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On a new notion of complexity on infinite words

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Abstract. The intuition according to which an infinite word is "complicated" all the more as it has many distinct factors can be translated into terms of "complexity function" of this word. In this paper, some properties of a new notion of complexity called "window complexity" are studied. A characterization of modulo-recurrent words via window complexity is given.

1 Introduction

The study of factors of infinite words goes back at least to the work of Thue [13, 14]. Among questions which have been addressed, is the problem of computing the complexity function P, where $\mathsf{P}(n)$ is the number of distinct factors of length n; it was introduced in 1975 by Ehrenfeucht et al. [6]. And since then,

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it has been abundantly used to study infinite words; in particular it allowed the classification of certain families of infinite words (see for instance [1, 4, 10]).

As is shown in [12], this classical definition of complexity does not always show how complicated an infinite word is. That is why other notions of complexity were introduced by many authors, such as arithmetic complexity [3] and palindromic complexity [2].

Our aim in this paper is to give some properties of the window complexity. This new complexity was introduced by two of the authors in [8].

2 Preliminaries

Let A^* be the free monoid generated by a non-empty finite set A called alphabet. The elements of A are called letters and those of A^*, words. For any word v in A^*, $|v|$ denotes the length of v, namely the number of its letters. The identity element of A^* denoted by ε is the empty word; it is the word of length 0.

An infinite word is a sequence of letters in A indexed by $\mathbb{N}$. We denote by A^ω the set of infinite words in A and we set $A^\infty = A^* \cup A^\omega$.

An infinite word u is said τ-periodic if τ is the least positive integer such that $u_{i+\tau} = u_i$ for all $i \geq 0$.

A finite word u of length n formed by repeating a single letter x is typically denoted x^n. We define the nth power of a finite word v as being the concatenation of n copies of v; we denote it v^n. We say that an infinite word u is eventually periodic if there exist two finite words v and w such that $u = wvvv\cdots$; then u is simply denoted wv^ω.

Let $u \in A^\infty$ and $v \in A^*$. The word v is said to be a factor of u if there exist $u_1 \in A^*$ and $u_2 \in A^\infty$ such that $u = u_1 v u_2$.

For any infinite word u in A^ω, we shall write $u = u_0u_1u_2u_3\cdots$ where $u_i \in A$ for all $i \geq 0$. Let $u \in A^\omega$. The language of length n of u, denoted by $L_n(u)$, is the set of factors of u of length n:

$$L_n(u) = \{u_k u_{k+1} \cdots u_{k+n-1} : k \geq 0\}.$$

The set of all the factors of u is simply denoted by $L(u)$. A factor v of length n of a word $u = u_0u_1u_2\cdots$ appears in u at position k if $v = u_k u_{k+1} \cdots u_{k+n-1}$. A word u is said to be recurrent if every factor of u appears infinitely many times in u.

The complexity function of the infinite word u is the map from $\mathbb{N}$ to $\mathbb{N}^*$ defined by $P(u, n) = \#L_n(u)$, where $\#L_n(u)$ is the number of elements in $L_n(u)$.

A Sturmian word is an infinite word u such that $P(u, n) = n + 1$ for every integer $n \geq 0$. Sturmian words are non-eventually periodic infinite words of minimal complexity, for more details see for instance [9, 11].

Let us recall the definition of window complexity, which was introduced in [8].

Definition 1 *Let* $u = u_0u_1u_2\cdots$ *be an infinite word. The window complexity function of* u *is the map* $P_f(u, .) : \mathbb{N} \longrightarrow \mathbb{N}^*$ *defined by*[1]

$$P_f(u, n) = \#\left\{u_{kn}u_{kn+1}\cdots u_{(k+1)n-1} : k \geq 0\right\} \quad .$$

Factors of length n occurring in u at a position multiple of n, as above, will be called "window factors of length n of u". The decomposition of u into such factors will be called the "window decomposition of size n of u" or simply "n-window decomposition of u".

3 Properties of the window complexity

3.1 Comparison of $P_f(u, .)$ and $P(u, .)$

Let us first compare the window complexity function with the usual complexity function.

Proposition 1 *For any infinite word* u*, we have:*

$$\forall n \geq 0,\ P_f(u, n) \leq P(u, n) \quad .$$

Proof. For any infinite word u, we have

$$\left\{u_{kn}u_{kn+1}\cdots u_{(k+1)n-1} : k \geq 0\right\} \subseteq L_n(u) \quad .$$

Thus $P_f(u, n) \leq P(u, n)$. □

We shall see in the next section that this proposition is sharp, *i.e.*, there exist infinite words for which $P_f(u, n) = P(u, n)$ for all $n \in \mathbb{N}$.

Proposition 2 *For any infinite word* u*, we have:*

$$\forall n \geq 2,\ P(u, n) \leq (n-1)\left(P_f(u, n-1)\right)^2 \quad .$$

[1] f in P_f is from "fenêtre", window in French

Proof. For all $n \geq 2$, let v, w be two window factors of length $n-1$ in u such that vw appears in the n-decomposition of u. Then, there are at most $n-1$ factors of u of length n contained in vw, and all factors of length n are obtained this way. The result follows. □

We deduce from this proposition that if P_f is bounded, then P is at most linear. Such infinite words actually exist, as we shall see in Proposition 7.

3.2 Window complexity and modulo-recurrent words

Let us now study the window complexity of a particular class of infinite words, introduced in [7]: modulo-recurrent words.

Definition 2 *An infinite word* $u = u_0u_1u_2\cdots$ *is said to be modulo-recurrent if, for any* $k \geq 1$*, every factor* w *of* u *appears in* u *at every position modulo* k*, i.e.,*

$$\forall i \in \{0, 1, \ldots, k-1\},\ \exists l_i \in \mathbb{N} : w = u_{kl_i+i}u_{kl_i+i+1}\cdots u_{kl_i+i+|w|-1}\ .$$

Note that all modulo-recurrent words are recurrent. The class of modulo-recurrent words includes words of diverse complexity, for instance Sturmian words or words with maximal complexity:

Proposition 3 [7] *Sturmian words are modulo-recurrent.*

Proposition 4 *Let* $u \in A^\omega$ *be an infinite word such that* $P(u, n) = (\#A)^n$ *for all* $n \in \mathbb{N}$*. Then* u *is modulo-recurrent.*

Proof. If $P(u, n) = (\#A)^n$ for all $n \in \mathbb{N}$, then $L(u) = A^*$. Let $w \in A^*$ and $k \geq 1$. Choose $j \in \mathbb{N}$ such that $|w| + j \equiv 1 \pmod{k}$, and $a \in A$. Then the word $(wa^j)^k$ occurs at some position n in u. It follows that w occurs at positions $n + i(|w| + j)$ in u for $i \in \{0, 1, \ldots, k-1\}$, hence at every position modulo k. □

A consequence of Proposition 4 is that almost every infinite word is modulo-recurrent, in the following sense: choose an infinite word u in A^ω at random, each letter being independently chosen in A according to a uniform law; then, with probability 1, the word u is modulo-recurrent. Indeed, it is known that $L(u) = A^*$ for almost every u.

Modulo-recurrent words can be characterized in terms of window complexity:

Theorem 1 *Let u be a recurrent infinite word. Then, the following assertions are equivalent:*

1. *The word u is modulo-recurrent.*
2. $\forall n \geq 0,\ P_f(u, n) = P(u, n)$.

Proof. Let u be a modulo-recurrent word. Since $P_f(u, n) \leq P(u, n)$ by Proposition 1, we need only to check that $P(u, n) \leq P_f(u, n)$. Let w be a factor of length n in u. Then, w appears in u at any position modulo n, in particular at a position $\equiv 0 \pmod{n}$. So, there exists $k \in \mathbb{N}$ such that $w = u_{kn}u_{kn+1}\cdots u_{(k+1)n-1}$. Hence, we have the inclusion

$$L_n(u) \subseteq \left\{u_{kn}u_{kn+1}\cdots u_{(k+1)n-1} : k \geq 0\right\}$$

and thus

$$P(u, n) \leq P_f(u, n).$$

Conversely, suppose that

$$\forall n \geq 0,\ P_f(u, n) = P(u, n).$$

Then, for every integer n, any factor of u of length n appears in u at least at one position $\equiv 0 \pmod{n}$. Let w be a factor of u of length n and k a positive integer. Let us consider an integer i such that $0 \leq i < k$. We have to show that w appears in u at a certain position $\equiv i \pmod{k}$. As u is a recurrent infinite word, we can find some words x and y such that xwy is a factor of u of length $|xwy| \equiv 0 \pmod{k}$, with $|x| = i$.

It follows that there exists an integer l such that xwy appears in u at position $l|xwy|$, *i.e.*, $xwy = u_{l|xwy|}u_{l|xwy|+1}\cdots u_{(l+1)|xwy|-1}$. Thus,

$$w = u_{l|xwy|+i}u_{l|xwy|+i+1}\cdots u_{l|xwy|+i+n-1}.$$

Therefore, w appears at a position $\equiv i \pmod{k}$. □

Note that Theorem 1 does not hold for non-recurrent words. Indeed, the word $u = ab^\omega$ satisfies $P_f(u, n) = P(u, n) = 2$ for all $n \geq 1$ (and of course $P_f(u, 0) = P(u, 0) = 1$), but it is not modulo-recurrent.

3.3 Window complexity and automatic words

One very interesting way to generate infinite words is to proceed by iterating a substitution on a letter.

