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Some classes of analytic and multivalent functions associated with q-derivative operators

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Abstract. By applying the q-derivative operator of order m $(m \in \mathbb{N}_0)$, we introduce two new subclasses of p-valently analytic functions of complex order. For these classes of functions, we obtain the coefficient inequalities and distortion properties. Some consequences of the main results are also considered.

1 Introduction and preliminaries

The theory of q-analysis in recent past has been applied in many areas of mathematics and physics, as for example, in the areas of ordinary fractional calculus, optimal control problems, q-difference and q-integral equations, and in q-transform analysis. One may refer to the books [5], [7], and the recent papers [1], [2], [3], [4], [8] and [12] on the subject. Purohit and Raina recently in [10], [11] have used the fractional q-calculus operators in investigating certain classes of functions which are analytic in the open disk. Purohit [9] also studied

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similar work and considered new classes of multivalently analytic functions in the open unit disk.

In the present paper, we aim at introducing some new subclasses of functions defined by applying the q-derivative operator of order m $(m \in \mathbb{N}_0)$ which are p-valent and analytic in the open unit disk. The results derived include the coefficient inequalities and distortion theorems for the subclasses defined and introduced below. Some consequences of the main results are also pointed out in the concluding section.

To make this paper self contained, we present below the basic definitions and related details of the q-calculus, which are used in the sequel.

The q-shifted factorial (see [5]) is defined for $\alpha, q \in \mathbb{C}$ as a product of n factors by

$$(\alpha ; q)_n = \begin{cases} 1; & n = 0 \\ (1-\alpha)(1-\alpha q)\ldots(1-\alpha q^{n-1}); & n \in \mathbb{N} \end{cases}, \tag{1}$$

and in terms of the basic analogue of the gamma function by

$$(q^{\alpha} ; q)_n = \frac{(1-q)^n \Gamma_q(\alpha+n)}{\Gamma_q(\alpha)} \quad (n > 0), \tag{2}$$

where the q-gamma function is defined by [5, p. 16, eqn. (1.10.1)]

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty} \ (0 < q < 1). \tag{3}$$

If $|q| < 1$, the definition (1) remains meaningful for $n = \infty$, as a convergent infinite product given by

$$(\alpha ; q)_\infty = \prod_{j=0}^{\infty} (1 - \alpha q^j) \, .$$

We recall here the following q-analogue definitions given by Gasper and Rahman [5]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = \frac{(1-q^x)\,\Gamma_q(x)}{1-q}, \tag{4}$$

and the q-binomial expansion is given by

$$(x-y)_\nu = x^\nu(-y/x;q)_\nu = x^\nu \prod_{n=0}^{\infty} \left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right]. \tag{5}$$

Also, the Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, defined by (see Gasper and Rahman [5, pp. 19, 22])

$$D_{q,z}f(z) = \frac{f(z) - f(zq)}{z(1-q)} \quad (z \neq 0, q \neq 1) \tag{6}$$

and

$$\int_0^z f(t)\, d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k). \tag{7}$$

Following the image formula for fractional q-derivative [10, pp. 58–59], namely:

$$D_{q,z}^{\alpha} z^{\lambda} = \frac{\Gamma_q(1+\lambda)}{\Gamma_q(1+\lambda-\alpha)}\, z^{\lambda-\alpha} \quad (\alpha \geq 0,\ \lambda > -1), \tag{8}$$

we have for $\alpha = m \ (m \in \mathbb{N})$:

$$D_{q,z}^{m} z^{\lambda} = \frac{\Gamma_q(1+\lambda)}{\Gamma_q(1+\lambda-m)}\, z^{\lambda-m} \quad (m \in \mathbb{N},\ \lambda > -1). \tag{9}$$

Further, in view of the relation that

$$\lim_{q \to 1^-} \frac{(q^{\alpha}; q)_n}{(1-q)^n} = (\alpha)_n, \tag{10}$$

we observe that the q-shifted factorial (1) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1) \ \ (n \in \mathbb{N})$.

2 New classes of functions

By $\mathcal{A}_p(n)$, we denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \qquad (n,\ p \in \mathbb{N}), \tag{11}$$

which are analytic and p-valent in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C},\ |z| < 1\}$. Also, let $\mathcal{A}_p^-(n)$ denote the subclass of $\mathcal{A}_p(n)$ consisting of analytic and p-valent functions expressed in the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0, n,\ p \in \mathbb{N}). \tag{12}$$

Differentiating (12) m times with respect to z and making use of (9), we get

$$D^m_{q,z}f(z) = \frac{\Gamma_q(1+p)}{\Gamma_q(1+p-m)}\, z^{p-m} - \sum_{k=n+p}^{\infty} a_k \frac{\Gamma_q(1+k)}{\Gamma_q(1+k-m)}\, z^{k-m} \tag{13}$$
$$(n,\ p \in \mathbb{N}, m \in \mathbb{N}_0, p > m).$$

By applying the q-derivative operator of order m to the function $f(z)$, we define here a new subclass $\mathcal{M}^m_{n,p}(\lambda, \delta, q)$ of the p-valent class $\mathcal{A}^-_p(n)$, which consist of functions $f(z)$ satisfying the inequality that

$$\left| \frac{1}{\delta} \left\{ \frac{z\, D^{1+m}_{q,z}f(z) + \lambda\, q\, z^2\, D^{2+m}_{q,z}f(z)}{\lambda\, z\, D^{1+m}_{q,z}f(z) + (1-\lambda)\, D^m_{q,z}f(z)} - [p-m]_q \right\} \right| < 1, \tag{14}$$
$$(m < p; p \in \mathbb{N}, m \in \mathbb{N}_0;\ 0 \leq \lambda \leq 1;\ \delta \in \mathbb{C} \setminus \{0\};\ 0 < q < 1;\ z \in \mathbb{U}),$$

where the q-natural number is expressed as

$$[n]_q = \frac{1-q^n}{1-q} \quad (0 < q < 1). \tag{15}$$

Also, let $\mathcal{N}^m_{n,p}(\lambda, \delta, q)$ denote the subclass of $\mathcal{A}^-_p(n)$ consisting of functions $f(z)$ which satisfy the inequality that

$$\left| \frac{1}{\delta} \left\{ D^{1+m}_{q,z}f(z) + \lambda\, z\, D^{2+m}_{q,z}f(z) - [p-m]_q \right\} \right| < [p-m]_q, \tag{16}$$
$$(m < p; p \in \mathbb{N}, m \in \mathbb{N}_0;\ 0 \leq \lambda \leq 1;\ \delta \in \mathbb{C} \setminus \{0\};\ 0 < q < 1;\ z \in \mathbb{U}).$$

The following results give the characterization properties for functions of the form (12) which belong to the classes defined above.

Theorem 1 *Let the function* $f(z)$ *be defined by* (12), *then* $f(z) \in \mathcal{M}^m_{n,p}(\lambda, \delta, q)$ *if and only if*

$$\sum_{k=n+p}^{\infty} (|\delta| - q^{k-m}[p-k]_q)\ \Delta(k, m, \lambda, q)\ a_k \leq |\delta|\, \Delta(p, m, \lambda, q), \tag{17}$$

where $\Delta(k, m, \lambda, q)$ *is given by*

$$\Delta(k, m, \lambda, q) = \frac{(1 + [k-m-1]_q\ q\ \lambda)\Gamma_q(1+k)}{\Gamma_q(1+k-m)}, \tag{18}$$

such that

$$\Delta(p, m, \lambda, q) - \sum_{k=n+p}^{\infty} \Delta(k, m, \lambda, q)\ a_k > 0. \tag{19}$$

The result is sharp.

Proof. Let $f(z) \in \mathcal{M}^m_{n,p}(\lambda, \delta, q)$, then on using (14), we get

$$\Re\left\{ \frac{z\ D^{1+m}_{q,z} f(z) + \lambda\ q\ z^2\ D^{2+m}_{q,z} f(z)}{\lambda\ z\ D^{1+m}_{q,z} f(z) + (1-\lambda)\ D^{m}_{q,z} f(z)} - [p-m]_q \right\} > -|\delta|. \tag{20}$$

Now, in view of (13), we have

$$\begin{aligned}
\mathcal{N} &\equiv z\ D^{1+m}_{q,z} f(z) + \lambda\ q\ z^2\ D^{2+m}_{q,z} f(z) \\
&= z\left[\frac{\Gamma_q(1+p)}{\Gamma_q(p-m)}\ z^{p-m-1} - \sum_{k=n+p}^{\infty} a_k \frac{\Gamma_q(1+k)}{\Gamma_q(k-m)}\ z^{k-m-1}\right] \\
&\quad + \lambda\ qz^2\left[\frac{\Gamma_q(1+p)}{\Gamma_q(p-m-1)}\ z^{p-m-2} - \sum_{k=n+p}^{\infty} a_k \frac{\Gamma_q(1+k)}{\Gamma_q(k-m-1)}\ z^{k-m-2}\right] \\
&= \Gamma_q(1+p) z^{p-m}\left[\frac{1}{\Gamma_q(p-m)} + \frac{\lambda\ q}{\Gamma_q(p-m-1)}\right] \\
&\quad - \sum_{k=n+p}^{\infty} a_k\ \Gamma_q(1+k)\ z^{k-m}\left[\frac{1}{\Gamma_q(k-m)} + \frac{\lambda\ q}{\Gamma_q(k-m-1)}\right] \\
&= \frac{[p-m]_q(1+[p-m-1]_q\ q\ \lambda)\Gamma_q(1+p)}{\Gamma_q(1+p-m)} z^{p-m} \\
&\quad - \sum_{k=n+p}^{\infty} a_k\ \frac{[k-m]_q(1+[k-m-1]_q\ q\ \lambda)\Gamma_q(1+k)}{\Gamma_q(1+k-m)}\ z^{k-m} \\
&= [p-m]_q\ \Delta(p,m,\lambda,q)\ z^{p-m} - \sum_{k=n+p}^{\infty} a_k\ [k-m]_q\ \Delta(k,m,\lambda,q)\ z^{k-m},
\end{aligned}$$

where $\Delta(k, m, \lambda, q)$ is given by (18).
Similarly, we can obtain

$$\begin{aligned}
\mathcal{D} \equiv \lambda\ z\ D^{1+m}_{q,z} f(z) + (1-\lambda)\ D^{m}_{q,z} f(z) &= \Delta(p,m,\lambda,q) z^{p-m} \\
&- \sum_{k=n+p}^{\infty} a_k\ \Delta(k,m,\lambda,q)\ z^{k-m}.
\end{aligned}$$

Hence

$$\mathcal{N} - [p-m]_q \mathcal{D} = \sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k\ z^{k-m}.$$

Therefore, from (20), we obtain the simplified form of the inequality that

$$\Re\left(\frac{\sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k\ z^{k-m}}{\Delta(p,m,\lambda,q)z^{p-m}-\sum_{k=n+p}^{\infty}\Delta(k,m,\lambda,q)\ a_k\ z^{k-m}}\right) > -|\delta|\,. \tag{21}$$

By putting $z = r$, the denominator of (21) (say $DN(r)$) becomes

$$\begin{aligned} DN(r) &= \Delta(p,m,\lambda,q)r^{p-m} - \sum_{k=n+p}^{\infty}\Delta(k,m,\lambda,q)\ a_k\ r^{k-m} \\ &= r^{p-m}\left(\Delta(p,m,\lambda,q) - \sum_{k=n+p}^{\infty}\Delta(k,m,\lambda,q)\ a_k\ r^{k-p}\right), \end{aligned}$$

which is positive for $r = 0$, and also remains positive for $0 < r < 1$, with the condition (19). So that on letting $r \to 1^-$ through real values, we get the desired assertion (17) of Theorem 1.

To prove the converse of Theorem 1, first we would show that

$$\begin{aligned} &\left|\frac{z\ D_{q,z}^{1+m}f(z) + \lambda\ q\ z^2\ D_{q,z}^{2+m}f(z)}{\lambda\ z\ D_{q,z}^{1+m}f(z) + (1-\lambda)\ D_{q,z}^{m}f(z)} - [p-m]_q\right| \\ &\quad \le \frac{\sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k}{\Delta(p,m,\lambda,q) - \sum_{k=n+p}^{\infty}\ \Delta(k,m,\lambda,q)\ a_k}. \end{aligned} \tag{22}$$

We have

$$\begin{aligned} &\left|\frac{z\ D_{q,z}^{1+m}f(z) + \lambda\ q\ z^2\ D_{q,z}^{2+m}f(z)}{\lambda\ z\ D_{q,z}^{1+m}f(z) + (1-\lambda)\ D_{q,z}^{m}f(z)} - [p-m]_q\right| \\ &= \frac{\left|\sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k z^{k-m}\right|}{\left|\Delta(p,m,\lambda,q)z^{p-m} - \sum_{k=n+p}^{\infty}\ \Delta(k,m,\lambda,q)\ a_k z^{k-m}\right|}. \end{aligned} \tag{23}$$

On the other hand if $|z| = 1$, then

$$\begin{aligned} &\left|\sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k z^{k-m}\right| \\ &\le \sum_{k=n+p}^{\infty}\left|q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k z^{k-m}\right| \\ &= \sum_{k=n+p}^{\infty} q^{k-m}\ [p-k]_q\ \Delta(k,m,\lambda,q)\ a_k \end{aligned} \tag{24}$$

and

$$\begin{aligned}
&\left|\Delta(p,m,\lambda,q)z^{p-m} - \sum_{k=n+p}^{\infty} \Delta(k,m,\lambda,q)a_k z^{k-m}\right| \\
&\geq \left|\Delta(p,m,\lambda,q)z^{p-m}\right| - \sum_{k=n+p}^{\infty} \left|\Delta(k,m,\lambda,q)a_k z^{k-m}\right| \qquad (25)\\
&= \Delta(p,m,\lambda,q) - \sum_{k=n+p}^{\infty} \Delta(k,m,\lambda,q)\; a_k.
\end{aligned}$$

Now (23), (24) and (25) imply (22), and then by applying the hypothesis (17), we find that

$$\begin{aligned}
&\left|\frac{z\, D_{q,z}^{1+m} f(z) + \lambda\, q\, z^2\, D_{q,z}^{2+m} f(z)}{\lambda\, z\, D_{q,z}^{1+m} f(z) + (1-\lambda)\, D_{q,z}^{m} f(z)} - [p-m]_q\right| \\
&\leq \frac{|\delta| \left\{\Delta(p,m,\lambda,q) - \sum_{k=n+p}^{\infty} \Delta(k,m,\lambda,q)\; a_k\right\}}{\Delta(p,m,\lambda,q) - \sum_{k=n+p}^{\infty} \Delta(k,m,\lambda,q)\; a_k} = |\delta|\,.
\end{aligned}$$

Hence, by the maximum modulus principle, we infer that

$$f(z) \in \mathcal{M}_{n,p}^{m}(\lambda,\delta,q).$$

It is easy to verify that the equality in (17) is attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{|\delta|\,\Delta(p,m,\lambda,q)}{(|\delta| + q^{p-m}[n]_q)\Delta(n+p,m,\lambda,q)}\, z^{n+p} \quad (m < p; p,n \in \mathbb{N}, m \in \mathbb{N}_0), \tag{26}$$

where $\Delta(p,m,\lambda,q)$ is given by (18). □

We now derive the following corollaries from Theorem 1.

From Theorem 1, we easily get the following corollary:

Corollary 1 *If the function* $f(z)$ *is defined by* (12) *and* $f(z) \in \mathcal{M}_{n,p}^{m}(\lambda,\delta,q)$, *then*

$$\sum_{k=n+p}^{\infty} a_k \;\leq |\delta|\;\; \Xi(p,n,m,\lambda,\delta,q), \tag{27}$$

where $\Xi(p,n,m,\lambda,\delta,q)$ *is defined by*

$$\Xi(p,n,m,\lambda,\delta,q)=\frac{\Delta(p,m,\lambda,q)}{(|\delta|+q^{p-m}[n]_q)\Delta(n+p,m,\lambda,q)},\tag{28}$$

and $\Delta(k,m,\lambda,q)$ *is given by* (18).

Corollary 2 *If* $f(z)\in\mathcal{M}^m_{n,p}(\lambda,\delta,q)$, *then*

$$\sum_{k=n+p}^{\infty}[k]_q[k-1]_q\cdots[k-p+1]_q\ a_k\le|\delta|\ \Theta(p,n,m,\lambda,\delta,q),\tag{29}$$

where $\Theta(p,n,m,\lambda,\delta,q)$ *is defined by*

$$\Theta(p,n,m,\lambda,\delta,q)=\frac{\Gamma_q(1+n+p-m)\ \Delta(p,m,\lambda,q)}{(|\delta|+q^{p-m}[n]_q)\ (1+[n+p-m-1]_q\ q\ \lambda)\Gamma_q(1+n)},\tag{30}$$

and $\Delta(k,m,\lambda,q)$ *is given by* (18).

Proof. Since $f(z)\in\mathcal{M}^m_{n,p}(\lambda,\delta,q)$, then under the hypotheses of Theorem 1, we have

$$\begin{aligned}&\sum_{k=n+p}^{\infty}\frac{(|\delta|-q^{k-m}[p-k]_q)\ (1+[k-m-1]_q\ q\ \lambda)\ \Gamma_q(1+k)}{\Gamma_q(1+k-m)}\ a_k\\&\le|\delta|\ \Delta(p,m,\lambda,q),\end{aligned}\tag{31}$$

where $\Delta(k,m,\lambda,q)$ is given by (18).

Using the recurrence relation (4) successively p times, we can write

$$\Gamma_q(1+k)=[k]_q[k-1]_q\dots[k-p+1]_q\ \Gamma_q(k-p+1).\tag{32}$$

We now show here that

$$\alpha_k\le\alpha_{k+1},$$

where

$$\begin{aligned}\alpha_k&=\frac{(|\delta|-q^{k-m}[p-k]_q)\ (1+q\ \lambda\ [k-m-1]_q)\Gamma_q(1+k-p)}{\Gamma_q(1+k-m)}\\&=(A_k)\,(B_k)\,(C_k)\,,\end{aligned}\tag{33}$$

$$\begin{aligned}A_k &= |\delta| - q^{k-m}[p-k]_q,\\ B_k &= 1 + q\ \lambda\ [k-m-1]_q \quad \text{and}\\ C_k &= \frac{\Gamma_q(1+k-p)}{\Gamma_q(1+k-m)}.\end{aligned}$$

It is sufficient to show that

$$\frac{\alpha_k}{\alpha_{k+1}} = \frac{(A_k)\ (B_k)\ (C_k)}{(A_{k+1})(B_{k+1})(C_{k+1})} \leq 1.$$

Evidently, for $k = n + p$, we have

$$\frac{A_k}{A_{k+1}} = \frac{|\delta| + q^{p-m}[n]_q}{|\delta| + q^{p-m}[n+1]_q},$$

and since $[n+1]_q > [n]_q$, hence A_k is positive and consequently

$$\frac{A_k}{A_{k+1}} < 1. \tag{34}$$

Also, it follows easily that

$$\frac{B_k}{B_{k+1}} = \frac{1 + q\ \lambda\ [n+p-m-1]_q}{1 + q\ \lambda\ [n+p-m]_q} < 1. \tag{35}$$

Further, upon using the familiar asymptotic formula ([6, pp. 311, eqn. (1.7)]) given by

$$\Gamma_q(x) \approx (1-q)^{1-x} \prod_{n=0}^{\infty}(1-q^{n+1}) \qquad (x \to \infty,\ 0 < q < 1),$$

it can be verified that

$$\begin{aligned}C_k = \frac{\Gamma_q(1+k-p)}{\Gamma_q(1+k-m)} &\approx \frac{(1-q)^{1-1-k+p}\prod_{n=0}^{\infty}(1-q^{n+1})}{(1-q)^{1-1-k+m}\prod_{n=0}^{\infty}(1-q^{n+1})}\\ &= (1-q)^{p-m} \qquad (k \to \infty,\ 0 < q < 1, m < p).\end{aligned} \tag{36}$$

Thus, for large k, we conclude that

$$\frac{\alpha_k}{\alpha_{k+1}} \leq 1.$$

We, therefore, from (31) and (32) infer that

$$\sum_{k=n+p}^{\infty} [k]_q[k-1]_q \ldots [k-p+1]_q \; a_k$$
$$\leq \frac{|\delta| \;\; \Delta(k,m,\lambda,q)\Gamma_q(1+n+p-m)}{(|\delta| + q^{p-m}[n]_q)(1+[n+p-m-1]_q \;\; q \; \lambda) \; \Gamma_q(1+n)},$$

which in view of (30) yields the desired inequality (31). □

Next, we prove the following result.

Theorem 2 *Let the function* $f(z)$ *be defined by* (12)*, then* $f(z) \in \mathcal{N}^m_{n,p}(\lambda,\delta,q)$ *if and only if*

$$\sum_{k=n+p}^{\infty} [k-m]_q\Omega(k,m,\lambda,q)a_k \leq [p-m]_q \left[\frac{|\delta|-1}{\Gamma_q(1+m)} + \Omega(p,m,\lambda,q)\right], \quad (37)$$

where $\Omega(k,m,\lambda,q)$ *is given by*

$$\Omega(k,m,\lambda,q) = \begin{bmatrix} k \\ m \end{bmatrix}_q (1+[k-m-1]_q \; \lambda). \quad (38)$$

The result is sharp with the extremal function given by

$$f(z) = z^p - \frac{[p-m]_q \, [|\delta|-1+\Gamma_q(1+m) \;\; \Omega(p,m,\lambda,q)]}{[n+p-m]_q \;\; \Gamma_q(1+m) \;\; \Omega(n+p,m,\lambda,q)} z^{n+p}. \quad (39)$$

Proof. Let $f(z) \in \mathcal{N}^m_{n,p}(\lambda,\delta,q)$, then on using (16), we get

$$\Re\left\{D^{1+m}_{q,z}f(z) + \lambda \; z \; D^{2+m}_{q,z}f(z) - [p-m]_q\right\} > -|\delta| \;\; [p-m]_q. \quad (40)$$

Now, in view of (13), we have

$$D^{1+m}_{q,z}f(z) + \lambda \; z \; D^{2+m}_{q,z}f(z) = \frac{\Gamma_q(1+p)}{\Gamma_q(p-m)} \; z^{p-m-1} - \sum_{k=n+p}^{\infty} a_k \frac{\Gamma_q(1+k)}{\Gamma_q(k-m)} \; z^{k-m-1}$$
$$+ \lambda \; z \left[\frac{\Gamma_q(1+p)}{\Gamma_q(p-m-1)} \; z^{p-m-2} - \sum_{k=n+p}^{\infty} a_k \frac{\Gamma_q(1+k)}{\Gamma_q(k-m-1)} \; z^{k-m-2}\right]$$
$$= \Gamma_q(1+p)z^{p-m-1}\left[\frac{1}{\Gamma_q(p-m)} + \frac{\lambda}{\Gamma_q(p-m-1)}\right]$$

$$-\sum_{k=n+p}^{\infty} a_k\ \Gamma_q(1+k)\ z^{k-m-1}\left[\frac{1}{\Gamma_q(k-m)}+\frac{\lambda}{\Gamma_q(k-m-1)}\right]$$
$$=\frac{[p-m]_q(1+[p-m-1]_q\ \lambda)\Gamma_q(1+p)}{\Gamma_q(1+p-m)}z^{p-m-1}$$
$$-\sum_{k=n+p}^{\infty} a_k\ \frac{[k-m]_q(1+[k-m-1]_q\ \lambda)\Gamma_q(1+k)}{\Gamma_q(1+k-m)}\ z^{k-m-1}.$$

From (40), we obtain a simplified form of the inequality which is given by

$$\Re\left\{-\sum_{k=n+p}^{\infty} a_k\ \frac{[k-m]_q(1+[k-m-1]_q\ \lambda)\Gamma_q(1+k)}{\Gamma_q(1+k-m)}\ z^{k-m-1}\right.$$
$$\left.-[p-m]_q\left(1-\frac{(1+[p-m-1]_q\ \lambda)\Gamma_q(1+p)}{\Gamma_q(1+p-m)}z^{p-m-1}\right)\right\}>-|\delta|\ [p-m]_q.$$

Now taking (38) into account, the above inequality yields

$$\Re\left\{-\sum_{k=n+p}^{\infty}[k-m]_q\Omega(k,m,\lambda,q)\Gamma_q(1+m)\ a_k\ z^{k-m-1}\right.$$
$$\left.-[p-m]_q\left(1-\Omega(k,m,\lambda,q)\Gamma_q(1+m)z^{p-m-1}\right)\right\}>-|\delta|\ [p-m]_q. \tag{41}$$

By putting $z=r$ in (41), and letting $r\to 1^-$ through real values, we get the desired assertion (37) of Theorem 2.

To prove the converse of Theorem 2, we have

$$\left|\left\{D_{q,z}^{1+m}f(z)+\lambda\ z\ D_{q,z}^{2+m}f(z)-[p-m]_q\right\}\right|$$
$$\le\left|\sum_{k=n+p}^{\infty}[k-m]_q\Omega(k,m,\lambda,q)\Gamma_q(1+m)\ a_k\ z^{k-m-1}\right|$$
$$+\left|[p-m]_q\left(1-\Omega(k,m,\lambda,q)\Gamma_q(1+m)z^{p-m-1}\right)\right|.$$

Letting $|z|=1$, we find that

$$\left|\left\{D_{q,z}^{1+m}f(z)+\lambda\ z\ D_{q,z}^{2+m}f(z)-[p-m]_q\right\}\right|$$
$$\le\sum_{k=n+p}^{\infty}[k-m]_q\Omega(k,m,\lambda,q)\Gamma_q(1+m)\ a_k$$
$$+[p-m]_q\left(1-\Omega(k,m,\lambda,q)\Gamma_q(1+m)\right),$$

then by applying the hypothesis (37), we find that

$$\left|\left\{D_{q,z}^{1+m}f(z)+\lambda\ z\ D_{q,z}^{2+m}f(z)-[p-m]_q\right\}\right|\leq|\delta|\ \ [p-m]_q.$$

Hence, by the maximum modulus principle, we infer that

$$f(z)\in\mathcal{N}_{n,p}^{m}(\lambda,\delta,q).$$

□

The following corollaries follow from Theorem 2 in the same manner as Corollaries 1 and 2 from Theorem 1.

Corollary 3 *If the function* $f(z)$ *be defined by* (12) *and* $f(z)\in\mathcal{N}_{n,p}^{m}(\lambda,\delta,q)$, *then*

$$\sum_{k=n+p}^{\infty}a_k\ \leq\ X(p,n,m,\lambda,\delta,q), \tag{42}$$

where $X(p,n,m,\lambda,\delta,q)$ *is given by*

$$X(p,n,m,\lambda,\delta,q)=\frac{[p-m]_q\,[|\delta|-1+\Gamma_q(1+m)\ \Omega(p,m,\lambda,q)]}{\Gamma_q(1+m)[n+p-m]_q\ \Omega(n+p,m,\lambda,q)}. \tag{43}$$

Corollary 4 *If* $f(z)\in\mathcal{N}_{n,p}^{m}(\lambda,\delta,q)$, *then*

$$\sum_{k=n+p}^{\infty}[k]_q[k-1]_q\cdots[k-p+1]_q\ a_k\ \leq\ \Psi(p,n,m,\lambda,\delta,q), \tag{44}$$

where $\Psi(p,n,m,\lambda,\delta,q)$ *is given by*

$$\Psi(p,n,m,\lambda,\delta,q)=\frac{[p-m]_q\,[|\delta|-1+\Gamma_q(1+m)\ \Omega(p,m,\lambda,q)]\,\Gamma_q(1+n+p-m)}{\Gamma_q(1+n)[n+p-m]_q\ (1+[n+p-m-1]_q\ \lambda)}. \tag{45}$$

3 Distortion theorems

In this section, we establish certain distortion theorems for the classes of functions defined above involving the q-differential operator.

Theorem 3 *Let* $\lambda\in\mathbb{R}$ *and* $\delta\in\mathbb{C}\setminus\{0\}\in\mathbb{N}$ *satisfy the inequalities:*

$$m<p;\ \ m\in\mathbb{N}_0;\ \ p,n\in\mathbb{N};\ \ 0\leq\lambda\leq 1,\ \ 0<q<1.$$

Also, let the function $f(z)$ *defined by* (12) *be in the class* $\mathcal{M}_{n,p}^{m}(\lambda, \delta, q)$, *then*

$$||f(z)| - |z|^p| \leq |\delta| \ \Xi(p, n, m, \lambda, \delta, q) \ |z|^{n+p} \quad (z \in \mathbb{U}), \tag{46}$$

where $\Xi(p, n, m, \lambda, \delta, q)$ *is given by* (28).

Proof. Since $f(z) \in \mathcal{M}_{n,p}^{m}(\lambda, \delta, q)$, then from the Corollary 1, we have

$$\sum_{k=n+p}^{\infty} a_k \ \leq |\delta| \ \Xi(p, n, m, \lambda, \delta, q),$$

where $\Xi(p, n, m, \lambda, \delta, q)$ is given by (28).

This inequality in conjunction with the following inequality (easily obtainable from (11)):

$$|z|^p - |z|^{n+p} \sum_{k=n+p}^{\infty} a_k \leq |f(z)| \leq |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} a_k, \tag{47}$$

yields the assertion (46) of Theorem 3. □

To obtain the distortion theorem for a normalized multivalent analytic function of the form (12), we define here a q-differential operator $\mathbb{D}_{q,z}^{m}$ which is expressed in the form

$$\mathbb{D}_{q,z}^{m} f(z) = \frac{\Gamma_q(1+p-m)}{\Gamma_q(1+p)} z^m D_{q,z}^{m} f(z). \tag{48}$$

Theorem 4 *Let* $m < p;\ m \in \mathbb{N}_0,\ p, n \in \mathbb{N},\ 0 \leq \lambda \leq 1,\ \delta \in \mathbb{C} \setminus \{0\} \in \mathbb{N},\ 0 < q < 1$, *and let the function* $f(z)$ *defined by* (12) *be in the class* $\mathcal{M}_{n,p}^{m}(\lambda, \delta, q)$. *Then*

$$\left| \left| \mathbb{D}_{q,z}^{m} f(z) \right| - |z|^p \right| \leq |\delta| \ \mathbb{A}(p, n, m, \lambda, \delta, q) \ |z|^{n+p}, \tag{49}$$

where

$$\mathbb{A}(p, n, m, \lambda, \delta, q) = \frac{1 + [p-m-1]_q \ q \ \lambda}{(|\delta| + q^{p-m}[n]_q) \ (1 + [n+p-m-1]_q \ q \ \lambda)}. \tag{50}$$

Proof. Since

$$\mathbb{D}_{q,z}^{m} f(z) = \frac{\Gamma_q(1+p-m)}{\Gamma_q(1+p)} z^m D_{q,z}^{m} f(z) = z^p - \sum_{k=n+p}^{\infty} a_k \frac{\Gamma_q(1+k)\Gamma_q(1+p-m)}{\Gamma_q(1+p)\Gamma_q(1+k-m)} z^k,$$

therefore, on using the relation (32), we can write

$$\mathbb{D}_{q,z}^{m} f(z) = z^p - \sum_{k=n+p}^{\infty} a_k \frac{[k]_q[k-1]_q \ldots [k-p+1]_q \Gamma_q(1+k-p)\Gamma_q(1+p-m)}{\Gamma_q(1+p)\Gamma_q(1+k-m)} z^k$$
$$= z^p - \sum_{k=n+p}^{\infty} a_k \, [k]_q[k-1]_q \ldots [k-p+1]_q \; \phi(k) \; z^k, \tag{51}$$

where

$$\phi(k) = \frac{\Gamma_q(1+k-p)\Gamma_q(1+p-m)}{\Gamma_q(1+p)\Gamma_q(1+k-m)}. \tag{52}$$

Now, we show that the function $\phi(k)$ ($m \in \mathbb{N}_0$, $k \geq n+p$; $p, n \in \mathbb{N}, m < p$) is a decreasing function of k for $m \in \mathbb{N}_0$, $0 < q < 1$.

