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Existence and uniqueness of a periodic solution to certain third order nonlinear delay differential equation with multiple deviating arguments

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Abstract. In this paper, we use Lyapunov's second method, by constructing a complete Lyapunov functional, sufficient conditions which guarantee existence and uniqueness of a periodic solution, uniform asymptotic stability of the trivial solution and uniform ultimate boundedness of solutions of Eq. (2). New results are obtained and proved, an example is given to illustrate the theoretical analysis in the work and to test the effectiveness of the method employed. The results obtained in this investigation extend many existing and exciting results on nonlinear third order delay differential equations.

1 Introduction

The importance of functional differential equations, in particular the delay differential equations, cannot be over emphasized as it creates a significant branch of nonlinear analysis and find numerous applications in physics, chemistry, biology, geography, economics, theory of nuclear reactors and in other fields of

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engineering and natural sciences to mention few. The existence, uniqueness, boundedness and stability of solutions of the models derived from these applications are paramount to researchers in various fields of research.

Many work has been done by distinguished authors see for instance Burton [4, 5], Diver [7], Hale [9], Yoshizawa [21, 22] which contain general results on the subject matters. Other remarkable authors worked on stability, boundedness, asymptotic behaviour of solutions of third order delay differential include Ademola *et al* [2, 3], Omeike [10], Sadek [11], Tunç *et al* [13, 15, 16, 18] and the reference cited therein.

To the best of our knowledge few authors have discussed the existence and uniqueness of a periodic solution to delay differential equations (see the paper of Chukwu [6], Gui [8] and Zhu [23]). Also, in 2000, Tejumola and Tchegnani [12] discussed criteria for the existence of periodic solutions of third order differential equation with constant deviating arguments $\tau > 0$:

$$\dddot{x}+f(t,x,\dot{x},\ddot{x})\ddot{x}+g(t,x(t-\tau),\dot{x}(t-\tau))+h(x(t-\tau)) = p(t,x,x(t-\tau),\dot{x},\dot{x}(t-\tau),\ddot{x}).$$

In 2010, Tunç [17] established conditions on the existence of periodic solution for the nonlinear differential equation of third order with constant deviating argument $\tau > 0$:

$$\dddot{x} + \psi(\dot{x})\ddot{x} + g(\dot{x}(t-\tau)) + f(x) = p(t,x,x(t-\tau),\dot{x},\dot{x}(t-\tau),\ddot{x}).$$

Recently, in 2012, Abo-El-Ela *et al.* [1] discussed the existence and uniqueness of a periodic solutions for third order delay differential equation with two deviating arguments.

$$\begin{aligned}&\dddot{x} + \psi(\dot{x})\ddot{x} + f(x)\dot{x} + g_1(t,\dot{x}(t-\tau_1(t))) + g_2(t,\dot{x}(t-\tau_2(t)))\\ &= p(t) = p(t,x,x(t-\tau),\dot{x},\dot{x}(t-\tau),\ddot{x}).\end{aligned}$$

Also, in 2012, Tunç [14] considered the existence of periodic solutions to nonlinear differential equations of third order with multiple deviating arguments τ_i, $(i = 1, 2, \ldots, n)$:

$$\begin{aligned}&\dddot{x} + \psi(\dot{x})\ddot{x} + \sum_{i=1}^{n} g_i(\dot{x}(t-\tau_i)) + f(x)\\ &= p(t,x,x(t-\tau_1),\ldots,x(t-\tau_n),\dot{x},\ldots x(t-\tau_1),\ldots,\dot{x}(t-\tau_n),\ldots x).\end{aligned}$$

However, the problem of uniform asymptotic stability, boundedness, existence and uniqueness of a periodic solution of third order neutral delay differential equation with multiple deviating arguments $\tau_i(t) \geq 0$ $(i = 1, 2, \dots, n)$ and for all $t \geq 0$, has not been investigated. Therefore, the purpose of this paper is to establish criteria for uniform stability, boundedness, existence and uniqueness of a periodic solution for the third order nonlinear delay differential equation with multiple deviating arguments $\tau_i(t) \geq 0$ $(i = 1, 2, \dots, n)$:

$$\begin{aligned} \dddot{x}(t) + f(t, x(t), \dot{x}(t), \ddot{x}(t))\ddot{x}(t) + \sum_{i=1}^{n} g_i(t, x(t - \tau_i(t)), \dot{x}(t - \tau_i(t))) \\ + \sum_{i=1}^{n} h_i(t, x(t - \tau_i(t))) = p(t, x, X, \dot{x}, \dot{X}, \ddot{x}), \end{aligned} \tag{1}$$

where $X = x(t-\tau_1(t)), \dots, x(t-\tau_n(t))$ and $\dot{X} = \dot{x}(t-\tau_1(t)), \dots, \dot{x}(t-\tau_n(t))$. Let $\dot{x}(t) = y(t)$ and $\ddot{x}(t) = z(t)$, (1) is equivalent to the system of first order differential equations

$$\begin{aligned} &\dot{x}(t) = y(t), \quad \dot{y}(t) = z(t) \\ &\dot{z}(t) = p(t, x(t), X, y(t), Y, z(t)) - f(t, x(t), y(t), z(t))z(t) \\ &- \sum_{i=1}^{n} g_i(t, x(t), y(t)) - \sum_{i=1}^{n} h_i(t, x(t)) + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_{it}(s, x(s), y(s))ds \\ &+ \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_{ix}(s, x(s), y(s))y(s)ds + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_{iy}(s, x(s), y(s))z(s)ds \\ &+ \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} h_{it}(s, x(s))ds + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} h_{ix}(s, x(s))y(s)ds, \end{aligned} \tag{2}$$

where $0 \leq \tau_i(t) \leq \gamma$, $\gamma > 0$ is a constant to be determined later, the functions f, g_i, h_i and p are continuous in their respective arguments on $\mathbb{R}^+ \times \mathbb{R}^3$, $\mathbb{R}^+ \times \mathbb{R}^2$, $\mathbb{R}^+ \times \mathbb{R}$ and $\mathbb{R}^+ \times \mathbb{R}^{2n+3}$ respectively with $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$, periodic in t of period ω, and the derivatives $f_t(t, x, y, z)$, $f_x(t, x, y, z)$, $f_z(t, x, y, z)$, $g_{it}(t, x, y)$, $g_{ix}(t, x, y)$, $g_{iy}(t, x, y)$, $h_{it}(t, x)$, $h_{ix}(t, x)$, with respect to t, x, y, z, for all i, $(i = 1, 2, \dots, n)$ exist and are continuous for all t, x, y, z with $h_i(t, 0) = 0$ for all t. The dots as usual, stands for differentiation with respect to t. Motivation for this study comes from the works [1, 8, 12, 14, 17] and the recent papers [2, 19]. These results are new and complement many existing and exciting latest results on third order delay differential equations.

2 Preliminary results

Consider the following general nonlinear non-autonomous delay differential equation

$$\dot{X} = \frac{dX}{dt} = F(t, X_t), \quad X_t = X(t+\theta), \quad -r \le \theta < 0, \quad t \ge 0, \tag{3}$$

where $F : \mathbb{R}^+ \times C_H \to \mathbb{R}^n$ is a continuous mapping, $F(t+\omega, \phi) = F(t, \phi)$ for all $\phi \in C$ and for some positive constant ω. We assume that F takes closed bounded sets into bounded sets in $\mathbb{R}^n$. $(C, \|\cdot\|)$ is the Banach space of continuous function $\varphi : [-r, 0] \to \mathbb{R}^n$ with supremum norm, $r > 0$; for $H > 0$, we define $C_H \subset C$ by $C_H = \{\varphi \in C : \|\varphi\| < H\}$, C_H is the open H-ball in C, $C = C([-r, 0], \mathbb{R}^n)$.

Lemma 1 [22] *Suppose that* $F(t, \phi) \in \overline{C}_0(\phi)$ *and* $F(t, \phi)$ *is periodic in* t *of period* ω, $\omega \ge r$, *and consequently for any* $\alpha > 0$ *there exists an* $L(\alpha) > 0$ *such that* $\phi \in C_\alpha$ *implies* $|F(t, \phi)| \le L(\alpha)$. *Suppose that a continuous Lyapunov functional* $V(t, \phi)$ *exists, defined on* $t \in \mathbb{R}^+$, $\phi \in S^*$, S^* *is the set of* $\phi \in C$ *such that* $|\phi(0)| \ge H$ (H *may be large) and that* $V(t, \phi)$ *satisfies the following conditions:*

(i) $a(|\phi(0)|) \le V(t, \phi) \le b(\|\phi\|)$, *where* $a(r)$ *and* $b(r)$ *are continuous, increasing and positive for* $r \ge H$ *and* $a(r) \to \infty$ *as* $r \to \infty$;

(ii) $\dot{V}_{(3)}(t, \phi) \le -c(|\phi(0)|)$, *where* $c(r)$ *is continuous and positive for* $r \ge H$.

Suppose that there exists an $H_1 > 0$, $H_1 > H$, *such that*

$$hL(\gamma^*) < H_1 - H, \tag{4}$$

where $\gamma^* > 0$ *is a constant which is determined in the following way: By the condition on* $V(t, \phi)$ *there exist* $\alpha > 0$, $\beta > 0$ *and* $\gamma > 0$ *such that* $b(H_1) \le a(\alpha)$, $b(\alpha) \le a(\beta)$ *and* $b(\beta) \le a(\gamma)$. γ^* *is defined by* $b(\gamma) \le a(\gamma^*)$. *Under the above conditions, there exists a periodic solution of* (3) *with period* ω. *In particular, the relation* (4) *can always be satisfied if* h *is sufficiently small.*

Lemma 2 [22] *Suppose that* $F(t, \phi)$ *is defined and continuous on* $0 \le t \le c$, $\phi \in C_H$ *and that there exists a continuous Lyapunov functional* $V(t, \phi, \varphi)$ *defined on* $0 \le t \le c$, $\phi, \varphi \in C_H$ *which satisfy the following conditions:*

(i) $V(t, \phi, \varphi) = 0$ *if* $\phi = \varphi$;

(ii) $V(t,\phi,\varphi) > 0$ *if* $\phi \neq \varphi$;

(iii) *for the associated system*

$$\dot{x}(t) = F(t,x_t), \quad \dot{y}(t) = F(t,y_t) \tag{5}$$

we have $V'_{(5)}(t,\phi,\varphi) \leq 0$, *where for* $\|\phi\| = H$ *or* $\|\varphi\| = H$, *we understand that the condition* $V'_{(5)}(t,\phi,\varphi) \leq 0$ *is satisfied in the case* V' *can be defined.*

Then, for given initial value $\phi \in C_{H_1}$, $H_1 < H$, *there exists a unique solution of* (3).

Lemma 3 [22] *Suppose that a continuous Lyapunov functional* $V(t,\phi)$ *exists, defined on* $t \in \mathbb{R}^+$, $\|\phi\| < H$, $0 < H_1 < H$ *which satisfies the following conditions:*

(i) $a(\|\phi\|) \leq V(t,\phi) \leq b(\|\phi\|)$, *where* $a(r)$ *and* $b(r)$ *are continuous, increasing and positive,*

(ii) $\dot{V}_{(3)}(t,\phi) \leq -c(\|\phi\|)$, *where* $c(r)$ *is continuous and positive for* $r \geq 0$,

then the zero solution of (3) *is uniformly asymptotically stable.*

Lemma 4 [4] *Let* $V : \mathbb{R}^+ \times C \to \mathbb{R}$ *be continuous and locally Lipschitz in* ϕ. *If*

(i) $W_0(|X_t|) \leq V(t,X_t) \leq W_1(|X_t|) + W_2\left(\int\limits_{t-r(t)}^{t} W_3(X_t(s))ds\right)$ *and*

(ii) $\dot{V}_{(3)}(t,X_t) \leq -W_4(|X_t|) + N$, *for some* $N > 0$, *where* W_i $(i = 0,1,2,3,4)$ *are wedges.*

Then X_t *of* (3) *is uniformly bounded and uniformly ultimately bounded for bound* B.

3 Main results

Theorem 1 In addition to the basic assumptions on the functions f, g_i, h_i, p and τ_i, suppose that $a, a_1, b_i, B_i, c_i, \delta_i, E_i, K_i, M_i, \gamma$ $(i = 1,2,\ldots,n)$ are positive constants and for all $t \geq 0$.

(i) $a \leq f(t,x,y,z) \leq a_1$ for all x,y,z;

(ii) $b_i \le \dfrac{g_i(t,x,y)}{y} \le \begin{cases} K_i t & \text{for all } t > 0, x \text{ and } y \neq 0, \\ B_i & \text{for all } t \ge 0, x \text{ and } y \neq 0, \end{cases}$ and $|g_{ix}(t,x,y)| \le M_i$;

(iii) $h_i(t,0) = 0$, $\delta_i \le \dfrac{h_i(t,x)}{x} \le \begin{cases} E_i t & \text{for all } t > 0 \neq x, \\ c_i & \text{for all } t \ge 0 \neq x, \end{cases}$ and $ab_i > c_i$;

(iv) $\tau_i(t) \le \gamma$, $\tau_i' \le \rho$, $\rho \in (0,1)$, $0 \le P(t) < \infty$;

if

$$\gamma < \min\left\{\frac{1}{2}\sum_{i=1}^{n} \beta\delta_i A_1^{-1}, \sum_{i=1}^{n}(\alpha b_i - c_i)A_2^{-1}, \frac{1}{2}(a-\alpha)A_3^{-2}\right\}, \tag{6}$$

where

$$A_1 := \frac{1}{2}\beta + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + (1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n} E_i$$

$$A_2 := \frac{1}{2}(\alpha+a) + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + (1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}(c_i + K_i + M_i)$$

and

$$A_3 := 1 + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + (1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n} B_i,$$

then (2) has a unique periodic solution of period ω.

Proof. Let (x_t, y_t, z_t) be any solution of (2) and the functional $V = V(t, x_t, y_t, z_t)$ be defined as

$$V = e^{-P(t)}U, \tag{7}$$

where

$$P(t) = \int_0^t |p(s,x,X,y,Y,z)|ds \tag{8a}$$

and $U = U(t, x_t, y_t, z_t)$ is defined as

$$2U = 2(\alpha + a)\sum_{i=1}^{n}\int_0^x h_i(t,\xi)d\xi + 4\sum_{i=1}^{n}\int_0^y g_i(t,x,\tau)d\tau + 4y\sum_{i=1}^{n} h_i(t,x)$$
$$+ 2(\alpha + a)yz + 2z^2 + 2(\alpha + a)\int_0^y \tau f(t,x,\tau,0)d\tau + \beta y^2 + \sum_{i=1}^{n} b_i x^2 + 2a\beta xy$$
$$+ 2\beta xz + \int_{-\tau(t)}^{0}\int_{t+s}^{t}(\lambda_0 x^2(\theta) + \lambda_1 y^2(\theta) + \lambda_2 z^2(\theta))d\theta ds, \tag{8b}$$

where α and β are fixed constants satisfying

$$\sum_{i=1}^{n} b_i^{-1}c_i < \alpha < a \tag{8c}$$

and

$$0 < \beta < \min\left\{\sum_{i=1}^{n} b_i, \sum_{i=1}^{n}(ab_i - c_i)A_4^{-1}, \frac{1}{2}(a-\alpha)A_5^{-1}\right\}, \tag{8d}$$

where

$$A_4 := 1 + a + \sum_{i=1}^{n}\delta_i^{-1}\left(\frac{g_i(t,x,y)}{y} - b_i\right)^2$$

and

$$A_5 := 1 + \sum_{i=1}^{n}\delta_i^{-1}\Big(f(t,x,y,z) - a\Big)^2.$$

Now, since $h_i(t,0) = 0$ for all $t \in \mathbb{R}^+$, (7) can be recast in the form

$$V = e^{-P(t)}\Bigg[\sum_{i=1}^{n} b_i^{-1}\int_0^x[(\alpha + a)b_i - 2h_{ix}(t,\xi)]h_i(t,\xi)d\xi + \frac{\beta}{2}y^2 + \frac{1}{2}(\alpha y + z)^2$$
$$+ 2\sum_{i=1}^{n}\int_0^y\left(\frac{g_i(t,x,\tau)}{\tau} - b_i\right)\tau d\tau + \sum_{i=1}^{n} b_i^{-1}\Big(h_i(t,x) + b_i y\Big)^2$$
$$+ \int_0^y\Big[(\alpha + a)f(t,x,\tau,0) - (\alpha^2 + a^2)\Big]\tau d\tau + \frac{1}{2}(\beta x + ay + z)^2$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\beta(b_i - \beta)x^2 + \frac{1}{2}\int_{-\tau(t)}^{0}\int_{t+s}^{t}\Big(\lambda_0 x^2(\theta) + \lambda_1 y^2(\theta) + \lambda_2 z^2(\theta)\Big)d\theta ds\Bigg], \tag{9}$$

where $P(t)$ is the function defined by (8a). In view of the assumptions of Theorem 1 and the fact that the double integrals are non-negative, there exists a positive constant d_0 such that

$$V \geq d_0(x^2 + y^2 + z^2) \tag{10a}$$

for all $t \geq 0, x, y$ and z, where

$$\begin{aligned} d_0 = e^{-P_0} \min \Bigg\{ & \frac{1}{2}\sum_{i=1}^{n} b_i^{-1}(\alpha b_i - c_i + a b_i - c_i)\delta_i + \sum_{i=1}^{n} b_i^{-1} \min\{b_i, \delta_i\} \\ & + \frac{1}{2}\min\{1, a, \beta\} + \frac{1}{2}\sum_{i=1}^{n} \beta(b_i - \beta), \quad \frac{1}{2}\beta + \sum_{i=1}^{n} b_i^{-1} \min\{b_i, \delta_i\} + \frac{1}{2}\min\{1, \alpha\} \\ & + \frac{1}{2}\min\{1, a, \beta\} + \frac{1}{2}\alpha(a - \alpha), \quad \frac{1}{2}\min\{1, \alpha\} + \frac{1}{2}\min\{1, a, \beta\} \Bigg\}. \end{aligned}$$

Clearly, from(10a), we have $V(t, x, y, z) = 0$ if and only if $x^2 + y^2 + z^2 = 0$, $V(t, x, y, z) > 0$ if and only if $x^2 + y^2 + z^2 \neq 0$, it follows that

$$V(t, x, y, z) \to +\infty \quad \text{as} \quad x^2 + y^2 + z^2 \to \infty. \tag{10b}$$

Moreover, from the hypotheses of Theorem 1 and the obvious inequality $2|x_1 x_2| \leq x_2^2 + x_2^2$, Eq. (7) turns out to be

$$V(t, x, y, z) \leq d_1(x^2 + y^2 + z^2) + d_2 \int_{-\tau(t)}^{0} \int_{t+s}^{t} d_3(x^2(\theta) + y^2(\theta) + z^2(\theta)) d\theta ds \tag{11}$$

for all $t \geq 0, x, y, z$, and s, where

$$\begin{aligned} d_1 = \max \Bigg\{ & \frac{1}{2}\sum_{i=1}^{n} \Big[(\alpha + a + 2)c_i + \beta(1 + a + b_i)\Big], \\ & \frac{1}{2}\Big(2\sum_{i=1}^{n}(B_i + c_i) + (\alpha + a)(1 + a_1) + \beta(1 + a)\Big), \frac{1}{2}(2, \alpha + \beta + a)\Bigg\}, \end{aligned}$$

$d_2 = \dfrac{1}{2}$ and

$d_3 = \max\{\lambda_0, \lambda_1, \lambda_2\}$.

Next, the derivative of the function V with respect to t along a solution (x_t, y_t, z_t) of (2) is given by

$$\dot{V}_{(2)} = -e^{-P(t)}\Big[U\dot{P}(t) - \dot{U}_{(2)}\Big], \tag{12}$$

where $P(t)$ and U are defined by (8a) and (8b) respectively,

$$\dot{P}(t) = |p(t, x, X, y, Y, z)| \tag{13a}$$

and

$$\dot{U}_{(2)} = a\beta y^2 + 2\beta yz + [\beta x + (\alpha + a)y + 2z]p(t, x, X, y, Y, z) + \sum_{j=1}^{3} U_j - \sum_{j=4}^{5} U_j$$
$$- \beta[f(t, x, y, z) - a]xz - \beta \sum_{i=1}^{n} \left(\frac{g_i(t, x, y)}{y} - b_i \right) xy, \tag{13b}$$

where:

$$U_1 := (\alpha + a) \sum_{i=1}^{n} \int_0^x h_{it}(t, \xi) d\xi + 2 \sum_{i=1}^{n} \int_0^y g_{it}(t, x, \tau) d\tau + 2y \sum_{i=1}^{n} h_{it}(t, x);$$

$$U_2 := (\alpha + a) \int_0^y \tau f_t(t, x, \tau, 0) d\tau + 2 \sum_{i=1}^{n} y \int_0^y g_{ix}(t, x, \tau) d\tau$$
$$+ (\alpha + a)y \int_0^y \tau f_x(t, x, \tau, 0) d\tau;$$

$$U_3 := (\beta x + (\alpha + a)y + 2z) \Bigg[\sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_{it}(s, x(s), y(s)) ds$$
$$+ \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} h_{it}(s, x(s)) ds + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_{iy}(s, x(s), y(s)) z(s) ds$$
$$+ \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} g_{ix}(s, x(s), y(s)) y(s) ds$$
$$+ \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} h_{ix}(s, x(s)) y(s) ds \Bigg] + \Big(\lambda_0 x^2(t) + \lambda_1 y^2(t) + \lambda_2 z^2(t) \Big) \tau_i(t)$$
$$- \frac{1}{2}(1 - \tau_i'(t)) \int_{t-\tau_i(t)}^{t} \Big(\lambda_0 x^2(\theta) + \lambda_1 y^2(\theta) + \lambda_2 z^2(\theta) \Big) d\theta$$

$$U_4 := \beta x \sum_{i=1}^{n} h_i(t, x) + \sum_{i=1}^{n} \left[(\alpha + a) \frac{g_i(t, x, y)}{y} - 2h_{ix}(t, x) \right] y^2$$
$$+ \Big[2f(t, x, y, z) - (\alpha + a) \Big] z^2$$

and

$$U_5 := (\alpha + a)yz\Big[f(t,x,y,z) - f(t,x,y,0)\Big].$$

Now, $h_{it}(t,x) \leq E_i x$ for all $t \geq 0 \neq x$ and $g_{it}(t,x,y) \leq K_i y$ for all $t \geq 0, x, y \neq 0$, these inequalities imply the existence of a positive constant q_0 such that

$$U_1 \leq q_0(x^2 + y^2 + z^2),$$

where $q_0 = \max\{1,\ \frac{1}{2}\sum_{i=1}^{n}[(\alpha+a)E_i+2c_i],\ \ \sum_{i=1}^{n}(K_i+c_i)\}$. Also, $f(t,x,y,z) \leq a_1$ and $g_i(t,x,y) \leq B_i y$ these inequalities imply that

$$U_2 \leq 0$$

for all $t \geq 0, x, y$ and z.
Furthermore, in view of the assumptions of the theorem and the obvious inequality $2mn \leq m^2 + n^2$, we obtain

$$\begin{aligned}
U_3 \leq{}& \Big[\frac{1}{2}\beta + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_0\Big]\tau_i(t)x^2 + \Big[\frac{1}{2}(\alpha + a) \\
&+ \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_1\Big]\tau_i(t)y^2 + \Big[1 + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) \\
&+ \lambda_2\Big]\tau_i(t)z^2 - \frac{1}{2}\Big[(1 - \tau_i'(t))\lambda_0 - (2 + \alpha + \beta + a)\sum_{i=1}^{n}E_i\Big]\int_{t-\tau_i(t)}^{t} x^2(s)ds \\
&- \frac{1}{2}\Big[(1 - \tau_i'(t))\lambda_1 - (2 + \alpha + \beta + a)\sum_{i=1}^{n}(c_i + K_i + M_i)\Big]\int_{t-\tau_i(t)}^{t} y^2(s)ds \\
&- \frac{1}{2}\Big[(1 - \tau_i'(t))\lambda_2 - (2 + \alpha + \beta + a)\sum_{i=1}^{n}B_i\Big]\int_{t-\tau_i(t)}^{t} z^2(s)ds, \\
U_4 \geq{}& \beta\sum_{i=1}^{n}\delta_i x^2 + \sum_{i=1}^{n}\Big(\alpha b_i - c_i + a b_i - c_i\Big)y^2 + (a - \alpha)z^2.
\end{aligned}$$

Finally, $f(t,x,y,z) \geq a$ implies that for $y > 0$, $yf_z(t,x,y,z) \geq 0$ for all $t \geq 0, x, y$ and z, so that

$$U_5 = (\alpha + a)yz^2 f_z(t,x,y,\theta_0 z) \geq 0$$

for all $t \geq 0, x, y, z$ and $(\alpha + a)yz^2 f_z(t, x, y, \theta_0 z) = 0$ when $z = 0$. Employing estimates U_i $(i = 1, \cdots, 5)$ in (13b), there exists a positive constant $q_1 = \max\{2, \alpha + a, \beta\}$ such that

$$
\begin{aligned}
U_{(2)} &\leq (a+1)\beta y^2 + \beta z^2 + q_1(|x| + |y| + |z|)|p(t, x, X, y, Y, z)| \\
&+ q_0(x^2 + y^2 + z^2) + \left[\frac{1}{2}\beta + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_0\right]\tau_i(t)x^2 \\
&- (a - \alpha)z^2 + \left[\frac{1}{2}(\alpha + a) + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_1\right]\tau_i(t)y^2 \\
&+ \left[1 + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_2\right]\tau_i(t)z^2 \\
&- \frac{1}{2}\left[(1 - \tau_i'(t))\lambda_0 - (2 + \alpha + \beta + a)\sum_{i=1}^{n} E_i\right]\int_{t-\tau_i(t)}^{t} x^2(s)ds \\
&- \frac{1}{2}\left[(1 - \tau_i'(t))\lambda_1 - (2 + \alpha + \beta + a)\sum_{i=1}^{n}(c_i + K_i + M_i)\right]\int_{t-\tau_i(t)}^{t} y^2(s)ds \\
&- \frac{1}{2}\left[(1 - \tau_i'(t))\lambda_2 - (2 + \alpha + \beta + a)\sum_{i=1}^{n} B_i\right]\int_{t-\tau_i(t)}^{t} z^2(s)ds - \beta\sum_{i=1}^{n}\delta_i x^2 \\
&- \sum_{i=1}^{n}\Big(\alpha b_i - c_i + a b_i - c_i\Big)y^2 - \beta[f(t, x, y, z) - a]xz \\
&- \beta\sum_{i=1}^{n}\left(\frac{g_i(t, x, y)}{y} - b_i\right)xy.
\end{aligned}
\tag{14}
$$

