cdots\omega_{g_j}\omega_{h_{i_1}^1}\ldots\omega_{h_{i_{k-1}}^{k-1}}(a_{i_k}^k) = 0$$

for each $1 \le j \le n$ and $1 \le i_r \le t_i, 1 \le r \le k$.

It is proved in [18, Theorem 1.6] M_R is reduced if and only if $M[x]_{R[x]}$ is reduced. In the following we extend this result to $M[S]_{R*S}$.

Proposition 3 *Let* R *be a ring,* S *a u.p. monoid and* $\omega : S \longrightarrow \mathrm{End}(R)$ *a monoid homomorphism. Then module* M_R *is reduced and* S*-compatible if and only if* $M[S]_{R*S}$ *is reduced.*

Proof. Assume that M_R is reduced and $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$, $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$ with $m(s)f(s) = 0$. Let $g(s) = b_1k_1 + \cdots + b_mk_m \in R * S$ and $k(s) = n_1s_1 + \cdots + n_ps_p \in M[S]$ such that $m(s)g(s) = k(s)f(s) \in m(s)(R * S) \cap M[S]f(s)$. From $m(s)f(s) = 0$ we have $m_i\omega_{g_i}(a_j) = 0 = m_ia_j$ for each $1 \leq i \leq n, 1 \leq j \leq t$, by Proposition 1 and S-compatibility assumption on M_R. Then by Lemma 1 we have $m_ira_j = 0$ for each $r \in R$ which implies that $0 = m(s)g(s)f(s) = k(s)f^2(s)$. Therefore $n_ia_ja_l = 0$ for each $1 \leq i \leq p$ and $1 \leq j, \ell \leq t$ by Proposition 2. Thus $n_ia_j^2 = 0$ and so $n_ia_j = 0$ for each $1 \leq i \leq p$ and $1 \leq j \leq t$ by Lemma 1. Therefore $k(s)f(s) = 0$ which implies that $m(s)(R * S) \cap M[S]f(s) = 0$ and hence $M[S]_{R*S}$ is reduced.

Conversely, assume that $M[S]_{R*S}$ is reduced and $m \in M, r \in R$ with $mr = 0$. Also assume that $n \in M, a \in R$ such that $ma = nr \in Mr \cap mR$. Put $m(s) = mg$ and $k(s) = nh$ for some $g, h \in S$. Thus $m(s)a = k(s)r \in M[S]r \cap m(s)(R * S)$. Since $M[S]_{R*S}$ is reduced $M[S]r \cap m(s)(R * S) = 0$ which implies that $ma = nr = 0$. Hence M_R is reduced. Now, assume that $mr = 0$ for some $m \in M$ and $r \in R$. For each $g \in S$ we have $mgr = m\omega_g(r)g \in M[S]r \cap m(R*S)$. Since $M[S]_{R*S}$ is reduced, $M[S]r \cap m(R * S) = 0$. Thus $m\omega_g(r) = 0$. Clearly, if $m\omega_g(r) = 0$ for each $g \in S$ we have $mr = 0$. Therefore M_R is S-compatible. □

Corollary 7 *Let* R *be a ring and* σ *an endomorphism of* R*. Then* M_R *is reduced and* σ*-compatible if and only if* $M[x]_{R[x;\sigma]}$ *is reduced.*

Corollary 8 *Let* R *be a ring and* σ *an endomorphism of* R*. Then* R *is reduced and* σ*-compatible if and only if* $R[x; \sigma]$ *is reduced.*

3 Extensions of Baer and quasi-Baer modules

In this section we study on the relationship between the Baerness and p.p-property of a module M_R and right $R * S$-module $M[S]$.

According to [5] a module M_R is called *quasi-Armendariz* if whenever $m(x) R[x]f(x) = 0$ for $m(x) = m_0 + m_1x + \cdots + m_nx^n \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_mx^m \in R[x]$, then $m_iRa_j = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Let S be a monoid. According to [12] a ring R is called S-*quasi Armendariz* if for each two elements $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m \in R[S]$ satisfy $\alpha R[s]\beta = 0$, implies that $a_iRb_j = 0$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$.