We note that

$$\frac{\phi(k+1)}{\phi(k)} = \frac{\Gamma_q(2+k-p)\Gamma_q(1+k-m)}{\Gamma_q(2+k-m)\Gamma_q(1+k-p)} \quad (k \geq n+p; n, p \in \mathbb{N}),$$

and it is sufficient here to consider the value $k = n + p$, so that on using (4), we get

$$\frac{\phi(k+1)}{\phi(k)} = \frac{1-q^{1+n}}{1-q^{1+n+p-m}} \quad (0 < q < 1).$$

The function $\phi(k)$ is a decreasing function of k if $\frac{\phi(k+1)}{\phi(k)} \leq 1$ $(n, p \in \mathbb{N})$, and this gives

$$\frac{1-q^{1+n}}{1-q^{1+n+p-m}} \leq 1 \quad (0 < q < 1).$$

Multiplying the above inequality both sides by $1 - q^{1+n+p-m}$ (provided that $m < p$), we are at once lead to the inequality $m \leq p$. Thus, $\phi(k)$ $(k \geq n+p; n, p \in \mathbb{N})$ is a decreasing function of k for $m < p, m \in \mathbb{N}_0, 0 < q < 1$.

Using (51), we observe that

$$\left|\mathbb{D}_{q,z}^{m} f(z)\right| \geq |z|^p - \sum_{k=n+p}^{\infty} [k]_q[k-1]_q \ldots [k-p+1]_q \; \phi(k) \, |a_k| \; |z|^k$$
$$\geq |z|^p - \phi(n+p) \, |z|^{n+p} \sum_{k=n+p}^{\infty} [k]_q[k-1]_q \ldots [k-p+1]_q \; |a_k| \, ,$$

which in view of (29) of Corollary 2 leads to

$$\left|\mathbb{D}^{m}_{q,z}f(z)\right| \geq |z|^{p} - |\delta| \;\; \phi(n+p)\;\Theta(p,n,m,\lambda,\delta,q)\;\; |z|^{n+p}$$

$$\geq |z|^{p} - |\delta| \;\; \mathbb{A}(p,n,m,\lambda,\delta,q)\;\; |z|^{n+p}, \tag{53}$$

where $\mathbb{A}(p,n,m,\lambda,\delta,q)$ is given by (50).

Similarly, it follows that

$$\left|\mathbb{D}^{m}_{q,z}f(z)\right| \leq |z|^{p} + \;\; |\delta| \;\; \mathbb{A}(p,n,m,\lambda,\delta,q)\;\; |z|^{n+p}, \tag{54}$$

and hence, (53) and (54) establish the assertion (49) of Theorem 4. □

The following distortion inequalities for the function $f(z) \in \mathcal{N}^{m}_{n,p}(\lambda,\delta,q)$ can be proved in the same manner as detailed in the proof of Theorem 4 above:

Theorem 5 *Let* $\lambda \in \mathbb{R}$ *and* $\delta \in \mathbb{C}\setminus\{0\} \in \mathbb{N}$ *satisfy the inequalities:*

$$m < p; m \in \mathbb{N}_0; p, n \in \mathbb{N};\;\; 0 \leq \lambda \leq 1,\;\; 0 < q < 1.$$

Also, let the function $f(z)$ *defined by* (12) *be in the class* $\mathcal{N}^{m}_{n,p}(\lambda,\delta,q)$*, then*

$$||f(z)| - |z|^{p}| \leq |\delta| \;\; X(p,n,m,\lambda,\delta,q)\;\; |z|^{n+p} \quad (z \in \mathbb{U}), \tag{55}$$

where $X(p,n,m,\lambda,\delta,q)$ *is given by* (43).

Theorem 6 *Let* $m < p;\;\; m \in \mathbb{N}_0,\;\; p, n \in \mathbb{N},\;\; 0 \leq \lambda \leq 1,\;\; \delta \in \mathbb{C}\setminus\{0\} \in \mathbb{N},\;\; 0 < q < 1$ *and let the function* $f(z)$ *defined by* (12) *be in the class* $\mathcal{N}^{m}_{n,p}(\lambda,\delta,q)$*. Then*

$$\left|\left|\mathbb{D}^{m}_{q,z}f(z)\right| - |z|^{p}\right| \leq |\delta| \;\; \mathbb{B}(p,n,m,\lambda,\delta,q)\;\; |z|^{n+p}, \tag{56}$$

where

$$\mathbb{B}(p,n,m,\lambda,\delta,q) = \frac{[p-m]_q\;[|\delta| - 1 + \Gamma_q(1+m)\;\Omega(p,m,\lambda,q)]\,\Gamma_q(1+p-m)}{\Gamma_q(1+p)[n+p-m]_q\;\;(1+[n+p-m-1]_q\;\lambda)}, \tag{57}$$

$\Omega(p,m,\lambda,q)$ *is given by* (38).

4 Some consequences of the main results

In this section, we briefly consider some special cases of the results derived in the preceding sections.

When $m = 0$ and $\delta = \gamma\beta$ $(\gamma \in \mathbb{C} \setminus \{0\},\ 0 < \beta \leq 1)$, the condition (14) reduces to the inequality:

$$\left| \frac{1}{\gamma} \left\{ \frac{z\, D_{q,z} f(z) + \lambda\, q z^2\, D^2_{q,z} f(z)}{\lambda\, z\, D_{q,z} f(z) + (1-\lambda) f(z)} - [p]_q \right\} \right| < \beta, \tag{58}$$

$$(p \in \mathbb{N},\ 0 \leq \lambda \leq 1;\ 0 < \beta \leq 1;\ \gamma \in \mathbb{C} \setminus \{0\};\ 0 < q < 1;\ z \in \mathbb{U})$$

and we write

$$\mathcal{M}^0_{n,p}(\lambda, \gamma\beta, q) = \mathcal{R}_{n,p}(\lambda, \beta, \gamma, q), \tag{59}$$

where $\mathcal{R}_{n,p}(\lambda, \beta, \gamma, q)$ represents a subclass of p-valently analytic functions which satisfy the condition (58).

Similarly, the condition (16) when $m = 0$ and $\delta = \gamma\beta$ reduces to the inequality:

$$\left| \frac{1}{\gamma} \left\{ D_{q,z} f(z) + \lambda\, z\, D^2_{q,z} f(z) - [p]_q \right\} \right| < \beta\ [p]_q, \tag{60}$$

$$(p \in \mathbb{N},\ 0 \leq \lambda \leq 1;\ 0 < \beta \leq 1;\ \gamma \in \mathbb{C} \setminus \{0\};\ 0 < q < 1;\ z \in \mathbb{U})$$

and we write

$$\mathcal{N}^0_{n,p}(\lambda, \gamma\beta, q) = \mathcal{L}_{n,p}(\lambda, \beta, \gamma, q), \tag{61}$$

where $\mathcal{L}_{n,p}(\lambda, \beta, \gamma, q)$ is another subclass of p-valently analytic functions which satisfy the condition (60).

Now, by setting $m = 0$, $\delta = \gamma\beta$, and making use of the relations (59) and (61), Theorems 1 and 2 give the following coefficient inequalities for the classes $\mathcal{R}_{n,p}(\lambda, \beta, \gamma, q)$ and $\mathcal{L}_{n,p}(\lambda, \beta, \gamma, q)$, respectively.

Corollary 5 *Let the function* $f(z)$ *be defined by* (12), *then* $f(z) \in \mathcal{R}_{n,p}(\lambda, \beta, \gamma, q)$ *if and only if*

$$\sum_{k=n+p}^{\infty} \left(\beta\ |\gamma| - q^k [p-k]_q\right) \left(1 + [k-1]_q\ q\ \lambda\right) a_k\ \leq \beta\ |\gamma| \left(1 + [p-1]_q\ q\ \lambda\right). \tag{62}$$

The result is sharp with the extremal function given by

$$f(z) = z^p - \frac{\beta\ |\gamma| \left(1 + [p-1]_q\ q\ \lambda\right)}{\left(\beta\ |\gamma| + q^p [n]_q\right) \left(1 + [n+p-1]_q\ q\ \lambda\right)}\ z^{n+p}. \tag{63}$$

Corollary 6 *Let the function* $f(z)$ *be defined by* (12)*, then* $f(z) \in \mathcal{L}_{n,p}(\lambda, \beta, \gamma, q)$ *if and only if*

$$\sum_{k=n+p}^{\infty} [k]_q \ (1 + [k-1]_q \ \lambda) \ a_k \ \leq [p]_q \, [\beta \ \ |\gamma| + [p-1]_q \ \lambda] \ . \tag{64}$$

The result is sharp with the extremal function given by

$$f(z) = z^p - \frac{[p]_q \, [\beta \ \ |\gamma| + [p-1]_q \ \lambda]}{[n+p]_q \ (1 + [n+p-1]_q \ \lambda)} z^{n+p}. \tag{65}$$

Again, if we put $m = 0$, $\delta = \gamma\beta$, then Theorem 3 and Theorem 5, respectively, yield the following distortion theorems for the classes $\mathcal{R}_{n,p}(\lambda, \beta, \gamma, q)$ and $\mathcal{L}_{n,p}(\lambda, \beta, \gamma, q)$.

Corollary 7 *Let* $\lambda, \beta \in \mathbb{R}$ *and* $\gamma \in \mathbb{C} \setminus \{0\} \in \mathbb{N}$ *satisfy the inequalities:*

$$p, n \in \mathbb{N}; \ 0 \leq \lambda \leq 1, \ 0 < q < 1.$$

Also, let the function $f(z)$ *defined by* (12) *be in the class* $\mathcal{R}_{n,p}(\lambda, \beta, \gamma, q)$*, then*

$$||f(z)| - |z|^p| \leq \beta \, |\gamma| \ \ \mathbb{E}(p, n, \lambda, \beta, \gamma, q) \, |z|^{n+p} \quad (z \in \mathbb{U}), \tag{66}$$

where

$$\mathbb{E}(p, n, \lambda, \beta, \gamma, q) = \frac{1 + [p-1]_q \ q \ \lambda}{(\beta \, |\gamma| + q^p [n]_q)(1 + [n+p-1]_q \ q \ \lambda)}. \tag{67}$$

Corollary 8 *Let* $\lambda, \beta \in \mathbb{R}$ *and* $\gamma \in \mathbb{C} \setminus \{0\} \in \mathbb{N}$ *satisfy the inequalities:*

$$p, n \in \mathbb{N}; \ 0 \leq \lambda \leq 1, \ 0 < q < 1.$$

Also, let the function $f(z)$ *defined by* (12) *be in the class* $\mathcal{L}_{n,p}(\lambda, \beta, \gamma, q)$*, then*

$$||f(z)| - |z|^p| \leq \mathbb{F}(p, n, \lambda, \beta, \gamma, q) \, |z|^{n+p} \quad (z \in \mathbb{U}), \tag{68}$$

where

$$\mathbb{F}(p, n, \lambda, \beta, \gamma, q) = \frac{[p]_q \, [\beta \, |\gamma| + [p-1]_q \ \lambda]}{[n+p]_q (1 + [n+p-1]_q \ \lambda)}. \tag{69}$$

Further, if we set $p = 1$, then from (59) and (61), we get

$$\mathcal{M}^0_{n,1}(\lambda, \gamma\beta, q) = \mathcal{R}_{n,1}(\lambda, \beta, \gamma, q) = \mathcal{H}_n(\lambda, \gamma, \beta, q) \tag{70}$$

and

$$\mathcal{N}^0_{n,1}(\lambda, \gamma\beta, q) = \mathcal{L}_{n,1}(\lambda, \beta, \gamma, q) = \mathcal{G}_n(\lambda, \gamma, \beta, q), \tag{71}$$

where $\mathcal{H}_n(\lambda, \gamma, \beta, q)$ and $\mathcal{G}_n(\lambda, \gamma, \beta, q)$ are precisely the subclass of analytic and univalent functions studied recently by Purohit and Raina [11]. Thus, if we set $p = 1$, and taking into consideration the relations (70) and (71), Corollary 5 to Corollary 8 yield the known results obtained recently by Purohit and Raina [11].

Finally, by letting $q \to 1^-$, and making use of the limit formula (10), we observe that the function classes $\mathcal{M}^m_{n,p}(\lambda, \delta, q)$, $\mathcal{N}^m_{n,p}(\lambda, \delta, q)$ and the inequalities (17) and (37) of Theorem 1 and Theorem 2 provide, respectively, the q-extensions of the known results due to Srivastava and Orhan [13, pp. 687-688, eqn. (11) and (14)].

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References

[1] M. H. Abu-Risha, M. H., Annaby, M. E. H., Ismail, Z. S. Mansour, Linear q-difference equations, *Z. Anal. Anwend.* **26** (2007), 481–494.

[2] D. Albayrak, S. D. Purohit, F. Uçar, On q-analogues of Sumudu transforms, *An. Ştiinţ. Univ. Ovidius Constanţa*, **21** (1) (2013), 239–260.

[3] D. Albayrak, S. D. Purohit, F. Uçar, On q-Sumudu transforms of certain q-polynomials, *Filomat*, **27** (2) (2013), 411–427.

[4] G. Bangerezako, Variational calculus on q-nonuniform lattices, *J. Math. Anal. Appl*, **306** (1) (2005), 161–179.

[5] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[6] M. E. H. Ismail, M. E. Muldoon, *Inequalities and monotonicity properties for gamma and* q*-gamma functions*, pp. 309–323 in R. V. M. Zahar, ed., Approximation and Computation: A Festschrift in Honor of Walter Gautschi, ISNM, vol. 119, Birkhäuser, Boston-Basel-Berlin, 1994.

[7] V. G. Kac, P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, New York, 2002.

[8] Z. S. I. Mansour, Linear sequential q-difference equations of fractional order, *Fract. Calc. Appl. Anal.*, **12** (2) (2009), 159–178.

[9] S. D. Purohit, A new class of multivalently analytic functions associated with fractional q-calculus operators, *Fractional Differ. Calc.*, **2** (2) (2012), 129–138.

[10] S. D. Purohit, R. K. Raina, Certain subclass of analytic functions associated with fractional q-calculus operators, *Math. Scand.*, **109** (1) (2011), 55–70.

[11] S. D. Purohit, R. K. Raina, *Fractional* q*-calculus and certain subclass of univalent analytic functions*, *Mathematica* (Cluj), **55** (1) (2014), 62–74.

[12] P. M. Rajković, S. D. Marinković, M. S. Stanković, Fractional integrals and derivatives in q-calculus, *Appl. Anal. Discrete Math.*, **1** (2007), 311–323.

[13] H. M. Srivastava, H. Orhan, Coefficient inequalities and inclusion relations for some families of analytic and multivalent functions, *Appl. Math. Lett.*, **20** (2007), 686–691.

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Hydromagnetic thermosolutal instability of Rivlin-Ericksen rotating fluid permeated with suspended particles and variable gravity field in porous medium

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Abstract. The thermosolutal instability of Rivlin-Ericksen elastico-viscous rotating fluid permeated with suspended particles (fine dust) and variable gravity field in porous medium in hydromagnetics is considered. By applying normal mode analysis method, the dispersion relation has been derived and solved analytically. It is observed that the rotation, magnetic field, gravity field, suspended particles and viscoelasticity introduce oscillatory modes. For stationary convection, the rotation and stable solute gradient has stabilizing effects and suspended particles are found to have destabilizing effect on the system whereas the medium permeability has stabilizing or destabilizing effect on the system under certain conditions. The magnetic field has destabilizing effect in the absence of rotation whereas in the presence of rotation, magnetic field has stabilizing or destabilizing effect under certain conditions. The effect of rotation, suspended particles, magnetic field, stable solute gradient and medium permeability has also been shown graphically.

1 Introduction

A detailed account of the thermal instability of a Newtonian fluid, under varying assumptions of hydrodynamics and hydromagnetics has been given by

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Chandrasekhar [3]. Bhatia and Steiner [1] have studied the thermal instability of a Maxwellian visco-elastic fluid in the presence of magnetic field while the thermal convection in Oldroydian visco-elastic fluid has been considered by Sharma [14]. Veronis [20] has investigated the problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient. The buoyancy forces can arise not only from density differences due to variations in solute concentration. Thermosolutal instability problems arise in oceanography, limnology and engineering.

The medium has been considered to be non-porous in all the above studies. Lapwood [5] has studied the convective flow in a porous medium using linearized stability theory. The Rayleigh instability of a thermal boundary layer in flow through a porous medium has been considered by Wooding [21] whereas Scanlon and Segel [13] have considered the effect of suspended particles on the onset of Be'nard convection and found that the critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by the particles. The suspended particles were thus found to destabilize the layer.

Sharma and Sunil [15] have studied the thermal instability of an Oldroydian viscoelastic fluid with suspended particles in hydromagnetics in a porous medium. There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relations or Oldroyd's constitutive relations. One such class of fluids is Rivlin-Ericksen [12] elastico-viscous fluid. Srivastava and Singh [18] have studied the unsteady flow of a dusty elastico-viscous Rivlin-Ericksen fluid through channels of different cross-sections in the presence of time-dependent pressure gradient. Garg et al. [4] has studied the rectilinear oscillations of a sphere along its diameter in conducting dusty Rivlin-Ericksen fluid in the presence of magnetic field.

Stommel and Fedorov [19] and Linden [6] have remarked that the length scalar characteristic of double diffusive convecting layers in the ocean may be sufficiently large that the Earth's rotation might be important in their formation. Moreover, the rotation of the Earth distorts the boundaries of a hexagonal convection cell in a fluid through a porous medium and the distortion plays an important role in the extraction of energy in the geothermal regions. Brakke [2] explained a double-diffusive instability that occurs when a solution of a slowly diffusing protein is layered over a denser solution of more rapidly diffusing sucrose. The problem of thermosolutal convection in fluids in a porous medium is of importance in geophysics, soil sciences, ground water hydrology and astrophysics. The scientific importance of the field has also increased because hydrothermal circulation is the dominant heat transfer mechanism in the development of young oceanic crust (Lister, [7]). Generally, it is accepted that comets consist of a dusty 'snowball' of a mixture of frozen gases which in the process of their journey change from solid to gas and vice-versa. The physical properties of comets, meteorites and inter-planetary dust strongly suggest the importance of porosity in the astrophysical context (McDonnel, [8]).

Thermal instability of a fluid layer under variable gravitational field heated from below or above is investigated analytically by Pradhan and Samal [9]. Although the gravity field of the Earth is varying with height from its surface, we usually neglect this variation for laboratory purposes and treat the field as constant. However, this may not the case for large scale flows in the ocean, the atmosphere or the mantle. It can become imperative to consider gravity as a quantity varying with distance from the centre.

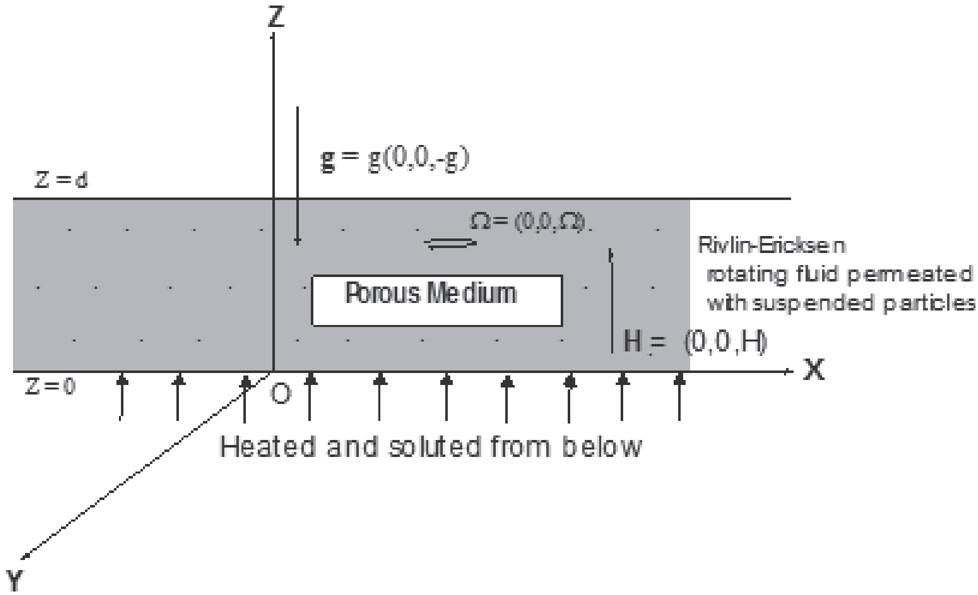
A porous medium is a solid with holes in it, and is characterized by the manner in which the holes are imbedded, how they are interconnected and the description of their location, shape and interconnection. However, the flow of a fluid through a homogeneous and isotropic porous medium is governed by Darcy's law which states that the usual viscous term in the equations of motion of Rivlin-Ericksen fluid is replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\right]\mathbf{q}$, where μ and μ' are the viscosity and viscoelasticity of the incompressible Rivlin-Ericksen fluid, k_1 is the medium permeability and q is the Darcian (filter) velocity of the fluid (Garg et al. [4], Sharma and Sunil [15] and Sharma and Rana [16, 17]).

Sharma and Rana [16] have studied thermal instability of Walters' (Model B') elastico-viscous in the presence of variable gravity field and rotation in porous medium. Sharma and Rana [17] have also studied the thermosolutal instability of incompressible Walters' (Model B') rotating fluid permeated with suspended particles and variable gravity field in porous medium. Recently, Rana and Kumar [11] have studied thermal instability of Rivlin-Ericksen elastico-viscous rotating fluid permeated with suspended particles and variable gravity field in porous medium and thermal instability of compressible Walters' (Model B') elastico-viscous rotating fluid a permeated with suspended dust particles in porous medium have been studied by Rana and Kango [10]. Keeping in mind the importance in various applications mentioned above, our interest, in the present paper is to study the thermosolutal instability of Rivlin-Ericksen elastico-viscous rotating fluid permeated with suspended particles and variable field in porous medium in hydromagnetics.

2 Mathematical formulation of the problem

Consider an infinite horizontal layer of an electrically conducting Rivlin-Ericksen elastico-viscous fluid of depth d in a porous medium bounded by the planes $z = 0$ and $z = d$ in an isotropic and homogeneous medium of porosity ϵ and permeability k_1, which is acted upon by a uniform rotation Ω $(0, 0, \Omega)$ uniform vertical magnetic field $\mathbf{H}$ $(0, 0, \mathbf{H})$ and variable gravity $\mathbf{g}$ $(0, 0, -g)$, $g = \lambda g_0$, g_0 (> 0) is the value of g at $z = 0$ and λ can be positive or negative as gravity increases or decreases upward from its value g_0. This layer is heated and soluted from below such that a uniform temperature gradient β $\left(= \left|\frac{dT}{dz}\right|\right)$ and a uniform solute gradient β' $\left(= \left|\frac{dC}{dz}\right|\right)$ are maintained as shown in schematic sketch of physical situation.

The character of equilibrium of this initial static state is determined by supposing that the system is slightly disturbed and then following its further evolution.



Schematic Sketch of Physical Situation

The hydromagnetic equations in porous medium (Chandrasekhar [3], Rivlin and Ericksen [12], Rana and Kumar [11]) relevant to the problem are

$$\frac{1}{\epsilon}\left[\frac{\partial \mathbf{q}}{\partial t}+\frac{1}{\epsilon}(\mathbf{q}\cdot\nabla)\mathbf{q}\right]=-\frac{1}{\rho_0}\nabla p+\mathbf{g}\left(1+\frac{\delta\rho}{\rho_0}\right)-\frac{1}{k_1}\left(\upsilon+\upsilon'\frac{\partial}{\partial t}\right)q$$
$$+\frac{2}{\epsilon}(q\times\Omega)+\frac{K'N}{\rho_0\epsilon}(\mathbf{q_d}-\mathbf{q})+\frac{\mu_e}{4\pi\rho_0}(\nabla\times\mathbf{H})\times\mathbf{H}, \tag{1}$$

$$\nabla\cdot\mathbf{q}=0, \tag{2}$$

$$E\frac{\partial T}{\partial t}+(\mathbf{q}\cdot\nabla)T+\frac{mNC_{pt}}{\rho_0 C_f}\left[\epsilon\frac{\partial}{\partial t}+\mathbf{q_d}.\nabla\right]T=\kappa\,\nabla^2 T, \tag{3}$$

$$E'\frac{\partial C}{\partial t}+(\mathbf{q}\cdot\nabla)T+\frac{mNC'_{pt}}{\rho_0 C'_f}\left[\epsilon\frac{\partial}{\partial t}+\mathbf{q_d}.\nabla\right]T=\kappa'\,\nabla^2 T \tag{4}$$

$$\nabla\cdot\mathbf{H}=0, \tag{5}$$

$$\epsilon\frac{\partial \mathbf{H}}{\partial t}=\nabla\times(\mathbf{q}\times\mathbf{H})+\epsilon\eta\nabla^2\mathbf{H}, \tag{6}$$

where $E=\epsilon+(1-\epsilon)\left(\frac{\rho_s c_s}{\rho_0 c_f}\right)$, ρ_s, c_s; ρ_0, c_f denote the density and heat capacity of solid (porous) matrix and fluid respectively and E' is a constant analogous to E but corresponding to solute rather than heat; κ, κ' are the thermal diffusivity and solute diffusivity respectively.

The equation of state is

$$\rho=\rho_0[1-\alpha(T-T_0)+\alpha'(C-C_0)], \tag{7}$$

where the suffix zero refers to values at the reference level $z=0$. Here ρ,υ,υ', p,ϵ,T,C,μ_e, α, α', $\mathbf{q}(0,0,0)$ and $\mathbf{H}(0,0,H)$ stand for density, kinematic viscosity, kinematic viscoelasticity, pressure, medium porosity, temperature, solute concentration, magnetic permeability, thermal coefficient of expansion, an analogous solvent coefficient of expansion, velocity of the fluid and magnetic field. Here $\mathbf{q_d}(\overline{x},t)$ and $N(\overline{x},t)$ denote the velocity and number density of the particles respectively, $K=6\pi\eta\rho\upsilon$, where η is particle radius, is the Stokes drag coefficient, $\mathbf{q_d}=(l,r,s)$ and $\overline{x}=(x,y,z)$.

If mN is the mass of particles per unit volume, then the equations of motion and continuity for the particles are

$$mN\left[\frac{\partial \mathbf{q_d}}{\partial t}+\frac{1}{\epsilon}(\mathbf{q_d}\cdot\nabla)\mathbf{q_d}\right]=K'N(\mathbf{q}-\mathbf{q_d}), \tag{8}$$

$$\epsilon\frac{\partial N}{\partial t}+\nabla\cdot(N\mathbf{q_d})=0. \tag{9}$$

The presence of particles adds an extra force term proportional to the velocity difference between particles and fluid and appears in the equation of motion (1). Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign, in the equations of motion for the particles (8). The buoyancy force on the particles is neglected. Interparticles reactions are not considered either since we assume that the distance between the particles are quite large compared with their diameters. These assumptions have been used in writing the equations of motion (8) for the particles.

The initial state of the system is taken to be quiescent layer (no settling) with a uniform particle distribution number. The initial state is

$$\mathbf{q} = (0,0,0), \mathbf{q_d} = (0,0,0),$$
$$T = -\beta z + T_0, C = -\beta^{'} z + C_0,$$

$$\rho = \rho_0(1 + \alpha\beta z - \alpha'\beta' z), N_0 = \text{constant} \tag{10}$$

is an exact solution to the governing equations.

3 Perturbation equations

Let $\mathbf{q}(u, v, w)$, $\mathbf{q}_d(l, r, s)$, θ, γ, δp and $\delta\rho$ denote, respectively, the perturbations in fluid velocity $\mathbf{q}(0,0,0)$, the perturbation in particle velocity $\mathbf{q}_d(0,0,0)$, temperature T, solute concentration C, pressure p and density ρ.

The change in density $\delta\rho$ caused by perturbation of temperature θ and solute concentration γ is given by

$$\delta\rho = -\rho_0(\alpha\theta - \alpha^{'}\gamma). \tag{11}$$

The linearized perturbation equations governing the motion of fluids are

$$\begin{aligned} \frac{1}{\epsilon}\frac{\partial \mathbf{q}}{\partial t} = &-\frac{1}{\rho_0}\Omega\delta p - \mathbf{g}\left(\alpha\theta - \alpha^{'}\gamma\right) - \frac{1}{k_1}\left(\upsilon + \upsilon^{'}\frac{\partial}{\partial t}\right) q \\ &+ \frac{K^{'}N}{\epsilon}\left(\mathbf{q_d} - \mathbf{q}\right) + \frac{2}{\epsilon}\left(\mathbf{q} \times \Omega\right) + \frac{\mu_e}{4\pi\rho_0}\left(\nabla \times \mathbf{h}\right) \times \mathbf{H}, \end{aligned} \tag{12}$$

$$\nabla \cdot \mathbf{q} = 0, \tag{13}$$

$$\left(\frac{m}{K'}\frac{\partial}{\partial t} + 1\right)\mathbf{q_d} = q, \tag{14}$$

$$(E + b\epsilon)\frac{\Omega\theta}{\partial t} = \beta\left(w + bs\right) + \ \kappa\nabla^2\theta, \tag{15}$$

$$\left(E' + b'\epsilon\right)\frac{\Omega\theta}{\partial t} = \beta'\left(w + b^{'}s\right) + \ \kappa'\nabla^2\gamma \tag{16}$$

$$\nabla \cdot \mathbf{h} = 0, \tag{17}$$

$$\epsilon\frac{\partial \mathbf{H}}{\partial t} = (\mathbf{H}\cdot\nabla)q + \epsilon\eta\nabla^2\mathbf{H}, \tag{18}$$

where $b = \frac{mNC_{pt}}{\rho_0 C_f}$, $b' = \frac{mNC'_{pt}}{\rho_0 C'_f}$ and w, s are the vertical fluid and particles velocity.

In the Cartesian form, equations (12)–(18) can be expressed as

$$\begin{aligned}\frac{1}{\epsilon}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{\partial u}{\partial t} &= -\frac{1}{\rho_0}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}(\delta p)\\ &-\frac{1}{k_1}\left(\upsilon+\upsilon^{'}\frac{\partial}{\partial t}\right)\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)u-\frac{mN}{\epsilon\rho_0}\frac{\partial u}{\partial t}\\ &+\frac{\mu_e H}{4\pi\rho_0}\left(\frac{\partial h_x}{\partial z}-\frac{\partial h_z}{\partial x}\right)+\frac{2}{\epsilon}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\Omega v,\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{1}{\epsilon}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{\partial v}{\partial t} &= -\frac{1}{\rho_0}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial y}(\delta p)\\ &-\frac{1}{k_1}\left(\upsilon+\upsilon^{'}\frac{\partial}{\partial t}\right)\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)v-\frac{mN}{\epsilon\rho_0}\frac{\partial v}{\partial t}\\ &+\frac{\mu_e H}{4\pi\rho_0}\left(\frac{\partial h_y}{\partial z}-\frac{\partial h_z}{\partial y}\right)\frac{2}{\epsilon}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\Omega u,\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{1}{\epsilon}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{\partial w}{\partial t} &= -\frac{1}{\rho_0}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial z}(\delta p)-\frac{1}{k_1}\left(\upsilon+\upsilon^{'}\frac{\partial}{\partial t}\right)\\ &\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)w-\frac{mN}{\epsilon\rho_0}\frac{\partial w}{\partial t}+g\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\alpha\theta,\end{aligned} \tag{21}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{22}$$

$$(E + b\epsilon)\frac{\Omega\theta}{\partial t} = \beta\left(w + bs\right) + \ \kappa\nabla^2\theta, \tag{23}$$

$$\left(E' + b'\epsilon\right)\frac{\Omega\theta}{\partial t} = \beta'\left(w + b^{'}s\right) + \ \kappa'\nabla^2\gamma \tag{24}$$

$$\epsilon\frac{\partial h_x}{\partial t} = H\frac{\partial u}{\partial z} + \epsilon\eta\nabla^2 h_x, \tag{25}$$

$$\epsilon\frac{\partial h_y}{\partial t}=H\frac{\partial v}{\partial z}+\epsilon\eta\nabla^2 h_y, \tag{26}$$
$$\epsilon\frac{\partial h_z}{\partial t}=H\frac{\partial w}{\partial z}+\epsilon\eta\nabla^2 h_z, \tag{27}$$
$$\frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}+\frac{\partial h_z}{\partial z}=0. \tag{28}$$

Operating equation (19) and (20) by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ respectively, adding and using equation (25)-(28), we get

$$\begin{aligned}\frac{1}{\epsilon}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right)&=\frac{1}{\rho_0}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\delta p-\\&-\frac{1}{k_1}\left(\upsilon+\upsilon'\frac{\partial}{\partial t}\right)\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\left(\frac{\partial w}{\partial z}\right)-\frac{mN}{\epsilon\rho_0}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right)+\\&+\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{{}_eH}{4\pi\rho_0}\nabla^2 h_z-\frac{2}{\epsilon}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\Omega\zeta,\end{aligned} \tag{29}$$

where $\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ is the z-component of vorticity.