Since $\tau_i(t) \leq \gamma$ and $\tau_i'(t) \leq \rho$ for all $t \geq 0$, estimate (14) can be rearranged in the form

$$
\begin{aligned}
U_{(2)} &\leq (a+1)\beta y^2 + \beta z^2 + q_1(|x| + |y| + |z|)|p(t, x, X, y, Y, z)| \\
&+ q_0(x^2 + y^2 + z^2) + \left[\frac{1}{2}\beta + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_0\right]\gamma x^2 \\
&- (a - \alpha)z^2 + \left[\frac{1}{2}(\alpha + a) + \sum_{i=1}^{n}(B_i + c_i + E_i + K_i + M_i) + \lambda_1\right]\gamma y^2
\end{aligned}
$$

$$+\left[1+\sum_{i=1}^{n}(B_i+c_i+E_i+K_i+M_i)+\lambda_2\right]\gamma z^2$$
$$-\frac{1}{2}\left[(1-\rho)\lambda_0-(2+\alpha+\beta+a)\sum_{i=1}^{n}E_i\right]\int_{t-\tau_i(t)}^{t}x^2(s)ds$$
$$-\frac{1}{2}\left[(1-\rho)\lambda_1-(2+\alpha+\beta+a)\sum_{i=1}^{n}(c_i+K_i+M_i)\right]\int_{t-\tau_i(t)}^{t}y^2(s)ds$$
$$-\frac{1}{2}\left[(1-\rho)\lambda_2-(2+\alpha+\beta+a)\sum_{i=1}^{n}B_i\right]\int_{t-\tau_i(t)}^{t}z^2(s)ds-\frac{\beta}{2}\sum_{i=1}^{n}\delta_i x^2 \tag{15}$$
$$-\sum_{i=1}^{n}\left(\alpha b_i-c_i+ab_i-c_i\right)y^2-\frac{\beta}{4}\sum_{i=1}^{n}\delta_i\left[x+2\delta_i^{-1}(f(t,x,y,z)-a)z\right]^2$$
$$-\frac{\beta}{4}\sum_{i=1}^{n}\delta_i\left[x+2\delta_i^{-1}\left(\frac{g_i(t,x,y)}{y}-b_i\right)y\right]^2$$
$$+\beta\sum_{i=1}^{n}\delta_i^{-1}\left(\frac{g_i(t,x,y)}{y}-b_i\right)^2y^2+\beta\sum_{i=1}^{n}\delta_i^{-1}\left(f(t,x,y,z)-a\right)^2z^2,$$

for all $t\geq 0, x, y, z$. Choosing $\lambda_0=(1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}E_i>0$, $\lambda_1=(1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}(c_i+K_i+M_i)>0$ and $\lambda_2=(1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}B_i>0$, and the fact that $[x+2\delta_i^{-1}(f(t,x,y,z)-a)z]^2\geq 0$ and $[x+2\delta_i^{-1}(\frac{g_i(t,x,y)}{y}-b_i)y]^2\geq 0$ for all $t\geq 0, x, y$ and z, the inequality in (15) yields

$$\begin{aligned}
U_{(2)} &\leq q_1(|x|+|y|+|z|)|p(t,x,X,y,Y,z)|+q_0(x^2+y^2+z^2)\\
&-\left\{\frac{1}{2}\sum_{i=1}^{n}\beta\delta_i-\left[\frac{1}{2}\beta+\sum_{i=1}^{n}(B_i+c_i+E_i+K_i+M_i)\right.\right.\\
&\left.\left.+(1-\rho)^{-1}(2+\alpha+\beta+a)\ \mathrm{sum}_{i=1}^{n}E_i\right]\gamma\right\}x^2\\
&-\left\{\sum_{i=1}^{n}(\alpha b_i-c_i)-\left[\frac{1}{2}(\alpha+a)+\sum_{i=1}^{n}(B_i+c_i+E_i+K_i+M_i)\right.\right.\\
&\left.\left.+(1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n}(c_i+K_i+M_i)\right]\gamma\right\}y^2\\
&-\left\{\frac{1}{2}(a-\alpha)-\left[1+\sum_{i=1}^{n}(B_i+c_i+E_i+K_i+M_i)\right.\right.
\end{aligned}$$

$$+ (1-\rho)^{-1}(2+\alpha+\beta+a)\sum_{i=1}^{n} B_i\Big]\gamma\Big\}z^2$$
$$-\left\{\sum_{i=1}^{n}(ab_i-c_i)-\beta\left[1+a+\sum_{i=1}^{n}\delta_i^{-1}\left(\frac{g_i(t,x,y)}{y}-b_i\right)^2\right]\right\}y^2$$
$$-\left\{\frac{1}{2}(a-\alpha)-\beta\left[1+\sum_{i=1}^{n}\delta_i^{-1}\Big(f(t,x,y,z)-a\Big)^2\right]\right\}z^2.$$

In view of the estimates (6) and (8d) there exists a positive constant q_2 such that

$$U_{(2)} \le q_1(|x|+|y|+|z|)|p(t,x,X,y,Y,z)|+q_0(x^2+y^2+z^2)-q_2(x^2+y^2+z^2) \quad (16)$$

for all $t \ge 0, x, y$ and z. Applying the assumptions of Theorem 1, estimates (8c), (8d) in (8b), there exists a constant q_3 such that

$$U \ge q_3(x^2+y^2+z^2) \quad (17)$$

for all $t \ge 0, x, y$ and z, where $q_3 = d_0e^{P_0} > 0$. Using (13a), (16) and (17) in (12) choosing $q_2 > q_0$ and $(x^2+y^2+z^2)^{1/2} \ge 3^{1/2}q_1q_3^{-1}$ sufficiently large, there exists a constant $d_3 > 0$ such that

$$\dot{V}_{(2)} \le -d_3(x^2+y^2+z^2) \quad (18)$$

for all $t \ge 0, x, y$ and z, where $d_3 = e^{-P_0}(q_2-q_0) > 0$. From inequalities (10a), (10b), (11) and (18), the assumptions of Lemma 1 hold, also by estimates (10) and (18) the hypotheses of Lemma 2 are satisfied. Hence by Lemma 1 and Lemma 2 Eq. (2) has a unique periodic solution of period ω. This completes the proof of Theorem 1. □

If $p(t,x,X,\dot{x},\dot{X},\ddot{x}) = 0$ in (1), Eq. (2) reduces to

$$\begin{aligned}
&\dot{x}(t) = y(t), \quad \dot{y}(t) = z(t) \quad \dot{z}(t) = -f(t,x(t),y(t),z(t))z(t)\\
&-\sum_{i=1}^{n} g_i(t,x(t),y(t)) - \sum_{i=1}^{n} h_i(t,x(t)) + \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} g_{it}(s,x(s),y(s))ds\\
&+\sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} g_{ix}(s,x(s),y(s))y(s)ds\\
&+\sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} g_{iy}(s,x(s),y(s))z(s)ds\\
&+\sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} h_{it}(s,x(s))ds + \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} h_{ix}(s,x(s))y(s)ds,
\end{aligned} \quad (19)$$

where f, g_i and h_i are the functions defined in section 1.

Theorem 2 If in addition to the hypotheses of Theorem 1, $g_i(t,0,0) = h_i(t,0) = p(t,x,X,y,Y,z) = 0$, then the trivial solution of (19) is uniformly asymptotically stable, provided that the inequality in (6) holds.

Proof. If $p(t,x,X,y,Y,z) = 0$, the function V defined in (7) reduces to $V = U$, where U is defined in (8b). With the assumptions of Theorem 2, it is not difficult to show that

$$V \geq d_4(x^2 + y^2 + z^2) \tag{20}$$

for all $t \geq 0, x, y, z$, where $d_4 = d_0 e^{P_0}$. Furthermore, in view of the assumptions of Theorem 2 estimate (11) holds.
Next, let (x_t, y_t, z_t) be any solution of (19), little calculation shows that

$$\dot{V}_{(19)} \leq -d_5(x^2 + y^2 + z^2) \tag{21}$$

for all $t \geq 0, x, y, z$, where $d_5 = d_3 e^{P_0}$. The inequalities in (11), (20) and (21) verify the assumptions of Lemma 3, thus by Lemma 3 the trivial solution of (19) is uniformly asymptotically stable. □

Theorem 3 If the hypothesis on the function p of Theorem 1 is replaced by

$$|p(t,x,X,y,Y,z)| \leq P_1, \qquad 0 < P_1 < \infty \tag{22}$$

for all $t \geq 0, x, X, y, Y$ and z, then the solutions of (2) is uniformly bounded and uniformly ultimately bounded.

Proof. If $t = 0$ in (8a), Eq. (7) becomes $V = U$, under the assumptions of Theorem 3, estimates (10a), (10b) and (11) hold. Let (x_t, y_t, z_t) be any solution of (2), the derivative of $V = U$ along a solution of (2) is estimated by (16). By (22), choosing q_2 sufficiently large such that $q_2 > q_0 + P_1 q_1$ there exist positive constants d_6 and d_7 such that

$$\dot{V}_{(2)} \leq -d_6(x^2 + y^2 + z^2) + d_7 \tag{23}$$

for all $t \geq 0, x, y, z$, where $d_6 = q_2 - q_0 - P_1 q_1 > 0$ and $d_7 = 3P_1 q_1 > 0$. In view of the inequality in (10), (11) and (23) all hypotheses of Lemma 4 hold true, thus by Lemma 4 the solutions of (2) are uniformly bounded and uniformly ultimately bounded. □

4 An example

Example 1 *Consider the following third order neutral delay differential equation*

$$
\begin{aligned}
&\dddot{x} + \frac{3}{2}\ddot{x} + \frac{\ddot{x}}{1 + \sin t + |x\dot{x}| + \exp[(1 + \dot{x}\ddot{x})^{-1}]} \\
&+ 4\sum_{i=1}^{n} \dot{x}(t - \tau_i(t)) + \sum_{i=1}^{n} x(t - \tau_i(t)) \\
&+ \sum_{i=1}^{n} \frac{\dot{x}(t - \tau_i(t))}{3 + \sin(t/2) + |x(t - \tau_i(t))\dot{x}(t - \tau_i(t))|} + \sum_{i=1}^{n} \frac{x(t - \tau_i(t))}{4 + \sin t} \\
&= \frac{1}{4 + \sin t + |x| + |X| + |\dot{x}| + |\dot{X}| + |\ddot{x}|}.
\end{aligned}
\tag{24}
$$

(24) *is equivalent to system of first order differential equations*

$$
\begin{aligned}
&\dot{x} = y, \quad \dot{y} = z, \\
&\dot{z} = \frac{1}{4 + \sin t + |x| + |X| + |y| + |Y| + |z|} - \left(1 + \frac{1}{4 + \sin t}\right) n x \\
&\quad + \left(\frac{3}{2} + \frac{1}{1 + \sin t + |xy| + \exp[(1 + |yz|)^{-1}]}\right) z \\
&\quad - \left(4 + \frac{1}{3 + \sin(t/2) + |xy| + y^2}\right) n y \\
&\quad + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \frac{y\cos(\mu/2)\,d\mu}{2[3 + \sin(\mu/2) + |xy| + y^2]^2} \\
&\quad - \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \frac{y^2 y(\mu)\,d\mu}{[3 + \sin(\mu/2) + |xy| + y^2]^2} \\
&\quad + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \left(4 + \frac{3 + \sin(\mu/2) - y^2}{[3 + \sin(\mu/2) + |xy| + y^2]^2}\right) z(\mu)\,d\mu \\
&\quad - \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \frac{x\cos\mu}{4 + \sin\mu}\,d\mu \\
&\quad + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} \left(1 + \frac{1}{4 + \sin\mu}\right) y(\mu)\,d\mu.
\end{aligned}
\tag{25}
$$

In view of (2) *and* (25) *we have the following relations and estimates:*

(A) *The function* $f(t,x,y,z) = \dfrac{3}{2} + \dfrac{1}{1+\sin t + |xy| + \exp[(1+|yz|)^{-1}]}$, *it is not difficult to show that for all* $t \geq 0, x, y$ *and* z:

(i) $\dfrac{3}{2} \leq f(t,x,y,z) \leq \dfrac{5}{2}$, *where* $a = \dfrac{3}{2} > 0$ *and* $a_1 = \dfrac{5}{2} > 0$;

(ii) $f_t(t,x,y,z) = \dfrac{-\cos t}{[1+\sin t + |xy| + \exp[(1+|yz|)^{-1}]]^2} \leq 0$;

(iii) *for* $x > 0$, $yf_x(t,x,y,z) = \dfrac{-y^2}{[1+\sin t + |xy| + \exp[(1+|yz|)^{-1}]]^2} \leq 0$ *and*

(iv) *for* $z > 0$,

$$yf_z(t,x,y,z) = \frac{y^2 \exp[(1+|yz|)^{-1}]}{[1+|yz|]^2[1+\sin t + |xy| + \exp[(1+|yz|)^{-1}]]^2} \geq 0.$$

(B) *The function* $g_i(t,x,y) = \left(4 + \dfrac{1}{3+\sin(t/2)+|xy|+y^2}\right)y$, *which for all* $t \geq 0, x$ *and* y *we have:*

(i) $4 \leq \dfrac{g_i(t,x,y)}{y} \leq 5$, *where* $b_i = 4 > 0$ *and* $B_i = 5 > 0$;

(ii) *for* $x > 0$, $g_{ix}(t,x,y) = \dfrac{-y^2}{[3+\sin(t/2)+|xy|+y^2]^2} \leq 0$ *and*

(iii) $g_{it}(t,x,y) = \dfrac{-y\cos(t/2)}{2[3+\sin(t/2)+|xy|+y^2]^2} \leq \dfrac{|y||1-2\sin^2(t/4)|}{2[3+\sin(t/2)+|xy|+y^2]^2}$.

Now since $0 < \dfrac{|1-2\sin^2(t/4)|}{2[3+\sin(t/2)+|xy|+y^2]^2} < 1$ *for all* $t \geq 0, x, y$, *where* $K_i = 1 > 0$ *so that*

$$g_{it}(t,x,y) \leq |y|.$$

(C) *The function* $h_i(t,x) = x + \dfrac{x}{4+\sin t}$, *it is not difficult to show that*

(i) $h_i(t,0) = 0$;

(ii) $\dfrac{h_i(t,x)}{x} \geq 1$, *where* $\delta_i = 1 > 0$;

(iii) $h_{ix}(t,x) \leq 2 = c_i$;

(iv) $ab_i - c_i$ *implies that* $2 > 0$;

(v) $h_{it}(t,x) = \dfrac{-x\cos t}{4+\sin t} \leq |x|$ *since* $0 < \dfrac{|1-2\sin^2(t/2)|}{|4+\sin t|} < 1$, *where* $E_i = 1 > 0$.

(D) $p(t, x, x(t-\tau_i(t)),$

$$y(t-\tau_i(t)), z) = \frac{1}{4+\sin t+|x|+|x(t-\tau_i(t))|+|y|+|y(t-\tau_i(t))|+|z|}.$$

It is not difficult to show that $|p| \leq 1 < \infty$, *where* $P_1 = 1 > 0$.

(E) *Finally, it can be shown that* $0 < 1 = \alpha$, $\rho = \dfrac{1}{2} < 1$, $0 < \beta < \dfrac{1}{4}$ *and* $\gamma < \dfrac{1}{234}$.

All assumptions of the theorems are verified and hence the conclusions of these theorems follow.

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Performance evaluation of two Markovian retrial queueing model with balking and feedback

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Abstract. In this paper, we consider the performance evaluation of two retrial queueing system. Customers arrive to the system, if upon arrival, the queue is full, the new arriving customers either move into one of the orbits, from which they make a new attempts to reach the primary queue, until they find the server idle or balk and leave the system, these later, and after getting a service may comeback to the system requiring another service. So, we derive for this system, the joint distribution of the server state and retrial queue lengths. Then, we give some numerical results that clarify the relationship between the retrials, arrivals, balking rates, and the retrial queue length.

1 Introduction

In the parlance of queueing theory, such a mechanism in which ejected (or rejected) customers return at random intervals until they receive service is called a retrial queue. Retrial queues have an important application in a wide variety of fields, they are likewise prevalent in the evaluation and design of

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computer networks as they are in telecommunications, computer networks, and particularly wireless networks.

A retrial queue is similar to any ordinary queueing system in that there is an arrival process and one or more servers. The fundamental differences are firstly, the entities who enter during a down or busy period of the server or servers may reattempt service at some random time in the future, and secondly a waiting room, which is known as a primary queue in the context of retrial queues, is not mandatory. In place of the ordinary waiting room is a buffer called an orbit to which entities proceed after an unsuccessful attempt at service, and from which they retry service according to a given probabilistic or deterministic policy.

Owing to the utility and interesting mathematical properties of retrial queueing models, a vast literature on the subject has emerged over the past several decades. For a general survey of retrial queues and a summary of many results, the reader is directed to the works of [6, 8, 7, 5, 12, 15] and references therein.

In [4] Choi and Kim considered the $M/M/c$ retrial queues with geometric loss and feedback when $c = 1, 2$, they found the joint generating function of the number of busy servers and the queue length by solving Kummer differential equation for $c = 1$, and by the method of series solution for $c = 1, 2$. Retrial queueing model $MMAP/M_2/1$ with two orbits was studied by Avrachenkov, Dudin and Klimenok [3], in their paper, authors considered a retrial single-server queueing model with two types of customers. In case of the server occupancy at the arrival epoch, the customer moves to the orbit depending on the type of the customer. One orbit is infinite while the second one is a finite. Joint distribution of the number of customers in the orbits and some performance measures are computed. An $M/M/1$ queue with customers balking was proposed by Haight [9], Sumeet Kumar Sharma [10] studied the $M/M/1/N$ queuing system with retention of reneged customers, Kumar and Sharma [11] studied a single server queueing system with retention of reneged customers and balking. Kumar and Sharma [14] consider a single server, finite capacity Markovian feedback queue with balking, balking and retention of reneged customers in which the inter-arrival and service times follow exponential distribution. In our paper, we consider a retrial queueing model with two orbits O_1 and O_2, balking and feedback. In case of the server occupancy at the arrival epoch, the arriving customers have to choose between the two orbits depending on their thresholds if they decide to stay for an attempt to get served or leave the system (balk), and after getting a service, customers may comeback to the system requiring another service. The main result in this work consists in deriving the approximate analysis of the system.

2 Mathematical model

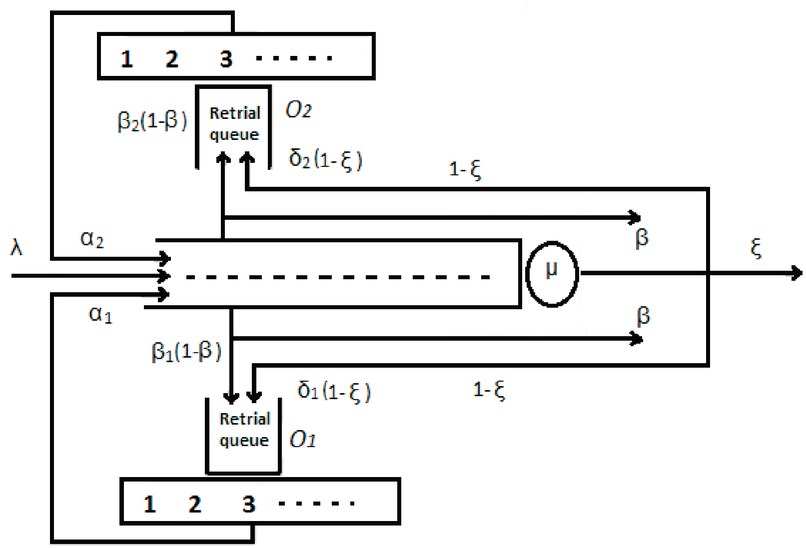

Figure 1: Retrial queueing model with balking and feedback

We consider a retrial queueing model with two orbits O_1 and O_2, new customers arrive from outside to the service node according to a poisson process with rate λ. If the queue is not full upon primary call arrivals, then the customers wait in the queue, thus will be served according to the FIFO order, where service times $B(t)$ are assumed to be independent and exponentially distributed with mean $1/\mu$. However, if upon arrival, the customers find the queue full, then they decide to stay for an attempt to get served with probability $\bar{\beta} = 1 - \beta$ or leave the system with probability β, $0 \leq \beta \leq 1$. The arriving customers who decide to stay for an attempt, they have to choose one of the orbits O_1, O_2; depending on their thresholds; if the number of customers in orbit O_1 is quite larger than that of orbit O_2, the customer will move into the orbit O_2 with probability $\bar{\beta}\beta_2$; $0 \leq \beta_2 \leq 1$, otherwise he/she removes into orbit O_1 with probability $\bar{\beta}\beta_1$; $0 \leq \beta_1 \leq 1$.

Notice that if the threshold of customers in orbit O_1 is quite larger than that of orbit O_2, the customers in orbit O_1 will make the attempts firstly and vice versa. Afterward, customers go in the retrial queues and make attempts to reach the primary queue, where the attempt times are assumed also to be independent and exponentially distributed with mean $1/\alpha_i$, $i = 1, 2$. Finally, after the customer is served completely, he/she may decide either to join the retrial groups O_1 or O_2 again for another service with probability $\bar{\xi}\delta_1$; (δ_1 is the probability that the customer chooses orbit O_1), with $0 \leq \delta_1 \leq 1$, or $\bar{\xi}\delta_2$; (δ_2 is the probability that the customer chooses orbit O_2), with $0 \leq \delta_2 \leq 1$, or leaves the system forever with probability ξ, $0 \leq \xi \leq 1$.

This sort of system abstracts and characterizes different practical situations in the telecommunication networks. For example, the mechanism based automatic repeat request protocol in data transmission systems may be modeled by a retrial queue system with feedback, since lost packets are negatively acknowledged by the receivers, then the senders send them once again.

In this paper we provide approximate expressions for our queueing performance measures; we investigate the joint distribution of the server state and queue length under the system steady state assumption. The condition of system stability is assumed to be hold, Further analysis around the stability of retrial queues can be found in [2], where E. Altman and A. A. Borovkov provided the general conditions under which ρ (system's load) < 1 is a sufficient condition for the stability of retrial queuing systems.

3 Main result

Theorem 1 *For our retrial queueing model with two orbits, balking and feedback in the steady state:*

1. *The average of the queue length along the idle period of the server is expressed by*

$$\mathbb{E}(N_i, S = 0) = m_i \left(\frac{\beta}{\bar{\beta}\beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda + \alpha_i) + \bar{\xi}\delta_i\mu}{\beta(\lambda + \alpha_i) + \mu(1 - \delta_i\bar{\xi})} \right) \right.$$
$$\left(\frac{\alpha_i\beta + \mu(1 - \delta_i\bar{\xi})}{\lambda} - \bar{\beta}\beta_i \right)$$
$$F\left\{ \frac{\beta\bar{\beta}_i(\lambda + 2\alpha_i) + \bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(2\alpha_i + \lambda) + \mu(1 - \delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} + \left(\frac{\alpha_i\beta^3}{\lambda\bar{\beta}^2\beta_i^2} \right)$$
$$\left(\frac{\bar{\beta}\beta_i(\lambda + \alpha_i) + \bar{\xi}\delta_i\mu}{\beta(\lambda + \alpha_i) + \mu(1 - \delta_i\bar{\xi})} \right) \left(\frac{\bar{\beta}\beta_i(\lambda + 2\alpha_i) + \bar{\xi}\delta_i\mu}{\beta(\lambda + 2\alpha_i) + \mu(1 - \delta_i\bar{\xi})} \right)$$
$$\left. -\bar{\beta}\beta_i F\left\{ \frac{\beta_i\bar{\beta}(\lambda + \alpha_i) + \bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\bar{\beta}\beta_i(\alpha_i + \lambda) + \mu\delta_i\bar{\xi}}{\bar{\beta}\beta_i\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta.} \right\} \right).$$

2. *The average of the queue length along the busy period of the server is expressed by*

$$\mathbb{E}(N_i, S=1) = m_i \left(\frac{\beta}{\bar{\beta}\beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})} \right) F\left\{ \frac{\beta\bar{\beta}_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} \right).$$

3. *The average of the queue length is given by*

$$\mathbb{E}(N, S=0)+\mathbb{E}(N, S=1) = \sum_{i=1}^{2} m_i \left(\frac{\beta}{\bar{\beta}\beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})} \right) \right.$$
$$\left(\frac{\alpha_i\beta+\mu(1-\delta_i\bar{\xi})}{\lambda} + 1 - \bar{\beta}\beta_i \right)$$
$$F\left\{ \frac{\beta\bar{\beta}_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\}$$
$$+\left(\frac{\alpha_i\beta^3}{\lambda\bar{\beta}^2\beta_i^2} \right) \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})} \right)$$
$$\left(\frac{\bar{\beta}\beta_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+2\alpha_i)+\mu(1-\delta_i\bar{\xi})} \right)$$
$$F\left\{ \frac{\beta\bar{\beta}_i(\lambda+3\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(3\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\}$$
$$\left. -\bar{\beta}\beta_i F\left\{ \frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\bar{\beta}\beta_i\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} \right).$$

Proof. To prove the theorem, we should firstly introduce the system statistical equilibrium equations for the system, so let us denote $N_1(t)$, $N_2(t)$ the number of repeated calls in the the retrial queue O_1 respectively O_2 at time t, and $S(t)$ represents the server state, where it takes two values 1 or 0 at time t when the server is busy or idle respectively. Thus, a process $\{S(t), N_1(t), N_2(t)\}$ which describes the number of customers in the system is the simplest and simultaneously the most important process associated with the retrial queueing system described in Fig.1.