A substitution is a map $f : A \longrightarrow A^*$. It can be naturally extended to a morphism from A^* to A^*, and to a map from A^∞ to A^∞.

If there exists a constant σ such that $|f(a)| = \sigma$ for all $a \in A$, then we say that f is σ-uniform (or just uniform, if σ is clear from the context). A 1-uniform morphism is called a coding.

Let f be a substitution on A^*. A word w on the alphabet A such that $f(w) = w$ is said to be a fixed point of f. If f is a non-erasing morphism and there exists a letter $a \in A$ such that $f(a) = am$ with $|m| > 0$, then we say that f is prolongable on a. In this case, the sequence a, $f(a)$, $f^2(a)$,... converges to the infinite word

$$u = amf(m)f^2(m)\dots f^k(m)\dots$$

which is a fixed point of f.

An infinite word is said to be σ-automatic if it is the image under a coding of a fixed point of a σ-uniform morphism, for $\sigma \geq 2$. Indeed, such a word is recognizable by a σ-automaton [5].

Proposition 5 *Let u be a σ-automatic infinite word. Then the sequence of integers $(P_f(u, n))_{n\in\mathbb{N}}$ is not strictly increasing.*

Proof. Let $u = g(v)$, where v is a fixed point of the σ-uniform morphism f and g is a coding. Then, for all $n \in \mathbb{N}$, we have $P_f(u, \sigma^n) \leq P_f(v, 1)$ since the window factors of length σ^n of u are the words $g(f^n(a))$ for $a \in L_1(v)$. Since the sequence $(P_f(u, n))_{n\in\mathbb{N}}$ contains a bounded subsequence, it is not strictly increasing. □

3.4 Bounded window complexity

We know that the complexity function of an eventually periodic word is bounded. By Proposition 1, it follows that the window complexity of an eventually periodic word is also bounded. More precisely:

Proposition 6

1. *If u is a τ-periodic word, then* $P_f(u, n) \leq \dfrac{\tau}{\gcd(n, \tau)}$.

2. *If u is eventually τ-periodic, then for n large enough,*

$$P_f(u, n) \leq 1 + \frac{\tau}{\gcd(n, \tau)}.$$

Proof.

1. Let $n \in \mathbb{N}$. A window factor of length n of u can be written as

$$u_{kn}u_{kn+1}\cdots u_{(k+1)n-1},$$

and it is entirely determined by $kn \bmod \tau$, which takes exactly $\dfrac{\tau}{\gcd(n,\,\tau)}$ different values.

2. If n is large enough, then $u = wv$ where $|w| = n$ and v is τ-periodic. Then $P_f(u,\,n) \leq 1 + P_f(v,\,n)$.

□

Since the window complexity of any eventually periodic word is bounded, a natural question is what happens for non-eventually periodic infinite words. Contrarily to the situation with the usual complexity function, bounded window complexity does not imply eventual periodicity. We present below a non-eventually periodic infinite word whose window complexity is bounded.

Consider the sequence $(n_i)_{i\geq 0}$ such that $n_0 = 0$ and for all $i \geq 0$, $n_{i+1} = n_i! + n_i$; and let $t = t_0t_1t_2\cdots$ be the infinite word defined by $t_n = 1$ if there exists $i \in \mathbb{N}$ such that $n = n_i$ and $t_n = 0$ otherwise.

The first few terms of (n_i) and t are:

i	0	1	2	3	4	5	$\cdots$
n_i	0	1	2	4	28	$28! + 28$	$\cdots$

$$t = 1110100000000000000000000001000\ldots.$$

Let us note that t is neither eventually periodic nor recurrent.

Proposition 7 *The window complexity of the infinite word* t *defined above satisfies* $P_f(t,\,0) = 1$, $P_f(t,\,1) = 2$, *and*

$$\forall n \geq 2,\ P_f(t,\,n) = 3\ .$$

Proof. Obviously, $P_f(t,\,0) = 1$ and $P_f(t,\,1) = \#A = 2$.

We see from the first terms of t that 11, 10, 00 all occur in the 2-window decomposition of t. Morever, since all n_i are even except $n_1 = 1$, we have $t_{2l+1} = 0$ for $l \geq 1$, therefore $t_{2l}t_{2l+1}$ cannot be equal to 01. So $P_f(t,\,2) = 3$.

More generally, let $n \geq 2$, let i be the smallest integer such that $n \leq n_i$, and let $r = n_i \bmod n$. Then $n_{i-1} < n \leq n_i$ and $i \geq 2$. We first prove by induction

on j that $n_j \equiv r \pmod{n}$ for all $j \geq i$. This obviously holds for $j = i$. Assume that $n_j \equiv r \pmod{n}$ for some $j \geq i$. Then $n_{j+1} - r = n_j! + (n_j - r)$, which is a multiple of n since $n_j \geq n$.

There are at least 3 window factors of length n: $t_0 t_1 \cdots t_{n-1}$, with 11 as a prefix, occurring at position 0; 0^n, occurring at position $n_{i+1} + n - r$ (since $n_{i+1} < n_{i+1} + n - r < n_{i+1} + n - r + n - 1 < n_{i+2}$); and $0^r 1 0^{n-r-1}$, occurring at position $n_i - r$.

Assume now that $w = t_{ln} t_{ln+1} \cdots t_{(l+1)n-1}$ is a window factor of length n of t. If it starts with 11, then it must be the prefix of length n, since 11 does not occur in t after position 1. Otherwise, $l \geq 1$. For $0 \leq k \leq n-1$, $ln + k = n_j$ is only possible if $k = r$, since $n_j \equiv r \pmod{n}$ if $j \geq i$, and $n_j < n$ if $j < i$. Hence w is either 0^n or $0^r 1 0^{n-r-1}$. We have shown that there is not other window factor of length n, *i.e.*, $P_f(t, n) = 3$. □

By Proposition 2, and since the word t is non-eventually periodic, we have $n + 1 \leq P(t, n) \leq 9(n-1)$ for $n \geq 2$. Actually, one can prove that $P(t, n) = n + o(\log n)$.

4 Some questions

We conclude with a few open questions.

- By Proposition 6, we know that if u is an eventually periodic infinite word then its window complexity function $P_f(u, \cdot)$ is bounded. Also, we have presented (Proposition 7) an infinite word, non-eventually periodic and non-recurrent, such that its window complexity function is bounded. Does there exist some infinite recurrent and non-eventually periodic word for which the window complexity function is bounded?

- Among infinite words with bounded window complexity, a subclass of particular interest is that of words with eventually constant window complexity, *i.e.*, verifying the following property:

$$\exists n_0, c \in \mathbb{N} : \forall n \geq n_0,\ P_f(u, n) = c. \tag{1}$$

  Eventually constant words have eventually constant window complexity. The example constructed in Proposition 7 shows that even non-eventually periodic words may have eventually constant window complexity.

  It would be interesting to see if there exist recurrent words, or better automatic words, that possess Property (1).

- We know by Proposition 5 that the window complexity function of an automatic word is not strictly increasing, and even contains a bounded subsequence. On the other hand, by Theorem 1, modulo-recurrent words have strictly increasing window complexity. Do there exist non-modulo-recurrent words with strictly increasing window complexity?

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Imbalances in directed multigraphs

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Abstract. In a directed multigraph, the imbalance of a vertex v_i is defined as $b_{v_i} = d^+_{v_i} - d^-_{v_i}$, where $d^+_{v_i}$ and $d^-_{v_i}$ denote the outdegree and indegree respectively of v_i. We characterize imbalances in directed multigraphs and obtain lower and upper bounds on imbalances in such digraphs. Also, we show the existence of a directed multigraph with a given imbalance set.

1 Introduction

A directed graph (shortly digraph) without loops and without multi-arcs is called a simple digraph [2]. The imbalance of a vertex v_i in a digraph as b_{v_i} (or simply b_i) $= d^+_{v_i} - d^-_{v_i}$, where $d^+_{v_i}$ and $d^-_{v_i}$ are respectively the outdegree and indegree of v_i. The imbalance sequence of a simple digraph is formed by listing the vertex imbalances in non-increasing order. A sequence of integers $F = [f_1, f_2, \ldots, f_n]$ with $f_1 \geq f_2 \geq \ldots \geq f_n$ is feasible if the sum of its elements is zero, and satisfies $\sum_{i=1}^{k} f_i \leq k(n-k)$, for $1 \leq k < n$.

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The following result [5] provides a necessary and sufficient condition for a sequence of integers to be the imbalance sequence of a simple digraph.

Theorem 1 *A sequence is realizable as an imbalance sequence if and only if it is feasible.*

The above result is equivalent to saying that a sequence of integers $B = [b_1, b_2, \ldots, b_n]$ with $b_1 \geq b_2 \geq \ldots \geq b_n$ is an imbalance sequence of a simple digraph if and only if

$$\sum_{i=1}^{k} b_i \leq k(n-k),$$

for $1 \leq k < n$, with equality when $k = n$.

On arranging the imbalance sequence in non-decreasing order, we have the following observation.

Corollary 1 *A sequence of integers* $B = [b_1, b_2, \ldots, b_n]$ *with* $b_1 \leq b_2 \leq \ldots \leq b_n$ *is an imbalance sequence of a simple digraph if and only if*

$$\sum_{i=1}^{k} b_i \geq k(k-n),$$

for $1 \leq k < n$ *with equality when* $k = n$.

Various results for imbalances in simple digraphs and oriented graphs can be found in [6], [7].