Operating equation (21) and (29) by $\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)$ and $\frac{\partial}{\partial z}$ respectively and adding to eliminate δp between equations (21) and (29), we get

$$\begin{aligned}\frac{1}{\epsilon}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial t}(\nabla^2 w)&-\frac{1}{k_1}\left(\upsilon-\upsilon'\frac{\partial}{\partial t}\right)\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\nabla^2 w+\\&+g\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\alpha\theta-\frac{mN}{\epsilon\rho_0}\frac{\partial}{\partial t}(\nabla^2 w)+\\&+\frac{{}_eH}{4\pi\rho_0}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial t}\nabla^2 h_z-\frac{2}{\epsilon}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\Omega\frac{\Omega\zeta}{\partial z},\end{aligned} \tag{30}$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

Operating equation (19) and (20) by $-\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$ respectively and adding, we get

$$\begin{aligned}\frac{1}{\epsilon}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{\Omega\zeta}{\partial t}&=-\frac{1}{k_1}\left(\upsilon-\upsilon'\frac{\partial}{\partial t}\right)\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\zeta-\\&-\frac{mN}{\epsilon\rho_0}\frac{\Omega\zeta}{\partial t}+\frac{2}{\epsilon}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\Omega\frac{\partial w}{\partial z}+\frac{\mu_e H}{4\pi\rho_0}\left(\frac{m}{K'}\frac{\partial}{\partial t}+1\right)\frac{\Omega\xi}{\partial t},\end{aligned} \tag{31}$$

where $\xi=\frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}$ is the z-component of current density.

Operating equations (25) and (26) by $-\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial x}$ respectively and adding, we get

$$\frac{1}{\epsilon}\frac{\Omega\xi}{\partial t} = H\frac{\Omega\xi}{\partial t} + \epsilon\eta\nabla^2\xi. \tag{32}$$

4 Dispersion relation

Analyzing the disturbances into normal modes, we assume that the perturbation quantities have x, y and t dependence of the form

$[w, s, \theta, \gamma, \zeta, h_z, \xi] = [W(z), S(z), \Theta(z), Z(z), \Gamma(z), K(z), X(z)]$

$$exp\left(ik_x x + ik_y y + nt\right), \tag{33}$$

where k_x and k_y are the wave numbers in the x and y directions, $k = \left({k_x}^2 + {k_y}^2\right)^{1/2}$ is the resultant wave number and n is the frequency of the harmonic disturbance, which is, in general, a complex constant.

Using expression (33) in (30)–(32), (27), (23), and (24) become

$$\begin{aligned}\frac{n}{\epsilon}\left[\frac{d^2}{dz^2} - k^2\right]W = -gk^2(\alpha\Theta - \alpha'\Gamma) - \frac{1}{k_1}\left(\upsilon + \upsilon' n\right)\left(\frac{d^2}{dz^2} - k^2\right)W \\ - \frac{mNn}{\epsilon\rho_0\left(\frac{m}{K'}n + 1\right)}\left(\frac{d^2}{dz^2} - k^2\right)W - \frac{2\Omega}{\epsilon}\frac{dZ}{dz} + \frac{\mu_e H}{4\pi\rho_0}\frac{d}{dz}\left(\frac{d^2}{dz^2} - k^2\right)K,\end{aligned} \tag{34}$$

$$\frac{n}{\epsilon}Z = -\frac{1}{k_1}\left(\upsilon + \upsilon' n\right) - \frac{mNn}{\epsilon\rho_0\left(\frac{m}{K'}n + 1\right)}Z + \frac{2\Omega}{\epsilon}\frac{dW}{dz} + \frac{{}_eH}{4\pi\rho_0}DX, \tag{35}$$

$$\epsilon nX = H\frac{dZ}{dz} + \epsilon\eta\left(\frac{d^2}{dz^2} - k^2\right)X, \tag{36}$$

$$\epsilon nK = H\frac{dW}{dz} + \epsilon\eta\left(\frac{d^2}{dz^2} - k^2\right)K, \tag{37}$$

$$(E + b\epsilon)\, n\Theta = \beta\left(W + bS\right) + \kappa\left(\frac{d^2}{dz^2} - k^2\right)\Theta, \tag{38}$$

$$\left(E' + b'\epsilon\right) n\Gamma = \beta'\left(W + b'S\right) + \kappa'\left(\frac{d^2}{dz^2} - k^2\right)\Gamma. \tag{39}$$

Equations (34)–(39) are in non dimensional form, become

$$\left[\frac{\sigma}{\epsilon}\left(1+\frac{M}{1+\tau_1\sigma}\right)+\frac{1+F\sigma}{P_l}\right]\left(D^2-a^2\right)W+\frac{ga^2d^2\alpha\Theta}{\upsilon}- \\ -\frac{ga^2d^2\alpha^{'}\Gamma}{\upsilon}+\frac{2\Omega d^3}{\epsilon\upsilon}DZ-\frac{\mu_e Hd}{4\pi\upsilon\rho_0}\left(D^2-a^2\right)DK=0, \tag{40}$$

$$\left[D^2-a^2-p_1\sigma\right]X=-\left(\frac{Hd}{\epsilon\eta}\right)DZ, \tag{41}$$

$$\left[D^2-a^2-p_2\sigma\right]K=-\left(\frac{Hd}{\epsilon\eta}\right)DW, \tag{42}$$

$$\left[\frac{\sigma}{\epsilon}\left(1+\frac{M}{1+\tau_1\sigma}\right)+\frac{1+F\sigma}{P_l}\right]Z=\left(\frac{2\Omega d^2}{\epsilon\upsilon}\right)DW+\frac{{}_eHd}{4\pi\upsilon\rho_0}DX, \tag{43}$$

$$\left[D^2-a^2-E_1p_1\sigma\right]\Theta=-\left(\frac{\beta d^2}{\kappa}\right)\left(\frac{B+\tau_1\sigma}{1+\tau_1\sigma}\right)W, \tag{44}$$

$$\left[D^2-a^2-E_1'p_1'\sigma\right]\Gamma=-\left(\frac{\beta^{'}d^2}{\kappa^{'}}\right)\left(\frac{B^{'}+\tau_1\sigma}{1+\tau_1\sigma}\right)W, \tag{45}$$

where we have put

$$a=kd,\sigma=\frac{nd^2}{\upsilon},\tau=\frac{m}{K'},\tau_1=\frac{\tau\upsilon}{d^2},M=\frac{mN}{\rho_0},$$

$E_1 = E + b\epsilon$, $B = b + 1$, $F = \frac{\upsilon^{'}}{d^2}$, $P_l = \frac{k_1}{d^2}$ is the dimensionless medium permeability, $p_1 = \frac{\upsilon}{\kappa}$ is the thermal Prandtl number, $p_1 = \frac{\upsilon}{\kappa'}$ is the Schmidt number, $p_2 = \frac{\upsilon}{\eta}$ is the magnetic Prandtl number and $D^* = d\frac{d}{dz}$ and the superscript * is suppressed.

Applying the operator $\left(D^2-a^2-p_2\sigma\right)$ to the equation (41) to eliminate X between equations (41) and (42), we get

$$\left\{\left[\frac{\sigma}{\epsilon}\left(1+\frac{M}{1+\tau_1\sigma}\right)+\frac{1+F\sigma}{P_l}\right]\left(D^2-a^2-p_2\sigma\right)+\frac{Q}{\epsilon}D^2\right\}W \\ =\frac{2\Omega d^2}{\upsilon}\left(D^2-a^2-p_2\sigma\right)DW. \tag{46}$$

Eliminating K, Θ and Z between equations (40)–(46), we obtain

$$\left[\frac{\sigma}{\epsilon}\left(1+\frac{M}{1+\tau_1\sigma}\right)+\frac{1+F\sigma}{P_l}\right](D^2-a^2)(D^2-a^2-E_1p_1\sigma)$$
$$(D^2-a^2-p_2\sigma)(D^2-a^2-E_1'p_1'\sigma)W-Ra^2\lambda\left(\frac{B+\tau_1\sigma}{1+\tau_1\sigma}\right)$$
$$(D^2-a^2-E_1'p_1'\sigma)(D^2-a^2-p_2\sigma)W+Sa^2\lambda\left(\frac{B'+\tau_1\sigma}{1+\tau_1\sigma}\right)$$
$$(D^2-a^2-E_1p_1\sigma)(D^2-a^2-p_2\sigma)W+\frac{Q}{\epsilon}$$
$$(D^2-a^2)(D^2-a^2-E_1'p_1'\sigma)(D^2-a^2-E_1p_1\sigma)W+$$
$$+\left[\frac{\frac{T_A}{\epsilon^2}(D^2-a^2-E_1p_1\sigma)(D^2-a^2-E_1'p_1'\sigma)(D^2-a^2-p_2\sigma)^2}{\left[\frac{\sigma}{\epsilon}\left(1+\frac{M}{1+\tau_1\sigma}\right)+\frac{1+F\sigma}{P_l}\right](D^2-a^2-p_2\sigma)+\frac{Q}{\epsilon}D^2}\right]D^2W=0, \tag{47}$$

where $R=\frac{g_0\alpha\beta d^4}{\upsilon\kappa}$, is the thermal Rayleigh number,
$S=\frac{g_0\alpha'\beta' d^4}{\upsilon\kappa'}$, is the analogous solute Rayleigh number,
$Q=\frac{\mu_e H^2 d^2}{4\pi\upsilon\rho_0\eta}$, is the Chandrasekhar number,
and $T_A=\left(\frac{2\Omega d^2}{\upsilon}\right)^2$, is the Taylor number.

Here we assume that the temperature at the boundaries is kept fixed, the fluid layer is confined between two boundaries and adjoining medium is electrically non conducting. The boundary conditions appropriate to the problem are [Chandrasekhar, (1981); Veronis, (1965)]

$$W=D^2 \quad W=DZ=\Gamma \quad =\Theta=0 \quad \text{at} \quad z=0 \;\text{ and }\; 1, \tag{48}$$

and the components of h are continuous. Since the components of the magnetic field are continuous and the tangential components are zero outside the fluid, we have

$$DK=0, \tag{49}$$

on the boundaries. Using the boundary conditions (48) and (49), we can show that all the even order derivatives of W must vanish for $z=0$ and $z=1$ and hence, the proper solution of equation (47) characterizing the lowest mode is

$$W=W_0\sin\pi z;\; W_0 \quad \text{is a constant.} \tag{50}$$

Substituting equation (50) in (47), we obtain the dispersion relation

$$\begin{aligned} R_1 x\lambda = & \left[\frac{i\sigma_1}{\epsilon}\left(1+\frac{M}{1+\tau_1\pi^2 i\sigma_1}\right)+\frac{1+F\pi^2 i\sigma_1}{P}\right](1+x) \\ & \left(1+x+E_1p_1i\sigma_1\right)\left(\frac{1+\tau_1\pi^2 i\sigma_1}{B+\tau_1\pi^2 i\sigma_1}\right) \\ & + \frac{S_1x\lambda\left(1+x+E_1p_1i\sigma_1\right)}{\left(D^2-a^2-E_1'p_1'\sigma\right)}\left(\frac{B'+\tau_1\pi^2 i\sigma_1}{B+\tau_1\pi^2 i\sigma_1}\right) \\ & + \frac{Q_1}{\epsilon}\frac{\left(1+x\right)\left(1+x+E_1p_1i\sigma_1\right)}{1+x+p_2i\sigma_1}\left(\frac{1+\tau_1\pi^2 i\sigma_1}{B+\tau_1\pi^2 i\sigma_1}\right) \\ & + \frac{\frac{T_{A_1}}{\epsilon^2}\left(1+x+E_1p_1i\sigma_1\right)}{\frac{i\sigma_1}{\epsilon}\left(1+\frac{M}{1+\tau_1\pi^2 i\sigma_1}\right)+\frac{1-F\pi^2 i\sigma_1}{P}}\left(\frac{1+\tau_1\pi^2 i\sigma_1}{B+\tau_1\pi^2 i\sigma_1}\right), \end{aligned} \tag{51}$$

where $R_1=\frac{R}{\pi^4}$, $S_1=\frac{S}{\pi^4}$, $T_{A_1}=\frac{T_A}{\pi^4}$, $x=\frac{a^2}{\pi^2}$, $i\sigma_1=\frac{\sigma}{\pi^2}$, $P=\pi^2 P_l$, $Q_1=\frac{Q}{\pi^4}$.

Equation (51) is required dispersion relation accounting for the effect of suspended particles, stable solute gradient, magnetic field, medium permeability, variable gravity field, rotation on thermosolutal instability of Rivlin-Ericksen elastico-viscous fluid in porous medium.

5 Stability of the system and oscillatory modes

Here we examine the possibility of oscillatory modes, if any, in Rivlin-Ericksen elastico-viscous fluid due to the presence of suspended particles, stable solute gradient, rotation, magnetic field, viscoelasticity and variable gravity field. Multiply equation (40) by W^* the complex conjugate of W, integrating over the range of z and making use of equations (41)–(44) with the help of boundary conditions (48) and (49), we obtain

$$\begin{aligned} & \left[\frac{\sigma}{\epsilon}\left(1+\frac{M}{1+\tau_1\sigma}\right)+\frac{1+F\sigma}{P_l}\right]I_1-\frac{\mu_e\epsilon\eta}{4\pi\upsilon\rho_0}\frac{1+\tau_1\sigma^*}{B+\tau_1\sigma^*}\left(I_2+p_2\sigma^*I_3\right) \\ & -\frac{\alpha a^2\lambda g_0\kappa}{\upsilon\beta}\frac{1+\tau_1\sigma^*}{B+\tau_1\sigma^*}\left(I_4+E_1p_1\sigma^*I_5\right) \\ & +d^2\left[\frac{\sigma^*}{\epsilon}\left(1+\frac{M}{1+\tau_1\sigma}\right)+\frac{1+F\sigma^*}{P_l}\right]I_6 \end{aligned}$$

$$
\begin{aligned}
&+\frac{\mu_e \epsilon \eta d^2}{4\pi \upsilon \rho_0}\frac{1+\tau_1\sigma^*}{B+\tau_1\sigma^*}\left(I_7+p_2\sigma^* I_8\right)\\
&+\frac{\alpha' a^2 \lambda g_0 \kappa'}{\upsilon \beta'}\frac{1+\tau_1\sigma^*}{B'+\tau_1\sigma^*}(I_9+E_1' p_1' \sigma^* I_{10})=0,
\end{aligned} \tag{52}
$$

where

$$I_1=\int_0^1\left(|DW|^2+a^2\,|W|^2\right)dz,$$

$$I_2=\int_0^1\left(\left|D^2K\right|^2+a^4|K|^2+2a^2|DK|^2\right)dz,$$

$$I_3=\int_0^1\left(|DK|^2+a^2|K|^2\right)dz,$$

$$I_4=\int_0^1\left(|D\Theta|^2+a^2|\Theta|^2\right)dz,$$

$$I_5=\int_0^1|\Theta|^2dz,$$

$$I_6=\int_0^1|Z|^2dz,$$

$$I_7=\int_0^1\left(|DX|^2+a^2|X|^2\right)dz,$$

$$I_8=\int_0^1|X|^2dz,$$

$$I_9=\int_0^1\left(|D\Gamma|^2+a^2|\Gamma|^2\right)dz,$$

$$I_{10}=\int_0^1|\Gamma|^2dz.$$

The integral parts I_1-I_{10} are all positive definite. Putting $\sigma=i\sigma_i$ in equation (52), where σ_i is real and equating the imaginary parts, we obtain

$$
\begin{aligned}
&\left[\frac{1}{\epsilon}\left(1+\frac{M}{1+\tau_1{}^2\sigma_i{}^2}\right)+\frac{F}{P_l}\right]\left(I_1-d^2I_4\right)\sigma_i\\
&-\frac{\mu_e\epsilon\eta}{4\pi\upsilon\rho_0}\left[\left(\frac{\tau_1(B-1)}{B^2+\tau_1{}^2\sigma_i{}^2}\right)I_2+\frac{B+\tau_1{}^2\sigma_i{}^2}{B^2+\tau_1{}^2\sigma_i{}^2}p_2I_3\right]\sigma_i\\
&+\frac{\alpha a^2\lambda g_0\kappa}{\upsilon\beta}\left[\left(\frac{\tau_1(B-1)}{B^2+\tau_1{}^2\sigma_i{}^2}\right)I_4+\frac{B+\tau_1{}^2\sigma_i{}^2}{B^2+\tau_1{}^2\sigma_i{}^2}E_1p_1I_5\right]\sigma_i\\
&+\frac{\alpha' a^2\lambda g_0\kappa'}{\upsilon\beta'}\left[\left(\frac{\tau_1(B'-1)}{B'^2+\tau_1{}^2\sigma_i{}^2}\right)I_9+\frac{B'+\tau_1{}^2\sigma_i{}^2}{B'^2+\tau_1{}^2\sigma_i{}^2}E_1'p_1'I_{10}\right]\sigma_i\\
&+\frac{\mu_e\epsilon\eta d^2}{4\pi\upsilon\rho_0}\left[\left(\frac{\tau_1(B-1)}{B^2+\tau_1{}^2\sigma_i{}^2}\right)I_6+\frac{B+\tau_1{}^2\sigma_i{}^2}{B^2+\tau_1{}^2\sigma_i{}^2}p_2I_8\right]\sigma_i=0
\end{aligned}
\tag{53}
$$

Equation (53) implies that $\sigma_i = 0$ or $\sigma_i \neq 0$ which mean that modes may be non oscillatory or oscillatory. The oscillatory modes introduced due to presence of rotation, stable solute gradient, magnetic field, suspended particles, viscoelasticity and variable gravity field.

6 The stationary convection

For stationary convection putting $\sigma = 0$ in equation (51) reduces it to

$$
R_1=\frac{1+x}{\lambda xB}\left[\frac{1+x}{P}+\frac{Q_1}{\epsilon}+\frac{T_{A_1}(1+x)P}{\{\epsilon(1+x)+Q_1P\}\epsilon}\right]+\frac{S_1B'}{B}, \tag{54}
$$

which expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters T_{A_1}, B, P, Q_1 and Rivlin-Ericksen elastico-viscous fluid behave like an ordinary Newtonian fluid since elastico-viscous parameter F vanishes with σ.

To study the effects of suspended particles, rotation and medium permeability, we examine the behavior of $\frac{dR_1}{dB}$, $\frac{dR_1}{dT_{A_1}}$, $\frac{dR_1}{dQ_1}$, $\frac{dR_1}{dS_1}$ and $\frac{dR_1}{dP}$ analytically.

Equation (54) yields

$$
\frac{dR_1}{dB}=-\frac{1+x}{\lambda xB^2}\left[\frac{1+x}{P}+\frac{Q_1}{\epsilon}+\frac{T_{A_1}(1+x)P}{\{\epsilon(1+x)+Q_1P\}\epsilon}\right]-\frac{S_1B'}{B^2}, \tag{55}
$$

which is negative implying thereby that the effect of suspended particles is to destabilize the system when the gravity increases upward from its value g_0

(i.e., $\lambda > 0$). This stabilizing effect is an agreement with the earlier work of Scanlon and Segel [13] and Rana and Kumar [11].

From equation (54), we get

$$\frac{dR_1}{dT_{A_1}} = \left(\frac{1+x}{\lambda x B}\right) \frac{(1+x)\,P}{\{\epsilon\,(1+x)+Q_1 P\}\,\epsilon}, \tag{56}$$

which shows that rotation has stabilizing effect on the system when gravity increases upwards from its value g_0 (i.e., $\lambda > 0$). This stabilizing effect is an agreement of the earlier work of Sharma and Rana [17], Rana and Kango [10].

From equation (54), we get

$$\frac{dR_1}{dQ_1} = \frac{1+x}{\lambda x B}\left[\frac{1}{\epsilon} - \frac{T_{A_1}\,(1+x)\,P^2}{\{\epsilon\,(1+x)+Q_1 P\}^2 \epsilon}\right], \tag{57}$$

which implies that magnetic field stabilizes the system, if

$$\{\epsilon\,(1+x)+Q_1 P\}^2 > T_{A_1}\,(1+x)\,P^2,$$

and destabilizes the system, if

$$\{\epsilon\,(1+x)+Q_1 P\}^2 < T_{A_1}\,(1+x)\,P^2,$$

when gravity increases upwards from its value g_0 (i.e., $\lambda > 0$).

In the absence of rotation, magnetic field has destabilizing effect on the system, when gravity increases upwards from its value g_0 (i.e., $\lambda > 0$). From equation (54), we get

$$\frac{dR_1}{dS_1} = \frac{B'}{B}, \tag{58}$$

which is positive implying thereby that the stable solute gradient has a stabilizing effect. This stabilizing effect is an agreement of the earlier work of Sharma and Rana [17],

It is evident from equation (54) that

$$\frac{dR_1}{dP} = -\frac{(1+x)^2}{\lambda x B}\left[\frac{1}{P^2} - \frac{T_{A_1}\,(1+x)}{\{\epsilon\,(1+x)+Q_1 P\}^2}\right], \tag{59}$$

From equation (58), we observe that medium permeability has destabilizing effect when $\{\epsilon\,(1+x)+Q_1 P\}^2 < T_{A_1}\,(1+x)\,P^2$ and medium permeability has a stabilizing effect when $\{\epsilon\,(1+x)+Q_1 P\}^2 > T_{A_1}\,(1+x)\,P^2$, when gravity increases upwards from its value g_0 (i.e. ,$\lambda > 0$).

In the absence of rotation and for constant gravity field $\frac{dR_1}{dP}$ is always negative implying thereby the destabilizing effect of medium permeability which is identical with the result as derived by Rana and Kumar [11], Rana and Kango [10].

The dispersion relation (54) is analyzed numerically. Graphs have been plotted by giving some numerical values to the parameters, to depict the stability characteristics.

In Fig. 1, Rayleigh number R_1 is plotted against suspended particles B for $\lambda = 2$, $T_{A_1} = 5$, $\epsilon = 0.5$, $P = 0.2$, $Q_1 = 10$, $S_1 = 10$, $B' = 2$ for fixed wave numbers $x = 0.2$, $x = 0.5$, and $x = 0.8$. For the wave numbers $x = 0.2, x = 0.5$, and $x = 0.8$, suspended particles have a destabilizing effect.

In Fig. 2, Rayleigh number R_1 is plotted against rotation T_{A_1} for $B = 3$, $\lambda = 2$, $\epsilon = 0.2$, $P = 0.2$, $Q_1 = 10$, $S_1 = 10$, $B' = 2$ for fixed wave numbers $x = 0.2, x = 0.5$, and $x = 0.8$. This shows that rotation has a stabilizing effect for fixed wave numbers $x = 0.2, x = 0.5$, and $x = 0.8$.

In Fig. 3, Rayleigh number R_1 is plotted magnetic field Q_1 for $B = 3$, $\lambda = 2$, $\epsilon = 0.2$, $T_{A_1} = 5$, $P = 0.2, S_1 = 10$, $B' = 2$ for fixed wave numbers $x = 0.2, x = 0.5$, and $x = 0.8$. This shows that magnetic field has a destabilizing effect for $Q_1 = 0.1$ to 1.5 and has a stabilizing effect for $Q_1 = 1.5$ to 10.

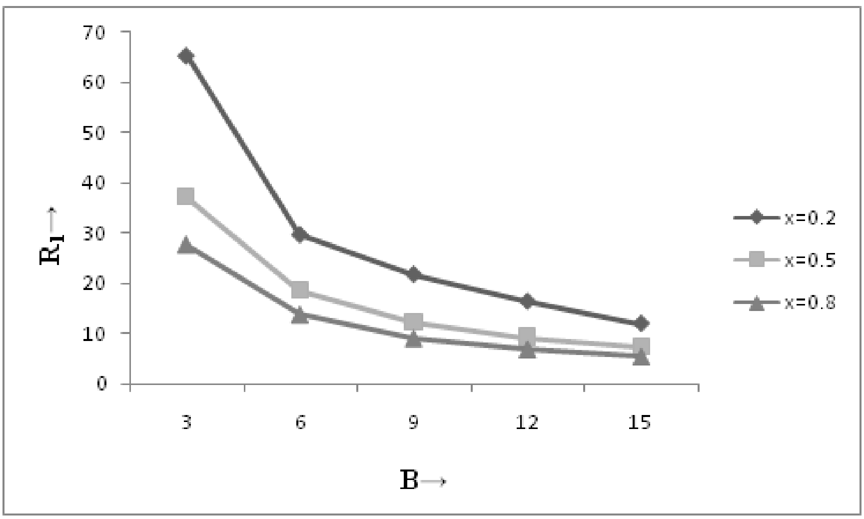


Figure 1: Variation of Rayleigh number R_1 with suspended particles B for $\lambda = 2$, $T_{A_1} = 5, Q_1 = 10$, $\epsilon = 0.2, P = 0.2, S_1 = 10$, $B' = 2$ for fixed wave numbers $x = 0.2, x = 0.5$, and $x = 0.8$.

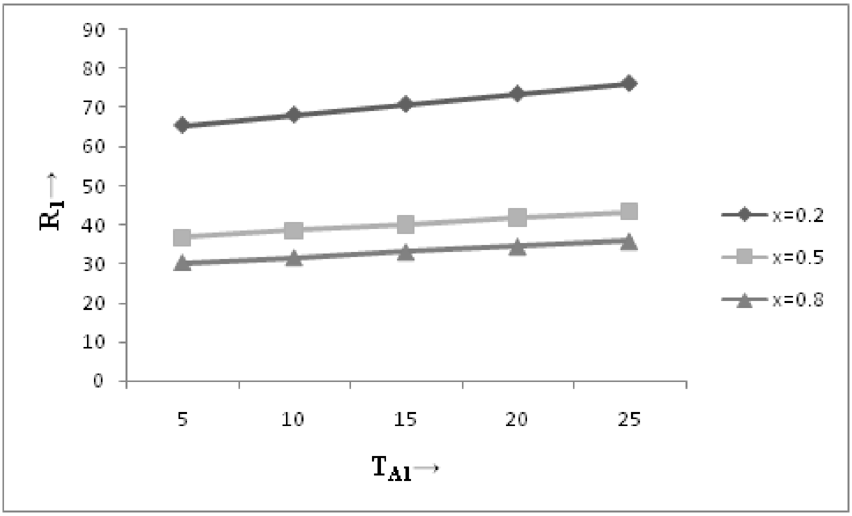


Figure 2: Variation of Rayleigh number R_1 with magnetic field S_1 for $B = 3$, $\lambda = 2$, $\epsilon = 0.2$, $P = 0.2$, $T_{A_1} = 5, Q_1 = 10$, for fixed wave numbers $x = 0.2, x = 0.5$, and $x = 0.8$.

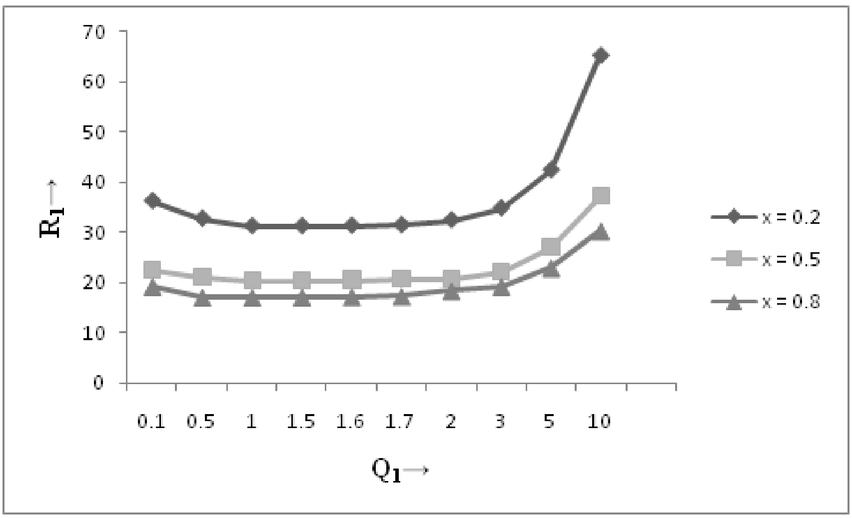


Figure 3: Variation of Rayleigh number R_1 with magnetic field Q_1 for $B = 3, \lambda = 2$, $\epsilon = 0.2$, $P = 0.2, T_{A_1} = 5$, $S_1 = 10, B' = 2$ for fixed wave numbers $x = 0.2, x = 0.5$, and $x = 0.8$.

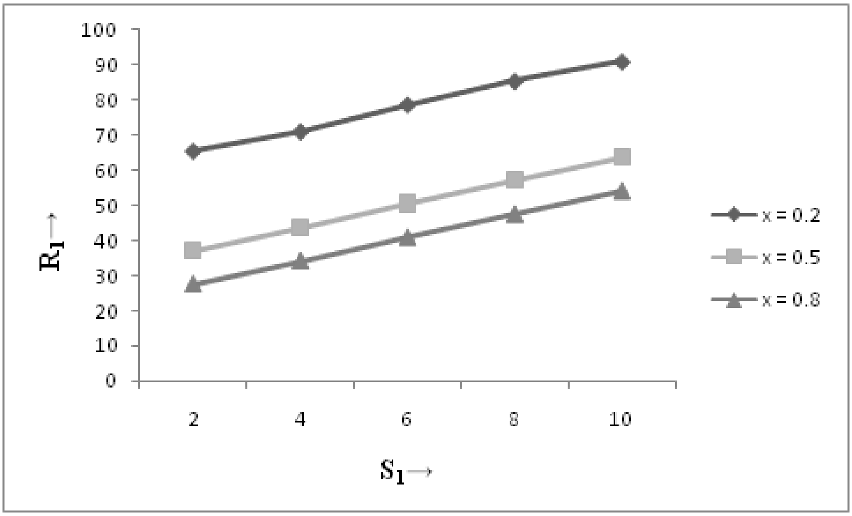


Figure 4: Variation of Rayleigh number R_1 with magnetic field S_1 for $B = 3, \lambda = 2$, $\epsilon = 0.2, P = 0.2, T_{A_1} = 5, Q_1 = 10$ for fixed wave numbers $x = 0.2, x = 0.5$, and $x = 0.8$.

In Fig. 4, Rayleigh number R_1 is plotted against stable solute gradient B' for $B = 3, \lambda = 2$, $\epsilon = 0.2, P = 0.2, Q_1 = 10$, $S_1 = 10$, for fixed wave numbers $x = 0.2, x = 0.5$, and $x = 0.8$. This shows that the stable solute gradient has a stabilizing effect for fixed wave numbers $x = 0.2, x = 0.5$ and $x = 0.8$.

