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Antal Bege
(1962–2012)

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Balancing diophantine triples

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Abstract. In this paper, we show that there are no three distinct positive integers a, b and c such that $ab+1, ac+1, bc+1$ all are balancing numbers.

1 Introduction

A *diophantine* m*-tuple* is a set $\{a_1, \ldots, a_m\}$ of positive integers such that $a_i a_j + 1$ is square for all $1 \le i < j \le m$. Diophantus investigated first the problem of finding *rational* quadruples, and he provided one example: $\{1/16, 33/16, 68/16, 105/16\}$. The first integer quadruple, $\{1, 3, 8, 120\}$ was found by Fermat. Infinitely many diophantine quadruples of integers are known and it is conjectured that there is no integer diophantine quintuple. This was almost proved by Dujella [2], who showed that there can be at most finitely many diophantine quintuples and all of them are, at least in theory, effectively computable.

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The following variant of the diophantine tuples problem was treated by [4]. Let A and B be two nonzero integers such that $D = B^2+4A \neq 0$. Let $(u_n)_{n=0}^{\infty}$ be a binary recursive sequence of integers satisfying the recurrence

$$u_{n+2} = Au_{n+1} + Bu_n \qquad \text{for all } n \geq 0.$$

It is well-known that if we write α and β for the two roots of the *characteristic equation* $x^2 - Ax - B = 0$, then there exist constants $\gamma, \delta \in \mathbb{Q}[\alpha]$ such that

$$u_n = \gamma\alpha^n + \delta\beta^n \qquad \text{for all } n \geq 0.$$

Assume further that the sequence $(u_n)_{n=0}^{\infty}$ is *non-degenerate* which means that $\gamma\delta \neq 0$ and α/β are not root of unity. We shall also make the convention that $|\alpha| \geq |\beta|$.

A diophantine triple with values in the set $U = \{u_n : n \geq 0\}$, is a set of three distinct positive integers $\{a, b, c\}$, such that $ab + 1, ac + 1, bc + 1$ are all in U. Note that if $u_n = 2^n + 1$ for all $n \geq 0$, then there are infinitely many such triples (namely, take a, b, c to be any distinct powers of two). The main result in [4] shows that only similar sequences can possess this property. The precise result proved there is the following.

Theorem 1 *Assume that $(u_n)_{n=0}^{\infty}$ is a non-degenerate binary recurrence sequence with $D > 0$, and suppose that there exist infinitely many nonnegative integers a, b, c with $1 \leq a < b < c$, and x, y, z such that*

$$ab + 1 = u_x, \quad ac + 1 = u_y, \quad bc + 1 = u_z.$$

Then $\beta \in \{\pm 1\}$, $\delta \in \{\pm 1\}$, $\alpha, \gamma \in \mathbb{Z}$. Furthermore, for all but finitely many of sixtuples $(a, b, c; x, y, z)$ as above one has $\delta\beta^z = \delta\beta^y = 1$ and one of the followings holds:

(i) $\delta\beta^x = 1$. In this case, one of δ or $\delta\alpha$ is a perfect square;

(ii) $\delta\beta^x = -1$. In this case, $x \in \{0, 1\}$.

No finiteness result was proved for the case when $D < 0$.

The first definition of balancing numbers is essentially due to Finkelstein [3], although he called them numerical centers. A positive integer n is called balancing number if

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)$$

holds for some positive integer r. Then r is called balancer corresponding to the balancing number n. The n^{th} term of the sequence of balancing numbers is denoted by B_n. The balancing numbers satisfy the recurrence relation

$$B_{n+2} = 6B_{n+1} - B_n,$$

where the initial conditions are $B_0 = 0$ and $B_1 = 1$. Let α and β denote the roots of the characteristic polynomial $b(x) = x^2 - 6x + 1$. Then the explicit formula for the terms B_n is given by

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{4\sqrt{2}}. \tag{1}$$

The first few terms of the balancing sequence are

$$0,\ 1,\ 6,\ 35,\ 204,\ 1189,\ 6930,\ 40391,\ 235416, \ldots .$$

Let denote the half of the associate sequence of the balancing numbers by C_n. Clearly, $C_n = (\alpha^n + \beta^n)/2$ satisfies $C_n = 6C_{n-1} - C_{n-2}$. Note that the terms C_n are odd positive integers:

$$1,\ 3,\ 17,\ 99,\ 577,\ 3363,\ 19601,\ 114243,\ 665857, \ldots .$$

Although Theorem 1 guarantees that there are at most finitely many Fibonacci and Lucas diophantine triples, it does not give a hint to find all of them. Luca and Szalay described a method to determine diophantine triples for Fibonacci numbers and Lucas numbers ([6] and [7], respectively). In this paper, we follow their method, although some new types of problems appeared when we proved the following theorem.

Theorem 2 *There do no exist positive integers* $a < b < c$ *such that*

$$ab + 1 = B_x, \qquad ac + 1 = B_y, \qquad bc + 1 = B_z, \tag{2}$$

where $0 < x < y < z$ *are natural numbers and* $(B_n)_{n=0}^{\infty}$ *is the sequence of balancing numbers.*

The main idea in the proof of Theorem 2 coincides the principal tool of [6], the details are different since the balancing numbers have less properties have been known then in case of Fibonacci and Lucas numbers.

2 Preliminary results

The proof of Theorem 2 uses the next lemma.

Lemma 1 *The following identities hold.*

1. $B_n = 35B_{n-2} - 6B_{n-3}$;
2. *If* $n \geq m$ *then* $(B_n - B_m)(B_n + B_m) = B_{n-m}B_{n+m}$, *especially* $(B_n - 1)(B_n + 1) = B_{n-1}B_{n+1}$;
3. $\gcd(B_n, B_m) = B_{\gcd(n,m)}$, *especially* $\gcd(B_n, B_{n-1}) = 1$;
4. $\gcd(B_n, C_n) = 1$;
5. $B_{n+m} = B_nC_m + C_nB_m$;
6. $B_{2n+1} - 1 = 2B_nC_{n+1}$.

Proof. The first property is a double application of the recurrence relation of balancing numbers. The second identity is Theorem 2.4.13 in [9], the next one is a specific case of a general statement described by [5]. The fourth feature can be found in the proof of Theorem VII in [1], the fifth property is given in [8]. Finally, the last one is coming easily from the explicit formulae for B_n and C_n. □

Lemma 2 *Any integer* $n \geq 2$ *satisfies the relation* $\gcd(B_n - 1, B_{n-2} - 1) \leq 34$.

Proof. Using the common tools in evaluating the greatest common divisor, the recurrence relation of balancing numbers, and Lemma 1 the statement is implied by the following rows. Put $Q_1 = \gcd(B_n - 1, B_{n-2} - 1)$. Then

$$\begin{aligned}
Q_1 &= \gcd(B_n - 1, B_n - B_{n-2}) = \gcd(B_n - 1, 6B_{n-1} - 2B_{n-2}) \leq \\
&\leq 2\gcd(B_n - 1, 3B_{n-1} - B_{n-2}) \leq 2\gcd(B_{n-1}B_{n+1}, 3B_{n-1} - B_{n-2}) \leq \\
&\leq 2\gcd(B_{n-1}, 3B_{n-1} - B_{n-2})\gcd(B_{n+1}, 3B_{n-1} - B_{n-2}) = \\
&= 2\gcd(B_{n-1}, B_{n-2})\gcd(35B_{n-1} - 6B_{n-2}, 3B_{n-1} - B_{n-2}) = \\
&= 2\gcd(-B_{n-1} + 6B_{n-2}, 3B_{n-1} - B_{n-2}) = \\
&= 2\gcd(-B_{n-1} + 6B_{n-2}, 17B_{n-2}) \leq \\
&\leq 34\gcd(-B_{n-1} + 6B_{n-2}, B_{n-2}) = 34\gcd(-B_{n-1}, B_{n-2}) = 34.
\end{aligned}$$

□

Lemma 3 *For any integer* $n \geq 2$ *we have* $\gcd(B_{2n-3} - 1, B_n - 1) \leq 1190$.

Proof. Similarly to the previous lemma, put $Q_2 = \gcd(B_{2n-3} - 1, B_n - 1)$. Then

$$\begin{aligned} Q_2 &= \gcd(2B_{n-2}C_{n-1}, B_n - 1) \leq 2\gcd(B_{n-2}, B_n - 1)\gcd(C_{n-1}, B_n - 1) \leq \\ &\leq 2\gcd(B_{n-2}, B_{n-1}B_{n+1})\gcd(C_{n-1}, B_{n-1}B_{n+1}) \leq \\ &\leq 2\gcd(B_{n-2}, B_{n-1})\gcd(B_{n-2}, B_{n+1})\gcd(C_{n-1}, B_{n-1})\gcd(C_{n-1}, B_{n+1}) \leq \\ &\leq 2 \cdot 1 \cdot 35 \cdot 1 \cdot 17 = 1190. \end{aligned}$$

For explaining that $\gcd(C_{n-1}, B_{n+1}) \leq 17$, by Lemma 1 we write

$$\gcd(C_{n-1}, B_{n+1}) = \gcd(C_{n-1}, B_{n-1}C_2 + C_{n-1}B_2) = \gcd(C_{n-1}, 17B_{n-1}) \leq 17.$$

□

Remark 1 For our purposes, it is sufficient to have upper bounds given by Lemma 2 and Lemma 3. Without proof we state that the possible values for Q_1 are only 1, 2 and 34, while $Q_2 \in \{1, 2, 5, 34\}$.

Lemma 4 *Let* $u_0 \geq 3$ *be a positive integer. Then for all integers* $u \geq u_0$ *the inequalities*

$$\alpha^{u-0.9831} < B_u < \alpha^{u-0.983} \tag{3}$$

hold.

Proof. Let $c_0 = 4\sqrt{2}$. Since $0 < \beta < 1 < \alpha$ then the inequalities $u \geq u_0 \geq 3$ imply

$$B_u \geq \frac{\alpha^u - \beta^{u_0}}{c_0} = \alpha^u \left(\frac{1 - \frac{\beta^{u_0}}{\alpha^u}}{c_0} \right) \geq \alpha^u \left(\frac{1 - \left(\frac{\beta}{\alpha}\right)^{u_0}}{c_0} \right) \geq \alpha^{u-0.9831}.$$

For any non-negative integer u,

$$B_u \leq \frac{\alpha^u}{c_0} < \alpha^{u-0.983}.$$

□

Lemma 5 *All positive integer solutions to the system (2) satisfy* $z \leq 2y - 1$.

Proof. The last two equations of the system (2) imply

$$c|\gcd(B_y - 1, B_z - 1). \tag{4}$$

Obviously, $B_z = bc + 1 < c^2$, hence $\sqrt{B_z} < c$. This, together with (4) gives $\sqrt{B_z} < B_y$. By (3) we obtain

$$\sqrt{\alpha^{z-0.9831}} < \sqrt{B_z} < B_y < \alpha^{y-0.983}.$$

It leads to

$$\alpha^{z-0.9831} < \alpha^{2y-1.966},$$

and then $z \le 2y - 1$. □

3 Proof of Theorem 2

Suppose that the integers $0 < a < b < c$ and $0 < x < y < z$ satisfy (2). Thus $1 \cdot 2 + 1 \le ab + 1 = B_x$ implies $2 \le x$. Thus $3 \le y$. The proof is split into two parts.

I. $z \le 449$.

In this case, we ran an exhaustive computer search to detect all positive integer solutions to the system (2). Observe that we have

$$a = \sqrt{\frac{(B_x - 1)(B_y - 1)}{(B_z - 1)}}, \qquad 2 \le x < y < z \le 449.$$

Going through all the eligible values for x, y and z, and checking if the above number a is an integer, we found no solution to the system (2).

II. $z > 449$.

Put $Q = \gcd(B_z - 1, B_y - 1)$. From the proof of Lemma 5 we know that $\sqrt{B_z} < Q$. Applying now Lemma 1,

$$\begin{aligned} Q &\le \gcd(B_{z-1}B_{z+1}, B_{y-1}B_{y+1}) \\ &\le \prod_{i,j\in\{\pm 1\}} \gcd(B_{z-i}, B_{y-j}) = \prod_{i,j\in\{\pm 1\}} B_{\gcd(z-i,y-j)}. \end{aligned} \tag{5}$$

Let $\gcd(z - i, y - j) = \frac{z-i}{k_{ij}}$. Suppose that $k_{ij} \ge 8$, for all the four possible pairs (i, j) in (5). Then Lemma 4, together with the previous two estimates, provides

$$\alpha^{\frac{z-0.9831}{2}} < \sqrt{B_z} < Q \le \left(B_{(z-1)/8}\right)^2 \left(B_{(z+1)/8}\right)^2 < \alpha^{4\cdot\left(\frac{z+1}{8} - 0.983\right)}$$

which leads to a contradiction if one compares the exponents of α.

Assume now that $k_{ij} \le 7$ fulfills for some i and j, let denote k this k_{ij}. Suppose further that

$$\frac{z-i}{k} = \frac{y-j}{l}$$

holds for a suitable positive integer l coprime to k.

If $l > k$, then according to $y < z$, the relation $z-i < y-j$ implies $z = y+1$. But this is impossible since

$$Q = \gcd(B_{y+1}-1, B_y-1) \le \gcd(B_{y+2}B_y, B_{y+1}B_{y-1}) = \gcd(B_{y+2}, B_{y-1}) \le B_3$$

follows in the virtue of Lemma 1. Thus

$$\alpha^{\frac{z-0.9831}{2}} < \sqrt{B_z} < Q \le B_3 = 35$$

leads to a contradiction by $z < 5.1$.

Suppose now that $k = l = 1$. Now $z-i = y-j$ can hold only if $z = y+2$. Thus, by Lemma 3, we have

$$Q = \gcd(B_{y+2}-1, B_y-1) \le 34 < B_3.$$

Hence, as in the previous part, we arrived at a contradiction.

In the sequel, we assume $l < k$. First suppose $3 \le k$. Taking any pair $(i_0, j_0) \ne (i,j)$ from the remaining three cases of $(-1,-1)$, $(-1,1)$, $(1,-1)$ and $(1,1)$, we have

$$y - j_0 = \frac{l}{k}(z-i) + j - j_0 = \frac{lz - li + kj - kj_0}{k}. \tag{6}$$

Thus

$$\begin{aligned} \gcd(z-i_0, y-j_0) &= \gcd\left(z-i_0, \frac{lz-li+kj-kj_0}{k}\right) \\ &\le \gcd(lz-li_0, lz-li+kj-kj_0) \\ &= \gcd(lz-li_0, li_0-li+kj-kj_0). \end{aligned}$$

Since $li_0 - li + kj - kj_0$ does not vanish, it follows that

$$\gcd(lz-li_0, li_0-li+kj-kj_0) \le |li_0-li+kj-kj_0| \le 2(k+l) \le 26.$$

Indeed, it is easy to see that $li_0 - li + kj - kj_0 = 0$, or equivalently $l(i_0-i) = k(j_0-j)$ leads to a contradiction since $2 \le k \le 7$ and $1 \le l \le k-1$ are coprime, further $i_0 - i$ and $j_0 - j$ are in the set $\{0, \pm 2\}$ meanwhile at least one of them is non-zero.

Then (5), together with Lemma 4, yields

$$\alpha^{\frac{z-0.9831}{2}} < B_{\frac{z+1}{3}} \cdot B_{26}^3 < \alpha^{\frac{z+1}{3}-0.983}\left(\alpha^{25.017}\right)^3.$$

Consequently, $z < 449.4$. It contradicts the condition separating Case 2 and 1.

Assume now that $k = k_{ij} = 2$ fulfills for some eligible pair (i, j). Thus $l = 1$. First suppose that $\gcd(z-1, y-1) = (z-1)/2$. It yields $z = 2y-1$, and we go back to the system

$$\begin{aligned} ab + 1 &= B_x, \\ ac + 1 &= B_y, \\ bc + 1 &= B_{2y-1}. \end{aligned}$$

First we obtain

$$\frac{B_{2y-1}}{B_y} = \frac{bc+1}{ac+1} < \frac{b}{a}$$

since $0 < a < b < c$. On the other hand, by Lemma 4,

$$\frac{B_{2y-1}}{B_y} > \frac{\alpha^{2y-1-0.9831}}{\alpha^{y-0.983}} = \alpha^{y-1.001}$$

follows. Consequently,

$$a\alpha^{y-1.001} < b,$$

and

$$a^2\alpha^{y-1.001} \le ab = B_x - 1 < B_x < \alpha^{x-0.983}.$$

Thus we arrived at a contradiction by

$$a^2 < \alpha^{x-y+0.018} \le \alpha^{-0.982} < 0.2.$$

If $\gcd(z-1, y+1) = (z-1)/2$ then $z = 2y+3$ contradicting Lemma 5. Similarly, $\gcd(z+1, y+1) = (z+1)/2$ leads to $z = 2y+1$. Finally, $\gcd(z+1, y-1) = (z+1)/2$ gives $z = 2y-3$, which is possible. But, in this case, by Lemma 3 we have

$$\alpha^{\frac{z-0.9831}{2}} < \sqrt{B_z} < c \le \gcd(B_{2y-3}-1, B_y-1) \le 1190,$$

and it results $z \le 9$ in the virtue of Lemma 4.

The proof of Theorem 2 is completed.

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The orthopole theorem in the Poincaré disc model of hyperbolic geometry

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Abstract. In this study we prove the orthopole theorem for a hyperbolic triangle.

1 Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of geometry. It is also known as a type of non-euclidean geometry, being in many respects similar to euclidean geometry. Hyperbolic geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. Several useful models of hyperbolic geometry are studied in the literature as, for instance, the Poincaré disc and ball models, the Poincaré half-plane model, and the Beltrami-Klein disc and ball models [5] etc. Following [8] and [9] and earlier discoveries, the Beltrami-Klein model is also known as the Einstein relativistic velocity model. Here, in this study, we give hyperbolic version of the orthopole theorem in the Poincaré disc model. The well-known orthopole theorem states that if A', B', C' be the projections of the vertices A, B, C of a triangle ABC on a straight line d, the perpendiculars from A' on

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BC, from B' on CA, and from C' on AB are concurrent at a point called the orthopole of d for the triangle ABC [4]. This result has a simple statement but it is of great interes. We just mention here few different proofs given by R. Goormaghtigh [3], J. Neuberg [6], W. Gallaty [2]. We use in this study the Poincaré disc model.

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z-plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the groupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus)$$

by the equation

$$gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then the following properties of $\oplus$ can be easy verified using algebraic calculation:

$a \oplus b = gyr[a, b](b \oplus a)$,	*gyrocommutative law*
$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c$,	*left gyroassociative law*
$(a \oplus b) \oplus c = a \oplus (b \oplus gyr[b, a]c)$,	*right gyroassociative law*
$gyr[a, b] = gyr[a \oplus b, b]$,	*left loop property*
$gyr[a, b] = gyr[a, b \oplus a]$,	*right loop property*

For more details, please see [7].

Definition 1 *The hyperbolic distance function in* D *is defined by the equation*

$$d(a, b) = |a \ominus b| = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

Here, $a \ominus b = a \oplus (-b)$, *for* $a, b \in D$.

Theorem 1 ***(The Möbius Hyperbolic Pythagorean Theorem)*** *Let* ABC *be a gyrotriangle in a Möbius gyrovector space* $(V_s, \oplus, \otimes)$, *with vertices* $A, B, C \in V_s$, *sides* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V_s}$ *and side gyrolenghts* $a, b, c \in (-s, s)$, $\mathbf{a} = -B \oplus C$, $\mathbf{b} = -C \oplus A$, $\mathbf{c} = -A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$ *and with gyroangles* α, β, *and* γ *at the vertices* A, B, *and* C. *If* $\gamma = \pi/2$, *then*

$$\frac{c^2}{s} = \frac{a^2}{s} \oplus \frac{b^2}{s}$$

(see [8, p. 290]).

For further details we refer to the recent book of A. Ungar [7].

Theorem 2 ***(Converse of Carnot's theorem for hyperbolic triangle)*** *Let* ABC *be a hyperbolic triangle in the Poincaré disc, whose vertices are the points* A, B *and* C *of the disc and whose sides (directed counterclockwise) are* $a = -B \oplus C, b = -C \oplus A$ *and* $c = -A \oplus B$. *Let the points* A', B' *and* C' *be located on the sides* a, b *and* c *of the hyperbolic triangle* ABC, *respectively. If the following holds*

$$\left|-A \oplus C'\right|^2 \ominus \left|-B \oplus C'\right|^2 \oplus \left|-B \oplus A'\right|^2 \ominus \left|-C \oplus A'\right|^2 \oplus \left|-C \oplus B'\right|^2 \ominus \left|-A \oplus B'\right|^2 = 0,$$

and two of the three perpendiculars to the sides of the hyperbolic triangle at the points A', B' *and* C' *are concurrent, then the three perpendiculars are concurrent (See [1]).*

2 Main results

In this section, we prove the orthopole theorem for a hyperbolic triangle.

Theorem 3 *Let* A', B', C' *be the projections of the vertices* A, B, C *of the gyrotriangle* ABC *on a straight gyroline* d. *If two of the three perpendiculars from* A' *on* BC, *from* B' *on* CA, *and from* C' *on* AB *are concurrent, then the three perpendiculars are concurrent.*

Proof. Let's note A'', B'', C'' the projections of the points A', B', C' on BC, CA, AB, respectively (See Figure 1).