2 Imbalances in r-graphs

A multigraph is a graph from which multi-edges are not removed, and which has no loops [2]. If $r \geq 1$ then an r-digraph (shortly r-graph) is an orientation of a multigraph that is without loops and contains at most r edges between the elements of any pair of distinct vertices. Clearly 1-digraph is an oriented graph. Let D be an f-digraph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and let d_v^+ and d_v^- respectively denote the outdegree and indegree of vertex v. Define b_{v_i} (or simply b_i) $= d_{v_i}^+ - d_{u_i}^-$ as imbalance of v_i. Clearly, $-r(n-1) \leq b_{v_i} \leq r(n-1)$. The imbalance sequence of D is formed by listing the vertex imbalances in non-decreasing order.

We remark that r-digraphs are special cases of (a, b)-digraphs containing at least a and at most b edges between the elements of any pair of vertices. Degree sequences of (a, b)-digraphs are studied in [3, 4].

Let u and v be distinct vertices in D. If there are f arcs directed from u to v and g arcs directed from v to u, we denote this by $u(f-g)v$, where $0 \leq f, g, f+g \leq r$.

A double in D is an induced directed subgraph with two vertices u, and v having the form $u(f_1f_2)v$, where $1 \leq f_1$, $f_2 \leq r$, and $1 \leq f_1 + f_2 \leq r$, and f_1 is the number of arcs directed from u to v, and f_2 is the number of arcs directed from v to u. A triple in D is an induced subgraph with tree vertices u, v, and w having the form $u(f_1f_2)v(g_1g_2)w(h_1h_2)u$, where $1 \leq f_1$, f_2, g_1, g_2, h_1, $h_2 \leq r$, and $1 \leq f_1 + f_2$, $g_1 + g_2$, $h_1 + h_2 \leq r$, and the meaning of f_1, f_2, g_1, g_2, h_1, h_2 is similar to the meaning in the definition of doubles. An oriented triple in D is an induced subdigraph with three vertices. An oriented triple is said to be transitive if it is of the form $u(1-0)v(1-0)w(0-1)u$, or $u(1-0)v(0-1)w(0-0)u$, or $u(1-0)v(0-0)w(0-1)u$, or $u(1-0)v(0-0)w(0-0)u$, or $u(0-0)v(0-0)w(0-0)u$, otherwise it is intransitive. An r-graph is said to be transitive if all its oriented triples are transitive. In particular, a triple C in an r-graph is transitive if every oriented triple of C is transitive.

The following observation can be easily established and is analogues to Theorem 2.2 of Avery [1].

Lemma 1 *If* D_1 *and* D_2 *are two* r*-graphs with same imbalance sequence, then* D_1 *can be transformed to* D_2 *by successively transforming (i) appropriate oriented triples in one of the following ways, either (a) by changing the intransitive oriented triple* $u(1-0)v(1-0)w(1-0)u$ *to a transitive oriented triple* $u(0-0)v(0-0)w(0-0)u$*, which has the same imbalance sequence or vice versa, or (b) by changing the intransitive oriented triple* $u(1-0)v(1-0)w(0-0)u$ *to a transitive oriented triple* $u(0-0)v(0-0)w(0-1)u$*, which has the same imbalance sequence or vice versa; or (ii) by changing a double* $u(1-1)v$ *to a double* $u(0-0)v$*, which has the same imbalance sequence or vice versa.*

The above observations lead to the following result.

Theorem 2 *Among all* r*-graphs with given imbalance sequence, those with the fewest arcs are transitive.*

Proof. Let B be an imbalance sequence and let D be a realization of B that is not transitive. Then D contains an intransitive oriented triple. If it is of

the form $u(1-0)v(1-0)w(1-0)u$, it can be transformed by operation *i(a)* of Lemma 3 to a transitive oriented triple $u(0-0)v(0-0)w(0-0)u$ with the same imbalance sequence and three arcs fewer. If D contains an intransitive oriented triple of the form $u(1-0)v(1-0)w(0-0)u$, it can be transformed by operation *i(b)* of Lemma 3 to a transitive oriented triple $u(0-0)v(0-0)w(0-1)u$ same imbalance sequence but one arc fewer. In case D contains both types of intransitive oriented triples, they can be transformed to transitive ones with certainly lesser arcs. If in D there is a double $u(1-1)v$, by operation *(ii)* of Lemme 4, it can be transformed to $u(0-0)v$, with same imbalance sequence but two arcs fewer. □

The next result gives necessary and sufficient conditions for a sequence of integers to be the imbalance sequence of some r-graph.

Theorem 3 *A sequence* $B = [b_1, b_2, \ldots, b_n]$ *of integers in non-decreasing order is an imbalance sequence of an* r*-graph if and only if*

$$\sum_{i=1}^{k} b_i \geq rk(k-n), \tag{1}$$

with equality when $k = n$.

Proof. Necessity. A multi subdigraph induced by k vertices has a sum of imbalances $rk(k-n)$.

Sufficiency. Assume that $B = [b_1, b_2, \ldots, b_n]$ be the sequence of integers in non-decreasing order satisfying conditions (1) but is not the imbalance sequence of any r-graph. Let this sequence be chosen in such a way that n is the smallest possible and b_1 is the least with that choice of n. We consider the following two cases.

Case (i). Suppose equality in (1) holds for some $k \leq n$, so that

$$\sum_{i=1}^{k} b_i = rk(k-n),$$

for $1 \leq k < n$.

By minimality of n, $B_1 = [b_1, b_2, \ldots, b_k]$ is the imbalance sequence of some r-graph D_1 with vertex set, say V_1. Let $B_2 = [b_{k+1}, b_{k+2}, \ldots, b_n]$.
Consider,

$$\begin{aligned}\sum_{i=1}^{f} b_{k+i} &= \sum_{i=1}^{k+f} b_i - \sum_{i=1}^{k} b_i \\ &\geq r(k+f)[(k+f)-n] - rk(k-n) \\ &= r(k_2 + kf - kn + fk + f_2 - fn - k_2 + kn) \\ &\geq r(f_2 - fn) \\ &= rf(f-n),\end{aligned}$$

for $1 \leq f \leq n-k$, with equality when $f = n-k$. Therefore, by the minimality for n, the sequence B_2 forms the imbalance sequence of some r-graph D_2 with vertex set, say V_2. Construct a new r-graph D with vertex set as follows.

Let $V = V_1 \cup V_2$ with, $V_1 \cap V_2 = \phi$ and the arc set containing those arcs which are in D_1 and D_2. Then we obtain the r-graph D with the imbalance sequence B, which is a contradiction.

Case (ii). Suppose that the strict inequality holds in (1) for some $k < n$, so that

$$\sum_{i=1}^{k} b_i > rk(k-n),$$

for $1 \leq k < n$. Let $B_1 = [b_1 - 1, b_2, \ldots, b_{n-1}, b_n + 1]$, so that B_1 satisfy the conditions (1). Thus by the minimality of b_1, the sequences B_1 is the imbalances sequence of some r-graph D_1 with vertex set, say V_1). Let $b_{v_1} = b_1 - 1$ and $b_{v_n} = a_n + 1$. Since $b_{v_n} > b_{v_1} + 1$, there exists a vertex $v_p \in V_1$ such that $v_n(0-0)v_p(1-0)v_1$, or $v_n(1-0)v_p(0-0)v_1$, or $v_n(1-0)v_p(1-0)v_1$, or $v_n(0-0)v_p(0-0)v_1$, and if these are changed to $v_n(0-1)v_p(0-0)v_1$, or $v_n(0-0)v_p(0-1)v_1$, or $v_n(0-0)v_p(0-0)v_1$, or $v_n(0-1)v_p(0-1)v_1$ respectively, the result is an r-graph with imbalances sequence B, which is again a contradiction. This proves the result. □

Arranging the imbalance sequence in non-increasing order, we have the following observation.

Corollary 2 *A sequence* $B = [b_1, b_2, \ldots, b_n]$ *of integers with* $b_1 \geq b_2 \geq \ldots \geq b_n$ *is an imbalance sequence of an* r*-graph if and only if*

$$\sum_{i=1}^{k} b_i \leq rk(n-k),$$

for $1 \leq k \leq n$*, with equality when* $k = n$*.*

The converse of an r-graph D is an r-graph D', obtained by reversing orientations of all arcs of D. If $B = [b_1, b_2, \ldots, b_n]$ with $b_1 \leq b_2 \leq \ldots \leq b_n$ is the imbalance sequence of an r-graph D, then $B' = [-b_n, -b_{n-1}, \ldots, -b_1]$ is the imbalance sequence of D.

The next result gives lower and upper bounds for the imbalance b_i of a vertex v_i in an r-graph D.

Theorem 4 *If* $B = [b_1, b_2, \ldots, b_n]$ *is an imbalance sequence of an* r*-graph* D*, then for each* i

$$r(i-n) \leq b_i \leq r(i-1).$$

Proof. Assume to the contrary that $b_i < r(i-n)$, so that for $k < i$,

$$b_k \leq b_i < r(i-n).$$

That is,

$$b_1 < r(i-n), b_2 < r(i-n), \ldots, b_i < r(i-n).$$

Adding these inequalities, we get

$$\sum_{k=1}^{i} b_k < ri(i-n),$$

which contradicts Theorem 3.

Therefore, $r(i-n) \leq b_i$.

The second inequality is dual to the first. In the converse r-graph with imbalance sequence $B = [b'_1, b'_2, \ldots, b'_n]$ we have, by the first inequality

$$\begin{aligned} b'_{n-i+1} &\geq r[(n-i+1)-n] \\ &= r(-i+1). \end{aligned}$$

Since $b_i = -b'_{n-i+1}$, therefore

$$b_i \leq -r(-i+1) = r(i-1).$$

Hence, $b_i \leq r(i-1)$. □

Now we obtain the following inequalities for imbalances in r-graphs.