In Fig. 5, Rayleigh number R_1 is plotted against medium permeability P for $B = 3, \lambda = 2$, $\epsilon = 0.2$, $T_{A_1} = 5$, $Q_1 = 2, S_1 = 10$, $B' = 2$ for fixed wave numbers $x = 0.2, x = 0.5$, and $x = 0.8$. This shows that medium permeability has a destabilizing effect for $P = 0.1$ to 0.8 and has a stabilizing effect for $P = 0.8$ to 2.0.

7 Conclusion

The thermosolutal instability of Rivlin-Ericksen elastico-viscous rotating fluid permeated with suspended particles and variable gravity field in porous medium in hydromagnetics has been investigated. For the stationary convection, it has been found that the rotation has stabilizing effect on the system as gravity increases upward from its value g_0 (i.e. for $\lambda > 0$). The stable solute gradient has stabilizing effect on the system and is independent of gravity field. The suspended particles are found to have destabilizing effect on the system

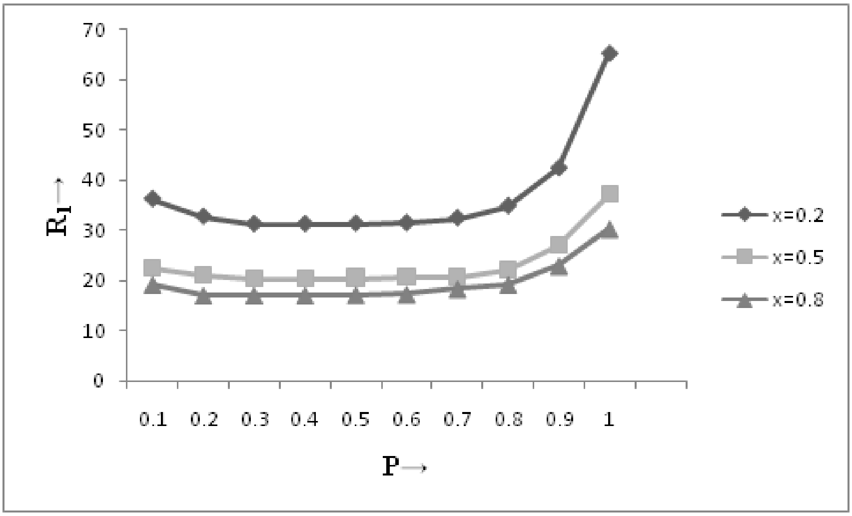


Figure 5: Variation of Rayleigh number R_1 with medium permeability P for $B = 3, \lambda = 2$, $Q_1 = 2$, $\epsilon = 0.2$, $T_{A_1} = 5, S_1 = 10, B' = 2$ for fixed wave numbers $x = 0.2, x = 0.5$ and $x = 0.8$.

as gravity increases upward from its value g_0 (i.e. for $\lambda > 0$) whereas the medium permeability has a stabilizing / destabilizing effect on the system for $\{\epsilon\,(1+x) + Q_1P\}^2 < T_{A_1}\,(1+x)\,P^2/\ \{\epsilon\,(1+x) + Q_1P\}^2 > T_{A_1}\,(1+x)\,P^2$, as gravity increases upward from its value g$_0$ (i.e. for $\lambda > 0$). The magnetic field has stabilizing destabilizing effecton the system for $\{\epsilon\,(1+x) + Q_1P\}^2 > T_{A_1}\,(1+x)\,P^2/\{\epsilon\,(1+x) + Q_1P\}^2 < T_{A_1}\,(1+x)\,P^2$, as gravity increases upward from its value g_0 (i.e. for $\lambda > 0$). The presence of rotation, gravity field, suspended particles and viscoelasticity introduces oscillatory modes. The effects of rotation, suspended particles and medium permeability on thermal instability have also been shown graphically.

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Nomenclature

$\mathbf{q}$	Velocity of fluid
q_d	Velocity of suspended particles
p	Pressure
$\mathbf{g}$	Gravitational acceleration vector
g	Gravitational acceleration
k_1	Medium permeability
T	Temperature
t	Time coordinate
c_f	Heat capacity of fluid
c_{pt}	Heat capacity of particles
mN	Mass of the particle per unit volume
k	Wave number of disturbance
k_x, k_y	Wave numbers in x and y directions
p_1	Thermal Prandtl number
P_l	Dimensionless medium permeability
Q	Magnetic field
T_A	Taylor number

Symbols

ϵ	Medium porosity
ρ	Fluid density
μ	Fluid viscosity
μ'	Fluid viscoelasticity
υ	Kinematic viscosity
υ'	Kinematic viscoelasticity
η	Particle radius
κ	Thermal diffusitivity
κ'	Solute diffusivity
α	Thermal coefficient of expansion
α'	Solvent coefficient of expansion
β	Adverse temperature gradient
β'	Solute gradient
Θ	Perturbation in temperature
n	Growth rate of the disturbance

δ Perturbation in respective physical quantity
ζ z-component of vorticity
ξ z-component of current density
Ω Rotation vector having components $(0, 0, \Omega)$
γ Perturbation in solute concentration
μ_e Magnetic permeability

References

[1] P. K. Bhatia, J. M. Steiner, Thermal instability of fluid layer in Hydromagnetics, *J. Math. Anal. Appl.* **41** (1973), 271-289.

[2] M. K. Brakke, Zone electrophoresis of dyes, proteins and viruses in density gradient columns of sucrose solutions, *Arch. Biochem. Biophys*, **55** (1955), 175–190.

[3] S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability*, Dover Publications, New York, 1981.

[4] A. Garg, R. K. Srivastava, K. K. Singh, Rectilinear oscillations of a sphere along its diameter in conducting dusty Rivlin-Ericksen fluid in the presence of magnetic field, *Proc. Nat. Acad. Sci. India*, **64** (1994), 355–372.

[5] E. R. Lapwood, Convection of a fluid in porous medium, *Math. Proc. Cambridge Phil. Soc.*, **44** (1948), 508–519.

[6] P. F. Linden, Salt fingers in a steady shear flow, *Geophysics Fluid Dynamics*, **6** (1974), 1–27.

[7] Lister C. R. B. Lister, On the thermal balance of a mid ocean ridge, *Geophysics, J. Roy. Astr. Soc.,* **26** (1972), 515–535.

[8] J. A. M. McDonnel, *Cosmic dust*, John Wiley and Sons, Toronto, 1978.

[9] G. K. Pradhan, P. C. Samal, Thermal instability of a fluid layer under variable body forces, *J. Math. Anal. Appl.*, **122** (1987), 487–498.

[10] G. C. Rana, S. K. Kango, Thermal instability of compressible Walters' (Model B') elastico-viscous rotating fluid a permeated with suspended dust particles in porous medium, *Advances in Applied Science Research*, **2** (2011), 586–597.

[11] G. C. Rana, S. Kumar, Thermal instability of a Rivlin-Ericksen elastico-viscous rotating fluid permeated with suspended particles and variable gravity field in porous medium, *Studia Geotechnica et Mechanica*, XXXII (2010), 39–54.

[12] R. S. Rivlin, J. L. Ericksen, Stress deformation relations for isotropic materials, *J. Rational Mech. Anal.,* **4** (1955), 323–334.

[13] J. W. Scanlon, L. A. Segel, Effect of suspended particles on onset of Be'nard convection, *Physics Fluids,* **16** (1973), 1573–78.

[14] R. C. Sharma, Effect of rotation on thermal instability of a viscoelastic fluid, *Acta Physica Hungarica*, **40** (1975), 11–17.

[15] R. C. Sharma, Sunil, Thermal instability of an Oldroydian viscoelastic fluid with suspended particles in hydromagnetics in porous medium, *J. of Polymer Plastic Technology and Engineering,* **33** (1994), 323–339.

[16] V. Sharma, G. C. Rana, Thermal instability of a Walters' ((Model B$'$) elastico-viscous fluid in the presence of variable gravity field and rotation in porous medium, *J. Non-Equilib. Thermodyn.*, **26** (2001), 31–40.

[17] V. Sharma, G. C. Rana, Thermosolutal instability of a Walters' ((Model B$'$) elastico-viscous rotating fluid in the presence of magnetic field and variable gravity field in porous medium, *Proc. Nat. Acad. Sci. India*, 73 (2003), 93–111.

[18] R. K. Srivastava, K. K. Singh, Unsteady flow of a dusty elastic-viscous Rivlin-Ericksen fluid through channels of different cross-sections in the presence of time-dependent pressure gradient, *Bull. Calcutta Math. Soc.*, **80** (1988), 286–308.

[19] H. Stomel, K. N. Fedorov, Small scale structure in temperature and salinity near Timor and Mindanao, *Tellus,* **19** (1967), 306–325.

[20] G. Veronis, On finite amplitude instability in thermohaline convection, *J. Marine Res.*, **23** (1967), 1–17.

[21] R. A. Wooding, Rayleigh instability of a thermal boundary layer in flow through a porous medium, *J. Fluid Mech.,* **9** (1960), 183–192.

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Some applications of differential subordination to certain subclass of p-valent meromorphic functions involving convolution

Abstract. By using the principle of differential subordination, we introduce subclass of p-valent meromorphic functions involving convolution and investigate various properties for this subclass. We also indicate relevant connections of the various results presented in this paper with those obtained in earlier works.

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1 Introduction

For any integer $m > -p$, let $\Sigma_{p,m}$ denote the class of all meromorphic functions f of the form

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1}$$

which are analytic and p-valent in the punctured disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \backslash \{0\}$. For convenience, we write $\Sigma_{p,-p+1} = \Sigma_p$. If f and g are analytic in

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U, we say that f is subordinate to g, written symbolically as, $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function g is univalent in U, we have the equivalence (see [10] and [11]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f \in \Sigma_{p,m}$, given by (1), and $g \in \Sigma_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \ \ (m > -p; p \in \mathbb{N}), \tag{2}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g * f)(z) \quad (m > -p; p \in \mathbb{N}). \tag{3}$$

For complex parameters

$$\alpha_1, \ldots, \alpha_q \text{ and } \beta_1, \ldots, \beta_s \ (\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\};\ j = 1, 2, \ldots, s),$$

we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by (see, for example, [14, p. 19])

$$\begin{gathered} {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} . \frac{z^k}{k!} \\ (q \le s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in U), \end{gathered} \tag{4}$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}), \\ \theta(\theta - 1) \ldots (\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} \tag{5}$$

Corresponding to the function $h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$, defined by

$$h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-p}\, {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z), \tag{6}$$

we consider a linear operator

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) : \Sigma_p \to \Sigma_p,$$

which is defined by the following Hadamard product (or convolution):

$$H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z) = h_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) * f(z). \quad (7)$$

We observe that, for a function $f(z)$ of the form (1), we have

$$H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z) = z^{-p} + \sum_{k=m}^{\infty} \Gamma_{p,q,s}(\alpha_1)\, a_k z^k, \quad (8)$$

where

$$\Gamma_{p,q,s}(\alpha_1) = \frac{(\alpha_1)_{k+p}\dots(\alpha_q)_{k+p}}{(\beta_1)_{k+p}\dots(\beta_s)_{k+p}(k+p)!}. \quad (9)$$

If, for convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s),$$

then one can easily verify from the definition (7) that (see [8])

$$z(H_{p,q,s}(\alpha_1)f(z))' = \alpha_1 H_{p,q,s}(\alpha_1+1)f(z) - (\alpha_1+p)H_{p,q,s}(\alpha_1)f(z). \quad (10)$$

For $m = -p+1\,(p\in\mathbb{N})$, the linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [8] and Aouf [2].

In particular, for $q = 2, s = 1, \alpha_1 > 0, \beta_1 > 0$ and $\alpha_2 = 1$, we obtain the linear operator

$$H_p(\alpha_1, 1; \beta_1)\, f(z) = \ell_p(\alpha_1,\beta_1)f(z) \quad (f\in\Sigma_p),$$

which was introduced and studied by Liu and Srivastava [7].

We note that, for any integer $n > -p$ and $f\in\Sigma_p$,

$$H_{p,2,1}(n+p,1;1)f(z) = D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z),$$

where D^{n+p-1} is the differential operator studied by Uralegaddi and Somanatha [16] and Aouf [1].

For functions $f, g \in \Sigma_{p,m}$, we define the linear operator $\mathcal{D}^n_{\lambda,p}(f*g):\Sigma_{p,m}\longrightarrow \Sigma_{p,m}$ $(\lambda\ge 0;\ p\in\mathbb{N};\ n\in\mathbb{N}_0)$ by

$$\mathcal{D}^0_{\lambda,p}(f*g)(z) = (f*g)(z), \quad (11)$$

$$\begin{aligned}\mathcal{D}^1_{\lambda,p}(f*g)(z) &= \mathcal{D}_{\lambda,p}(f*g)(z)\\ &= (1-\lambda)(f*g)(z) + \lambda z^{-p}\,(z^{p+1}(f*g)(z))' \quad (12)\\ &= z^{-p} + \sum_{k=m}^{\infty}[1+\lambda(k+p)]a_k b_k z^k\ (\lambda\ge 0;\ p\in\mathbb{N}),\end{aligned}$$

$$\begin{aligned}\mathcal{D}^2_{\lambda,p}(f*g)(z) &= \mathcal{D}(\mathcal{D}^1_{\lambda,p}(f*g))(z)\\ &= (1-\lambda)\mathcal{D}^1_{\lambda,p}(f*g)(z)+\lambda z^{-p}\ (z^{p+1}\mathcal{D}^1_{\lambda,p}(f*g)(z))'\\ &= z^{-p}+\sum_{k=m}^{\infty}[1+\lambda(k+p)]^2 a_k b_k z^k\ (\lambda\geq 0;\ p\in\mathbb{N}),\end{aligned}$$

and (in general)

$$\begin{aligned}\mathcal{D}^n_{\lambda,p}(f*g)(z) &= \mathcal{D}(\mathcal{D}^{n-1}_{\lambda,p}(f*g)(z))\\ &= z^{-p}+\sum_{k=m}^{\infty}[1+\lambda(k+p)]^n a_k b_k z^k\ (\lambda\geq 0).\end{aligned} \tag{13}$$

From (13) it is easy to verify that:

$$z(\mathcal{D}^n_{\lambda,p}(f*g)(z))' = \frac{1}{\lambda}\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z)-(p+\frac{1}{\lambda})\mathcal{D}^n_{\lambda,p}(f*g)(z)\ (\lambda>0). \tag{14}$$

For $m=0$ the linear operator $\mathcal{D}^n_{\lambda,p}(f*g)$ was introduced by Aouf et al. [4].

Making use of the principle of differential subordination as well as the linear operator $D^n_{\lambda,p}(f*g)$, we now introduce a subclass of the function class $\Sigma_{p,m}$ as follows:

For fixed parameters A and B $(-1\leq B<A\leq 1)$, we say that a function $f\in\Sigma_{p,m}$ is in the class $\Sigma^n_{\lambda,p,m}(f*g;A,B)$, if it satisfies the following subordination condition:

$$-\frac{z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'}{p}\prec\frac{1+Az}{1+Bz}. \tag{15}$$

In view of the definition of subordination, (15) is equivalent to the following condition:

$$\left|\frac{z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'+p}{Bz^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'+pA}\right|<1\ \ (z\in U).$$

For convenience, we write

$$\Sigma^n_{\lambda,p}\left(f*g;1-\frac{2\zeta}{p},-1\right)=\Sigma^n_{\lambda,p}(f*g;\zeta),$$

where $\Sigma^n_{\lambda,p}(f*g;\zeta)$ denotes the class of functions $f(z)\in\Sigma_{p,m}$ satisfying the following inequality:

$$\Re\left\{-z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'\right\}>\zeta\quad(0\leq\zeta<p;z\in U).$$

We note that:

(i) For $b_k = \lambda = 1$ in (15), the class $\Sigma^n_{\lambda,p,m}(f * g; A, B)$ reduces to the class $\Sigma^n_{p,m}(A, B)$ introduced and studied by Srivastava and Patel [15];

(ii) For $b_k = \Gamma_{p,q,s}(\alpha_1)$, where $\Gamma_{p,q,s}(\alpha_1)$ is given by (9), and $n = 0$ in (15), we have $\Sigma^n_{\lambda,p}(f*g; A, B) = \Sigma^m_{p,q,s}(\alpha_1, A, B)$, where the class $\Sigma^m_{p,q,s}(\alpha_1, A, B)$ introduced and studied by Aouf [3].

(iii) For $q = 2$, $s = 1$, $\alpha_1 = a > 0$, $\beta_1 = c > 0$ and $\alpha_2 = 1$, we have $\Sigma^m_{p,q,s}(\alpha_1, A, B) = \Sigma_{a,c}(p; m, A, B)$, where the class $\Sigma_{a,c}(p; m, A, B)$ was studied by Patel and Cho [13].

2 Preliminary lemmas

In order to establish our main results, we need the following lemmas.

Lemma 1 [6]. *Let the function* h *be analytic and convex (univalent) in* U *with* $h(0) = 1$. *Suppose also that the function* φ *given by*

$$\varphi(z) = 1 + c_{p+m}z^{p+m} + c_{p+m+1}z^{p+m+1} + \cdots \tag{16}$$

in analytic in U. *If*

$$\varphi(z) + \frac{z\varphi'(z)}{\gamma} \prec h(z) \quad (\Re(\gamma) \geq 0; \gamma \neq 0), \tag{17}$$

then

$$\varphi(z) \prec \psi(z) = \frac{\gamma}{p+m} z^{\frac{-\gamma}{p+m}} \int_0^z t^{\frac{\gamma}{p+m}-1} h(t)dt \prec h(z),$$

and ψ *is the best dominant.*

For real or complex numbers a, b and c $(c \notin \mathbb{Z}_0^-)$, the Gaussian hypergeometric function is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \cdots. \tag{18}$$

We note that the above series converges absolutely for $z \in U$ and hence represents an analytic function in U (see, for details [17, Chapter 14]).

Each of the identities (asserted by Lemma 2 below) is well-known (cf., e.g., [17, Chapter 14]).

Lemma 2 [17, Chapter 14]. *For real or complex parameters* a, b *and* c ($c \notin \mathbb{Z}_0^-$),

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a,b;c;z) \qquad (\Re(c) > \Re(b) > 0), \tag{19}$$

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1(a, c-b; c; \frac{z}{z-1}), \tag{20}$$

$$ {}_2F_1(a,b;c;z) = {}_2F_1(a, b-1; c; z) + \frac{az}{c}\, {}_2F_1(a+1, b; c+1; z). \tag{21}$$

3 Main results

Unless otherwise mentioned, we assume throughout this paper that $\lambda, \mu > 0, m > -p,\ p \in \mathbb{N}, n \in \mathbb{N}_0$ and g is given by (2).

Theorem 1 *Let the function* f *defined by* (1) *satisfying the following subordination condition:*

$$-\frac{(1-\mu)z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))' + \mu z^{p+1}(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z))'}{p} \prec \frac{1+Az}{1+Bz}.$$

Then

$$-\frac{z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'}{p} \prec \mathcal{G}(z) \prec \frac{1+Az}{1+Bz}, \tag{22}$$

where the function $\mathcal{G}$ *given by*

$$\mathcal{G}(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}\, {}_2F_1(1,1;\frac{1}{\lambda\mu(p+m)}+1;\frac{Bz}{1+Bz}) & (B \neq 0) \\ 1 + \frac{A}{\lambda\mu(p+m)+1} z & (B = 0) \end{cases}$$

is the best dominant of (22). Furthermore,

$$\Re\left\{-\frac{z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'}{p}\right\} > \xi \quad (z \in U), \tag{23}$$

where

$$\xi = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}\, {}_2F_1(1,1;\frac{1}{\lambda\mu(p+m)}+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{A}{\lambda\mu(p+m)+1} & (B = 0)\ . \end{cases}$$

The estimate in (23) is the best possible.

Proof. Consider the function φ defined by

$$\varphi(z) = -\frac{z^{p+1}\left(\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)'}{p} \quad (z \in \mathbb{U})\,. \tag{24}$$

Then φ is of the form (16) and is analytic in $\mathbb{U}$. Differentiating (24) with respect to z and using (14), we obtain

$$-\frac{(1-\mu)z^{p+1}\left(\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)' + \mu z^{p+1}\left(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z)\right)'}{p}$$
$$= \varphi(z) + \lambda\mu z\varphi'(z) \prec \frac{1+Az}{1+Bz}\,.$$

Now, by using Lemma 1 for $\beta = \frac{1}{\lambda\mu}$, we obtain

$$-\frac{z^{p+1}\left(\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)'}{p} \prec \mathcal{G}(z) = \frac{1}{\lambda\mu(p+m)} z^{-\frac{1}{\lambda\mu(p+m)}} \int_0^z t^{\frac{1}{\lambda\mu(p+m)}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

$$= \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}\,{}_2F_1(1,1;\frac{1}{\lambda\mu(p+m)}+1;\frac{Bz}{1+Bz}) & (B \neq 0) \\ 1 + \frac{A}{\lambda\mu(p+m)+1}z & (B = 0), \end{cases}$$

by change of variables followed by the use of the identities (19), (20) and (21) (with $a = 1$, $c = b+1$, $b = \frac{1}{\lambda\mu(p+m)}$). This proves the assertion (22) of Theorem 1.

Next, in order to prove the assertion (23) of Theorem 1, it suffices to show that

$$\inf_{|z|<1} \{\Re(\mathcal{G}(z))\} = \mathcal{G}(-1)\,. \tag{25}$$

Indeed we have, for $|z| \le r < 1$,

$$\Re\left(\frac{1+Az}{1+Bz}\right) \ge \frac{1-Ar}{1-Br}\,.$$

Upon setting

$$g(\zeta, z) = \frac{1+A\zeta z}{1+B\zeta z} \text{ and } d\nu(\zeta) = \frac{1}{\lambda\mu(p+m)}\zeta^{\frac{1}{\lambda\mu(p+m)}-1}d\zeta\ (0 \le \zeta \le 1),$$

which is a positive measure on the closed interval $[0,1]$, we get

$$\mathcal{G}(z) = \int_0^1 g(\zeta, z)\,d\nu(\zeta),$$

so that

$$\Re\{\mathcal{G}(z)\} \geq \int_0^1 \left(\frac{1-A\zeta r}{1-B\zeta r}\right) d\nu(\zeta) = \mathcal{G}(-r) \quad (|z| \leq r < 1)\,.$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (23) of Theorem 1.

Finally, the estimate in (23) is the best possible as the function $\mathcal{G}$ is the best dominant of (22). □

Taking $\mu = 1$ in Theorem 1, we obtain the following corollary.

Corollary 1 *The following inclusion property holds for the function class* $\Sigma^n_{\lambda,p}(f*g;A,B)$:

$$\Sigma^{n+1}_{\lambda,p,m}(f*g;A,B) \subset \Sigma^{n}_{\lambda,p,m}(f*g;\beta) \subset \Sigma^{n}_{\lambda,p,m}(f*g;A,B),$$

where

$$\beta = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}\,{}_2F_1(1,1;\frac{1}{\lambda(p+m)}+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{A}{\lambda(p+m)+1} & (B = 0). \end{cases}$$

The result is the best possible.

Taking $\mu = 1$, $A = 1 - \frac{2\sigma}{p}$ $(0 \leq \sigma < p)$ and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 2 *The following inclusion property holds for the function class* $\Sigma^n_{\lambda,p,m}(f*g;\sigma)$:

$$\Sigma^{n+1}_{\lambda,p,m}(f*g;\sigma) \subset \Sigma^{n}_{\lambda,p,m}(f*g;\beta)) \subset \Sigma^{n}_{\lambda,p,m}(f*g;\sigma),$$

where

$$\beta = \sigma + (p-\sigma)\left\{{}_2F_1(1,1;\frac{1}{\lambda\,(p+m)}+1;\frac{1}{2}) - 1\right\}.$$

The result is the best possible.

Theorem 2 *If* $f \in \Sigma^n_{\lambda,p,m}(f*g;\theta)$ $(0 \leq \theta < p)$, *then*

$$\Re\left\{-z^{p+1}\left[(1-\mu)(\mathcal{D}^n_{\lambda,p}(f*g)(z))' + \mu(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z))'\right]\right\} > \theta \quad (|z| < R), \quad (26)$$

where

$$R = \left\{\sqrt{1+\lambda^2\mu^2(p+m)^2} - \lambda\mu(p+m)\right\}^{\frac{1}{p+m}}.$$

The result is the best possible.

Proof. Since $f \in \Sigma^{n}_{\lambda,p}(f * g;\theta)$, we write

$$-z^{p+1}(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))' = \theta + (p-\theta)u(z) \quad (z \in U). \tag{27}$$

Then, clearly, u is of the form (16), is analytic in U, and has a positive real part in U. Differentiating (27) with respect to z and using (14), we obtain

$$-\frac{z^{p+1}\left[(1-\mu)(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))' + \mu(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z))'\right] + \theta}{p-\theta} = u(z) + \lambda\mu z u'(z). \tag{28}$$

Now, by applying the well-known estimate [5]

$$\frac{|zu'(z)|}{\Re\{u(z)\}} \le \frac{2(p+m)r^{p+m}}{1-r^{2(p+m)}} \quad (|z| = r < 1)$$

in (28), we obtain

$$\begin{aligned}&\Re\left\{-\frac{z^{p+1}\left[(1-\mu)(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))' + \mu(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z))'\right] + \theta}{p-\theta}\right\}\\ &\ge \Re\{u(z)\}.\left(1 - \frac{2\lambda\mu(p+m)r^{p+m}}{(1-r^{2(p+m)})}\right).\end{aligned} \tag{29}$$

It is easily seen that the right-hand side of (29) is positive provided that $r < R$, where R is given as in Theorem 2. This proves the assertion (26) of Theorem 2.

In order to show that the bound R is the best possible, we consider the function $f \in \Sigma_{p,m}$ defined by

$$-z^{p+1}(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))' = \theta + (p-\theta)\frac{1+z^{p+m}}{1-z^{p+m}} \quad (0 \le \theta < p; p \in \mathbb{N}; z \in U).$$

Noting that

$$\begin{aligned}&-\frac{z^{p+1}\left[(1-\mu)(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))' + \mu(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z))'\right] + \theta}{p-\theta}\\ &= \frac{1-z^{2(p+m)} + 2\lambda\mu(p+m)z^{p+m}}{\alpha_1(1-z^{p+m})^2} = 0\end{aligned}$$

for $z = R^{\frac{1}{p+m}} \exp\left(\frac{i\pi}{p+m}\right)$, we complete the proof of Theorem 2. □

Putting $\mu = 1$ in Theorem 2, we obtain the following result.

Corollary 3 *If* $f \in \Sigma^{n}_{\lambda,p,m}(f * g; \theta)$ $(0 \leq \theta < p; p \in \mathbb{N})$, *then* f *satisfies the condition of* $\Sigma^{n+1}_{\lambda,p,m}(f * g; \theta)$ *for* $|z| < R^*$, *where*

$$R^* = \left\{ \sqrt{1 + \lambda^2 (p+m)^2} - \lambda(p+m) \right\}^{\frac{1}{p+m}} .$$

The result is the best possible.

Theorem 3 *Let* $f \in \Sigma^{n}_{\lambda,p,m}(f * g; A, B)$ *and let*

$$F_{\delta,p}(f)(z) = \frac{\delta}{z^{\delta+p}} \int_0^z t^{\delta+p-1} f(t)dt \quad (\delta > 0; z \in U) . \tag{30}$$

Then

$$-\frac{z^{p+1}(\mathcal{D}^{n}_{\lambda,p}(F_{\delta,p}(f) * g)(z))'}{p} \prec \Phi(z) \prec \frac{1 + Az}{1 + Bz}, \tag{31}$$

where the function Φ *given by*

$$\Phi(z) = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 + Bz)^{-1} {}_2F_1(1, 1; \frac{\delta}{p+m} + 1; \frac{Bz}{Bz+1}) & (B \neq 0) \\ 1 + \frac{\delta}{\delta+p+m} Az & (B = 0), \end{cases}$$

is the best dominant of (31). Furthermore,

$$\Re \left\{ -\frac{z^{p+1}(\mathcal{D}^{n}_{\lambda,p}(F_{\delta,p}(f) * g)(z))'}{p} \right\} > \xi^* \quad (z \in U), \tag{32}$$

where

$$\xi^* = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 - B)^{-1} {}_2F_1(1, 1; \frac{\delta}{p+m} + 1; \frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{\delta}{\delta+p+m} A & (B = 0) . \end{cases}$$

The result is the best possible.

Proof. Defining the function φ by

$$\varphi(z) = -\frac{z^{p+1}(\mathcal{D}^{n}_{\lambda,p}(F_{\delta,p}(f) * g)(z))'}{p} \quad (z \in U), \tag{33}$$

we note that φ is of the form (16) and is analytic in U. Using the following operator identity:

$$z(\mathcal{D}^{n}_{\lambda,p}(F_{\delta,p}(f) * g)(z))' = \delta \mathcal{D}^{n}_{\lambda,p}(f * g)(z) - (\delta + p)\mathcal{D}^{n}_{\lambda,p}(F_{\delta,p}(f) * g)(z) \tag{34}$$

in (33) and differentiating the resulting equation with respect to z, we find that

$$-\frac{z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'}{p} = \varphi(z) + \frac{z\varphi'(z)}{\delta} \prec \frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 3 follows by employing the techniques that we used in proving Theorem 1 above. □

Remark 1 *By observing that*

$$z^{p+1}(\mathcal{D}^n_{\lambda,p}(F_{\delta,p}(f)*g)(z))' = \frac{\delta}{z^\delta}\int_0^z t^{\delta+p}(\mathcal{D}^n_{\lambda,p}(f*g)(t))'dt \quad (f \in \Sigma_{p,m}; z \in \mathbb{U}). \tag{35}$$

the following statement holds. If $\delta > 0$ and $f \in \Sigma^n_{\lambda,p,m}(f*g;A,B)$, then

$$\Re\left\{-\frac{\delta}{pz^\delta}\int_0^z t^{\delta+p}(\mathcal{D}^n_{\lambda,p}(f*g)(t))'dt\right\} > \xi^* \quad (z \in \mathbb{U}),$$

ξ^* is given as in Theorem 3.

In view of (35), Theorem 3 for $A = 1 - \frac{2\theta}{p}$ $(0 \le \theta < p; p \in \mathbb{N})$ and $B = -1$ yields.

Corollary 4 *If $\delta > 0$ and if $f \in \Sigma_{p,m}$ satisfies the following inequality*

$$\Re\left\{-z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'\right\} > \theta \quad (0 \le \theta < p; p \in \mathbb{N}; z \in \mathbb{U}),$$

then

$$\Re\left\{\frac{-\delta}{z^\delta}\int_0^z (\mathcal{D}^n_{\lambda,p}(f*g)(t))'dt\right\}$$
$$> \theta + (p-\theta)\left[{}_2F_1\left(1,1;\frac{\delta}{p+m}+1;\frac{1}{2}\right) - 1\right] \ (z \in \mathbb{U}).$$

The result is the best possible.