If we use Theorem 1 in the gyrotriangles $AA'B'$ and $AA'C'$, we get

$$\left|-A \oplus B'\right|^2 = \left|-B' \oplus A'\right|^2 \oplus \left|-A' \oplus A\right|^2 \tag{1}$$

and

$$\left|-C' \oplus A\right|^2 = \left|-A \oplus A'\right|^2 \oplus \left|-A' \oplus C'\right|^2 \tag{2}$$

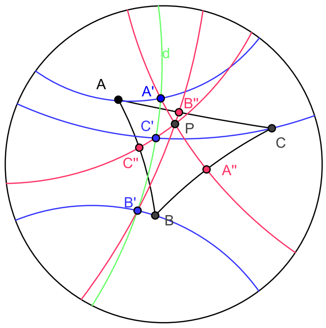

Figure 1: Projections of the points

Because $|-A' \oplus A|^2 = |-A \oplus A'|^2$, from the relations (1) and (2) we have

$$\left|-A \oplus B'\right|^2 \ominus \left|-B' \oplus A'\right|^2 = \left|-C' \oplus A\right|^2 \ominus \left|-A' \oplus C'\right|^2$$

i.e.

$$\alpha = \left|-A \oplus B'\right|^2 \ominus \left|-A \oplus C'\right|^2 = \left|-A' \oplus B'\right|^2 \ominus \left|-A' \oplus C'\right|^2 = \alpha' \tag{3}$$

Similary we prove that

$$\beta = \left|-B \oplus C'\right|^2 \ominus \left|-B \oplus A'\right|^2 = \left|-B' \oplus C'\right|^2 \ominus \left|-B' \oplus A'\right|^2 = \beta' \tag{4}$$

respectively

$$\gamma = \left|-C \oplus A'\right|^2 \ominus \left|-C \oplus B'\right|^2 = \left|-C' \oplus A'\right|^2 \ominus \left|-C' \oplus B'\right|^2 = \gamma'. \tag{5}$$

From the relations (3), (4) and (5) result

$$(\alpha \oplus \beta) \oplus \gamma = (\alpha' \oplus \beta') \oplus \gamma'.$$

Since $((-1, 1), \oplus)$ is a commutative group, we immediately obtain

$$\begin{aligned} &\left|-A \oplus B'\right|^2 \ominus \left|-A \oplus C'\right|^2 \oplus \left|-B \oplus C'\right|^2 \ominus \left|-B \oplus A'\right|^2 \\ &\oplus \left|-C \oplus A'\right|^2 \ominus \left|-C \oplus B'\right|^2 = 0. \end{aligned} \tag{6}$$

If we use the Theorem 1 in the gyrotriangles $AB'B''$, $AC'C''$, $BC'C''$, $BA'A''$, $CA'A''$ and $CB'B''$, we get

$$\left|-A \oplus B'\right|^2 = \left|-B' \oplus B''\right|^2 \oplus \left|-B'' \oplus A\right|^2, \tag{7}$$

$$\left|-A \oplus C'\right|^2 = \left|-C' \oplus C''\right|^2 \oplus \left|-C'' \oplus A\right|^2, \tag{8}$$

$$\left|-B \oplus C'\right|^2 = \left|-C' \oplus C''\right|^2 \oplus \left|-C'' \oplus B\right|^2, \tag{9}$$

$$\left|-B \oplus A'\right|^2 = \left|-A' \oplus A''\right|^2 \oplus \left|-A'' \oplus B\right|^2, \tag{10}$$

$$\left|-C \oplus A'\right|^2 = \left|-A' \oplus A''\right|^2 \oplus \left|-A'' \oplus C\right|^2, \tag{11}$$

$$\left|-C \oplus B'\right|^2 = \left|-B' \oplus B''\right|^2 \oplus \left|-B'' \oplus C\right|^2. \tag{12}$$

Now, from the relations (6) - (12), result

$$\begin{aligned}\left|-A \oplus B''\right|^2 \ominus \left|-A \oplus C''\right|^2 \oplus \left|-B \oplus C''\right|^2 \ominus \left|-B \oplus A''\right|^2 \oplus \left|-C \oplus A''\right|^2 \\ \ominus \left|-C \oplus B''\right|^2 = 0,\end{aligned}$$

and by Theorem 2 we obtain that the gyrolines $A'A'', B'B''$, and $C'C''$ are concurrent. $\square$

Many of the theorems of Euclidean geometry have relatively similar form in the Poincare disc model, the orthopole theorem for a hyperbolic triangle is an example in this respect.

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Mills' ratio: Reciprocal convexity and functional inequalities

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Dedicated to my children Boróka and Koppány

Abstract. This note contains sufficient conditions for the probability density function of an arbitrary continuous univariate distribution, supported on $(0,\infty)$, such that the corresponding Mills ratio to be reciprocally convex (concave). To illustrate the applications of the main results, the reciprocal convexity (concavity) of Mills ratio of the gamma distribution is discussed in details.

1 Introduction

By definition (see [7]) a function $f:(0,\infty)\to\mathbb{R}$ is said to be (strictly) reciprocally convex if $x\mapsto f(x)$ is (strictly) concave and $x\mapsto f(1/x)$ is (strictly) convex on $(0,\infty)$. Merkle [7] showed that f is reciprocally convex if and only if for all $x,y>0$ we have

$$f\left(\frac{2xy}{x+y}\right)\leq\frac{f(x)+f(y)}{2}\leq f\left(\frac{x+y}{2}\right)\leq\frac{xf(x)+yf(y)}{x+y}. \tag{1}$$

We note here that in fact the third inequality follows from the fact that the function $x\mapsto f(1/x)$ is convex on $(0,\infty)$ if and only if $x\mapsto xf(x)$ is convex on $(0,\infty)$. In what follows, similarly as in [7], a function $g:(0,\infty)\to\mathbb{R}$ is said

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to be (strictly) reciprocally concave if and only if $-g$ is (strictly) reciprocally convex, i.e. if $x \mapsto g(x)$ is (strictly) convex and $x \mapsto g(1/x)$ is (strictly) concave on $(0, \infty)$. Observe that if f is differentiable, then $x \mapsto f(1/x)$ is (strictly) convex (concave) on $(0, \infty)$ if and only if $x \mapsto x^2 f'(x)$ is (strictly) increasing (decreasing) on $(0, \infty)$.

As it was shown by Merkle [7], reciprocally convex functions defined on $(0, \infty)$ have a number of interesting properties: they are increasing on $(0, \infty)$ or have a constant value on $(0, \infty)$, they have a continuous derivative on $(0, \infty)$ and they generate a sequence of quasi-arithmetic means, with the first one between harmonic and arithmetic mean and others above the arithmetic mean. Some examples of reciprocally convex functions related to the Euler gamma function were given in [7].

By definition (see [9]) a function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotonic, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0$$

for all $x > 0$ and $n \in \{0, 1, \dots\}$. Note that strict inequality always holds above unless f is constant. It is known (Bernstein's Theorem) that f is completely monotonic if and only if [9, p. 161]

$$f(x) = \int_0^\infty e^{-xt} d\nu(t),$$

where ν is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. An important subclass of completely monotonic functions consists of the Stieltjes transforms defined as the class of functions $g : (0, \infty) \to \mathbb{R}$ of the form

$$g(x) = \alpha + \int_0^\infty \frac{d\nu(t)}{x + t},$$

where $\alpha \geq 0$ and ν is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$.

It was pointed out in [7] that if a function $h : (0, \infty) \to \mathbb{R}$ is a Stieltjes transform, then $-h$ is reciprocally convex, i.e. h is reciprocally concave. We note that some known reciprocally concave functions comes from probability theory. For example, the Mills ratio of the standard normal distribution is a reciprocally concave function. For this let us see some basics. The probability density function $\varphi : \mathbb{R} \to (0, \infty)$, the cumulative distribution function $\Phi : \mathbb{R} \to (0, 1)$ and the reliability function $\overline{\Phi} : \mathbb{R} \to (0, 1)$ of the standard normal law, are defined by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

$$\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt$$

and

$$\overline{\Phi}(x) = 1 - \Phi(x) = \int_{x}^{\infty} \varphi(t)dt.$$

The function $m : \mathbb{R} \to (0, \infty)$, defined by

$$m(x) = \frac{\overline{\Phi}(x)}{\varphi(x)} = e^{x^2/2} \int_{x}^{\infty} e^{-t^2/2}dt = \int_{0}^{\infty} e^{-xt}e^{-t^2/2}dt,$$

is known in literature as Mills' ratio [8, Sect. 2.26] of the standard normal law, while its reciprocal $r = 1/m$, defined by $r(x) = 1/m(x) = \varphi(x)/\overline{\Phi}(x)$, is the so-called failure (hazard) rate. For Mills' ratio of other distributions, like gamma distribution, we refer to [6] and to the references therein.

It is well-known that Mills' ratio of the standard normal distribution is convex and strictly decreasing on $\mathbb{R}$, at the origin takes on the value $m(0) = \sqrt{\pi/2}$. Moreover, it can be shown (see [2]) that $x \mapsto m'(x)/m(x)$ is strictly increasing and $x \mapsto x^2 m'(x)$ is strictly decreasing on $(0, \infty)$. With other words, the Mills ratio of the standard normal law is strictly reciprocally concave on $(0, \infty)$. Some other monotonicity properties and interesting functional inequalities involving the Mills ratio of the standard normal distribution can be found in [2]. The following complements the above mentioned results.

Theorem 1 *Let* m *be the Mills ratio of the standard normal law. Then the function* $x \mapsto m(\sqrt{x})/\sqrt{x}$ *is a Stieltjes transform and consequently it is strictly completely monotonic and strictly reciprocally concave on* $(0, \infty)$. *In particular, if* $x, y > 0$, *then the following chain of inequalities holds*

$$\sqrt{\frac{x+y}{2xy}}\, m\left(\sqrt{\frac{2xy}{x+y}}\right) \geq \frac{\sqrt{y}m(\sqrt{x}) + \sqrt{x}m(\sqrt{y})}{2\sqrt{xy}}$$
$$\geq \sqrt{\frac{2}{x+y}}\, m\left(\sqrt{\frac{x+y}{2}}\right) \geq \frac{\sqrt{x}m(\sqrt{x}) + \sqrt{y}m(\sqrt{y})}{x+y}.$$

In each of the above inequalities equality holds if and only if $x = y$.

Proof. For $x > 0$ the Mills of the standard normal distribution can be represented as [5, p. 145]

$$m(x) = \int_{-\infty}^{\infty} \frac{x}{x^2+t^2}\varphi(t)dt = 2\int_{0}^{\infty} \frac{x}{x^2+t^2}\varphi(t)dt.$$

From this we obtain that

$$\frac{m(\sqrt{x})}{\sqrt{x}} = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{1}{x+s}\frac{e^{-s/2}}{\sqrt{s}}ds,$$

which shows that the function $x \mapsto m(\sqrt{x})/\sqrt{x}$ is in fact a Stieltjes transform and owing to Merkle [7, p. 217] this implies that the function $x \mapsto -m(\sqrt{x})/\sqrt{x}$ is reciprocally convex on $(0,\infty)$, i.e. the function $x \mapsto m(\sqrt{x})/\sqrt{x}$ is reciprocally concave on $(0,\infty)$. The rest of the proof follows easily from (1). We note that the strictly complete monotonicity of the function $x \mapsto m(\sqrt{x})/\sqrt{x}$ can be proved also by using the properties of completely monotonic functions. Mills ratio m of the standard normal distribution is in fact a Laplace transform and consequently it is strictly completely monotonic (see [2]). On the other hand, it is known (see [9]) that if u is strictly completely monotonic and v is nonnegative with a strictly completely monotone derivative, then the composite function $u\circ v$ is also strictly completely monotonic. Now, since the function m is strictly completely monotonic on $(0,\infty)$ and $x \mapsto 2(\sqrt{x})' = 1/\sqrt{x}$ is strictly completely monotonic on $(0,\infty)$, we obtain that $x \mapsto m(\sqrt{x})$ is also strictly completely monotonic on $(0,\infty)$. Finally, by using the fact that the product of completely monotonic functions is also completely monotonic, the function $x \mapsto m(\sqrt{x})/\sqrt{x}$ is indeed strictly completely monotonic on $(0,\infty)$. $\square$

Now, since the Mills ratio of the standard normal distribution is reciprocally concave a natural question which arises here is the following: under which conditions does the Mills ratio of an arbitrary continuous univariate distribution, having support $(0,\infty)$, will be reciprocally convex (concave)? The goal of this paper is to find some sufficient conditions for the probability density function of an arbitrary continuous univariate distribution, supported on the semi-infinite interval $(0,\infty)$, such that the corresponding Mills ratio to be reciprocally convex (concave). The main result of this paper, namely Theorem 2 in section 2, is based on some recent results of the author [3] and complement naturally the results from [2, 3]. To illustrate the application of the main result, the Mills ratio of the gamma distribution is discussed in details in section 3.

We note that although the reciprocal convexity (concavity) of Mills ratio is interesting in his own right, the convexity of the Mills ratio of continuous distributions has important applications in monopoly theory, especially in static pricing problems. For characterizations of the existence or uniqueness of global maximizers we refer to [4] and to the references therein. Another application can be found in [10], where the convexity of Mills ratio is used to show that the price is a sub-martingale.

2 Reciprocal convexity (concavity) of Mills ratio

In this section our aim is to find some sufficient conditions for the probability density function such that the corresponding Mills ratio to be reciprocally convex (concave). As in [3] the proof is based on the monotone form of l'Hospital's rule [1, Lemma 2.2].

Theorem 2 *Let* $f : (0, \infty) \to (0, \infty)$ *be a probability density function and let* $\omega : (0, \infty) \to \mathbb{R}$, *defined by* $\omega(x) = f'(x)/f(x)$, *be the logarithmic derivative of* f. *Let also* $\overline{F} : (0, \infty) \to (0, 1)$, *defined by* $\overline{F}(x) = \int_x^\infty f(t)dt$, *be the survival function and* $m : (0, \infty) \to (0, \infty)$, *defined by* $m(x) = \overline{F}(x)/f(x)$, *be the corresponding Mills ratio. Then the following assertions are true:*

(a) *If* $f(x)/\omega(x) \to 0$ *as* $x \to \infty$, ω'/ω^2 *is (strictly) decreasing (increasing) on* $(0, \infty)$ *and the function*

$$x \mapsto \frac{x^3\omega'(x)}{x\omega^2(x) - x\omega'(x) - 2\omega(x)}$$

is (strictly) increasing (decreasing) on $(0, \infty)$, *then Mills ratio* m *is (strictly) reciprocally convex (concave) on* $(0, \infty)$.

(b) *If* $xf(x)/(1 - x\omega(x)) \to 0$, $f(x)/\omega(x) \to 0$ *as* $x \to \infty$, ω'/ω^2 *is (strictly) decreasing (increasing) on* $(0, \infty)$ *and the function*

$$x \mapsto \frac{x^2\omega'(x) - x\omega(x) + 2}{x\omega^2(x) - x\omega'(x) - 2\omega(x)}$$

is (strictly) increasing (decreasing) on $(0, \infty)$, *then Mills ratio* m *is (strictly) reciprocally convex (concave) on* $(0, \infty)$.

Proof. (a) By definition Mills ratio m is (strictly) reciprocally convex (concave) if m is (strictly) concave (convex) and $x \mapsto m(1/x)$ is (strictly) convex (concave). It is known (see [3, Theorem 2]) that if $f(x)/\omega(x)$ tends to zero as x tends to infinity and the function ω'/ω^2 is (strictly) increasing (decreasing), then m is (strictly) convex (concave). Thus, we just need to find conditions for the (strict) convexity (concavity) of the function $x \mapsto m(1/x)$. This function is (strictly) convex (concave) on $(0, \infty)$ if and only if the function $x \mapsto x^2m'(x)$ is (strictly) increasing (decreasing) on $(0, \infty)$. On the other hand, observe that Mills ratio m satisfies the differential equation

$$m'(x) = -\omega(x)m(x) - 1.$$

Thus, by using the monotone form of l'Hospital's rule (see [1, Lemma 2.2]) to prove that the function

$$x \mapsto x^2 m'(x) = -\frac{\left(\overline{F}(x) + f(x)/\omega(x)\right) - \lim\limits_{x\to\infty}\left(\overline{F}(x) + f(x)/\omega(x)\right)}{f(x)/(x^2\omega(x)) - \lim\limits_{x\to\infty} f(x)/(x^2\omega(x))}$$
$$= -\frac{\overline{F}(x) + f(x)/\omega(x)}{f(x)/(x^2\omega(x))}$$

is (strictly) increasing (decreasing) on $(0,\infty)$ it is enough to show that

$$x \mapsto -\frac{\left(\overline{F}(x) + f(x)/\omega(x)\right)'}{\left(f(x)/(x^2\omega(x))\right)'} = \frac{x^3\omega'(x)}{x\omega^2(x) - x\omega'(x) - 2\omega(x)}$$

is (strictly) increasing (decreasing) on $(0,\infty)$.

(b) Observe that according to [7, Lemma 2.2] the function $x \mapsto m(1/x)$ is (strictly) convex (concave) if and only if $x \mapsto xm(x)$ is (strictly) convex (concave) on $(0,\infty)$. Now, by using the monotone form of l'Hospital's rule (see [1, Lemma 2.2]) the function

$$x \mapsto (xm(x))' = m(x) - x - x\omega(x)m(x) = \frac{\overline{F}(x) - xf(x)/(1 - x\omega(x))}{f(x)/(1 - x\omega(x))}$$
$$= \frac{\left(\overline{F}(x) - xf(x)/(1 - x\omega(x))\right) - \lim\limits_{x\to\infty}\left(\overline{F}(x) - xf(x)/(1 - x\omega(x))\right)}{\left(f(x)/(1 - x\omega(x))\right) - \lim\limits_{x\to\infty}\left(f(x)/(1 - x\omega(x))\right)}$$

is (strictly) increasing (decreasing) on $(0,\infty)$ if the function

$$x \mapsto \frac{\left(\overline{F}(x) - xf(x)/(1 - x\omega(x))\right)'}{\left(f(x)/(1 - x\omega(x))\right)'} = \frac{x^2\omega'(x) - x\omega(x) + 2}{x\omega^2(x) - x\omega'(x) - 2\omega(x)}$$

is (strictly) increasing (decreasing) on $(0,\infty)$. Note that we used tacitly the fact that if $xf(x)/(1 - x\omega(x)) \to 0$ as $x \to \infty$, then $f(x)/(1 - x\omega(x)) \to 0$ as $x \to \infty$. $\square$

We note here that the reciprocal concavity of the Mills ratio of the standard normal distribution can be verified easily by using part (a) or part (b) of Theorem 2. More precisely, in the case of the standard normal distribution we have $\omega(x) = -x$, $\omega'(x) = -1$. Consequently $\varphi(x)/\omega(x) = -\varphi(x)/x \to 0$ as $x \to \infty$, the function $x \mapsto \omega'(x)/\omega^2(x) = -1/x^2$ is strictly increasing and

$$x \mapsto \frac{x^3\omega'(x)}{x\omega^2(x) - x\omega'(x) - 2\omega(x)} = -\frac{x^2}{x^2 + 3}$$

is strictly decreasing on $(0,\infty)$. This is turn implies that by using part (a) of Theorem 2 the Mills ratio of the standard normal distribution is strictly reciprocally concave on $(0,\infty)$.

Similarly, since $\varphi(x)/(1+x^2)\to 0$, $x\varphi(x)/(1+x^2)\to 0$, $-\varphi(x)/x\to 0$ as $x\to\infty$, the function $x\mapsto \omega'(x)/\omega^2(x) = -1/x^2$ is strictly increasing and

$$x\mapsto \frac{x^2\omega'(x)-x\omega(x)+2}{x\omega^2(x)-x\omega'(x)-2\omega(x)} = \frac{2}{x^3+3x}$$

is strictly decreasing on $(0,\infty)$, part (b) of Theorem 2 also implies that the Mills ratio of the standard normal distribution is strictly reciprocally concave on $(0,\infty)$.

Thus, Theorem 2 in fact generalizes some of the main results of [2].

3 Reciprocal convexity (concavity) of Mills ratio of the gamma distribution

The gamma distribution has support $(0,\infty)$, probability density function, cumulative distribution function and survival function as follows

$$f(x) = f(x;\alpha) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)},$$

$$F(x) = F(x;\alpha) = \frac{\gamma(\alpha,x)}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-t}dt$$

and

$$\overline{F}(x) = \overline{F}(x;\alpha) = \frac{\Gamma(\alpha,x)}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}\int_x^\infty t^{\alpha-1}e^{-t}dt,$$

where Γ is the Euler gamma function, $\gamma(\cdot,\cdot)$ and $\Gamma(\cdot,\cdot)$ denote the lower and upper incomplete gamma functions, and $\alpha>0$ is the shape parameter. As we can see below, *the Mills ratio of the gamma distribution* $m:(0,\infty)\to(0,\infty)$, defined by

$$m(x) = m(x;\alpha) = \frac{\Gamma(\alpha,x)}{x^{\alpha-1}e^{-x}},$$

is reciprocally convex on $(0,\infty)$ *for all* $0<\alpha\leq 1$ *and reciprocally concave on* $(0,\infty)$ *for all* $1\leq\alpha\leq 2$. In [3] it was proved that if $\alpha\geq 1$, then the Mills ratio m is decreasing and log-convex, and consequently convex on $(0,\infty)$. We note that the convexity of Mills ratio of the gamma distribution actually can be verified directly (see [10]), since

$$m(x) = \int_x^{\infty} \left(\frac{t}{x}\right)^{\alpha-1} e^{x-t} dt = \int_1^{\infty} x u^{\alpha-1} e^{(1-u)x} du$$

and

$$\begin{aligned} m'(x) &= \int_1^{\infty} \left((\alpha-1)u^{\alpha-2}\right)\left((1-u)e^{(1-u)x}\right) du \\ &= \int_1^{\infty} u^{\alpha-1} e^{(1-u)x} du + \int_1^{\infty} x u^{\alpha-1}(1-u)e^{(1-u)x} du, \end{aligned}$$

where the last equality follows from integration by parts. From this we clearly have that

$$m''(x) = \int_1^{\infty} (\alpha-1)(1-u)^2 u^{\alpha-2} e^{(1-u)x} du$$

and consequently m is convex on $(0,\infty)$ if $\alpha \geq 1$ and is concave on $(0,\infty)$ if $0 < \alpha \leq 1$. The concavity of the function m can be verified also by using [3, Theorem 2]. Namely, if let $\omega(x) = f'(x)/f(x) = (\alpha-1)/x - 1$, then $f(x)/\omega(x)$ tends to zero as x tends to infinity and the function $x \mapsto \omega'(x)/\omega^2(x) = (1-\alpha)/(\alpha-1-x)^2$ is decreasing on $(0,\infty)$ for all $0 < \alpha \leq 1$. Consequently in view of [3, Theorem 2] m is indeed concave on $(0,\infty)$ for all $0 < \alpha \leq 1$.

Now let us focus on the reciprocal convexity (concavity) of the Mills ratio of gamma distribution. Since

$$\frac{x^3\omega'(x)}{x\omega^2(x) - x\omega'(x) - 2\omega(x)} = \frac{(1-\alpha)x^2}{(\alpha-1-x)^2 + 2x + 1 - \alpha},$$

we obtain that

$$\left(\frac{x^3\omega'(x)}{x\omega^2(x) - x\omega'(x) - 2\omega(x)}\right)' = \frac{2(\alpha-1)(\alpha-2)(x^2 + (1-\alpha)x)}{((\alpha-1-x)^2 + 2x + 1 - \alpha)^2}.$$

This last expression is clearly positive if $0 < \alpha \leq 1$ and $x > 0$, and thus, by using part (a) of Theorem 2 we conclude that Mills ratio m is reciprocally convex on $(0,\infty)$ for all $0 < \alpha \leq 1$.