To simplify our analysis, we suppose that the service time function $B(t)$ is exponentially distributed. Thus, $\{S(t), N_1(t), N_2(t)\}$ forms a markov process, where we can consider the markov chain of this process representing this system is embedded at jump customers arrival times rather than a chain embedded at service completion epochs. Hence, the process $\{(S(t), N_1(t), N_2(t)) : t \succ$

$0\}$ forms a Markov chain with a state space $\{0,1\}\times\{0,1,...,N_1\}\times\{0,1,...,N_2\}$, where $\{S,N_1,N_2\}\approx\lim_{t\to\infty}\{S(t),N_1(t),N_2(t)\}$ in the steady state.
As a result, in the steady state the joint probabilities of server state S and the retrial queue lengths N_1, N_2, $P_{0n_1n_2}=\mathbb{P}\{S=0,N_1=n_1,N_2=n_2\}$, and $P_{1n_1n_2}=\mathbb{P}\{S=1,N_1=n_1,N_2=n_2\}$, can be characterized through the corresponding partial generating functions for $|z_1|\le 1$, $|z_2|\le 1$ by $P_0(z_1)=\sum_{n_1=0}^{\infty}P_{0n_1n_2}z_1^{n_1}$, $P_0(z_2)=\sum_{n_2=0}^{\infty}P_{0n_1n_2}z_2^{n_2}$ and $P_1(z_1)=\sum_{n_1=0}^{\infty}P_{1n_1n_2}z_1^{n_1}$, $P_1(z_2)=\sum_{n_2=0}^{\infty}P_{1n_1n_2}z_2^{n_2}$. Consequently, we can describe the set of statistical equilibrium equations for these probabilities $(P_{0n_1n_2},P_{1n_1n_2})$ as follows:

$$(\lambda+n_1\alpha_1)P_{0n_1n_2}=\xi\mu P_{1n_1n_2}+\bar{\xi}\delta_1\mu P_{1n_1-1n_2} \tag{1}$$

$$\begin{aligned}(\lambda\beta\beta_1+\mu+n_1\beta\alpha_1)P_{1n_1n_2}&=\bar{\beta}\beta_1\lambda P_{1n_1-1n_2}+(n_1+1)\beta\alpha_1P_{1n_1+1n_2}\\&\quad+(n_1+1)\alpha_1P_{0n_1+1n_2}+\lambda P_{0n_1n_2}\end{aligned} \tag{2}$$

$$(\lambda+n_2\alpha_2)P_{0n_1n_2}=\xi\mu P_{1n_1n_2}+\bar{\xi}\delta_2\mu P_{1n_1n_2-1} \tag{3}$$

$$\begin{aligned}(\lambda\bar{\beta}\beta_2+\mu+n_2\beta\alpha_2)P_{1n_1n_2}&=\bar{\beta}\beta_2\lambda P_{1n_1n_2-1}+(n_2+1)\beta\alpha_2P_{1n_1n_2+1}\\&\quad+(n_2+1)\alpha_2P_{0n_1n_2+1}+\lambda P_{0n_1n_2}.\end{aligned} \tag{4}$$

Now to continue in deriving the joint distribution, we multiply the equations (1), (2), (3) and (4) by $\sum_{i=n_i}^{\infty}z_i^{n_i}$, $i=1,2$ which yields to the following equations :

$$\lambda P_0(z_i)+\alpha_iz_iP_0'(z_i)=\xi\mu P_1(z_i)+\bar{\xi}\delta_i\mu z_iP_1(z_i) \tag{5}$$

$$\left[\lambda\bar{\beta}\beta_i(1-z_i)+\mu)\right]P_1(z_i)+\alpha_i\beta(z_i-1)P_1'(z_i)=\alpha_iP_0'(z_i)+\lambda P_0(z_i). \tag{6}$$

By taking the sum of equation (5) and (6), then divide the sum by (z_i-1) we obtain

$$\alpha_iP_0'(z_i)+\alpha_i\beta P_1'(z_i)=(\bar{\beta}\beta_i\lambda+\bar{\xi}\delta_i\mu)P_1(z_i). \tag{7}$$

By substituting equation (7) into (6), we can express $P_0(z_i)$ in terms of $P_1'(z_i)$, $P_1(z_i)$ as follows:

$$P_0(z_i)=(\tfrac{\alpha_i\beta}{\lambda})z_iP_1'(z_i)+\left(\tfrac{\mu}{\lambda}(1-\delta_i\bar{\xi})-\bar{\beta}\beta_iz_i\right)P_1(z_i). \tag{8}$$

By differentiating equation (8), we get

$$P_0'(z_i)=\tfrac{\alpha_i\beta}{\lambda}z_iP_1''(z_i)+\left(\tfrac{\mu(1-\delta_i\bar{\xi})+\alpha_i\beta}{\lambda}-\bar{\beta}\beta_iz_i\right)P_1'(z_i)-\bar{\beta}\beta_iP_1(z_i). \tag{9}$$

By substituting equations (8) and (9) into (5), we obtain a differential equation of $P_1(z_i)$

$$\begin{aligned} &z_i P_1''(z_i) + \left(\frac{\mu(1-\delta_i\bar{\xi}) + (\lambda+\alpha_i)\beta}{\alpha_i\beta} - \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} z_i\right) P_1'(z_i) \\ &- \frac{\lambda\left(\bar{\beta}\beta_i(\lambda+\alpha_i) + (1-\bar{\xi}\delta_i)\mu\right)}{\alpha_i^2\beta} P_1(z_i) = 0. \end{aligned} \tag{10}$$

Consequently, we transform the equation (10) into Kummer's differential equation, since it has already a solution.
Let

$$Y(x_i) = P_1(z_i(x_i)) \text{ and } z_i = \frac{\beta\alpha_i}{\bar{\beta}\beta_i\lambda} x_i, \quad i = 1, 2$$

which transforms (10) into

$$x_i Y_i''(x_i) + \left(\tfrac{(\lambda+\alpha_i)\beta + \mu(1-\delta_i\bar{\xi})}{\beta\alpha_i} - x_i\right) Y_i'(x_i) - \left(\tfrac{\beta\bar{\beta}(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\alpha_i\beta\bar{\beta}}\right) Y_i(x_i) = 0. \tag{11}$$

The equation (11) can be rewritten as follows

$$x_i Y_i''(x_i) + (d_i - x_i) Y'(x) - a_i Y(x_i) = 0 \tag{12}$$

such that $a_i = \frac{\beta\bar{\beta}(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\alpha_i\beta\bar{\beta}}$ and $d_i = \frac{(\lambda+\alpha_i)\beta+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}$. Referring to [1], [13], the equation (12) has a regular singular point at $x_i = 0$, and an irregular singularity at $x_i = \infty$. Furthermore, the solution of equation (12) is found analytically in a unite circle, $U = \{x : |x| \prec 1\}$ which represents in turn the solution of kummer's function $Y(x_i)$ and expressed by $Y(x_i) = m_i \times F(a_i; d_i; x_i)$, $m_i \neq 0$ so, equation (10) is solved for $P_1(z_i)$ as follows

$$\begin{aligned} P_1(z_i) = m_i \times F\Bigg\{ &\frac{\bar{\beta}\beta_i(\lambda+\alpha_i) + \bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}; \\ &\frac{(\lambda+\alpha_i)\beta + \mu(1-\delta_i\bar{\xi})}{\alpha_i\beta}; \frac{\bar{\beta}\beta_i\lambda}{\alpha_i\beta} z_i \Bigg\}, \quad |z_i| \leq 1. \end{aligned} \tag{13}$$

Referring to [13], the first derivative of Kummer's function $F(a_i; d_i; x_i)$ is defined as follows: $\frac{dF}{dx_i} = \frac{a_i}{d_i} F(a_i+1; d_i+1; x_i)$, hence $P_1'(z_i)$ is expressed as follows:

$$\begin{aligned} P_1'(z_i) = m_i \Bigg\{ &\frac{\beta}{\bar{\beta}\beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i) + \bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i) + \mu(1-\delta_i\bar{\xi})}\right) F\Bigg\{ \frac{\bar{\beta}\beta_i(\lambda+2\alpha_i) + \bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}; \\ &\frac{(\lambda+2\alpha_i)\beta + \mu(1-\delta_i\bar{\xi})}{\alpha_i\beta}; \frac{\bar{\beta}\beta_i\lambda}{\alpha_i\beta} z_i \Bigg\}\Bigg\}, \quad |z_i| \leq 1. \end{aligned} \tag{14}$$

Then we replace into equation (8) for $P_0(z_i)$, $P_1(z_i)$ and $P_1'(z_i)$ by their equivalence in equations (13) and (14), and hence $P_0(z_i)$ is expressed as follows:

$$\begin{aligned} P_0(z_i) = m_i \Bigg[& \frac{\alpha_i \beta^2}{\lambda \bar{\beta} \beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i) + \bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i) + \mu(1-\delta_i\bar{\xi})} \right) \\ & F\left\{ \frac{\beta\bar{\beta}_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} \\ & + \left(\frac{\mu(1-\delta_1\bar{\xi})}{\lambda} - \bar{\beta}\beta_i \right) \\ & F\left\{ \frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} \Bigg] \end{aligned} \tag{15}$$

Then at the boundary condition, where $z_i = 1, i = 1, 2$ we can ge the value of m through $P_0(1) + P_1(1) = 1$

$$\begin{aligned} m_i = \Bigg[& \frac{\alpha_i \beta^2}{\lambda \bar{\beta} \beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i) + \bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i) + \mu(1-\delta_i\bar{\xi})} \right) \\ & F\left\{ \frac{\beta\bar{\beta}_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} \\ & + \left(\frac{\mu(1-\delta_1\bar{\xi})}{\lambda} - \bar{\beta}\beta_i + 1 \right) \\ & F\left\{ \frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} \Bigg]^{-1} \end{aligned} \tag{16}$$

So, the generating functions of the joint distribution of server state S and queue length N_i are given by

$$\begin{aligned} P_0(z_i) = \mathbb{E}(z_i^{N_i}, S=0) = m_i \Bigg[& \frac{\alpha_i \beta^2}{\lambda \bar{\beta} \beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i) + \bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i) + \mu(1-\delta_i\bar{\xi})} \right) \\ & F\left\{ \frac{\beta\bar{\beta}_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} \\ & + \left(\frac{\mu(1-\delta_1\bar{\xi})}{\lambda} - \bar{\beta}\beta_i \right) \\ & F\left\{ \frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} z_i \right\} \Bigg] \\ P_1(z_i) = \mathbb{E}(z_i^{N_i} : S=1) = m_i \cdot F \Bigg\{ & \frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}; \\ & \frac{\beta(\lambda+\alpha_i)+(1-\delta_i\bar{\xi})\mu}{\beta\alpha_i}; \frac{\bar{\beta}\beta_i\lambda}{\alpha_i\beta} z_i \Bigg\}, \; |z_i| \le 1. \end{aligned}$$

Consequently, the average of the queue length along the idle period of the server is equivalent to $P_0'(1)$, which is expressed by

$$\mathbb{E}(N_i, S=0) = m_i \left(\frac{\beta}{\bar{\beta}\beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})} \right) \left(\frac{\alpha_i\beta+\mu(1-\delta_i\bar{\xi})}{\lambda} - \bar{\beta}\beta_i \right) \right.$$
$$F\left\{ \frac{\beta\bar{\beta}_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} + \left(\frac{\alpha_i\beta^3}{\lambda\bar{\beta}^2\beta_i^2} \right)$$
$$\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})} \right) \left(\frac{\bar{\beta}\beta_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+2\alpha_i)+\mu(1-\delta_i\bar{\xi})} \right)$$
$$F\left\{ \frac{\beta\bar{\beta}_i(\lambda+3\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(3\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\}$$
$$\left. -\bar{\beta}\beta_i F\left\{ \frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\bar{\beta}\beta_i\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} \right)$$

And the average of the queue length along the busy period of the server is equivalent to $P_1'(1)$, which is expressed by

$$\begin{aligned} \mathbb{E}(z_i^{N_i} : S=1) = m_i &\left\{ \frac{\beta}{\bar{\beta}\beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})} \right) \right. \\ F&\left\{ \frac{\bar{\beta}\beta_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}; \frac{(\lambda+2\alpha_i)\beta+\mu(1-\delta_i\bar{\xi})}{\alpha_i\beta}; \frac{\bar{\beta}\beta_i\lambda}{\alpha_i\beta} \right\} \left. \vphantom{\frac{\beta}{\bar{\beta}}} \right\} \end{aligned} \tag{17}$$

Thus the average of the queue length in the retrial queuing system is the sum of $P_0'(1)$ and $P_1'(1)$, which is given by

$$\mathbb{E}(N, S=0) + \mathbb{E}(N, S=1) = \sum_{i=1}^{2} m_i \left(\frac{\beta}{\bar{\beta}\beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})} \right) \right.$$
$$\left(\frac{\alpha_i\beta+\mu(1-\delta_i\bar{\xi})}{\lambda} + 1 - \bar{\beta}\beta_i \right)$$
$$F\left\{ \frac{\beta\bar{\beta}_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\}$$
$$+ \left(\frac{\alpha_i\beta^3}{\lambda\bar{\beta}^2\beta_i^2} \right) \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})} \right) \left(\frac{\bar{\beta}\beta_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+2\alpha_i)+\mu(1-\delta_i\bar{\xi})} \right)$$
$$F\left\{ \frac{\beta\bar{\beta}_i(\lambda+3\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(3\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\}$$
$$\left. - \bar{\beta}\beta_i F\left\{ \frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\bar{\beta}\beta_i\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} \right)$$

□

4 Numerical results

The average waiting time W in the steady state is often considered to be the most important of performance measures in retrial queuing systems. However, W is an average over all primary calls, including those calls which receive immediate service and really do not wait at all. A better grasp of understanding the waiting time process can be obtained by studying first the relationship between the retrial queue length $\mathbb{E}(N) = \mathbb{E}(N_1) + \mathbb{E}(N_2)$ and other inputs,outputs and feedback parameters. We have conducted some preliminary analysis through some simulations done on the queue lengths, in order to show the impact of the different parameters and its relationship with the retrial queue length $\mathbb{E}(N)$. The primary objective behind this was to understand what does happen at some telecommunication systems where redials or connection retrials arise naturally.

These analysis involved three scenarios "figure 2-figure 4" in order to clarify the relations in different situations among the input, output, balk and feedback parameters. These scenarios are realized through simulations via Matlab program. To begin, we chose a significant values for the parameters so as to meet the requirements of the phase-merging algorithm.

For the first figure, for each value of $\bar{\xi}$ ($\bar{\xi}$ = 0; 0.2; 0.4; 0.6; 0.8; 1) selected, we vary μ from 0 to 1 in increments of 0.1, where we evaluate $\mathbb{E}(N)$ at different values of service completion probability while $\beta_1 = \beta_2 = \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0.5$, $\beta = 0.7$, $\lambda = 0.7$. The numerical results are summarized in the following table:

μ	Average Retrial Queue Length	$\xi = 1$	$\xi = 0.8$	$\xi = 0.6$	$\xi = 0.4$	$\xi = 0.2$	$\xi = 0$
0	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	2.2963	2.2963	2.2963	2.2963	2.2963	2.2963
0.1	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	2.6852	2,5577	2,4302	2,3026	2,1749	2,0472
0.2	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3.0167	2,7798	2,5434	2,3077	2,0726	1,8383
0.3	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3.3060	2,9723	2,6409	2,3119	1,9857	1,6625
0.4	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3.5629	3,1417	2,7258	2,3154	1,9111	1,5135
0.5	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3.7937	3,2926	2,8005	2,3183	1,8467	1,3864
0.6	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4.0031	3,4280	2,8670	2,3208	1,7905	1,2771
0.7	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4.1945	3,5506	2,9264	2,3229	1,7413	1,1826
0.8	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4.3705	3,6623	2,9800	2,3247	1,6977	1,1002
0.9	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4.5332	3,7645	3,0284	2,3262	1,6591	1,0278
1	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4.6843	3,8586	3,0726	2,3276	1,6245	0,9639

The first figure shows that along the increase of μ the retrial queue lengths increase when the values of $\bar{\xi}$ become larger; for instance when $\bar{\xi}$ = 1; 0.8; 0.6 and decrease when $\bar{\xi}$ become smaller; for instance when $\bar{\xi}$ = 0; 0.2; 0.4. Obviously, this refers to the possibility of accepting repeated and primary calls becomes large. This figure shows us also that when ξ becomes greater than 0.6 or the feedback probability becomes less than 0.6, then $\mathbb{E}(N)$ is not affected remarkably or it decreases very slowly.

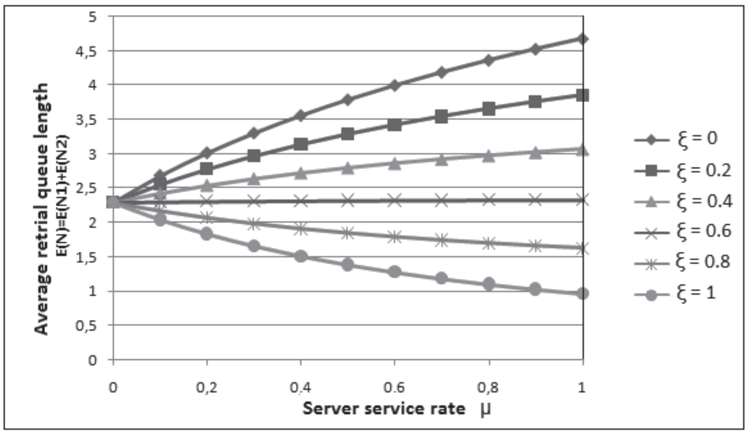

Figure 2: Average retrial queue length $\mathbb{E}(N)$ & service server rate μ

For the second figure, for each value of $\bar{\beta}$ such that

- For $\beta = 0,5$ we choose a significant parameters $\alpha_1 = \alpha_2 = 0.7$, $\delta_1 = 0.1$, $\delta_2 = 0.9$ and $\beta_1 = \beta_2 = 0.5$.
- For $\beta = 0.7$, we choose a significant parameters $\alpha_1 = \alpha_2 = 0.7$, $\delta_1 = 0.1$, $\delta_2 = 0.9$, and $\beta_1 = 0.6, \beta_2 = 0.4$,

we vary ξ from 0 to 1 in increments of 0.1, where we evaluate $E(N)$ at different values of balking probability β, while $\mu = 0.8$ and $\lambda = 0.7$. The numerical results are summarized in the following table:

ξ	Average Retrial Queue Length	$\bar{\beta} = 0.3$	$\bar{\beta} = 0.5$
0	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4,8845	2,9873
0.1	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4,4893	2,7386
0.2	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4,1030	2,4980
0.3	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3,7245	2,2662
0.4	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3,3533	2,0440
0.5	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	2,9887	1,8319
0.6	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	2,6304	1,6302
0.7	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	2,2782	1,4391
0.8	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	1,9322	1,2588
0.9	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	1,5926	1,0889
1	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	1,2598	0,9294

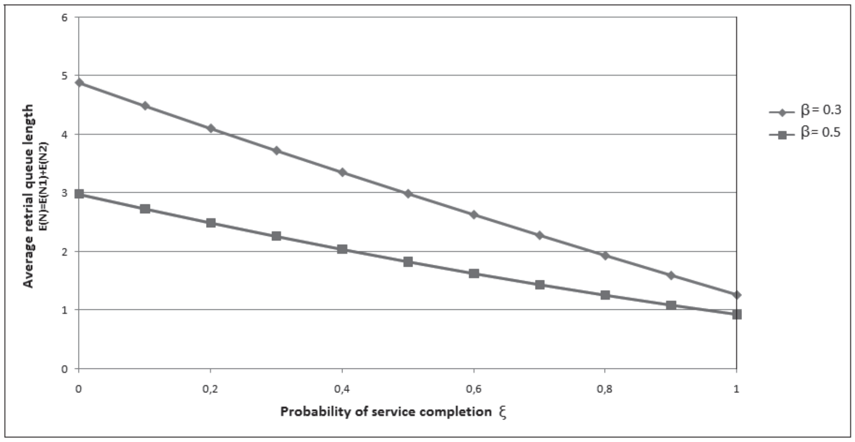

Figure 3: Average retrial queue length $\mathbb{E}(N)$ & probability of service completion ξ

The second figure shows that $\mathbb{E}(N)$ for our model with balking and feedback is not affected by feedback probability $\bar{\xi}$ when the probability $\bar{\beta}$ of non-balking or returning to retrial group after customer attempt's failure becomes less than 0.5. However, $\mathbb{E}(N)$ increases rapidly as $\bar{\xi}$ and $\bar{\beta}$ become high.

For the third figure, For each value of α_i ($\alpha_1 = \alpha_2 = 0.1$ and $\alpha_1 = \alpha_2 = 0.8$) selected, we vary $\bar{\beta}$ from 0.1 to 0.9 in increments of 0.1, such that for a good requirement we choose

for $\bar{\beta} = 0.1$	$\beta_1 = 0.7$	$\beta_2 = 0.3$
for $\bar{\beta} = 0.2$	$\beta_1 = 0.9$	$\beta_2 = 0.1$
for $\bar{\beta} = 0.3$	$\beta_1 = 0.95$	$\beta_2 = 0.05$
for $\bar{\beta} = 0.4$	$\beta_1 = 0.97$	$\beta_2 = 0.03$
for $\bar{\beta} = 0.5$	$\beta_1 = 0.98$	$\beta_2 = 0.02$
for $\bar{\beta} = 0.6$	$\beta_1 = 0.99$	$\beta_2 = 0.01$
for $\bar{\beta} = 0.7$	$\beta_1 = 0.993$	$\beta_2 = 0.007$
for $\bar{\beta} = 0.8$	$\beta_1 = 0.996$	$\beta_2 = 0.004$
for $\bar{\beta} = 0.9$	$\beta_1 = 0.998$	$\beta_2 = 0.002$

Then, we evaluate $\mathbb{E}(N)$ at different values of retrial probability α_i, while $\delta_1 = \delta_2 = 0.5$, $\xi = 0.5$, $\mu = 0.8$ and $\lambda = 0.7$. The numerical results are summarized in the following table:

$\bar{\beta}$	Average Retrial Queue Length	$\alpha_1 = \alpha_2 = 0.1$	$\alpha_1 = \alpha_2 = 0.8$
0.1	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	7.3610	5,8755
0.2	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	7,8458	6,1070
0.3	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	8,9262	6,7783
0.4	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	9,7914	7,3964
0.5	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	10,2588	7,8443
0.6	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	14,2576	10,7133
0.7	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	14,3527	11,1251
0.8	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	15,7566	13,0344
0.9	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	16,3376	13,9861

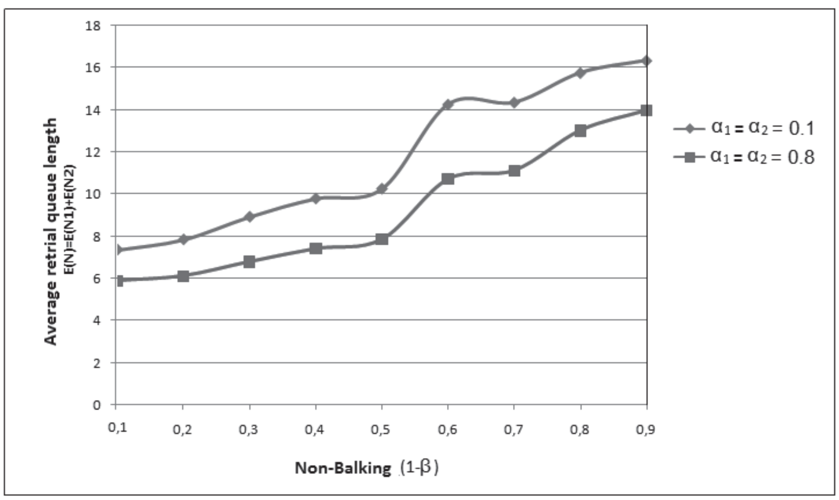

Figure 4: Average retrial queue length $\mathbb{E}(N)$ & non-balking rate $\bar{\beta}$

Figure 4 shows that along the design of retrial queuing system, we have to assign equivalent values for the non-balking probability $\bar{\beta}$ and the retrial probability α_i in order to keep the retrial queue length as short as possible. This can be concluded from the figure since when α_i takes values greater or equal to 0.5, and $\bar{\beta}$ gets values less than 0.5 $\mathbb{E}(N)$ becomes small.

As a conclusion, we conclude that Figures 2 through 4 indicate that the phase-merging algorithm is reasonably effective in approximating $\mathbb{E}(N)$, for all values of $\mu, \xi, \beta,$ and α.

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Related fixed point theorem for four complete metric spaces

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Abstract. In the present paper, we obtain a new result on fixed point theorem for four metric spaces. Here we choose continuous mappings. In fact our result is the generalization of many results of fixed point theorem on two and three metric spaces. We also give some illustrative examples to justify our result.

1 Introduction

Related fixed point theorems on two metric spaces have been studied by B. Fisher [2]. Also some fixed point theorems on three metric spaces have been studied by B. Fisher *et al* [3], R. K. Jain *et al.* [4], R. K. Namdeo and B. Fisher [7], K. Kikina *et al.* [6], Z. Ansari *et al.* [1], and V. Gupta [8]. Also, the fixed point theorems on four metric spaces have been studied by L. Kikina *et al.* [5]. In the present paper a generalization is given for four complete metric space. Our theorem improves Theorem (2.1) of R. K. Jain *et al.* [4].

The following fixed point theorem was proved by R. K. Jain, H. K. Sahu, B. Fisher [4].

Theorem 1 *Let* (X, d), (Y, ρ) *and* (Z, σ) *be complete metric spaces. If* T *is continuous mapping of* $X \mapsto Y$, S *is a continuous mapping of* $Y \mapsto Z$ *and* R *is mapping of* $Z \mapsto X$ *satisfying the inequalities*

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$$\begin{aligned} d(RSTx, RSTx') \leqslant c \max\{&d(x, x'), d(x, RSTx), \\ &d(x', RSTx'), \rho(Tx, Tx'), \sigma(STx, STx')\}, \end{aligned} \tag{1}$$

$$\begin{aligned} \rho(TRSy, TRSy') \leqslant c \max\{&\rho(y, y'), \rho(y, TRSy), \\ &\rho(y', TRSy'), \sigma(Sy, Sy'), d(RSy, RSy')\}, \end{aligned} \tag{2}$$

$$\begin{aligned} \sigma(STRz, STRz') \leqslant c \max\{&\sigma(z, z'), \sigma(z, STRz) \\ &\sigma(z', STRz'), d(Rz, Rz'), \rho(TRz, TRz')\}, \end{aligned} \tag{3}$$

$\forall\ x, x' \in X$, $y, y' \in Y$ and $z, z' \in Z$, where $0 \leqslant c < 1$, then RST has a unique fixed point $u \in X$, TRS has a unique fixed point $v \in Y$ and STR has a unique fixed point $w \in Z$. Further $Tu = v$, $Sv = w$ and $Rw = u$.