Definition 3 *Let R be a ring, S a monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. A module M_R is called S-skew quasi-Armendariz, if for any $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$ and $f(s) = a_1h_1 + \cdots + a_th_t \in R * S$ satisfy $m(s)(R * S)f(s) = 0$ implies that $m_ig_iRga_jh_j = 0$ for each $1 \leq i \leq n$, $1 \leq j \leq t$ and $g \in S$.*

Clearly a ring R is S-skew quasi-Armendariz if and only if R_R is S-skew quasi-Armendariz.

Birkenmeier and Park in [9, Theorem 1.2] proved that for a u.p. monoid S the monoid ring $R[S]$ is quasi-Baer (resp. right p.q.-Baer) if and only if R is quasi-Baer (resp. right p.q.-Baer). In the following we extend these results to $M[S]$ as a right $R * S$-module.

Theorem 2 *Let R be a ring, S a u.p. monoid, $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism. If M_R is S-compatible, then we have the following:*

(i) *M_R is right p.q.-Baer if and only if $M[S]_{R*S}$ is right p.q.-Baer.*

(ii) *M_R is quasi-Baer if and only if $M[S]_{R*S}$ is quasi-Baer.*

In this case, M_R is S-skew quasi-Armendariz.

Proof. (i) Assume that R is right p.q.-Baer. Let $m(s) = m_1g_1 + \cdots + m_ng_n \in M[S]$. There exists $e_i \in S_\ell(R)$ such that $ann_R(m_iR) = e_iR$ for $1 \leq i \leq n$. Then $e = e_1e_2 \cdots e_n \in S_\ell(R)$ and $eR = \bigcap_{i=1}^n ann_R(m_iR)$. Since every compatible automorphism is idempotent stabilizing by [3, Theorem 2.14] we have $e(R*S) \subseteq ann_{R*S}(m(s)R*S)$. Note that $ann_{R*S}(m(s)R*S) \subseteq ann_{R*S}(m(s)R)$. Now we show that $ann_{R*S}(m(s)R) \subseteq e(R*S)$. Let $g(s) = b_1h_1 + \cdots + b_mh_m \in ann_{R*S}(m(s)R)$. Then $m(s)Rg(s) = 0$. We proceed by induction on n to show that $g(s) \in e(R * S)$. Let $n = 1$. Then $m_1g_1R(b_1h_1 + \cdots + b_th_t) = 0$. Thus $m_1g_1Rb_jh_j = 0$ for each $1 \leq j \leq t$, since S is cancellative, by [9, Lemma 1.1]. Since ω_{g_1} is automorphism $m_1R\omega_{g_1}(b_j) = 0$ and so $\omega_{g_1}(b_j) \in ann_R(m_1R) = e_1R$ for each $1 \leq j \leq t$. Thus $\omega_{g_1}(b_j) = e_1\omega_{g_1}(b_j)$ and so $b_j = e_1b_j$ for each $1 \leq j \leq t$, since ω_{g_1} is a compatible automorphism of R. Therefore $b_j \in e_1R = eR$. Hence $g(s) = eg(s) \in e(R * S)$, as desired. Now assume that

$$(*) \qquad (m_1g_1 + \cdots + m_ng_n)R(b_1h_1 + \cdots + b_th_t) = 0.$$

Since S is u.p. monoid there exist $1 \leq i \leq n, 1 \leq j \leq t$ such that $g_i h_j$ is u.p. element in the product of two subsets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_t\}$ of S. Without loss of generality we can assume that $i = n, j = t$. Thus $m_n g_n R b_t h_t = 0$. That is $\omega_{g_n}(b_t) \in ann_R(m_n R) = e_n R$ and $\omega_{g_n}(b_t) = e_n \omega_{g_n}(b_t)$. Since ω_{g_n} is a compatible automorphism of R, $b_t = e_n b_t$ and $b_t \in e_n R$. Replacing R by Re_n in the equation $(*)$ we have $(m_1 g_1 + \cdots + m_{n-1} g_{n-1}) R (e_n b_1 h_1 + \cdots + e_n b_t h_t) = 0$. By induction on n we have $e_n b_j \in e_1 R \cap e_2 R \cap \cdots \cap e_{n-1} R$ for each $1 \leq j \leq t$. In particular, $b_t \in e_1 R \cap \cdots \cap e_{n-1} R$. Therefore $b_t = e_n b_t \in e_1 R \cap \cdots \cap e_n R = eR = \bigcap_{i=1}^n ann_R(m_i R)$. Since ω_{g_i} is a compatible automorphism of R for each $1 \leq i \leq n$ we have

$$(**) \qquad (m_1 g_1 + \cdots + m_n g_n) R (b_1 h_1 + \cdots + b_{t-1} h_{t-1}) = 0.$$