Similarly, since

$$\frac{x^2\omega'(x) - x\omega(x) + 2}{x\omega^2(x) - x\omega'(x) - 2\omega(x)} = \frac{x^2 + 2(2-\alpha)x}{x^2 + 2(2-\alpha)x + (\alpha-1)(\alpha-2)},$$

we get

$$\left(\frac{x^2\omega'(x) - x\omega(x) + 2}{x\omega^2(x) - x\omega'(x) - 2\omega(x)}\right)' = \frac{2(\alpha-1)(\alpha-2)(x+2-\alpha)}{(x^2+2(2-\alpha)x+(\alpha-1)(\alpha-2))^2}$$

and this is negative if $1 \leq \alpha \leq 2$ and $x > 0$. Consequently, by using part (b) of Theorem 2 we get that the Mills ratio of the gamma distribution is indeed reciprocally concave for $1 \leq \alpha \leq 2$. Here we used that if x tends to ∞, then the expressions $f(x)/\omega(x)$ and $xf(x)/(1 - x\omega(x))$ tend to 0.

Finally, we note that the convexity (concavity) of $x \mapsto m(1/x)$ can be verified also by using the integral representation of Mills ratio of the gamma distribution. More precisely, if we rewrite $m(x)$ as

$$m(x) = \int_0^\infty \left(1 + \frac{u}{x}\right)^{\alpha-1} e^{-u} du,$$

then

$$x^2 m'(x) = -\int_0^\infty (\alpha-1)\left(1 + \frac{u}{x}\right)^{\alpha-2} u e^{-u} du$$

and

$$[x^2 m'(x)]' = \int_0^\infty (\alpha-1)(\alpha-2)\left(1 + \frac{u}{x}\right)^{\alpha-3} \frac{u^2}{x^2} e^{-u} du.$$

This shows that $x \mapsto x^2 m'(x)$ is decreasing on $(0, \infty)$ if $1 \leq \alpha \leq 2$ and increasing on $(0, \infty)$ if $0 < \alpha \leq 1$ or $\alpha \geq 2$. Summarizing, the Mills ratio of the gamma distribution is reciprocally convex on $(0, \infty)$ if $0 < \alpha \leq 1$ and reciprocally concave on $(0, \infty)$ if $1 \leq \alpha \leq 2$. When $\alpha > 2$ the functions $x \mapsto m(x)$ and $x \mapsto m(1/x)$ are convex on $(0, \infty)$, thus in this case m is nor reciprocally convex and neither reciprocally concave on its support.

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Existence and generalized duality of strong vector equilibrium problems

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Abstract. In this paper, with the help of the duality operator and K. Fan's lemma, we present existence results for strong vector equilibrium problems, under pseudomonotonicity assumptions and without any pseudomonotonicity assumptions, respectively. Then, as an application, the main results allow us to state existence theorems for strong vector variational inequality problems.

1 Introduction and mathematical tools

Let A be a nonempty subset of a topological space E, let C be a nontrivial pointed convex cone of a real topological linear space Z, and let $\varphi : A \times A \to Z$ be a given bifunction. In [1], the scalar equilibrium problem was extended to vector-valued bifunctions in the following way:

$$\text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in A. \tag{VEP}$$

Throughout this paper we deal with (VEP), which is called *the strong vector equilibrium problem.* In the last decade, the study of strong vector equilibrium problems and their particular cases received a special attention from many authors, see, for instance: [1, 3, 7, 8, 12, 14, 15, 16, 24, 26].

Most of the existence results for vector equilibrium problems, are given in the hypothesis of a cone with nonempty interior, but, there are important

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ordered topological linear spaces whose ordering cones have an empty interior. For example, when $Z := L^p(T, \mu)$, where (T, μ) is a σ-finite measure space and $p \in [1, +\infty[$, the cone

$$C := \{u \in L^p(T, \mu) \mid u(t) \geq 0 \text{ a.e. in } [0, T]\}$$

has an empty interior. Therefore, for optimization problems stated in infinite dimensional spaces, several generalized interior-point conditions were given in order to assure strong duality. To this purpose some generalizations of the classical interior have been introduced (see, for example, [5, 6, 17, 20, 25]).

Another way to overcome this problem is to introduce approximative solutions. H. W. Kuhn and A. W. Tucker [22] observed that some efficient solutions of optimization problems are not satisfactorily characterized by a scalar minimization problem. To eliminate such anomalous efficient points various concepts of proper efficiency for optimization problems have been introduced (see, for instance, [4, 13, 18, 19]).

The aim of this paper is to present new existence results for (VEP) without any solidness assumption for the cone C.

The paper is organized as follows. In Section 2 with the help of the duality operator, we attach to problem (VEP) a generalized dual strong vector equilibrium problem. By introducing a new generalization of the maximal g-pseudomonotonicity due to W. Oettli [24], new existence results for the strong vector equilibrium problem (VEP) are given. To see which convexity notion satisfies assumption (iv) of Theorem 1, in Corollary 1 this assumption is replaced by the one in which we demand the C-quasiconvexity of the vector-function $\varphi(a, \cdot)$ for all $a \in A$.

Then, in Section 3, considering two particular cases of the duality operator, we present existence results for (VEP) under pseudomonotonicity assumptions, respectively, without any pseudomonotonicity assumptions. The results allow us to recover already established results from the literature. For example we recover results from [10] and [24].

Section 4 is devoted to applications. Thus, we give existence results for strong vector variational inequalities under pseudomonotonicity assumptions, and without any pseudomonotonicity assumptions, respectively. In Example 1 is showed that the set of operators which satisfies the assumptions of Theorem 2 is nonempty. Furthermore, from this example we see that there exist maximal pseudomonotone operators which are not strongly pseudomonotone in the sense of Definition 5.

In what follows, by using Ky Fan's lemma we will present new existence results for (VEP) without any solidness assumption on the cone C.

Definition 1 *Let A be a nonempty subset of a real topological linear space E. A multifunction $T : A \to 2^A$ is said to be a KKM-operator if, for every finite subset $\{a_1, a_2, \ldots, a_n\}$ of A, the following inclusion holds:*

$$co\{a_1, a_2, \ldots, a_n\} \subseteq \bigcup_{i=1}^{n} T(a_i).$$

In finite dimensional spaces the next lemma was given by B. Knaster, C. Kuratowski, S. Mazurkiewicz in [21], while in infinite dimensional spaces it was established by Ky Fan.

Lemma 1 ([9]) *Let A be a nonempty subset of a real Hausdorff topological linear space E, and let $T : A \to 2^A$ be a KKM-operator satisfying the following conditions:*

(i) $T(a)$ *is closed for all* $a \in A$*;*

(ii) *there is* $\bar{a} \in A$ *such that* $T(\bar{a})$ *is a compact set.*

Then

$$\bigcap_{a \in A} T(a) \neq \emptyset.$$

Now, let us recall the following weakened convexity notion, which can be found, by example, in [11].

Definition 2 Let E and Z be real topological linear spaces, let A be a nonempty subset of E, and let $C \subseteq Z$ be a convex cone. A function $f : A \to Z$ is said to be C-quasiconvex if A is convex and, for all $a_1, a_2 \in A$ and all $\lambda \in [0, 1]$, we have

$$f(\lambda a_1 + (1 - \lambda) a_2) \leq_C f(a_1)$$

or

$$f(\lambda a_1 + (1 - \lambda) a_2) \leq_C f(a_2).$$

2 Existence results via Ky Fan's lemma

From now on, E is considered to be a real Hausdorff topological linear space, $A \subseteq E$ is a nonempty convex subset, and C is a pointed convex cone of the real topological linear space Z.

With the help of an operator, we attach to problem (VEP) a dual problem. Let $\mathcal{D}$ be an operator from $\mathcal{F}(A, Z) := \{\psi \mid \psi : A \times A \to Z\}$ into itself, which is called the duality operator. In fact, $\mathcal{D}$ is a set of fixed rules applied to problem (VEP). By means of $\mathcal{D}$ we introduce the following *generalized dual strong vector equilibrium problem*:

$$\text{find } \bar{a} \in A \text{ such that } \mathcal{D}(\varphi)(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in A. \tag{DVEP}$$

The following proposition shows that, under a certain hypothesis, the generalized dual of this dual problem becomes the initial problem. Its proof is straightforward.

Proposition 1 *If*

$$\mathcal{D} \circ \mathcal{D}(\varphi) = \varphi,$$

then the generalized dual problem of (DVEP) *is problem* (VEP)*.*

Let $G : A \times A \to Z$ be defined by

$$G(a, b) := -\mathcal{D}(\varphi)(b, a) \text{ for all } a, b \in A.$$

In this framework, problem (DVEP) can be written as:

$$\text{find } \bar{a} \in A \text{ such that } G(b, \bar{a}) \notin C \setminus \{0\} \text{ for all } b \in A. \tag{GVEP}$$

The next notions are generalizations of the g-monotonicity and maximal g-monotonicity, respectively, introduced by W. Oettli [24] in the scalar case.

Definition 3 The bifunction $\varphi : A \times A \to Z$ is said to be:

(i) G-pseudomonotone if, for all $a, b \in A$,

$$\varphi(a, b) \notin -C \setminus \{0\} \text{ implies } G(b, a) \notin C \setminus \{0\};$$

(ii) maximal G-pseudomonotone if it is G-pseudomonotone and, for all $a, b \in A$,

$$G(x, a) \notin C \setminus \{0\} \text{ for all } x \in]a, b] \text{ implies } \varphi(a, b) \notin -C \setminus \{0\}.$$

Proposition 2 *If* $\varphi : A \times A \to Z$ *is maximal* G*-pseudomonotone, then the sets of solutions of problems* (VEP) *and* (GVEP) *coincide.*

Proof. Let $\bar{a} \in A$ be a solution of problem (VEP), i.e.

$$\varphi(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in A.$$

By the G-pseudomonotonicity of φ we deduce that $G(b, \bar{a}) \notin C \setminus \{0\}$ for all $b \in A$, which assures that $\bar{a}$ is a solution of problem (GVEP).

For the converse inclusion, suppose that $\bar{a} \in A$ is a solution of problem (GVEP), i.e.

$$G(b, \bar{a}) \notin C \setminus \{0\} \text{ for all } b \in A.$$

Take $b \in A$. Thus, by the convexity of the set A we have $]\bar{a}, b] \subseteq A$. Therefore

$$G(x, \bar{a}) \notin C \setminus \{0\} \text{ for all } x \in]\bar{a}, b].$$

Since φ is maximal G-pseudomonotone, we get $\varphi(\bar{a}, b) \notin -C \setminus \{0\}$. Taking into account that $b \in A$ was arbitrarily chosen, it results that $\bar{a}$ is a solution of problem (VEP).

Now, let us consider the case when the set of solutions of problem (GVEP) is empty, i.e. for any $a \in A$ there exists $b_a \in A$ such that

$$G(b_a, a) \in C \setminus \{0\}.$$

Due to fact that φ is maximal G-pseudomonotone we have

$$\varphi(a, b_a) \in -C \setminus \{0\},$$

which means that the set of solutions of problem (VEP) is the empty set. Thus, we proved that the sets of solutions of problems (GVEP) and (VEP) coincide also for this particular case, and this completes the proof. □

By using the dual formulation (GVEP) of problem (VEP) we obtain the following existence results for solutions of problem (VEP).

Theorem 1 *Suppose that the bifunctions* $\varphi : A \times A \to Z$ *and* $G : A \times A \to Z$ *satisfy the following conditions:*

(i) $\varphi(a, a) \in C$ *for all* $a \in A$*;*

(ii) φ *is maximal* G*-pseudomonotone;*

(iii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid G(b, a) \notin C \setminus \{0\}\}$ *is closed;*

(iv) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a, b) \in -C \setminus \{0\}\}$ *is convex;*

(v) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) \in -C \setminus \{0\} \textit{ for all } x \in A \setminus D.$$

Then problem (VEP) *admits a solution.*

Proof. First, we show that the multifunction $T : A \to 2^A$, defined by

$$T(b) := \mathrm{cl}\{a \in A \mid \varphi(a, b) \notin -C \setminus \{0\}\},$$

is a KKM-operator. In view of assumption (i), it results that the set $T(b)$ is nonempty for each $b \in A$.

By contradiction, we suppose that T is not a KKM-operator, i.e. there exist a finite subset $\{b_1, b_2, \dots, b_n\}$ of A and numbers $\lambda_1, \dots, \lambda_n \geq 0$ with $\lambda_1 + \cdots + \lambda_n = 1$ such that

$$\bar{b} := \sum_{i=1}^{n} \lambda_i b_i \notin T(b_j) \text{ for all } j \in \{1, 2, \dots, n\}.$$

This relation gives

$$\varphi(\bar{b}, b_j) \in -C \setminus \{0\} \text{ for all } j \in \{1, \dots, n\}.$$

So, $b_j \in W(\bar{b})$ for all $j \in \{1, \dots, n\}$. But, by assumption (iv), the set $W(\bar{b})$ is convex. Consequently, it follows that $\bar{b} \in W(\bar{b})$. This is a contradiction to assumption (i).

Assumption (v) assures the existence of an element $\tilde{b} \in D$ such that

$$\varphi(x, \tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Thus, $T(\tilde{b}) \subseteq D$. Because D is compact and $T(\tilde{b})$ is closed, it follows that $T(\tilde{b})$ is a compact set. The assumptions of Lemma 1 are satisfied and, by this we get the existence of a point $\bar{a} \in D$ such that $\bar{a} \in T(b)$ for all $b \in A$. The G-pseudomonotonicity of φ and the closedness of the set $S(b)$ imply that

$$T(b) \subseteq S(b) \text{ for all } b \in A.$$

Therefore, we obtain $\bar{a} \in S(b)$ for all $b \in A$, i.e. $\bar{a}$ is a solution of problem (GVEP). By Proposition 2, $\bar{a}$ is a solution of problem (VEP). □

Remark 1 It is worth to underline that our result is different from Theorem 1 established in [1]. Assumption (i) of Theorem 1 is stronger than condition (i) of Theorem 1 from [1] (we just have to take $B = A$ and T to be the identity operator in condition (i) of Theorem 1 of [1]), while our coercivity condition is weaker than the compactness assumption for the set A. Further, the part of the maximal G-pseudomonotonicity which assures the inclusion between the set of solutions of the dual problem and the set of solution of the initial problem, is different from the one considered in condition (v) of the Theorem 1 from [1].

Corollary 1 *Suppose that the bifunctions* $\varphi : A \times A \to Z$ *and* $G : A \times A \to Z$ *satisfy the following conditions:*

(i) $\varphi(a, a) \in C$ *for all* $a \in A$*;*

(ii) φ *is maximal* G*-pseudomonotone;*

(iii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid G(b, a) \notin C \setminus \{0\}\}$ *is closed;*

(iv) *for each* $a \in A$*, the function* $\varphi(a, \cdot) : A \to Z$ *is* C*-quasiconvex;*

(v) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem (VEP) *admits a solution.*

Proof. For the proof of this corollary, we have to show that the assumptions of Theorem 1 are satisfied. It is obvious that the assumptions (i), (ii), (iii) and (v) are satisfied. To verify assumption (iv), fix $a \in A$, and let $b_1, b_2 \in A$ and $\lambda \in [0, 1]$ be such that $b_1, b_2 \in W(a)$, i.e.

$$\varphi(a, b_1) \in -C \setminus \{0\} \text{ and } \varphi(a, b_2) \in -C \setminus \{0\}.$$

By the C-quasiconvexity of $\varphi(a, \cdot)$ there is an index $i_0 \in \{1, 2\}$ with the property

$$\varphi(a, b_{i_0}) \in \varphi(a, tb_1 + (1 - t)b_2) + C.$$

So, there exists $c \in C$ such that

$$\varphi(a, b_{i_0}) = \varphi(a, tb_1 + (1 - t)b_2) + c. \tag{1}$$

Because $d := \varphi(a, b_{i_0}) \in -C \setminus \{0\}$, by (1) we get

$$\varphi(a, tb_1 + (1-t)b_2) = -c + d \in -C \setminus \{0\}.$$

Thus, the set $W(a)$ is convex. $\square$

Remark 2 Assumption (iv) in Theorem 1 does not imply assumption (iv) of Corollary 1. Indeed, let $E = Z$, let $C \subsetneq Z$ be a pointed convex cone such that the ordering defined by C is not total on A, and let $\varphi : A \times A \to Z$ be defined by

$$\varphi(a, b) := b \text{ for all } a, b \in A.$$

In order to verify assumption (iv) of Theorem 1, fix $a \in A$ and take $b_1, b_2 \in W(a)$. Thus $b_1, b_2 \in -C \setminus \{0\}$. Because $-C \setminus \{0\}$ is convex, we have

$$\lambda b_1 + (1-\lambda)b_2 \in -C \setminus \{0\} \text{ for every } \lambda \in [0, 1].$$

So, $W(a)$ is a convex set.

Now, let $b_1, b_2 \in A$ and $\lambda \in [0, 1]$. Suppose that $\varphi(a, \cdot) : A \to Z$ is C-quasiconvex. Thus, we obtain

$$b_1 \in b_2 + C \text{ or } b_2 \in b_1 + C.$$

Since b_1 and b_2 were arbitrarily chosen and the ordering induced by C on A is not total, it follows that the function $\varphi(a, \cdot)$ is not C-quasiconvex.

3 Particular cases of the generalized dual problem

In what follows we consider two particular cases of the operator $\mathcal{D}$. Firstly we define $\mathcal{D} : \mathcal{F}(A, Z) \to \mathcal{F}(A, Z)$ by

$$\mathcal{D}(\psi)(a, b) := -\psi(b, a) \text{ for all } a, b \in A. \tag{2}$$

So, the generalized dual strong vector equilibrium problem becomes:

$$\text{find } \bar{a} \in A \text{ such that } \varphi(b, \bar{a}) \notin C \setminus \{0\} \text{ for all } b \in A. \tag{DVEP1}$$

Under pseudomonotonicity assumptions we will give an existence result for the strong vector equilibrium problem (VEP). For this, we recall some monotonicity notions, used in the past for vector-valued bifunctions. Taking into consideration that the vector-valued bifunction $G : A \times A \to Z$, associated with the operator $\mathcal{D} : \mathcal{F}(A, Z) \to \mathcal{F}(A, Z)$ defined by (2), coincides with φ, Definition 3 yields the following definition.

Definition 4 The bifunction $\varphi : A \times A \to Z$ is said to be:

(i) pseudomonotone if, for all $a, b \in A$,

$$\varphi(a, b) \notin -C \setminus \{0\} \text{ implies } \varphi(b, a) \notin C \setminus \{0\};$$

(ii) maximal pseudomonotone if it is pseudomonotone and, for all $a, b \in A$,

$$\varphi(x, a) \notin C \setminus \{0\} \text{ for all } x \in]a, b] \text{ implies } \varphi(a, b) \notin -C \setminus \{0\}.$$

Proposition 3 *If $\varphi : A \times A \to Z$ is maximal pseudomonotone, then the sets of solutions of problems* (VEP) *and* $(\mathsf{DVEP1})$ *coincide.*

Proof. Take

$$G(b, a) := \varphi(b, a) \text{ for all } a, b \in A$$

in Proposition 2. □

Theorem 1 provides the next existence result of solutions of (VEP) under a pseudomonotonicity assumption.

Corollary 2 *Suppose that the bifunction $\varphi : A \times A \to Z$ satisfies the following conditions:*

(i) $\varphi(a, a) \in C$ *for all* $a \in A$*;*

(ii) φ *is maximal pseudomonotone;*

(iii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid \varphi(b, a) \notin C \setminus \{0\}\}$ *is closed;*

(iv) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a, b) \in -C \setminus \{0\}\}$ *is convex;*

(v) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) \in -C \setminus \{0\} \textit{ for all } x \in A \setminus D.$$

Then problem (VEP) *admits a solution.*

Now, if we define $\mathcal{D} : \mathcal{F}(A, Z) \to \mathcal{F}(A, Z)$ by $\mathcal{D}(\psi) := \psi$, we obtain an existence result for problem (VEP) without pseudomonotonicity assumptions. It is easy to verify that the assumption of φ to be maximal G-pseudomonotone is fulfilled.

In this case, the generalized dual problem of problem (VEP) is exactly:

$$\text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in A. \qquad (\mathsf{VEP})$$

Corollary 3 *Suppose that the bifunction* $\varphi : A \times A \to Z$ *satisfies the following conditions:*

(i) $\varphi(a, a) \in C$ *for all* $a \in A$*;*

(ii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid \varphi(a, b) \notin -C \setminus \{0\}\}$ *is closed;*

(iii) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a, b) \in -C \setminus \{0\}\}$ *is convex;*

(iv) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem (VEP) *admits a solution.*

Theorem 1 and Corollary 3 allow us to reobtain Lemma 1 and Theorem 2 established by W. Oettli [24], which are existence results for scalar equilibrium problems. Indeed, in what follows assume that $Z := \mathbb{R}$ and $C := \mathbb{R}_+$.

Corollary 4 ([24]) *Let the bifunctions* $\varphi : A \times A \to \mathbb{R}$ *and* $G : A \times A \to \mathbb{R}$ *satisfy the following conditions:*

(i) $\varphi(a, a) \geq 0$ *for all* $a \in A$*;*

(ii) φ *is maximal* G*-pseudomonotone;*

(iii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid G(b, a) \leq 0\}$ *is closed;*

(iv) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a, b) < 0\}$ *is convex;*

(v) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) < 0 \text{ for all } x \in A \setminus D.$$

Then the scalar equilibrium problem admits a solution.

Corollary 5 ([24]) *Suppose that the bifunction* $\varphi : A \times A \to \mathbb{R}$ *satisfies the following conditions:*

(i) $\varphi(a, a) \geq 0$ *for all* $a \in A$*;*

(ii) *for each* $b \in A$*, the set* $S(b) := \{a \in A \mid \varphi(a, b) \geq 0\}$ *is closed;*

(iii) *for each* $a \in A$*, the set* $W(a) := \{b \in A \mid \varphi(a,b) < 0\}$ *is convex;*

(iv) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\varphi(x, \tilde{b}) < 0 \text{ for all } x \in A \setminus D.$$

Then the scalar equilibrium problem admits a solution.

Corollary 5 is a slight generalization of an existence result established in [10] and recovered by [23]. In the sequel we deduce Fan's result.

Corollary 6 ([10]) *Let* A *be a compact set, and let* $\varphi : A \times A \to \mathbb{R}$ *satisfy the following conditions:*

(i) $\varphi(a,a) \geq 0$ *for all* $a \in A$*;*

(ii) $\varphi(\cdot, b) : A \to \mathbb{R}$ *is upper semicontinuous for all* $b \in A$*;*

(iii) $\varphi(a, \cdot) : A \to \mathbb{R}$ *is quasiconvex for all* $a \in A$*.*

Then the scalar equilibrium problem admits a solution.