2 Main result

Theorem 2 *Let $(Z_1, d_1), (Z_2, d_2), (Z_3, d_3)$, and (Z_4, d_4) be complete metric spaces. If A_1 is a continuous mapping of $Z_1 \mapsto Z_2$, A_2 is continuous mapping of $Z_2 \mapsto Z_3$, A_3 is continuous mapping of $Z_3 \mapsto Z_4$ and A_4 is a mappings of $Z_4 \mapsto Z_1$, satisfying the inequalities*

$$\begin{aligned} &d_1(A_4A_3A_2A_1z_1, A_4A_3A_2A_1z_1') \\ &\quad \leqslant c \max\{d_1(z_1, z_1'), d_1(z_1, A_4A_3A_2A_1z_1), \\ &\qquad d_1(z_1', A_4A_3A_2A_1z_1'), d_2(A_1z_1, A_1z') \\ &\qquad d_3(A_2A_1z_1, A_2A_1z_1'), d_4(A_3A_2A_1z_1, A_3A_2A_1z_1')\}, \end{aligned} \tag{4}$$

$$\begin{aligned} &d_2(A_1A_4A_3A_2z_2, A_1A_4A_3A_2z_2') \\ &\quad \leqslant\ c \max\{d_2(z_2, z_2'), d_2(z_2, A_1A_4A_3A_2z_2), \\ &\qquad d_2(z_2', A_1A_4A_3A_2z_2'), d_3(A_2z_2, A_2z_2'), \\ &\qquad d_4(A_3A_2z_2, A_3A_2z_2'), d_1(A_4A_3A_2z_2, A_4A_3A_2z_2')\}, \end{aligned} \tag{5}$$

$$\begin{aligned} &d_3(A_2A_1A_4A_3z_3, A_2A_1A_4A_3z_3') \\ &\quad \leqslant\ c \max\{d_3(z_3, z_3'), d_3(z_3, A_2A_1A_4A_3z_3), \\ &\qquad d_3(z_3', A_2A_1A_4A_3z_3'), d_4(A_3z_3, A_3z_3'), \\ &\qquad d_1(A_4A_3z_3, A_4A_3z_3'), d_2(A_1A_4A_3z_3, A_1A_4A_3z_3')\}, \end{aligned} \tag{6}$$

$$\begin{aligned} &d_4(A_3A_2A_1A_4z_4, A_3A_2A_1A_4z_4') \\ &\quad \leqslant\ c \max\{d_4(z_4, z_4'), d_4(z_4, A_3A_2A_1A_4z_4), \\ &\qquad d_4(z_4', A_3A_2A_1A_4z_4'), d_1(A_4z_4, A_4z_4'), \\ &\qquad d_2(A_1A_4z_4, A_1A_4z_4'), d_3(A_2A_1A_4z_4, A_2A_1A_4z_4')\}, \end{aligned} \tag{7}$$

$\forall\, z_1, z_1' \in Z_1,\ z_2, z_2' \in Z_2,\ z_3, z_3' \in Z_3$ *and* $z_4, z_4' \in Z_4$*, where* $0 \leqslant c < 1$*, then* $A_4A_3A_2A_1$ *has a unique fixed point* $\alpha_1 \in Z_1$*,* $A_1A_4A_3A_2$ *has a unique fixed point* $\alpha_2 \in Z_2$*,* $A_2A_1A_4A_3$ *has a unique fixed point* $\alpha_3 \in Z_3$ *and* $A_3A_2A_1A_4$ *has a unique fixed point* $\alpha_4 \in Z_4$*.*

Further $A_1\alpha_1 = \alpha_2,\ A_2\alpha_2 = \alpha_3,\ A_3\alpha_3 = \alpha_4,\ A_4\alpha_4 = \alpha_1$*.*

Proof. Let z_1^0 be an arbitrary point in Z_1.

Define the sequence $\{z_n^1\}$, $\{z_n^2\}$, $\{z_n^3\}$ and $\{z_n^4\}$ in Z_1, Z_2, Z_3 and Z_4 respectively by

$$\begin{aligned}
(A_4A_3A_2A_1)^n z_1^0 &= z_n^1\\
A_1 z_{n-1}^1 &= z_n^2\\
A_2 z_n^2 &= z_n^3\\
A_3 z_n^3 &= z_n^4\\
A_4 z_n^4 &= z_n^1 \qquad \text{for } n = 1, 2, \ldots
\end{aligned}$$

Applying inequality (5), we get,

$$\begin{aligned}
d_2(z_n^2, z_{n+1}^2) &= d_2(A_1A_4A_3A_2z_{n-1}^2, A_1A_4A_3A_2z_n^2)\\
&\leqslant c\max\{d_2(z_{n-1}^2, z_n^2), d_2(z_{n-1}^2, A_1A_4A_3A_2z_{n-1}^2),\\
&\quad d_2(z_n^2, A_1A_4A_3A_2z_n^2), d_3(A_2z_{n-1}^2, A_2z_n^2),\\
&\quad d_4(A_3A_2z_{n-1}^2, A_3A_2z_n^2), d_1(A_4A_3A_2z_{n-1}^2, A_4A_3A_2z_n^2)\}\\
d_2(z_n^2, z_{n+1}^2) &\leqslant c\max\{d_2(z_{n-1}^2, z_n^2), d_2(z_{n-1}^2, z_n^2), d_2(z_n^2, z_{n+1}^2),\\
&\quad d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4), d_1(z_{n-1}^1, z_n^1)\}\\
d_2(z_n^2, z_{n+1}^2) &\leqslant c\max\{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2), d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\}
\end{aligned} \tag{8}$$

Using inequality (6), we get,

$$\begin{aligned}
d_3(z_n^3, z_{n+1}^3) &= d_3(A_2A_1A_4A_3z_{n-1}^3, A_2A_1A_4A_3z_n^3)\\
&\leqslant \max\{d_3(z_{n-1}^3, z_n^3), d_3(z_{n-1}^3, A_2A_1A_4A_3z_{n-1}^3),\\
&\quad d_3(z_n^3, A_2A_1A_4A_3z_n^3), d_4(A_3z_{n-1}^3, A_3z_n^3),\\
&\quad d_1(A_4A_3z_{n-1}^3, A_4A_3z_n^3), d_2(A_1A_4A_3z_{n-1}^3, A_1A_4A_3z_n^3)\}\\
d_3(z_n^3, z_{n+1}^3) &\leqslant c\max\{d_3(z_{n-1}^3, z_n^3), d_3(z_{n-1}^3, z_n^3), d_3(z_n^3, z_{n+1}^3),\\
&\quad d_4(z_{n-1}^4, z_n^4), d_1(z_{n-1}^1, z_n^1), d_2(z_n^2, z_{n+1}^2)\}\\
d_3(z_n^3, z_{n+1}^3) &\leqslant c\max\{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2),\\
&\quad d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\}
\end{aligned} \tag{9}$$

Using inequality (7), we have,

$$\begin{aligned}
d_4(z_n^4, z_{n+1}^4) &= d_4(A_3A_2A_1A_4z_{n-1}^4, A_3A_2A_1A_4z_n^4)\\
&\leqslant c\max\{d_4(z_{n-1}^4, z_n^4), d_4(z_{n-1}^4, A_3A_2A_1A_4z_{n-1}^4),\\
&\quad d_4(z_n^4, A_3A_2A_1A_4z_n^4), d_1(A_4z_{n-1}^4, A_4z_n^4),\\
&\quad d_2(A_1A_4z_{n-1}^4, A_1A_4z_n^4), d_3(A_2A_1A_4z_{n-1}^4, A_2A_1A_4z_n^4)\}\\
d_4(z_n^4, z_{n+1}^4) &\leqslant c\max\{d_4(z_{n-1}^4, z_n^4), d_4(z_{n-1}^4, z_n^4), d_4(z_n^4, z_{n+1}^4),\\
&\quad d_1(z_{n-1}^1, z_n^1), d_2(z_n^2, z_{n+1}^2), d_3(z_n^3, z_{n+1}^3)\}\\
d_4(z_n^4, z_{n+1}^4) &\leqslant c\max\{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2),\\
&\quad d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\}
\end{aligned} \tag{10}$$

Using inequality (4), we have,

$$\begin{aligned}
d_1(z_n^1, z_{n+1}^1) &= d_1(A_4A_3A_2A_1z_{n-1}^1, A_4A_3A_2A_1z_n^1)\\
&\leqslant c\max\{d_1(z_{n-1}^1, z_n^1), d_1(z_{n-1}^1, A_4A_3A_2A_1z_{n-1}^1),\\
&\quad d_1(z_n^1, A_4A_3A_2A_1z_n^1), d_2(A_1z_{n-1}^1, A_1z_n^1),\\
&\quad d_3(A_2A_1z_{n-1}^1, A_2A_1z_n^1), d_4(A_3A_2A_1z_{n-1}^1, A_3A_2A_1z_n^1)\}\\
d_1(z_n^1, z_{n+1}^1) &\leqslant c\max\{d_1(z_{n-1}^1, z_n^1), d_1(z_{n-1}^1, z_n^1), d_1(z_n^1, z_{n+1}^1),\\
&\quad d_2(z_n^2, z_{n+1}^2), d_3(z_n^3, z_{n+1}^3), d_4(z_n^4, z_{n+1}^4)\}\\
d_1(z_n^1, z_{n+1}^1) &\leqslant c\max\{d_1(z_{n-1}^1, z_n^1), d_2(z_{n-1}^2, z_n^2),\\
&\quad d_3(z_{n-1}^3, z_n^3), d_4(z_{n-1}^4, z_n^4)\}
\end{aligned} \tag{11}$$

By induction on using inequalities (8), (9), (10) and (11), we have,

$$\begin{aligned}
d_1(z_n^1, z_{n+1}^1) &\leqslant c^{n-1}\max\{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2),\\
&\quad d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\}\\
d_2(z_n^2, z_{n+1}^2) &\leqslant c^{n-1}\max\{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2),\\
&\quad d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\}\\
d_3(z_n^3, z_{n+1}^3) &\leqslant c^{n-1}\max\{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2),\\
&\quad d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\}\\
d_4(z_n^4, z_{n+1}^4) &\leqslant c^{n-1}\max\{d_1(z_1^1, z_2^1), d_2(z_1^2, z_2^2),\\
&\quad d_3(z_1^3, z_2^3), d_4(z_1^4, z_2^4)\}
\end{aligned}$$

Since $c < 1$, it follows that $\{z_n^1\}$,$\{z_n^2\}$,$\{z_n^3\}$ and $\{z_n^4\}$ are Cauchy sequences with limit α_1, α_2, α_3 and α_4 in Z_1, Z_2, Z_3 and Z_4 respectively.

Since A_1, A_2 and A_3 are continuous, we have,

$$\begin{aligned}
\lim_{n\to\infty} z_n^2 &= \lim_{n\to\infty} A_1 z_n^1 = A_1\alpha_1 = \alpha_2\\
\lim_{n\to\infty} z_n^3 &= \lim_{n\to\infty} A_2 z_n^2 = A_2\alpha_2 = \alpha_3\\
\lim_{n\to\infty} z_n^4 &= \lim_{n\to\infty} A_3 z_n^3 = A_3\alpha_3 = \alpha_4
\end{aligned}$$

Using inequality (4), again, we get,

$$\begin{aligned}
d_1(A_4A_3A_2A_1\alpha_1, z_n^1) &= d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1 z_{n-1}^1)\\
&\leqslant c\max\{d_1(\alpha_1, z_{n-1}^1), d_1(\alpha_1, A_4A_3A_2A_1\alpha_1),\\
&\quad d_1(z_{n-1}^1, A_4A_3A_2A_1 z_{n-1}^1), d_2(A_1\alpha_1, A_1 z_{n-1}^1),\\
&\quad d_3(A_2A_1\alpha_1, A_2A_1 z_{n-1}^1), d_4(A_3A_2A_1\alpha_1, A_3A_2A_1 z_{n-1}^1)\}
\end{aligned}$$

Since A_1, A_2 and A_3 are continuous, it follows on letting $n \to \infty$ that

$$d_1(A_4A_3A_2A_1\alpha_1, \alpha_1) \leqslant\ c\max\{d_1(\alpha_1, A_4A_3A_2A_1\alpha_1)\}$$

Thus, we have, $A_4A_3A_2A_1\alpha_1 = \alpha_1$. Since $c < 1$ and α_1 is the fixed point of $A_4A_3A_2A_1$

$$A_1A_4A_3A_2\alpha_2 = A_1A_4A_3A_2A_1\alpha_1 = A_1\alpha_1 = \alpha_2$$

and $$A_2A_1A_4A_3\alpha_3 = A_2A_1A_4A_3A_2\alpha_2 = A_2\alpha_2 = \alpha_3$$

and $$A_3A_2A_1A_4\alpha_4 = A_3A_2A_1A_4A_3\alpha_3 = A_3\alpha_3 = \alpha_4$$

Hence α_2, α_3 and α_4 are fixed points of $A_1A_4A_3A_2$, $A_2A_1A_4A_3$ and $A_3A_2A_1A_4$ respectively.

2.1 Uniqueness

Suppose that $A_4A_3A_2A_1$ has a second fixed point α_1'. Then, using inequality (4), we have,

$$\begin{aligned}
d_1(\alpha_1, \alpha_1') &= d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha_1')\\
&\leqslant c\max\{d_1(\alpha_1, \alpha_1'), d_1(\alpha_1, A_4A_3A_2A_1\alpha_1),\\
&\quad d_1(\alpha_1', A_4A_3A_2A_1\alpha_1'), d_2(A_1\alpha_1, A_1\alpha_1'),\\
&\quad d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\
&\quad d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\}
\end{aligned}$$

$$\begin{aligned}
d_1(\alpha_1,\alpha_1') \leqslant c\max\{&d_1(\alpha_1,\alpha_1'), d_1(\alpha_1,\alpha_1), d_1(\alpha_1',\alpha_1'),\\
&d_2(A_1\alpha_1,A_1\alpha_1'), d_3(A_2A_1\alpha_1,A_2A_1\alpha_1'),\\
&d_4(A_3A_2A_1\alpha_1,A_3A_2A_1\alpha_1')\} \qquad (12)\\
d_1(\alpha_1,\alpha_1') \leqslant c\max\{&d_2(A_1\alpha_1,A_1\alpha_1'), d_3(A_2A_1\alpha_1,A_2A_1\alpha_1'),\\
&d_4(A_3A_2A_1\alpha_1,A_3A_2A_1\alpha_1')\}
\end{aligned}$$

Using inequality (5), we have

$$\begin{aligned}
d_2(A_1\alpha_1,A_1\alpha_1') &= d_2(A_1A_4A_3A_2A_1\alpha_1, A_1A_4A_3A_2A_1\alpha_1')\\
&\leqslant c\max\{d_2(A_1\alpha_1,A_1\alpha_1'),\\
&\quad d_2(A_1\alpha_1, A_1A_4A_3A_2A_1\alpha_1),\\
&\quad d_2(A_1\alpha_1', A_1A_4A_3A_2A_1\alpha_1'),\\
&\quad d_3(A_2A_1\alpha_1,A_2A_1\alpha_1'),\\
&\quad d_4(A_3A_2A_1\alpha_1,A_3A_2A_1\alpha_1'),\\
&\quad d_4(A_4A_3A_2A_1\alpha_1,A_4A_3A_2A_1\alpha_1')\}\\
d_2(A_1\alpha_1,A_1\alpha_1') &\leqslant c\max\{d_2(A_1\alpha_1,A_1\alpha_1'), d_2(A_1\alpha_1,A_1\alpha_1'),\\
&\quad d_1(A_1\alpha_1',A_1\alpha_1'),\\
&\quad d_3(A_2A_1\alpha_1,A_2A_1\alpha_1'),\\
&\quad d_4(A_3A_2A_1\alpha_1,A_3A_2A_1\alpha_1'),\\
&\quad d_1(A_4A_3A_2A_1\alpha_1,A_4A_3A_2A_1\alpha_1')\}\\
d_2(A_1\alpha_1,A_1\alpha_1') &\leqslant c\max\{d_2(A_1\alpha_1,A_1\alpha_1'), d_3(A_2A_1\alpha_1,A_2A_1\alpha_1'),\\
&\quad d_4(A_3A_2A_1\alpha_1,A_3A_2A_1\alpha_1'), d_1(\alpha_1,\alpha_1')\}\\
d_2(A_1\alpha_1,A_1\alpha_1') &\leqslant c\max\{d_3(A_2A_1\alpha_1,A_2A_1\alpha_1'),\\
&\quad d_4(A_3A_2A_1\alpha_1,A_3A_2A_1\alpha_1'), d_1(\alpha_1,\alpha_1')\}
\end{aligned}$$

Now, we have

$$\begin{aligned}
d_2(A_1\alpha_1,A_1\alpha_1') \leqslant c\max\{&d_3(A_2A_1\alpha_1,A_2A_1\alpha_1'),\\
&d_4(A_3A_2A_1\alpha_1,A_3A_2A_1\alpha_1'),\\
&cd_2(A_1\alpha_1,A_1\alpha_1'), cd_3(A_2A_1\alpha_1,A_2A_1\alpha_1'),\\
&cd_4(A_3A_2A_1\alpha_1,A_3A_2A_1\alpha_1')\} \qquad (13)\\
d_2(A_1\alpha_1,A_1\alpha_1') \leqslant c\max\{&d_3(A_2A_1\alpha_1,A_2A_1\alpha_1')\\
&d_4(A_3A_2A_1\alpha_1,A_3A_2A_1\alpha_1')\}
\end{aligned}$$

Similarly on using inequality (6), we get

$$\begin{aligned}
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leqslant c\max\{&d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\
&d_3(A_2A_1\alpha_1, A_2A_1A_4A_3A_2A_1\alpha_1),\\
&d_3(A_2A_1\alpha_1', A_2A_1A_4A_3A_2A_1\alpha_1'),\\
&d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),\\
&d_1(A_4A_3A_2A_1\alpha_1, A_4A_3A_2A_1\alpha_1'),\\
&d_2(A_1A_4A_3A_2A_1\alpha_1, A_1A_4A_3A_2A_1\alpha_1')\}\\
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leqslant c\max\{&d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\
&d_3(A_2A_1\alpha_1, A_2A_1\alpha_1'),\\
&d_3(A_2A_1\alpha_1', A_2A_1\alpha_1'),\\
&d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),\\
&d_1(\alpha_1, \alpha_1'), d_2(A_1\alpha_1, A_1\alpha_1')\}\\
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leqslant c\max\{&d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),\\
&d_1(\alpha_1, \alpha_1'), d_2(A_1\alpha_1, A_1\alpha_1')\}\\
d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leqslant c\max\{&d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1'),\\
&cd_2(A_1\alpha_1, A_1\alpha_1'), cd_3(A_2A_1\alpha_1, A_2A_1\alpha_1')\\
&cd_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\}
\end{aligned} \tag{14}$$

Using inequality (13) and (14), we have

$$d_3(A_2A_1\alpha_1, A_2A_1\alpha_1') \leqslant c\max\{d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\} \tag{15}$$

Similarly on using inequality (7), (13) and (15), we have,

$$d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')) \leqslant cd_1(\alpha_1, \alpha_1') \tag{16}$$

Using inequality (12), (13), (15) and (16), we have

$$\begin{aligned}
d_1(\alpha_1, \alpha_1') &\leqslant cd_2(A_1\alpha_1, A_1\alpha_1')\\
&\leqslant c^2 d_3(A_2A_1\alpha_1, A_2A_1\alpha_1')\\
&\leqslant c^3 d_4(A_3A_2A_1\alpha_1, A_3A_2A_1\alpha_1')\\
&\leqslant c^4 d_1(\alpha_1, \alpha_1')
\end{aligned}$$

Now we have

$$d_1(\alpha_1, \alpha_1') \leqslant c^4 d_1(\alpha_1, \alpha_1')$$

Since $0 \leqslant c < 1$, we have

$$d_1(\alpha_1, \alpha_1') = 0$$

$\Rightarrow \quad \alpha_1 = \alpha_1'$, proving the uniqueness of α_1.

We can similarly prove that $A_1A_4A_3A_2$ has a unique fixed point $d_2 \in Z_2$ and $A_2A_1A_4A_3$ has a unique fixed point $\alpha_3 \in Z_3$ and $A_3A_2A_1A_4$ has unique fixed point $\alpha_4 \in Z_4$. □

Now, in support of our result, we give some examples.

Example 1 *Let suppose* $X = [0,1], Y = [1,2], Z = [2,3]$ *and* $L = [3,4]$ *be complete metric spaces with usual metric. If* $T : [0,1] \to [1,2]$, $S : [1,2] \to [2,3]$ *and* $R : [2,3] \to [3,4]$ *are continuous mappings and* $U : [3,4] \to [0,1]$ *is a mapping satisfying given conditions (in Theorem 2.1), where*

$$T(x) = \begin{cases} 1, & \text{if} \quad 0 \le x \le \frac{3}{4} \\ \frac{4}{3}x, & \text{if} \quad \frac{3}{4} < x \le 1 \end{cases}, \quad S(y) = \begin{cases} 2, & \text{if} \quad 1 \le y \le \frac{3}{2} \\ \frac{4}{3}y, & \text{if} \quad \frac{3}{2} < y \le 2 \end{cases}$$

$$R(z) = \begin{cases} 3, & \text{if} \quad 2 \le z \le \frac{5}{2} \\ \frac{6}{5}z, & \text{if} \quad \frac{5}{2} < z \le 3 \end{cases}, \quad U(u) = \begin{cases} 1, & \text{if} \quad 3 \le u \le \frac{7}{2} \\ \frac{3}{5}, & \text{if} \quad \frac{7}{2} < u \le 4 \end{cases}$$

then $URST$ *has fixed point 1 such that* $URST(1) = 1$, $TURS$ *has fixed point 4/3 such that* $TURS(4/3) = 4/3$, $STUR$ *has fixed point 2 such that* $STUR(2) = 2$ *and* $RSTU$ *has fixed point 3 such that* $RSTU(3) = 3$. *Also* $T(1) = 4/3$, $S(4/3) = 2$, $R(2) = 3$ *and* $U(3) = 1$.

Remark 1 *Below we give an example which satisfies all the condition of Theorem 2.1 but does not satisfies the condition of Theorem 1.1.*

Example 2 *Let* $X = [0,1], Y = [1,2], Z = [2,3]$ *and* $L = [3,4]$ *be complete metric space with usual metric. If* $T : [0,1] \to [1,2]$, $S : [1,2] \to [2,3]$ *and* $R : [2,3] \to [3,4]$ *are continuous mappings and* $U : [3,4] \to [0,1]$ *is a mapping satisfying given conditions (in Theorem 2.1), where*

$$T(x) = \begin{cases} 1, & \text{if} \quad 0 \le x \le \frac{1}{4} \\ 2x + \frac{1}{2}, & \text{if} \quad \frac{1}{4} \le x \le \frac{3}{4} \\ 2, & \text{if} \quad \frac{3}{4} \le x \le 1 \end{cases}, \quad S(y) = \begin{cases} 2, & \text{if} \quad 1 \le y \le \frac{5}{4} \\ \frac{4}{5}y + 1, & \text{if} \quad \frac{5}{4} \le y \le \frac{7}{4} \\ 2.4, & \text{if} \quad \frac{7}{4} \le y \le 2 \end{cases}$$

$$R(z) = \begin{cases} 3, & \text{if} \quad 2 \le z \le \frac{9}{4} \\ z + \frac{3}{4}, & \text{if} \quad \frac{9}{4} \le z \le \frac{11}{4} \\ 3.5, & \text{if} \quad \frac{11}{4} \le z \le 3 \end{cases}, \quad U(u) = \begin{cases} 0, & \text{if} \quad 3 \le u \le \frac{7}{2} \\ \frac{2}{7}u - 1, & \text{if} \quad \frac{7}{2} \le u \le 4 \end{cases}$$

then $URST$ *has fixed point* 0 *such that* $URST(0) = 0$*,* $TURS$ *has fixed point* 1 *such that* $TURS(1) = 1$*,* $STUR$ *has fixed point 2 such that* $STUR(2) = 2$ *and* $RSTU$ *has fixed point 3 such that* $RSTU(3) = 3$*. Also* $T(0)$*=1,* $S(1)$*=2,* $R(2)$*=3 and* $U(3)$*=0.*

Example 3 *Let suppose* $X = [0,2], Y = [1,5], Z = [0,10]$ *and* $L = [1,12]$ *be complete metric spaces with usual metric. If* $T : [0,2] \to [1,5]$*,* $S : [1,5] \to [0,10]$ *and* $R : [0,10] \to [1,12]$ *are continuous mappings and* $U : [1,12] \to [0,2]$ *is a mapping satisfying given conditions (in Theorem 2.1), where*

$$T(x) = [1 + x, 2], \qquad S(y) = [2y + 1, 5]$$

$$R(z) = [1 + z, 10], \qquad U(u) = \begin{cases} \left[\frac{u}{6}, 5\right] & \text{if} \quad 1 \le z \le 6 \\ \left[1, \frac{u}{6}\right] & \text{if} \quad 6 < z \le 12 \end{cases}$$

then $URST$ *has fixed point* 1 *such that* $URST(1) = 1$*,* $TURS$ *has fixed point* 2 *such that* $TURS(2) = 2$*,* $STUR$ *has fixed point 5 such that* $STUR(5) = 5$ *and* $RSTU$ *has fixed point 6 such that* $RSTU(6) = 6$*. Also* $T(1) = 2$*,* $S(2) = 5$*,* $R(5) = 6$ *and* $U(6) = 1$*.*

Example 4 *Let suppose* $X = [0,3], Y = [1,4], Z = [4,7]$ *and* $L = [3,10]$ *be complete metric spaces with usual metric. If* $T : [0,3] \to [1,4]$*,* $S : [1,4] \to [4,7]$ *and* $R : [4,7] \to [3,10]$ *be a continuous mappings and* $U : [3,10] \to [0,3]$ *be a mapping satisfying given conditions (in Theorem 2.1), where*

$$T(x) = 1 + x, \qquad S(y) = y + 2,$$

$$R(z) = z + 3, \qquad U(u) = \begin{cases} \frac{u}{7} & \text{if} \quad 3 \le z \le 5 \\ \frac{u+1}{8} & \text{if} \quad 5 < z \le 10 \end{cases}$$

then $URST$ *has fixed point* 1 *such that* $URST(1) = 1$*,* $TURS$ *has fixed point* 2 *such that* $TURS(2) = 2$*,* $STUR$ *has fixed point 4 such that* $STUR(4) = 4$ *and* $RSTU$ *has fixed point 7 such that* $RSTU(7) = 7$*. Also* $T(1) = 2$*,* $S(2) = 4$*,* $R(5) = 7$ *and* $U(7) = 1$*.*

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Contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold

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Abstract. In this paper we prove some properties of the indefinite Lorentzian para-Sasakian manifolds. Section 1 is introductory. In Section 2 we define D-totally geodesic and $D^{\perp}$-totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold and deduce some results concerning such a manifold. In Section 3 we state and prove some results on mixed totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold. Finally, in Section 4 we obtain a result on the anti-invariant distribution of totally umbilic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold.