Theorem 5 *If* $B = [b_1, b_2, \ldots, b_n]$ *is an imbalance sequence of an* r*-graph with* $b_1 \geq b_2 \geq \ldots \geq b_n$*, then*

$$\sum_{i=1}^{k} b_i^2 \leq \sum_{i=1}^{k} (2rn - 2rk - b_i)^2,$$

for $1 \leq k \leq n$ *with equality when* $k = n$.

Proof. By Theorem 3, we have for $1 \leq k \leq n$ with equality when $k = n$

$$rk(n-k) \geq \sum_{i=1}^{k} b_i,$$

implying

$$\sum_{i=1}^{k} b_i^2 + 2(2rn-2rk)rk(n-k) \geq \sum_{i=1}^{k} b_i^2 + 2(2rn-2rk)\sum_{i=1}^{k} b_i,$$

from where

$$\sum_{i=1}^{k} b_i^2 + k(2rn-2rk)^2 - 2(2rn-2rk)\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} b_i^2,$$

and so we get the required

$$\begin{aligned} b_1^2 + b_2^2 + \ldots + b_k^2 + (2rn-2rk)^2 + (2rn-2rk)^2 + \ldots + (2rn-2rk)^2 \\ - 2(2rn-2rk)b_1 - 2(2rn-2rk)b_2 - \ldots - 2(2rn-2rk)b_k \\ \geq \sum_{i=1}^{k} b_i^2, \end{aligned}$$

or

$$\sum_{i=1}^{k} (2rn-2rk-b_i)^2 \geq \sum_{i=1}^{k} b_i^2.$$

□

The set of distinct imbalances of vertices in an r-graph is called its imbalance set. The following result gives the existence of an r-graph with a given imbalance set. Let $(p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n)$ denote the greatest common divisor of $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n$.

Theorem 6 *If* $P = \{p_1, p_2, \ldots, p_m\}$ *and* $Q = \{-q_1, -q_2, \ldots, -q_n\}$ *where* $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n$ *are positive integers such that* $p_1 < p_2 < \ldots < p_m$ *and* $q_1 < q_2 < \ldots < q_n$ *and* $(p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n) = t$, $1 \leq t \leq r$, *then there exists an* r*-graph with imbalance set* $P \cup Q$.

Proof. Since $(p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n) = t$, $1 \leq t \leq r$, there exist positive integers $f_1, f_2, \ldots, f_m$ and $g_1, g_2, \ldots, g_n$ with $f_1 < f_2 < \ldots < f_m$ and $g_1 < g_2 < \ldots < g_n$ such that

$$p_i = tf_i$$

for $1 \leq i \leq m$ and

$$q_i = tg_i$$

for $1 \leq j \leq n$.

We construct an r-graph D with vertex set V as follows.

Let

$$V = X_1^1 \cup X_2^1 \cup \ldots \cup X_m^1 \cup X_1^2 \cup X_1^3 \cup \ldots \cup X_1^n \cup Y_1^1 \cup Y_2^1 \cup \ldots \cup Y_m^1 \cup Y_1^2 \cup Y_1^3 \cup \ldots \cup Y_1^n,$$

with $X_i^j \cap X_k^l = \phi$, $Y_i^j \cap Y_k^l = \phi$, $X_i^j \cap Y_k^l = \phi$ and

$|X_i^1| = g_1$, for all $1 \leq i \leq m$,

$|X_1^i| = g_i$, for all $2 \leq i \leq n$,

$|Y_i^1| = f_i$, for all $1 \leq i \leq m$,

$|Y_1^i| = f_1$, for all $2 \leq i \leq n$.

Let there be t arcs directed from every vertex of X_i^1 to each vertex of Y_i^1, for all $1 \leq i \leq m$ and let there be t arcs directed from every vertex of X_1^i to each vertex of Y_1^i, for all $2 \leq i \leq n$ so that we obtain the r-graph D with imbalances of vertices as under.

For $1 \leq i \leq m$, for all $x_i^1 \in X_i^1$

$$b_{x_i^1} = t|Y_i^1| - 0 = tf_i = p_i,$$

for $2 \leq i \leq n$, for all $x_1^i \in X_1^i$

$$b_{x_1^i} = t|Y_1^i| - 0 = tf_1 = p_1,$$

for $1 \leq i \leq m$, for all $y_i^1 \in Y_i^1$

$$b_{y_i^1} = 0 - t|X_i^1| = -tg_i = -q_i,$$

and for $2 \leq i \leq n$, for all $y_1^i \in Y_1^i$

$$b_{y_1^i} = 0 - t|X_1^i| = -tg_i = -q_i.$$

Therefore imbalance set of D is $P \cup Q$. □

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The connected vertex detour number of a graph

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Abstract. For a connected graph G of order $p \geq 2$ and a vertex x of G, a set $S \subseteq V(G)$ is an x-detour set of G if each vertex $v \in V(G)$ lies on an $x - y$ detour for some element y in S. The minimum cardinality of an x-detour set of G is defined as the x-detour number of G, denoted by $d_x(G)$. An x-detour set of cardinality $d_x(G)$ is called a d_x-set of G. A connected x-detour set of G is an x-detour set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-detour set of G is defined as the connected x-detour number of G and is denoted by $cd_x(G)$. A connected x-detour set of cardinality $cd_x(G)$ is called a cd_x-set of G. We determine bounds for the connected x-detour number and find the same for some special classes of graphs. If a, b and c are positive integers such that $3 \leq a \leq b+1 < c$, then there exists a connected graph G with detour number $dn(G) = a, d_x(G) = b$ and $cd_x(G) = c$ for some vertex x in G. For positive integers R, D and $n \geq 3$ with $R < D \leq 2R$, there exists a connected graph G with $rad_D G = R$, $diam_D G = D$ and $cd_x(G) = n$ for some vertex x in G. Also, for each triple D, n and p of integers with $4 \leq D \leq p - 1$ and $3 \leq n \leq p$, there is a connected graph G of order p, detour diameter D and $cd_x(G) = n$ for some vertex x of G.

1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q

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respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices x and y in a connected graph G, the distance $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a branch of G at v. The closed interval $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a geodetic set if $I[S] = V$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set. The geodetic number of a graph was introduced in [1, 7] and further studied in [3].

The concept of vertex geodomination number was introduced by Santhakumaran and Titus in [8] and further studied in [9]. Let x be a vertex of a connected graph G. A set S of vertices of G is an x-geodominating set of G if each vertex v of G lies on an $x - y$ geodesic in G for some element y in S. The minimum cardinality of an x-$\mathsf{geodominating\ set}$ of G is defined as the x-geodomination number of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a g_x-set. The connected vertex geodomination number was introduced and studied by Santhakumaran and Titus in [11]. A connected x-geodominating set of G is an x-geodominating set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-geodominating set of G is the connected x-geodomination number of G and is denoted by $cg_x(G)$. A connected x-geodominating set of cardinality $cg_x(G)$ is called a cg_x-set of G.

For vertices x and y in a connected graph G, the detour distance $D(x, y)$ is the length of a longest $x - y$ path in G. For any vertex u of G, the detour eccentricity of u is $e_D(u) = \max\{D(u, v) : v \in V\}$. A vertex v of G such that $D(u, v) = e_D(u)$ is called a detour eccentric vertex of u. The detour radius R and detour diameter D of G are defined by $R = \mathsf{rad}_D G = \min\{e_D(v) : v \in V\}$ and $D = \mathsf{diam}_D G = \max\{e_D(v) : v \in V\}$ respectively. An $x - y$ path of length $D(x, y)$ is called an $x - y$ detour. The closed interval $I_D[x, y]$ consists of all vertices lying on some $x - y$ detour of G, while for $I_D[S] = \bigcup_{x,y \in S} I_D[x, y]$. A set S of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. A detour set of cardinality $dn(G)$ is called a minimum detour set. The detour number of a graph was introduced in [4] and further studied in [5].

The concept of vertex detour number was introduced by Santhakumaran

and Titus in [10]. Let x be a vertex of a connected graph G. A set S of vertices of G is an x-detour set if each vertex v of G lies on an $x-y$ detour in G for some element y in S. The minimum cardinality of an x-detour set of G is defined as the x-detour number of G and is denoted by $d_x(G)$. An x-detour set of cardinality $d_x(G)$ is called a d_x-set of G.

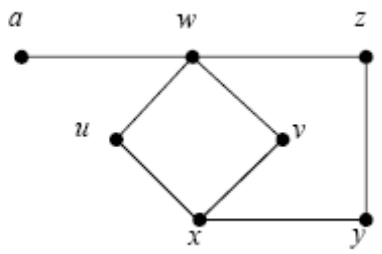

Figure 1

For the graph G given in Figure 1, $\{a, y\}$ and $\{a, z\}$ are the minimum x-detour sets of G and so $d_x(G) = 2$. It was proved in [10] that for any vertex x in G, $1 \leq d_x(G) \leq p - 1$. An elaborate study of results in vertex detour number with several interesting applications is given in [10].

The following theorems will be used in the sequel.