Proof. Because A is compact and $\varphi(\cdot, b) : A \to \mathbb{R}$ is upper semicontinuous on A for all $b \in A$, the assumptions (ii) and (iv) of Corollary 5 are satisfied.

It remains to show the convexity of the set $W(a)$ for each $a \in A$. So, let $b_1, b_2 \in A$ and $\lambda \in [0,1]$. By the quasiconvexity of $\varphi(a, \cdot) : A \to \mathbb{R}$ and the inequality

$$\max\{\varphi(a, b_1), \varphi(a, b_2)\} < 0$$

we deduce that

$$\varphi(a, \lambda b_1 + (1-\lambda) b_2) < 0.$$

Thus, assumption (iii) of Corollary 5 is also satisfied and the proof is completed. □

4 Applications to strong vector variational inequalities

Strong vector variational inequality problems are particular cases of the strong vector equilibrium problem (VEP). Let A be a nonempty convex subset of a real topological linear space E, and let $F : A \to L(E, Z)$ be a mapping, where $L(E, Z)$ denotes the set of all continuous linear functions from E to a real Hausdorff topological linear space Z. Further, let $C \subseteq Z$ be a nontrivial pointed convex cone. Using these notations, in this section we will study the following variational inequalities:

$$\text{find } \bar{a} \in A \text{ such that } \langle F(b), b - \bar{a} \rangle \notin -C \setminus \{0\} \text{ for all } b \in A; \tag{MVI}$$

$$\text{find } \bar{a} \in A \text{ such that } \langle F(\bar{a}), b - \bar{a} \rangle \notin -C \setminus \{0\} \text{ for all } b \in A. \tag{SVI}$$

As in the previous section, for all $a, b \in A$, $\langle F(b), b - a \rangle$ denotes the value of the function $F(b)$ at the point $b - a$. Problem (MVI) is called *the strong Minty vector variational inequality*, while (SVI) is called *the strong Stampacchia vector variational inequality*.

Using the generalized duality theory presented in the main section we deduce that the strong Stampacchia vector variational inequality (SVI) admits as a generalized dual the strong Minty vector variational inequality (MVI). We notice that the vice-versa also holds, i.e. the generalized dual problem of (MVI) is (SVI).

In [12] there is presented an existence result for (SVI) under the following monotonicity property. The mapping $F : A \to L(E, Z)$ is said to be strongly pseudomonotone if, for all $a, b \in A$,

$$\langle F(a), b - a \rangle \notin -C \setminus \{0\} \text{ implies } \langle F(b), b - a \rangle \in C.$$

In what follows we work with the notion of pseudomonotonicity, which is weaker than the above one. To see this, we will give an example.

Definition 5 ([12]) The mapping $F : A \to L(E, Z)$ is said to be pseudomonotone if, for all $a, b \in A$,

$$\langle F(a), b - a \rangle \notin -C \setminus \{0\} \text{ implies } \langle F(b), b - a \rangle \notin -C \setminus \{0\}.$$

Example 1 Let $E := \mathbb{R}^2$, $A := [0, 1] \times [0, 1]$, $Z := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, and define $F : A \to L(\mathbb{R}^2, \mathbb{R}^2)$ by

$$\langle F(a), x \rangle := (x_1 + x_2)(a_1 - 2, a_2 + 2) \text{ for all } a := (a_1, a_2) \in A,\ x := (x_1, x_2) \in \mathbb{R}^2.$$

Let $a := (a_1, a_2)$ and $b := (b_1, b_2)$ be points from A. Since $a_1 - 2 < 0$ and $a_2 + 2 > 0$, it follows from

$$\langle F(a), b - a \rangle = (b_1 + b_2 - a_1 - a_2)(a_1 - 2, a_2 + 2)$$

that $\langle F(a), b-a\rangle \notin -\mathbb{R}^2_+ \setminus \{0\}$. Similarly, taking into consideration that $b_1-2<0$ and $b_2+2>0$, we obtain from

$$\langle F(b), b-a\rangle = (b_1+b_2-a_1-a_2)(b_1-2, b_2+2) \tag{3}$$

that $\langle F(b), b-a\rangle \notin -\mathbb{R}^2_+ \setminus \{0\}$. Consequently, F is pseudomonotone. On the other hand, when $b_1+b_2-a_1-a_2 \neq 0$, then (3) implies that

$$\langle F(b), b-a\rangle \notin \mathbb{R}^2_+.$$

Thus F is not strongly pseudomonotone.

The following notion is a particular case of Definition 3.

Definition 6 The mapping $F : A \to L(E, Z)$ is said to be maximal pseudomonotone if the following conditions are satisfied:

(i) F is pseudomonotone;

(ii) for all $a, b \in A$ the following implication holds: if $\langle F(x), a-x\rangle \notin C \setminus \{0\}$ for all $x \in]a, b]$, then $\langle F(a), a-b\rangle \notin C \setminus \{0\}$.

The next statement follows by Proposition 3.

Proposition 4 *If* F *is maximal pseudomonotone, then the solution sets of problems* (SVI) *and* (MVI) *coincide.*

Using Corollary 2, we obtain the following existence result for (SVI).

Theorem 2 *Suppose that the following conditions are satisfied:*

(i) *F is maximal pseudomonotone;*

(ii) *the set* $S(b) := \{a \in A \mid \langle F(b), b-a\rangle \notin -C \setminus \{0\}\}$ *is closed for all* $b \in A$*;*

(iii) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\langle F(x), \tilde{b}-x\rangle \in -C \setminus \{0\} \textit{ for all } x \in A \setminus D.$$

Then the problem (SVI) *admits a solution.*

Proof. Let $\varphi : A \times A \to Z$ be defined by $\varphi(a, b) := \langle F(a), b - a\rangle$. We show that φ satisfies the assumptions of Corollary 2. Indeed, the assumptions (ii), (iii) and (v) are satisfied by the hypothesis (i), (ii) and (iii), respectively. It remains to verify the assumptions (i) and (iv) of Corollary 2. Let $a \in A$. Since $F(a) \in L(E, Z)$, it follows that $\varphi(a, a) = 0$ and that the set

$$W(a) := \{b \in A \mid \langle F(a), b - a\rangle \in -C \setminus \{0\}\}$$

is convex. So, all the assumptions of Corollary 2 are fulfilled. Hence, there exists $\bar{a} \in A$ which is a solution for (SVI). $\square$

Example 2 To show that there exists mappings which satisfy the assumptions of Theorem 2, let $E := \mathbb{R}^2$, $A := [0, 1] \times [0, 1]$, $Z := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, and define $F : [0, 1] \times [0, 1] \to L(\mathbb{R}^2, \mathbb{R}^2)$ by

$$\langle F(a), x\rangle := (x_1 + x_2)(a_1 + 1, a_2 + 1) \tag{4}$$

for all $a := (a_1, a_2) \in A$ and all $x := (x_1, x_2) \in \mathbb{R}^2$.

Since

$$(a_1 + 1, a_2 + 1) \in \mathbb{R}^2_+ \setminus \{0\} \text{ for each } a := (a_1, a_2) \in A,$$

it results from (4) that

$$\forall\, a \in A \,:\, \{x \in \mathbb{R}^2 \mid \langle F(a), x\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1{+}_2 \le 0\}. \tag{5}$$

This inequality implies that

$$\forall\, a, b \in A \,:\, \{x \in \mathbb{R}^2 \mid \langle F(a), x\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\} = \{x \in \mathbb{R}^2 \mid \langle F(b), x\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\}. \tag{6}$$

Let $a := (a_1, a_2)$ and $b := (b_1, b_2)$ be points from A. Suppose that

$$\langle F(a), b - a\rangle \notin -\mathbb{R}^2_+ \setminus \{0\}.$$

Then we have $\langle F(a), a - b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}$. By virtue of (6) we obtain

$$\langle F(b), a - b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}, \text{ whence } \langle F(b), b - a\rangle \notin -\mathbb{R}^2_+ \setminus \{0\}.$$

Thus F is a pseudomonotone mapping

Next suppose that

$$\langle F(x), a - x\rangle \notin \mathbb{R}^2_+ \setminus \{0\} \text{ for all } x \in]a, b].$$

In particular, we have

$$\langle F(b), a - b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}.$$

By virtue of (6) we get $\langle F(a), a - b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}$. Hence F is a maximal pseudomonotone mapping. In other words, condition (i) in Theorem 2 is satisfied.

From (5) it follows that

$$S(b) = \{a \in A \mid \langle F(b), a - b\rangle \notin \mathbb{R}^2_+ \setminus \{0\}\} = \{(a_1, a_2) \in A \mid a_1 + a_2 \leq b_1 + b_2\}$$

for each $b := (b_1, b_2) \in A$. Consequently, condition (ii) in Theorem 2 is also satisfied.

Finally, it is obvious that condition (iii) in Theorem 2 is satisfied for $D := A$.

By Corollary 3 we obtain an existence result for the strong Stampacchia vector variational inequality without monotonicity assumptions. This new existence result is a slight generalization of Theorem 2.1 from [12].

Theorem 3 *Suppose that the following conditions are satisfied:*

(i) *for all* $b \in A$ *the set* $S(b) := \{a \in A \mid \langle F(a), b - a\rangle \notin -C \setminus \{0\}\}$ *is closed;*

(ii) *there exist a nonempty, compact and convex set* $D \subseteq A$ *as well as an element* $\tilde{b} \in D$ *such that*

$$\langle F(x), \tilde{b} - x\rangle \in -C \setminus \{0\} \text{ for all } x \in A \setminus D.$$

Then problem (SVI) *admits a solution.*

Proof. Define the bifunction $\varphi : A \times A \to Z$ by

$$\varphi(a, b) := \langle F(a), b - a\rangle \text{ for all } a, b \in A.$$

Let $a \in A$. Since $F(a) \in L(E, Z)$, it follows that $\varphi(a, a) = 0$ and that the set

$$W(a) := \{b \in A \mid \langle F(a), b - a\rangle \in -C \setminus \{0\}\}$$

is convex. By virtue of this observation, all the assumptions of Corollary 3 are satisfied. So, the strong Stampacchia vector variational inequality admits a solution. □

Corollary 7 ([12]) *Let* E *be a real Banach space, let A be a compact subset, let* Z *be a real Banach space ordered by a nonempty pointed solid convex cone C. Suppose that* $F : A \to L(E, Z)$ *is a mapping such that for every* $b \in A$ *the set*

$$\{a \in A \mid \langle F(a), b - a\rangle \in -C \setminus \{0\}\}$$

is open in A. Then problem (SVI) *is solvable.*

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Mark sequences in bipartite multidigraphs and constructions

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Abstract. A bipartite r-digraph is an orientation of a bipartite multigraph without loops and contains at most r edges between any pair of vertices from distinct parts. In this paper, we obtain necessary and sufficient conditions for a pair of sequences of non-negative integers in non-decreasing order to be a pair of sequences of numbers, called marks (or r-scores), attached to the vertices of a bipartite r-digraph. One of the characterizations is combinatorial and the other is recursive. As an application, these characterizations provide algorithms to construct a bipartite r-digraph with given mark sequences.

1 Introduction

An r-digraph is an orientation of a multigraph without loops and contains at most r edges between any pair of distinct vertices. So, 1-digraph is an oriented graph, and a complete 1-digraph is a tournament. Let D be an r-digraph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and let $d^+_{v_i}$ and $d^-_{v_i}$ denote the outdegree and indegree, respectively, of a vertex v_i. Define p_{v_i} (or simply p_i)$= r(n-1) + d^+_{v_i} - d^-_{v_i}$ as the mark (or r-score) of v_i, implying $0 \le p_{v_i} \le 2r(n-1)$. Then the sequence $P = [p_i]_1^n$ in non-decreasing order is called the mark sequence of D.

The following criterion for marks in r-digraphs due to Pirzada et al. [8] is analogous to a result on scores in tournaments given by Landau [6].

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Theorem 1 *A sequence* $P = [p_i]_1^n$ *of non-negative integers in non-decreasing order is the mark sequence of an* r*-digraph if and only*

$$\sum_{i=1}^{t} p_i \geq rt(t-1),$$

for $1 \leq t \leq n$*, with equality when* $t = n$*.*

Many results on marks in digraphs can be seen in [7, 9, 12, 14]. Also results for scores in oriented graphs can be found in [1, 11], while on tournaments we refer to [3, 4, 5]. Also it is important to mention here that the concept of scores has been extended to hypertournaments [15, 16, 17].

A bipartite r-digraph is an orientation of a bipartite multigraph without loops and contains at most r edges between any pair of vertices from distinct parts. So bipartite 1-digraph is an oriented bipartite graph and a complete bipartite 1-digraph is a bipartite tournament. Let $D(X, Y, A)$ be a bipartite r-digraph with vertex sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ and arc set A with each arc having one end in X and the other end in Y. For any vertex v_i in $D(X, Y)$, let $d^+_{v_i}$ and $d^-_{v_i}$ be the outdegree and indegree, respectively, of v_i. Define p_{x_i} (or simply p_i) $= rn + d^+_{x_i} - d^-_{x_i}$ and q_{y_j} (or simply q_j)$=$ $rm + d^+_{y_j} - d^-_{y_j}$ as the marks (or r-scores) of x_i in X and y_j in Y respectively. Clearly, $0 \leq p_{x_i} \leq 2rn$ and $0 \leq q_{y_j} \leq 2rm$. Then the sequences $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ in non-decreasing order are called the mark sequences of $D(X, Y, A)$.

A bipartite r-digraph can be interpreted as the result of a competition between two teams in which each player of one team plays with every player of the other team at most r times in which ties (draws) are allowed. A player receives two points for each win, and one point for each tie. With this marking system, player x_i (respectively y_j) receives a total of p_{x_i} (respectively q_{y_j}) points. The sequences P and Q of non-negative integers in non-decreasing order are said to be realizable if there exists a bipartite r-digraph with mark sequences P and Q.

In a bipartite r-digraph $D(X, Y, A)$, if there are a_1 arcs directed from a vertex $x \in X$ to a vertex $y \in Y$ and a_2 arcs directed from vertex y to vertex x, with $0 \leq a_1, a_2 \leq r$ and $0 \leq a_1 + a_2 \leq r$, we denote it by $x(a_1 - a_2)y$. For example, if there are exactly r arcs directed from $x \in X$ to $y \in Y$ and no arc directed from y to x, this is denoted by $x(r-0)y$, and if there is no arc directed from x to y and no arc directed from y to x, this is denoted by $x(0-0)y$.

The following characterization of mark sequences in bipartite 2-digraphs [13] is analogous to a result on scores in bipartite tournaments due to Beineke and Moon [2].

Theorem 2 *Let* $P = [p_i]_1^m$ *and* $Q = [q_j]_1^n$ *be sequences of non-negative integers in non-decreasing order. Then* P *and* Q *are mark sequences of some bipartite 2-digraph if and only if*

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j \geq 4fg,$$

for $1 \leq f \leq m$ *and* $1 \leq g \leq n$ *with equality when* $f = m$ *and* $g = n$.

Analogous results for scores in oriented bipartite graphs can be found in [10].

An oriented tetra in a bipartite r-digraph is an induced 1-subdigraph with two vertices from each part. Define oriented tetras of the form $x(1-0)y(1-0)x'$ $(1-0)y'(1-0)x$ and $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ to be of α-type and all other oriented tetras to be of β-type. A bipartite r-digraph is said to be of α-type or β-type according as all of its oriented tetras are of α-type or β-type respectively. We assume, without loss of generality, that β-type bipartite r-digraphs have no pair of symmetric arcs because symmetric arcs $x(a-a)y$, where $1 \leq a \leq \frac{r}{2}$, can be transformed to $x(0-0)y$ with the same marks. A transmitter is a vertex with indegree zero.

2 Criteria for realizability and construction algorithms

We start with the following observations.

Lemma 1 *Among all bipartite* r*-digraphs with given mark sequences, those with the fewest arcs are of* β*-type.*

Proof. Let $D(X,Y)$ be a bipartite r-digraph with mark sequences P and Q. Assume $D(X,Y)$ is not of β-type. Then $D(X,Y)$ has an oriented tetra of α-type, that is, $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ or $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ where $x, x' \in X$ and $y, y' \in Y$. Since $x(1-0)y(1-0)x'(1-0)y'(1-0)x$ can be transformed to $x(0-0)y(0-0)x'(0-0)y'(0-0)x$ with the same mark sequences and four arcs fewer, and $x(1-0)y(1-0)x'(1-0)y'(0-0)x$ can be transformed to $x(0-0)y(0-0)x'(0-0)y'(0-1)x$ with the same mark sequences and two arcs fewer, therefore, in both cases we obtain a bipartite r-digraph having same mark sequences P and Q with fewer arcs. Note that if there are symmetric arcs between x and y, that is $x(a-a)y$, where $1 \leq a \leq \frac{r}{2}$, then these can be transformed to $x(0-0)y$ with the same mark sequences and a arcs fewer. Hence the result follows. $\square$

Lemma 2 *Let* $P=[p_i]_1^m$ *and* $Q=[q_j]_1^n$ *be mark sequences of a* β*-type bipartite* r*-digraph. Then either the vertex with mark* p_m*, or the vertex with mark* q_n*, or both can act as transmitters.*

We now have some observations about bipartite r-digraphs, as these will be required in application of Theorem 2.10. If $P=[p_1,p_2,\dots,p_m]$ and $Q=[q_1,q_2,\dots,q_n]$ are mark sequences of a bipartite r-digraph, then $p_i\le 2rn$ and $q_j\le 2rm$, where $1\le i\le m$ and $1\le j\le m$.

Lemma 3 *If* $P=[p_1,p_2,\dots,\ p_{m-1},p_m]$ *and* $Q=[0,0,\dots,0,0]$ *with each* $p_i=2rn$ *are mark sequences of some bipartite* r*-digraph, then* $P'=[p_1,p_2,\dots,p_{m-1}]$ *and* $Q'=[0,0,\dots,0]$ *are also mark sequences of some bipartite* r*-digraph.*

Proof. Let P and Q as given above be mark sequences of bipartite r-digraph D with parts $X=\{x_1,x_2,\dots,x_{m-1},x_m\}$ and $Y=\{y_1,y_2,\dots,y_{n-1},x_n\}$. Since mark of each x_i is $2rn$, so $x_i(r-0)y_j$ for each x_i and each y_j, $1\le i\le m$ and $1\le j\le n$. Deleting x_m will neither change the marks of the vertices x_i, for all $1\le i\le m-1$ nor will change the marks of the vertices y_j, for all $1\le j\le n$. Hence $P'=[p_1,p_2,\dots,p_{m-1}]$ and $Q'=[0,0,\dots,0]$ are the mark sequences of the bipartite r-digraph with parts $\{x_1,x_2,\dots,x_{m-1}\}$ and $\{y_1,y_2,\dots,y_{n-1},x_n\}$, that is the bipartite r-digraph $D-x_n$. $\square$

Lemma 4 *If* $P=[p_1,p_2,\dots,p_{m-1},p_m]$ *and* $Q=[0,0,\dots,0,q_n]$ *with* $4n-p_m=3$ *and* $q_n\ge 3$ *are mark sequences of some bipartite* r*-digraph, then* $P'=[p_1,p_2,\dots,p_{m-1}]$ *and* $Q'=[0,0,\dots,0,q_n-3]$ *are also mark sequences of some bipartite* r*-digraph.*

Proof. Let P and Q as given above be mark sequences of bipartite r-digraph D with parts $X=\{x_1,x_2,\dots,x_{m-1},x_m\}$ and $Y=\{y_1,y_2,\dots,y_{n-1},x_n\}$. Since $4n-p_m=3$ and $3\le q_n\le 4m$, therefore in D necessarily $x_m(2-0)y_i$, for all $1\le i\le n-1$. Also $y_n(1-0)x_m$, because if $y_n(0-0)x_m$, or $y_n(0-2)x_m$,or $y_n(0-1)x_m$, then in all these cases $p_{x_m}\ge 4(n-1)+2$, a contradiction to our assumption. Also $y_n(2-0)x_m$ is not possible because in that case $p_{x_m}=4(n-1)<4n-3$.

Now delete x_m, obviously this keeps marks of y_1,y_2, $\dots$,y_{n-1} as zeros and reduces mark of y_m by 3, and we obtain a bipartite r-digraph with mark sequences $P'=[p_1,p_2,\dots,p_{m-1}]$ and $Q'=[0,0,\dots,0,q_n-3]$, as required. $\square$

Lemma 5 *If* $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \ldots, 0, q_n]$ *with* $4n - p_m = 4$ *and* $q_n \geq 4$ *are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \ldots, p_{m-1}]$ *and* $Q' = [0, 0, \ldots, 0, q_n - 4]$ *are also mark sequences of some bipartite* r*-digraph.*

Proof. Let P and Q as given above be mark sequences of a bipartite r-digraph D with parts $X = \{x_1, x_2, \ldots, x_{m-1}, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_{n-1}, x_n\}$. Since $4n - p_m = 4$ and $4 \leq q_n \leq 4m$, therefore in D necessarily $x_l(2-0)y_i$, for all $1 \leq i \leq n-1$. Also $y_n(2-0)x_l$, because if $y_n(0-0)x_m$, or $y_n(1-0)x_m$, or $y_n(0-2)x_m$,or $y_n(0-1)x_m$, then in all these cases $p_{x_m} \geq 4(n-1)+1$, a contradiction to our assumption.