1 Introduction

Many valuable and essential results were given on differential geometry with contact and almost contact structure. In 1970 the geometry of cosymplectic

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manifold was studied by G. D. Ludden [14]. After them, in 1973 and 1974, B. Y. Chen and K. Ogive introduced the geometry of submanifolds and totally real submanifolds in [8], [17], [7]. Then K. Ogive expressed the differential geometry of Kaehler submanifolds in [17]. In 1976 contact manifolds in Riemannian geometry were discussed by D. E. Blair [5]. Later on, A. Bejancu discussed CR-submanifolds of a Kaehler manifold [1], [2], [4], and then, K. Yano and M. Kon gave the notion of invariant and anti invariant submanifold in [13] and [21]. M. Kobayashi studied CR-submanifolds of a Sasakian manifold in 1981 [12]. New classes of almost contact metric structures and normal contact manifold in [18], [6] were studied by J. A. Oubina, C. Calin and I. Mihai. A. Bejancu and K. L. Duggal introduced (ϵ)-Sasakian manifolds. Lightlike submanifold of semi Riemannian manifolds was introduced by K. L. Duggal and A. Bejancu [10], [9]. In 2003 and 2007, lightlike submanifolds and hypersurfaces of indefinite Sasakian manifolds were introduced [11]. Lastly, LP-Sasakian manifolds were studied by many authors in [15], [16], [19], [20].

In this paper we define D-totally and $D^{\perp}$- totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold and prove some interesting results.

An n-dimensional differentiable manifold is called indefinite Lorentzian para-Sasakian manifold if the following conditions hold

$$\phi^2 X = X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{1}$$

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2}$$

$$\tilde{g}(X, \xi) = \epsilon\eta(X), \tag{3}$$

for all vector fields X, Y on $\tilde{M}$ [5] and where ϵ is 1 or -1 according to ξ is space-like or time-like vector field.

An indefinite almost metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite Lorentzian para-Sasakian manifold if

$$(\tilde{\nabla}_X \phi)Y = g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi, \tag{4}$$

where $\tilde{\nabla}$ is the Levi-Civita $(L - C)$ connection for a semi-Riemannian metric $\tilde{g}$. Also we have

$$\tilde{\nabla}_X \xi = \epsilon\phi X, \tag{5}$$

where $X \in T\tilde{M}$.

From the definition of contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold we have

Definition 1 *An n-dimensional Riemannian submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is called a contact CR-submanifold if*

i) ξ *is tangent to* M,

ii) *there exists on M a differentiable distribution $D : x \longrightarrow D_x \subset T_x(M)$, such that D_x is invariant under ϕ; i.e., $\phi D_x \subset D_x$, for each $x \in M$ and the orthogonal complementary distribution $D^\perp : x \longrightarrow D_x^\perp \subset T_x{}^\perp(M)$ of the distribution D on M is totally real; i.e., $\phi D_x^\perp \subset T_x^\perp(M)$, where $T_x(M)$ and $T_x^\perp(M)$ are the tangent space and the normal space of M at x.*

D (*resp.* $D^\perp$) *is the horizontal* (*resp. vertical*) *distribution. The contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold is called ξ-horizontal* (*resp. ξ-vertical*) *if $\xi_x \in D_x$* (*resp. $\xi_x \in D_x^\perp$*) *for each $x \in M$ by* [12].

The Gauss and Weingarten formulae are as follows

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{6}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{7}$$

for any $X, Y \in TM$ and $N \in T^\perp M$, where $\nabla^\perp$ is the connection on the normal bundle $T^\perp M$, h is the second fundamental form and A_N is the Weingarten map associated with N via

$$g(A_N X, Y) = g(h(X, Y), N). \tag{8}$$

The equation of Gauss is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)), \tag{9}$$

where $\tilde{R}$ (resp. R) is the curvature tensor of $\tilde{M}$ (resp. M).

For any $x \in M, X \in T_x M$ and $N \in T_x^\perp M$, we write

$$X = PX + QX \tag{10}$$

$$\phi N = BN + CN, \tag{11}$$

where PX (resp. BN) denotes the tangential part of X (resp. ϕN) and QX (resp. CN) denotes the normal part of X (resp. ϕN) respectively.

Using (6), (7), (10), (11) in (4) after a brief calculation we obtain on comparing the horizontal, vertical and normal parts

$$P\nabla_X\phi PY - PA_{\phi QY}X = \phi P\nabla_X Y + g(PX,Y)\xi + \epsilon\eta(Y)PX + 2\epsilon\eta(Y)\eta(X), \quad (12)$$

$$Q\nabla_X\phi PY + QA_{\phi QY}X = Bh(X,Y) + g(QX,Y)\xi + \epsilon\eta(Y)QX, \quad (13)$$

$$h(X,\phi PY) + \nabla_X^{\perp}\phi QY = \phi Q\nabla_X Y + Ch(X,Y). \quad (14)$$

From (5) we have

$$\nabla_X\xi = \epsilon\phi PX, \quad (15)$$

$$h(X,\xi) = \epsilon\phi QX. \quad (16)$$

Also we have

$$h(X,\xi) = 0 \quad \text{if} \quad X \in D, \quad (17)$$

$$\nabla_X\xi = 0, \quad (18)$$

$$h(\xi,\xi) = 0, \quad (19)$$

$$A_N\xi \in D^{\perp}. \quad (20)$$

2 D-totally geodesic and $D^{\perp}$-totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold

First we define the D-totally (resp. $D^{\perp}$-totally) geodesic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold.

Definition 2 *A contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is called D-totally geodesic (resp. $D^{\perp}$-totally geodesic) if $h(X,Y) = 0$, $\forall\ X,Y \in D$ (resp. $X,Y \in D^{\perp}$).*

From the above definition, the following propositions follow immediately.

Proposition 1 *Let M be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. Then M is a D-totally geodesic if and only if $A_N X \in D^{\perp}$ for each $X \in D$ and N a normal vector field to M.*

Proof. Let M be D-totally geodesic. Then from (8) we get

$$g(h(X,Y),N) = g(A_N X, Y) = 0.$$

So if
$$h(X,Y)=0, \quad \forall \;\; X, \;\; Y \in \; D$$
i.e.,
$$A_N X \in \; D^{\perp}.$$
Conversely, let $A_N X \in \; D^{\perp}$. Then for $X, \;\; Y \in \; D$ we can obtain
$$g(A_N X, Y) = 0 = g(h(X,Y), N)$$
i.e.,
$$h(X,Y)=0$$
$\forall \;\; X, \;\; Y \in \; D$, which implies that M is D-totally geodesic. Thus our proof is complete. □

Proposition 2 *Let M be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. Then M is $D^{\perp}$-totally geodesic if and only if $A_N X \in D$ for each $X \in D^{\perp}$ and N a normal vector field to M.*

Proof. The proof follows immediately from the above proposition. □

Concerning the integrability of the horizontal distribution D and vertical distribution $D^{\perp}$ on M, we can state the following theorem:

Theorem 1 *Let M be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. If M is ξ-horizontal, then the distribution D is integrable iff*
$$h(X, \phi Y) = h(\phi X, Y) \tag{21}$$
$\forall \;\; X, \;\; Y \in \; D$. *If M is ξ-vertical then the distribution $D^{\perp}$ is integrable iff*
$$A_{\phi X} Y - A_{\phi Y} X = \epsilon[\eta(Y)X - \eta(X)Y] \tag{22}$$
$\forall \;\; X, \;\; Y \in \; D^{\perp}$.

Proof. If M is ξ-horizontal, then using (14) we get
$$h(X, \phi PY) = \phi Q \nabla_X Y + Ch(X,Y)$$
$\forall \;\; X, \;\; Y \in \; D$. Therefore $[X,Y] \in D \quad$ iff $\; h(X, \phi Y) = h(Y, \phi X)$

Hence, if M is ξ-horizontal, $[X,Y] \in \; D$ iff $h(X, \phi Y) = h(\phi X, Y)$.

Again using (14) we get

$$\nabla^{\perp}_{X}\phi Y = Ch(X,Y) + \phi Q \nabla_X Y \tag{23}$$

for $X,\ Y\ \in\ D^{\perp}$.

After some calculations we see that

$$\begin{aligned}\tilde{\nabla}_X \phi Y = {} & g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(Y)\eta(X)\xi + \phi P \nabla_X Y \\ & + \phi Q \nabla_X Y + Bh(X,Y) + Ch(X,Y).\end{aligned} \tag{24}$$

Again from (7) and (24) we get

$$\begin{aligned}\nabla^{\perp}_{X}\phi Y = {} & A_{\phi Y}X + g(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(Y)\eta(X)\xi \\ & + \phi P \nabla_X Y + \phi Q \nabla_X Y + Bh(X,Y) + Ch(X,Y)\end{aligned} \tag{25}$$

for $X, Y \in D^{\perp}$. From (24) and (25) we can write

$$\phi P \nabla_X Y = -A_{\phi Y}X - g(X,Y)\xi - \epsilon\eta(Y)X - 2\epsilon\eta(Y)\eta(X)\xi - Bh(X,Y). \tag{26}$$

Interchanging X and Y in (26) we get

$$\phi P \nabla_Y X = -A_{\phi X}Y - g(X,Y)\xi - \epsilon\eta(X)Y - 2\epsilon\eta(Y)\eta(X)\xi - Bh(X,Y). \tag{27}$$

Substracting (27) from (26) we have

$$\phi P[X,Y] = -A_{\phi Y}X + A_{\phi X}Y - \epsilon\eta(Y)X + \epsilon\eta(X)Y. \tag{28}$$

Now since M is ξ-vertical, $[X,Y] \in\ D^{\perp}$ iff

$$A_{\phi X}Y - A_{\phi Y}X = \epsilon[\eta(Y)X - \eta(X)Y].$$

So the proof is complete. □

D-umbilic (resp. $D^{\perp}$-umbilic) contact CR-submanifold of indefinite Lorentzian para-Sasakian manifold is defined as follows:

Definition 3 *A contact CR-submanifold* M *of an indefinite Lorentzian para-Sasakian manifold is said to be* D*-umbilic (resp.* $D^{\perp}$*-umbilic) if* $h(X,Y) = g(X,Y)L$ *holds for all* $X, Y \in D$ *(resp.* $X, Y \in D^{\perp}$*),* L *being some normal vector field.*

In view of the above definition we state and prove the following proposition:

Proposition 3 *Suppose* M *is a* D*-umbilic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold* $\tilde{M}$*. If* M *is* ξ*-horizontal (resp.* ξ*-vertical) then* M *is* D*-totally geodesic (resp.* $D^{\perp}$*-totally geodesic).*

Proof. Consider M as D-umbilic ξ-horizontal contact CR-submanifold. Then we have from Definition 3

$$h(X, Y) = g(X, Y)L \qquad \forall \;\; X, Y \in \; D,$$

L being some normal vector field on M. By putting $X = Y = \xi$ and using (19) we have

$$h(\xi, \xi) = g(\xi, \xi)L$$

i.e.

$$L = 0,$$

and consequently we get $h(X, Y) = 0$, which proves that M is D-totally geodesic.

Similarly, it can be easily shown that if M is $D^{\perp}$-umbilic ξ-vertical contact CR-submanifold then it is $D^{\perp}$-totally geodesic. □

3 Mixed totally geodesic contact CR-submanifolds of indefinite Lorentzian para-Sasakian manifold

In this section we define mixed totally geodesic contact CR-submanifolds of an indefinite Lorentzian para-Sasakian manifold (followed [12]).

Definition 4 *A contact CR-submanifold* M *of an indefinite Lorentzian para-Sasakian manifold* $\tilde{M}$ *is said to be mixed totaly geodesic if* $h(X, Y) = 0 \; \forall \;\; X \in D$ *and* $Y \in D^{\perp}$.

Then we extract the following lemma and theorem

Lemma 1 *Let* M *be a contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold. Then* M *is mixed totally geodesic iff*

$$A_N X \in D, \qquad \forall \;\; X \in D, \quad \text{and} \;\; \forall \;\; \text{normal vector field } N, \tag{29}$$

$$A_N X \in D^{\perp}, \qquad \forall \;\; X \in D^{\perp} \quad \text{and} \;\; \forall \;\; \text{normal vector field } N. \tag{30}$$

Proof. If M is mixed totally geodesic, then from (8), we get

$$h(X,Y)=0,$$

i.e., iff $A_N X \in D$, $\forall\, X \in D$ and $\forall$ normal vector field N. Conversely, if M is mixed totally geodesic, then using (8) we easily observe that $A_N X \in D^{\perp}$, $\forall\, X \in D^{\perp}$ and $\forall$ normal vector field N.
Hence the lemma is proved. □

Using condition (29) we obtain the following theorem

Theorem 2 *If M is a mixed totally geodesic contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold, then*

$$A_{\phi N}X = -\phi A_N X, \tag{31}$$

$$\nabla^{\perp}_X \phi N = \phi \nabla^{\perp}_X N \tag{32}$$

$\forall\, X \in D$ *and* $\forall$ *normal vector field* N.

Proof. We get from (29), (6), (7) and after having some calculations we derive

$$\nabla_X \phi N = \phi \nabla^{\perp}_X N - \phi A_N X, \tag{33}$$

$$\nabla_X \phi N = -A_{\phi N}X + \nabla^{\perp}_X \phi N. \tag{34}$$

Comparing the above two equations we have the required theorem. Hence the proof follows. □

Again we have the following definition

Definition 5 *A contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$ is called foliate contact CR-submanifold $\tilde{M}$ if D is involute. If M is a foliate ξ-horizontal contact CR-submanifold, we know from* [3]

$$h(\phi X, \phi Y) = h(\phi^2 X, Y) = -h(X,Y). \tag{35}$$

Considering the above definition we give the following proposition.

Proposition 4 *If M is a foliate ξ-horizontal mixed totally geodesic contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold, then*

$$\phi A_N X = A_N \phi X \tag{36}$$

for all $X \in D$ *and normal vector field* N.

Proof. From (21) and (8) we compute the following:

$$g(h(X, \phi Y), N) = g(\phi A_N X, Y),$$

i.e.

$$g(h(\phi X, Y), N) = g(A_N \phi X, Y).$$

Therefore

$$\phi A_N X = A_N \phi X.$$

Hence the proof follows. □

4 Anti-invariant distribution $D^{\perp}$ on totally umbilical contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold

Here we consider a contact CR-submanifold M of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. Then we establish the following theorem.

Theorem 3 *Let M be a totally umbilical contact CR-submanifold of an indefinite Lorentzian para-Sasakian manifold $\tilde{M}$. Then the anti invariant distribution $D^{\perp}$ is one dimensional, i.e.* $\dim D^{\perp}=1$.

Proof. For an indefinite Lorentzian para-Sasakian structure we have

$$(\tilde{\nabla}_Z \phi)W = g(Z, W)\xi + \epsilon\eta(W)Z + 2\epsilon\eta(W)\eta(Z)\xi. \tag{37}$$

Also by the covariant derivative of tensor fields (for any $Z, W \in \Gamma(D^{\perp})$ we know

$$\tilde{\nabla}_Z \phi W = (\tilde{\nabla}_Z \phi)W + \phi\tilde{\nabla}_Z W. \tag{38}$$

Using (37), (38), (6), (7) and (4) we obtain

$$\begin{aligned} \nabla^{\perp}_Z \phi W - g(H, \phi W)Z = \phi[\nabla_Z W + g(Z, W)H] + g(Z, W)\xi \\ + \epsilon\eta(W)Z + 2\epsilon\eta(W)\eta(Z)\xi \end{aligned} \tag{39}$$

for any $Z, W \in \Gamma(D^{\perp})$.

Taking the inner product with $Z \in \Gamma(D^{\perp})$ in (39) we obtain

$$\begin{aligned} -g(H, \phi W)\|Z\|^2 = g(Z, W)g(\phi H, Z) + \epsilon\eta(W)\|Z\|^2 + g(Z, W)g(\xi, Z) \\ + 2\eta(W)\eta(Z)g(Z, \xi). \end{aligned} \tag{40}$$

Using (2) after a brief calculation we have

$$\begin{aligned} g(H,\phi W) = &-\frac{g(Z,W)g(\phi H,Z)}{||Z||^2} - \frac{g(Z,W)g(\xi,Z)}{||Z||^2} \\ &- \epsilon g(W,\xi) - 2\frac{g(Z,\xi)^2 g(W,\xi)}{||Z||^2}. \end{aligned} \tag{41}$$

Interchanging Z and W we have

$$\begin{aligned} g(H,\phi Z) = &-\frac{g(Z,W)g(\phi H,W)}{||W||^2} - \frac{g(Z,W)g(\xi,W)}{||W||^2} \\ &- \epsilon g(Z,\xi) - 2\frac{g(W,\xi)^2 g(Z,\xi)}{||W||^2}. \end{aligned} \tag{42}$$

Substituting (41) in (40) and simplifying we get

$$\begin{aligned} & g(H,\phi W)\left[1 - \frac{g(Z,W)^2}{||Z||^2||W||^2}\right] - \frac{g(Z,W)}{||Z||^2}[\frac{g(Z,W)g(\xi,W)}{||W||^2} - g(Z,\xi)] \\ & - \epsilon\left[\frac{g(Z,W)g(\xi,Z)}{||Z||^2} - g(W,\xi)\right] \\ & - 2g(z,\xi)g(W,\xi)\left[\frac{g(Z,W)g(W,\xi)}{||W||^2||Z||^2} - \frac{g(Z,W)}{||Z||^2}\right] = 0. \end{aligned} \tag{43}$$

The equation (43) has a solution if $Z \parallel W$, i.e. dim $D^{\perp}$=1.
Hence the theorem is proved. □

Example 1 *Let* $\mathbf{R}^3$ *be a* 3*-dimensional Euclidean space with rectangular coordinates* (x,y,z)*. In* $\mathbf{R}^3$ *we define*

$$\eta = -dz - y\,dx \qquad \xi = \frac{\partial}{\partial z}$$

$$\phi(\tfrac{\partial}{\partial x}) = \tfrac{\partial}{\partial y}, \qquad \phi(\tfrac{\partial}{\partial y}) = \tfrac{\partial}{\partial x} - y\tfrac{\partial}{\partial z}, \qquad \phi(\tfrac{\partial}{\partial z}) = 0.$$

The Lorentzian metric g *is defined by the matrix:*

$$\begin{pmatrix} -\epsilon y^2 & 0 & \epsilon y \\ 0 & 0 & 0 \\ \epsilon y & 0 & -\epsilon \end{pmatrix}.$$

Then it can be easily seen that (ϕ, ξ, η, g) *forms an indefinite Lorentzian para-Sasakian structure in* $\mathbf{R}^3$ *and the above results can be verified for this example.*

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A hyperbolic variant of the Nelder–Mead simplex method in low dimensions

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Abstract. The Nelder–Mead simplex method is a widespread applied numerical optimization method with a vast number of practical applications, but very few mathematically proven convergence properties. The original formulation of the algorithm is stated in $\mathbb{R}^n$ using terms of Euclidean geometry. In this paper we introduce the idea of a hyperbolic variant of this algorithm using the Poincaré disk model of the Bolyai–Lobachevsky geometry. We present a few basic properties of this method and we also give a Matlab implementation in 2 and 3 dimensions.

1 Introduction

The Nelder–Mead simplex method [10] was published in 1965 and since then it has been applied in an enormous amount of practical optimization problems basically in every area of applied science. It has became one of the most widely known direct search methods for function minimization, it is also incorporated in the `fminsearch` command of numerical computational software systems such as Matlab or Scilab. Also the method's mathematical study has gained much attention, unfortunately there is very little known about its convergence properties. A breakthrough in lack of proven properties appeared in [5] in 1998, unfortunately the most useful theorems are only stated in 1 and 2 dimensions. Immediately on the following pages of the same volume [9] a counterexample is given in 2 dimensions where the method would fail to converge

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to the unique minimizer given a tricky (but smooth and convex) function and a well-established initialization of the method. There are also quite a number of attempts to modify or restrict the method to enable convergence proofs, see e.g. [6, 11], or further related experiments in [3, 4]. A nice overview on the history of this method is given in [12]. Some of our recent works also summarizes some application experiences using this algorithm [2, 7].

Today also the non-Euclidean geometries are well known and accepted, we should mention the names of János Bolyai and Nikolai Lobachevsky who have independently clarified the notions of hyperbolic geometry in the early nineteenth century and after whom it is sometimes referred to as the Bolyai–Lobachevsky geometry. Later many models of hyperbolic geometry have been developed, one of these being the Poincaré disk model. Hyperbolic geometry is many times introduced or studied throughout this model, where the points of the plane are those inside the unit circle and the lines are the circular arcs intersecting the unit circle perpendicularly.[1] Even nowadays works appear with adapting some Euclidean notions and theorems to this model of hyperbolic geometry, see e.g. [1].

Our current work is also a member of this family. Specifically we adapt the Nelder–Mead simplex method (originally formulated in $\mathbb{R}^n$ using terms of Euclidean geometry) to the hyperbolic space, to the Poincaré disk model (in 2 dimensions) and its analogue using a unit sphere (in 3 dimensions). The motivation for this research came from our practise. We had the problem to choose some adequate points ("poles") inside the unit circle, and we have found the Nelder–Mead method to be the first to find a suitable set of poles without any a priori knowledge of their location. But this way we had to map the natural domain ($\mathbb{R}^n$) of the Nelder–Mead algorithm inside the unit circle, for details see e.g. [2, 7]. Along the way the adaptation of this method to the natural domain of our problem (which corresponds to the Poincaré disk model) seemed to be another promising path. In this paper we summarize the current results of our efforts, we introduce the hyperbolic Nelder–Mead method. We present a few basic properties of this method and we also give a Matlab implementation. Our hope is that with this approach some new directions will be opened both to the application and to the mathematical study of this algorithm.

The software tools (Matlab programs) can be downloaded from `http://numanal.inf.elte.hu/~locsi/hypnm/` in order to enable the Reader to reproduce the results presented in this paper.

[1] And of course diameters of the circle are also considered as lines.

2 The original method by Nelder and Mead

In this section we describe 'a simplex method for function minimization' following the original publication [10]. The statement of the algorithm shall now be given so that the specialities and calculations of $\mathbb{R}^n$ are skipped, only geometric terms shall be used, and therefore we may immediately imagine both the original idea and the hyperbolic realization.

The method relies on the comparison of the function values at the vertices of a non-degenerate simplex in our n-dimensional space $\mathbb{X}$. Let us call the vertices of the simplex $x_1, x_2, \ldots, x_{n+1} \in \mathbb{X}$, the real valued function to be minimized f, and $y_i := f(x_i)$ $(i = 1, 2, \ldots, n+1)$. We may start with an arbitrary simplex, which is usually chosen as a point $x_s \in \mathbb{X}$ and some 'nearby' points.

One *step* of the algorithm is basically a substitution of one point of the simplex, with a better one. Let us define the indices h and l such that y_h and y_l are respectively the *highest* (worst) and *lowest* (best) function values, and $\overline{x}$ the *centroid* of the points x_i with $i \neq h$. To carry out an update of the simplex, four operations are used.

1. *Reflection.* Reflect x_h across the point $\overline{x}$ to get x_r, $y_r := f(x_r)$. If $y_l \leq y_r < y_h$ then replace x_h with x_r and continue with the next step.

2. *Expansion.* If $y_r < y_l$, then reflect $\overline{x}$ across the point x_r to get x_e, $y_e := f(x_e)$. Now replace x_h with the x_e if $y_e < y_r$ or with x_r if $y_e \geq y_r$ and continue with the next step. (Thus we either stick with a new simplex gained with reflection, or create an expanded simplex.)

3. *Contraction.* If $y_r \geq y_i$ for all $i \neq h$ then define x_b as x_r if $y_r < y_h$ or as x_h if $y_r \geq y_h$ (so the 'better' of x_r and x_h), and find the midpoint x_c of $\overline{x}$ and x_b, $y_c := f(x_c)$. Now replace x_h with x_c and continue with the next step (with this contracted simplex); unless $y_c > \min\{y_h, y_r\}$, in this case perform the following operation.

4. *Shrink.* Leave only the point x_l and for all $i \neq l$ replace x_i with the midpoint of x_i and x_l; continue with the next step. (This operation is reported in [5] to be needed very rarely.)

With these steps our initial simplex 'adapts itself to the local landscape' (defined by the function f) and finally 'contracts on to the final minimum' as [10] summarizes the behaviour of the algorithm.[2]

[2] In the original description of the algorithm these four operations each depend on a given

We may stop the iteration e.g. if the function values have smaller standard deviation than a small given $\varepsilon > 0$ value or if we have made more steps than a prescribed limit etc.

Figure 1 presents the operations possible in one step of the Nelder–Mead algorithm in 2 dimensions, on the Euclidean plane. In this case the simplex is a triangle. On Figure 2 an example is shown for the progress of the algorithm optimizing the $\mathbb{R}^2 \to \mathbb{R}$ quadratic function $f(x, y) = x^2 + 6y^2 + 2xy$ with the initial simplex of coordinates $(1.2, 0.7), (1.1, 1.4)$ and $(1.7, 1.1)$. The first 10 steps are presented.