Theorem 4 *Let $f \in \Sigma_{p,m}$. Suppose also that $h \in \Sigma_{p,m}$ satisfies the following inequality:*

$$\Re\left\{z^p(\mathcal{D}^n_{\lambda,p}(h*g)(z))\right\} > 0 \quad (z \in \mathbb{U}).$$

If

$$\left|\frac{\mathcal{D}^n_{\lambda,p}(f*g)(z)}{\mathcal{D}^n_{\lambda,p}(h*g)(z)}-1\right|<1 \quad (z\in U),$$

then

$$\Re\left\{-\frac{z(\mathcal{D}^n_{\lambda,p}(f*g)(z))'}{\mathcal{D}^n_{\lambda,p}(f*g)(z)}\right\}>0 \quad (|z|<R_0),$$

where

$$R_0=\left[\frac{\sqrt{9(p+m)^2+4p(2p+m)}-3(p+m)}{2(2p+m)}\right]^{\frac{1}{p+m}}.$$

Proof. Letting

$$w(z)=\frac{\mathcal{D}^n_{\lambda,p}(f*g)(z)}{\mathcal{D}^n_{\lambda,p}(h*g)(z)}-1=t_{p+m}z^{p+m}+t_{p+m+1}z^{p+m+1}+\cdots, \tag{36}$$

we note that w is analytic in U, with $w(0)=0$ and $|w(z)|\leq|z|^{p+m}$ $(z\in U)$. Then, by applying the familiar Schwarz's lemma [12], we obtain

$$w(z)=z^{p+m}\Psi(z),$$

where the functions Ψ is analytic in U and $|\Psi(z)|\leq 1\,(z\in U)$. Therefore, (36) leads us to

$$\mathcal{D}^n_{\lambda,p}(f*g)(z)=\mathcal{D}^n_{\lambda,p}(h*g)(z)\,\left(1+z^{p+m}\Psi(z)\right) \quad (z\in U). \tag{37}$$

Differentiating (37) logarithmically with respect to z, we obtain

$$\frac{z(\mathcal{D}^n_{\lambda,p}(f*g)(z))'}{\mathcal{D}^n_{\lambda,p}(f*g)(z)}=\frac{z(\mathcal{D}^n_{\lambda,p}(h*g)(z))'}{\mathcal{D}^n_{\lambda,p}(h*g)(z)}+\frac{z^{p+m}\left\{(p+m)\Psi(z)+z\Psi'(z)\right\}}{1+z^{p+m}\Psi(z)}. \tag{38}$$

Putting $\varphi(z)=z^p\mathcal{D}^n_{\lambda,p}(h*g)(z)$, we see that the function φ is of the form (16), is analytic in U, $\Re\{\varphi(z)\}>0\,(z\in U)$ and

$$\frac{z(\mathcal{D}^n_{\lambda,p}(h*g)(z))'}{\mathcal{D}^n_{\lambda,p}(h*g)(z)}=\frac{z\varphi'(z)}{\varphi(z)}-p,$$

so that we find from (38) that

$$\Re\left\{-\frac{z(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))'}{\mathcal{D}^{n}_{\lambda,p}(f*g)(z)}\right\} \geq p-\left|\frac{z\varphi'(z)}{\varphi(z)}\right|-\left|\frac{z^{p+m}\left\{(p+m)\Psi(z)+z\Psi'(z)\right\}}{1+z^{p+m}\Psi(z)}\right| \quad (z\in\mathbb{U}). \tag{39}$$

Now, by using the following known estimates [9]

$$\left|\frac{\varphi'(z)}{\varphi(z)}\right| \leq \frac{2(p+m)r^{p+m-1}}{1-r^{2(p+m)}} \quad (|z|=r<1)$$

and

$$\left|\frac{(p+m)\Psi(z)+z\Psi'(z)}{1+z^{p+m}\Psi(z)}\right| \leq \frac{(p+m)}{1-r^{p+m}} \quad (|z|=r<1)$$

in (39), we obtain

$$\Re\left\{-\frac{z(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))'}{\mathcal{D}^{n}_{\lambda,p}(f*g)(z)}\right\} \geq \frac{p-3(p+m)r^{p+m}-(2p+m)r^{2(p+m)}}{1-r^{2(p+m)}} \quad (|z|=r<1),$$

which is certainly positive, provided that $r<R_0$, R_0 being given as in Theorem 4. □

Theorem 5 *If $f\in\Sigma_{p,m}$ satisfies the following subordination condition*

$$(1-\mu)z^p\mathcal{D}^{n}_{\lambda,p}(f*g)(z)+\mu z^p\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z)\prec\frac{1+Az}{1+Bz},$$

then

$$\Re\left\{z^p\mathcal{D}^{n}_{\lambda,p}(f*g)(z)\right\}^{\frac{1}{d}}>\xi^{\frac{1}{d}} \quad (d\in\mathbb{N};z\in\mathbb{U}),$$

where ξ is given as in Theorem 1. The result is the best possible.

Proof. Defining the function φ by

$$\varphi(z)=z^p\mathcal{D}^{n}_{\lambda,p}(f*g)(z) \quad (f\in\Sigma_{p,m};z\in\mathbb{U}), \tag{40}$$

we see that the function φ is of the form (16) and is analytic in $\mathbb{U}$. Differentiating (40) with respect to z and using the identity (14), we obtain

$$(1-\mu)z^p\mathcal{D}^{n}_{\lambda,p}(f*g)(z)+\mu z^p\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z)=\varphi(z)+\lambda\mu z\varphi'(z)\prec\frac{1+Az}{1+Bz}.$$

Now, by following the lines of the proof of Theorem 1 mutatis mutandis, and using the elementary inequality:

$$\Re\left(w^{\frac{1}{d}}\right) \geq (\Re w)^{\frac{1}{d}} \quad (\Re(w) > 0; d \in \mathbb{N}),$$

we arrive at the result asserted by Theorem 5. □

Remark 2 (i) *Taking* $b_k = \lambda = 1$ *in the above results, we obtain the results obtained by Srivastava and Patel* [15]*;*

(ii) *Taking* $b_k = \Gamma_{p,q,s}(\alpha_1)$, *where* $\Gamma_{p,q,s}(\alpha_1)$ *is given by* (9)*, and* $n = 0$ *in the above results, we obtain the results obtained by Aouf* [3].

References

[1] M. K. Aouf, New criteria for multivalent meromorphic starlike functions of order alpha, *Proc. Japan Acad. Ser. A Math. Sci.*, **69** (1993), 66–70.

[2] M. K. Aouf, Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function, *Comput. Math. Appl.*, **55** (3) (2008), 494–509.

[3] M. K. Aouf, Applications of differential subodination to certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function, *Indian J. Math.*, **52** (2) (2010), 357–380.

[4] M. K. Aouf, A. Shamandy, A. O. Mostafa, S. M. Madian, Properties of some families of meromorphic p-valent functions involving certain differential operator, *Acta Univ. Apulensis Math. Inform.*, **20** (2009), 7–16.

[5] S. D. Bernardi, New distortion theorems for functions of positive real part and applications to the partial sums of univalent convex functions, *Proc. Amer. Math. Soc.*, **45** (1) (1974), 113–118.

[6] D. J. Hallenbeck, St. Ruscheweyh, Subordination by convex functions, *Proc. Amer. Math. Soc.*, **52** (1975), 191–195.

[7] J. -L. Liu, H. M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, *J. Math. Anal. Appl.*, **259** (2000), 566–581.

[8] J. -L. Liu, H. M. Srivastava, Classes of meromorphically multivalent functions associated with the generalized hypergeometric function, *Math. Comput. Modelling*, **39** (2004), 21–34.

[9] T. H. MaGregor, Radius of univalence of certain analytic functions, *Proc. Amer. Math. Soc.*, **14** (1963), 514–520.

[10] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.*, **28** (1981), 157–171.

[11] S. S. Miller, P. T. Mocanu, *Differential subordinations: theory and applications*, Series on Monographs and Texbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.

[12] Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952.

[13] J. Patel, N. E. Cho, Certain subclasses of meromorphically p-valent functions involving a linear operator, *Math. Sci. Res. J.*, **7** (4) (2003), 117–128.

[14] H. M. Srivastava, P. W. Karlsson, *Multiple Gaussian hypergeometric series*, Halsted Press, Ellis Horwood Limited, Chichester, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1985.

[15] H. M. Srivastava, J. Patel, Applications of differential subodination to certain subclasses of meromorphically multivalent functions, *J. Inequal. Pure Appl. Math.* **6** (3) Art. 88 (2005), 1–15.

[16] B. A. Uralegaddi, C. Somanatha, Certain classes of meromorphic multivalent functions, *Tamkang J. Math.* **23** (1992), 223–231.

[17] E. T. Whittaker, G. N. Watson, *A course on modern analysis: an introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions*, Fourth Edition (Reprinted), Cambridge University Press, Cambridge, 1927.

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On some Ringel-Hall numbers in tame cases

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Abstract. Let k be a finite field and consider the finite dimensional path algebra kQ, where Q is a quiver of tame type i.e. of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. Let $\mathcal{H}(kQ)$ be the corresponding Ringel-Hall algebra. We are going to determine the Ringel-Hall numbers of the form $F^{P'}_{XP}$ with P, P' preprojective indecomposables of defect -1 and $F^{I'}_{IX}$ with I, I' preinjective indecomposables of defect 1. It turns out that these numbers are either 1 or 0.

1 Introduction

Let k be a finite field with q elements and consider the path algebra kQ where Q is a quiver of tame type i.e. of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. When Q is of type $\tilde{A}_n$ we exclude the cyclic orientation. So kQ is a finite dimensional tame hereditary algebra with the category of finite dimensional (hence finite) right modules denoted by mod-kQ. Let $[M]$ be the isomorphism class of $M \in$ mod-kQ. The category mod-kQ can and will be identified with the category rep-kQ of the finite dimensional k-representations of the quiver $Q = (Q_0 = \{1, 2, \dots, n\}, Q_1)$. Here $Q_0 = \{1, 2, \dots, n\}$ denotes the set of vertices of the quiver, Q_1 the set of arrows and for an arrow α we denote by $s(\alpha)$ the starting point of the arrow and by $e(\alpha)$ its endpoint. Recall that a k-representation of Q is defined as a set of finite dimensional k-spaces $\{V_i | i = \overline{1, n}\}$ corresponding to the vertices together with k-linear maps $V_\alpha : V_{s(\alpha)} \to V_{e(\alpha)}$ corresponding to the arrows. The dimension of a module $M = (V_i, V_\alpha) \in$ mod-kQ = rep-kQ

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is then $\underline{\dim}M = (\dim_k V_i)_{i=\overline{1,n}} \in \mathbb{Z}^n$. For $a = (a_i), b = (b_i) \in \mathbb{Z}^n$ we say that $a \le b$ iff $b_i - a_i \ge 0\ \forall i$.

Let $P(i)$ and $I(i)$ be the projective and injective indecomposable corresponding to the vertex i and consider the Cartan matrix C_Q with the j-th column being $\underline{\dim}P(j)$. We have a bilinear form on $\mathbb{Z}^n$ defined as $\langle a, b\rangle = aC_Q^{-t}b^t$. Then for two modules $X, Y \in \text{mod-}kQ$ we get

$$\langle \underline{\dim}X, \underline{\dim}Y\rangle = \dim_k \operatorname{Hom}(X, Y) - \dim_k \operatorname{Ext}^1(X, Y).$$

We denote by $\mathbf{q}$ the quadratic form defined by $\mathbf{q}(a) = \langle a, a\rangle$. Then $\mathbf{q}$ is positive semi-definite with radical $\mathbb{Z}\delta$, that is $\{a \in \mathbb{Z}^n | \mathbf{q}(a) = 0\} = \mathbb{Z}\delta$. Here δ is known for each type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ (see [4]). A vector $a \in \mathbb{N}^n$ is called positive real root of $\mathbf{q}$ if $\mathbf{q}(a) = 1$. It is known (see [4]) that for all positive roots a there is a unique indecomposable module $M \in \text{mod-}kQ$ (unique up to isomorphism) with $\underline{\dim}M = a$. The rest of the indecomposables are of dimension $t\delta$, with t positive integers. The defect of a module M is $\partial M = \langle \delta, \underline{\dim}M\rangle = -\langle \underline{\dim}M, \delta\rangle$. For a short exact sequence $0 \to X \to Y \to Z \to 0$ we have that $\partial Y = \partial X + \partial Z$.

Consider the Auslander-Reiten translates $\tau = D\operatorname{Ext}^1(-, kQ)$ and $\tau^{-1} = \operatorname{Ext}^1(D(kQ), -)$, where $D = \operatorname{Hom}_k(-, k)$. An indecomposable module M is preprojective (preinjective) if exists a positive integer m such that $\tau^m(M) = 0$ ($\tau^{-m}(M) = 0$). Otherwise M is said to be regular. Note that the dimension vectors of preprojective and preinjective indecomposables are positive real roots of $\mathbf{q}$. A module is preprojective (preinjective, regular) if every indecomposable component is preprojective (preinjective, regular). Note that an indecomposable module M is preprojective (preinjective, regular) iff $\partial M < 0$ ($\partial M > 0$, $\partial M = 0$). Moreover if Q is of type $\tilde{A}_n$ then $\partial M = -1$ for M preprojective indecomposable and $\partial M = 1$ for M preinjective indecomposable.

We consider now the rational Ringel-Hall algebra $\mathcal{H}(kQ)$ of the algebra kQ. Its $\mathbb{Q}$-basis is formed by the isomorphism classes $[M]$ from mod-kQ and the multiplication is defined by

$$[N_1][N_2] = \sum_{[M]} F^M_{N_1 N_2}[M].$$

The structure constants $F^M_{N_1 N_2} = |\{M \supseteq U|\ U \cong N_2,\ M/U \cong N_1\}|$ are called Ringel-Hall numbers. It is well-known that Ringel-Hall algebras play an important role in linking representation theory with the theory of quantum groups. They also appear in cluster theory. This is why it is important to know the structure of these algebras, by deriving formulas for Ringel-Hall numbers.

When Q is the Kronecker quiver (i.e. of type $\tilde{A}_1$) then the Ringel-Hall numbers were determined in [7] and [3]. It was shown that for P, P' preprojective indecomposables the Ringel-Hall numbers $F_{XP}^{P'}$ are 0 or 1. A dual statement could be formulated for preinjectives. This result was important since it played a crucial role in obtaining other formulas for Ringel-Hall numbers.

Our main theorem generalizes this result for every tame case. More precisely we show that the Ringel-Hall numbers of the form $F_{XP}^{P'}$ with P, P' preprojective indecomposables of defect -1 and $F_{IX}^{I'}$ with I, I' preinjective indecomposables of defect 1 are either 1 or 0. We also describe the modules X for which these Ringel-Hall numbers are 1.

We should remark that the main result of this paper is a fundamental tool in obtaining other important formulas for the Ringel-Hall products in tame cases (see also the paper [9]).

Finally we note that the left to right implication part of Lemma 4 appears as main result in [10], however for the sake of completeness we include the full proof of it.

2 Facts on tame hereditary algebras

For a detailed description of the forthcoming notions we refer to [1],[2],[4],[6] and [12].

Let k be a finite field with q elements and consider the path algebra kQ where Q is a quiver of tame type.

The vertices of the Auslander-Reiten quiver of kQ are the isomorphism classes of indecomposables and its arrows correspond to the so-called irreducible maps. It will have a preprojective component (with all the isoclasses of preprojective indecomposables), a preinjective component (with all the isoclasses of preinjective indecomposables). All the other components (containing the isoclasses of regular indecomposables) are "tubes" of the form $\mathbb{Z}A_\infty/m$, where m is the rank of the tube. The tubes are indexed by the points of the scheme $\mathbb{P}_k^1$, the degree of a point $x \in \mathbb{P}_k^1$ being denoted by $\deg x$. A tube of rank 1 is called homogeneous, otherwise it is called non-homogeneous. We have at most 3 non-homogeneous tubes indexed by points x of degree $\deg x = 1$. All the other tubes are homogeneous. Notice that the number of points $x \in \mathbb{P}_k^1$ of degree 1 is $q+1$ and there are $N(q,l) = \frac{1}{l}\sum_{d|l}\mu(\frac{l}{d})q^d$ points of degree $l \geq 2$, where μ is the Möbius function and $N(q,l)$ is the number of monic, irreducible polynomials of degree l over a field with q elements (see [12]).

Indecomposables from different tubes have no nonzero homomorphisms and no non-trivial extensions. Note that all regular modules form an extension-closed abelian subcategory of mod-kQ, the simple objects in this subcategory being called quasi-simple modules; any indecomposable regular module is regular uniserial and hence it is uniquely determined by its quasi-socle and quasi-length, and also by its quasi-top and quasi-length.

In case of a homogeneous tube τ_x we have a single quasi-simple regular denoted by $R_x[1]$ with $\underline{\dim} R_x[1] = (\deg x)\delta$, which lies on the "mouth" of the tube. $R_x[t]$ will denote the regular indecomposable with quasi-socle $R_x[1]$ and quasi-length t. In case of a non-homogeneous tube τ_x of rank m on the mouth of the tube we have m quasi-simples denoted by $R_x^i[1]$ $i = \overline{1,m}$ such that $\sum_{i=1}^{m} \underline{\dim} R_x^i[1] = \delta$. $R_x^i[t]$ will denote the regular indecomposable with quasi-socle $R_x^i[1]$ and quasi-length t.

The following lemma is well-known.

Lemma 1 a) *For* P *preprojective,* I *preinjective, and* R *regular module we have* $\mathrm{Hom}(R,P) = \mathrm{Hom}(I,P) = \mathrm{Hom}(I,R) = \mathrm{Ext}^1(P,R) = \mathrm{Ext}^1(P,I) = \mathrm{Ext}^1(R,I) = 0$.

b) *If* $x \neq x'$ *and* R_x $(R_{x'})$ *is a regular with components from the tube* τ_x $(\tau_{x'})$*, then* $\mathrm{Hom}(R_x, R_{x'}) = \mathrm{Ext}^1(R_x, R_{x'}) = 0$.

c) *For* τ_x *homogeneous and* $R_x[t]$*,* $R_x[t']$ *indecomposables from* τ_x *we have* $\dim_k \mathrm{Hom}(R_x[t], R_x[t']) = \dim_k \mathrm{Ext}^1(R_x[t], R_x[t']) = \min(t,t') \deg x$.

d) *For* τ_x *non-homogeneous of rank* m *and* $R_x^i[t]$ *an indecomposable from* τ_x *such that* $lm < t \leq (l+1)m$ *we have* $\dim_k \mathrm{End}(R_x^i[t]) = l+1$.

e) *For* τ_x *non-homogeneous of rank* m *and* $R_x^i[t]$ *an indecomposable from* τ_x *such that* $lm \leq t < (l+1)m$ *we have* $\dim_k \mathrm{Ext}^1(R_x^i[t], R_x^i[t]) = l$.

f) *For* P *preprojective and* I *preinjective indecomposable modules we have* $\mathrm{End}(P) \cong k$*,* $\mathrm{End}(I) \cong k$*,* $|\mathrm{Aut}(P)| = |\mathrm{Aut}(I)| = q-1$ *and* $\mathrm{Ext}^1(P,P) = \mathrm{Ext}^1(I,I) = 0$.

3 Some Ringel-Hall numbers

Consider the Ringel-Hall numbers of the form $F^{P'}_{XP}$ with P, P' preprojetive indecomposables of defect -1 and $F^{I'}_{IX}$ with I, I' preinjective indecomposables of defect 1. We are going to show that these numbers are either 1 or 0.

We consider the preprojective case, the preinjective case being dual. We begin with some lemmas. The first lemma is well known (see for example in [11]).

Lemma 2 *Let* P *be a preprojective indecomposable with defect* $\partial P = -1$*,* P' *a preprojective module and* R *a regular indecomposable. Then we have:*

a) *Every nonzero morphism* $f : P \to P'$ *is a monomorphism.*

b) *For every nonzero morphism* $f : P \to R$*,* f *is either a monomorphism or* $\operatorname{Im} f$ *is regular. In particular if* R *is quasi-simple and* $\operatorname{Im} f$ *is regular then* f *is an epimorphism.*

Proof. a) Consider the short exact sequence $0 \to \operatorname{Ker} f \to P \to \operatorname{Im} f \to 0$. Since $\operatorname{Ker} f \subseteq P$ and $\operatorname{Im} f \subseteq P'$ we have that $\operatorname{Ker} f$ and $\operatorname{Im} f$ are either preprojective (so with negative defect) or 0. Moreover we have that $\partial \operatorname{Ker} f + \partial \operatorname{Im} f = \partial P = -1$ and we know that $\operatorname{Im} f \neq 0$ (since f is nonzero). It follows that $\operatorname{Ker} f = 0$.

b) Consider the short exact sequence $0 \to \operatorname{Ker} f \to P \to \operatorname{Im} f \to 0$. Since $\operatorname{Ker} f \subseteq P$ we have that $\operatorname{Ker} f$ is either preprojective (so with negative defect) or 0. On the other hand $\operatorname{Im} f \subseteq R$ implies that $\operatorname{Im} f$ can contain preprojectives and regulars as direct summands (and it is nonzero since f is nonzero). The equality $\partial \operatorname{Ker} f + \partial \operatorname{Im} f = \partial P = -1$ gives us two cases. When $\partial \operatorname{Ker} f = 0$ then $\operatorname{Ker} f$ is 0 so f is monomorphism. In the second case (when $\partial \operatorname{Ker} f = -1$) $\partial \operatorname{Im} f = 0$, so $\operatorname{Im} f$ can contain just regular direct summands. □

Lemma 3 *Let* P *be a preprojective indecomposable with defect* $\partial P = -1$ *(Then* $\underline{\dim} P \neq \delta$ *since* $\underline{\dim} P$ *is a positive real root of* $\mathbf{q}$*).*

a) *Suppose that* $\underline{\dim} P > \delta$*. Then* P *projects to the quasi-simple regular* $R_x[1]$ *from each homogeneous tube* τ_x *with* $(\deg x)\delta < \dim P$*. Also* P *projects to a unique quasi-simple regular from the mouth of each non-homogeneous tube* τ_x*. We will denote these quasi-simple regulars by* $R_x^P[1]$ *where for* τ_x *homogeneous with* $(\deg x)\delta < \dim P$ *we have* $R_x^P[1] = R_x[1]$*.*

b) *Suppose that* $\underline{\dim} P < \delta$*. Then* P *projects at most to a single quasi-simple regular from each non-homogeneous tube* τ_x *denoted by* $R_x^P[1]$*.*

Proof. a) Suppose that $R_x[1]$ denotes the quasi-simple regular from the mouth of the homogeneous tube τ_x with $\underline{\dim} R_x[1] = (\deg x)\delta < \underline{\dim} P$. Then we have $\operatorname{Ext}^1(P, R_x[1]) = 0$ (see Lemma 1) so

$$\dim_k \operatorname{Hom}(P, R_x[1]) = \langle \underline{\dim} P, \underline{\dim} R_x[1] \rangle = \langle \underline{\dim} P, (\deg x)\delta \rangle =$$

$$(\deg x)(-\partial P) = \deg x \neq 0.$$

This means that we have a nonzero morphism $f : P \to R_x[1]$ with $\underline{\dim} P > \underline{\dim} R_x[1]$. Using Lemma 2 we deduce that f is not a monomorphism, so $\operatorname{Im} f$ is regular and $R_x[1]$ is quasi-simple, which means that f is an epimorphism.

Denote by $R_x^i[1]$, $i = \overline{1, m}$ the i-th quasi-simple regular from the mouth of the non-homogeneous tube τ_x of rank $m \geq 2$. Notice that this time $\deg x = 1$,

$\sum_{i=1}^{m} \underline{\dim} R_x^i[1] = \delta$ and $\mathrm{Ext}^1(P, R_x^i[1]) = 0$, so we have

$$\sum_{i=1}^{m} \dim_k \mathrm{Hom}(P, R_x^i[1]) = \sum_{i=1}^{m} \langle \underline{\dim} P, \underline{\dim} R_x^i[1] \rangle$$

$$= \langle \underline{\dim} P, \sum_{i=1}^{m} \underline{\dim} R_x^i[1] \rangle = \langle \underline{\dim} P, \delta \rangle = -\partial P = 1.$$

It follows that $\exists ! i_0$ such that $\mathrm{Hom}(P, R_x^{i_0}[1]) \neq 0$, so we have a nonzero morphism $f : P \to R_x^{i_0}[1]$ with $\underline{\dim} P > \delta > \underline{\dim} R_x^{i_0}[1]$. Using Lemma 2 we deduce that f is not a monomorphism, so $\mathrm{Im}\, f$ is regular and $R_x^{i_0}[1]$ is quasi-simple, which means that f is an epimorphism. Let $R_x^P[1] := R_x^{i_0}[1]$.

b) Since $\underline{\dim} P < \delta$ clearly P could project only on quasi-simple regulars from non-homogeneous tubes. Denote again by $R_x^i[1]$, $i = \overline{1, m}$ the i-th quasi-simple regular on the mouth of the non-homogeneous tube τ_x of rank $m \geq 2$. As above we can deduce that $\exists ! i_0$ such that $\mathrm{Hom}(P, R_x^{i_0}[1]) \neq 0$, so we have a nonzero morphism $f : P \to R_x^{i_0}[1]$. But if $\underline{\dim} P \not> \underline{\dim} R_x^{i_0}[1]$ then f is a monomorphism and not an epimorphism. □

Remark 1 *Notice that* $\dim_k \mathrm{Hom}(P, R_x^P[1]) = \deg x$.

Lemma 4 *Let* $P \ncong P'$ *be preprojective indecomposables with defect* -1. *Then* $F_{XP}^{P'} \neq 0$ *iff* X *satisfies the following conditions:*

i) *it is a regular module with* $\underline{\dim} X = \underline{\dim} P' - \underline{\dim} P$*;*

ii) *if it has an indecomposable component from a tube* τ_x *then the quasi-top of this component is the quasi-simple regular* $R_x^{P'}[1]$*;*

iii) *its indecomposable components are taken from pairwise different tubes.*

Proof. "⇒" Suppose $F_{XP}^{P'} \neq 0$. We will check the conditions i), ii) and iii).

Condition i). Since $F_{XP}^{P'} \neq 0$ we have a short exact sequence $0 \to P \to P' \to X \to 0$. Then $\underline{\dim} X = \underline{\dim} P' - \underline{\dim} P$ and $\partial P' = \partial P + \partial X$, but $\partial P' = \partial P = -1$, so $\partial X = 0$. Notice that X can't have preprojective components, for if P'' would be such a component then $P' \twoheadrightarrow P'' \ncong P'$ which is impossible due to Lemma 2 a). So X is regular.

Condition ii). Let R be an indecomposable component of X taken from the tube τ_x. Denote by $\mathrm{top} R$ its quasi-top which must be quasi-simple due to uniseriality. Then $P' \twoheadrightarrow X \twoheadrightarrow R \twoheadrightarrow \mathrm{top} R$ so using Lemma 3 $\mathrm{top} R \cong R_x^{P'}[1]$.

Condition iii). Suppose $X = X' \oplus R_1 \oplus \ldots \oplus R_l$, where $R_1, \ldots, R_l$ are taken from the same tube τ_x. Then by Condition ii) they have the same quasi-top

$R_x^{P'}[1]$ and we have the monomorphism

$$0 \to \mathrm{Hom}(X, R_x^{P'}[1]) \to \mathrm{Hom}(P', R_x^{P'}[1]).$$

It follows that

$$\dim_k \mathrm{Hom}(X, R_x^{P'}[1]) \le \dim_k \mathrm{Hom}(P', R_x^{P'}[1]) = \deg x.$$

We can conclude that

$$\dim_k \mathrm{Hom}(X, R_x^{P'}[1]) = \dim_k \mathrm{Hom}(X', R_x^{P'}[1]) + \sum_{i=1}^{l} \dim_k \mathrm{Hom}(R_i, R_x^{P'}[1]) \le \deg x,$$

$$\dim_k \mathrm{Hom}(R_i, R_x^{P'}[1]) = \deg x \text{ for } \tau_x \text{ homogeneous}$$

and

$$\dim_k \mathrm{Hom}(R_i, R_x^{P'}[1]) \ge 1 = \deg x \text{ for } \tau_x \text{ non-homogeneous.}$$

It follows that $l = 1$.

"$\Leftarrow$" Let R be an indecomposable regular module with $\underline{\dim}R < \underline{\dim}P'$ satisfying condition ii). By Lemma 2 b) it follows that for a nonzero morphism $f : P' \to R$, $\mathrm{Im}\, f$ is regular. We will show that P' projects on R. Observe that if $R = R_x^{P'}[1]$ the assertion is true due to Lemma 3. Suppose now that R is not a quasi-simple.

If R is from a homogeneous tube τ_x then $R = R_x[t]$, $\underline{\dim}R = t(\deg x)\delta$ and $\mathrm{Hom}(P', R) \ne 0$ since $\dim_k \mathrm{Hom}(P', R) = \langle \underline{\dim}P', t(\deg x)\delta\rangle = -t(\deg x)\partial P' = t \deg x$. Notice that in the case when there are no epimorphisms in $\mathrm{Hom}(P', R)$ then using Lemma 2 b) and the uniseriality of regulars we would have $\mathrm{Hom}(P', R) = \mathrm{Hom}(P', R_x[t]) \cong \mathrm{Hom}(P', R_x[t-1])$, a contradiction. So we have an epimorphism $P' \to R$.

If R is from a non-homogeneous tube τ_x of rank m then $\deg x = 1$, $R = R_x^j[t]$ and $\mathrm{top}R = R_x^{P'}[1] = R_x^i[1]$ (condition ii)). We have that $\underline{\dim}R = \underline{\dim}R_x^j[t-1] + \underline{\dim}(\mathrm{top}R)$, so $\dim_k \mathrm{Hom}(P', R) = \langle \underline{\dim}P', \underline{\dim}R_x^j[t-1]\rangle + \langle \underline{\dim}P', \underline{\dim}(\mathrm{top}R)\rangle = \dim_k \mathrm{Hom}(P', R_x^j[t-1]) + 1 > 0$. If there is no epimorphism $P' \to R$ then using uniseriality and Lemma 2 b) for nonzero $f \in \mathrm{Hom}(P', R)$ we have that $\mathrm{Im}\, f = R_x^j[l]$ with $1 \le l < t$ and P' projects on $\mathrm{top}\, \mathrm{Im}\, f$ so $\mathrm{top}R_x^j[l] = \mathrm{top}R_x^j[t] = \mathrm{top}R$ (see Lemma 3). But this means that $t - l = sm$ with $s \ge 1$ so if $t \le m$ we have a contradiction and if $t > m$ as in the homogeneous case we would have that $\mathrm{Hom}(P', R_x^j[t]) \cong \mathrm{Hom}(P', R_x^j[t-m])$ that is

$$0 = \langle \underline{\dim}P', \underline{\dim}R_x^j[t] - \underline{\dim}R_x^j[t-m]\rangle = \langle \underline{\dim}P', \delta\rangle = 1,$$

again a contradiction.

Suppose now that the module $X = R_1 \oplus \ldots \oplus R_l$ satisfies conditions i), ii) and iii). From the discussion above we have the epimorphisms $f_i : P' \to R_i$. Let $f : P' \to X$, $f(x) = \sum f_i(x)$ the diagonal map. Due to Lemma 2 b) we have that $\operatorname{Im} f$ is regular, so due to uniseriality $\operatorname{Im} f = R'_1 \oplus \ldots \oplus R'_l$ with $R'_i \subseteq R_i$. Since $f_i = p_i f$ are epimorphisms we have that $R'_i = R_i$, so f is an epimorphism. Notice that $\operatorname{Ker} f \subseteq P'$ hence it is preprojective, $\partial \operatorname{Ker} f = \partial P' - \partial X = -1$ therefore $\operatorname{Ker} f$ is an indecomposable preprojective with $\underline{\dim} \operatorname{Ker} f = \underline{\dim} P$. It follows that $\operatorname{Ker} f \cong P$, so we have an exact sequence $0 \to P \to P' \to X \to 0$ which implies that $F^{P'}_{XP} \neq 0$. □

Lemma 5 *Let* $P \ncong P'$ *be preprojective indecomposables with defect* -1 *such that* $\operatorname{Hom}(P, P') \neq 0$. *Suppose the points* $y_i \in \mathbb{P}^1_k$, $i = \overline{1, s}$ $(s = 0, 1, 2, 3)$ *are indexing the non-homogeneous tubes (in case* $s = 0$ *we have only homogeneous tubes). Then* $\underline{\dim} P' - \underline{\dim} P = t_0\delta + \sum_{i=1}^{s} \sigma_i^0$, *where* $0 \leq \sigma_i^0 < \delta$ *and* σ_i^0 *(in case it is nonzero) is the dimension of a regular from the non-homogeneous tube* τ_{y_i} *with top* $R^{P'}_{y_i}[1]$. *In this case* $\dim_k \operatorname{Hom}(P, P') = t_0 + 1$ *so* t_0 *is unique.*

Proof. Since $\operatorname{Hom}(P, P') \neq 0$ we have a monomorphism $P \to P'$ with factor X satisfying conditions i),ii),iii) from the previous lemma. It follows that $\underline{\dim} P' - \underline{\dim} P = \underline{\dim} X = t\delta + \sum_{i=1}^{s} \sigma_i$, where $0 \leq \sigma_i$ (in case it is nonzero) is the dimension of a regular R_i from the non-homogeneous tube τ_{y_i} with top $R^{P'}_{y_i}[1]$. Suppose $\sigma_i = t_i\delta + \sigma_i^0$ with $0 \leq \sigma_i^0 < \delta$ and $0 \leq t_i$. If $t_i \neq 0$ then there is a unique regular R_{t_i} of dimension $t_i\delta$ from the non-homogeneous tube τ_{y_i} which embeds into R_i; the factor will be of dimension σ_i^0 with top $R^{P'}_{y_i}[1]$ (if $\sigma_i^0 \neq 0$). Let $t_0 = t + \sum_{i=1}^{s} t_i$.