Theorem 1 [6] *Let v be a vertex of a connected graph G. The following statements are equivalent:*

(i) *v is a cut vertex of G.*

(ii) *There exist vertices u and w distinct from v such that v is on every $u-w$ path.*

(iii) *There exists a partition of the set of vertices $V-\{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every $u-w$ path.*

Theorem 2 [4] *Every end-vertex of a nontrivial connected graph G belongs to every detour set of G.*

Theorem 3 [4] *If T is a tree with k end-vertices, then $dn(T) = k$.*

Theorem 4 [10] *Let x be any vertex of a connected graph G. Then every end-vertex of G other than the vertex x (whether x is end-vertex or not) belongs to every d_x-set.*

Theorem 5 [10] *Let* T *be a tree with* k *end-vertices. Then* $d_x(T) = k - 1$ *or* $d_x(T) = k$ *according as* x *is an end-vertex or not.*

Theorem 6 [10] *For any vertex* x *in* G, $dn(G) \leq d_x(G) + 1$.

Theorem 7 [10] *If* G *is the complete graph* K_p $(p \geq 2)$, *the cycle* C_p $(p \geq 3)$, *the complete bipartite graph* $K_{m,n}$ $(m, n \geq 2)$, *the* n*-cube* Q_n $(n \geq 2)$ *or the wheel* $W_n = K_1 + C_{n-1}$ $(n \geq 4)$, *then* $d_x(G) = 1$ *for every vertex* x *in* G.

Throughout this paper G denotes a connected graph with at least two vertices.

2 Connected vertex detour number

Definition 1 *Let* x *be any vertex of a connected graph* G. *A* connected x-detour set *of* G *is an* x*-detour set* S *such that the subgraph* $G[S]$ *induced by* S *is connected. The minimum cardinality of a connected* x*-detour set of* G *is the* connected x-detour number *of* G *and is denoted by* $cd_x(G)$. *A connected* x*-detour set of cardinality* $cd_x(G)$ *is called a* cd_x-set *of* G.

Example 1 *For the graph* G *given in Figure 2, the minimum vertex detour sets, the vertex detour numbers, the minimum connected vertex detour sets and the connected vertex detour numbers are given in Table 1.*

It is observed in [10] that x is not an element of any d_x-set of G. However, x may belong to a cd_x-set of G. For the graph G given in Figure 2, the vertex v is an element of a cd_v-set and the vertex t is not an element of any cd_t-set.

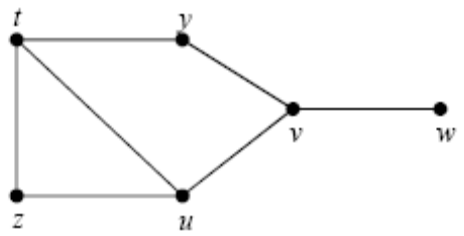

Figure 2

Table 1

Vertex x	d_x-sets	$d_x(G)$	cd_x-sets	$cd_x(G)$
t	$\{y,w\},\{z,w\},\{u,w\}$	2	$\{y,v,w\},\{u,v,w\}$	3
y	$\{w\}$	1	$\{w\}$	1
z	$\{w\}$	1	$\{w\}$	1
u	$\{w\}$	1	$\{w\}$	1
v	$\{y,w\},\{z,w\},\{u,w\}$	2	$\{y,v,w\},\{u,v,w\}$	3
w	$\{y\},\{z\},\{u\}$	1	$\{y\},\{z\},\{u\}$	1

Theorem 8 *Let x be any vertex of a connected graph G. If $y \neq x$ is an end vertex of G, then y belongs to every x-detour set of G.*

Proof. Let x be any vertex of G and let $y \neq x$ be an end-vertex of G. Then y is the terminal vertex of an $x-y$ detour and y is not an internal vertex of any detour so that y belongs to every x-detour set of G. □

Theorem 9 *Let G be a connected graph with cut vertices and let S_x be a connected x-detour set of G. If v is a cut vertex of G, then every component of $G-\{v\}$ contains an element of $S_x \bigcup \{x\}$.*

Proof. Suppose that there is a component B of $G-\{v\}$ such that B contains no vertex of $S_x \bigcup \{x\}$. Then clearly, $x \in V - V(B)$. Let $u \in V(B)$. Since S_x is a connected x-detour set, there exists an element $y \in S_x$ such that u lies in some $x-y$ detour $P : x = u_0, u_1, \ldots, u, \ldots, u_n = y$ in G. By Theorem 1, the $x-u$ subpath of P and the $u-y$ subpath of P both contain v, it follows that P is not a path, contrary to assumption. □

Corollary 1 *Let G be a connected graph with cut vertices and let S_x be a connected x-detour set of G. Then every branch of G contains an element of $S_x \bigcup \{x\}$.*

Theorem 10 (i) *If T is any tree, then $cd_x(T) = p$ for any cut vertex x of T.*

(ii) *If T is any tree which is not a path, then for an end vertex x, $cd_x(T) = p - D(x,y)$, where y is the vertex of T with deg $y \geq 3$ such that $D(x,y)$ is minimum.*

(iii) *If T is a path, then $cd_x(T) = 1$ for any end vertex x of T.*

Proof. (i) Let x be a cut vertex of T and let S be any connected x-detour set of T. By Theorem 8, every connected x-detour set of T contains all end vertices. If $S \neq V(T)$, there exists a cut vertex v of T such that $v \notin S$. Let u and w be two end vertices belonging to different components of $T - \{v\}$. Since v lies on the unique path joining u and w, it follows that the subgraph $G[S]$ induced by S is disconnected, which is a contradiction. Hence $cd_x(T) = p$.

(ii) Let T be a tree which is not a path and x an end vertex of T. Let $S = (V(T) - I_D[x, y]) \bigcup \{y\}$. Clearly S is a connected x-detour set of T and so $cd_x(T) \leq | S | = p - D(x, y)$. We claim that $cd_x(T) = p - D(x, y)$. Otherwise, there is a connected x-detour set M of T with $| M | < p - D(x, y)$. By Theorem 8, every connected x-detour set of T contains all end vertices except possibly x and hence there exists a cut vertex v of T such that $v \in S$ and $v \notin M$. Let $B_1, B_2, \ldots, B_m (m \geq 3)$ be the components of $T - \{y\}$. Assume that x belongs to B_1.

Case 1. Suppose $v = y$. Let $z \in B_2$ and $w \in B_3$ be two end vertices of T. By Theorem 1, v lies on the unique $z - w$ detour. Since z and w belong to M and $v \notin M$, $G[M]$ is not connected, which is a contradiction.

Case 2. Suppose $v \neq y$. Let $v \in B_i (i \neq 1)$. Now, choose an end vertex $u \in B_i$ such that v lies on the $y - u$ detour. Let $a \in B_j (j \neq i, 1)$ be an end vertex of T. By Theorem 1, y lies on the $u - a$ detour. Hence it follows that v lies on the $u - a$ detour. Since u and a belong to M and $v \notin M$, $G[M]$ is not connected, which is a contradiction.

(iii) Let T be a path. For an end vertex x in T, let y be the eccentric vertex of x. Clearly every vertex of T lies on the $x - y$ detour and so $\{y\}$ is a connected x-detour set of T so that $cd_x(T) = 1$. □

Corollary 2 *For any tree* T, $cd_x(T) = p$ *if and only if* x *is a cut vertex of* T.

Proof. This follows from Theorem 10. □

Theorem 11 *For any vertex* x *in a connected graph* G,

$$1 \leq d_x(G) \leq cd_x(G) \leq p.$$

Proof. It is clear from the definition of x-detour number that $d_x(G) \geq 1$. Since every connected x-detour set is also an x-detour set, it follows that $d_x(G) \leq cd_x(G)$. Also, since $V(G)$ induces a connected x-detour set of G, it is clear that $cd_x(G) \leq p$. □

Remark 1 *The bounds in Theorem 11 are sharp. For the cycle* C_n, $d_x(C_n) = 1$ *for every vertex* x *in* C_n. *For any non-trivial tree* T *with* $p \geq 3$, $cd_x(T) = p$ *for any cut vertex* x *in* T. *For the graph* G *given in Figure 3,* $d_x(G) = cd_x(G) = 2$ *for the vertex* x. *Also, all the inequalities in the theorem are strict. For an end vertex* x *in the star* $G = K_{1,n}(n \geq 3)$, $d_x(G) = n - 1$, $cd_x(G) = n$ *and* $p = n + 1$ *so that* $1 < d_x(G) < cd_x(G) < p$.

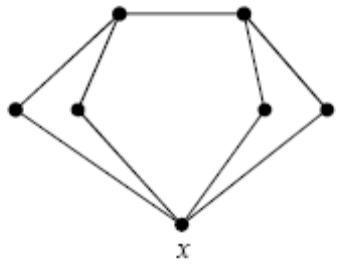

Figure 3

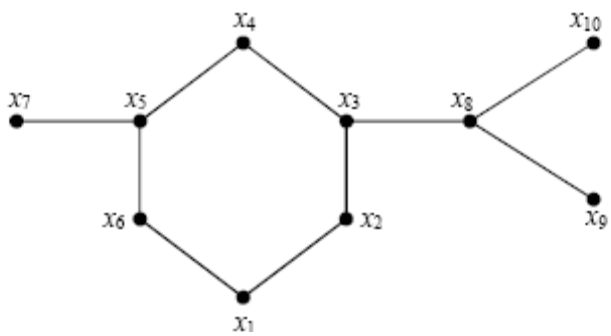

Figure 4

The following theorem gives a characterization for $cd_x(G) = 1$. For this, we introduce the following definition. Let x be any vertex in G. A vertex y in G is said to be an x-*detour superior vertex* if for any vertex z with $D(x, y) < D(x, z)$, z lies on an $x - y$ detour. For the graph G given in Figure 4, x_9 and x_{10} are the only x_1-detour superior vertices.