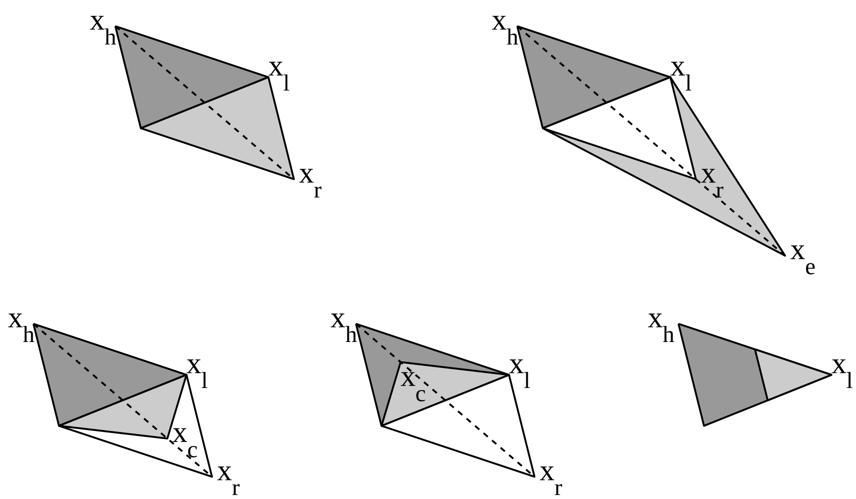


Figure 1: Operations on an Euclidean simplex in 2 dimensions. Respectively: reflection, expansion, (outside and inside) contraction and shrink. The old simplex is marked with dark gray, the new simplex with light grey, the reflected simplex is shown in white for the expansion and contraction operations.

Note that midpoints and reflected points can be calculated through simple linear combinations. This makes the (Euclidean) Nelder–Mead algorithm easy to implement, and quite effective, even in higher dimensions.

parameter. It is also shown that the 'natural choice' of these parameters, which are equivalent of calculating reflections and midpoints (as used above), are also the most efficient choice in practise.

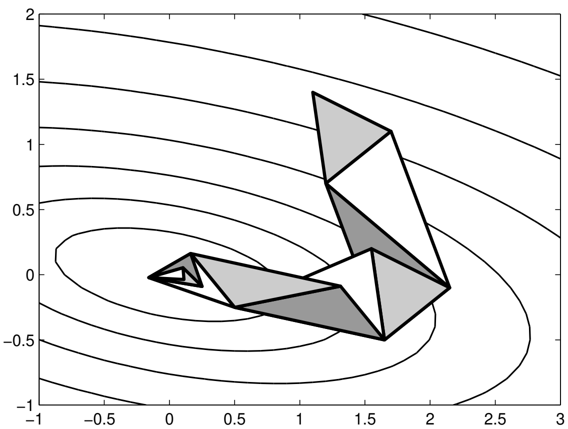


Figure 2: The Nelder–Mead algorithm optimizing a quadratic function on the Euclidean plane.

3 Constructions in hyperbolic spaces

We have given the statement of the Nelder–Mead simplex method so that in each step of the iteration some simple geometric calculations shall be done: finding centroid, midpoint, reflection across a point. These are valid constructions not only in Euclidean geometry, but also in hyperbolic geometry. In this section we give an overview of some possible approaches to numerically calculate the locations of the required points. Using these we will be able to put together the hyperbolic version of the Nelder–Mead algorithm in 2 and 3 dimensions.

3.1 In 2 dimensions

We will use the Poincaré disk model of hyperbolic geometry. It is useful to identify this model with the complex unit disk $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$. So the points of the plane are the complex numbers $z \in \mathbb{D}$. Let us also define the unit torus $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$, and $\mathbb{D}^* := \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$. The isometric transforms in this model (except for reflecting through a line) can be written by means of Blaschke functions, defined as

$$B_{a,d}(z) := d \cdot \frac{z - a}{1 - \overline{a}z} \qquad (a \in \mathbb{D}, d \in \mathbb{T}, z \in \mathbb{C}).$$

One approach makes use of the fact that for any $w_1, w_2 \in \mathbb{D}$, $w_1 \neq w_2$

there exists a unique set of values $(a, d, p) \in \mathbb{D} \times \mathbb{T} \times (0, 1)$ such that $B_{a,d}(0) = w_1, B_{a,d}(p) = w_2$ and $B_{a,d}$ maps the interval $[0, p]$ onto the hyperbolic line segment connecting w_1 and w_2. This way the calculation of midpoints and reflected points can be reduced to finding the appropriate points on $(0, 1)$. More on this method can be found in [2]. The advantage of this approach is the elegant and straightforward calculation with complex functions, the downside is that it is too much bound to the complex domain, to two dimensions, the generalization to higher dimensions is troublesome, if not impossible.

In contrast to the analytic techniques of the first approach, the second approach arises from geometric considerations. (Of course in two dimensions sometimes the use of complex expressions and Blaschke functions again makes the calculations easier.) We give a more detailed overview here, because our implementation relies on this second approach. Basically we have to imitate the regular constructions numerically.

- A *hyperbolic line* is basically a circular arc intersecting the unit circle perpendicularly.[3] It turns out that for the centre $c \in \mathbb{C}$ and radius $r \in \mathbb{R}$, $r > 0$ of such circles $c \in \mathbb{D}^*$ and $c\overline{c} = |c|^2 = 1 + r^2$ holds.

- Given two points $a, b \in \mathbb{D}$, $a \neq b$ we can *fit a hyperbolic line* on these two points by finding the centre of the circle which intersects the unit circle perpendicularly and passes through a and b. We know that the inverse image of a with respect to the unit circle can be expressed as $1/\overline{a}$ and also lies on the circle in question. (The same holds for b.) Now this circle can be found as the one fitted on the three points a, b and $1/\overline{b}$. This can be done e.g. by solving a linear system of equations.

- Finding *the intersection of two hyperbolic lines* translates to finding the intersection points of two circles (if they exist) and choosing the one inside $\mathbb{D}$. Note that also in hyperbolic geometry it cannot occur that two lines have exactly two intersections.

- The *perpendicular bisector* of a line segment between a and b can be found as a circle with centre both on the Euclidean line on a, b and the radical axis of the unit circle and the hyperbolic line (as Euclidean circle) on a and b.

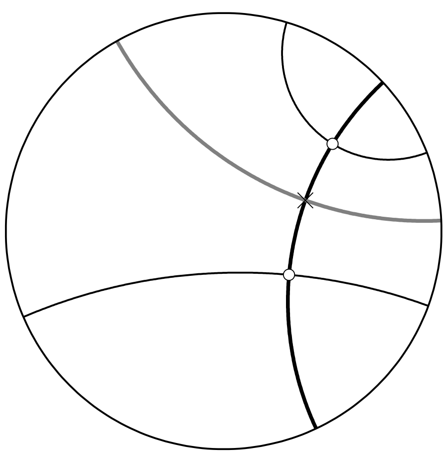
[3]Since the diameters are also considered as lines, they should be handled as special cases in an implementation. The calculation is also easier in that case. These special cases will also occur in 3 dimensions, but we will not treat them here in more detail.

- The above three constructions allow us to find the *midpoint* of a line segment, which is also a *centroid* of two points: find the intersection of the hyperbolic line fitted on the two points and the perpendicular bisector of the line segment between the two points.

- The *reflection across a point* $a \in \mathbb{D}$ can be formalized e.g. using Blaschke functions. The formula $B_{-a,0}(-B_{a,0}(z))$ gives the reflected image of $z \in \mathbb{D}$ through a. (The idea is to reduce to reflect across the origin.)

Now we have all the constructions which are needed to adapt the Nelder–Mead method to the hyperbolic plane. But further geometric constructions can also be formulated. We have the hyperbolic analogue of: translation, reflection through a line, rotation around a point, perpendicular line at a point.

Figure 3 presents some basic elements in the Poincaré disk model of hyperbolic geometry, and the operations possible in one step of the Nelder–Mead algorithm (c.f. Figure 1 in the Euclidean case, and to the outline of the algorithm given in Section 2). In this case the simplex is a hyperbolic triangle.



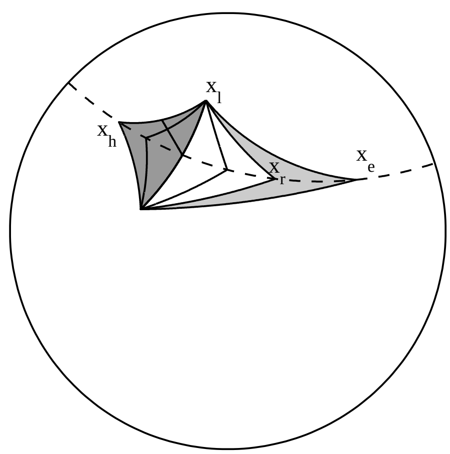


Figure 3: Left: Some basic elements of hyperbolic geometry. The line fitted on two points, perpendicular lines from the points, and a perpendicular bisector. Right: Operations on a hyperbolic simplex in 2 dimensions: reflection, expansion, (outside and inside) contraction and shrink. The old simplex is marked with dark gray, the reflected simplex with white, the further simplices are either shown in light grey or just their outlines.

3.2 In 3 dimensions

To define the analogue of the Poincaré disk model in 3 dimensions, i.e. a hyperbolic space, we will use $\mathbb{S} := \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \right\}$, the unit sphere. The points will be the ones inside $\mathbb{S}$, the lines will be the circular arcs intersecting the surface of $\mathbb{S}$ perpendicularly, the planes will be the spherical caps intersecting $\mathbb{S}$ perpendicularly.

It turns out that the usual basic constructions in Euclidean space (such as fitting a line on two points, fitting a plane on three points, finding the intersection line of two planes, finding the perpendicular bisector plane of a line segment etc.) can be also done and calculated in this hyperbolic space. Of course now we can not use the help of complex analysis, we have to deal with terms of analytic geometry in $\mathbb{R}^3$ by translating the required notions of hyperbolic lines and planes to circular arcs and spherical caps.

We refer to the program codes referenced in this paper (see end of Section 1) for construction and implementation details. Figure 4 presents some basic elements in the geometry of this three-dimensional hyperbolic space and the possible moves of a simplex in one step of the Nelder–Mead algorithm. In this case the simplex is a hyperbolic tetrahedron.

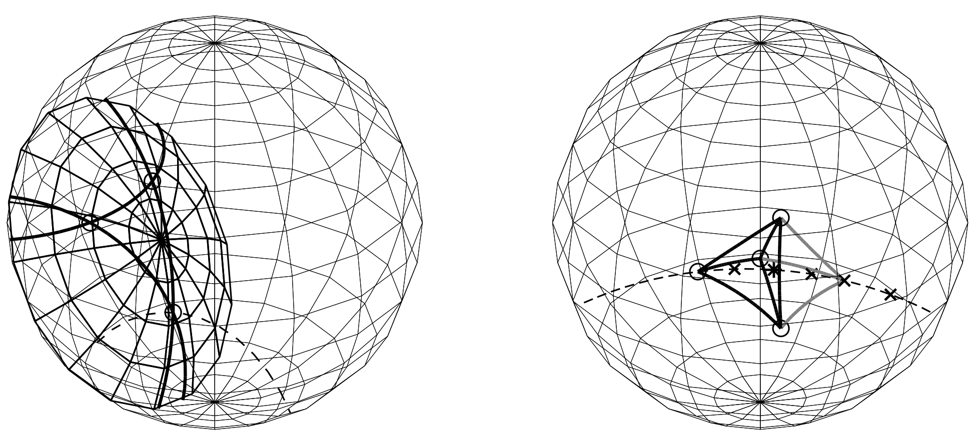


Figure 4: Left: Some basic elements of geometry in hyperbolic space. Three points on a plane, the lines of the arising triangle's edges and a perpendicular line at one of the points. Right: Operations on a hyperbolic simplex in 3 dimensions (without shrink), circles denote the vertices of the original simplex, a star the centroid of one side, exes the possible new vertices of a new simplex.

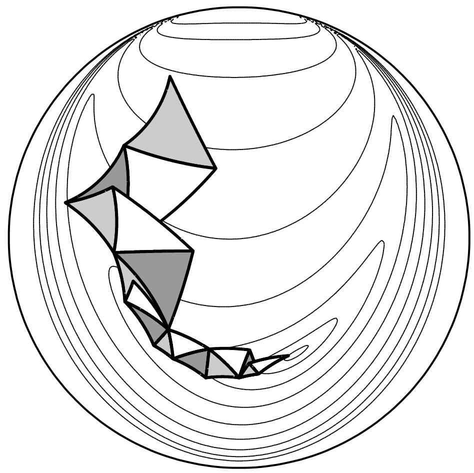


Figure 5: The Nelder–Mead algorithm adapted to the hyperbolic plane.

4 The hyperbolic simplex method

Having defined the Nelder–Mead simplex method in geometric terms (Section 2), and the needed constructions being present on the Poincaré disk model (and its three-dimensional analogue) of hyperbolic geometry (Section 3), now we have the hyperbolic realization of the method in our hands.

Figure 5 gives an example of the Nelder–Mead method optimizing a function on the hyperbolic plane. The function being minimized is similar to the famous Rosenbrock-function (or banana function, see [10]), with its natural domain $\mathbb{R}^2$ being mapped onto $\mathbb{D}$ using a map detailed in e.g. [7]. This function earned his fame, because numerical methods prior to the one proposed by Nelder and Mead were unable to determine its minimum. Now we see that the hyperbolic version is also capable of tending towards the optimum.

Furthermore an optimization process can be observed on Figure 6 in case of a quadratic function on the hyperbolic space as carried out by the Nelder–Mead algorithm.

The Reader is encouraged to download the collection of Matlab programs from the referenced homepage (see Section 1) and experiment with the algorithm, specific functions to optimize and constructions in hyperbolic geometry. Especially the three-dimensional graphics are more comprehensible when the user can interact with the figures, not just observe a printed planar projection.

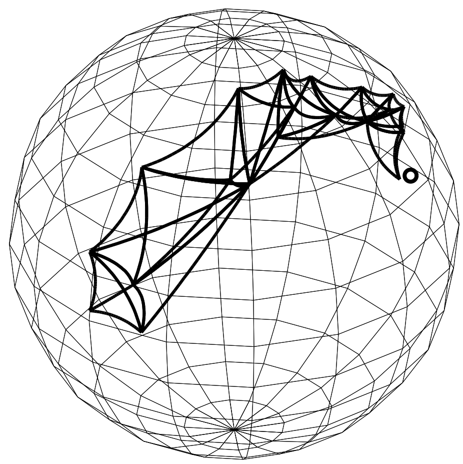


Figure 6: The Nelder–Mead algorithm adapted to the hyperbolic space.

5 Some basic properties

The mathematical study of the original Nelder–Mead simplex method (without any restrictions or modifications) has quite few proven properties or convergence theorems. Basically all known results are summarized in [5]. Some of its general results easily translate also to the hyperbolic versions introduced above. In this section we revisit these straightforward properties of the algorithm.

Proposition 1 (Nondegeneracy[4] of hyperbolic simplices) *If the initial simplex is nondegenerate, so are all subsequent simplices produced by the hyperbolic version of the Nelder–Mead algorithm.* (C.f. [5, Lemma 3.1.(1)].)

Proof. By construction, each of the trial points x_r, x_e and x_c (either inside or outside) lies strictly outside the face defined by the n best vertices, along the line joining x_h and $\bar{x}$. If a nonshrink operation occurs, the worst vertex is replaced by one of these trial points, thus the simplex remains nondegenerate. If a shrink operation occurs, then each vertex (except the best) is replaced by the midpoint of the line segment defined by the current and the best vertex.

[4] We understand by nondegeneracy that the vertices of a simplex are not collinear on the hyperbolic plane, not coplanar in the hyperbolic space, and in general: no lower-dimensional hyperspace can be found which contains all vertices of the simplex.

Also in this case it is clear from the geometry that the simplex' nondegeneracy is again preserved. □

The function values at the vertices of the simplex were denoted by y_i, now let us also mark the number of iterations, and require that at the beginning of each iteration step the vertices are ordered, i.e. in step k

$$y_1^{(k)} \leq y_2^{(k)} \leq \ldots \leq y_{n+1}^{(k)}$$

holds in case of a simplex in n dimensions.

Proposition 2 (Convergence of function values at vertices) *Let f be a function defined on the n-dimensional hyperbolic space $\mathbb{X}$, that is bounded from below. When the Nelder–Mead algorithm is applied to minimize f, starting with a nondegenerate simplex, then* (c.f. [5, Lemma 3.3])

1. *the sequence $\left(y_1^{(k)}\right)$ always converges;*
2. *at every nonshrink iteration k, $y_i^{(k+1)} \leq y_i^{(k)}$ $(1 \leq i \leq n+1)$ with strict inequality for at least one value of i;*
3. *if there are only a finite number of shrink steps, then*
   (a) *each sequence $\left(y_i^{(k)}\right)$ $(1 \leq i \leq n+1)$ converges as $k \to \infty$,*
   (b) $\lim_{k\to\infty} y_i^{(k)} =: y_i^* \leq y_i^{(k)}$ *for $1 \leq i \leq n+1$ and all k,*
   (c) $y_1^* \leq y_2^* \leq \ldots \leq y_{n+1}^*$.

Note that this proposition does not state that the algorithm will converge to a global (or even local) minimum point. This is unfortunately not true in general (see counterexample in [9]).

Proof.

1. Since the algorithm never replaces the best vertex with a point of higher function value, the sequence $\left(y_1^{(k)}\right)$ is monotonically non-increasing and bounded from below (like f), thus it is convergent.
2. A shrink step could result in higher function values at the simplex' vertices, but other operations always replace the worst value with a better one, thus—taking to account also the ordering at the beginning of each step—some values will be strictly lower and none of them will increase.

3. Shrink steps are reported to be taken extremely rarely, so assuming their finiteness is a very weak restriction. Otherwise these statements are immediate consequences of the previous arguments and the properties of convergent sequences.

□

Now we will show that shrink steps will not occur at all if the method is applied to a strictly convex function on the hyperbolic space $\mathbb{X}$, endowed with the metric $\rho\colon \mathbb{X} \times \mathbb{X} \to \mathbb{R}$.[5]

Definition 1 (Strict convexity) *The function* f *defined on the points of hyperbolic space* $\mathbb{X}$ *is called* strictly convex *if for every* $a, b \in \mathbb{X}$, $a \neq b$*, and for every point* p *on the line segment connecting* a *and* b *(with endpoints excluded) the following formula holds:*

$$f(p) < \lambda \cdot f(a) + (1 - \lambda) \cdot f(b), \quad \textit{with} \quad \lambda := \frac{\rho(p, b)}{\rho(a, b)}.$$

Basically this is the usual definition of strict convexity, but now, because of dealing with hyperbolic spaces, the usual terms with linear combinations also with the points of the metric space had to be omitted.

It is easy to see that in case of a strictly convex function f, for every point p on the open line segment connecting a and b

$$f(p) < \max\{ f(a), f(b) \}$$

holds[6], specifically also a centroid of 2, 3 (or more) points has lower function value than the maximum of the function values at the given points.

Proposition 3 (No shrink for strictly convex functions) *Assume that* f *is a strictly convex function defined on the points of the hyperbolic space* $\mathbb{X}$ *and that the Nelder–Mead algorithm is applied to minimize* f*, starting with a non-degenerate simplex. Then no shrink steps will be taken.* (C.f. [5, Lemma 3.5].)

Proof. A shrink should be performed when we fail to accept the relevant contraction point x_c. We will show now that this can not happen.

[5]For instance the usual metric on the Poincaré disk model can be expressed using Blaschke functions as $\rho(a, b) = |B_a(b)|$.

[6]Furthermore it turns out that this property would have been sufficient to prove Proposition 3.

It follows from the statement of the algorithm that if we are considering a contraction point, then $y_n \le y_r$ and of course $y_n \le y_{n+1} = y_h$ holds. We can assume that $y_r < y_{n+1}$ (i.e. $y_n \le y_r < y_{n+1}$ holds), the other case is settled similarly. Now $\bar{x}$ is the centroid of $x_1, \dots, x_n$, so by the strict convexity of f, $f(\bar{x}) < y_n$ holds. The point x_c is the midpoint of $\bar{x}$ and x_r, so $y_c < \max\{f(\bar{x}), y_r\} = y_r$. So now $y_c < y_r < y_h$ holds, hence x_c will be accepted, a contraction shall be made and a shrink step will not be taken. □

6 Summary

In this paper we have introduced a hyperbolic variant of the Nelder–Mead simplex method. The algorithm was adapted to the Poincaré disk model of the Bolyai–Lobachevsky geometry (in two dimensions), as well as its three-dimensional analogue.

Matlab implementations (and resulting graphics) were presented about the necessary geometric constructions in the hyperbolic spaces at hand, which are—together with the adapted variant of the Nelder–Mead method—available to download at http://numanal.inf.elte.hu/~locsi/hypnm/.

Finally some straightforward mathematical properties of the original simplex method were translated to the hyperbolic case.

7 Directions of further research

Apart from the Poincaré disk model, it might be interesting to adapt the Nelder–Mead simplex method to other models (such as the Klein model, Poincaré half-plane model etc.) or other geometries.

Naturally also the higher-dimensional cases should be investigated and implemented.

Of course the detailed analysis of both

- the practical convergence properties of this method (compared to the original Nelder–Mead method), and
- the mathematical convergence properties of the hyperbolic Nelder–Mead method

awaits to be carried out, with the first task requiring more of an engineering approach, and the second one likely to be quite unpromising taking to account the similar efforts in the case of the original algorithm. Nevertheless we may have some hope at least in low dimensions.

Our current investigation lacks the most basic special case: optimization in 1 dimension, on a hyperbolic line—a line or line segment endowed with a hyperbolic metric. Note that [5] contains results also in 1 dimensions.

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Meromorphic functions sharing fixed points and poles with finite weights

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Abstract. In the paper, with the aid of weighted sharing method we study the problems of meromorphic functions that share fixed points (or a nonzero finite value) and poles with finite weights. The results of the paper improve some recent results due to Y. H. Cao and X. B. Zhang [Journal of Inequalities and Applications, 2012:100].

1 Introduction, definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [8], [15] and [16]. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function $\alpha(z)(\not\equiv \infty)$ is called a small function with respect to f, provided that $T(r, \alpha) = S(r, f)$.

We say that two meromorphic functions f and g share a small function $a(z)$ CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share $a(z)$ IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, and we say that f and g share ∞

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IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM. A finite value z_0 is a fixed point of $f(z)$ if $f(z_0) = z_0$ and we define

$$E_f = \{z \in \mathbb{C} : f(z) = z, \text{counting multiplicities}\}.$$

In 1995, W. Bergweiler and A. Eremenko, H. H. Chen and M. L. Fang, L. Zalcman respectively proved the following result.

Theorem A (see ([3], Theorem 2), ([5], Theorem 1) and [17]) *Let f be a transcendental meromorphic function and $n(\geq 1)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.*

In 1997, C. C. Yang and X. H. Hua proved the following result, which corresponded to Theorem A.

Theorem B (see [14], Theorem 1) *Let f and g be two nonconstant meromorphic functions, $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2000, M. L. Fang proved the following result.

Theorem C (see [6], Theorem 2) *Let f be a transcendental meromorphic function, and let n be a positive integer. Then $f^n f' - z = 0$ has infinitely many solutions.*

In 2002, M. L. Fang and H. L. Qiu proved the following result, which corresponded to Theorem C.

Theorem D (see [7], Theorem 1) *Let f and g be two nonconstant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n g' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

In 2009, J. F. Xu, H. X. Yi and Z. L. Zhang proved the following result.

Theorem E (see [12]) *Let f be a transcendental meromorphic function, $n(\geq 2)$, k be two positive integers. Then $f^n f^{(k)}$ takes every finite nonzero value infinitely many times or has infinitely many fixed points.*

Regarding Theorem E, it is natural to ask the following question:

Question 1 *Is there a corresponding uniqueness theorem to Theorem E?*

Recently, Y. H. Cao and X. B. Zhang proved the following results which deal with Question 1.

Theorem F (see [4], Theorem 1.1) *Let* f *and* g *be two transcendental meromorphic functions, whose zeros are of multiplicities at least* k*, where* k *is a positive integer. Let* $n > \max\{2k-1, k+4/k+4\}$ *be a positive integer. If* $f^n f^{(k)}$ *and* $g^n g^{(k)}$ *share* z *CM,* f *and* g *share* ∞ *IM, then one of the following two conclusions hold:*

(i) $f^n f^{(k)} = g^n g^{(k)}$*;*

(ii) $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$*, where* c_1*,* c_2 *and* c *are constants satisfying* $4(c_1c_2)^{n+1}c^2 = -1$*.*

Theorem G (see [4], Theorem 1.2) *Let* f *and* g *be two nonconstant meromorphic functions, whose zeros are of multiplicities at least* k*, where* k *is a positive integer. Let* $n > \max\{2k-1, k+4/k+4\}$ *be a positive integer. If* $f^n f^{(k)}$ *and* $g^n g^{(k)}$ *share* 1 *CM,* f *and* g *share* ∞ *IM, then one of the following two conclusions hold:*

(i) $f^n f^{(k)} = g^n g^{(k)}$*;*

(ii) $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$*, where* c_3*,* c_4 *and* d *are constants satisfying* $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$*.*

Regarding Theorem F and Theorem G, one may ask the following questions which are the motive of the author.

Question 2 *Is it really possible in any way to relax the nature of sharing the fixed point (*1*-point) in Theorem F (Theorem G) without increasing the lower bound of* n*?*

Question 3 *What will be the IM-analogous of Theorems F and G?*

In the paper, we will prove two theorems first one of which improves Theorem F and second one improves Theorem G and dealt with Question 2 and Question 3. To state the main results of the paper we need the following notion of weighted sharing of values introduced by I. Lahiri [9, 10] which measure how close a shared value is to being shared CM or to being shared IM.

Definition 1 *Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all* a*-points of* f *where an* a*-point of multiplicity* *m is counted m times if* $m \leq k$ *and k+1 times if* $m > k$. *If* $E_k(a; f) = E_k(a; g)$, *we say that* f, g *share the value* a *with weight k.*

The definition implies that if f, g *share a value* a *with weight* k, *then* z_0 *is an* a*-point of* f *with multiplicity* $m(\leq k)$ *if and only if it is an* a*-point of* g *with multiplicity* $m(\leq k)$ *and* z_0 *is an a-point of* f *with multiplicity* $m(> k)$ *if and only if it is an a-point of* g *with multiplicity* $n(> k)$, *where m is not necessarily equal to n.*

We write f, g *share* (a, k) *to mean that* f, g *share the value* a *with weight k. Clearly if* f, g *share* (a, k) *then* f, g *share* (a, p) *for any integer* p, $0 \leq p < k$. *Also we note that* f, g *share a value* a *IM or CM if and only if* f, g *share* $(a, 0)$ *or* (a, ∞) *respectively.*

We now state the main results of the paper.