Since S is u.p. monoid there exist $1 \leq i \leq n, 1 \leq j \leq t-1$ such that $g_i h_j$ is u.p. element in the product of two subsets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_{t-1}\}$ of S. Without loss of generality we can assume that $i = n, j = t-1$. Thus $m_n g_n R b_{t-1} h_{t-1} = 0$ which implies that $\omega_{g_n}(b_t) \in ann(m_n R) = e_n R$ and $\omega_{g_n}(b_{t-1}) = e_n \omega_{g_n}(b_{t-1})$. Therefore $b_{t-1} = e_n b_{t-1}$, since ω_{g_n} is an idempotent stabilizing automorphism of R. Replacing R by Re_n in the equation $(**)$ we have $(m_1 g_1 + \cdots + m_{n-1} g_{n-1}) R e_n (b_1 h_1 + \cdots + b_{t-1} h_{t-1}) = 0$. Then by induction on n we can conclude that $e_n b_j \in ann_R(m_1 R) \cap \cdots \cap ann_R(m_{n-1} R)$ for each $1 \leq j \leq t-1$ and hence $b_{t-1} = e_n b_{t-1} \in \cap_{i=1}^n ann_R(m_i R) = eR$. Therefore from the equation $(**)$ we have $0 = (m_1 g_1 + \cdots + m_n g_n) R (b_1 h_1 + \cdots + b_{t-2} h_{t-2})$. By continuing this process we can conclude that $b_j \in \cap_{i=1}^n ann_R(m_i R) = eR$ for each $1 \leq j \leq t$ which implies that $g(s) = eg(s)$. Thus $ann_R(m(s)R) \subseteq e(R * S)$. So we have $ann_{R*S}(m(s)(R * S)) \subseteq ann_R(m(s)R) \subseteq e(R * S)$. Hence $ann_{R*S}(m(s)R * S) = e(R * S)$. Therefore $M[S]_{R*S}$ is p.q.-Baer.

Conversely assume that $M[S]_{R*S}$ is p.q.-Baer. Take $m \in M$. Then ann_{R*S} $(m(R * S)) = e(s)(R * S)$ for some idempotent $e(s) = e_1 s_1 + \cdots + e_n s_n$ in $R * S$. Let $a \in ann_R(mR)$. Since M_R is S-compatible, $ann_R(mR) \subseteq ann_{R*S}(m(R * S)) = e(s)(R * S)$. Therefore $a = e(s)a = (e_1 g_1 + \cdots + e_n g_n)a$. Thus there exist $1 \leq i_0 \leq n$ such that $a = e_{i_0} \omega_{g_{i_0}}(a)$ and so $ann_R(mR) \subseteq e_{i_0} R$. Since $e(s) \in ann_{R*S}(m(R * S))$ then $0 = mRe(s) = mR(e_1 s_1 + \cdots + e_n g_n)$. Since S is cancellative $mRe_i = 0$ for each $1 \leq i \leq n$. Thus $e_{i_0} \in ann_R(mR)$ and hence $ann_R(mR) = e_{i_0} R$. Also, e_{i_0} is idempotent, since $e_{i_0} \in ann_R(mR), a = e_{i_0} \omega_{g_{i_0}}(a)$ for each $a \in ann_R(mR)$ and $\omega_{g_{i_0}}$ is idempotent stabilizing, we have $e_{i_0} = e_{i_0} \omega_{g_{i_0}}(e_{i_0}) = e_{i_0}^2$. Therefore R is p.q.-Baer.