Now delete x_m, obviously this keeps marks of y_1,y_2, $\ldots$,y_{n-1} as zeros and reduces mark of y_n by 4, and we obtain a bipartite r-digraph with mark sequences $P' = [p_1, p_2, \ldots, p_{m-1}]$ and $Q' = [0, 0, \ldots, 0, q_n - 4]$, as required. □

Lemma 6 *If* $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \ldots, 0, q_n]$ *with* $4n - p_m = 4$ *and* $q_n \geq 3$ *are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \ldots, p_{m-1}]$ *and* $Q' = [0, 0, \ldots, 0, q_n - 3]$ *are also mark sequences of some bipartite* r*-digraph.*

Proof. The proof follows by using the same argument as in Lemma 5. □

Lemma 7 *If* $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \ldots, 0, 1, 3]$ *with* $4n - p_m = 4$*, are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \ldots, p_{m-1}]$ *and* $Q' = [0, 0, \ldots, 0, 0, 0]$ *are also mark sequences of some bipartite* r*-digraph.*

Lemma 8 *If* $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \ldots, 0, 1, 1, 2]$ *with* $4n - p_m = 4$*, are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \ldots, p_{m-1}]$ *and* $Q' = [0, 0, \ldots, 0, 0, 0]$ *are also mark sequences of some bipartite* r*-digraph.*

Lemma 9 *If* $P = [p_1, p_2, \ldots, p_{m-1}, p_m]$ *and* $Q = [0, 0, \ldots, 0, 1, 1, 1, 1]$ *with* $4n - p_m = 4$*, are mark sequences of some bipartite* r*-digraph, then* $P' = [p_1, p_2, \ldots, p_{m-1}]$ *and* $Q' = [0, 0, \ldots, 0, 0, 0]$ *are also mark sequences of some bipartite* r*-digraph.*

Remarks. We note that the sequences of non-negative integers $[p_1]$ and $[q_1, q_2, \ldots, q_n]$, with $p_1 + q_1 + q_2 + \cdots + q_n = 2rn$, are always mark sequences of some bipartite r-digraph. We observe that the bipartite r-digraph

$D(X, Y)$, with vertex sets $X = \{x_1\}$ and $Y = \{y_1, y_2, \dots, y_n\}$, where for q_i even, say $2t$, we have $x_1((r-t)-t)y_i$ and for q_i odd, say $2t+1$, we have $x_1((r-t-1)-t)y_i$, has mark sequences $[p_1]$ and $[q_1, q_2, \dots, q_n]$. Also the sequences $[0]$ and $[2r, 2r, \dots, 2r]$ are mark sequences of some bipartite r-digraph.

The next result provides a useful recursive test whether or not a pair of sequences is realizable.

Theorem 3 *Let* $P = [p_i]_1^m$ *and* $Q = [q_j]_1^n$ *be sequences of non-negative integers in non-decreasing order with* $p_m \geq q_n$ *and* $rn \leq p_m \leq 2rn$.
(A) *If* $q_n \leq 2r(m-1)+1$*, let* P' *be obtained from* P *by deleting one entry* p_m*, and* Q' *be obtained as follows.*

For $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$*, reducing* $[2r-(i-1)]n - p_m$ *largest entries of* Q *by* i *each, and reducing* $p_m - (2r-i)n$ *next largest entries by* $i-1$ *each.*
(B) *In case* $q_n > 2r(m-1)+1$*, say* $q_n = 2r(m-1)+1+h$*, where* $1 \leq h \leq r-1$*, then let* P' *be obtained from* P *by deleting one entry* p_m*, and* Q' *be obtained from* Q *by reducing the entry* q_n *by* $h+1$.

Then P *and* Q *are the mark sequences of some bipartite* r*-digraph if and only if* P' *and* Q' *(arranged in non-decreasing order) are the mark sequences of some bipartite* r*-digraph.*

Proof. Let P' and Q' be the mark sequences of some bipartite r-digraph $D'(X', Y')$. First suppose Q' is obtained from Q as in A. Construct a bipartite r-digraph $D(X, Y)$ as follows. Let $X = X' \cup x$, $Y = Y'$ with $X' \cap x = \phi$. Let $x((r-i)-0)y$ for those vertices y of Y' whose marks are reduced by i in going from P to P' and Q to Q', and $x(r-0)y$ for those vertices y of Y' whose marks are not reduced in going from P to P' and Q to Q'. Then $D(X, Y)$ is the bipartite r-digraph with mark sequences P and Q. Now, if Q' is obtained from Q as in B, then construct a bipartite r-digraph $D(X, Y)$ as follows. Let $X = X' \cup x$, $Y = Y'$ with $X' \cap x = \phi$. Let $x((r-h-1)-0)y$ for that vertex y of Y' whose marks are reduced by h in going from P and Q to P' and Q'. Then $D(X, Y)$ is the bipartite r-digraph with mark sequences P and Q.

Conversely, suppose P and Q be the mark sequences of a bipartite r-digraph $D(X, Y)$. Without loss of generality, we choose $D(X, Y)$ to be of β-type. Then by Lemma 2.2, any of the vertex $x \in X$ or $y \in Y$ with mark p_m or q_n respectively can be a transmitter. Let the vertex $x \in X$ with mark p_m be a transmitter. Clearly, $p_m \geq rn$ and because if $p_m < rn$, then by deleting p_m we have to reduce more than n entries from Q, which is absurd.
(A) Now $q_n \leq 2r(m-1)+1$ because if $q_n > 2r(m-1)+1$, then on reduction

$q'_n = q_n - 1 > 2r(m-1) + 1 - 1 = 2r(m-1)$, which is impossible.

Let $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, let V be the set of $[2r-(i-1)]n - p_m$ vertices of largest marks in Y, and let W be the set of $p_m - (2r-i)n$ vertices of next largest marks in Y and let $Z = Y - \{V, W\}$. Construct $D(X, Y)$ such that $x((r-i)-0)v$ for all $v \in V$, $x((r-i-1)-0)w$ for all $w \in W$ and $x(r-0)z$ for all $z \in Z$. Clearly, $D(X, Y) - x$ realizes P' and Q' (arranged in non-decreasing order).

(B) Now in D, let $q_n > 2r(m-1)+1$, say $q_n = 2r(m-1)+1+h$, where $1 \leq h \leq r-1$. This means $y_m(r-0)x_i$, for all $1 \leq i \leq m-1$. Since x_m is a transmitter, so there cannot be an arc from y_n to x_m. Therefore $x_m((r-h-1)-0)y_n$, since y_n needs $h+1$ more marks. Now delete x_m, it will decrease the mark of y_n by $h+1$, and the resulting bipartite r-digraph will have mark sequences P' and Q' as desired. □

Theorem 2.10 provides an algorithm of checking whether or not the sequences P and Q of non-negative integers in non-decreasing order are mark sequences, and for constructing a corresponding bipartite r-digraph. Let $P = [p_1, p_2, \ldots, p_m]$ and $Q = [q_1, q_2, \ldots, q_n]$, where $p_m \geq q_n$, $rn \leq p_m \leq 2rn$ and $q_n \leq 2r(m-1)+1$, be the mark sequences of a bipartite r-digraph with parts $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ respectively. Deleting p_m and performing A of Theorem 2.10 if $[2r-(i-1)]n \geq p_m \geq (2r-i)n$, $1 \leq i \leq r$, we get $Q' = [q'_1, q'_2, \ldots, q'_n]$. If the marks of the vertices y_j were decreased by i in this process, then the construction yielded $x_m((r-i)-0)y_j$, if these were decreased by $i-1$, then the construction yielded $x_m((r-i-1)-0)y_j$. If we perform B of Theorem 2.10, the mark of y_n was decreased by $h+1$, the construction yielded $x_m((r-h-1)-0)y_n$. For vertices y_j whose marks remained unchanged, the construction yielded $x_m(r-0)y_j$. Note that if the condition $p_m \geq rn$ does not hold, then we delete q_n for which the conditions get satisfied and the same argument is used for defining arcs. If this procedure is applied recursively, then it tests whether or not P and Q are the mark sequences, and if P and Q are the mark sequences, then a bipartite r-digraph with mark sequences P and Q is constructed.

We illustrate this reduction and the resulting construction with the following examples.

Example 1. Consider the sequences of non-negative integers $P = [14, 14, 15]$ and $Q = [6, 6, 8, 9]$. We check whether or not P and Q are mark sequences of some bipartite 3-digraph.

1. $P = [14, 14, 15]$, $Q = [6, 6, 8, 9]$.

We delete 15. Clearly $[2r-(i-1)]n = [2.3-(3-1)]4 = 16 \geq 15 \geq (2r-i)n =$

$(2.3-3)4=12$. So reduce $[2r-(i-1)]n-p_m=[2.3-(3-1]4-15=16-15=1$ largest entry of Q by $i=3$ and $p_m-(2r-i)n=15-(2.3-3)4=15-12=3$ next largest entries of Q by $i-1=3-1=2$ each, we get $P_1=[14,14]$, $Q_1=[4,4,6,6]$, and arcs are defined as $x_3(0-0)y_4$, $x_3(1-0)y_3$, $x_3(1-0)y_2$, $x_3(1-0)y_1$.

2. $P_1=[14,14]$, $Q_1=[4,4,6,6]$.

We delete 14. Here $[2r-(i-1)]n=[2.3-(3-1)]4=16\geq 14\geq (2r-i)n=(2.3-3)4=12$. Reduce $[2r-(i-1)]n-p_m=[2.3-(3-1]4-14=16-14=2$ largest entries of Q_1 by $i=3$ and $p_m-(2r-i)n=14-(2.3-3)4=14-12=2$ next largest entries of Q_1 by $i-1=3-1=2$ each, we get $P_2=[14]$, $Q_2=[2,2,3,3]$, and arcs are defined as $x_2(0-0)y_4$, $x_2(0-0)y_3$, $x_2(1-0)y_2$, $x_2(1-0)y_1$.

3. $P_2=[14]$, $Q_2=[2,2,3,3]$.

We delete 14. Here $[2r-(i-1)]n=[2.3-(3-1)]4=16\geq 14\geq (2r-i)n=(2.3-3)4=12$. Reduce $[2r-(i-1)]n-p_m=[2.3-(3-1]4-14=16-14=2$ largest entries of Q_2 by $i=3$ and $p_m-(2r-i)n=14-(2.3-3)4=14-12=2$ next largest entries of Q_2 by $i-1=3-1=2$ each, we get $P_3=\phi$, $Q_3=[0,0,0,0]$, and arcs are defined as $x_1(0-0)y_4$, $x_1(0-0)y_3$, $x_1(1-0)y_2$, $x_1(1-0)y_1$.

The resulting bipartite 3-digraph has mark sequences $P=[14,14,15]$ and $Q=[6,6,8,9]$ with vertex sets $X=\{x_1,x_2,x_3\}$ and $Y=\{y_1,y_2,y_3,y_4\}$ and arcs as $x_3(0-0)y_4$, $x_3(1-0)y_3$, $x_3(1-0)y_2$, $x_3(1-0)y_1$, $x_2(0-0)y_4$, $x_2(0-0)y_3$, $x_2(1-0)y_2$, $x_2(1-0)y_1$, $x_1(0-0)y_4$, $x_1(0-0)y_3$, $x_1(1-0)y_2$, $x_1(1-0)y_1$.

Example 2. Consider the two sequences of non-negative integers given by $P=[13,16,22,24]$ and $Q=[5,6,10]$. We check whether or not P and Q are mark sequences of some bipartite 4-digraph.

1. $P=[13,16,22,24]$ and $Q=[5,6,10]$.

We delete 24. Here $[2r-(i-1)]n=[2.4-(1-1)]3=24$, so reduce $[2r-(i-1)]n-p_m=[2.4-(1-1]3-24=24-24=0$ largest entries of Q by $i=1$, and obviously we reduce $p_m-(2r-i)n=24-(2.4-1)3=24-21=3$ next largest entries of Q by $i-1=1-1=0$ each, we get $P_1=[13,16,22]$ and $Q_1=[5,6,10]$, and arcs are $x_4(4-0)y_3$, $x_4(4-0)y_2$, $x_4(4-0)y_1$.

2. $P_1=[13,16,22]$ and $Q_1=[5,6,10]$.

We delete 22. Here $[2r-(i-1)]n=[2.4-(1-1)]3=24\geq 22\geq (2r-i)n=(2.4-1)3=21$. Reduce $[2r-(i-1)]n-p_m=[2.4-(1-1]3-22=24-22=2$ largest entries of Q_1 by $i=1$ and $p_m-(2r-i)n=22-(2.4-1)3=22-21=1$ next largest entries of Q_1 by $i-1=1-1=0$ each, we get $P_2=[13,16]$, $Q_2=[5,5,9]$, and arcs are defined as $x_3(3-0)y_3$, $x_3(3-0)y_2$, $x_3(4-0)y_1$.

3. $P_2=[13,16]$, $Q_2=[5,5,9]$.

We delete 16. Here $[2r-(i-1)]n = [2.4-(3-1)]3 = 18 \geq 16 \geq (2r-i)n = (2.4-3)3 = 15$. Reduce $[2r-(i-1)]n - p_m = [2.4-(3-1]3-16 = 18-16 = 2$ largest entries of Q_2 by $i = 3$ and $p_m-(2r-i)n = 16-(2.4-3)3 = 16-15 = 1$ next largest entry of Q_2 by $i-1 = 3-1 = 2$, we get $P_3 = [13]$, $Q_3 = [3, 2, 6]$, and arcs are defined as $x_2(3-0)y_3$, $x_2(3-0)y_2$, $x_2(2-0)y_1$.

4. $P_3 = [13]$, $Q_3 = [3, 2, 6]$.

Here $13 + 3 + 2 + 6 = 24$ which is same as $2rn = 2.4.3 = 24$. Thus by the argument as discussed in the remarks, P_3 and Q_3 are mark sequences of some bipartite 4-digraph. Here arcs are $x_1(1-3)y_3$, $x_1(3-1)y_2$, $x_1(2-1)y_1$.

The resulting bipartite 4-digraph with mark sequences $P = [13, 16, 22, 24]$ and $Q = [5, 6, 10]$ has vertex sets $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3\}$ and arcs as $x_4(4-0)y_3$, $x_4(4-0)y_2$, $x_4(4-0)y_1$, $x_3(3-0)y_3$, $x_3(3-0)y_2$, $x_3(4-0)y_1$, $x_2(3-0)y_3$, $x_2(3-0)y_2$, $x_2(2-0)y_1$, $x_1(1-3)y_3$, $x_1(3-1)y_2$, $x_1(2-1)y_1$.

Now we give a combinatorial criterion for determining whether the sequences of non-negative integers are realizable as marks. This is analogous to Landau's theorem [6] on tournament scores and similar to the result by Beineke and Moon [2] on bipartite tournament scores.

Theorem 4 *Let $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ be the sequences of non-negative integers in non-decreasing order. Then P and Q are the mark sequences of some bipartite r-digraph if and only if*

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j \geq 2rfg, \tag{1}$$

for $1 \leq f \leq m$ and $1 \leq g \leq n$, with equality when $f = m$ and $g = n$.

Proof. The necessity of the condition follows from the fact that the sub-bipartite r-digraph induced by f vertices from the first part and g vertices from the second part has a sum of marks $2rfg$.

For sufficiency, assume that $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ are the sequences of non-negative integers in non-decreasing order satisfying conditions (2.1) but are not mark sequences of any bipartite r-digraph. Let these sequences be chosen in such a way that m and n are the smallest possible and p_1 is the least with that choice of m and n. We consider the following two cases.

Case (a). Suppose the equality in (2.1) holds for some $f \leq m$ and $g \leq n$, so that

$$\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j = 2rfg.$$

By the minimality of m and n, $P_1 = [p_i]_1^f$ and $Q_1 = [q_j]_1^g$ are the mark sequences of some bipartite r-digraph $D_1(X_1, Y_1)$. Let $P_2 = [p_{f+1} - 2rg, p_{f+2} - 2rg, \dots, p_m - 2rg]$ and $Q_2 = [q_{g+1} - 2rf, q_{g+2} - 2rf, \dots, q_n - 2rf]$.
Consider the sum

$$\begin{aligned}\sum_{i=1}^{s}(p_{f+i} - 2rg) + \sum_{j=1}^{t}(q_{g+j} - 2rf) &= \sum_{i=1}^{f+s} p_i + \sum_{j=1}^{g+t} q_j - \left(\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j\right)\\ &\quad - 2rsg - 2rtf\\ &\geq 2r(f+s)(g+t) - 2rfg - 2rsg - 2rtf\\ &= 2r(fg + ft + sg + st - fg - sg - tf)\\ &= 2rst,\end{aligned}$$

for $1 \leq s \leq m - f$ and $1 \leq t \leq n - g$, with equality when $s = m - f$ and $t = n - g$. Thus, by the minimality of m and n, the sequences P_2 and Q_2 form the mark sequences of some bipartite r-digraph $D_2(X_2, Y_2)$. Now construct a new bipartite r-digraph $D(X, Y)$ as follows.

Let $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$ with $X_1 \cap X_2 = \phi$, $Y_1 \cap Y_2 = \phi$. Let $x_2(r-0)y_1$ and $y_2(r-0)x_1$ for all $x_i \in X_i, y_i \in Y_i$, where $1 \leq i \leq 2$, so that we get the bipartite r-digraph $D(X, Y)$ with mark sequences P and Q, which is a contradiction.

Case (b). Suppose the strict inequality holds in (2.1) for some $f \neq m$ and $g \neq n$. Also, assume that $p_1 > 0$. Let $P_1 = [p_1 - 1, p_2, \dots, p_{m-1}, p_m + 1]$ and $Q_1 = [q_1, q_2, \dots, q_n]$. Clearly, P_1 and Q_1 satisfy the conditions (2.1). Thus, by the minimality of p_1, the sequences P_1 and Q_1 are the mark sequences of some bipartite r-digraph $D_1(X_1, Y_1)$. Let $p_{x_1} = p_1 - 1$ and $p_{x_m} = p_m + 1$. Since $p_{x_m} > p_1 + 1$, therefore there exists a vertex $y \in Y_1$ such that $x_m(1-0)y(1-0)x_1$, or $x_m(0-0)y(1-0)x_1$, or $x_m(1-0)y(0-0)x_1$, or $x_m(0-0)y(0-0)x_1$, is an induced sub-bipartite 1-digraph in $D_1(X_1, Y_1)$, and if these are changed to $x_m(0-0)y(0-0)x_1$, or $x_m(0-1)y(0-0)x_1$, or $x_m(0-0)y(0-1)x_1$, or $x_m(0-1)y(0-1)x_1$ respectively, the result is a bipartite r-digraph with mark sequences P and Q, which is a contradiction. Hence the result follows. □

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On (κ, μ)-contact metric manifolds with certain curvature restrictions

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Abstract. In this paper we give a classification of (κ, μ)-contact metric manifolds with certain curvature restrictions.

1 Introduction

In 1995, Blair, Koufogiorgos and Papantoniou [3] introduced a type of contact metric manifolds $M^{(2n+1)}(\phi, \xi, \eta, g)$ whose curvature tensor R satisfies

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \ \forall\, X, Y \in \chi(M).$$

Here, (κ, μ) are real constants and $2h$ denotes the Lie-Derivative in the direction of ξ. In this case we say that the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution and the class of contact metric manifolds satisfying this condition are called (κ, μ)-contact metric manifolds. In case the vector field ξ is Killing, this class of manifolds are called Sasakian manifolds. In 1999, Boeckx [5] proved that a (κ, μ)-contact metric manifolds is either Sasakian or locally ϕ-symmetric. Later in 2000, Boeckx [6] gave a full classification of non-Sasakian (κ, μ)-contact metric manifolds. In 2008, Ghosh [7] proved that all conformally recurrent (κ, μ)-contact metric manifolds are locally isometric either to the unit sphere S^{2n+1} or to $E^{n+1} \times S^n$. In this paper, we study (κ, μ)-contact metric manifolds with different curvature restrictions and classify such manifolds.

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2 Preliminaries

Let (M^{2n+1}, g) be an almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g). Then we have

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(X, \xi) = \eta(X); \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) = d\eta(X, Y) = -g(X, \phi Y) \tag{2}$$

for all $X, Y \in \chi(M)$. The operator h satisfies the following results [2], [3], [4]:

$$h\phi = -\phi h, \quad \eta \circ h = 0, \quad g(hX, Y) = g(X, hY), \quad h^2 = (\kappa - 1)\phi^2; \tag{3}$$

$$h\xi = 0, \quad g(X, \phi hZ) = g(\phi hX, Z); \tag{4}$$

$$\nabla_X \xi = -\phi X - \phi hX, \quad (\nabla_X \eta)(Y) = g(X + hX, \phi Y), \tag{5}$$

where ∇ is the Riemannian connection of g. In a (κ, μ)-contact metric manifolds we have the following [3], [4]:

$$R(\xi, X)Y = \kappa\{g(X, Y)\xi - \eta(Y)X\} + \mu\{g(hX, Y)\xi - \eta(Y)hX\}; \tag{6}$$

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}; \tag{7}$$

$$\begin{aligned} S(X, Y) = &\{2(n-1) - n\mu\}g(X, Y) + \{2(n-1) + \mu\}g(hX, Y) \\ &+ \{2(1-n) + n(2\kappa + \mu)\}\eta(X)\eta(Y); \end{aligned} \tag{8}$$

$$S(X, \xi) = 2n\kappa\eta(X), \qquad Q\xi = 2n\kappa\xi; \tag{9}$$

$$r = 2n(2n - 2 + \kappa - n\mu); \tag{10}$$

and

$$\begin{aligned} (\nabla_X h)(Y) - (\nabla_Y h)(X) = &(1-\kappa)\{2g(X, \phi Y)\xi + \eta(X)\phi Y - \eta(Y)\phi X\} \\ &+ (1-\mu)\{\eta(X)\phi hY - \eta(Y)\phi hX\} \end{aligned} \tag{11}$$

for all $X, Y \in \chi(M)$ where S, r are respectively the Ricci tensor and the scalar curvature of M.

For (κ, μ)-contact metric manifolds with $h = 0$, we have $\kappa = 1$, and in this case the manifold reduces to a Sasakian one. The following relations hold in a Sasakian manifold [2]:

$$(i)\ \nabla_X \xi = -\phi X, \qquad (ii)\ (\nabla_X \eta)(Y) = g(X, \phi Y), \tag{12}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{13}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{14}$$

$$S(X, \xi) = 2n\eta(X), \tag{15}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2n\eta(X)\eta(Y), \tag{16}$$

for all $X, Y \in \chi(M)$. The above formulae will be used in the sequel.