Theorem 12 *Let* x *be any vertex of a connected graph* G. *Then the following are equivalent:*

(i) $cd_x(G) = 1$

(ii) $d_x(G) = 1$

(iii) *There exists an* x*-detour superior vertex* y *in* G *such that every vertex of* G *is on an* $x - y$ *detour.*

Proof.

(i) $\Rightarrow$ (ii) Let $cd_x(G) = 1$. Then it follows from Theorem 11 that $d_x(G) = 1$.

(ii) $\Rightarrow$ (iii) Let $d_x(G) = 1$ and $S_x = \{y\}$ be a d_x-set of G. If y is not an x-detour superior vertex, then there is a vertex z in G with $D(x, y) < D(x, z)$ and z does not lie on any $x - y$ detour. Thus S_x is not a d_x-set of G, which is a contradiction.

(iii) $\Rightarrow$ (i) Let y be an x-detour superior vertex of G such that every vertex of G is on an $x - y$ detour. Then $\{y\}$ is a connected x-detour set of G so that $cd_x(G) = 1$. □

Corollary 3 (i) *For the complete graph* K_p, $cd_x(K_p) = 1$ *for any vertex* x *in* K_p.

(ii) *For any cycle* C_p, $cd_x(C_p) = 1$ *for any vertex* x *in* C_p.

(iii) *For the wheel* $W_p = K_1 + C_{p-1}(p \geq 5)$, $cd_x(W_p) = 1$ *for any vertex* x *in* W_p.

(iv) *For any cube* Q_n, $cd_x(Q_n) = 1$ *for any vertex* x *in* Q_n.

(v) *For the complete bipartite graph* $K_{m,n}(m, n \geq 2)$, $cd_x(K_{m,n}) = 1$ *for any vertex* x *in* $K_{m,n}$.

Proof. This follows from Theorems 7 and 12. □

Theorem 13 *For any vertex* x *in a connected graph* G, $dn(G) \leq d_x(G) + 1 \leq cd_x(G) + 1$.

Proof. This follows from Theorem 6 and Theorem 11. □

The following theorem gives a realization for the detour number, the vertex detour number and the connected vertex detour number when

$$3 \leq a \leq b + 1 < c.$$

Theorem 14 *For any three integers* a, b *and* c *with* $3 \leq a \leq b + 1 < c$, *there exists a connected graph* G *with* $dn(G) = a, d_x(G) = b$ *and* $cd_x(G) = c$ *for some vertex* x *in* G.

Proof. We prove this theorem by considering two cases.

Case 1. $3 \leq a = b + 1 < c$. Let $k > c$ be any integer and let P_{k-a+2} : $u_1, u_2, \ldots, u_{k-a+2}$ be a path of order $k - a + 2$. Add $a - 2$ new vertices $v_1, v_2, \ldots, v_{a-2}$ to P_{k-a+2} and join these to u_{k-c+1}, thereby producing the graph G of Figure 5. Then G is a tree of order k with a end vertices. By Theorem 3, $dn(G) = a$ and it follows from Theorem 5 and Theorem 10 (ii) that $d_x(G) = b$ and $cd_x(G) = c$ respectively, for the vertex $x = u_1$.

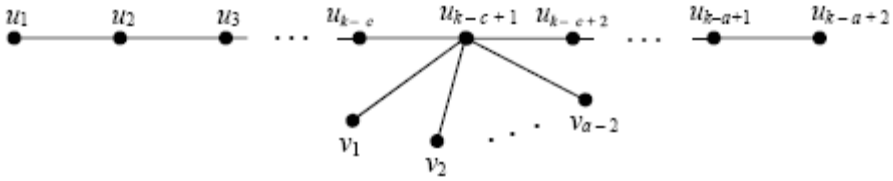

Figure 5

Case 2. $3 \leq a < b + 1 < c$. Let $F = (K_3 \bigcup P_2 \bigcup (b - a + 1)K_1) + \overline{K_2}$, where $U = V(K_3) = \{u_1, u_2, u_3\}$, $W = V(P_2) = \{w_1, w_2\}$, $X = V((b - a + 1)K_1) = \{x_1, x_2, \ldots, x_{b-a+1}\}$ and $V(\overline{K_2}) = \{x, y\}$. Let $P_{c-b-1} : v_1, v_2, \ldots, v_{c-b-1}$ be the path of order $c - b - 1$. Let H be the graph obtained from P_{c-b-1} by adding $a - 1$ new vertices $z_1, z_2, \ldots, z_{a-1}$ and joining each $z_i (1 \leq i \leq a - 1)$ to v_1. Now, let G be the graph obtained from F and H by identifying u_1 in F and v_{c-b-1} in H. The graph G is shown in Figure 6. Let $Z = \{z_1, z_2, \ldots, z_{a-1}\}$ be the set of all end vertices of G.

First, we show that $dn(G) = a$. By Theorem 2, every detour set of G contains Z. Since $I_D[Z] = Z \bigcup \{v_1\} \neq V(G)$, it follows that Z is not a detour set of G and so $dn(G) > | Z | = a-1$. On the other hand, let $S = Z \bigcup \{w_1\}$. Then, for each i with $1 \leq i \leq b - a + 1$, the path $P : z_1, v_1, v_2, \ldots, v_{c-b-2}, u_1, u_2, u_3, y, x_i, x, w_2, w_1$ is a $z_1 - w_1$ detour in G of length $c - b + 6$. Hence S is a detour set of G and so $dn(G) \leq | S | = a$. Therefore, $dn(G) = a$.

Next, we show that $d_x(G) = b$ for the vertex x. Let S_x be any x-detour set of G. By Theorem 8, $Z \subseteq S_x$. It is clear that $D(x, z_i) = c - b + 5$ for $1 \leq i \leq a - 1$ and no $x_j (1 \leq j \leq b - a + 1)$ lies on an $x - z_i$ detour for any $z_i \in Z$. Thus Z is not an x-detour set of G. Now we claim that $X \subseteq S_x$. Assume, to the contrary, $X \supset S_x$. Then there exists an $x_i \in X$ such that $x_i \notin S_x (1 \leq i \leq b - a + 1)$. Now, it is clear that this x_i does not lie on any $x - v$ detour for any $v \in S_x$, which is a contradiction to S_x is an x-detour set. Hence $X \subseteq S_x$. Thus we see that every x-detour set S_x contains $X \bigcup Z$. Now, since $X \bigcup Z$ is an x-detour set

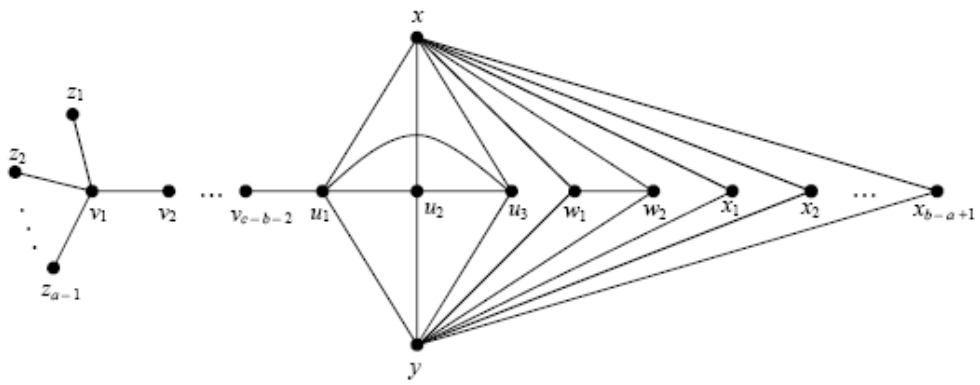

Figure 6

of G, it follows that $X \bigcup Z$ is the unique minimum x-detour set of G so that $d_x(G) = \mid X \bigcup Z \mid = b$.

Now, we show that $cd_x(G) = c$. Let T_x be any connected x-detour set of G. Since any connected x-detour set of G is also an x-detour set of G, it follows that T_x contains $X \bigcup Z$ as in the above paragraph. Now, since the induced subgraph $G[T_x]$ is connected, $M = \{v_1, v_2, \ldots, v_{c-b-1}\} \subseteq T_x$. Thus $M \bigcup X \bigcup Z \subseteq T_x$. It is clear that $M \bigcup X \bigcup Z$ is an x-detour set of G and the induced subgraph $G[M \bigcup X \bigcup Z]$ is not connected. Let $T = M \bigcup X \bigcup Z \bigcup \{x\}$. It is clear that T is a minimum connected x-detour set of G and so $cd_x(G) = c$. $\square$

For every connected graph G, $rad_D G \leq diam_D G \leq 2rad_D G$. Chartrand, Escuadro and Zhang [2] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the detour radius and detour diameter, respectively, of some connected graph. This theorem can also be extended so that the connected vertex detour number can be prescribed when $a < b \leq 2a$.

Theorem 15 *For positive integers* R, D *and* $n \geq 3$ *with* $R < D \leq 2R$, *there exists a connected graph* G *with* $rad_D G = R, diam_D G = D$ *and* $cd_x(G) = n$ *for some vertex* x *in* G.