We show that $\dim_k \operatorname{Hom}(P, P') = t_0 + 1$. Suppose first that we don't have non-homogeneous tubes, so we are in the Kronecker case (see [12]). In this case $\delta = (1, 1)$, $\underline{\dim} P' - \underline{\dim} P = t_0\delta$ and then $\dim_k \operatorname{Hom}(P, P') = t_0 + 1$. (see for example [7] Lemma 2.1). Consider now the case when we do have non-homogeneous tubes, so $s \geq 1$ and suppose $t_0\delta + \sigma_1^0 \neq 0$. Then there are unique regular indecomposables $R_1 \in \tau_{y_1}$ of dimension $t_0\delta + \sigma_1^0$ and top $R^{P'}_{y_1}[1]$ and $R_i \in \tau_{y_i}$ of dimension σ_i^0 and top $R^{P'}_{y_i}[1]$ for $i \in I = \{i = \overline{2, s} | \sigma_i^0 \neq 0\}$. Suppose that $I' = \{i = \overline{1, s} | \sigma_i^0 \neq 0\}$ and $|I'| = l$ (where we can have $l = 0$). Let $R = R_1 \oplus (\bigoplus_{i \in I} R_i)$. It follows from the previous lemma that $F^{P'}_{RP} \neq 0$ so we have a short exact sequence $0 \to P \to P' \to R \to 0$ which induces the exact sequences

$$0 \to \operatorname{End}(P) \to \operatorname{Hom}(P, P') \to \operatorname{Hom}(P, R) \to \operatorname{Ext}^1(P, P)$$

and

$$0 \to \mathrm{End}(R) \to \mathrm{Hom}(P', R) \to \mathrm{Hom}(P, R) \to \mathrm{Ext}^1(R, R) \to \mathrm{Ext}^1(P', R).$$

We deduce using Lemma 1 and Remark 1 that

$$\dim_k \mathrm{Hom}(P, P') = \dim_k \mathrm{Hom}(P, R) + 1 =$$

$$\dim_k \mathrm{Hom}(P', R) + \dim_k \mathrm{Ext}^1(R, R) - \dim_k \mathrm{End}(R) + 1,$$

where

$$\dim_k \mathrm{Hom}(P', R) = \langle \underline{\dim} P', \underline{\dim} R \rangle = \langle \underline{\dim} P', t_0\delta + \sum_{i=1}^{s} \sigma_i^0 \rangle = t_0 + l,$$

$$\dim_k \mathrm{Ext}^1(R, R) = \dim_k \mathrm{Ext}^1(R_1, R_1) + \sum_{i \in I} \dim_k \mathrm{Ext}^1(R_i, R_i) = t_0,$$

$$\dim_k \mathrm{End}(R) = \dim_k \mathrm{End}(R_1) + \sum_{i \in I} \dim_k \mathrm{End}(R_i) = t_0 + l,$$

so it results that $\dim_k \mathrm{Hom}(P, P') = t_0 + 1$. □

The following lemma can be found in [5] or in [8].

Lemma 6 *For* t_0 *nonnegative integer we have that*

$$\sum_{\substack{(t_x)_{x \in \mathbb{P}^1_k} \\ t_x \in \mathbb{Z},\, t_x \geq 0 \\ \sum_x t_x (\deg x) = t_0}} 1 = \frac{q^{t_0+1} - 1}{q - 1}.$$

Now we are ready to prove the main theorem.

Theorem 1 *Let* $P \not\cong P'$ *be preprojective indecomposables with defect* -1*. If* $\mathrm{Hom}(P, P') = 0$ *then* $F^{P'}_{XP} = 0$ *for every* X*. If* $\mathrm{Hom}(P, P') \neq 0$ *then* $F^{P'}_{XP} = 1$ *for any* X *satisfying conditions i), ii) and iii) from Lemma 4, otherwise* $F^{P'}_{XP} = 0$*.*

Proof. Suppose $\mathrm{Hom}(P, P') \neq 0$. Then using the notation from Lemma 5 $\underline{\dim} P' - \underline{\dim} P = t_0\delta + \sum_{i=1}^{s} \sigma_i^0$, where $0 \leq \sigma_i^0 < \delta$ and σ_i^0 (in case it is nonzero) is the dimension of a regular from the non-homogeneous tube τ_{y_i} with top $R^{P'}_{y_i}[1]$; also we have $\dim_k \mathrm{Hom}(P, P') = t_0 + 1$. Since by Lemma 2 every nonzero morphism in $\mathrm{Hom}(P, P')$ is a monomorphism and $|\mathrm{Aut}(P)| = q - 1$. We have that the number of submodules of P' which are isomorphic to P is

$$u_P^{P'} = \frac{|\mathrm{Hom}(P, P')| - 1}{|\mathrm{Aut}(P)|} = \frac{q^{t_0+1} - 1}{q - 1}.$$

A regular module X satisfying conditions i), ii) and iii) from Lemma 4 will be called of good type. By Lemma 4 we have that

$$u_P^{P'} = \sum_{[X]} F_{XP}^{P'} = \sum_{\substack{[X] \\ X \text{ of good type}}} F_{XP}^{P'},$$

the terms in the last sum being nonzero. We will count now the number of nonisomorphic regulars of good type. For τ_x a homogeneous tube and $t \geq 1$ denote by $R_x(t)$ the regular $R_x[t]$ of quasi-length t and let $R_x(0) = 0$. For τ_{y_i} $(i = \overline{1, s})$ a non-homogeneous tube and $t \neq 0$ denote by $R_{y_i}(t)$ the unique indecomposable from τ_{y_i} of dimension $t\delta + \sigma_i^0$ with top $R_{y_i}^{P'}[1]$. For $t = 0$ and $\sigma_i^0 \neq 0$ let $R_{y_i}(0)$ be the unique indecomposable from τ_{y_i} of dimension σ_i^0 with top $R_{y_i}^{P'}[1]$. For $t = 0$ and $\sigma_i^0 = 0$ let $R_{y_i}(0) = 0$. Then the modules

$$\bigoplus_{\substack{(t_x)_{x \in \mathbb{P}_k^1} \\ t_x \in \mathbb{Z},\, t_x \geq 0 \\ \sum_x t_x(\deg x) = t_0}} R_x(t_x)$$

are nonisomorphic regulars of good type, so by the previous lemma we have at least $\frac{q^{t_0+1}-1}{q-1}$ of them. It follows that

$$\frac{q^{t_0+1} - 1}{q - 1} = \sum_{\substack{[X] \\ X \text{ of good type}}} F_{XP}^{P'},$$

the number of nonzero terms in the sum being at least $\frac{q^{t_0+1}-1}{q-1}$, so the assertion of the theorem follows. □

Remark 2 *It follows from the previous theorem that for* $P \not\cong P'$ *preprojective indecomposables with defect* -1 *such that* $\mathrm{Hom}(P, P') \neq 0$ *the decomposition from Lemma 5* $\underline{\dim}P' - \underline{\dim}P = t_0\delta + \sum_{i=1}^{s} \sigma_i^0$ *(where* $0 \leq \sigma_i^0 < \delta$ *and* σ_i^0 *(in case it is nonzero) is the dimension of a regular from the non-homogeneous tube* τ_{y_i} *with top* $R_{y_i}^{P'}[1]$*) is unique, so both* t_0 *and* σ_i^0 *are unique.*

We can dualize the previous results for preinjective modules. So we have the dual of Lemma 3.

Lemma 7 *Let* I *be a preinjective indecomposable with defect* $\partial I = 1$*.*

a) *Suppose that* $\underline{\dim} I > \delta$*. Then the quasi-simple regular* $R_x[1]$ *from each homogeneous tube* τ_x *with* $(\deg x)\delta < \dim I$ *embeds into* I*. Also, a unique quasi-simple regular from the mouth of each non-homogeneous tube* τ_x *embeds into* I*. We will denote these quasi-simple regulars by* $R^I_x[1]$*, where for* τ_x *homogeneous with* $(\deg x)\delta < \dim I$ *we have* $R^I_x[1] = R_x[1]$*.*

b) *Suppose that* $\underline{\dim} I < \delta$*. Then at most a single quasi-simple regular from each non-homogeneous tube* τ_x *embeds into* I*. We denote this quasi-simple regular by* $R^I_x[1]$*.*

The dual of Theorem 1 is

Theorem 2 *Let* $I \ncong I'$ *be preinjective indecomposables with defect* 1*.*

If $\mathrm{Hom}(I', I) = 0$ *then* $F^{I'}_{IX} = 0$ *for every* X*. If* $\mathrm{Hom}(I', I) \neq 0$ *then* $F^{I'}_{IX} = 1$ *for* X *satisfying the conditions i), ii) and iii) below, otherwise* $F^{I'}_{IX} = 0$*.*

i) X *is a regular module with* $\underline{\dim} X = \underline{\dim} P' - \underline{\dim} P$*;*

ii) *If* X *has an indecomposable component from a tube* τ_x *then the quasi-socle of this component is the quasi-simple regular* $R^{I'}_x[1]$*;*

iii) *The indecomposable components of* X *are taken from pairwise different tubes.*

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References

[1] I. Assem, D. Simson, A. Skowronski, *Elements of representation theory of associative algebras, Volume 1: Techniques of representation theory*, LMS Student Texts (No. 65), Cambridge Univ. Press 2006.

[2] M. Auslander, I. Reiten, S. Smalø, *Representation theory of Artin algebras*, Cambridge Stud. in Adv. Math. 36, Cambridge Univ. Press 1995.

[3] P. Baumann, C. Kassel, The Hall algebra of the category of coherent sheaves on the projective line, *J. Reine Angew. Math.*, **533** (2001), 207–233.

[4] V. Dlab, C. M. Ringel, Indecomposable representations of graphs and algebras, *Mem. Am. Math. Soc.*, **173** (1976).

[5] A. Hubery, The composition algebra of an affine quiver, preprint, http://arxiv.org/abs/math/0403206

[6] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lect. Notes Math. 1099, Springer 1984.

[7] Cs. Szántó, Hall numbers and the composition algebra of the Kronecker algebra, *Algebr. Represent. Theory*, **9** (2006), 465–495.

[8] Cs. Szántó, Hall coefficients in the Kronecker case, *Proceedings of the Algebra Symposium, "Babeş-Bolyai" University Cluj-Napoca*, November 23–24 (2001), 253–260.

[9] Cs. Szántó, On some Ringel-Hall products in tame cases, *J. Pure Appl. Algebra*, **216** (2012), 2069–2078.

[10] Cs. Szántó, On some nonzero Ringel-Hall numbers in tame cases, *Mathematica (Cluj)*, **53** (76) (2011), 189–195.

[11] P. Zhang, Composition algebras of affine type, *J. Algebra*, **206** (1998), 505–540.

[12] P. Zhang, Y. B. Zhang, J. Y. Guo, Minimal generators of RingelHall algebras of affine quivers, *J. Algebra*, **239** (2001), 675–704.

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A particular Galois connection between relations and set functions

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Abstract. Motivated by a recent paper of U. Höhle and T. Kubiak on regular sup-preserving maps, we investigate a particular Galois-type connection between relations on one set X to another Y and functions on the power set $\mathcal{P}(X)$ to $\mathcal{P}(Y)$.

Since relations can largely be identified with union-preserving set functions, the results obtained can be used to provide some natural generalizations of most of the former results on relations and relators (families of relations). The results on inverses seem to be the only exceptions.

1 Introduction

In this paper, a subset R of a product set $X \times Y$ is called a relation on X to Y. And, a function U on the power set $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ is called a corelation on X to Y.

Motivated by a recent paper of Höhle and Kubiak [9], for any relation R on X to Y, we define a correlation $R^{\star}$ on X to Y such that $R^{\star}(A) = R[A]$ for all $A \subset X$. Moreover, for any correlation U on X to Y, we define a relation U^{*} on X to Y such that $U^{*}(x) = U(\{x\})$ for all $x \in X$.

And, we show that the functions $\star$ and $*$ establish an interesting Galois-type connection between the family $\mathcal{P}(X \times Y)$ of all relations on X to Y and the family $\mathcal{Q}(X, Y)$ of all correlations on X to Y, whenever $\mathcal{P}(X \times Y)$ is

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considered to be partially ordered by the ordinary set inclusion and $\mathcal{Q}(X, Y)$ by the pointwise one.

Since relations can largely be identified with union-preserving correlations, the results obtained can be used to provide some natural generalizations of most of the former results on relations and relators (families of relations). (The most relevant ones are in [21] and [16].) The results on inverse relations and relators seem to be the only exceptions.

To keep the paper almost completely self-contained, the most important definitions concerning relations, functions, ordered sets and Galois connections [5, p. 155] will be briefly listed in the next two preparatory sections in somewhat novel forms. They will clarify our subsequent results and show the way to further investigations on Galois-type connections.

2 Relations and functions

A subset F of a product set $X{\times}Y$ is called a *relation on* X *to* Y. If in particular $F \subset X^2$, with $X^2 = X{\times}X$, then we may simply say that F is a *relation on* X. In particular, $\Delta_X = \{(x, x):\ x \in X\}$ is called the *identity relation on* X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{y \in Y:\ (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images of* x *and* A *under* F respectively. If $(x, y) \in F$, then we may also write $x F y$.

Moreover, the sets $D_F = \{x \in X:\ F(x) \neq \emptyset\}$ and $R_F = F[X]$ are called the *domain and range of* F, respectively. If in particular $D_F = X$, then we say that F is a *relation of* X *to* Y, or that F *is a non-partial relation on* X *to* Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{y\}$.

Moreover, a function $\star$ of X to itself is called a *unary operation on* X. While, a function $*$ of X^2 to X is called a *binary operation on* X. And, for any $x, y \in X$, we usually write $x^\star$ and $x * y$ instead of $\star(x)$ and $*\big((x, y)\big)$.

For any relation F on X to Y, we may naturally define a *set-valued function* $F^\diamond$ on X such that $F^\diamond(x) = F(x)$ for all $x \in X$. This $F^\diamond$ can be identified with F. However, thus in contrast to $F \subset X{\times}Y$ we already have $F^\diamond \subset X{\times}\mathcal{P}(Y)$.

Therefore, instead of $F^\diamond$, it is usually more convenient to work with F or its selection functions. A function f of D_F to Y is called a *selection of* F if $f \subset F$, i.e., $f(x) \in F(x)$ for all $x \in D_F$.

Thus, the Axiom of Choice can be briefly expressed by saying that every relation has at least one selection function. Moreover, it can be easily seen that each relation is the union of its selection functions.

If F is a relation on X to Y, then $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can be naturally defined by specifying $F(x)$ for all $x \in X$.

For instance, the *complement relation* F^c can be naturally defined such that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$. The latter notation will not cause confusions, since thus we also have $F^c = X \times Y \setminus F$.

Quite similarly, the *inverse relation* F^{-1} can be naturally defined such that $F^{-1}(y) = \{x \in X : \ y \in F(x)\}$ for all $y \in Y$. Thus, the operations c and -1 are compatible in the sense $\left(F^c\right)^{-1} = \left(F^{-1}\right)^c$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation* $G \circ F$ can be naturally defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G\big[F[A]\big]$ for all $A \subset X$.

On the other hand, if G is a relation on Z to W, then the *box product relation* $F \boxtimes G$ can be naturally defined such that $(R \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$.

The box product relation, whose origin seems to go back to a thesis of J. Riquet in 1951, has been mainly investigated in [21]. In that, for instance, we have proved that $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subset X \times Z$.

Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can see that the box and composition products are actually equivalent tools. However, the box product can immediately be defined for an arbitrary family of relations.

3 Generalized ordered sets and Galois connections

Now, a relation R on X may be called *reflexive* if $\Delta_X \subset R$, and *transitive* if $R \circ R \subset R$. Moreover, R may be called *symmetric* if $R^{-1} \subset R$, and *antisymmetric* if $R \cap R^{-1} \subset \Delta_X$.

Thus, a reflexive and transitive (symmetric) transitive relation may be called a *preorder* (*tolerance*) *relation*. And, a symmetric (antisymmetric) preorder relation may be called an *equivalence* (*partial order*) *relation*.

For instance, for $A \subset X$, the *Pervin relation* $P_A = A^2 \cup A^c \times X$ [18] is a preorder relation on X. While, for a pseudo-metric d on X and $r > 0$, the *surrounding* $B_r^d = \big\{(x, y) \in X^2 : \ d(x, y) < r\big\}$ is a tolerance relation on X.

Moreover, we may recall that if $\mathcal{A}$ is a partition of X, i. e., a family of pairwise disjoint, nonvoid subsets of X such that $X = \bigcup \mathcal{A}$, then $E_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is an equivalence relation on X, which can be identified with $\mathcal{A}$.

According to [15], an ordered pair $X(\leq) = \big(X, \leq\big)$, consisting of a set X and a relation $\leq$ on X, will be called a *generalized ordered set*, or an *ordered set without axioms*. And, we shall usually write X in place of $X(\leq)$.

Now, a generalized ordered set $X(\leq)$ may, for instance, be called reflexive if the relation $\leq$ is reflexive. Moreover, the generalized ordered set $X'(\leq') = X(\leq^{-1})$ may be called the dual of $X(\leq)$.

Having in mind the terminology of Birkhoff [1, p. 1], a generalized ordered set will be briefly called a *goset*. Moreover, a preordered (partially ordered) set will be called a *proset* (*poset*).

Thus, every set X is a proset with the universal relation X^2. Moreover, X is a poset with the identity relation Δ_X. And every subfamily of the power set $\mathcal{P}(X)$ is a poset with the ordinary set inclusion $\subset$.

The usual definitions on posets can be naturally extended to gosets [15]. (And also to *relator spaces* [14] which include *formal context* [7, p. 17] as an important particular case).

For instance, for any subset A of a goset X, we may naturally define

$$\mathrm{lb}(A) = \big\{ x \in X : \quad \forall \ a \in A : \quad x \leq a \big\},$$
$$\mathrm{ub}(A) = \big\{ x \in X : \quad \forall \ a \in A : \quad a \leq x \big\},$$

and

$$\min(A) = A \cap \mathrm{lb}(A), \qquad \max(A) = A \cap \mathrm{ub}(A),$$
$$\inf(A) = \max\big(\mathrm{lb}(A)\big), \qquad \sup(A) = \min\big(\mathrm{ub}(A)\big).$$

Thus, for instance, min may be considered as a relation on $\mathcal{P}(X)$ to X, or as a function of $\mathcal{P}(X)$ of to itself. However, if X is antisymmetric, then $\mathrm{card}\big(\min(A)\big) \leq 1$ for all $A \subset X$. Therefore, min is actually a function.

Now, a goset X may, for instance, be naturally called inf-complete if $\inf(A) \neq \emptyset$ for all $A \subset X$. In [3], as an obvious extension of [1, Theorem 3, p. 112], we have proved that thus "inf-complete" is also equivalent to "sup-complete".

However, it now more important to note that, for any two subsets A and B of a goset X, we also have

$$\mathrm{lb}(A) \subset' B \iff B \subset \mathrm{lb}(A) \iff A \subset \mathrm{ub}(B).$$

Therefore, the set-functions lb and ub form a Galois connection between the poset $\mathcal{P}(X)$ and its dual in the sense of [5, Definition 7.23], suggested by Schmidt's reformulation [12, p. 209] of Ore's Galois connexion [10].

Instead of Galois connections, it is usually more convenient to use residuated mappings of Blyth and Janowitz [2] in some modified and generalized forms suggested by the present author in [19, 17, 24, 22].

However, now for a function f of one goset X to another Y and a function g of Y to X, we shall say that

(1) f and g form an *increasing upper Galois connection* between X and Y if $f(x) \leq y$ implies $x \leq g(y)$ for all $x \in X$ and $y \in Y$,

(2) f and g form an *increasing lower Galois connection* between X and Y if $x \leq g(y)$ implies $f(x) \leq y$ for all $x \in X$ and $y \in Y$.

Now, if both (1) and (2) hold, then we may naturally say that the functions f and g form an *increasing Galois connection* between X and Y. Important examples for Galois connections can be found in [6]. (See also [13, 16, 4].)

In the theory of relator spaces, it has turned out that the increasing upper and lower Galois connections are actually particular cases of upper and lower semicontinuous pairs of relations [20].

Therefore, they can be naturally extended to relators between relator spaces [23]. For this, it is enough to study first these connections only for functions between power sets instead of those between gosets.

4 Functions on one power set to another

Definition 1 If U is a function on one power set $\mathcal{P}(X)$ to another $\mathcal{P}(Y)$, then we simply say that U is a correlation on X to Y.

Remark 1 According to Birkhoff [1, p. 111], the term "operation on X" could also be used. However, this may cause some confusions because of the customary meaning of this expression.

Definition 2 A correlation U on X to Y, is called

(1) increasing if $U(A) \subset U(B)$ for all $A \subset B \subset X$,

(2) quasi-increasing if $U(\{x\}) \subset U(A)$ for all $x \in A \subset X$,

(3) union-preserving if $U\left(\bigcup \mathcal{A}\right) = \bigcup_{A \in \mathcal{A}} U(A)$ for all $\mathcal{A} \subset \mathcal{P}(X)$.

Remark 2 In the $X = Y$ particular case, U may also be naturally called extensive, intensive, involutiv, and idempotent if $A \subset U(A)$, $U(A) \subset A$, $U\big(U(A)\big) = A$, and $U\big(U(A)\big) = U(A)$ for all $A \subset X$, respectively.

Moreover, in particular an increasing and idempotent correlation may be called a projection or modification operation. And an extensive (intensive) projection operation may be called a closure (interior) operation.

Simple reformulations of properties (2) and (1) in Definition 1 give the following two theorems.

Theorem 1 *For a correlation* U *on* X *to* Y, *the following assertions are equivalent:*

(1) U *is quasi-increasing*,

(2) $\bigcup_{x \in A} U(\{x\}) \subset U(A)$ *for all* $A \subset X$.

Theorem 2 *For a corelation* U *on* X *to* Y, *the following assertions are equivalent:*

(1) U *is increasing*,

(2) $\bigcup_{A \in \mathcal{A}} U(A) \subset U(\bigcup \mathcal{A})$ *for all* $\mathcal{A} \subset \mathcal{P}(X)$,

(3) $U(A) \cup U(B) \subset U(A \cup B)$ *for all* $A, B \subset X$.

Hence, it is clear that in particular we also have

Corollary 1 *A correlation* U *on* X *to* Y *is union-preserving if and only if it is increasing and* $U(\bigcup \mathcal{A}) \subset \bigcup_{A \in \mathcal{A}} U(A)$ *for all* $\mathcal{A} \subset \mathcal{P}(X)$.

However, it now more important to note that we also have the following theorem which has also been proved, in a different way, by Pataki [11].

Theorem 3 *For a correlation* U *on* X *to* Y, *the following assertions are equivalent:*

(1) U *is uninon-preserving*,

(2) $U(A) = \bigcup_{x \in A} U(\{x\})$ *for all* $A \subset X$.

Proof. Since $A = \bigcup_{x \in A} \{x\}$ for all $A \subset X$, it is clear that (1) implies (2).

On the other hand, if (2) holds, then we can note that U is already increasing. Therefore, to obtain (1), by Corollary 1, we need only prove that $U(\bigcup \mathcal{A}) \subset \bigcup_{A \in \mathcal{A}} U(A)$ for every $\mathcal{A} \subset \mathcal{P}(X)$.

For this, note that if $\mathcal{A} \subset \mathcal{P}(X)$, then by (2) we have

$$U\Big(\bigcup \mathcal{A}\Big) = \bigcup_{x \in \bigcup \mathcal{A}} U(\{x\}).$$

Therefore, if $y \in U(\bigcup \mathcal{A})$, then there exists $x \in \bigcup \mathcal{A}$ such that $y \in U(\{x\})$. Thus, in particular there exists $A_0 \in \mathcal{A}$ such that $x \in A_0$, and so $\{x\} \subset A_0$. Hence, by using the increasingness of U, we can already infer that

$$y \in U(\{x\}) \subset U(A_0) \subset \bigcup_{A \in \mathcal{A}} U(A).$$

Therefore, the required inclusion is also true. □

From this theorem, by Theorem 1, it is clear that in particular we also have

Corollary 2 *A correlation* U *on* X *to* Y *is union-preserving if and only if it is quasi-increasing and* $U(A) \subset \bigcup_{x \in A} U(\{x\})$ *for all* $A \subset X$.

Definition 3 For any two correlations U and V on X to Y, we write

$$U \leq V \iff U(A) \subset V(A) \quad \text{for all} \quad A \subset X.$$

Remark 3 Note that if in particular $U \subset V$, then $U(A) = V(A)$ for all $A \in D_U$ and $U(A) = \emptyset \subset V(A)$ for all $A \subset X$ with $A \notin D_U$. Therefore, we have $U(A) \subset V(A)$ for all $A \subset X$, and thus $U \leq V$.

Theorem 4 *With the inequality considered in Definition 3, the family* $\mathcal{Q}(X, Y)$ *of all correlations on* X *to* Y, *forms a complete poset.*

Proof. It can be easily seen that if $\mathcal{U}$ is a family of correlations on X to Y and

$$V(A) = \bigcup_{U \in \mathcal{U}} U(A)$$

for all $A \subset X$, then $V \in \mathcal{Q}(X, Y)$ such that $V = \sup(\mathcal{U})$.

Therefore, $\mathcal{Q}(X, Y)$ is sup-complete, and hence it is also inf-complete. □

Remark 4 Note that if in particular each member of $\mathcal{U}$ is increasing (quasi-increasing), then V is also increasing (quasi-increasing).

Therefore, with the inequality given in Definition 3, the family $\mathcal{Q}_1(X, Y)$ of all quasi-increasing correlations on X to Y is also a complete poset.

5 A Galois connection between relations and correlations

According to the corresponding definitions of Höhle and Kubiak [9], we may also naturally introduce the following

Definition 4 For any relation R on X to Y, we define a correlation $R^\star$ on X to Y such that

$$R^\star(A) = R[A]$$

for all $A \subset X$.

Conversely, for any correlation U on X to Y, we define a relation U^* on X to Y such that

$$U^*(x) = U(\{x\})$$

for all $x \in X$.

Now, by using the corresponding definitions, we can easily prove the following two theorems.

Theorem 5 *If* U *is a correlation on* X *to* Y, *then* $R^\star \leq U$ *implies* $R \subset U^*$ *for any relation* R *on* X *to* Y.

Proof. If $R^\star \leq U$, then by the corresponding definitions

$$R(x) = R\big[\{x\}\big] = R^\star\big(\{x\}\big) \subset U\big(\{x\}\big) = U^*(x)$$

for all $x \in X$. Therefore, $R \subset U^*$ also holds. □

Theorem 6 *For a correlation* U *on* X *to* Y, *the following assertions are equivalent:*

(1) U *is quasi-increasing,*

(2) $R \subset U^*$ *implies* $R^\star \leq U$ *for any relation* R *on* X *to* Y.

Proof. If (1) holds and $R \subset U^*$, then

$$R^\star(A) = R[A] = \bigcup_{x\in A} R(x) \subset \bigcup_{x\in A} U^*(x) = \bigcup_{x\in A} U\big(\{x\}\big) \subset U(A)$$

for all $A \subset X$. Therefore, $R^\star \leq U$, and thus (2) also holds.

Conversely, if (2) holds, then because of $U^* \subset U^*$ we have $\big(U^*\big)^\star \leq U$. Therefore, for any $A \subset X$, we have

$$U^{*\star}(A) \subset U(A).$$

Moreover, by using the corresponding definitions, we can see that

$$U^{*\star}(A) = U^*[A] = \bigcup_{x\in A} U^*(x) = \bigcup_{x\in A} U\big(\{x\}\big).$$

Therefore, $\bigcup_{x\in A} U\big(\{x\}\big) \subset U(A)$, and thus (1) also holds. □

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 3 *For an arbitrary relation* R *and a quasi-increasing correlation* U *on* X *to* Y, *we have*

$$R^\star \leq U \iff R \subset U^*.$$

Remark 5 This corollary shows that the operation $\star$ and the restriction of $*$ to $\mathcal{Q}_1(X, Y)$ establish an increasing Galois connection between the posets $\mathcal{P}(X \times Y)$ and $\mathcal{Q}_1(X, Y)$.

Therefore, the extensive theory of Galois connections (see [2, 8, 5]) could be applied here. However, because of the simplicity of Definition 4, it seems now more convenient to use some elementary, direct proofs.

6 Some further properties of the operations $\star$ and $*$

By the corresponding definitions, we evidently have the following

Theorem 7 *Under the notation of Definition 4,*

(1) $R \subset S$ *implies* $R^\star \leq S^\star$ *for any relations* R *and* S *on* X *to* Y,

(2) $U \leq V$ *implies* $U^* \subset V^*$ *for any correlations* U *and* V *on* X *to* Y.

Remark 6 Note that, by using Corollary 3, instead of assertion (2), we could only prove that the restriction of the operation $*$ to $\mathcal{Q}_1(X, Y)$ is increasing.

From (2), by using Remark 3, we can immediately derive

Corollary 4 $U \subset V$ *also implies* $U^* \subset V^*$ *for any correlations* U *and* V *on* X *to* Y.

Moreover, we can also easily prove the following theorem whose first statement has also been established by Höhle and Kubiak [9].

Theorem 8 *For any two relations* R *and* S *on* X *to* Y,

(1) $R^{\star *} = R$,

(2) $R^\star \leq S^\star$ *implies* $R \subset S$.

Proof. By the corresponding definitions, we have

$$R^{\star *}(x) = \left(R^\star\right)^*(x) = R^\star\big(\{x\}\big) = R\left[\{x\}\right] = R(x)$$

for all $x \in X$. Therefore, (1) is also true.

To prove (2), note that if $R^\star \leq S^\star$ holds, then by Theorem 7 we also have $R^{\star *} \subset S^{\star *}$. Hence, by using (1), we can see that $R \subset S$ also holds. $\square$

Remark 7 The above theorem shows that the function $\star$ is injective, $*$ is onto $\mathcal{P}(X, Y)$, and $\star *$ is the identity function of $\mathcal{P}(X \times Y)$.

Moreover, by Theorems 7 and 8, we can also at once state

Corollary 5 *For any two relations* R *and* S *on* X *to* Y*, we have* $R \subset S$ *if and only if* $R^\star \leq S^\star$.

Concerning the dual operation $*\star$, we can only prove the following theorem which, to some extent, has also been established by Höhle and Kubiak [9] and Pataki [11].

Theorem 9 *For a corelation* U *on* X *to* Y*, the following assertions are equivalent:*

(1) $U^{*\star} = U$,

(2) U *is union-preserving*,

(3) $U = R^\star$ *for some relation* R *on* X *to* Y.