3 On a class of (κ, μ)-contact metric manifolds

A Riemannian manifold (M, g) is called a hyper-generalized recurrent manifold (for details we refer to [8]) if and only if its curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_W R)(X, Y)Z = {} & A(W)R(X, Y)Z \\ & + B(W)\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\}\end{aligned} \tag{17}$$

for all $X, Y, Z \in \chi(M)$; where A and B are two non-zero 1-forms metrically equivalent to two vector fields σ and ρ, respectively. Moreover, if the the scalar curvature r is a non-zero constant, then these associated 1-forms are related by

$$A + 4nB = 0. \tag{18}$$

Consequently we have

$$\sigma + 4n\rho = 0. \tag{19}$$

Before proceeding for the main theorems of the paper, we are to state the following lemma [7]:

Lemma 1 *For a (κ, μ)-contact metric space, the relation $\nabla_\xi h = \mu h\phi$ holds.*

We are now going to prove the main theorems of the paper:

By contracting (17) with respect to W, we obtain

$$\begin{aligned}(\operatorname{div} R)(X, Y)Z = {} & g(R(X, Y)Z, \sigma) + \big\{S(Y, Z)g(X, \rho) - S(X, Z)g(Y, \rho) \\ & + g(Y, Z)S(X, \rho) - g(X, Z)S(Y, \rho)\big\}.\end{aligned} \tag{20}$$

Using the result

$$(\operatorname{div} R)(X, Y)Z = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z),$$

in (20), one obtains

$$\begin{aligned}(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = {} & g(R(X, Y)Z, \sigma) + \big[S(Y, Z)g(X, \rho) - S(X, Z)g(Y, \rho) \\ & + g(Y, Z)S(X, \rho) - g(X, Z)S(Y, \rho)\big].\end{aligned} \tag{21}$$

Setting $Z = \xi$, yields on using (9)

$$\begin{aligned} & 2n\kappa\big[g(X + hX, \phi Y) - g(Y + hY, \phi X)\big] \\ + \;& S(Y, \phi X) - S(X, \phi Y) + S(Y, \phi hX) - S(X, \phi hY) \\ = \;& g(R(X, Y)\xi, \sigma) + 2n\kappa[\eta(Y)g(X, \rho) - \eta(X)g(Y, \rho)] \\ - \;& [\eta(Y)S(X, \rho) + \eta(X)S(Y, \rho)]. \end{aligned} \tag{22}$$

Replacing X by ϕX and Y by ϕY and using (1) and (3), we have

$$2\kappa + \mu + n\mu - \mu\kappa = 0. \tag{23}$$

In a (κ, μ)-contact metric manifold, the scalar curvature r is a non-zero constant, therefore using (18) in (17) and thereby contraction over W yields

$$\begin{aligned} (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = g(R(X, Y)Z, \sigma) - \frac{1}{4n}\big[S(Y, Z)g(X, \sigma) \\ - S(X, Z)g(Y, \sigma) + g(Y, Z)S(X, \sigma) \\ - g(X, Z)S(Y, \sigma)\big]. \end{aligned} \tag{24}$$

Using (8) and (11) on (24), one obtains

$$\begin{aligned} & (3\mu + 2n\kappa - n\mu - \mu^2)\big(\eta(X)g(\phi hY, Z) - \eta(Y)g(\phi hX, Z)\big) \\ & = g(R(X, Y)Z, \sigma) - \frac{1}{4n}\big[S(Y, Z)g(X, \sigma) - S(X, Z)g(Y, \sigma) \\ & \quad + g(Y, Z)S(X, \sigma) - g(X, Z)S(Y, \sigma)\big]. \end{aligned} \tag{25}$$

Putting $X = \xi$ and setting $Z = \sigma$ in the above equality, we have by (6)

$$(3\mu + 2n\kappa - n\mu - \mu^2)g(\phi hY, \sigma) = 0. \tag{26}$$

Two cases arise from above

$$\text{(i)} \quad 3\mu + 2n\kappa - n\mu - \mu^2 = 0, \tag{27}$$

$$\text{(ii)} \quad \phi h\sigma = 0. \tag{28}$$

From (23) we have

$$-\kappa(\mu - 2) + (n + 1)\mu = 0.$$

$$\text{or,} \quad \kappa = (n + 1)\frac{\mu}{\mu - 2}. \tag{29}$$

Putting this value of κ in (23) we have

$$\mu(\mu-n-3)(\mu+2n-2)=0. \tag{30}$$

From (29) and (30) we get the following set of corresponding values of μ and κ:

μ	κ
0	0
$n+3$	$n+3$
$2-2n$	$n-\frac{1}{n}$

Since, $\kappa<1$ and $n>1$, therefore only the case $\kappa=0=\mu$ is admissible and other possibilities will be ignored. For $\kappa=0=\mu$, from (11) we have, $R(X,Y)\xi=0$, for all X, Y. Therefore by [1], a (κ,μ)-contact metric manifold (M^{2n+1},g) admitting such a structure is locally isometric to either (i) the unit sphere $S^{2n+1}(1)$ or (ii) to the product space $E^{n+1}\times S^{n}(4)$.

Next let us consider the case (ii). From (28) we have the following:

$$\begin{aligned}
&\phi g\sigma=0\\
\Rightarrow\ &\phi^{2}h\sigma=-h\sigma+\eta(h\sigma)\xi,\\
\Rightarrow\ &h\sigma=0,\ \text{by (3)}\\
\Rightarrow\ &h^{2}\sigma=(\kappa-1)\phi^{2}\sigma=0.
\end{aligned}$$

Since, $\kappa<1$, it follows that $\phi^{2}\sigma=0$ and consequently $\sigma=\eta(\sigma)\xi$ i.e. for all vector field W on M, $A(W)=\eta(\sigma)\eta(W)$. Applying (18), we find $B(W)=\eta(\rho)\eta(W)$. Hence putting the values of A and B in (17), one obtains

$$\begin{aligned}
&(\nabla_{W}R)(X,Y)Z\\
=\ &\eta(\sigma)\eta(W)R(X,Y)Z\\
-\ &\eta(\rho)\eta(W)\big\{S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY\big\}.
\end{aligned} \tag{31}$$

Placing $\phi^{2}W$ in lieu of W and thereby contracting over W in the resulting equation, we find

$$(\nabla_{X}S)(Y,Z)-(\nabla_{Y}S)(X,Z)=g\big((\nabla_{\xi}R)(X,Y)Z,\xi\big). \tag{32}$$

Replacing Y by ξ, the above equation reduces to

$$(\nabla_{X}S)(\xi,Z)-(\nabla_{\xi}S)(X,Z)=g\big((\nabla_{\xi}R)(X,Y)Z,\xi\big). \tag{33}$$

We have,

$$\begin{aligned}(\nabla_X S)(\xi, Z) &= 2n\kappa(\nabla_X \eta)(Z) + S(\phi X, Z) + S(\phi h X, Z)\\ &= (2n\kappa + n\mu + \mu)g(hX, \phi Z).\end{aligned} \tag{34}$$

Again, from (8) we have

$$\begin{aligned}(\nabla_\xi S)(X, Z) &= (2n - 2 + \mu)g((\nabla_\xi h)(X), Z)\\ &= \mu\{2(n-1) + \mu\}g(h\phi X, Z).\end{aligned} \tag{35}$$

Moreover, applying covariant differentiation with respect to the vector field ξ, we obtain

$$\begin{aligned}(\nabla_\xi R)(X, \xi)Z &= -\mu g((\nabla_\xi h)(X), Z)\xi + \mu(\nabla_\xi \eta)(Z)hX\\ &= -\mu g(\mu(h\phi)(X), Z)\xi + \mu\eta(Z)\mu(h\phi)(X), \text{ by Lemma 3.1}\\ &= -\mu^2 g(h\phi X, Z).\end{aligned} \tag{36}$$

Combining the results (33), (34), (35) and (36) we finally obtain

$$(2n\kappa + n\mu + \mu)g(hX, \phi Z) - \mu\{2(n-1) + \mu\}g(h\phi X, Z) = \mu^2 g(h\phi X, Z).$$

$$\text{i.e., } (2n\kappa + n\mu + \mu)g(h\phi X, Z) = 0.$$

$$\text{i.e., } (2n\kappa + n\mu + \mu) = 0, \text{ since } g(h\phi X, Z) \not\equiv 0\ . \tag{37}$$

From (37) we get $\kappa = \frac{n-3}{2n}\mu$. Putting this value of κ in (23) we find

$$\mu\{\mu(n-3) - 2(n-1)(n+3)\} = 0. \tag{38}$$

So, either $\mu = 0$ or $\mu = \frac{2(n-1)(n+3)}{n-3}$. Hence we obtain the following set of values for κ and μ:

μ	κ
0	0
$\frac{2(n-1)(n+3)}{n-3}$	$\frac{(n-1)(n+3)}{n}$, unless $n = 3$

In case $n = 3$ from (37) we find $\kappa = 0$. Hence from (23) we find $\mu = 0$, whenever $n = 3$.

By similar argument, as explained earlier, we are to consider $\kappa = 0 = \mu$. Hence the same result follows for case (ii). Thus we can state:

Theorem 1 *A hyper-generalized recurrent* (κ, μ)*-contact metric manifold* (M^{2n+1}, g) *is locally isometric to either (i) the unit sphere* $S^{2n+1}(1)$ *or (ii) to the product space* $E^{n+1} \times S^n(4)$.

Recalling Theorem (2.1) (viii) of [8], a hyper-generalized recurrent (κ,μ)-contact metric manifold is generalized 2-Ricci recurrent. Hence we can state as follows:

Corollary 1 *A generalized 2-Ricci recurrent (κ,μ)-contact metric manifold is locally isometric to either (i) the unit sphere $S^{2n+1}(1)$ or (ii) to the product space $E^{n+1}\times S^n(4)$.*

Again by virtue of Theorem (2.1) (v) of [8], a hyper-generalized recurrent (κ,μ)-contact metric manifold is generalized conharmonically recurrent. Thus we have the following:

Corollary 2 *A generalized conharmonically recurrent (κ,μ)-contact metric manifold is locally isometric to either (i) the unit sphere $S^{2n+1}(1)$ or (ii) to the product space $E^{n+1}\times S^n(4)$.*

Again, in a (κ,μ)-contact metric manifold, if $\kappa = 1$, then it reduces to a Sasakian manifold. Now we are going to find the consequences of the above theorem for $\kappa = 1$ i.e. for the case of Sasakian manifolds.

Taking $X=\xi$ in (21) gives

$$\begin{aligned} & (\nabla_\xi S)(Y,Z)-(\nabla_Y S)(\xi,Z) \\ = \;& g(R(\xi,Y)Z,\sigma) \\ + \;& \big[S(Y,Z)\eta(\rho)-2n\eta(Z)g(Y,\rho)+2ng(Y,Z)\eta(\rho)-\eta(Z)S(Y,\rho)\big]. \end{aligned}$$

Since, for a Sasakian manifold ξ is a Killing vector field, therefore $\pounds_\xi S = 0$ and hence $\nabla_\xi S = 0$. Thereby from the above we obtain

$$\begin{aligned} & -S(\phi Y,Z)+2ng(\phi Y,Z) \\ = \;& g(R(\xi,Y)Z,\sigma) \\ + \;& \big[S(Y,Z)\eta(\rho)-2n\eta(Z)g(Y,\rho)+2ng(Y,Z)\eta(\rho)-\eta(Z)S(Y,\rho)\big]. \end{aligned} \tag{39}$$

Replacing Y and Z by ϕY and ϕZ respectively and using (1) and (2) yields

$$\begin{aligned} & S(Y,\phi Z)-2ng(Y,\phi Z) \\ = \;& \eta(\sigma)\{g(Y,Z)-\eta(Y)\eta(Z)\} \\ + \;& \eta(\rho)\big\{S(Y,Z)+2ng(Y,Z)-4n\eta(Y)\eta(Z)\big\}. \end{aligned} \tag{40}$$

Again replacing ϕY for Y in (40), we obtain

$$S(Y,Z)-2ng(Y,Z)=\eta(\sigma)g(\phi Y,Z)+\eta(\rho)\big\{S(\phi Y,Z)+2ng(\phi Y,Z)\big\}. \tag{41}$$

Since, S and g are symmetric, the left hand side of (41) is symmetric with respect to Y and Z. Hence we have

$$S(Y,Z) = 2ng(Y,Z). \tag{42}$$

Thus a hyper-generalized recurrent Sasakian manifold is an Einstein manifold with non-vanishing scalar curvature $r = 2n(2n+1)$. Using (42) in (17) and thereafter by (18), we acquire

$$(\nabla_W R)(X,Y)Z = -4nB(W)\{R(X,Y)Z - g(Y,Z)X + g(X,Z)Y\}. \tag{43}$$

On cyclic transposition of the last equation twice over X, Y, W and thereafter summing up these resulting equations we get by virtue of the second Bianchi identity,

$$\begin{aligned} & B(W)\{R(X,Y)Z - g(Y,Z)X + g(X,Z)Y\} \\ + \; & B(Y)\{R(W,X)Z - g(X,Z)W + g(W,Z)X\} \\ + \; & B(X)\{R(Y,W)Z - g(W,Z)Y + g(Y,Z)W\} = 0. \end{aligned} \tag{44}$$

On contraction with respect to W and using (42), we obtain

$$R(X,Y)\rho = B(Y)X - B(X)Y. \tag{45}$$

In a similar fashion, we can also find

$$R(Z,\rho)X = B(X)Z - g(X,Z)\rho. \tag{46}$$

Assigning $W = \rho$ in (44) and utilizing (45) and (46) one determines

$$g(\rho,\rho)\{R(X,Y)Z - g(Y,Z)X + g(X,Z)Y\} = 0.$$

Since $\rho \neq 0$, one must have for arbitrary vector fields X, Y and Z on M

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{47}$$

This implies the space under consideration 1 is of constant curvature 1 and hence locally isometric to the unit sphere. This gives the following theorem:

Theorem 2 *A hyper-generalized recurrent Sasakian manifold* (M^{2n+1}, g) *is of constant curvature 1 and hence locally isometric to a unit sphere* $S^{2n+1}(1)$.

Also by virtue of Theorem (2.1) (v) of [8], a hyper-generalized recurrent Sasakian manifold is generalized conharmonically recurrent. Hence we state the following:

Corollary 3 *A generalized conharmonically recurrent Sasakian manifold is of constant curvature 1 and hence locally isometric to a unit sphere* $S^{2n+1}(1)$.

Retrieving the Theorem (2.1) (viii) of [8], a hyper-generalized recurrent Sasakian manifold is generalized 2-Ricci recurrent. Thus one obtains,

Corollary 4 *A generalized 2-Ricci recurrent Sasakian manifold is of constant curvature 1 and hence locally isometric to a unit sphere* $S^{2n+1}(1)$.

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Inclusion relations for multiplier transformation

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Abstract. Due to widely study of K-uniformly typed of functions, we establish here the inclusion relations for K-uniformly starlike, K-uniformly convex, close to convex and quasi-convex functions under the $D^{\lambda;m}_{\mu,a}$ operator introduced by the authors [1].

1 Introduction

Let $U = \{z : z \in C \ \ |z| < 1\}$ be the open unit disk and A denotes the class of functions f normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which is analytic in the open unit disk U and satisfies the condition $f(0) = f'(0)-1 = 0$. A function $f \in A$ is said to be in $UST(k, \alpha)$, the class of k-uniformly starlike functions of order $\alpha,\ 0 \le \alpha < 1$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) - \alpha \ge k\left|\frac{zf'(z)}{f(z)} - 1\right|, \qquad k \ge 0,\ \ 0 \le \alpha < 1.$$

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Similarly, a function $f \in A$ is said to be in $UCV(k, \alpha)$, the class of k-uniformly convex functions of order α, $0 \leq \alpha < 1$ if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) - \alpha \geq k\left|\frac{zf''(z)}{f'(z)}\right|, \qquad k \geq 0,\ \ 0 \leq \alpha < 1.$$

The classes of uniformly convex and uniformly starlike were introduced by Goodman [3,4] and later generalized by Kanas and Wisniowska ([14],[15]) (see also the work of Kanas and Srivastava [16], Ronning ([7],[8]), Ma and Minda [20] and Gangadharan et al. [2]).

Let F and G be analytic functions in the unit disk U. The function F is subordinate to G written $F \prec G$. If G is univalent, then $F(0) = G(0)$ and $F(U) \subset G(U)$.
In general, given two functions F and G which are analytic in U, the function F is said to be subordinate to G if there exist a function w analytic in U with

$$w(0) = 0 \ \text{ and } (\forall z \in U) :\ |w(z)| < 1,$$

such that

$$(\forall z \in U) :\ F(z) = G(w(z)).$$

For arbitrarily chosen $k \in [0, \infty[$ and $0 \leq \alpha < 1$, let $\Omega_{k,\alpha}$ denote the domain

$$\Omega_{k,\alpha} = \{u + iv,\ \ (u-\alpha)^2 > k^2(u-1)^2 + k^2v^2\}.$$

This characterization enables us to designate precisely the domain $\Omega_{k,\alpha}$ as a convex domain contain in the right half-plane. Moreover, $\Omega_{k,\alpha}$ is an elliptic region for $k > 1$, parabolic for $k = 1$, hyperbolic for $0 < k < 1$ and finally $\Omega_{0,0}$ is the whole right half-plane.

Let $q_{k,\alpha}(z) : U \to \Omega_{k,\alpha}$ denote the conformal mapping of U onto $\Omega_{k,\alpha}$ so that $q_{k,\alpha}(0) = 0$, ${q_{k,\alpha}}'(0) > 0$. The explicit forms of $q_{k,\alpha}(z)$, were obtained in [13] as follows:

$$q_{k,\alpha}(z) = \begin{cases} \frac{1+(1-2\alpha)z}{1-z} & \text{for } k = 0, \\ \frac{1-\alpha}{1-k^2}\cos\left\{\frac{2}{\pi}\arccos(k) i \log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right\} - \frac{k^2-\alpha}{1-k^2} & \text{for } k \in (0,1) \\ 1 + \frac{2(1-\alpha)}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 & \text{for } k = 1, \\ \frac{1-\alpha}{k^2-1}\sin\left\{\frac{\pi}{2K(x)}\int_0^{\frac{u(z)}{\sqrt{x}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\right\} + \frac{k^2-\alpha}{k^2-1} & \text{for } k > 1, \end{cases}$$

where $u(z) = \frac{z-\sqrt{x}}{1-\sqrt{x}z}$, $x \in (0,1)$ and K is such $k = \cosh\frac{\pi K'(x)}{4K(x)}$.

Let P denote the class of Caratheodory functions analytic in U e.g.

$$P = \{p : \mathrm{p\,analytic\,in}\,U,\ p(0) = 1,\ \Re\, p(z) > 0\}.$$

The characterization of the classes $UST(k, \alpha)$ and $UCV(k, \alpha)$, can be expressed in terms of subordination as follows,

$$f \in UST(k, \alpha) \Leftrightarrow p(z) = \frac{zf'(z)}{f(z)} \prec q_{k,\alpha}(z), \quad z \in U,$$

and

$$f \in UCV(k, \alpha) \Leftrightarrow p(z) = \frac{zf''(z)}{f'(z)} + 1 \prec q_{k,\alpha}(z), \quad z \in U.$$

So that

$$\Re\, p(z) > \Re\, q_{k,\alpha}(z) > \frac{k+\alpha}{k+1}. \tag{1}$$

Define $UCC(k, \alpha, \beta)$ to be the family of functions $f \in A$ such that

$$\frac{zf'(z)}{g(z)} \prec q_{k,\alpha}(z), \quad z \in U,$$

for some $g(z) \in UST(k, \beta)$. On the other hand, let $UQC(k, \alpha, \beta)$ be the family of functions $f \in A$ such that

$$\frac{(zf'(z))'}{g'(z)} \prec q_{k,\alpha}(z), \quad z \in U,$$

for some $g(z) \in UCV(k, \beta)$.

We observe that, $UCC(0, \alpha, \beta)$ is the class of close-to-convex functions of order α and type β and $UQC(0, \alpha, \beta)$ is the class of quasi-convex functions of order α and type β.

We now state the following definition.

Definition 1 ([1]) *Let the function $f \in A$, then for $\mu, m \in C$, $a \in C/\{-1, -2, \dots\}$, and $\lambda > -1$, we define the following operator:*

$$D^{\lambda,m}_{\mu,a} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+a}{1+a}\right)^m \frac{(\lambda+1)_{k-1}}{(\mu)_{k-1}} a_k z^k. \tag{2}$$

Here $(x)_k$ is Pochhammer symbol (or the shifted factorial), defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} \begin{cases} 1, k=0 & \text{and } x \in C\backslash\{0\}; \\ x(x+1)\ldots(x+k-1) \text{ if} k \in N & \text{and } x \in C, \end{cases}$$

and $\Gamma(x),\ (x \in C)$ denotes the Gamma function.

It should be noted that the operator $D^{\lambda,m}_{\mu,a} f(z)$ is a generalization of many operators considered earlier. For $m \in Z$, $a \geq 1$, $\mu = 1 \,\text{and}\, \lambda = 0$ the operator $D^{\lambda,m}_{\mu,a}$ were studied by Cho and Srivastava [6], for $m = -1$, $\mu = 1 \,\text{and}\, \lambda = 0$ the operator is the integral operator studied by Owa and Srivastava [17], for any negative real number m and $\mu = 1$, $a = 1$, $\lambda = 0$ the operator $D^{\lambda,m}_{\mu,a}$ is the integral operator studied by Jung et. al [5], for any nonnegative integer number m and $\mu = 1$, $a = 0$, $\lambda = 0$ the operator $D^{\lambda,m}_{\mu,a}$ is the differential operator defined by Salagean [9], for $m = 0$, $\mu = 1$, $\lambda > -1$ the operator $D^{\lambda,m}_{\mu,a}$ is the differential operator defined by Ruscheweyh [19], for $\mu = 1 \,\text{and}\, \lambda > -1$ the operator $D^{\lambda,m}_{\mu,a}$ is the multiplier transformations defined by Al-Shaqsi and Darus [10] and for $D^{\lambda,m}_{\mu,a}$ the operator $D^{\lambda,m}_{\mu,a}$ is the derivative operator given by Al-Shaqsi and Darus [11]. In particular, we note that $D^{0,0}_{1,a} = f(z)$ and $D^{0,1}_{1,0} = zf'(z)$.

It is readily verified from (2) that

$$z(D^{\lambda,m}_{\mu+1,a} f(z))' = \mu D^{\lambda,m}_{\mu,a} f(z) - (\mu-1) D^{\lambda,m}_{\mu+1,a} f(z) \tag{3}$$

$$z(D^{\lambda,m}_{\mu,a} f(z))' = (\lambda+1) D^{\lambda+1,m}_{\mu,a} f(z) - \lambda D^{\lambda,m}_{\mu,a} f(z) \tag{4}$$

$$z(D^{\lambda,m}_{\mu,a} f(z))' = (a+1) D^{\lambda,m+1}_{\mu,a} f(z) - a D^{\lambda,m}_{\mu,a} f(z). \tag{5}$$

2 Main results

The main object of this paper is to study the inclusion properties of the above-mentioned classes under the multiplier transformation $D^{\lambda,m}_{\mu,a} f(z)$.