Theorem 1 *Let* f *and* g *be two transcendental meromorphic functions, whose zeros are of multiplicities at least* k, *where* k *is a positive integer. If* $f^n f^{(k)}$ *and* $g^n g^{(k)}$ *share* (z, l), *where* l, n *are positive integers;* f *and* g *share* ∞ *IM, then conclusions of Theorem F hold provided one of the following holds:*

(i) $l \geq 2$ *and* $n > \max\{2k - 1, k + 4/k + 4\}$;

(ii) $l = 1$ *and* $n > \max\{2k - 1, 3k/2 + 5/k + 5\}$;

(iii) $l = 0$ *and* $n > \max\{2k - 1, 4k + 10/k + 10\}$.

Theorem 2 *Let* f *and* g *be two nonconstant meromorphic functions, whose zeros are of multiplicities at least* k, *where* k *is a positive integer. If* $f^n f^{(k)}$ *and* $g^n g^{(k)}$ *share* $(1, l)$, *where* l, n *are positive integers;* f *and* g *share* ∞ *IM, then conclusions of Theorem G hold provided one of the following holds:*

(i) $l \geq 2$ *and* $n > \max\{2k - 1, k + 4/k + 4\}$;

(ii) $l = 1$ *and* $n > \max\{2k - 1, 3k/2 + 5/k + 5\}$;

(iii) $l = 0$ *and* $n > \max\{2k - 1, 4k + 10/k + 10\}$.

We now explain some definitions and notations which are used in the paper.

Definition 2 [8] *For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $N(r, a; f \mid = 1)$ *the counting functions of simple* a*-points of* f. *For a positive integer* p *we denote by* $N(r, a; f \mid \geq p)$ *the counting function of those* a*-points of* f *(counted with proper multiplicities) whose multiplicities are not less than* p. *By* $\overline{N}(r, a; f \mid \geq p)$ *we denote the corresponding reduced counting function.*

Analogously we can define $N(r, a; f \mid \leq p)$ *and* $\overline{N}(r, a; f \mid \leq p)$.

Definition 3 [10] *Let* k *be a positive integer or infinity. We denote by* $N_k(r,a;f)$ *the counting function of* a*-points of* f*, where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and* k *times if* $m > k$*. Then*

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + \cdots + \overline{N}(r,a;f \mid\geq k).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 4 [1] *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f *with multiplicity* p *and also a* 1*-point of* g *with multiplicity* q*. We denote by* $\overline{N}_L(r,1;f)$ *the counting function of those* 1*-points of* f *and* g*, where* $p > q$*, by* $N_E^{(k}(r,1;f)$ $(k \geq 2$ *is an integer) the counting function of those* 1*-points of* f *and* g*, where* $p = q \geq k$*, where each point in these counting functions is counted only once. In the same manner we can define* $\overline{N}_L(r,1;g)$ *and* $N_E^{(k}(r,1;g)$.

Definition 5 [9, 10] *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share the value* a *IM. We denote by* $\overline{N}_*(r,a;f,g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g*. Clearly* $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ *and* $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two nonconstant meromorphic functions defined in $\mathbb{C}$. We shall denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 1 [13] *Let* f *be a nonconstant meromorphic function and let* $a_n(z)(\not\equiv 0)$, $a_{n-1}(z), \ldots, a_0(z)$ *be meromorphic functions such that* $T(r,a_i(z)) = S(r,f)$ *for* $i = 0,1,2,\ldots,n$*. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2 [16] *Let* f *be a nonconstant meromorphic function, and let* k *be positive integer. Suppose that* $f^{(k)} \not\equiv 0$*. Then*

$$N\left(r,0;f^{(k)}\right) \leq N(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f). \quad (1)$$

Lemma 3 [18] *Let* f *be a nonconstant meromorphic function, and* p, k *be positive integers. Then*

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2}$$

Lemma 4 [2] *Let* F, G *be two nonconstant meromorphic functions sharing* $(1, 2)$, $(\infty, 0)$ *and* $H \not\equiv 0$. *Then*

(i) $T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_*(r, \infty; F, G) - m(r, 1; G) - N_E^{(3}(r, 1; F) - \overline{N}_L(r, 1; G) + S(r, F) + S(r, G)$;

(ii) $T(r, G) \leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \overline{N}_*(r, \infty; F, G) - m(r, 1; F) - N_E^{(3}(r, 1; G) - \overline{N}_L(r, 1; F) + S(r, F) + S(r, G)$.

Lemma 5 [11] *Let* F, G *be two nonconstant meromorphic functions sharing* $(1, 1)$, $(\infty, 0)$ *and* $H \not\equiv 0$. *Then*

(i) $T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + \frac{3}{2}\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + \frac{1}{2}\overline{N}(r, 0; F) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G)$;

(ii) $T(r, G) \leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \frac{3}{2}\overline{N}(r, \infty; G) + \frac{1}{2}\overline{N}(r, 0; G) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G)$.

Lemma 6 [11] *Let* F, G *be two nonconstant meromorphic functions sharing* $(1, 0)$, $(\infty, 0)$ *and* $H \not\equiv 0$. *Then*

(i) $T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + 3\overline{N}(r, \infty; F) + 2\overline{N}(r, \infty; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G)$;

(ii) $T(r, G) \leq N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) + 3\overline{N}(r, \infty; G) + \overline{N}(r, 0; F) + 2\overline{N}(r, 0; G) + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G)$.

Lemma 7 [4] *Let* f *and* g *be nonconstant meromorphic functions, whose zeros are of multiplicities at least* k, *where* k *is a positive integer. Let* $n > 2k - 1$ *be a positive integer. If* f, g *share* ∞ *IM and if* $f^n f^{(k)} g^n g^{(k)} = z^2$, *then* $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, *where* c_1, c_2 *and* c *are three constants satisfying* $4(c_1 c_2)^{n+1} c^2 = -1$.

Lemma 8 [4] *Let* f *and* g *be nonconstant meromorphic functions, whose zeros are of multiplicities at least* k, *where* k *is a positive integer. Let* $n > 2k - 1$ *be a positive integer. If* f, g *share* ∞ *IM and if* $f^n f^{(k)} g^n g^{(k)} = 1$, *then* $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where* c_3, c_4 *and* d *are three constants satisfying* $(-1)^k (c_3 c_4)^{n+1} d^{2k} = 1$.

3 Proof of the theorems

Proof of Theorem 1. We consider $F(z) = f^n f^{(k)}$, $G(z) = g^n g^{(k)}$, $F_1(z) = F(z)/z$ and $G_1(z) = G(z)/z$. Then F_1, G_1 are transcendental meromorphic functions that share $(1, l)$ and f, g share $(\infty, 0)$. Since f and g are transcendental, z is a small function with respect to both F and G. We now discuss the following two cases separately.

Case 1 *We assume that* $H \not\equiv 0$. *Now we consider the following three subcases.*

Subcase 1 *Suppose that* $l \geq 2$. *Then using Lemma 4 we obtain*

$$\begin{aligned} T(r, F) &\leq T(r, F_1) + S(r, F) \\ &\leq N_2(r, 0; F_1) + N_2(r, 0; G_1) + \overline{N}(r, \infty; F_1) + \overline{N}(r, \infty; G_1) \\ &\quad + \overline{N}_*(r, \infty; F_1, G_1) - m(r, 1; G_1) - N_E^{(3}(r, 1; F_1) \\ &\quad - \overline{N}_L(r, 1; G_1) + S(r, F_1) + S(r, G_1) \\ &\leq N_2(r, 0; F) + N_2(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) \\ &\quad + \overline{N}_*(r, \infty; F, G) + S(r, F) + S(r, G). \end{aligned} \tag{3}$$

Noting that

$$\begin{aligned} \overline{N}_*(r, \infty; F, G) &= \overline{N}_L(r, \infty; F) + \overline{N}_L(r, \infty; G) \\ &\leq \overline{N}(r, \infty; F) = \overline{N}(r, \infty; G), \end{aligned} \tag{4}$$

we obtain from (3) that

$$\begin{aligned} T(r, F) \leq\ & N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) \\ & + S(r, F) + S(r, G). \end{aligned} \tag{5}$$

Obviously,

$$N(r, \infty; F) = (n+1)N(r, \infty; f) + k\overline{N}(r, \infty; f) + S(r, f). \tag{6}$$

Again

$$\begin{aligned} n m(r, f) &= m(r, F/f^{(k)}) \leq m(r, F) + m(r, 1/f^{(k)}) + S(r, f) \\ &= m(r, F) + T(r, f^{(k)}) - N(r, 0; f^{(k)}) + S(r, f) \\ &\leq m(r, F) + T(r, f) + k\overline{N}(r, \infty; f) - N(r, 0; f^{(k)}) + S(r, f). \end{aligned} \tag{7}$$

From (6) and (7) we obtain

$$(n-1)T(r,f) \quad \leq \quad T(r,F) - N(r,\infty;f) - N(r,0;f^{(k)}) + S(r,f). \tag{8}$$

Similarly,

$$(n-1)T(r,g) \quad \leq \quad T(r,G) - N(r,\infty;g) - N(r,0;g^{(k)}) + S(r,g). \tag{9}$$

Using (6), Lemma 2 we obtain from (8)

$$\begin{aligned}
(n-1)T(r,f) &\leq N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\
&\quad - N(r,\infty;f) - N(r,0;f^{(k)}) + S(r,f) + S(r,g) \\
&\leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,0;f^{(k)}) + N_2(r,0;g^{(k)}) \\
&\quad + 2\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) - N(r,\infty;f) \\
&\quad - N(r,0;f^{(k)}) + S(r,f) + S(r,g) \\
&\leq 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N(r,0;f^{(k)}) + N(r,0;g^{(k)}) \\
&\quad + 2N(r,\infty;f) + \overline{N}(r,\infty;g) - N(r,\infty;f) \\
&\quad - N(r,0;f^{(k)}) + S(r,f) + S(r,g) \\
&\leq 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N(r,0;g) + N(r,\infty;f) \\
&\quad + (k+1)\overline{N}(r,\infty;g) + S(r,f) + S(r,g) \\
&\leq \frac{2}{k}N(r,0;f) + \frac{2}{k}N(r,0;g) + N(r,0;g) + N(r,\infty;g) \\
&\quad + (k+1)\overline{N}(r,\infty;g) + S(r,f) + S(r,g) \\
&\leq \frac{2}{k}(T(r,f) + T(r,g)) + (k+3)T(r,g) \\
&\quad + S(r,f) + S(r,g).
\end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned}
(n-1)T(r,g) \quad &\leq \quad \frac{2}{k}(T(r,f) + T(r,g)) + (k+3)T(r,f) \\
&\quad\quad + S(r,f) + S(r,g).
\end{aligned} \tag{11}$$

Combining (10) and (11) we get

$$(n-k-4/k-4)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),$$

a contradiction with the fact that $n > k + 4/k + 4$.

Subcase 2 *Let* $l = 1$. *Then using (4) and Lemma 5 we obtain*

$$\begin{aligned}
T(r,F) &\leq T(r,F_1) + S(r,F) \\
&\leq N_2(r,0;F_1) + N_2(r,0;G_1) + \frac{3}{2}\overline{N}(r,\infty;F_1) + \overline{N}(r,\infty;G_1) \\
&\quad + \overline{N}_*(r,\infty;F_1,G_1) + \frac{1}{2}\overline{N}(r,0;F_1) + S(r,F_1) + S(r,G_1) \\
&\leq N_2(r,0;F) + N_2(r,0;G) + \frac{3}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\
&\quad + \overline{N}_*(r,\infty;F,G) + \frac{1}{2}\overline{N}(r,0;F) + S(r,F) + S(r,G) \\
&\leq N_2(r,0;F) + N_2(r,0;G) + \frac{5}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\
&\quad + \frac{1}{2}\overline{N}(r,0;F) + S(r,F) + S(r,G).
\end{aligned} \tag{12}$$

Using (12), Lemma 2 and Lemma 3 we obtain from (8)

$$\begin{aligned}
(n-1)T(r,f) \quad \leq \quad & N_2(r,0;F) + N_2(r,0;G) + \frac{5}{2}\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) \\
& + \frac{1}{2}\overline{N}(r,0;F) - N(r,\infty;f) - N(r,0;f^{(k)}) \\
& + S(r,f) + S(r,g) \\
\leq \quad & N_2(r,0;f) + N_2(r,0;g) + N_2(r,0;f^{(k)}) + N_2(r,0;g^{(k)}) \\
& + \frac{5}{2}\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \frac{1}{2}\overline{N}(r,0;f) \\
& + \frac{1}{2}\overline{N}(r,0;f^{(k)}) - N(r,\infty;f) - N(r,0;f^{(k)}) \\
& + S(r,f) + S(r,g) \\
\leq \quad & \frac{5}{2}\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + N(r,0;f^{(k)}) + N(r,0;g^{(k)}) \\
& + \frac{5}{2}N(r,\infty;f) + \overline{N}(r,\infty;g) + \frac{1}{2}N_{k+1}(r,0;f) \\
& + \frac{k}{2}\overline{N}(r,\infty;f) - N(r,\infty;f) - N(r,0;f^{(k)}) \\
& + S(r,f) + S(r,g) \\
\leq \quad & \frac{k+6}{2}\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + (k+1)\overline{N}(r,\infty;g) \\
& + N(r,0;g) + \frac{k+3}{2}N(r,\infty;f) + S(r,f) + S(r,g)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{k+6}{2k}N(r,0;f)+\frac{k+2}{k}N(r,0;g)+\frac{3k+5}{2}N(r,\infty;g)\\
&\quad +S(r,f)+S(r,g)\\
&\leq \left(\frac{2}{k}+\frac{1}{2}\right)(T(r,f)+T(r,g))+\frac{1}{k}T(r,f)+\frac{3k+6}{2}T(r,g)\\
&\quad +S(r,f)+S(r,g).
\end{aligned}
$$

This implies

$$
\begin{aligned}
\left(n-1-\frac{1}{k}\right)T(r,f) \leq \left(\frac{2}{k}+\frac{1}{2}\right)(T(r,f)+T(r,g))\\
+\frac{3k+6}{2}T(r,g)+S(r,f)+S(r,g).
\end{aligned}
\tag{13}
$$

Similarly,

$$
\begin{aligned}
\left(n-1-\frac{1}{k}\right)T(r,g) \leq \left(\frac{2}{k}+\frac{1}{2}\right)(T(r,f)+T(r,g))\\
+\frac{3k+6}{2}T(r,f)+S(r,f)+S(r,g).
\end{aligned}
\tag{14}
$$

From (13) and (14) we obtain

$$
\left(n-\frac{3k}{2}-\frac{5}{k}-5\right)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),
$$

a contradiction with our assumption that $n > 3k/2+5/k+5$.

Subcase 3 *Let* $l = 0$. *Then using (4) and Lemma 6 we obtain*

$$
\begin{aligned}
T(r,F) &\leq T(r,F_1)+S(r,F)\\
&\leq N_2(r,0;F_1)+N_2(r,0;G_1)+3\overline{N}(r,\infty;F_1)+2\overline{N}(r,\infty;G_1)\\
&\quad +\overline{N}_*(r,\infty;F_1,G_1)+2\overline{N}(r,0;F_1)+\overline{N}(r,0;G_1)\\
&\quad +S(r,F_1)+S(r,G_1)\\
&\leq N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)\\
&\quad +\overline{N}_*(r,\infty;F,G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G)\\
&\leq N_2(r,0;F)+N_2(r,0;G)+4\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)\\
&\quad +2\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G).
\end{aligned}
\tag{15}
$$

Using (15), Lemma 2 and Lemma 3 we obtain from (8)

$$\begin{aligned}
(n-1)T(r,f) &\leq N_2(r,0;F)+N_2(r,0;G)+4\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)\\
&\quad +2\overline{N}(r,0;F)+\overline{N}(r,0;G)-N(r,\infty;f)-N(r,0;f^{(k)})\\
&\quad +S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,0;f)+3\overline{N}(r,0;g)+N_2(r,0;f^{(k)})+N_2(r,0;g^{(k)})\\
&\quad +4\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+2\overline{N}(r,0;f^{(k)})+\overline{N}(r,0;g^{(k)})\\
&\quad -N(r,\infty;f)-N(r,0;f^{(k)})+S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,0;f)+3\overline{N}(r,0;g)+N(r,0;g^{(k)})+2\overline{N}(r,0;f^{(k)})\\
&\quad +\overline{N}(r,0;g^{(k)})+3N(r,\infty;f)+2\overline{N}(r,\infty;g)\\
&\quad +S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,0;f)+3\overline{N}(r,0;g)+N(r,0;g)+2N_{k+1}(r,0;f)\\
&\quad +N_{k+1}(r,0;g)+(2k+3)N(r,\infty;f)+(2k+2)\overline{N}(r,\infty;g)\\
&\quad +S(r,f)+S(r,g)\\
&\leq (2k+6)\overline{N}(r,0;f)+(k+4)\overline{N}(r,0;g)+N(r,0;g)\\
&\quad +(2k+3)N(r,\infty;f)+(2k+2)\overline{N}(r,\infty;g)\\
&\quad +S(r,f)+S(r,g)\\
&\leq \frac{k+4}{k}(N(r,0;f)+N(r,0;g))+\frac{k+2}{k}N(r,0;f)\\
&\quad +N(r,0;g)+(2k+3)N(r,\infty;f)+(2k+2)\overline{N}(r,\infty;g)\\
&\quad +S(r,f)+S(r,g)\\
&\leq \left(1+\frac{4}{k}\right)(T(r,f)+T(r,g))+\left(2k+\frac{2}{k}+4\right)T(r,f)\\
&\quad +(2k+3)T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

This gives

$$\begin{aligned}
\left(n-2k-\frac{2}{k}-5\right)T(r,f) \leq \left(1+\frac{4}{k}\right)(T(r,f)+T(r,g))\\
+(2k+3)T(r,g)+S(r,f)+S(r,g).
\end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned}
\left(n-2k-\frac{2}{k}-5\right)T(r,g) \leq \left(1+\frac{4}{k}\right)(T(r,f)+T(r,g))\\
+(2k+3)T(r,f)+S(r,f)+S(r,g).
\end{aligned} \tag{17}$$

In view of (16) and (17) we obtain

$$\left(n-4k-\frac{10}{k}-10\right)(T(r,f)+T(r,g)) \quad \leq \quad S(r,f)+S(r,g),$$

which contradicts our assumption that $n > 4k + 10/k + 10$.

Case 2 *We now assume that* $H = 0$. *That is*

$$\left(\frac{F_1''}{F_1'}-\frac{2F_1'}{F_1-1}\right)-\left(\frac{G_1''}{G_1'}-\frac{2G_1'}{G_1-1}\right)=0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F_1-1}=\frac{A}{G_1-1}+B, \tag{18}$$

where $A(\neq 0)$ *and* B *are constants. From (18) it is clear that* F_1 *and* G_1 *share* 1 *CM and hence they share the value* 1 *with weight* 2, *and therefore,* $n > k + 4/k + 4$. *Now we consider the following three subcases.*

Subcase 4 *Let* $B \neq 0$ *and* $A = B$. *Then from (18) we get*

$$\frac{1}{F_1-1}=\frac{BG_1}{G_1-1}. \tag{19}$$

If $B = -1$, *then from (19) we obtain*

$$F_1G_1 = 1,$$

i.e.,

$$f^nf^{(k)}g^ng^{(k)} = z^2.$$

Therefore by Lemma 7 we obtain $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, *where* c_1, c_2 *and* c *are three constants satisfying* $4(c_1c_2)^{n+1}c^2 = -1$. *If* $B \neq -1$, *from (19) we have* $\frac{1}{F_1} = \frac{BG_1}{(1+B)G_1-1}$, *and therefore,* $\overline{N}(r, \frac{1}{1+B}; G_1) = \overline{N}(r, 0; F_1)$. *Now using the second fundamental theorem of Nevanlinna, we get*

$$\begin{aligned}
T(r,G) &\leq T(r,G_1)+S(r,G)\\
&\leq \overline{N}(r,0;G_1)+\overline{N}\left(r,\frac{1}{1+B};G_1\right)+\overline{N}(r,\infty;G_1)+S(r,G)\\
&\leq \overline{N}(r,0;F_1)+\overline{N}(r,0;G_1)+\overline{N}(r,\infty;G_1)+S(r,G)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,G).
\end{aligned} \tag{20}$$

Using (20), Lemma 2 and Lemma 3 we obtain from (9)

$$\begin{aligned}(n-1)T(r,g) &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)-N(r,\infty;g)\\ &\quad -N(r,0;g^{(k)})+S(r,g)\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,0;f^{(k)})\\ &\quad +\overline{N}(r,0;g^{(k)})-N(r,\infty;g)-N(r,0;g^{(k)})\\ &\quad +S(r,f)+S(r,g)\\ &\leq \overline{N}(r,0;f)+\overline{N}(r,0;g)+N_{k+1}(r,0;f)+k\overline{N}(r,\infty;f)\\ &\quad +S(r,f)+S(r,g)\\ &\leq \frac{k+2}{k}N(r,0;f)+\frac{1}{k}N(r,0;g)+k\overline{N}(r,\infty;f)\\ &\quad +S(r,f)+S(r,g)\\ &\leq \frac{1}{k}(T(r,f)+T(r,g))+(k+\frac{1}{k}+1)T(r,f)\\ &\quad +S(r,f)+S(r,g).\end{aligned}$$

Thus we obtain

$$\left(n-k-\frac{3}{k}-2\right)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g),$$

a contradiction as $n>k+4/k+4$.

Subcase 5 *Let* $B\neq 0$ *and* $A\neq B$. *Then from (18) we get* $F_1=\frac{(B+1)G_1-(B-A+1)}{BG_1+(A-B)}$, *and so,* $\overline{N}(r,\frac{B-A+1}{B+1};G_1)=\overline{N}(r,0;F_1)$. *Proceeding as in Subcase 4 we obtain a contradiction.*

Subcase 6 *Let* $B=0$ *and* $A\neq 0$. *Then from (18) we get* $F_1=\frac{G_1+A-1}{A}$ *and* $G_1=AF_1-(A-1)$. *If* $A\neq 1$, *we have* $\overline{N}(r,\frac{A-1}{A};F_1)=\overline{N}(r,0;G_1)$ *and* $\overline{N}(r,1-A;G_1)=\overline{N}(r,0;F_1)$. *Using the similar arguments as in Subcase 4 we obtain a contradiction. Thus* $A=1$ *which implies* $F_1=G_1$, *and therefore,* $f^nf^{(k)}=g^ng^{(k)}$.

This completes the proof of Theorem 1. □

Proof of Theorem 2. Using Lemma 8 and proceeding similarly as in the proof of Theorem 1, we can prove Theorem 2. □

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Some applications of differential subordination to certain subclass of p-valent meromorphic functions involving convolution

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Abstract. By using the principle of differential subordination, we introduce subclass of p-valent meromorphic functions involving convolution and investigate various properties for this subclass. We also indicate relevant connections of the various results presented in this paper with the obtained results in earlier works.

1 Introduction

For any integer $m > -p$, let $\Sigma_{p,m}$ denote the class of all meromorphic functions f of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1}$$

which are analytic and p-valent in the punctured disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \backslash \{0\}$. For convenience, we write $\Sigma_{p,-p+1} = \Sigma_p$. If f and g are analytic in

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$\mathbb{U}$, we say that f is subordinate to g, written symbolically as, $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1\,(z \in \mathbb{U})$ such that $f(z) = g(w(z))\,(z \in \mathbb{U})$. In particular, if the function g is univalent in $\mathbb{U}$, we have the equivalence (see [10] and [11]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f \in \Sigma_{p,m}$, given by (1), and $g \in \Sigma_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \;\; (m > -p;\; p \in \mathbb{N}), \tag{2}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g * f)(z) \quad (m > -p;\; p \in \mathbb{N}). \tag{3}$$

For complex parameters

$$\alpha_1, \ldots, \alpha_q \text{ and } \beta_1, \ldots, \beta_s \;(\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\};\; j = 1, 2, \ldots, s),$$

we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by (see, for example, [14, p.19])

$$\begin{gathered} {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k}{(\beta_1)_k \ldots (\beta_s)_k} \cdot \frac{z^k}{k!} \\ (q \leq s+1;\; q,\; s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\, z \in \mathbb{U}), \end{gathered} \tag{4}$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}), \\ \theta(\theta - 1) \ldots (\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C})\,. \end{cases} \tag{5}$$

Corresponding to the function $h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$, defined by

$$h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-p}\, {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)\,, \tag{6}$$

we consider a linear operator

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) : \Sigma_p \to \Sigma_p\,,$$

which is defined by the following Hadamard product (or convolution):

$$H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z) = h_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z) * f(z). \tag{7}$$

We observe that, for a function $f(z)$ of the form (1), we have

$$H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s)f(z) = z^{-p} + \sum_{k=m}^{\infty} \Gamma_{p,q,s}(\alpha_1)\, a_k z^k. \tag{8}$$

where

$$\Gamma_{p,q,s}(\alpha_1) = \frac{(\alpha_1)_{k+p}\dots(\alpha_q)_{k+p}}{(\beta_1)_{k+p}\dots(\beta_s)_{k+p}(k+p)!}. \tag{9}$$

If, for convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s),$$

then one can easily verify from the definition (7) that (see [8])

$$z(H_{p,q,s}(\alpha_1)f(z))' = \alpha_1 H_{p,q,s}(\alpha_1+1)f(z) - (\alpha_1+p)H_{p,q,s}(\alpha_1)f(z). \tag{10}$$

For $m = -p+1\,(p \in \mathbb{N})$, the linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [8] and Aouf [2].

In particular, for $q = 2, s = 1, \alpha_1 > 0, \beta_1 > 0$ and $\alpha_2 = 1$, we obtain the linear operator

$$H_p(\alpha_1, 1; \beta_1)\, f(z) = \ell_p(\alpha_1, \beta_1)f(z) \quad (f \in \Sigma_p),$$

which was introduced and studied by Liu and Srivastava [7].