Proof. If (2) holds, then by the proof of Theorem 6, and Theorem 3, we have

$$U^{*\star}(A) = \bigcup_{x \in A} U(\{x\}) = U(A)$$

for all $A \subset X$. Therefore, (1) also holds.

Now, since (1) trivially implies (3), we need only show that (3) also implies (2). For this, note that if (3) holds, then

$$U(A) = R^\star(A) = R[A] = \bigcup_{x \in A} R(x) = \bigcup_{x \in A} R[\{x\}] = \bigcup_{x \in A} R^\star(\{x\}) = \bigcup_{x \in A} U(\{x\})$$

for all $A \subset X$. Therefore, by Theorem 3, assertion (2) also holds. □

Remark 8 The above theorem shows that the function $\star$ maps $\mathcal{P}(X \times Y)$ onto the family $\mathcal{Q}_3(X, Y)$ of all union-preserving correlations on X to Y.

Moreover, the restriction of $*$ to $\mathcal{Q}_3(X, Y)$ is injective and that of $*\star$ is the identity function of $\mathcal{Q}_3(X, Y)$. Therefore, the Galois connection mentioned in Remark 5 is rather particular.

Now, as an immediate consequence of Theorems 7 and 9, we can also state

Corollary 6 *For any two union-preserving correlations* U *and* V *on* X *to* Y*, we have* $U \leq V$ *if and only if* $U^* \subset V^*$.

Proof. Note that if $U^* \subset V^*$ holds, then by Theorem 7 we also have $U^{*\star} \leq V^{*\star}$. Hence, by Theorem 9, we can see that $U \leq V$ also holds. □

Moreover, in addition to Theorem 9, we can also prove the following

Theorem 10 *Under the notation* $\circ = *\star$*, for any two correlations* U *and* V *on* X *to* Y*, we have*

(1) $U^{\circ\circ} = U^{\circ}$,

(2) $U \leq V$ *implies* $U^{\circ} \leq V^{\circ}$,

(3) $U^{\circ} \leq U$ *if and only if* U *is quasi-increasing*.

Proof. Assertion (2) is immediate from Theorem 7. While, from the proof of Theorem 6, we know that

$$U^{\circ}(A) = U^{*\star}(A) = \bigcup_{x \in A} U(\{x\})$$

for all $A \subset X$. Hence, by Definition 2 and Theorem 1, it is clear that (3) is true.

Moreover, from the above equality, we can also see that

$$U^{\circ\circ}(A) = \bigcup_{x \in A} U^{\circ}(\{x\}) = \bigcup_{x \in A} U(\{x\}) = U^{\circ}(A)$$

for all $A \subset X$. Therefore, (1) is also true. □

Remark 9 The above theorem shows that the function $\circ$ is a projection operation operation on $\mathcal{Q}(X, Y)$ such that its restriction to $\mathcal{Q}_1(X, Y)$ is already an interior operation.

Moreover, from Theorem 9, we can see that, for any correlation U on X to Y, we have $U^{\circ} = U$ if and only if U is union-preserving. Therefore, $\mathcal{Q}_3(X, Y)$ is the family of all open elements of $\mathcal{Q}(X, Y)$.

Now, as some useful consequences of our former results, we can also easily prove the following two theorems.

Theorem 11 *If* R *is a relation on* X *to* Y *and* $U = R^{\star}$*, then*

(1) U *is the smallest quasi-increasing correlation on* X *to* Y *such that* $R \subset U^*$,

(2) U *is the largest union-preserving correlation on* X *to* Y *such that* $U^* \subset R$.

Proof. From Theorems 9 and 8, we can see that U is union-preserving and $\mathsf{U}^{*}=\mathsf{R}^{\star *}=\mathsf{R}$.

Moreover, if V is a quasi-increasing corelation on X to Y such that $\mathsf{R}\subset\mathsf{V}^{*}$, then by Theorem 6 we also have $\mathsf{R}^{\star}\leq\mathsf{V}$, and thus $\mathsf{U}\leq\mathsf{V}$. Therefore, (1) is true.

On the other hand, if V is a correlation on X to Y such that $\mathsf{V}^{*}\subset\mathsf{R}$, then by Theorem 7 we also have $\mathsf{V}^{*\star}\leq\mathsf{R}^{\star}$, and thus $\mathsf{V}^{*\star}\leq\mathsf{U}$. Hence, if in particular V is union-preserving, then by Theorem 9 we can see that $\mathsf{V}\leq\mathsf{U}$. Therefore, (2) is also true. □

Theorem 12 *If* U *is a correlation on* X *to* Y *and* $\mathsf{R}=\mathsf{U}^{*}$, *then*

(1) R *is the largest relation on* X *to* Y *such that* $\mathsf{R}^{\star}\leq\mathsf{U}$ *whenever* U *is quasi-increasing,*

(2) R *is the smallest relation on* X *to* Y *such that* $\mathsf{U}\leq\mathsf{R}^{\star}$ *whenever* U *is union-preserving.*

Proof. If U is quasi-increasing, then by Theorem 10 we have $\mathsf{R}^{\star}=\mathsf{U}^{*\star}=\mathsf{U}^{\circ}\leq\mathsf{U}$. While, if U is union-preserving, then by Theorem 9 we have $\mathsf{R}^{\star}=\mathsf{U}^{*\star}=\mathsf{U}$.

Moreover, if S is a relation on X to Y such that $\mathsf{S}^{\star}\leq\mathsf{U}$, then by Theorem 5 we also have $\mathsf{S}\subset\mathsf{U}^{*}$, and thus $\mathsf{S}\subset\mathsf{R}$ even if U is not supposed to be quasi-increasing. Thus, in particular (1) is true.

While, if S is a relation on X to Y such that $\mathsf{U}\leq\mathsf{S}^{\star}$, then by Theorem 7, we also have $\mathsf{U}^{*}\subset\mathsf{S}^{\star *}$. Hence, by the definition of R and Theorem 8, we can see that $\mathsf{R}\subset\mathsf{S}$ even if U is not supposed to be union-preserving. Thus, in particular (2) is also true. □

Remark 10 Concerning the operations $\star$ and $*$, it is also worth noticing that if R is relation and U is a correlation on X to Y, then by the corresponding definitions of [14] we have

(1) $\mathsf{R}^{\star}(A)=\mathrm{cl}_{\mathsf{R}^{-1}}(A)$ for all $A\subset\mathsf{X}$,

(2) $\mathsf{R}^{\star}\leq\mathsf{U}\iff A\in\mathrm{Int}_{\mathsf{R}}\big(\mathsf{U}(A)\big)$ for all $A\subset\mathsf{X}$.

Moreover, if U is quasi-increasing, then under the notation

$$\mathrm{Int}_{\star}(\mathsf{U})=\left\{\,\mathsf{S}\subset\mathsf{X}\times\mathsf{Y}:\quad\mathsf{S}^{\star}\leq\mathsf{U}\,\right\}$$

we have $\mathsf{U}^{*}=\max\big(\mathrm{Int}_{\star}(\mathsf{U})\big)=\bigcup\mathrm{Int}_{\star}(\mathsf{U})$ by assertion (1) in Theorem 12.

7 Compatibility of the operation $\star$ with some set and relation theoretic ones

Now, as some immediate consequence of the corresponding results on relations, we can also state the following theorems.

Theorem 13 *If* R *is a relation on* X *to* Y*, then for any family* $\mathcal{A}$ *of subsets of* X *we have*

(1) $R^\star(\bigcup\mathcal{A}) = \bigcup_{A\in\mathcal{A}} R^\star(A)$, (2) $R^\star(\bigcap\mathcal{A}) \subset \bigcap_{A\in\mathcal{A}} R^\star(A)$.

Theorem 14 *If* R *is a relation on* X *to* Y*, then for any* $A, B \subset X$ *we have*

(1) $R^\star(A)\setminus R^\star(B) \subset R^\star(A\setminus B)$, (2) $R^\star(A)^c \subset R^\star(A^c)$ *if* $Y = R[X]$.

Remark 11 If in particular R^{-1} is a function, then the corresponding equalities are also true in the above two theorems.

Theorem 15 *If* $\mathcal{R}$ *is a family of relations on* X *to* Y*, then for any* $A \subset X$ *we have*

(1) $(\bigcup\mathcal{R})^\star(A) = \bigcup_{R\in\mathcal{R}} R^\star(A)$, (2) $(\bigcap\mathcal{R})^\star(A) \subset \bigcap_{R\in\mathcal{R}} R^\star(A)$.

Theorem 16 *If* R *and* S *are relations on* X *to* Y*, then for any* $A \subset X$ *we have*

(1) $R^\star(A)\setminus S^\star(A) \subset (R\setminus S)^\star(A)$, (2) $R^\star(A)^c \subset R^{c\star}(A)$ *if* $A \neq \emptyset$.

Theorem 17 *If* R *is a relation on* X *to* Y*, then for any* $A \subset X$ *we have*

$$R^{c\star}(A)^c = \bigcap_{x\in A} R(x).$$

Moreover, we can also easily prove the following theorem which has also been established by Höhle and Kubiak [9].

Theorem 18 *For any two relations* R *on* X *to* Y *and* S *on* Y *to* Z*, we have*

$$(S\circ R)^\star = S^\star \circ R^\star.$$

Proof. By the corresponding definitions, we have

$$(S\circ R)^\star(A) = (S\circ R)[A] = S\big[R[A]\big] = S^\star\big(R^\star(A)\big) = \big(S^\star\circ R^\star\big)(A)$$

for all $A \subset X$. Therefore, the required equality is also true. □

From this theorem, by using Theorem 9, we can immediately derive

Corollary 7 *For an arbitrary relation on* R *on* X *to* Y *and a union-preserving correlation* V *on* Y *to* Z*, we have*

$$(V^* \circ R)^\star = V \circ R^\star .$$

In addition to Theorem 18, we can also easily prove the following correction of a false statement of Höhle and Kubiak [9].

Theorem 19 *For an arbitrary correlation* U *on* X *to* Y *and a union-preserving correlation* V *on* Y *to* Z*, we have*

$$(V \circ U)^* = V^* \circ U^* .$$

Proof. By the corresponding definitions and Theorem 9, we have

$$\begin{aligned} (V \circ U)^*(x) &= (V \circ U)\big(\{x\}\big) = V\,\big(U\big(\{x\}\big)\big) \\ &= V\,\big(U^*(x)\big) = V^{*\star}\big(U^*(x)\big) = V^*\big[\,U^*(x)\big] = \big(V^* \circ U^*\big)(x) \end{aligned}$$

for all $x \in X$. Therefore, the required equality is also true. □

From this theorem, by using Theorems 9 and 8, we can immediately derive

Corollary 8 *For a correlation* U *on* X *to* Y *and a relation* S *on* Y *to* Z*, we have*

$$(S^\star \circ U)^* = S \circ U^* .$$

Remark 12 In addition to Theorem 18, it is also worth mentioning that if R is a relation on X to Y and S is a relation on Z to W, then for any $A \subset X{\times}Z$ we have

$$(R \boxtimes S)^\star(A) = S \circ A \circ R^{-1} .$$

8 Partial compatibility of the operation $\star$ with the relation theoretic inversion

Theorem 20 *For a relation* R *on* X *to* Y*, the following assertions are equivalent:*

(1) $R^{-1} \circ R = \Delta_X$,

(2) $\big(R^\star\big)^{-1} \subset \big(R^{-1}\big)^\star$,

(3) R^{-1} *is a function on* Y *onto* X*.*

Proof. For any $x \in X$, we have

$$R^\star\big(\{x\}\big) = R\big[\{x\}\big] = R(x)\,, \qquad \text{and thus} \qquad \{x\} \in \big(R^\star\big)^{-1}\big(R(x)\big)\,.$$

Hence, if (2) holds, we can infer that

$$\{x\} \in \big(R^{-1}\big)^\star\big(R(x)\big)\,, \qquad \text{and thus} \qquad \big(R^{-1}\big)^\star\big(R(x)\big) = \{x\}.$$

Therefore,

$$R^{-1}\big[R(x)\big] = \{x\}, \qquad \text{and thus} \qquad \big(R^{-1} \circ R\big)(x) = \Delta_X(x)\,.$$

Hence, we can see that (1) also holds.

To prove the converse implication, note that if $A \subset X$ and $B \subset Y$ such that $A \in \big(R^\star\big)^{-1}(B)$, then we also have

$$R^\star(A) = B\,, \qquad \text{and thus} \qquad R\,[A] = B\,.$$

Hence, we can infer that

$$R^{-1}\big[R\,[A]\big] = R^{-1}[B]\,, \qquad \text{and thus} \qquad \big(R^{-1} \circ R\big)[A] = R^{-1}[B]\,.$$

Therefore, if (1) holds, then

$$\Delta_X[A] = R^{-1}[B]\,, \qquad \text{and thus} \qquad A = \big(R^{-1}\big)^\star(B)\,.$$

Hence, it is clear that (2) also holds.

Therefore, (1) and (2) are equivalent. The proof of the equivalence of (1) and (3) will be left to the reader. □

From Theorem 20, by writing R^{-1} in place of R we can immediately derive the following

Theorem 21 *For a relation* R *on* X *to* Y*, the following assertions are equivalent:*

(1) $R \circ R^{-1} = \Delta_Y$ *,*

(2) $\big(R^{-1}\big)^\star \subset \big(R^\star\big)^{-1}$ *,*

(3) R *is a function on* X *onto* Y*.*

Proof. Note that now R^{-1} is a relation on Y to X. Therefore, by Theorem 20, the following assertions are equivalent:

(a) $\big(R^{-1}\big)^{-1} \circ R^{-1} = \Delta_Y$;

(b) $\left(\left(R^{-1}\right)^{\star}\right)^{-1} \subset \left(\left(R^{-1}\right)^{-1}\right)^{\star}$;

(c) $\left(R^{-1}\right)^{-1}$ is a function on X onto Y.

Hence, since $R = \left(R^{-1}\right)^{-1}$, and

$$\left(R^{-1}\right)^{\star} \subset \left(R^{\star}\right)^{-1} \quad \Longleftrightarrow \quad \left(\left(R^{-1}\right)^{\star}\right)^{-1} \subset R^{\star},$$

it is clear that assertions (1), (2) and (3) are also equivalent. □

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 9 *For a relation* R *on* X *to* Y*, the following assertions are equivalent:*

(1) $\left(R^{\star}\right)^{-1} = \left(R^{-1}\right)^{\star}$,

(2) $R^{-1} \circ R = \Delta_X$ *and* $R \circ R^{-1} = \Delta_Y$,

(3) R *is an injective function of* X *onto* Y*.*

9 Partial compatibility of the operation $*$ with the relation theoretic inversion

From Theorem 20, by writing U^* in place of R, we can easily derive

Theorem 22 *If* U *is a union-preserving correlation on* X *to* Y *such that* $\left(U^*\right)^{-1}$ *is a function on* Y *onto* X*, then*

$$\left(U^{-1}\right)^* \subset \left(U^*\right)^{-1}.$$

Proof. Now, by Theorems and , we have

$$U^{-1} = \left(U^{*\star}\right)^{-1} = \left(\left(U^*\right)^{\star}\right)^{-1} \subset \left(\left(U^*\right)^{-1}\right)^{\star}.$$

Hence, by using Corollary and Theorem , we can infer that

$$\left(U^{-1}\right)^* \subset \left(\left(\left(U^*\right)^{-1}\right)^{\star}\right)^* = \left(\left(U^*\right)^{-1}\right)^{\star *} = \left(U^*\right)^{-1}.$$

□

From Theorem 21, we can quite similarly derive the following

Theorem 23 *If* U *is a union-preserving correlation on* X *to* Y *such that* U^* *is a function on* X *onto* Y, *then*

$$(U^*)^{-1} \subset (U^{-1})^*.$$

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 10 *If* U *is a union-preserving correlation on* X *to* Y *such that* U^* *is an injective function of* X *onto* Y, *then*

$$(U^*)^{-1} = (U^{-1})^*.$$

Moreover, by using Corollary 9, we can also easily prove the following

Theorem 24 *If* U *is an injective, union-preserving correlation on* X *to* Y *such that* U^{-1} *is also union-preserving, then the following assertions are equivalent:*

(1) $(U^*)^{-1} = (U^{-1})^*$,

(2) U^* *is an injective function of* X *onto* Y.

Proof. Now, since the implication (2) $\Longrightarrow$ (1) has already been established in Corollary 10, we need only prove that (1) also implies (2).

For this note that if (1) holds, then by Theorem 9 we also have

$$\left((U^*)^\star\right)^{-1} = (U^{*\star})^{-1} = U^{-1} = \left(U^{-1}\right)^{*\star} = \left(\left(U^{-1}\right)^*\right)^\star = \left((U^*)^{-1}\right)^\star.$$

Therefore, by Corollary 9, assertion (2) also holds. □

From Corollary 9, we can also immediately derive the following

Theorem 25 *For a symmetric relation* R *on* X, *the following assertions are equivalent:*

(1) $R^2 = \Delta_X$,

(2) $R^\star$ *is an involution*,

(3) R *is an injective function of* X *onto* Y.

Remark 13 Moreover, by Theorem 18, we can at once see that, for an arbitrary relation R on X, the correlation $R^\star$ is an involution if and only if $R^2 = \Delta_X$. That is, for any $x, y \in X$, we have $R(x) \cap R^{-1}(y) \neq \emptyset$ if and only if $x = y$.

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References

[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ. **25**, Providence, Rhode Island, 1967.

[2] T. S. Blyth, M. F. Janowitz, *Residuation theory*, Pergamon Press, Oxford, 1972.

[3] Z. Boros, Á. Száz, Infimum and supremarkum completeness properties of ordered sets without axioms, *An. Stiinţ. Univ. Ovidius Constanta Ser. Mat.*, **16** (2008), 1–7.

[4] S. Buglyó, Á. Száz, A more important Galois connection between distance functions and inequality relations, *Sci. Ser. A Math. Sci.* (N.S.), **18** (2009), 17–38.

[5] B. A. Davey, H. A. Priestley, *Introduction to lattices and order*, Cambridge University Press, Cambridge, 2002.

[6] K. Denecke, M. Erné, S. L. Wismath (Eds.) *Galois connections and applications*, Kluwer Academic Publisher, Dordrecht, 2004.

[7] B. Ganter, R. Wille, *Formal concept analysis*, Springer-Verlag, Berlin, 1999.

[8] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott, *A compendium of continuous lattices*, Springer-Verlag, Berlin, 1980.

[9] U. Höhle, T. Kubiak, On regularity of sup-preserving maps: generalizing Zareckiĭ's theoremark, *Semigroup Forum*, **83** (2011), 313–319.

[10] O. Ore, Galois connexions, *Trans. Amer. Math. Soc.*, **55** (1944), 493–513.

[11] G. Pataki, On a generalized infimal convolution of set functions, *Ann. Math. Sil.*, **27** (2013), 99–106.

[12] J. Schmidt, Beiträge zur filtertheorie II, *Math. Nachr.*, **10** (1953), 197–232.

[13] Á. Száz, A Galois connection between distance functions and inequality relations, *Math. Bohem.*, **127** (2002), 437–448.

[14] Á. Száz, Upper and lower bounds in relator spaces, *Serdica Math. J.*, **29** (2003), 239–270.

[15] Á. Száz, *Lower and upper bounds in ordered sets without axioms*, Tech. Rep., Inst. Math., Univ. Debrecen, **2004/1**, 11 pp.

[16] Á. Száz, *Galois-type connections on power sets and their applications to relators*, Tech. Rep., Inst. Math., Univ. Debrecen, **2005/2**, 38 pp.

[17] Á. Száz, Supremum properties of Galois–type connections, *Comment. Math. Univ. Carolin.*, **47** (2006), 569–583.

[18] Á. Száz, Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities, *Filomat*, **21** (2007), 87–97.

[19] Á. Száz, Galois type connections and closure operations on preordered sets, *Acta Math. Univ. Comenian.*, **78** (2009), 1–21.

[20] Á. Száz, Galois-type connections and continuities of pairs of relations, *J. Int. Math. Virtual Inst.*, **2** (2012), 39–66.

[21] Á. Száz, Inclusions for composition and box products of relations, *J. Int. Math. Virt. Inst.*, **3** (2013), 97–125.

[22] Á. Száz, Galois and Pataki connections revisited, *Tech. Rep.*, Inst. Math., Univ. Debrecen, **2013/3**, 20 pp.

[23] Á. Száz, Generalizations of Galois and Pataki connections to relator spaces, *Tech. Rep.*, Inst. Math., Univ. Debrecen, **2014/1**, 32 pp.

[24] Á. Száz, J. Túri, Comparisons and compositions of Galois–type connections, *Miskolc Math. Notes*, **7** (2006), 189–203.

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On the combinatorics of extensions of preinjective Kronecker modules

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Abstract. We explore the combinatorial properties of a particular type of extension monoid product of preinjective Kronecker modules. The considered extension monoid product plays an important role in matrix completion problems. We state theorems which characterize this product in both implicit and explicit ways and we prove that the conditions given in the definition of the generalized majorization are equivalent with our criteria. Generalized majorization is a purely combinatorial construction introduced by its authors in a different setting.

1 Introduction

In order to understand the motivation behind our work we need to recall briefly the notion of matrix pencil and the problem of matrix subpencil. Kronecker modules and related notions will be presented in Section 3.

A *matrix pencil* over a field κ is a matrix $A + \lambda B$ where A, B are matrices over κ of the same size and λ is an indeterminate. Two pencils $A+\lambda B$, $A'+\lambda B'$ are *strictly equivalent*, denoted by $A+\lambda B \sim A'+\lambda B'$, if and only if there exists invertible, constant (λ independent) matrices P, Q such that $P(A' + \lambda B')Q = A + \lambda B$.

Every matrix pencil is strictly equivalent to a canonical diagonal form, described by the *classical Kronecker invariants*, namely *the minimal indices for columns, the minimal indices for rows, the finite elementary divisors* and *the infinite elementary divisors* (see [7] for all the details).

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A pencil $A'+\lambda B'$ is called *subpencil* of $A+\lambda B$ if and only if there are pencils $A_{12}+\lambda B_{12}$, $A_{21}+\lambda B_{21}$, $A_{22}+\lambda B_{22}$ such that

$$A+\lambda B \sim \begin{pmatrix} A'+\lambda B' & A_{12}+\lambda B_{12} \\ A_{21}+\lambda B_{21} & A_{22}+\lambda B_{22} \end{pmatrix}.$$

In this case we also say that the subpencil can be completed to the bigger pencil.

There is an unsolved challenge in pencil theory with lots of applications in control theory (problems related to pole placement, non-regular feedback, dynamic feedback etc. may be formulated in terms of matrix pencils, for details see [9]). This important open problem can be formulated in the following way: if $A+\lambda B$, $A'+\lambda B'$ are pencils over $\mathbb{C}$, find a necessary and sufficient condition in terms of their classical Kronecker invariants for $A'+\lambda B'$ to be a subpencil of $A+\lambda B$. Also construct the completion pencils $A_{12}+\lambda B_{12}$, $A_{21}+\lambda B_{21}$, $A_{22}+\lambda B_{22}$.

Han Yang was the first to give a representation theoretical modular approach to the matrix subpencil problem, the connection being detailed in [8]. Also, the Kronecker invariants of a module correspond to the classical Kronecker invariants of the associated pencil. In this way a one-to-one correspondence can be made between isoclasses of Kronecker modules and equivalence classes of matrix pencils (with respect to the strict equivalence relation mentioned earlier). In particular, preinjective Kronecker modules correspond to matrix pencils having only minimal indices for columns. This correspondence between matrix pencils and Kronecker modules allows us to deal with the matrix subpencil problem on a module theoretical level, armed with new tools and insights, in addition to the "classical" approach (linear algebra, matrix theory, combinatorics). The matrix subpencil problem itself can be formulated in a very elegant and succinct way in terms of the extension monoid product of certain Kronecker modules (see [17]).

Particular cases of the matrix subpencil problem were considered by Dodig and Stošić in a series of articles (e.g. [3, 4, 6]). One can see that one of the central notions of their work is the so-called generalized majorization, a generalization of the dominance of partitions (which is a well-known notion in partition combinatorics).

Generalized majorization seems to be inevitable when dealing with pencil completion problems. In this paper we give a module theoretical interpretation of this purely combinatorial construction in the form of a particular extension monoid product, together with equivalent formulations and a simple linear-time algorithm to work with in practice.

The paper is organized in the following way:

- In Section 2 we recall some elementary notions of partition combinatorics, and also present the notion of generalized majorization. Generalized majorization was introduced in [5] and is intensively studied and used by the authors in dealing with technical difficulties of matrix completion problems (e.g. [3, 4, 6]).

- Section 3 is dedicated to a brief survey of the category of Kronecker modules, presenting in some detail the preinjective (and dually preprojective) Kronecker modules.

- In Section 4 we present the notion of extension monoid product, as it applies in the case of preinjective Kronecker modules. Also, this is the place for our new results: Theorem 6 and Theorem 7 giving an implicit and respectively an explicit combinatorial description, followed by an easy linear-time algorithm. Corollary 2 establishes the link between the extension monoid product of preinjective Kronecker modules and the generalized majorization.

We emphasize that all our new results are valid in a field independent context and can be dualized to preinjective modules in a natural way.

From now on, throughout the paper empty sums are considered to be zero. In case of integers a and b, by $\{a, \dots, b\}$ we mean the set of all integers x, such that $a \leq x \leq b$, so if $a > b$, then $\{a, \dots, b\} = \emptyset$. We will usually denote sequences of integers like $(a_1, a_2, \dots, a_n)$. If in a certain sequence or subsequence the index of the first element is strictly greater than the index of the last one, the sequence is regarded as being empty.

2 Some elementary notions of partition combinatorics

An *integer sequence* is a sequence $\mathbf{a} = (a_1, a_2, \dots)$ of integers, with only finitely many nonzero elements. The largest integer $l \geq 0$ with $a_l \neq 0$ is called the *length* of $\mathbf{a}$, denoted by $\ell(\mathbf{a})$ (if $\mathbf{a}$ is a sequence consisting only of zeros, then $\ell(\mathbf{a}) = 0$). We will not distinguish between integer sequences which differ only in the number of zero elements after the l^{th} position, therefore we regard $(a_1, a_2, \dots, a_l)$, $(a_1, a_2, \dots, a_l, 0)$, $(a_1, a_2, \dots, a_l, 0, \dots, 0)$ and $(a_1, a_2, \dots, a_l, 0, \dots)$ as being the same integer sequence. Clearly, $\mathbf{a} \in \mathbb{Z}^n$ for some $n \geq \max\{\ell(\mathbf{a}), 1\}$. The *weight* of an integer sequence is the sum of its elements, denoted by $|\mathbf{a}| = a_1 + a_2 + \cdots$.

A *raising operator* R is defined in the following way (on the set of integer sequences having length at most n):

$$R : \mathbb{Z}^n \to \mathbb{Z}^n, \quad R = \prod_{i<j} R_{ij}^{r_{ij}},$$

where $r_{ij} \in \mathbb{N}$ and $R_{ij} : \mathbb{Z}^n \to \mathbb{Z}^n$,

$$R_{ij}(\mathbf{a}) = (a_1, \ldots, a_i + 1, a_{i+1}, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots, a_n)$$

for any pair of integers i and j with $1 \leq i < j \leq n$ and any $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. Note that the terms in the product above commute with each other.

If $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$ are integer sequences and $\sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \lambda_i$ for all $1 \leq k$, we say that $\boldsymbol{\mu}$ is dominated (or majored) by $\boldsymbol{\lambda}$. The *dominance* (or *majorization*) relation is a partial order on the set of integer sequences and is denoted by $\boldsymbol{\mu} \prec \boldsymbol{\lambda}$.

Another natural order on the set of integer sequences is the *lexicographical ordering*. If $\boldsymbol{\mu} \neq \boldsymbol{\lambda}$ then $\boldsymbol{\lambda}$ is lexicographically strictly greater than $\boldsymbol{\mu}$ if for the smallest i such that $\mu_i \neq \lambda_i$ one has $\lambda_i > \mu_i$. The lexicographical order is a total order on the set of integer sequences.

The following two theorems make the connection between raising operators and dominance relation (for proofs see [10]).

Theorem 1 *Let* $\mathbf{a} \in \mathbb{Z}^n$ *and* R *a raising operator. Then* $\mathbf{a} \prec R\mathbf{a}$.

Conversely, we have:

Theorem 2 *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$ *be such that* $\mathbf{a} \prec \mathbf{b}$ *and* $|\mathbf{a}| = |\mathbf{b}|$. *Then there exists a raising operator* R *such that* $\mathbf{b} = R\mathbf{a}$.

If the elements of the integer sequence $\mathbf{a}$ are weakly ordered and nonnegative (i.e. $a_1 \geq a_2 \geq \cdots a_l \geq 0$), then we call $\mathbf{a}$ a *partition* of $m = |\mathbf{a}|$. Naturally, everything said so far about integer sequences applies in the case of partitions as well. In particular, the dominance (or majorization) relation is a partial order on the set of partitions. If we denote by $\mathcal{P}_m$ the set of partitions of m, we can put together the two previous theorems in form of the following corollary:

Corollary 1 *Let* $\mathbf{a}, \mathbf{b} \in \mathcal{P}_m$ *be two partitions of* m. *Then* $\mathbf{a} \prec \mathbf{b}$ *if and only if there is a raising operator* R *such that* $\mathbf{b} = R\mathbf{a}$.

In [5] the authors consider a generalization of the dominance relation, the so-called generalized majorization, defined as follows:

Definition 1 *Consider the partitions* $\mathbf{d} = (d_1, \dots, d_x)$, $\mathbf{a} = (a_1, \dots, a_y)$ *and* $\mathbf{g} = (g_1, \dots, g_{x+y})$. *Then* $\mathbf{g}$ *is said to be majorized by* $\mathbf{d}$ *and* $\mathbf{a}$ *if the following conditions hold:*

$$d_i \geq g_{i+y}, \quad i = 1, \dots x, \tag{1}$$

$$\sum_{i=1}^{h_j} g_i - \sum_{i=1}^{h_j - j} d_i \leq \sum_{i=1}^{j} a_i, \quad j = 1, \dots, y, \tag{2}$$

$$\sum_{i=1}^{x+y} g_i = \sum_{i=1}^{x} d_i + \sum_{i=1}^{y} a_i. \tag{3}$$

Here $h_j := \min\{i | d_{i-j+1} < g_i\}$, $j = 1, \dots, y$. *This relation is called the* generalized majorization *and is denoted in the following way:* $\mathbf{g} \prec' (\mathbf{d}, \mathbf{a})$.

Remark 1 *Observe that in the previous definition we have* $0 < h_1 < h_2 < \dots < h_y < x + y + 1$ *for the values* h_j. *Also, this strictly increasing sequence determines another one, denoted by* $0 < h'_1 < h'_2 < \dots < h'_x < x + y + 1$, *in the following way:*

$$h'_i = \begin{cases} \min\{l \in \{1, \dots, x+y\} | l \neq h_j, 1 \leq j \leq y\} & i = 1 \\ \min\{l \in \{h'_{i-1} + 1, \dots, x+y\} | l \neq h_j, 1 \leq j \leq y\} & 1 < i \leq x \end{cases}.$$

The elements of these two sequences form disjoint sets, moreover we have $d_i \geq g_{h'_i}$ *for all* $i \in \{1, \dots, h_y - y\}$, *the sequence* $(h'_1, h'_2, \dots, h'_{h_y - y})$ *being lexicographically the smallest one with this property. Conversely, if there are sequences* $(h'_1, h'_2, \dots, h'_x)$ *satisfying* $d_i \geq g_{h'_i}$ *for all* $i \in \{1, \dots, x\}$ *we can define the sequence* $(h_1, h_2, \dots, h_y)$ *in terms of lexicographically the smallest such sequence* $(h'_1, h'_2, \dots, h'_x)$ *in the following way:*

$$h_i = \begin{cases} \min\{l \in \{1, \dots, x+y\} | l \neq h'_j, 1 \leq j \leq x\} & i = 1 \\ \min\{l \in \{h_{i-1} + 1, \dots, x+y\} | l \neq h'_j, 1 \leq j \leq x\} & 1 < i \leq y \end{cases},$$

then we get back exactly the sequence $(h_1, h_2, \dots, h_y)$ *given in Definition 1.*

3 The category of Kronecker modules

In this section we present a short compilation of definitions and well-known facts about the category of Kronecker modules, with emphasis on preinjective (and dually preprojective) Kronecker modules. The calculations, justifications and proofs leading to these results can be found in many standard textbooks on representation theory of algebras (e.g. [1, 2, 12, 13]).