(ii) Assume that M_R is quasi-Baer. First we show that M_R is S-skew quasi-Armendariz. Suppose that $m(s) = m_1 g_1 + \cdots + m_n g_n \in M[S]$ and $f(s) =$

$a_1h_1 + \cdots + a_th_t \in R * S$ such that $m(s)(R * S)f(s) = 0$. Thus $m(s)rgf(s) = 0$ for each $r \in R, g \in S$. We proceed by induction on $\ell(m(s)) + \ell(f(s)) = n + t$. If $\ell(m(s)) = 1$, then $m_1g_1rg(a_1h_1 + \cdots + a_th_t) = 0$. Since S is cancellative $m_1g_1rga_jh_j = 0$, as desired. Also if $\ell(f(s)) = 1$ the result is clear. From

$$(*) \qquad (m_1g_1 + \cdots + m_ng_n)rg(a_1h_1 + \cdots + a_th_t) = 0$$

there exist $1 \le i \le n, 1 \le j \le t$ such that g_ih_j is u.p. element in the product of two subsets $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_t\}$ of S. Without loss of generality we can assume that $i = n, j = t$. Then $m_ng_nrga_th_t = 0$ and so $m_n\omega_{g_n}(r)\omega_{g_n}\omega_g(a_t) = 0 = m_nr'\omega_{g_n}\omega_g(a_t)$. Thus $\omega_{g_n}\omega_g(a_t) \in ann_R(m_nR) = eR$ such that $e^2 = e \in R$ and so $\omega_{g_n}\omega_g(a_t) = e\omega_{g_n}\omega_g(a_t)$. Replacing rg by reg in the equation $(*)$ we have

$$(m_1g_1 + \cdots + m_{n-1}g_{n-1})reg(a_1h_1 + \cdots + a_th_t) = 0$$

since ω_g is idempotent stabilizing by [3, Theorem 2.14]. Then by induction we can conclude that $m_ig_irega_jh_j = 0$ for $1 \le i \le n-1, 1 \le j \le t$. Thus $m_ig_irega_th_t = 0$ and so $m_ig_ire\omega_g(a_t)gh_t = 0$ for each $1 \le i \le n-1$. Since $\omega_{g_n}\omega_g(a_t) = e\omega_{g_n}\omega_g(a_t)$ and ω_{g_n} is a compatible automorphism of R, $\omega_g(a_t) = e\omega_g(a_t)$. Thus $0 = m_ig_ire\omega_g(a_t)gh_t = m_ig_ir\omega_g(a_t)gh_t$ for each $1 \le i \le n-1$. On the other hand $m_ng_nrega_th_t = 0$ and hence $m_ig_irga_th_t = 0$ for each $1 \le i \le n$. Thus $0 = m(s)rgf(s) = (m_1g_1 + \cdots + m_ng_n)rg(a_1h_1 + \cdots + a_{t-1}h_{t-1})$. Then by induction hypothesis $m_ig_irga_jh_j = 0$ for each $1 \le i \le n, 1 \le j \le t-1$. Therefore $m_ig_iRga_jh_j = 0$ for each $1 \le i \le n, 1 \le j \le t$. Hence M_R is S-skew quasi-Armendariz. Let V be a submodule of $M[S]$. Let U be a right R-submodule of M generated by all coefficients of elements of V. Since M_R is quasi-Baer $ann_R(U) = eR$ for some $e^2 = e \in R$. Thus $e(R * S) \subseteq ann_{R*S}(V)$, since ω_s is compatible automorphism for each $s \in S$. Suppose that $g(s) = b_1h_1 + \cdots + b_th_t \in ann_{R*S}(V)$. Thus for each $m(s) = m_1g_1 + \cdots + m_ng_n \in V$, $m(s)(R * S)g(s) = 0$ and hence $m_ig_iRgb_jh_j = 0$ for each $1 \le i \le n$, $1 \le j \le t$ since M_R is S-skew quasi-Armendariz. Therefore $\omega_{g_i}\omega_g(b_j) \in ann_R(U) = eR$ which implies that $\omega_{g_i}\omega_g(b_j) = e\omega_{g_i}\omega_g(b_j)$ for each $1 \le i \le n$, $1 \le j \le t$. Since ω_s is compatible automorphism of R for each $s \in S$, $b_j = eb_j$ for each $1 \le j \le t$. That is $g(s) \in e(R * S)$ and so $ann_{R*S}(V) \subseteq e(R * S)$. Hence $M[S]_{R*S}$ is quasi-Baer.

Conversely, assume that $M[S]_{R*S}$ is quasi-Bear and U is a right R-submodule of M_R. Then as in the proof of the sufficiently of (i), one can show that $ann_R(U)$ is generated as a right R-submodule, by an idempotent of R.Therefore M is quasi-Baer. □

Now we obtain the following results as a corollary of Theorem 2.