Proof. If $R = 1$, then $D = 2$. Let $G = K_{1,n}$. Then by Theorem 10 (ii), $cd_x(G) = n$ for an end vertex x in G. Now, let $R \geq 2$. We construct a graph G with the desired properties as follows:

Let $C_{R+1} : v_1, v_2, \ldots, v_{R+1}, v_1$ be a cycle of order $R + 1$ and let $P_{D-R+1} : u_0, u_1, \ldots, u_{D-R}$ be a path of order $D - R + 1$. Let H be the graph obtained

from C_{R+1} and P_{D-R+1} by identifying v_1 in C_{R+1} and u_0 in P_{D-R+1}. Now, add $n-2$ new vertices $w_1, w_2, \ldots, w_{n-2}$ to H and join each vertex $w_i (1 \leq i \leq n-2)$ to the vertex u_{D-R-1} to obtain the graph G of Figure 7.

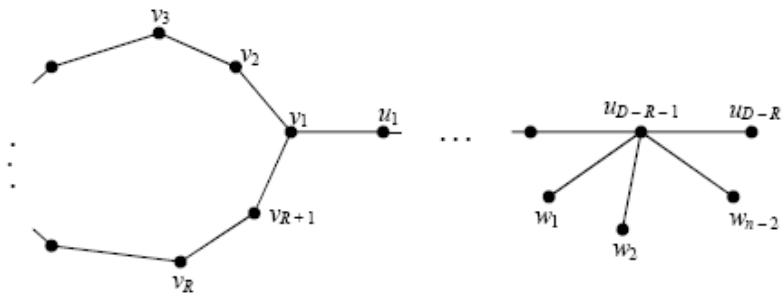

Figure 7

Now $\mathrm{rad}_D G = R$, $\mathrm{diam}_D G = D$ and G has $n-1$ end vertices. Let $S = \{w_1, w_2, \ldots, w_{n-2}, u_{D-R}\}$ be the set of all end vertices of G. Then by Theorem 8, every connected x-detour set of G contains S for the vertex $x = v_2$. It is clear that S is an x-detour set of G and the induced subgraph $G[S]$ is not connected so that $\mathrm{cd}_x(G) > n-1$. Let $S' = S \bigcup \{u_{D-R-1}\}$. Then S' is a connected x-detour set of G and so $\mathrm{cd}_x(G) = n$. □

The graph G of Figure 7 is the smallest graph with the properties described in Theorem 15. We leave the following problem as an open question.

Problem 1 *For positive integers* R *and* $n \geq 3$, *does there exist a connected graph* G *with* $\mathrm{rad}_D G = \mathrm{diam}_D G = R$ *and* $\mathrm{cd}_x(G) = n$ *for some vertex* x *of* G?

In the following, we construct a graph of prescribed order, detour diameter and vertex detour number under suitable conditions.

Theorem 16 *For each triple* D, n *and* p *of integers with* $4 \leq D \leq p-1$ *and* $3 \leq n \leq p$, *there is a connected graph* G *of order* p, *detour diameter* D *and* $\mathrm{cd}_x(G) = n$ *for some vertex* x *of* G.

Proof. We prove this theorem by considering three cases.

Case 1. Suppose $3 \leq n \leq p - D + 2$. Let G be a graph obtained from the cycle $C_D : u_1, u_2, \ldots, u_D, u_1$ of order D by (i) adding $n-2$ new vertices $v_1, v_2, \ldots, v_{n-2}$ and joining each vertex $v_i (1 \leq i \leq n-2)$ to u_1 and (ii) adding

$p - D - n + 2$ new vertices $w_1, w_2, \ldots, w_{p-D-n+2}$ and joining each vertex $w_i (1 \leq i \leq p - D - n + 2)$ to both u_1 and u_3. The graph G has order p and detour diameter D and is shown in Figure 8. Let $S = \{v_1, v_2, \ldots, v_{n-2}\}$ be the set of all end vertices of G. Then by Theorem 8, every connected x-detour set of G contains S for the vertex $x = u_1$. It is clear that S is not an x-detour set of G. Also any connected x-detour set of G must contain $S \bigcup \{u_1\}$. Since $S \bigcup \{u_1\}$ is not an x-detour set of G, $cd_x(G) > n - 1$. Let $S' = S \bigcup \{u_1, u_D\}$. Then S' is a connected x-detour set of G and so $cd_x(G) = n$.

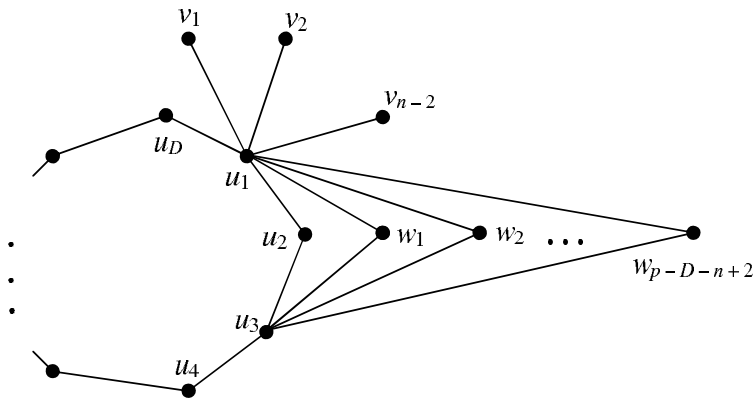

Figure 8

Case 2. Suppose $p - D + 3 \leq n \leq p - 1$. Let $P_{D+1} : u_0, u_1, u_2, \ldots, u_D$ be a path of length D. Add $p - D - 1$ new vertices $v_1, v_2, \ldots, v_{p-D-1}$ to P_{D+1} and join each $v_i (1 \leq i \leq p - D - 1)$ to u_{p-n}, so by producing the graph G of Figure 9. The graph G has order p and detour diameter D. Then by Theorem 10 (ii), $cd_x(G) = p - (p - n) = n$ for the vertex $x = u_0$.

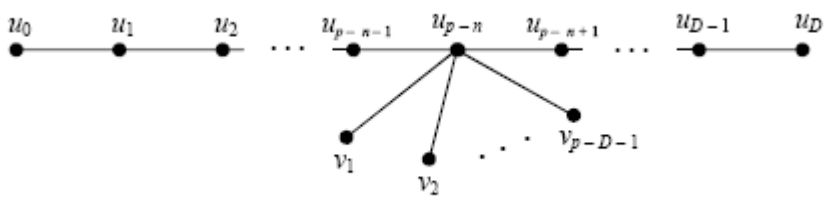

Figure 9

Case 3. Suppose $n = p$. Let G be any tree of order p and detour diameter

D. Then by Theorem 10 (i), $cd_x(G) = p$ for any cut vertex x in G. □

Theorem 17 *For any two integers* n *and* p *with* $3 \leq n \leq p$, *there exists a connected graph* G *with order* p *and* $cd_x(G) = n$ *for some vertex* x *of* G.

Proof. We prove this theorem by considering two cases.

Case 1. Let $3 \leq n \leq p-2$. Then $p-n+1 \geq 3$. Let G be the graph obtained from the cycle $C_{p-n+1} : u_1, u_2, \ldots, u_{p-n+1}, u_1$ by adding the $n-1$ new vertices $v_1, v_2, \ldots, v_{n-1}$ and joining these to u_1. The graph G has order p and is shown in Figure 10. Let $S = \{v_1, v_2, \ldots, v_{n-1}\}$ be the set of all end vertices of G. Then by Theorem 8, every connected x-detour set of G contains S for the vertex $x = u_2$. It is clear that S is an x-detour set of G and the induced subgraph $G[S]$ is not connected so that $cd_x(G) > n-1$. Let $S' = S \bigcup \{u_1\}$. It is clear that S' is a connected x-detour set of G and so $cd_x(G) = n$.

Case 2: Let $n = p-1$ or p. Let $G = K_{1,p-1}$. Then by Theorem 10, $cd_x(G) = p-1$ or p according as x is an end vertex or the cut vertex. □

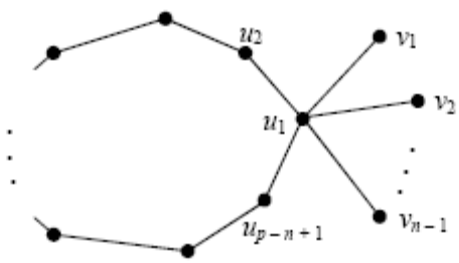

Figure 10

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Generalized GCD matrices

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Abstract. Let f be an arithmetical function. The matrix $[f(i,j)]_{n\times n}$ given by the value of f in greatest common divisor of (i,j), $f\big((i,j)\big)$ as its i, j entry is called the greatest common divisor (GCD) matrix. We consider the generalization of this matrix where the elements are in the form $f\big(i,(i,j)\big)$.