Let K be the *Kronecker quiver*

$$K:\ 1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$$

and κ an arbitrary field. The path algebra of the Kronecker quiver is the *Kronecker algebra* and we will denote it by κK. A finite dimensional right module over the Kronecker algebra is called a *Kronecker module*. We denote by mod-κK the category of finite dimensional right modules over the Kronecker algebra.

A (finite dimensional) κ-linear representation of the quiver K is a quadruple $M = (V_1, V_2; \varphi_\alpha, \varphi_\beta)$ where V_1, V_2 are finite dimensional κ-vector spaces (corresponding to the vertices) and $\varphi_\alpha, \varphi_\beta : V_2 \to V_1$ are κ-linear maps (corresponding to the arrows). Thus a κ-linear representation of K associates vector spaces to the vertices and compatible κ-linear functions (or equivalently, matrices) to the arrows. Let us denote by rep-κK the category of finite dimensional κ-representations of the Kronecker quiver. There is a well-known equivalence of categories between mod-κK and rep-κK, so that every Kronecker module can be identified with a representation of K.

The *simple Kronecker modules* (up to isomorphism) are

$$S_1 : \kappa \leftleftarrows 0 \quad \text{and} \quad S_2 : 0 \leftleftarrows \kappa.$$

For a Kronecker module M we denote by $\underline{\dim} M$ its *dimension* and by $[M]$ the isomorphism class of M. The dimension of M is a vector

$$\underline{\dim} M = ((\dim M)_1, (\dim M)_2) = (m_{S_1}(M), m_{S_2}(M)),$$

where $m_{S_i}(M)$ is the number of factors isomorphic with the simple module S_i in a composition series of M, $i = \overline{1,2}$. As a representation $M : V_1 \underset{\varphi_\beta}{\overset{\varphi_\alpha}{\leftleftarrows}} V_2$, we have that $\underline{\dim} M = (\dim_\kappa V_1, \dim_\kappa V_2)$.

The *defect* of $M \in$ mod-κK with $\underline{\dim} M = (a, b)$ is defined in the Kronecker case as $\partial M = b - a$.

An indecomposable module $M \in$ mod-κK is a member in one of the following three families: preprojectives, preinjectives and regulars. In what follows we give some details on the first two of these families.

The *preprojective indecomposable Kronecker module*s are determined up to isomorphism by their dimension vector. For $n \in \mathbb{N}$ we will denote by P_n the indecomposable preprojective module of dimension $(n+1, n)$. So P_0 and P_1 are the projective indecomposable modules ($P_0 = S_1$ being simple). It is known that (up to isomorphism) $P_n = (\kappa^{n+1}, \kappa^n; f, g)$, where choosing the canonical basis in κ^n and κ^{n+1}, the matrix of $f : \kappa^n \to \kappa^{n+1}$ (respectively of $g : \kappa^n \to \kappa^{n+1}$) is $\begin{pmatrix} \mathbb{I}_n \\ 0 \end{pmatrix}$ (respectively $\begin{pmatrix} 0 \\ \mathbb{I}_n \end{pmatrix}$). Thus in this case

$$P_n : \ \kappa^{n+1} \underset{\left(\begin{smallmatrix} \mathbb{I}_n \\ 0 \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 0 \\ \mathbb{I}_n \end{smallmatrix}\right)}{\leftleftarrows}} \kappa^n \ ,$$

where $\mathbb{I}_n$ is the identity matrix. We have for the defect $\partial P_n = -1$.

We define a *preprojective Kronecker module* P as being a direct sum of indecomposable preprojective modules: $P = P_{a_1} \oplus P_{a_2} \oplus \cdots \oplus P_{a_l}$, where we use the convention that $a_1 \le a_2 \le \cdots \le a_l$.

The *preinjective indecomposable Kronecker module*s are also determined up to isomorphism by their dimension vector. For $n \in \mathbb{N}$ we will denote by I_n the indecomposable preinjective module of dimension $(n, n+1)$. So I_0 and I_1 are the injective indecomposable modules ($P_0 = S_2$ being simple). It is known that (up to isomorphism) $I_n = (\kappa^n, \kappa^{n+1}; f, g)$, where choosing the canonical basis in κ^{n+1} and κ^n, the matrix of $f : \kappa^{n+1} \to \kappa^n$ (respectively of $g : \kappa^{n+1} \to \kappa^n$) is $\begin{pmatrix} \mathbb{I}_n & 0 \end{pmatrix}$ (respectively $\begin{pmatrix} 0 & \mathbb{I}_n \end{pmatrix}$). Thus in this case

$$I_n : \ \kappa^n \underset{(0\, \mathbb{I}_n)}{\overset{(\mathbb{I}_n\, 0)}{\leftleftarrows}} \kappa^{n+1} \ ,$$

where $\mathbb{I}_n$ is the identity matrix. We have for the defect $\partial I_n = 1$.

We define a *preinjective Kronecker module* I as being a direct sum of indecomposable preinjective modules: $I = I_{a_1} \oplus I_{a_2} \oplus \cdots \oplus I_{a_l}$, where we use the convention that $a_1 \ge a_2 \ge \cdots \ge a_l$.

The sequence $(a_1, a_2, \ldots, a_l)$ determines the preinjective (respectively) preprojective Kronecker module up to isomorphism therefore this sequence is called a *Kronecker invariant* of the module.

The category of Kronecker modules has been extensively studied because the Kronecker algebra is a very important example of a tame hereditary algebra. Moreover, the category has also a geometric interpretation, since it is derived equivalent with the category of coherent sheaves on the projective line. In addition, Kronecker modules correspond to matrix pencils in linear algebra, so the Kronecker algebra relates representation theory with numerical linear algebra and matrix theory.

4 The extension monoid product of preinjective Kronecker modules

For $d \in \mathbb{N}^2$ let $M_d = \{[M] | M \in \text{mod-}\kappa K, \underline{\dim} M = d\}$ be the set of isomorphism classes of Kronecker modules of dimension d. Following Reineke in [11] for subsets $\mathcal{A} \subset M_d$, $\mathcal{B} \subset M_e$ we define

$$\mathcal{A} * \mathcal{B} = \{[X] \in M_{d+e} \,|\, \exists\, 0 \to N \to X \to M \to 0 \text{ exact for some } [M] \in \mathcal{A}, [N] \in \mathcal{B}\}.$$

So the product $\mathcal{A} * \mathcal{B}$ is the set of isoclasses of all extensions of modules M with $[M] \in \mathcal{A}$ by modules N with $[N] \in \mathcal{B}$. This is in fact Reineke's extension monoid product using isomorphism classes of modules instead of modules. It is important to know (see [11]) that the product above is associative, i.e. for $\mathcal{A} \subset M_d$, $\mathcal{B} \subset M_e$, $\mathcal{C} \subset M_f$, we have $(\mathcal{A} * \mathcal{B}) * \mathcal{C} = \mathcal{A} * (\mathcal{B} * \mathcal{C})$. Also $\{[0]\} * \mathcal{A} = \mathcal{A} * \{[0]\} = \mathcal{A}$. We will call the operation "$*$" simply the *extension monoid product.*

Remark 2 *For* $M, N \in \text{mod-}\kappa K$ *and* κ *finite, the product* $\{[M]\} * \{[N]\}$ *coincides with the set* $\{[M][N]\}$ *of terms in the Ringel-Hall product* $[M][N]$ *(see Section 4 from* [18]*).*

From now on we deal only with the extension monoid product of preinjective Kronecker modules. It is very important to mention that all results can be dualized in natural way to preprojective Kronecker modules as well.

According to the main result from [16] (Theorem 3.3), the possible middle terms in preprojective (and dually preinjective) short exact sequences do not depend on the base field. This allows us to describe the combinatorial rules governing the extension monoid product of preinjective Kronecker modules in a field independent way. Specifically, this allows us to restate the main result from [18] involving the Ringel-Hall product (valid only over finite fields) in terms of the extension monoid product (in a field independent manner). The following theorem gives an implicit description of the extension monoid product of two arbitrary preinjective Kronecker modules over an arbitrary field:

Theorem 3 *If* $a_1 \geq \cdots a_p \geq 0$*,* $b_1 \geq \cdots \geq b_n \geq 0$ *and* $c_1 \geq \cdots \geq c_r \geq 0$ *are nonnegative integers, then*

$$[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a_1} \oplus \cdots \oplus I_{a_p}]\} * \{[I_{b_1} \oplus \cdots \oplus I_{b_n}]\}$$

if and only if $r = n + p$*,* $\exists \beta : \{1, \dots, n\} \to \{1, \dots, n + p\}$*,* $\exists \alpha : \{1, \dots, p\} \to \{1, \dots, n+p\}$ *both functions strictly increasing with* $\mathrm{Im}\alpha \cap \mathrm{Im}\beta = \emptyset$ *and* $\exists m_j^i \geq 0$*,* $1 \leq i \leq n$*,* $1 \leq j \leq p$*, such that* $\forall\ \ell \in \{1, \dots, n + p\}$

$$c_\ell = \begin{cases} b_i - \sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq j\leq p}} m_j^i, & \textit{where } i = \beta^{-1}(\ell) \quad \ell \in \mathrm{Im}\beta \\ a_j + \sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq i\leq n}} m_j^i, & \textit{where } j = \alpha^{-1}(\ell) \quad \ell \in \mathrm{Im}\alpha \end{cases}. \tag{4}$$

We can formulate another version of the previous theorem, based on Lemma 4 from [19], giving another equivalent characterization of the considered extension monoid product:

Theorem 4 *If* $a_1 \geq \dots a_p \geq 0$*,* $b_1 \geq \cdots \geq b_n \geq 0$ *and* $c_1 \geq \cdots \geq c_r \geq 0$ *are nonnegative integers, then*

$$[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a_1} \oplus \cdots \oplus I_{a_p}]\} * \{[I_{b_1} \oplus \cdots \oplus I_{b_n}]\}$$

if and only if $r = n + p$*,* $\sum_{i=1}^{n+p} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$*,* $\exists \beta : \{1, \dots, n\} \to \{1, \dots, n+p\}$*,* $\exists \alpha : \{1, \dots, p\} \to \{1, \dots, n+p\}$ *both functions strictly increasing with* $\mathrm{Im}\alpha \cap \mathrm{Im}\beta = \emptyset$ *such that* $b_i \geq c_{\beta(i)}$ *and* $a_j \leq c_{\alpha(j)}$ *for* $1 \leq i \leq n$*,* $1 \leq j \leq p$ *and for any* $j \in \{1, \dots, p\}$ *the following inequality is satisfied:*

$$\sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq i\leq n}} (b_i - c_{\beta(i)}) \geq \sum_{k=1}^{j} (c_{\alpha(k)} - a_k).$$

The following combinatorial rule describes products of the form $\{[I_{a_n}]\} * \{[I_{a_{n-1}}]\} * \cdots * \{[I_{a_1}]\}$ with $0 \leq a_n \leq a_{n-1} \leq \cdots \leq a_1$ increasing. It has been proved in [15] for finite fields and also in [20] (in a field independent context):

Theorem 5 *Suppose that* $a_1 \geq \cdots \geq a_n \geq 0$ *and* $c_1 \geq \cdots \geq c_n \geq 0$*. Then*

$$[I_{c_1} \oplus I_{c_2} \oplus \cdots \oplus I_{c_n}] \in \{[I_{a_n}]\} * \{[I_{a_{n-1}}]\} \cdots * \{[I_{a_1}]\}$$

if and only if $\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} a_i$ *for all* $k \in \{1, \dots, n\}$ *with* $\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} a_i$*.*

Remark 3 *The condition* $\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} a_i$ *in Theorem 5 says that the partition* $\mathbf{c} = (c_1, c_2, \dots, c_n)$ *is dominated by* $\mathbf{a} = (a_1, a_2, \dots, a_n)$, *i.e.* $\mathbf{c} \prec \mathbf{a}$.

Using Theorem 4 and Theorem 5 we are ready now to prove our first result, which is a characterization of the products of the form

$$\{[I_{a_p}]\} * \{[I_{a_{p-1}}]\} * \cdots * \{[I_{a_1}]\} * \{[I_{b_1}]\} * \{[I_{b_2}]\} * \cdots * \{[I_{b_n}]\},$$

where the integers $0 \leq a_p \leq a_{p-1} \leq \cdots \leq a_1$ are increasingly ordered, whereas the integers $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ are decreasing.

Theorem 6 *If* $a_1 \geq \cdots a_p \geq 0$, $b_1 \geq \cdots \geq b_n \geq 0$ *and* $c_1 \geq \cdots \geq c_r \geq 0$ *are nonnegative integers, then*

$$[I_{c_1} \oplus I_{c_2} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a_p}]\} * \{[I_{a_{p-1}}]\} * \cdots * \{[I_{a_1}]\} * \{[I_{b_1}]\} * \{[I_{b_2}]\} * \cdots * \{[I_{b_n}]\}$$

if and only if $r = n + p$, $\sum_{i=1}^{n+p} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$, $\exists\, \beta : \{1, \dots, n\} \to \{1, \dots, n+p\}$, $\exists\, \alpha : \{1, \dots, p\} \to \{1, \dots, n+p\}$ *both functions strictly increasing with* $\mathrm{Im}\alpha \cap \mathrm{Im}\beta = \emptyset$ *such that* $b_i \geq c_{\beta(i)}$ *for* $1 \leq i \leq n$ *and for any* $j \in \{1, \dots, p\}$ *the following inequality is satisfied:*

$$\sum_{\substack{\beta(i)<\alpha(j) \\ 1 \leq i \leq n}} (b_i - c_{\beta(i)}) \geq \sum_{k=1}^{j} (c_{\alpha(k)} - a_k). \tag{5}$$

Proof. As a first step, observe that $\{[I_{a_p}]\} * \{[I_{a_{p-1}}]\} * \cdots * \{[I_{a_1}]\} * \{[I_{b_1}]\} * \{[I_{b_2}]\} * \cdots * \{[I_{b_n}]\} = \left(\{[I_{a_p}]\} * \{[I_{a_{p-1}}]\} * \cdots * \{[I_{a_1}]\}\right) * \left(\{[I_{b_1}]\} * \{[I_{b_2}]\} * \cdots * \{[I_{b_n}]\}\right) = \left(\{[I_{a_p}]\} * \{[I_{a_{p-1}}]\} * \cdots * \{[I_{a_1}]\}\right) * \{[I_{b_1} \oplus I_{b_2} \oplus \cdots \oplus I_{b_n}]\}$. Here we have used the associativity of the extension monoid product and we have applied repeatedly Theorem 3 on the second part of the product to get the equality $\{[I_{b_1}]\} * \{[I_{b_2}]\} * \cdots * \{[I_{b_n}]\} = \{[I_{b_1} \oplus I_{b_2} \oplus \cdots \oplus I_{b_n}]\}$.

As for the first part of the product we use Theorem 5 to write $\{[I_{a_p}]\} * \cdots * \{[I_{a_1}]\} = \{[I_{a'_1} \oplus \cdots \oplus I_{a'_p}] \mid (a'_1, \dots, a'_p) \prec (a_1, \dots, a_p),\ \sum_{i=1}^{p} a'_i = \sum_{i=1}^{p} a_i\}$. Hence we have that $[I_{c_1} \oplus I_{c_2} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a_p}]\} * \{[I_{a_{p-1}}]\} * \cdots * \{[I_{a_1}]\} * \{[I_{b_1}]\} * \{[I_{b_2}]\} * \cdots * \{[I_{b_n}]\}$ if and only if there exists a partition $\mathbf{a'} = (a'_1, \dots, a'_p)$ such that $|\mathbf{a'}| = |\mathbf{a}|$, $\mathbf{a'} \prec \mathbf{a} = (a_1, \dots, a_p)$ and $[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a'_1} \oplus \cdots \oplus I_{a'_p}]\} * \{[I_{b_1} \oplus \cdots \oplus I_{b_n}]\}$. For the rest of the proof we will work with this equivalent statement.

"$\Longrightarrow$" Suppose there is a partition $\mathbf{a'} = (a'_1, \dots, a'_p)$ such that $|\mathbf{a'}| = |\mathbf{a}|$, $\mathbf{a'} \prec \mathbf{a}$ and $[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a'_1} \oplus \cdots \oplus I_{a'_p}]\} * \{[I_{b_1} \oplus \cdots \oplus I_{b_n}]\}$. Using Theorem 4 we immediately get the equalities $r = n+p$, respectively $\sum_{i=1}^{n+p} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$ and the existence of the strictly increasing functions $\beta : \{1,\dots,n\} \to \{1,\dots,n+p\}$ and $\alpha : \{1,\dots,p\} \to \{1,\dots,n+p\}$ with disjoint images such that $b_i \geq c_{\beta(i)}$ for $1 \leq i \leq n$ and $\sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq i\leq n}} (b_i - c_{\beta(i)}) \geq \sum_{k=1}^{j} (c_{\alpha(k)} - a'_k)$ for all $j \in \{1,\dots,p\}$. By reordering the last inequality and using the fact that $\mathbf{a}' \prec \mathbf{a}$ we obtain

$$\sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq i\leq n}} b_i + \sum_{k=1}^{j} a_k \geq \sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq i\leq n}} b_i + \sum_{k=1}^{j} a'_k \geq \sum_{k=1}^{j} c_{\alpha(k)} + \sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq i\leq n}} c_{\beta(i)},$$

leading to $\sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq i\leq n}} (b_i - c_{\beta(i)}) \geq \sum_{k=1}^{j} (c_{\alpha(k)} - a_k)$ as desired.

"$\Longleftarrow$" Conversely, suppose that the inequalities (5) and all other conditions from the right-to-left implication are satisfied. If in addition $a_j \leq c_{\alpha(j)}$ for $1 \leq j \leq p$, then using Theorem 4 we are done (in this case $\mathbf{a}' = \mathbf{a}$). If $a_j > c_{\alpha(j)}$ for some $j \in \{1,\dots,p\}$, then there exists a raising operator R and a partition $\mathbf{a}' = (a'_1,\dots,a'_p)$ such that $\mathbf{a} = R\mathbf{a}'$ and $a'_j \leq c_{\alpha(j)}$ for $1 \leq j \leq p$ (using the fact that $\sum_{i=1}^{n+p} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$ and $b_i \geq c_{\beta(i)}$ for $1 \leq i \leq n$). Suppose in addition that $\mathbf{a}'$ is lexicographically the greatest partition with the mentioned property. Then the inequality

$$\sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq i\leq n}} (b_i - c_{\beta(i)}) \geq \sum_{k=1}^{j} (c_{\alpha(k)} - a'_k)$$

is satisfied for all $j \in \{1,\dots,p\}$. Since $\mathbf{a} = R\mathbf{a}'$, by Corollary 1 we have $\mathbf{a}' \prec \mathbf{a}$ and therefore $[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a'_1} \oplus \cdots \oplus I_{a'_p}]\} * \{[I_{b_1} \oplus \cdots \oplus I_{b_n}]\}$ (since all the conditions from Theorem 4 are fulfilled). □

As one can see, all we had to do to obtain the characterization of products of the form $\{[I_{a_p}]\} * \{[I_{a_{p-1}}]\} * \cdots * \{[I_{a_1}]\} * \{[I_{b_1}]\} * \{[I_{b_2}]\} * \cdots * \{[I_{b_n}]\}$ with $0 \leq a_p \leq a_{p-1} \leq \cdots \leq a_1$ and $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ was a relaxation of Theorem 4 by dropping the condition $a'_j \leq c_{\alpha(j)}$ for $1 \leq j \leq p$. We can do the very same thing with the explicit version of Theorem 4 (which is Theorem 6 from [19]). We state the following:

Theorem 7 *Let* $a_1 \geq \cdots a_p \geq 0$, $b_1 \geq \cdots \geq b_n \geq 0$, $c_1 \geq \cdots \geq c_r \geq 0$ *be decreasing sequences of nonnegative integers and let* $B_j = \{l \in \{0,\dots,n\} | \sum_{k=1}^{l} b_k + \sum_{k=1}^{j} a_k \geq \sum_{k=1}^{l+j} c_k\}$ *for* $1 \leq j \leq p$. *Then*

$$[I_{c_1} \oplus I_{c_2} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a_p}]\} * \{[I_{a_{p-1}}]\} * \cdots * \{[I_{a_1}]\} * \{[I_{b_1}]\} * \{[I_{b_2}]\} * \cdots * \{[I_{b_n}]\}$$

if and only if $r = p + n$, $\sum_{i=1}^{r} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$, $B_j \neq \emptyset$, $b_i \geq c_{\beta_i}$ *for* $1 \leq j \leq p$ *and* $1 \leq i \leq n$, *where*

$$\alpha_j = \begin{cases} \min(B_1 + 1), & j = 1 \\ \max(\alpha_{j-1} + 1, \min B_j + j), & 1 < j \leq p \end{cases}$$

and

$$\beta_i = \begin{cases} \min(l \in \{1, \dots, r) | l \neq \alpha_j, 1 \leq j \leq p\}, & i = 1 \\ \min(l \in \{\beta_{i-1} + 1, \dots, r\} | l \neq \alpha_j, 1 \leq j \leq p), & 1 < i \leq n \end{cases}.$$

In this case all we had to do was to drop the condition $a_j \leq c_{\alpha_j}$.

This also leads to a very simple linear-time algorithm (in the number of indecomposables), a slightly modified version of the algorithm given in [19]. Given the preinjective modules $I_{c_1} \oplus I_{c_2} \oplus \cdots \oplus I_{c_r}, I_{a_1}, \dots, I_{a_p}, I_{b_1}, \dots, I_{b_n} \in$ mod-κK (with $0 \leq a_p \leq a_{p-1} \leq \cdots \leq a_1$ and $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$) this is a method one could follow in practice to decide whether $[I_{c_1} \oplus I_{c_2} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a_p}]\} * \{[I_{a_{p-1}}]\} * \cdots * \{[I_{a_1}]\} * \{[I_{b_1}]\} * \{[I_{b_2}]\} * \cdots * \{[I_{b_n}]\}$:

First check the conditions $r = n+p$ *and* $\sum_{i=1}^{r} c_i = \sum_{i=1}^{p} a_i + \sum_{i=1}^{n} b_i$. *If these are not fulfilled stop with a negative answer, otherwise set the initial values* $j = i = k = 1$ *for the integers used to index elements from the sequences* $(a_1, \dots, a_p)$, $(b_1, \dots, b_n)$ *respectively* $(c_1, \dots, c_r)$. *Repeat the following steps for all successive values of* $1 \leq k \leq r$*:*

1. *If* $j \leq p$ *and* $(a_1 + \cdots + a_{j-1}) + (b_1 + \cdots + b_{i-1}) + a_j \geq c_1 + \cdots + c_k$, *then increase* j *by one.*

2. *Else, if* $i \leq n$ *and* $b_i \geq c_k$ *and* $(a_1 + \cdots + a_{j-1}) + (b_1 + \cdots + b_{i-1}) + b_i \geq c_1 + \cdots + c_k$, *then increase* i *by one.*

3. *If none of the steps above can be carried out than stop with a negative answer.*

Finally, if one of the first two steps can be made for $k = r$ *too, then return a positive answer, i.e. we have* $[I_{c_1} \oplus I_{c_2} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a_p}]\} * \{[I_{a_{p-1}}]\} * \cdots * \{[I_{a_1}]\} * \{[I_{b_1}]\} * \{[I_{b_2}]\} * \cdots * \{[I_{b_n}]\}$.

It is trivial to see that the algorithm is linear in the number of indecomposables (i.e. in $r = n + p$), since the only cycle in the algorithm runs at most r times and the partial sums $a_1 + \cdots + a_j$, $b_1 + \cdots + b_i$ and $c_1 + \cdots + c_k$ can be computed one term at a time at every iteration.

Finally, we show that the conditions given in Theorem 6 are equivalent to the conditions of the generalized majorization, described in Section 2, establishing a module theoretical background for this notion.

Corollary 2 *Let* $\mathbf{d} = (d_1, \dots, d_x)$, $\mathbf{a} = (a_1, \dots, a_y)$, *and* $\mathbf{g} = (g_1, \dots, g_{x+y})$ *be partitions. Then*

$$[I_{g_1} \oplus \cdots \oplus I_{g_{x+y}}] \in \{[I_{a_y}]\} * \cdots * \{[I_{a_1}]\} * \{[I_{d_1}]\} * \cdots * \{[I_{d_x}]\}$$

if and only if $\mathbf{g} \prec' (\mathbf{d}, \mathbf{a})$, *i.e.* $\mathbf{g}$ *is majorized by* $\mathbf{d}$ *and* $\mathbf{a}$.

Proof. The proof is obviously based on Theorem 6, hence let us begin by highlighting the equivalent notations: $x = n$, $y = p$, $x + y = r$, $\mathbf{a} = (a_1, \dots, a_y) = (a_1, \dots, a_p)$, $\mathbf{d} = (d_1, \dots, d_x) = (b_1, \dots, b_n)$, $\mathbf{g} = (g_1, \dots, g_{x+y}) = (c_1, \dots, c_r)$, where n, p, r, $(a_1, \dots, a_p)$, $(b_1, \dots, b_n)$ and $(c_1, \dots, c_r)$ are the corresponding variables used in the statement of Theorem 6.

"$\Longrightarrow$" If $[I_{g_1} \oplus \cdots \oplus I_{g_{x+y}}] \in \{[I_{a_y}]\} * \cdots * \{[I_{a_1}]\} * \{[I_{d_1}]\} * \cdots * \{[I_{d_x}]\}$ then condition (3) from Definition 1 is immediate. We also know that we have a strictly increasing function $\beta : \{1, \dots, x\} \to \{1, \dots, x+y\}$ such that $d_i \geq g_{\beta(i)}$ for $1 \leq i \leq x$. Among all these functions let β be the function for which the sequence $(\beta(1), \beta(2), \dots, \beta(x))$ is lexicographically the smallest one with this property. We immediately have for the corresponding function $\alpha : \{1, \dots, y\} \to \{1, \dots, x+y\}$ that $(\alpha(1), \dots, \alpha(y)) = (h_1, \dots, h_y)$ (see Remark 1 at the end of Section 2). By reordering the inequality corresponding to (5) from Theorem 6 we get

$$\sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq i\leq x}} d_i + \sum_{k=1}^{j} a_k \geq \sum_{k=1}^{j} g_{\alpha(k)} + \sum_{\substack{\beta(i)<\alpha(j)\\ 1\leq i\leq x}} g_{\beta(i)} = \sum_{i=1}^{\alpha(j)} g_i,$$

which is equivalent with

$$\sum_{i=1}^{\alpha(j)} g_i \leq \sum_{i=1}^{\alpha(j)-j} d_i + \sum_{i=1}^{j} a_i$$

for all $j \in \{1, \dots, y\}$. Since $(\alpha(1), \dots, \alpha(y)) = (h_1, \dots, h_y)$, this is exactly condition (2) from the definition. Condition (1) also follows easily, since $\beta(i) \leq i + y$ and therefore $d_i \geq g_{\beta(i)} \geq g_{i+y}$ for all $i \in \{1, \dots, x\}$.

"$\Longleftarrow$" Conversely, let $\mathbf{g}$ to be majorized by $\mathbf{d}$ and $\mathbf{a}$. Set $(\alpha(1), \dots, \alpha(y)) = (h_1, \dots, h_y)$ and $(\beta(1), \dots, \beta(x)) = (h'_1, \dots, h'_x)$ as described in Remark 1. Then condition (2) is equivalent with the inequalities (5) from Theorem 6.

Condition (3) transfers as it is, and we also know that $d_i \geq g_{\beta(i)}$ for all $i \in \{1, \ldots, \alpha(y) - y\}$. If $\alpha(y) - y = x$, we are done, otherwise we must have $(\beta(\alpha(y) - y + 1), \ldots, \beta(x)) = (\alpha(y) + 1, \ldots, x + y)$. Considering now condition (1) we can write $d_i \geq g_{i+y} = g_{\beta(i)}$ for all $i \in \{\alpha(y) - y + 1, \ldots, x\}$, so $d_i \geq g_{\beta(i)}$ is fulfilled on the whole range $1 \leq i \leq x$ and the implication now follows by Theorem 6. □

References

[1] I. Assem, D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras, Vol. 1, Techniques of representation theory*, London Mathematical Society Student Texts 65, Cambridge University Press, Cambridge, 2006.

[2] M. Auslander, I. Reiten, S. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, No. 36, Cambridge University Press, 1995.

[3] M. Dodig, Completion up to a matrix pencil with column minimal indices as the only nontrivial Kronecker invariants, *Linear Algebra Appl.*, **438** (2013), 3155–3173.

[4] M. Dodig, M. Stošić, Combinatorics of column minimal indices and matrix pencil completion problems, *SIAM J. Matrix Anal. Appl.*, **31** (2010), 2318–2346.

[5] M. Dodig, M. Stošić, On convexity of polynomial paths and generalized majorizations, *Electron. J. Combin.*, **17** (1) (2010), R61.

[6] M. Dodig, M. Stošić, On properties of the generalized majorization, *Electron. J. Linear Algebra*, **26** (2013), 471–509.

[7] F. R. Gantmacher, *Matrix theory*, Vol. 1 and 2, Chelsea, New York, 1974.

[8] Y. Han, Subrepresentations of Kronecker representations, *Linear Algebra Appl.*, **402** (2005), 150–164.

[9] J. J Loiseau, S. Mondié, I. Zaballa, P. Zagalak, Assigning the Kronecker invariants of a matrix pencil by row or column completions, *Linear Algebra Appl.*, **278** (1998), 327–336.

[10] I. Macdonald, *Symmetric Functions and Hall Polynomials*, Second edition, Clarendon Press, Oxford, 1995.

[11] M. Reineke, The monoid of families of quiver representations, *Proc. Lond. Math. Soc.*, **84** (2002), 663–685.

[12] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics, No. 1099, Springer-Verlag, Berlin, 1984.

[13] D. Simson, A. Skowroński, *Elements of the representation theory of associative algebras, Vol. 2, Tubes and Concealed Algebras of Euclidean Type*, London Mathematical Society Student Texts 71, Cambridge University Press, Cambridge, 2007.

[14] Cs. Szántó, Hall numbers and the composition algebra of the Kronecker algebra, *Algebr. Represent. Theory*, **9** (2006), 465–495.

[15] Cs. Szántó, On the Hall product of preinjective Kronecker modules, *Mathematica (Cluj)*, **48** (71), No. 2 (2006), 203–206.

[16] Cs. Szántó, I. Szöllősi, On preprojective short exact sequences in the Kronecker case, *J. Pure Appl. Algebra*, **216** (2012), no. 5, 1171–1177.

[17] Cs. Szántó, I. Szöllősi, Preinjective subfactors of preinjective Kronecker modules, *arXiv:1309.4710* [math.RT], 13 September 2013, 13 pages.

[18] Cs. Szántó, I. Szöllősi, The terms in the Ringel-Hall product of preinjective Kronecker modules, *Period. Math. Hungar.*, **63** (2) (2011), 75–92.

[19] I. Szöllősi, Computing the extensions of preinjective and preprojective Kronecker modules, *J. Algebra*, **408** (2014) 205–221.

[20] I. Szöllősi, The extension monoid product of preinjective Kronecker modules, *Mathematica (Cluj)*, **55** (78), No. 1 (2013), 75–88.

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