We shall need the following lemmas to prove our theorems:

Lemma 1 ([12]) *Let σ, ν be complex numbers. Suppose also that $m(z)$ be convex univalent in U with $m(0) = 1$ and $\Re[\sigma m(z) + \nu] > 0$, $z \in U$. If $u(z)$ is analytic in U with $u(0) = 1$, then*

$$u(z) + \frac{zu'(z)}{\sigma\, u(z) + \nu} \prec m(z) \ \Rightarrow u(z) \prec m(z)\ .$$

Lemma 2 ([18]) *Let $h(z)$ be the convex in the unit disk U and let $E \geq 0$. suppose $B(z)$ is analytic in U with $\Re B(z) > E$. If $g(z)$ is analytic in U and $g(0) = h(0)$. Then*

$$E\, z^2 g''(z) + B(z) z g'(z) + g(z) \prec h(z) \Rightarrow g(z) \prec h(z).$$

Our first result is the following:

Theorem 1 *Let $f(z) \in A$.*
If $D^{\lambda,m}_{\mu,a} f(z) \in UST(k,\alpha)$, and $\Re\mu > \frac{1-\alpha}{1+k}$, then $D^{\lambda,m}_{\mu+1,a} f(z) \in UST(k,\alpha)$.

Proof. Let $p(z) = \frac{z\left(D^{\lambda,m}_{\mu+1,a} f(z)\right)'}{D^{\lambda,m}_{\mu+1,a} f(z)}$. In view of (3), we can write

$$\frac{\mu D^{\lambda,m}_{\mu,a} f(z)}{D^{\lambda,m}_{\mu+1,a} f(z)} = p(z) + \mu - 1.$$

Differentiating the above expression yields

$$\frac{z\left(D^{\lambda,m}_{\mu,a} f(z)\right)'}{D^{\lambda,m}_{\mu,a} f(z)} = p(z) + \frac{zp'(z)}{p(z) + \mu - 1}.$$

From this and argument given in the introduction we may write

$$p(z) + \frac{zp'(z)}{p(z) + \mu - 1} \prec q_{k,\alpha}(z).$$

Therefore, the theorem follows by Lemma 1 and the condition (1) since $q_{k,\alpha}(z)$ is univalent and convex in U and $\Re(q_{k,\alpha}(z)) > \frac{k+\alpha}{k+1}$. $\square$

Theorem 2 *Let $f(z) \in A$.*
If $D^{\lambda,m}_{\mu,a} f(z) \in UCV(k,\alpha)$, then $D^{\lambda,m}_{\mu+1,a} f(z) \in UCV(k,\alpha)$.

Proof.

$$\begin{aligned} D^{\lambda,m}_{\mu,a} f(z) \in UCV(k,\alpha) &\Leftrightarrow z\left(D^{\lambda,m}_{\mu,a} f(z)\right)' \in UST(k,\alpha) \\ &\Leftrightarrow D^{m}_{\mu,a}(zf'(z)) \in UST(k,\alpha) \\ &\Leftrightarrow D^{m}_{\mu+1,a}(zf'(z)) \in UST(k,\alpha) \\ &\Leftrightarrow D^{m}_{\mu+1,a} f(z) \in UCV(k,\alpha), \end{aligned}$$

and the proof is complete. $\square$

Theorem 3 *Let* $f(z) \in A$.
If $D_{\mu,a}^{\lambda,m} f(z) \in UCC(k,\alpha,\beta)$, *and* $\Re\mu > \frac{1-\alpha}{1+k}$, *then* $D_{\mu+1,a}^{\lambda,m} f(z) \in UCC(k,\alpha,\beta)$.

Proof. Since $D_{\mu,a}^{\lambda,m} f(z) \in UCC(k,\alpha,\beta)$, by definition, we can write

$$\frac{z\left(D_{\mu,a}^{\lambda,m} f(z)\right)'}{K(z)} \prec q_{k,\alpha}(z),$$

for some $K(z) \in UST(k,\beta)$. For $g(z)$ such that $D_{\mu,a}^{\lambda,m} g(z) = K(z)$, we have

$$\frac{z\left(D_{\mu,a}^{\lambda,m} f(z)\right)'}{D_{\mu,a}^{\lambda,m} g(z)} \prec q_{k,\alpha}(z). \tag{6}$$

Letting $r(z) = \frac{z\left(D_{\mu+1,a}^{\lambda,m} f(z)\right)'}{D_{\mu+1,a}^{\lambda,m} g(z)}$ *and* $R(z) = \frac{z\left(D_{\mu+1,a}^{\lambda,m} g(z)\right)'}{D_{\mu+1,a}^{\lambda,m} g(z)}$, we observe that r and R are analytic in U and $r(0) = R(0) = 1$. Now, by Theorem 1, $D_{\mu+1,a}^{\lambda,m} g(z) \in UST(k,\beta)$ and so $\Re(R(z)) > \frac{k+\alpha}{k+1}$,also, note that

$$z\left(D_{\mu+1,a}^{\lambda,m} f(z)\right)' = \left(D_{\mu+1,a}^{\lambda,m} g(z)\right) r(z). \tag{7}$$

Differentiating both sides of (7) yields

$$z\frac{\left(z\left(D_{\mu+1,a}^{\lambda,m} f(z)\right)'\right)'}{D_{\mu+1,a}^{\lambda,m} g(z)} = \frac{z\left(D_{\mu+1,a}^{\lambda,m} g(z)\right)'}{D_{\mu+1,a}^{\lambda,m} g(z)} r(z) + zr'(z) = R(z)r(z) + zr'(z).$$

Now using the identity (3), we obtain

$$\begin{aligned}
\frac{z\left(D_{\mu,a}^{\lambda,m} f(z)\right)'}{D_{\mu,a}^{\lambda,m} g(z)} &= \frac{D_{\mu,a}^{\lambda,m}\left(zf'(z)\right)}{D_{\mu,a}^{\lambda,m} g(z)} \\
&= \frac{z(D_{\mu+1,a}^{\lambda,m} zf'(z))' + (\mu-1)D_{\mu+1,a}^{\lambda,m}\left(zf'(z)\right)}{z(D_{\mu+1,a}^{\lambda,m} g(z))' + (\mu-1)D_{\mu+1,a}^{\lambda,m} g(z)} \\
&= \frac{\frac{z(D_{\mu+1,a}^{\lambda,m} zf'(z))'}{D_{\mu+1,a}^{\lambda,m} g(z)} + (\mu-1)\frac{D_{\mu+1,a}^{\lambda,m}(zf'(z))}{D_{\mu+1,a}^{\lambda,m} g(z)}}{\frac{z(D_{\mu+1,a}^{\lambda,m} g(z))'}{D_{\mu+1,a}^{\lambda,m} g(z)} + (\mu-1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\dfrac{z\left(z\left(D^{\lambda,m}_{\mu+1,a}f(z)\right)'\right)'}{D^{\lambda,m}_{\mu+1,a}g(z)} + (\mu-1)\dfrac{z\left(D^{\lambda,m}_{\mu+1,a}f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}g(z)}}{\dfrac{z(D^{\lambda,m}_{\mu+1,a}g(z))'}{D^{\lambda,m}_{\mu+1,a}g(z)} + (\mu-1)} \\
&= \frac{R(z)r(z) + zr'(z) + (\mu-1)r(z)}{R(z)+(\mu-1)} \\
&= r(z) + \frac{zr'(z)}{R(z)+(\mu-1)}.
\end{aligned} \tag{8}
$$

From (6), (7) and (8), we conclude that

$$r(z) + \frac{zr'(z)}{R(z)+(\mu-1)} \prec Q_{k,\alpha}(z).$$

In order to apply Lemma 2, Let $E = 0$ *and* $B(z) = \frac{1}{R(z)+(\mu-1)}$, we obtain

$$\Re(B(z)) = \frac{1}{|R(z)+(\mu-1)|^2}\Re(R(z)+(\mu-1)) > 0.$$

Then we conclude that $r(z) \prec q_{k,\alpha}(z)$ and so the proof is complete. □

Using a similar argument in Theorem 2, we can prove

Theorem 4 *Let* $f(z) \in A$.
If $D^{\lambda,m}_{\mu,a}f(z) \in UQC(k,\alpha,\beta)$, *then* $D^{\lambda,m}_{\mu+1,a}f(z) \in UQC(k,\alpha,\beta)$.

Now, we examine the closure property of the above classes of functions under the generalized Bernardi-Libera-Livingston operator $\Psi_c(f)$ *which is defined by*

$$\Psi_c(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \ \ (c > -1), \quad f(z) \in A. \tag{9}$$

Theorem 5 *Let* $c > \frac{-(k+\alpha)}{k+1}$.
If $D^{\lambda,m}_{\mu+1,a}f(z) \in UST(k,\alpha)$, *then* $D^{\lambda,m}_{\mu+1,a}\Psi_c(f(z)) \in UST(k,\alpha)$, *where* Ψ_c *is the integral operator defined by (9).*

Proof. From (3) and (9), we have

$$z(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z))' = (c+1)D^{\lambda,m}_{\mu+1,a}f(z) - cD^{\lambda,m}_{\mu+1,a}\Psi_c f(z). \tag{10}$$

Substituting $p(z) = \frac{z\left(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)}$ in (10), we can write

$$(c+1)\frac{D^{\lambda,m}_{\mu+1,a}f(z)}{D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)} = p(z) + c \tag{11}$$

Differentiating (11) yields

$$\frac{z\left(D^{\lambda,m}_{\mu+1,a}f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}f(z)} = p(z) + \frac{zp'(z)}{p(z)+c}$$

Applying Lemma 1, it follows that $p(z) \prec q_{k,\alpha}(z)$, that is,

$$\frac{z\left(D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}\Psi_c f(z)} \prec q_{k,\alpha}(z),$$

and so

$$D^{\lambda,m}_{\mu+1,a}\Psi_c f(z) \in UST(k,\alpha).$$

□

A similar argument leads to:

Theorem 6 *Let* $c > \frac{-(k+\alpha)}{k+1}$.

If $D^{\lambda,m}_{\mu+1,a}f(z) \in UCV(k,\alpha)$, *then* $D^{\lambda,m}_{\mu+1,a}\Psi_c(f(z)) \in UCV(k,\alpha)$, *where* Ψ_c *is the integral operator defined by (9).*

Theorem 7 *Let* $c > \frac{-(k+\alpha)}{k+1}$.

If $D^{\lambda,m}_{\mu+1,a}f(z) \in UCC(k,\alpha)$, *then* $D^{\lambda,m}_{\mu+1,a}\Psi_c(f(z)) \in UCC(k,\alpha)$.

Proof. By definition, there exists a function $K(z) \in UST(k,\beta)$ and for $g(z)$ such that $D^{\lambda,m}_{\mu+1,a}g(z) = K(z)$, we have

$$\frac{z\left(D^{\lambda,m}_{\mu+1,a}f(z)\right)'}{D^{\lambda,m}_{\mu+1,a}g(z)} \prec Q_{k,\alpha}(z). \tag{12}$$

Now from (10) we have

$$\begin{aligned}
\frac{z\left(D_{\mu+1,a}^{\lambda,m}f(z)\right)'}{D_{\mu+1,a}^{\lambda,m}g(z)} &= \frac{D_{\mu+1,a}^{\lambda,m}\left(zf'(z)\right)}{D_{\mu+1,a}^{\lambda,m}g(z)} \\
&= \frac{z(D_{\mu+1,a}^{\lambda,m}\Psi_c zf'(z))' + cD_{\mu+1,a}^{\lambda,m}\Psi_c zf'(z)}{z(D_{\mu+1,a}^{\lambda,m}\Psi_c g(z))' + cD_{\mu+1,a}^{\lambda,m}\Psi_c g(z)} \\
&= \frac{z\left(z\left(D_{\mu+1,a}^{\lambda,m}\Psi_c f(z)\right)'\right)' + cz\left(D_{\mu+1,a}^{\lambda,m}\Psi_c f(z)\right)'}{z(D_{\mu+1,a}^{\lambda,m}\Psi_c g(z))' + cD_{\mu+1,a}^{\lambda,m}\Psi_c g(z)} \qquad (13) \\
&= \frac{\frac{z\left(z\left(D_{\mu+1,a}^{\lambda,m}\Psi_c f(z)\right)'\right)'}{D_{\mu+1,a}^{\lambda,m}\Psi_c g(z)} + c\frac{z\left(D_{\mu+1,a}^{\lambda,m}\Psi_c f(z)\right)'}{D_{\mu+1,a}^{\lambda,m}\Psi_c g(z)}}{\frac{z(D_{\mu+1,a}^{\lambda,m}\Psi_c g(z))'}{D_{\mu+1,a}^{\lambda,m}\Psi_c g(z)} + c}.
\end{aligned}$$

Since $D_{\mu+1,a}^{\lambda,m}g(z) \in UST(k,\beta)$, by Theorem 6, we have $D_{\mu+1,a}^{\lambda,m}\Psi_c(g(z)) \in UST(k,\beta)$. Letting $r(z) = \frac{z\left(D_{\mu+1,a}^{\lambda,m}\Psi_c f(z)\right)'}{D_{\mu+1,a}^{\lambda,m}\Psi_c g(z)}$ *and* $R(z) = \frac{z\left(D_{\mu+1,a}^{\lambda,m}\Psi_c g(z)\right)'}{D_{\mu+1,a}^{\lambda,m}\Psi_c g(z)}$, we observe that $\Re\{R(z)\} > \frac{k+\beta}{k+1}$. Also, note that

$$z\left(D_{\mu+1,a}^{\lambda,m}\Psi_c f(z)\right)' = (D_{\mu+1,a}^{\lambda,m}\Psi_c g(z))r(z) \qquad (14)$$

Differentiating both sides of (14) yields

$$z\frac{\left(z\left(D_{\mu+1,a}^{\lambda,m}\Psi_c f(z)\right)'\right)'}{D_{\mu+1,a}^{\lambda,m}\Psi_c g(z)} = z\frac{\left(D_{\mu+1,a}^{\lambda,m}\Psi_c g(z)\right)'}{D_{\mu+1,a}^{\lambda,m}\Psi_c g(z)}r(z)+zr'(z) = R(z)r(z)+zr'(z). \qquad (15)$$

Therefore from (13) and (10), we obtain

$$\frac{z\left(D_{\mu+1,a}^{\lambda,m}f(z)\right)'}{D_{\mu+1,a}^{\lambda,m}g(z)} = \frac{R(z)r(z)+zr'(z)+cr(z)}{R(z)+c} = r(z) + \frac{zr'(z)}{R(z)+c}.$$

From (12), (14) and (15), we conclude that

$$r(z) + \frac{zr'(z)}{R(z)+c} \prec q_{k,\alpha}(z).$$

In order to apply Lemma 2, Let $E = 0$ *and* $B(z) = \frac{1}{R(z)+c}$, we note that

$$\Re\{B(z)\} > 0 \text{ if } c > -\frac{k+\beta}{k+1}.$$

Then we conclude that $r(z) \prec Q_{k,\alpha}(z)$ and so the proof is complete. A similar argument yields. $\square$

Theorem 8 *Let* $c > \frac{-(k+\alpha)}{k+1}$.
If $D^{\lambda,m}_{\mu+1,a} f(z) \in UQC(k,\alpha,\beta)$, *then* $D^{\lambda,m}_{\mu+1,a}\Psi_c(f(z)) \in UQC(k,\alpha,\beta)$.

Similarly by using (4) and (5) we obtain the following results. Since the proof of the results is similar to the proof of Theorems 1-8, it will be omitted.

Theorem 9 *Let* $f(z) \in A$.
If $D^{\lambda+1,m}_{\mu,a} f(z) \in UST(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a} f(z) \in UST(k,\alpha)$.

Theorem 10 *Let* $f(z) \in A$.
If $D^{\lambda+1,m}_{\mu,a} f(z) \in UCV(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a} f(z) \in UCV(k,\alpha)$.

Theorem 11 *Let* $f(z) \in A$.
If $D^{\lambda+1,m}_{\mu,a} f(z) \in UCC(k,\alpha,\,\beta)$, *then* $D^{\lambda,m}_{\mu,a} f(z) \in UCC(k,\alpha,\,\beta)$.

Theorem 12 *Let* $f(z) \in A$.
If $D^{\lambda+1,m}_{\mu,a} f(z) \in UQC(k,\alpha,\,\beta)$, *then* $D^{\lambda,m}_{\mu,a} f(z) \in UQC(k,\alpha,\,\beta)$.

Theorem 13 *Let* $f(z) \in A$.
If $D^{\lambda,m+1}_{\mu,a} f(z) \in UST(k,\alpha)$, *and* $\Re\{a\}\,\frac{-(k+\alpha)}{k+1}$, *then* $D^{\lambda,m}_{\mu,a} f(z) \in UST(k,\alpha)$.

Theorem 14 *Let* $f(z) \in A$.
If $D^{\lambda,m+1}_{\mu,a} f(z) \in UCV(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a} f(z) \in UCV(k,\alpha)$.

Theorem 15 *Let* $f(z) \in A$.
If $D^{\lambda,m+1}_{\mu,a} f(z) \in UCC(k,\alpha,\,\beta)$, *and* $\Re\{a\} > \frac{-(k+\alpha)}{k+1}$, *then* $D^{\lambda,m}_{\mu,a} f(z) \in UCC(k,\alpha,\,\beta)$.

Theorem 16 *Let* $f(z) \in A$.
If $D^{\lambda,m+1}_{\mu,a} f(z) \in UQC(k,\alpha,\,\beta)$, *then* $D^{\lambda,m}_{\mu,a} f(z) \in UQC(k,\alpha,\,\beta)$.

Theorem 17 *Let* $c > \frac{-(k+\alpha)}{k+1}$.
If $D^{\lambda,m}_{\mu,a} f(z) \in UST(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a}\Psi_c(f(z)) \in UST(k,\alpha)$.

Theorem 18 *Let* $c > \frac{-(k+\alpha)}{k+1}$.
If $D^{\lambda,m}_{\mu,a} f(z) \in UCV(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a} \Psi_c(f(z)) \in UCV(k,\alpha)$.

Theorem 19 *Let* $c > \frac{-(k+\alpha)}{k+1}$.
If $D^{\lambda,m}_{\mu,a} f(z) \in UCC(k,\alpha)$, *then* $D^{\lambda,m}_{\mu,a} \Psi_c(f(z)) \in UCC(k,\alpha)$.

Theorem 20 *Let* $c > \frac{-(k+\alpha)}{k+1}$.
If $D^{\lambda,m}_{\mu,a} f(z) \in UQC(k,\alpha,\beta)$, *then* $D^{\lambda,m}_{\mu,a} \Psi_c(f(z)) \in UQC(k,\alpha,\beta)$.

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Minimal digraphs with given imbalance sequence

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Abstract. Let a and b be integers with $0 \le a \le b$. An (a,b)-*graph* is such digraph D in which any two vertices are connected at least a and at most b arcs. The *imbalance* $a(v)$ of a vertex v in an (a,b)-graph D is defined as $a(v) = d^+(v) - d^-(v)$, where $d^+(v)$ is the outdegree and $d^-(v)$ is the indegree of v. The *imbalance sequence* A of D is formed by listing the imbalances in nondecreasing order. A sequence of integers is (a,b)-*realizable*, if there exists an (a,b)-graph D whose imbalance sequence is A. In this case D is called a *realization* of A. An (a,b)-realization D of A is *connection minimal* if does not exist (a,b')-realization of D with $b' < b$. A digraph D is *cycle minimal* if it is a connected digraph which is either acyclic or has exactly one oriented cycle whose removal disconnects D. In this paper we present algorithms which construct connection minimal and cycle minimal realizations having a given imbalance sequence A.

1 Introduction

Let a, b and n be nonnegative integers with $0 \le a \le b$ and $n \ge 1$. An (a,b)-*graph* is a digraph D in which any two vertices are connected at least a and at most b arcs. If $d^-(v)$ denotes the outdegree and $d^+(v)$ denotes of vertex v in an (a,b)-graph D then the *imbalance* [14] of v is defined as

$$a(v) = d^+(v) - d^-(v).$$

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Since loops have no influence on the imbalances therefore for the simplicity we suppose everywhere in this paper that the investigated graphs are loopless.

The *imbalance sequence* of D is formed by listing its imbalances in nondecreasing order (although imbalances can be listed in nonincreasing order as well). The set of distinct imbalances of a digraph is called its *imbalance set*. Mostly the literature on imbalance sequences is concerned with obtaining necessary and sufficient conditions for a sequence of integers to be an imbalance sequence of different digraphs [9, 10, 11, 14, 18, 19, 20, 21], although there are papers on the imbalance sets too [15, 16, 17, 19].

If D is an (a, b)-digraph and A is its imbalance sequence then D is a *realization* of A. If we wish to find a realization of A in any set of directed graphs then

$$\sum_{i=1}^{n} a_i = 0 \tag{1}$$

is a natural necessary condition. If we allow parallel arcs then this simple condition is sufficient to find a realization. If parallel arcs are not allowed then the simple examle $A = [-3, 3]$ shows that (1) is *not sufficient* to find a realization.

Mubayi et al. [14] characterized imbalance sequences of simple digraphs (digraphs without loops and parallel arcs [2, 9, 25]) proving the following necessary and sufficient condition. We remark that simple digraphs are such $(0, 2)$-graphs which do not contain loops and parallel arcs.

Theorem 1 (Mubayi, Will, West, 2001 [14]) *A nondecreasing sequence* $A = [a_1, \ldots, a_n]$ *of integers is the imbalance sequence of a simple digraph iff*

$$\sum_{i=1}^{k} a_i \leq k(n-k) \tag{2}$$

for $1 \leq k \leq n$ *with equality when* $k = n$.

Proof. See [14]. □

Mubayi et al. [14] provided a Havel-Hakimi type [3, 4, 7, 8] greedy algorithm Greedy for constructing a simple realization.

The pseudocode of Greedy follows the conventions used in [1].

The *input data* of Greedy are n: the number of elements A ($n \geq 2$); $A = (a_1, \ldots, a_n)$: a nondecreasing sequence of integers satisfying (2). Its *output* is M: the $n \times n$ sized incidence matrix of a simple directed graph D

whose imbalance sequence is A. The *working variables* are the cycle variables i, j, k, l, x and y.

Greedy(n, A)

```
01 for i ← 1 to n                                   // line 01–03: initialization of M
02     for j ← 1 to n
03         M_ij ← 0
04 i = 1                                            // line 04–10: computation of M
05 while a_i > 0
06        Let k = a_i, j_1 < ··· < j_k, further let a_{j_1}, ..., a_{j_k} be
          the k smallest elements among a_{i+1}, ..., a_n, where
          a_x smaller a_y means that a_x < a_y or a_x = a_y and x < y
07        for l ← 1 to k
08            a_l ← a_l + 1
09            M_{i,a_l} ← 1
10        i ← i + 1
11 return M                                         // line 11: return of the result
```

The running time of Greedy is $\Theta(n^2)$ since the lines 1–3 require $\Theta(n^2)$ time, the **while** cycle executes $O(n)$ times and in the cycle line 06 and line 07 require $O(n)$ time.