We note that, for any integer $n > -p$ and $f \in \Sigma_p$,

$$H_{p,2,1}(n+p,1;1)f(z) = D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z),$$

where D^{n+p-1} is the differential operator studied by Uralegaddi and Somanatha [16] and Aouf [1].

For functions $f,\, g \in \Sigma_{p,m}$, we define the linear operator $\mathcal{D}^n_{\lambda,p}(f*g) : \Sigma_{p,m} \longrightarrow \Sigma_{p,m}$ $(\lambda \geq 0;\ p \in \mathbb{N};\ n \in \mathbb{N}_0)$ by

$$\mathcal{D}^0_{\lambda,p}(f*g)(z) = (f*g)(z), \tag{11}$$

$$\begin{aligned}\mathcal{D}^1_{\lambda,p}(f*g)(z) &= \mathcal{D}_{\lambda,p}(f*g)(z)\\ &= (1-\lambda)(f*g)(z)+\lambda z^{-p}\left(z^{p+1}(f*g)(z)\right)'\\ &= z^{-p}+\sum_{k=m}^{\infty}[1+\lambda(k+p)]a_kb_kz^k\ (\lambda\geq 0;\ p\in\mathbb{N}),\end{aligned} \tag{12}$$

$$\begin{aligned}\mathcal{D}^2_{\lambda,p}(f*g)(z) &= \mathcal{D}(\mathcal{D}^1_{\lambda,p}(f*g))(z)\\ &= (1-\lambda)\mathcal{D}^1_{\lambda,p}(f*g)(z)+\lambda z^{-p}\left(z^{p+1}\mathcal{D}^1_{\lambda,p}(f*g)(z)\right)'\\ &= z^{-p}+\sum_{k=m}^{\infty}[1+\lambda(k+p)]^2a_kb_kz^k\ (\lambda\geq 0;\ p\in\mathbb{N}),\end{aligned}$$

and (in general)

$$\begin{aligned}\mathcal{D}^n_{\lambda,p}(f*g)(z) &= \mathcal{D}(\mathcal{D}^{n-1}_{\lambda,p}(f*g)(z)) =\\ &= z^{-p}+\sum_{k=m}^{\infty}[1+\lambda(k+p)]^na_kb_kz^k\ (\lambda\geq 0).\end{aligned} \tag{13}$$

From (13) it is easy to verify that:

$$z(\mathcal{D}^n_{\lambda,p}(f*g)(z))' = \frac{1}{\lambda}\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z)-(p+\frac{1}{\lambda})\mathcal{D}^n_{\lambda,p}(f*g)(z)\ (\lambda>0). \tag{14}$$

For $m=0$ the linear operator $\mathcal{D}^n_{\lambda,p}(f*g)$ was introduced by Aouf et al. [4].

Making use of the principle of differential subordination as well as the linear operator $D^n_{\lambda,p}(f*g)$, we now introduce a subclass of the function class $\Sigma_{p,m}$ as follows:

For fixed parameters A and B $(-1\leq B<A\leq 1)$, we say that a function $f\in\Sigma_{p,m}$ is in the class $\Sigma^n_{\lambda,p,m}(f*g;A,B)$, if it satisfies the following subordination condition:

$$-\frac{z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'}{p}\prec\frac{1+Az}{1+Bz}. \tag{15}$$

In view of the definition of subordination, (15) is equivalent to the following condition:

$$\left|\frac{z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'+p}{Bz^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'+pA}\right|<1\ \ (z\in U).$$

For convenience, we write

$$\Sigma^n_{\lambda,p}\left(f*g;1-\frac{2\zeta}{p},-1\right)=\Sigma^n_{\lambda,p}\left(f*g;\zeta\right),$$

where $\Sigma^{n}_{\lambda,p}(f*g;\zeta)$ denotes the class of functions $f(z)\in\Sigma_{p,m}$ satisfying the following inequality:

$$\Re\left\{-z^{p+1}\left(\mathcal{D}^{n}_{\lambda,p}(f*g)(z)\right)'\right\}>\zeta \quad (0\le\zeta<p;\ z\in U).$$

We note that:

(i) For $b_k=\lambda=1$ in (15), the class $\Sigma^{n}_{\lambda,p,m}(f*g;A,B)$ reduces to the class $\Sigma^{n}_{p,m}(A,B)$ was introduced and studied by Srivastava and Patel [15];

(ii) For $b_k=\Gamma_{p,q,s}(\alpha_1)$, where $\Gamma_{p,q,s}(\alpha_1)$ is given by (9), and $n=0$ in (15), we have $\Sigma^{n}_{\lambda,p}(f*g;A,B)=\Sigma^{m}_{p,q,s}(\alpha_1,A,B)$, where the class $\Sigma^{m}_{p,q,s}(\alpha_1,A,B)$ introduced and studied by Aouf [3];

(iii) For $q=2$, $s=1$, $\alpha_1=a>0$, $\beta_1=c>0$ and $\alpha_2=1$, we have $\Sigma^{m}_{p,q,s}(\alpha_1,A,B)=\Sigma_{a,c}(p;m,A,B)$, where the class $\Sigma_{a,c}(p;m,A,B)$ was studied by Patel and Cho [13].

2 Preliminary lemmas

In order to establish our main results, we need the following lemmas.

Lemma 1 [6]. *Let the function* h *be analytic and convex (univalent) in* U *with* $h(0)=1$. *Suppose also that the function* φ *given by*

$$\varphi(z)=1+c_{p+m}z^{p+m}+c_{p+m+1}z^{p+m+1}+\dots \tag{16}$$

in analytic in U. *If*

$$\varphi(z)+\frac{z\varphi'(z)}{\gamma}\prec h(z) \quad (\Re(\gamma)\ge 0;\gamma\ne 0), \tag{17}$$

then

$$\varphi(z)\prec\psi(z)=\frac{\gamma}{p+m}z^{\frac{-\gamma}{p+m}}\int_0^z t^{\frac{\gamma}{p+m}-1}h(t)dt\prec h(z),$$

and ψ *is the best dominant.*

For real or complex numbers a, b and $c\,(c \notin \mathbb{Z}_0^-)$, the Gaussian hypergeometric function is defined by

$$_2F_1(a,b;c;z) = 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \dots\ . \tag{18}$$

We note that the above series converges absolutely for $z \in U$ and hence represents an analytic function in U (see, for details [17, Chapter 14]).

Each of the identities (asserted by Lemma 2 below) is well-known (cf., e.g., [17, Chapter 14]).

Lemma 2 [17, Chapter 14]. *For real or complex parameters* a, b *and* $c\,(c \notin \mathbb{Z}_0^-)$,

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a,b;c;z) \;\; (\Re(c) > \Re(b) > 0)\,; \tag{19}$$

$$_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1(a, c-b; c; \frac{z}{z-1})\,; \tag{20}$$

$$_2F_1(a,b;c;z) = {}_2F_1(a, b-1; c; z) + \frac{az}{c}\, {}_2F_1(a+1, b; c+1; z)\,; \tag{21}$$

3 Main results

Unless otherwise mentioned, we assume throughout this paper that $\lambda, \mu > 0, m > -p,\ p \in \mathbb{N}, n \in \mathbb{N}_0$ and g is given by (2).

Theorem 1 *Let the function* f *defined by (1) satisfying the following subordination condition:*

$$-\frac{(1-\mu)z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))' + \mu z^{p+1}(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z))'}{p} \prec \frac{1+Az}{1+Bz}.$$

Then

$$-\frac{z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'}{p} \prec \mathcal{G}(z) \prec \frac{1+Az}{1+Bz}\,, \tag{22}$$

where the function $\mathcal{G}$ *given by*

$$\mathcal{G}(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\, {}_2F_1\left(1, 1; \frac{1}{\lambda\mu(p+m)} + 1; \frac{Bz}{1+Bz}\right) & (B \neq 0) \\ 1 + \frac{A}{\lambda\mu(p+m)+1} z & (B = 0) \end{cases}$$

is the best dominant of (22). Furthermore,

$$\Re\left\{-\frac{z^{p+1}\left(\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)'}{p}\right\} > \xi \quad (z \in U), \tag{23}$$

where

$$\xi = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1;\frac{1}{\lambda\mu(p+m)}+1;\frac{B}{B-1}\right) & (B \neq 0) \\ 1 - \frac{A}{\lambda\mu(p+m)+1} & (B = 0)\,. \end{cases}$$

The estimate in (23) is the best possible.

Proof. Consider the function φ defined by

$$\varphi(z) = -\frac{z^{p+1}\left(\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)'}{p} \quad (z \in U). \tag{24}$$

Then φ is of the form (16) and is analytic in U. Differentiating (24) with respect to z and using (14), we obtain

$$-\frac{(1-\mu)z^{p+1}\left(\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)' + \mu z^{p+1}\left(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z)\right)'}{p}$$

$$= \varphi(z) + \lambda\mu z\varphi'(z) \prec \frac{1+Az}{1+Bz}\,.$$

Now, by using Lemma 1 for $\beta = \frac{1}{\lambda\mu}$, we obtain

$$-\frac{z^{p+1}\left(\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)'}{p} \prec \mathcal{G}(z) = \frac{1}{\lambda\mu(p+m)} z^{-\frac{1}{\lambda\mu(p+m)}} \int_0^z t^{\frac{1}{\lambda\mu(p+m)}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

$$= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\frac{1}{\lambda\mu(p+m)}+1;\frac{Bz}{1+Bz}\right) & (B \neq 0) \\ 1 + \frac{A}{\lambda\mu(p+m)+1}z & (B = 0)\,, \end{cases}$$

by change of variables followed by the use of the identities (19), (20) and (21) (with $a = 1$, $c = b+1$, $b = \frac{1}{\lambda\mu(p+m)}$). This proves the assertion (22) of Theorem 1.

Next, in order to prove the assertion (23) of Theorem 1, it suffices to show that

$$\inf_{|z|<1}\left\{\Re(\mathcal{G}(z))\right\} = \mathcal{G}(-1)\,. \tag{25}$$

Indeed we have, for $|z| \leq r < 1$,

$$\Re\left(\frac{1+Az}{1+Bz}\right) \geq \frac{1-Ar}{1-Br}.$$

Upon setting

$$g(\zeta, z) = \frac{1+A\zeta z}{1+B\zeta z} \text{ and } d\nu(\zeta) = \frac{1}{\lambda\mu(p+m)}\zeta^{\frac{1}{\lambda\mu(p+m)}-1}d\zeta\,(0 \leq \zeta \leq 1),$$

which is a positive measure on the closed interval $[0, 1]$, we get

$$\mathcal{G}(z) = \int_0^1 g(\zeta, z)d\nu(\zeta),$$

so that

$$\Re\{\mathcal{G}(z)\} \geq \int_0^1 \left(\frac{1-A\zeta r}{1-B\zeta r}\right) d\nu(\zeta) = \mathcal{G}(-r) \quad (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (23) of Theorem 1.

Finally, the estimate in (23) is the best possible as the function $\mathcal{G}$ is the best dominant of (22). □

Taking $\mu = 1$ in Theorem 1, we obtain the following corollary.

Corollary 1 *The following inclusion property holds for the function class* $\Sigma^n_{\lambda,p}(f * g; A, B)$*:*

$$\Sigma^{n+1}_{\lambda,p,m}(f * g; A, B) \subset \Sigma^n_{\lambda,p,m}(f * g; \beta) \subset \Sigma^n_{\lambda,p,m}(f * g; A, B),$$

where

$$\beta = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1, 1; \frac{1}{\lambda(p+m)} + 1; \frac{B}{B-1}\right) & (B \neq 0) \\ 1 - \frac{A}{\lambda(p+m)+1} & (B = 0). \end{cases}$$

The result is the best possible.

Taking $\mu = 1$, $A = 1 - \frac{2\sigma}{p}$ $(0 \leq \sigma < p)$ and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 2 *The following inclusion property holds for the function class* $\Sigma^{n}_{\lambda,p,m}(f*g;\sigma)$:

$$\Sigma^{n+1}_{\lambda,p,m}(f*g;\sigma)\subset\Sigma^{n}_{\lambda,p,m}(f*g;\beta))\subset\Sigma^{n}_{\lambda,p,m}(f*g;\sigma),$$

where

$$\beta=\sigma+(p-\sigma)\left\{{}_2F_1(1,1;\frac{1}{\lambda\,(p+m)}+1;\frac{1}{2})-1\right\}.$$

The result is the best possible.

Theorem 2 *If* $f\in\Sigma^{n}_{\lambda,p,m}(f*g;\theta)\ (0\le\theta<p)$, *then*

$$\Re\left\{-z^{p+1}\left[(1-\mu)(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))'+\mu(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z))'\right]\right\}>\theta\quad(|z|<R),\tag{26}$$

where

$$R=\left\{\sqrt{1+\lambda^2\mu^2(p+m)^2}-\lambda\mu(p+m)\right\}^{\frac{1}{p+m}}.$$

The result is the best possible.

Proof. Since $f\in\Sigma^{n}_{\lambda,p}(f*g;\theta)$, we write

$$-z^{p+1}(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))'=\theta+(p-\theta)u(z)\quad(z\in U).\tag{27}$$

Then, clearly, u is of the form (16), is analytic in U, and has a positive real part in U. Differentiating (27) with respect to z and using (14), we obtain

$$-\frac{z^{p+1}\left[(1-\mu)(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))'+\mu(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z))'\right]+\theta}{p-\theta}=u(z)+\lambda\mu zu'(z)\,.\tag{28}$$

Now, by applying the well-known estimate [5]

$$\frac{\left|zu'(z)\right|}{\Re\{u(z)\}}\le\frac{2(p+m)r^{p+m}}{1-r^{2(p+m)}}\quad(|z|=r<1)$$

in (28), we obtain

$$\Re\left\{-\frac{z^{p+1}\left[(1-\mu)(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))'+\mu(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z))'\right]+\theta}{p-\theta}\right\}$$

$$\geq \Re\{u(z)\} \cdot \left(1 - \frac{2\lambda\mu(p+m)r^{p+m}}{(1-r^{2(p+m)})}\right). \tag{29}$$

It is easily seen that the right-hand side of (29) is positive provided that $r < R$, where R is given as in Theorem 2. This proves the assertion (26) of Theorem 2.

In order to show that the bound R is the best possible, we consider the function $f \in \Sigma_{p,m}$ defined by

$$-z^{p+1}\left(\mathcal{D}^n_{\lambda,p}(f*g)(z)\right)' = \theta + (p-\theta)\frac{1+z^{p+m}}{1-z^{p+m}} \quad (0 \leq \theta < p; p \in \mathbb{N}; z \in U).$$

Noting that

$$\begin{aligned}
&-\frac{z^{p+1}\left[(1-\mu)(\mathcal{D}^n_{\lambda,p}(f*g)(z))' + \mu(\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z))'\right] + \theta}{p-\theta} \\
&= \frac{1 - z^{2(p+m)} + 2\lambda\mu(p+m)z^{p+m}}{\alpha_1(1-z^{p+m})^2} = 0
\end{aligned}$$

for $z = R^{\frac{1}{p+m}} \exp\left(\frac{i\pi}{p+m}\right)$, we complete the proof of Theorem 2. □

Putting $\mu = 1$ in Theorem 2, we obtain the following result.

Corollary 3 *If* $f \in \Sigma^n_{\lambda,p,m}(f*g;\theta)$ $(0 \leq \theta < p; p \in \mathbb{N})$, *then* f *satisfies the condition of* $\Sigma^{n+1}_{\lambda,p,m}(f*g;\theta)$ *for* $|z| < R^*$, *where*

$$R^* = \left\{\sqrt{1+\lambda^2(p+m)^2} - \lambda(p+m)\right\}^{\frac{1}{p+m}}.$$

The result is the best possible.

Theorem 3 *Let* $f \in \Sigma^n_{\lambda,p,m}(f*g;A,B)$ *and let*

$$F_{\delta,p}(f)(z) = \frac{\delta}{z^{\delta+p}}\int_0^z t^{\delta+p-1}f(t)dt \quad (\delta > 0; z \in U). \tag{30}$$

Then

$$-\frac{z^{p+1}\left(\mathcal{D}^n_{\lambda,p}(F_{\delta,p}(f)*g)(z)\right)'}{p} \prec \Phi(z) \prec \frac{1+Az}{1+Bz}, \tag{31}$$

where the function Φ *given by*

$$\Phi(z)=\begin{cases}\frac{A}{B}+(1-\frac{A}{B})(1+Bz)^{-1}\,{}_2F_1(1,1;\frac{\delta}{p+m}+1;\frac{Bz}{Bz+1}) & (B\neq 0)\\ 1+\frac{\delta}{\delta+p+m}Az & (B=0)\,,\end{cases}$$

is the best dominant of (31). Furthermore,

$$\Re\left\{-\frac{z^{p+1}(\mathcal{D}^n_{\lambda,p}(F_{\delta,p}(f)*g)(z))'}{p}\right\}>\xi^* \quad (z\in U), \tag{32}$$

where

$$\xi^*=\begin{cases}\frac{A}{B}+(1-\frac{A}{B})(1-B)^{-1}\,{}_2F_1(1,1;\frac{\delta}{p+m}+1;\frac{B}{B-1}) & (B\neq 0)\\ 1-\frac{\delta}{\delta+p+m}A & (B=0)\,.\end{cases}$$

The result is the best possible.

Proof. Defining the function φ by

$$\varphi(z)=-\frac{z^{p+1}(\mathcal{D}^n_{\lambda,p}(F_{\delta,p}(f)*g)(z))'}{p} \quad (z\in U)\,, \tag{33}$$

we note that φ is of the form (16) and is analytic in U. Using the following operator identity:

$$z(\mathcal{D}^n_{\lambda,p}(F_{\delta,p}(f)*g)(z))'=\delta\mathcal{D}^n_{\lambda,p}(f*g)(z)-(\delta+p)\mathcal{D}^n_{\lambda,p}(F_{\delta,p}(f)*g)(z) \tag{34}$$

in (33) and differentiating the resulting equation with respect to z, we find that

$$-\frac{z^{p+1}(\mathcal{D}^n_{\lambda,p}(f*g)(z))'}{p}=\varphi(z)+\frac{z\varphi'(z)}{\delta}\prec\frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 3 follows by employing the techniques that we used in proving Theorem 1 above. □

Remark 1 *By observing that*

$$z^{p+1}(\mathcal{D}^n_{\lambda,p}(F_{\delta,p}(f)*g)(z))'=\frac{\delta}{z^\delta}\int_0^z t^{\delta+p}(\mathcal{D}^n_{\lambda,p}(f*g)(t))'dt \quad (f\in\Sigma_{p,m}; z\in U). \tag{35}$$

If $\delta > 0$ *and* $f \in \Sigma^{n}_{\lambda,p,m}(f * g; A, B)$, *then*

$$\Re\left\{-\frac{\delta}{pz^{\delta}}\int_{0}^{z} t^{\delta+p}(\mathcal{D}^{n}_{\lambda,p}(f*g)(t))' dt\right\} > \xi^{*} \ \ (z \in U),$$

where ξ^{*} *is given as in Theorem 3.*

In view of (35), Theorem 3 for $A = 1 - \frac{2\theta}{p}$ $(0 \le \theta < p; p \in \mathbb{N})$ *and* $B = -1$ *yields*

Corollary 4 *If* $\delta > 0$ *and if* $f \in \Sigma_{p,m}$ *satisfies the following inequality:*

$$\Re\left\{-z^{p+1}(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))'\right\} > \theta \quad (0 \le \theta < p; p \in \mathbb{N}; z \in U),$$

then

$$\Re\left\{\frac{-\delta}{z^{\delta}}\int_{0}^{z}(\mathcal{D}^{n}_{\lambda,p}(f*g)(t))' dt\right\} > \theta+(p-\theta)\left[{}_2F_1(1,1;\frac{\delta}{p+m}+1;\frac{1}{2})-1\right] \ \ (z \in U).$$

The result is the best possible.

Theorem 4 *Let* $f \in \Sigma_{p,m}$. *Suppose also that* $h \in \Sigma_{p,m}$ *satisfies the following inequality:*

$$\Re\left\{z^{p}(\mathcal{D}^{n}_{\lambda,p}(h*g)(z))\right\} > 0 \quad (z \in U).$$

If

$$\left|\frac{\mathcal{D}^{n}_{\lambda,p}(f*g)(z)}{\mathcal{D}^{n}_{\lambda,p}(h*g)(z)} - 1\right| < 1 \quad (z \in U),$$

then

$$\Re\left\{-\frac{z(\mathcal{D}^{n}_{\lambda,p}(f*g)(z))'}{\mathcal{D}^{n}_{\lambda,p}(f*g)(z)}\right\} > 0 \quad (|z| < R_0),$$

where

$$R_0 = \left[\frac{\sqrt{9(p+m)^2+4p(2p+m)}-3(p+m)}{2(2p+m)}\right]^{\frac{1}{p+m}}.$$

Proof. Letting

$$w(z) = \frac{\mathcal{D}^{n}_{\lambda,p}(f*g)(z)}{\mathcal{D}^{n}_{\lambda,p}(h*g)(z)} - 1 = t_{p+m}z^{p+m} + t_{p+m+1}z^{p+m+1} + \dots, \tag{36}$$

we note that w is analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| \leq |z|^{p+m}$ $(z \in \mathbb{U})$. Then, by applying the familiar Schwarz's lemma [12], we obtain

$$w(z) = z^{p+m}\Psi(z)\,,$$

where the functions Ψ is analytic in $\mathbb{U}$ and $|\Psi(z)| \leq 1\,(z \in \mathbb{U})$. Therefore, (36) leads us to

$$\mathcal{D}^n_{\lambda,p}(f * g)(z) = \mathcal{D}^n_{\lambda,p}(h * g)(z)\ \left(1 + z^{p+m}\Psi(z)\right) \quad (z \in \mathbb{U})\,. \tag{37}$$

Differentiating (37) logarithmically with respect to z, we obtain

$$\frac{z\left(\mathcal{D}^n_{\lambda,p}(f * g)(z)\right)'}{\mathcal{D}^n_{\lambda,p}(f * g)(z)} = \frac{z\left(\mathcal{D}^n_{\lambda,p}(h * g)(z)\right)'}{\mathcal{D}^n_{\lambda,p}(h * g)(z)} + \frac{z^{p+m}\left\{(p+m)\Psi(z) + z\Psi'(z)\right\}}{1 + z^{p+m}\Psi(z)}\,. \tag{38}$$

Putting $\varphi(z) = z^p\mathcal{D}^n_{\lambda,p}(h * g)(z)$, we see that the function φ is of the form (16), is analytic in $\mathbb{U}$, $\Re\{\varphi(z)\} > 0\,(z \in \mathbb{U})$ and

$$\frac{z\left(\mathcal{D}^n_{\lambda,p}(h * g)(z)\right)'}{\mathcal{D}^n_{\lambda,p}(h * g)(z)} = \frac{z\varphi'(z)}{\varphi(z)} - p\,,$$

so that we find from (38) that

$$\begin{aligned}\Re\left\{-\frac{z\left(\mathcal{D}^n_{\lambda,p}(f * g)(z)\right)'}{\mathcal{D}^n_{\lambda,p}(f * g)(z)}\right\} &\geq p \\ &- \left|\frac{z\varphi'(z)}{\varphi(z)}\right| - \left|\frac{z^{p+m}\left\{(p+m)\Psi(z) + z\Psi'(z)\right\}}{1 + z^{p+m}\Psi(z)}\right| \quad (z \in \mathbb{U})\,.\end{aligned} \tag{39}$$

Now, by using the following known estimates [9]:

$$\left|\frac{\varphi'(z)}{\varphi(z)}\right| \leq \frac{2(p+m)r^{p+m-1}}{1 - r^{2(p+m)}} \qquad (|z| = r < 1)$$

and

$$\left|\frac{(p+m)\Psi(z) + z\Psi'(z)}{1 + z^{p+m}\Psi(z)}\right| \leq \frac{(p+m)}{1 - r^{p+m}} \qquad (|z| = r < 1)$$

in (39), we obtain

$$\Re\left\{-\frac{z\left(\mathcal{D}^n_{\lambda,p}(f * g)(z)\right)'}{\mathcal{D}^n_{\lambda,p}(f * g)(z)}\right\} \geq \frac{p - 3(p+m)r^{p+m} - (2p+m)r^{2(p+m)}}{1 - r^{2(p+m)}} \quad (|z| = r < 1)\,,$$

which is certainly positive, provided that $r < R_0$, R_0 being given as in Theorem 4. $\square$

Theorem 5 *If* $f \in \Sigma_{p,m}$ *satisfies the following subordination condition:*

$$(1-\mu)z^p\mathcal{D}^n_{\lambda,p}(f*g)(z) + \mu z^p\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z) \prec \frac{1+Az}{1+Bz}\,,$$

then

$$\Re\left\{z^p\mathcal{D}^n_{\lambda,p}(f*g)(z)\right\}^{\frac{1}{d}} > \xi^{\frac{1}{d}} \quad (d \in \mathbb{N}; z \in U)\,,$$

where ξ *is given as in Theorem 1. The result is the best possible.*

Proof. Defining the function φ by

$$\varphi(z) = z^p\mathcal{D}^n_{\lambda,p}(f*g)(z) \quad (f \in \Sigma_{p,m}; z \in U), \tag{40}$$

we see that the function φ is of the form (16) and is analytic in U. Differentiating (40) with respect to z and using the identity (14), we obtain

$$(1-\mu)z^p\mathcal{D}^n_{\lambda,p}(f*g)(z) + \mu z^p\mathcal{D}^{n+1}_{\lambda,p}(f*g)(z) = \varphi(z) + \lambda\mu z\varphi^{'}(z) \prec \frac{1+Az}{1+Bz}\,.$$

Now, by following the lines of the proof of Theorem 1 mutatis mutandis, and using the elementary inequality:

$$\Re\left(w^{\frac{1}{d}}\right) \geq (\Re w)^{\frac{1}{d}} \quad (\Re(w) > 0; d \in \mathbb{N})\,,$$

we arrive at the result asserted by Theorem 5. $\square$

Remark 2 (i) *Taking* $b_k = \lambda = 1$ *in the above results, we obtain the results obtained by Srivastava and Patel [15];*

(ii) *Taking* $b_k = \Gamma_{p,q,s}(\alpha_1)$*, where* $\Gamma_{p,q,s}(\alpha_1)$ *is given by (9), and* $n = 0$ *in the above results, we obtain the results obtained by Aouf [3].*

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