Corollary 9 *Let* R *be a ring,* S *a u.p. monoid,* $\omega : S \longrightarrow \mathrm{Aut}(R)$ *a monoid homomorphism and* M_R *is a* S*-compatible module. Then we have the following:*

(i) M_R *is a reduced p.p.- module if and only if* $M[S]_{R*S}$ *is a reduced p.p.-module.*

(ii) M_R *is a reduced Baer module if and only if* $M[S]_{R*S}$ *is a reduced Baer module.*

Proof. (i) Clearly reduced p.p.- modules are p.q.-Baer. Then the result follows from Theorem 2 and Proposition 3.

(ii) The result follows from Theorem 2 and the fact that a reduced quasi-Baer module is Baer. □

Corollary 10 *Let* R *be a ring and* S *a u.p. monoid. Then we have the following:*

(i) [6, Theorem 1.2] R *is quasi-Baer (resp. right p.q.-Baer) if and only if* $R[S]$ *is quasi-Baer (resp. right p.q.-Baer).*

(ii) [6, Corollary 1.3] R *is reduced Baer (resp. p.p.- ring) if and only if* $R[S]$ *is a reduced Baer (resp. p.p.- ring).*

Corollary 11 *Let* M_R *be a module. Then the following are equivalent:*

(i) M_R *is quasi-Baer (resp. p.q.-Baer).*

(ii) $M[x]_{R[x]}$ *is quasi-Baer (resp. p.q.-Baer).*

(iii) $M[x, x^{-1}]_{R[x,x^{-1}]}$ *is quasi-Baer (resp. p.q.-Baer).*

Corollary 12 *Let* R *be a* σ*-compatible ring for an automorphism* σ *of* R*. Then the following are equivalent:*

(i) R *is quasi-Baer (resp. p.q.-Baer).*

(ii) $R[x; \sigma]$ *is quasi-Baer (resp. p.q.-Baer).*

(iii) $R[x, x^{-1}; \sigma]$ *is quasi-Baer (resp. p.q.-Baer).*

(iv) $R[x]$ *is quasi-Baer (resp. p.q.-Baer).*

(v) $R[x, x^{-1}]$ *is quasi-Baer (resp. p.q.-Baer).*

Birkenmeier et al. [6, Example 1.5] showed that the "u.p. monoid" condition on S in Theorem 2 is not superfluous.

The next example shows that the "S-compatibility" assumption on R_R in Theorem 2 is not superfluous.

Example 4 [15, Example 2] *Let K be a field, $A = K[s,t]$ a commutative polynomial ring, and consider the ring $R = A/(st)$. Then R is reduced. Let $\bar{s} = s + (st)$ and $\bar{t} = t + (st)$ in $R = A/(st)$. Define an automorphism σ of R by $\sigma(\bar{s}) = \bar{t}$ and $\sigma(\bar{t}) = \bar{s}$. Hirano in [15] showed that $R[x;\sigma]$ is quasi-Baer but R is not quasi-Baer. Since $\sigma(\bar{s}\bar{t}) = 0$ but $\bar{s}\sigma(\bar{t}) = \bar{s}^2 \neq 0$ (since R is reduced), hence σ is not compatible. Therefore the "compatibility" assumption on σ is not superfluous.*

Theorem 3 *Let R be a ring, S a u.p. monoid and $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism. If M_R is a S-compatible and S-skew Armendariz module, then M_R is Baer if and only if $M[S]_{R*S}$ is Baer.*

Proof. The proof is similar to that of Theorem 2. □

Corollary 13 *Let R be a ring, S a u.p. monoid and $\omega : S \longrightarrow \mathrm{Aut}(R)$ a monoid homomorphism. Let M_R is S-compatible reduced module. Then M_R is Baer if and only if $M[S]_{R*S}$ is Baer.*

Proof. This follows from Proposition 1 and Theorem 3. □

Corollary 14 *Let R be a σ-compatible ring for an automorphism σ of R. If R is σ-skew Armendariz, then the following are equivalent:*