Kleitman and Wang in 1973 [12] proposed a new version of Havel-Hakimi algorithm, where instead of the recursive choosing the largest remaining element of the investigated degree sequence it is permitted to choose *arbitrary element.* Mubayi et al. [14] point out an interesting difference between the directed and undirected graphs. Let us consider the imbalance sequence $A = [-3, 1, 1, 3]$ of a transitive tournament. Deleting the element 1 and adding 1 to the smallest imbalance leaves us trying to realize $[-2, -1, 3]$, which has no realization among the simple digraphs although it has among the $(0, 2)$-graphs.

Let $\alpha(D)$ denote the number of edges of D. It is easy to see the following assertion.

Lemma 1 *If a directed graph* D *is a realization of a sequence* $A = [a_1, \ldots, a_n]$ *then*

$$\alpha(D) \geq \frac{1}{2} \sum_{i=1}^{n} |a_i|. \tag{3}$$

Proof. Any realization has to contain at least so many outgoing arcs as the sum of the positive elements of A. Since S is realizable for the corresponding set, according to (1) the sum of the absolute values of the negative elements of A equals to the sum of the positive elements, therefore we have to divide the sum in (3) by 2. □

A realization D of A is called *arc minimal* (for a given set of digraphs) if A has no realization (in the given set) containing less arcs than D. Mubayi et al. [14] proved the following characterization of GREEDY.

Lemma 2 (Mubayi et al., 2001 [14]) *If A is realizable for the simple graphs then the realization generated by* GREEDY *contains the minimal number of arcs.*

Proof. See in [14]. □

Lemma 1 and Lemma 2 imply the following assertion.

Theorem 2 *If A is realizable for simple digraphs then the realization generated by* GREEDY *is arc minimal and the number of arcs contained by the realization is given by the lower bound* (3)*.*

Wang [22] gave an asymptotic formula for the number of labeled simple realizations of an imbalance sequence.

In this paper we deal with the more general problem of (a, b)-graphs. The following recent paper characterizes the imbalance sequences of $(0, b)$-graphs.

Theorem 3 (Pirzada, Naikoo, Samee, Iványi, 2010 [19]) *A nondecreasing sequence $A = [a_1, \dots, a_n]$ of integers is the imbalance sequence of a $(0, b)$-graph iff*

$$\sum_{i=1}^{k} a_i \leq bk(n-k) \tag{4}$$

for $1 \leq k \leq n$ with equality when $k = n$.

Proof. See [19]. □

We say that a realization D is *cycle minimal* if D is connected and does not contain a nonempty set of arcs S such that deleting S keeps the digraph connected but preserves imbalances of all vertices. Obviously such a set S, if it exists, must add 0 to the imbalances of vertices incident to it and hence must be a union of oriented cycles. Thus a cycle minimal digraph is either acyclic or has exactly one oriented cycle whose removal disconnects the digraph. For the sake of brevity we shall use the phrase *minimal realization* to refer a cycle minimal realization of A. We denote the set of all minimal realizations of A by $\mathcal{M}(\mathcal{A})$.

A realization D of A is called *connection minimal* (for a given set of directed graphs) if the maximal number of arcs $\gamma(D)$ connecting two different vertices of D is minimal.

The aim of this paper is to construct a connection and a cycle minimal digraph D having a prescribed imbalance sequence A. At first we determine the minimal b which allows to reconstruct the given A. Then we apply a series of arithmetic operations called *contractions* to the imbalance sequence A. This gives us a *chain* $C(A)$ of imbalance sequences. Then by the recursive transformations of $C(A)$ we get a required D.

The structure of the paper is as follows. After the introductory Section 1 in Section 2 we present an algorithm which determines the minimal number of arcs which are necessary between the neighboring vertices to realize a given imbalance sequence then in Section 3 we define a contraction operation and show that the contraction of an imbalance sequence produces another imbalance sequence. Finally in Section 4 we present an algorithm which constructs a connection minimal realization of an imbalance sequence.

2 Computation of the minimal r

According to (1) the sum of elements of any imbalance sequence equals to zero. Let us suppose that according to (1) the sum of the elements of a potential imbalance sequence $P = [p_1, \ldots, p_n]$ is zero and $b = \max(-a_1, a_n)$. Then it is easy to construct such $(0, b)$-digraph D whose imbalance sequence is A connecting the vertices having positive imbalance with the vertices having negative imbalance using the prescribed number of arcs. It is a natural question the value $b_{\text{min}}(P)$ defined as the minimal value of b sufficient for a potential imbalance sequence P to be the imbalance sequence of some $(0, b)$-graph.

$b_{\text{min}}(P)$ has the following natural bounds.

Lemma 3 *If* $A = [a_1, \ldots, a_n]$ *is an imbalance sequence, then*

$$\left\lceil \frac{a_n - a_1}{n} \right\rceil \le b_{\min} \le \min(-a_1, a_n). \tag{5}$$

The following algorithm Bmin computes $b_{\min}(A)$ for a sequence $A = [a_1, \ldots, a_n]$ satisfying (4). Bmin is based on Theorem 3, on the bounds given by Lemma 3 and on the logarithmic search algorithm described by D. E. Knuth [13, page 410] and is similar to algorithm MinF-MaxG [6, Section 4.2].

Input. n: the number of elements A ($n \ge 2$);
$A = [a_1, \ldots, a_n]$: a nondecreasing sequence of integers satisfying (4).

Output. $b_{\min}(A)$: the smallest sufficient value of b.

Working variables. k: cycle variable;
l: current value of the lower bound of $b_{\min}(A)$;
u: current value of the upper bound of $b_{\min}(A)$;
S: the current sum of the first k elements of A.

BMIN(n, A)

01 $l \leftarrow \lceil a_n - a_1 \rceil$ // line 01–02: initialization of l and u
02 $u \leftarrow \min(a_n, -a_i)$
03 **while** $l < u$ // line 03–14: computation of the minimal necessary b
04 $\quad b \leftarrow \lfloor \frac{l+u}{2} \rfloor$
05 $\quad S = S \leftarrow 0$
06 $\quad$ **for** $k \leftarrow 1$ **to** $n - 1$
07 $\qquad S \leftarrow S + a_i$
08 $\qquad$ **if** $S < bk(n-k)$
09 $\qquad\quad l \leftarrow r$
10 $\qquad$ **if** $l == r + 1$
11 $\qquad\quad b_{\min} \leftarrow l + 1$
12 $\qquad\quad$ **return** b
13 $\quad$ **go to** 03
14 $\quad u \leftarrow b$
15 $b_{\min} \leftarrow l$ // line 15–16: return of the computed minimal b
16 **return** $b_{\min}$

The next assertion characterizes BMIN.

Lemma 4 *Algorithm* BMIN *computes* $b_{\min}$ *for a sequence* $A = [a_1, \ldots, a_n]$ *satisfying* (4) *in* $\Theta(n \log n)$ *time.*

Proof. BMIN computes $b_{\min}$ on the base of Theorem 3 therefore it is correct. Running time of BMIN is $\Theta(n \log n)$ since the **while** cycle executes $\Theta(\log n)$ times and the **for** cycle in it requires $\Theta(n)$ time. □

3 Contraction of an imbalance sequence

Let D be a digraph having n vertices and m arcs. Throughout we assume that the vertices of D are labeled $v_1, \ldots, v_n$ according to their imbalances in nondecreasing order while the arcs of D are labeled $e_1, \ldots, e_m$ arbitrarily. Let $A = [a_1, \ldots, a_n] = [a_{n1}, \ldots, a_{nn}]$ be the imbalance sequence of D, where

$a_i = a_{ni}$ is the imbalance of vertex $v_i = v_{ni}$. We define an arithmetic operation, called *contraction* on A as follows.

Let $n \geq 2$, an imbalance sequence $A = [a_1, \dots, a_n]$ and an ordered pair (a_i, a_j) with $1 \leq i,\ j \leq n$ and $i \neq j$ be given. Then the *contraction* of (a_i, a_j) means that we delete a_i and a_j from A, add a new element $a_i' = a_i + a_j$ and sort nonincreasingly the received sequence using Counting-Sort [1] so that the indices of the elements are updated and the updated index of $a_i + a_j$ is denoted by k_i. The new sequence is denoted by $A/(i, j)$.

Note that $j < i$ is permitted. We refer to $A/(i, j)$ as a *minor* of A. Our terminology is inspired by the concept of minor and edge contraction from graph theory [23, 24]. The proof of Theorem 5 explains our choice of terminology.

An imbalance sequence A is a $(0, b)$-*imbalance sequence* if at least one of its realizations is a $(0, b)$-graph. We also observe that if $b' > b$ then a $(0, b)$-imbalance sequence is also a $(0, b')$-imbalance sequence.

The next assertion allows us to construct imbalance sequences and their realizations recursively. It also establishes a relation between the arithmetic operation of contraction discussed above and the edge contraction operation of graphs.

Theorem 4 *If A is a $(0, b)$-imbalance sequence, then all its minors are $(0, 2b)$-imbalance sequences.*

Proof. Let A be the imbalance sequence of a $(0, b)$-graph. Suppose that $B = A/(p, q)$ and let a_p and a_q be both negative with $a_p \leq a_q$. Then $b_l = a_p + a_q$ so that $b_l < a_p$. Thus, for all $k \leq q$, we have

$$\begin{aligned}
\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{k} b_i + (q-k)a_q, \quad \text{(since all these elements are negative)} \\
&\geq \sum_{i=1}^{q} a_i, \qquad\qquad \text{(since } A \text{ is a nondecreasing sequence).}
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{k} b_i &\geq \sum_{i=1}^{q} a_i - (q-k)a_q \\
&\geq \sum_{i=1}^{k} a_i \geq bk(k-n), \quad \text{(since } A \text{ is an imbalance sequence)} \\
&\geq (2b)k(k-n+1) \qquad \text{(since } n \geq k+2\text{),}
\end{aligned}$$

For $q < k \le n-1$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &= \sum_{i=1}^{k} a_i \\ &\ge bk(k-n), \qquad \text{(since } A \text{ is an imbalance sequence)} \\ &\ge (2b)k(k-n+1)\end{aligned}$$

for $n \ge k = 2$ and equality holds when $k = n-1$. Thus in either case B is an imbalance sequence of a $(0, 2b)$-graph by Theorem 3.

By symmetry, we have that Theorem 4 holds if a_p and a_q are both positive.

Now suppose that $a_p \le 0$ and $a_q \ge 0$ with $|a_p| \ge |a_q|$. If $b_l = a_p + a_q$, then $b_l \le 0$. For all $k \le p$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &\ge \sum_{i=1}^{k} a_i \ge bk(k-n), \qquad \text{(since } A \text{ is an imbalance sequence)} \\ &\ge (2b)k(k-n+1), \qquad \text{(since } n \ge k+2).\end{aligned}$$

For all $p < k \le l$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &\ge \sum_{i=1}^{k} a_i - a_p \ge \sum_{i=1}^{k} a_i \\ &\ge bk(k-n), \qquad \text{(since } A \text{ is an imbalance sequence)} \\ &\ge (2b)k(k-n+1), \qquad \text{(since } n \ge k+2).\end{aligned}$$

For all $k > l$, we have

$$\begin{aligned}\sum_{i=1}^{k} b_i &\ge \sum_{i=1}^{k} a_i + a_q \ge \sum_{i=1}^{k} a_i \\ &\ge bk(k-n), \qquad \text{(since } A \text{ is an imbalance sequence)} \\ &\ge (2b)k(k-n+1).\end{aligned}$$

The last inequality holds if $n \ge k = 2$, with equality when $k = n-1$. Thus once again B is an imbalance sequence of a $(0, b)$-graph by Theorem 3. By symmetry, Theorem 4 holds if $|a_p| \le |a_q|$. □

Suppose that D' is a cycle minimal realization of $A/(1, n)$. Then the following algorithm algorithm VERTEX constructs D'', a cycle minimal realization of A.

Input parameters of VERTEX are $n \geq 2$: the number of elements of A; $b \geq 1$: the connection parameter of D'; $A = [a_1, \ldots, a_n]$: the imbalance sequence; $A' = A/(1,n) = [a'_1, \ \ldots, \ a'_{n-1}]$; k: index of the element $a'_1 = a_1 + a_n$ in the minor A'; D': a $(0,b)$-graph, which is a cycle minimal realization of $A/(1,n)$ (D' is given by an $(n-1) \times (n-1)$ sized incidence matrix $\mathcal{X} = [x_{i,j}]$).

The output of VERTEX is D'': a $(0,q)$-graph, which is a cycle minimal realization of A, where $q = \max(1, b, a_n)$.

```
VERTEX(n, A, k, X)
01 read n                               // line 01–04: read of the input data
02 for i ← 1 to n − 1
03     for j ← 1 to n − 1
04         read x_ij
05 for i ← 1 to n − 1                   // line 05–08: add an isolated vertex to D′
06     x_in ← 0
07 for i ← 1 to n
08     x_ni ← 0
09 if a_1 = 0 and a_n = 0               // line 09–12: if all a's are equal to zero
10     for i ← 1 to n
11         x_i,i+1 ← 1                  // line 12: i + 1 is taken mod n
12         return X
13 x_nk ← a_n                           // line 13: if a_1 < 0
14 return X                             // line 14: return the incidence matrix of D″
```

We now show that VERTEX gives a cycle minimal realization of A.

Theorem 5 *The realization* D'' *obtained by* VERTEX *is a cycle minimal* $(0, \max(b, 1, a_n))$*-graph. The running time of* VERTEX *is* $\Theta(n^2)$.

Proof. If $a_1 = a_n = 0$, then D'' is constructed in lines 09–12 and contains exactly one cycle and is a 1-digraph. If we remove this cycle then remain isolated vertices that is a not connected graph.

If $a_1 < 0$, then due to Theorem 3 $a_n > 0$. In this case D'' is constructed in line 14 connecting the isolated vertex v'_n with the contracted vertex v'_k. So D'' contains a cycle only if the cycle minimal D' contained a cycle, and removing this cycle changes D'' to a not connected graph. In this case D'' is a $\max r, a_n$-graph.

So D'' is a $(0,q)$-graph, where $q = \max(b, 1, a_n)$.

The double cycle in lines 02–04 requires $\Theta(n^2)$ time, and the remaining part of the program requires only $O(n)$ time, so the running time of VERTEX is $\Theta(n^2)$. □

4 Construction of a cycle minimal realization

A *chain* of an imbalance sequence $A = A_n = [a_{n1}, \dots, a_{nn}]$ is a sequence of imbalance sequences $\mathcal{C}(A_n) = [A_n,\ A_{n-1},\ \dots,\ A_1]$ with $A_n = A$ and $A_{i-1}(A)$ being a minor of $A_i(A)$ for every $1 \le i \le n-1$. *The simple chain* $\mathcal{S}(\mathcal{A})$ of an imbalance sequence A is the sequence of imbalance sequences $[A_n,\ A_{n-1},\ \dots,\ A_1]$ with $A_n = A$ and $A_{i-1} = [a_{i-1,1}, \dots, a_{i-1,i-1}]$ being the minor of $A_i = [a_{i,1}, \dots, a_{i,i}]$ received by the contraction of the first and last element of A_i. It is worth to remark that the simple chain of an imbalance sequence is unique.

Chain is an algorithm for constructing the simple recursion chain $\mathcal{C}(A)$ of A.

The *input data* of Chain are $n \ge 2$: the length of an imbalance sequence $A = [a_{n1}, \dots, a_{nn}]$; an imbalance sequence A. The *output* of Chain is $\mathcal{C}$: the simple chain of A. *Working variable* is the cycle variable i.

Chain(n, A_n)
01 **read** n // line 01–03: read of the input data
02 **for** $i \leftarrow 1$ **to** n
03 **read** a_{ni}
04 **for** $i \leftarrow n$ **downto** 2 // line 04–05: construction of $\mathcal{C}$
05 delete the first and last elements of A_i, add a new element $a_{i1} + a_{ii}$, sort nondecreasingly the received sequence and denote by k_i the index of the new element
06 **return** $\mathcal{C}$ and k // line 06: return of the results

Now, since each contraction in Step 05 of Chain reduces the number of elements of the corresponding imbalance sequence by 1, the last element $A_1(A)$ of the chain contains exactly one element and so the length of the chain is equal to the number of elements n of the imbalance sequence A. Thus for all $1 \le i \le n$ the sequence $A_i(A)$ contains i elements. To every chain of an imbalance sequence A of length n we can associate bijectively a chain of $n-1$ ordered pairs with i element equal to (j, k), where $A_{n-i} = A_{n-i+1}/(j, k)$. That is (v_j, v_k) is contracted to obtain A_{n-i} from A_{n-i+1}. This bijection allows us to represent every chain of imbalance sequences by the sequence of pairs (j, k).

We present a simple algorithm Realization for associating a small cycle minimal realization D'' to any imbalance sequence A.

Input values are $n \ge 2$: the number of elements of A; A: an imbalance sequence; D': a cycle minimal $(0, b)$-graph which is a realization of $A/(1, n)$

and is given by its incidence matrix X.

The *output* of REALIZATION is D'', a $(0, q)$-graph, which is a cycle minimal realization of A; $k = [k_1, \ldots, k_{n-1}]$: the sequence of the updated indices of the elements received by contraction. D'' is represented by its incidence matrix X, and $q = \max(b, 1, a_n)$. *Working variable* is i: cyclic variable.

```
REALIZATION(n, A)
01 read n                                    // line 01–03: read of the input data
02 for i ← 1 to n
03      read a_ij
04 CHAIN(n, A)                    // line 04: construction the simple chain C(𝒜)
05 x_11 ← 0                                    // line 05: construction of D''_1
06 for i ← 2 to n                 // line 06–07: recursive construction of D''_n
07      VERTEX(i, A_i, k, 𝒳_{i−1})
08 return D''_n and k   // line 08: return of the constructed minimal digraph
```

The next assertion shows that REALIZATION is correct and constructs a cycle minimal realization of an imbalance sequence in polynomial time.

Theorem 6 *Let* A *be a* $(0, b)$*-imbalance sequence having* n *entries and let* $C(A) = [(a_1, b_1) \ldots (a_n, b_n)]$ *be a chain of* A. *Then there exists a cycle minimal digraph* D *having* n *vertices such that* D *is reconstructible from* C *and* D *is a* $q = \max(r, 1, a_n)$*-realization of* A. *Moreover, this reconstruction requires* $O(n^2)n$ *time.*

Proof. By VERTEX, the digraph D_n which is the output of REALIZATION, is assured to be a cycle minimal realization of A. Now, VERTEX constructs D_i from D_{i-1} in $O(n)$ time and there are $n - 1$ such constructions. Thus REALIZATION runs in $O(n^2)$ time. □

The following example illustrate the work of algorithms VERTEX, CHAIN and REALIZATION.

Example 1 *Let* $A = [-2, -2, -2, -1, 3, 4]$. *Figure 1 shows a realization of* A, *therefore* A *is an imbalance sequence.*

Figure 1 also shows that this realization is a 1-digraph. Since there are nonzero imbalances therefore all realizations have to contain arcs so this realization is connection minimal. Since all realization of A has to contain at least

$$m_{\min} = \frac{\sum_{i=1}^{n} a_i}{2}$$

arcs, and now $m_{\min} = 7$, so D is also an arc minimal realization.

Vertex/Vertex	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	0	0	0
v_2	0	0	0	0	0	0
v_3	0	0	0	0	0	0
v_4	0	0	0	0	0	0
v_5	1	1	1	0	0	0
v_6	1	1	1	1	0	0

Figure 1: Incidence matrix of a realization of $A = [-2, -2, -2, -1, 3, 4]$

Now we construct a cycle minimal realization of A using Realization. After the reading of the input data in lines 01–03 Chain constructs the simple chain $\mathcal{S} = [A_1, \ldots, A_6]$ and $k = [k_1, k_2, k_3, k_4, k_5] = [1, 1, 2, 3, 4]$, where $A_6 = A = [-2, -2, -2, -1, 3, 4]$, $A5 = [-2, -2, -1, 2, 3]$, $A_4 = [-2, -1, 1, 2]$, $A_3 = [-1, 0, 1]$, $A_2 = [0, 0]$ and $A_1 = [0]$.

After the construction of C Realization sets $x_{11} = 0$ in Step 5 and so it defines $\mathcal{X}_1$, the incidence matrix of D_1 consisting of an isolated vertex v_1. Then it constructs D_2, ..., D_6 in lines 06–07 calling Vertex recursively: at first $k_1 = 1$ helps to construct D_2 having the incidence matrix X_2 which is shown in Figure 2.

Vertex/Vertex	v_1	v_2
v_1	0	1
v_2	1	0

Figure 2: Incidence matrix of D_2 (X_2)

Now using $k_2 = 1$ D_3 is constructed. The result is the incidence matrix X_3 shown in Figure 3.

The next step is the construction of D_4 using $k_3 = 2$. Figure 4 shows X_4.

The next step is the construction of D_5 using $k_4 = 3$. Figure 5 shows X_5.

The final step is the construction of D_6 using $k_5 = 4$. Figure 6 shows X_6.

It is worth to remark that D_6 contains 9 arcs while the realization of A whose incidence matrix is shown in Figure 6 contains only the necessary 7 arcs and is also a cycle minimal realization of A.

The graph D_6' whose incidence matrix X_6' shown in Figure 7 contains only

Vertex/Vertex	v_1	v_2	v_3
v_1	0	1	0
v_2	1	0	0
v_3	1	0	0

Figure 3: Incidence matrix of D_3 (X_3)

Vertex/Vertex	v_1	v_2	v_3	v_4
v_1	0	1	0	0
v_2	1	0	0	0
v_3	1	0	0	0
v_4	2	0	0	0

Figure 4: Incidence matrix of D_4 (X_4)

Vertex/Vertex	v_1	v_2	v_3	v_4	v_5
v_1	0	1	0	0	0
v_2	1	0	0	0	0
v_3	1	0	0	0	0
v_4	2	0	0	0	0
v_5	3	0	0	0	0

Figure 5: Incidence matrix of D_5 (X_5)

Vertex/Vertex	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	1	0	0	0	0
v_2	1	0	0	0	0	0
v_3	1	0	0	0	0	0
v_4	2	0	0	0	0	0
v_5	3	0	0	0	0	0
v_6	0	0	0	4	0	0

Figure 6: Incidence matrix of D_6 (X_6)

7 arcs and is also a cycle minimal realization of A.

Vertex/Vertex	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	0	0	0
v_2	0	0	0	0	0	0
v_3	0	0	0	0	0	0
v_4	0	0	0	0	0	0
v_5	1	1	1	0	0	0
v_6	1	1	1	1	0	0

Figure 7: Incidence matrix of D'_6 (X'_6)

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