(i) *R is Baer.*

(ii) *$R[x;\sigma]$ is Baer .*

(iii) *$R[x,x^{-1};\sigma]$ is Baer.*

(iv) *$R[x]$ is Baer.*

(v) *$R[x,x^{-1}]$ is Baer.*

Theorem 4 *Let R be a ring, S a monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. If M_R is S-compatible and S-skew quasi-Armendariz, then M_R satisfies the ascending chain condition on annihilator of submodules if and only if so does $M[S]_{R*S}$.*

Proof. Assume that M_R satisfies the ascending chain condition on annihilator of submodules. Let $V_1 \subseteq V_2 \subseteq \dots$ be a chain of annihilator of submodules of $M[S]_{R*S}$. Then there exist submodules K_i of $M[S]_{R*S}$ such that $ann_{R*S}(K_i) = V_i$ and $K_i \supseteq K_{i+1}$ for each $i \geq 1$. Let U_i be a submodule of M generated by all coefficients of elements of K_i. Clearly $U_1 \supseteq U_2 \supseteq \cdots$. Then $ann_R(U_1) \subseteq ann_R(U_2) \subseteq \cdots$ is a chain of annihilator of submodules of M_R. Since M_R satisfies the ascending chain condition on annihilator of submodules there exists $n \geq 1$ such that $ann_R(U_n) = ann_R(U_i)$ for all $i \geq n$. We show that $ann_{R*S}(K_n) = ann_{R*S}(K_i)$ for all $i \geq n$. Let $f(s) = a_1h_1 + a_2h_2 + \cdots + a_th_t \in ann_{R*S}(K_i)$. For each $m(s) = m_1g_1 + \cdots + m_ng_n \in K_i$, $m(s)(R * S)f(s) = 0$. Therefore $m_ig_iRga_ph_p = 0$ for each $1 \leq j \leq n, 1 \leq p \leq t$ since $M[S]$ is S-skew quasi-Armendariz. Thus $m_jR\omega_{g_j}\omega_g(a_p) = 0$ and so $m_jRa_p = 0$, since M_R is S-compatible. Therefore $a_p \in ann(U_i) = ann(U_n)$ for each $1 \leq p \leq t$ and hence $f(s) \in ann_{R*S}(K_n)$. Thus $ann_{R*S}(K_n) = ann_{R*S}(K_i)$. Now assume that $M[S]_{R*S}$ satisfies the ascending chain condition on annihilator of submodules. Let $U_1 \subseteq U_2 \subseteq \cdots$ be a chain of annihilator of submodules of M_R. Then there exist submodules M_i of M such that $ann_R(M_i) = U_i$. Thus $M_1 \supseteq M_2 \supseteq \cdots$. Hence $M_i[S]$ is a submodule of $M[S]_{R*S}$, $M_i[S] \supseteq M_{i+1}[S]$ and $ann_{R*S}(M_i[S]) \subseteq ann_{R*S}(M_{i+1}[S])$ for all $i \geq 1$. Thus $ann_{R*S}(M_1[S]) \subseteq ann_{R*S}(M_2[S]) \subseteq \cdots$ is a chain of annihilator of submodules of $M[S]$ and so there exists $n \geq 1$ such that $ann_{R*S}(M_n[S]) = ann_{R*S}(M_i[S])$. We show that $ann_R(M_n) = ann_R(M_i)$ for $i \geq n$. Assume that $r \in ann_R(M_i)$. Since M is S-compatible, $r \in ann_{R*S}(M_i[S]) = ann_{R*S}(M_n[S])$ for all $i \geq n$. For each $m(s) \in M_n[S]$ and $r \in R$, $m(s)(R * S)r = 0$ which implies that $m_pg_pRgr = 0$ for each $1 \leq p \leq t, g \in S$, since M_R is S-skew quasi-Armendariz. Thus $m_pR\omega_{g_p}\omega_g(r) = 0 = m_pRr$, since M_R is S-compatible, and so $r \in ann_R(M_n)$. Therefore $ann_R(M_i) = ann_R(M_n)$. □

Corollary 15 *Let M_R be a module and σ a compatible automorphism of R. The following are equivalent:*

(i) M_R *satisfies the ascending chain condition on annihilator of submodules.*

(ii) $M[x]_{R[x;\sigma]}$ *satisfies the ascending chain condition on annihilator of submodules.*

(iii) $M[x, x^{-1}]_{R[x,x^{-1};\sigma]}$ *satisfies the ascending chain condition on annihilator of submodules.*

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