1 Introduction

The classical Smith determinant was introduced in 1875 by H. J. S. Smith [14] who also proved that

$$\det[(i,j)]_{n\times n} = \begin{vmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \cdots & \cdots & \cdots & \cdots \\ (n,1) & (n,2) & \cdots & (n,n) \end{vmatrix} = \varphi(1)\varphi(2)\cdots\varphi(n), \qquad (1)$$

where (i,j) represents the greatest common divisor of i and j, and $\varphi(n)$ denotes the Euler's totient function.
The GCD matrix with respect to f is

$$[f(i,j)]_{n\times n} = \begin{bmatrix} f((1,1)) & f((1,2)) & \cdots & f((1,n)) \\ f((2,1)) & f((2,2)) & \cdots & f((2,n)) \\ \cdots & \cdots & \cdots & \cdots \\ f((n,1)) & f((n,2)) & \cdots & f((n,n)) \end{bmatrix}.$$

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If we consider the GCD matrix $[f(i,j)]_{n\times n}$, where

$$f(n) = \sum_{d|n} g(d),$$

H. J. Smith proved that

$$\det[f(i,j)]_{n\times n} = g(1)\cdot g(2)\cdots g(n).$$

For $g = \varphi$

$$f(i,j) = \sum_{d|(i,j)} \varphi(d) = (i,j),$$

this formula reduces to (1). Many generalizations of Smith determinants have been presented in literature, see [1, 5, 7, 10, 13].
If we consider the GCD matrix $[f(i,j)]_{n\times n}$ where $f(n) = \sum_{d|n} g(d)$ Pólya and Szegő [12] proved that

$$[f(i,j)]_{n\times n} = G\cdot C^T, \tag{2}$$

where G and C are lower triangular matrices given by

$$g_{ij} = \begin{cases} g(j), & j \mid i \\ 0, & \text{otherwise} \end{cases}$$

and

$$c_{ij} = \begin{cases} 1, & j \mid i \\ 0, & \text{otherwise} \end{cases}.$$

L. Carlitz [4] in 1960 gave a new form of (2)

$$[f(i,j)]_{n\times n} = C_n \operatorname{diag}\big(g(1),\ g(2),\ldots,g(n)\big)\ C_n^T, \tag{3}$$

where $C_n = [c_{ij}]_{n\times n}$,

$$c_{ij} = \begin{cases} 1, & j \mid i \\ 0, & j \nmid i \end{cases},$$

$D = [d_{ij}]_{n\times n}$ diagonal matrix

$$d_{ij} = \begin{cases} g(i), & i = j \\ 0, & i \neq j \end{cases}.$$

From (3) it follows that the value of the determinant is

$$\det[f(i,j)]_{n\times n} = g(1)g(2)\cdots g(n). \tag{4}$$

There are quite a few generalized forms of GCD matrices, which can be found in several references [2, 3, 6, 8, 9, 11].
In this paper we study matrices which have as variables the gratest common divisor and the indices:

$$[f(i,j)]_{n\times n} = \begin{bmatrix} f(1,(1,1)) & f(1,(1,2)) & \cdots & f(1,(1,n)) \\ f(2,(2,1)) & f(2(2,2)) & \cdots & f(2,(2,n)) \\ \cdots & \cdots & \cdots & \cdots \\ f(n,(n,1)) & f(n,(n,2)) & \cdots & f(n,(n,n)) \end{bmatrix}.$$

2 Generalized GCD matrices

Theorem 1 *For a given arithmetical function* g *let*

$$f(n,m) = \sum_{d|n} g(d) - \sum_{d|(n,m)} g(d).$$

Then

$$[f(i,j)]_{n\times n} = C_n \operatorname{diag}[g(1), g(2), \ldots, g(n)]\, D_n^T,$$

where $C_n = [c_{ij}]_{n\times n}$,

$$c_{ij} = \begin{cases} 1, & j \mid i \\ 0, & j \not| i \end{cases},$$

$D_n = [d_{ij}]_{n\times n}$,

$$d_{ij} = \begin{cases} 1, & j \not| i \\ 0, & j \mid i \end{cases}.$$

Proof. After multiplication, the general element of

$$A = [a_{ij}]_{n\times n} = C \operatorname{diag}[g(1), g(2), \ldots, g(n)] D^T$$

is

$$a_{ij} = \sum_{\substack{k \mid i \\ k \not| j}} g(k) = \sum_{d|n} g(d) - \sum_{d|(n,m)} g(d) = f(i,j).$$

□

Particular cases
1. If $g(n) = \varphi(n)$ then

$$f(n,m) = \sum_{d|n} \varphi(d) - \sum_{d|(n,m)} \varphi(d) = n - (n,m).$$

We have the following decomposition:

$$[i-(i,j)]_{n\times n} = \begin{bmatrix} 1-(1,1) & 1-(1,2) & \cdots & 1-(1,n) \\ 2-(2,1) & 2-(2,2) & \cdots & 2-(2,n) \\ \cdots & \cdots & \cdots & \cdots \\ n-(n,1) & n-(n,2) & \cdots & n-(n,n) \end{bmatrix}.$$

2. If $g(n)=1$ then

$$f(n,m)=\tau(n)-\tau(n,m)$$

and

$$[\tau(i)-\tau(i,j)]_{n\times n} = C_n \operatorname{diag}\big(1,\ 1,\ldots,1\big)\ D_n^T.$$

3. Let $g(n)=\mu(n)$. From

$$f(n,m)=\sum_{d|n}\mu(d)-\sum_{d|(n,m)}\mu(d)=\begin{cases} 0, & n=1 \\ 0, & n>1, m>1, (n,m)>1 \\ -1, & \text{othewise} \end{cases}.$$

we have

$$[f(i,j)]_{n\times n} = C_n \operatorname{diag}\big(\mu(1),\ \mu(2),\ldots,\mu(n)\big)\ D_n^T.$$

4. For $g(n)=n$, $f(n,m)=\sigma(n)-\sigma((n,m))$ and

$$[f(i,j)]_{n\times n} = C_n \operatorname{diag}\big(1,\ 2,\ldots,n\big)\ D_n^T.$$

Remarks

1. Due to the fact that the first line of the matrix $[f(i,j)]_{n\times n}$ contains only 0-s, the determinant of the matrix will always be 0.

2. We can determine the value of the matrix associated with f, if the function f is of the form

$$f(n,m)=h(n)-h\big((n,m)\big).$$

By using the Möbius inversion formula, we get

$$g(n)=\sum_{d|n}\mu(d)h\left(\frac{n}{d}\right),$$

consequently by using Theorem 1, the matrix can be decomposed according to the function $h(n)$:

$$[f(i,j)]_{n\times n} = C_n \operatorname{diag}[(\mu*h)(1),(\mu*h)(2),\ldots,(\mu*h)(n)]\ D_n^T.$$

Theorem 2 *For a given arithmetical function* g *let*

$$f(i,j) = \sum_{k=1}^{n} g(k) - \sum_{d|i} g(d) - \sum_{d|j} g(d) + \sum_{d|(i,j)} g(d).$$

Then

$$[f(i,j)]_{n\times n} = D_n \; diag[g(1), g(2), \ldots, g(n)] \, D_n^T,$$

where $D_n = [d_{ij}]_{n\times n}$,

$$d_{ij} = \begin{cases} 1, & j \nmid i \\ 0, & j \mid i \end{cases}.$$

Proof. After multiplication, the general element of the matrix

$$A = [a_{ij}]_{n\times n} = D_n \; \mathrm{diag}[g(1), g(2), \ldots, g(n)] \, D_n^T$$

is

$$\begin{aligned} a_{ij} &= \sum_{\substack{k \nmid n \\ k \nmid m}} g(k) = \sum_{k=1}^{n} g(k) - \sum_{k|n \text{ or } k|m} g(k) = \\ &= \sum_{k=1}^{n} g(k) - \sum_{k|n} g(k) - \sum_{k|m} g(k) + \sum_{k|(n,m)} g(k) = f(i,j). \end{aligned}$$

□

Particular cases

1. If $g(n) = \varphi(n)$ then

$$f(i,j) = \sum_{k=1}^{n} \varphi(k) - i - j + (i,j),$$

$$[f(i,j)]_{n\times n} = D_n \; \mathrm{diag}[\varphi(1), \varphi(2), \ldots, \varphi(n)] \, D_n^T.$$

2. If $g(n) = 1$ then

$$f(i,j) = n - \tau(i) - \tau(j) + \tau(i,j)$$

and

$$[f(i,j)]_{n\times n} = D_n \; \mathrm{diag}\big(1,\ 1, \ldots, 1\big) \; D_n^T.$$

3. $g(n) = n$. Then

$$f(i,j) = \frac{n(n+1)}{2} - \sigma(n) - \sigma(m) + \sigma((n,m))$$

and

$$[f(i,j)]_{n\times n} = D_n \operatorname{diag}\big(1,\ 2, \ldots, n\big)\, D_n^T.$$

Another generalization is the following:

Theorem 3 *For a given arithmetical function* g *let*

$$f(i,j) = \sum_{k=1}^{n} g(k) - \sum_{d|i} g(d) - \sum_{d|j} g(d) + \sum_{d|(i,j)} g(d).$$

We define the following $A = [a_{ij}]_{n\times n}$ *matrix*

$$a_{ij} = \begin{cases} f(i,j), & i,j > 1 \\ g(1) + f(i,j), & i = 1 \text{ or } j = 1 \end{cases}.$$

Then

$$A = D'_n \operatorname{diag}[g(1), g(2), \ldots, g(n)]\, D'^T_n,$$

where $D'_n = [d'_{ij}]_{n\times n}$,

$$d'_{ij} = \begin{cases} 1, & i = j = 1 \\ d_{ij}, & ij \neq 1 \end{cases}.$$

Proof.
We calculate the general element of the matrix

$$B = [a_{ij}]_{n\times n} = D'_n \operatorname{diag}[g(1), g(2), \ldots, g(n)]\, D'^T_n.$$

If $i > 1$ or $j > 1$ we have

$$\begin{aligned} b_{ij} &= \sum_{\substack{k \nmid n \\ k \nmid m}} g(k) = \sum_{k=1}^{n} g(k) - \sum_{k|n \text{ or } k|m} g(k) = \\ &= \sum_{k=1}^{n} g(k) - \sum_{k|n} g(k) - \sum_{k|m} g(k) + \sum_{k|(n,m)} g(k) = a_{ij}. \end{aligned}$$

If $i = j = 1$

$$b_{11} = g(1) = a_{11}.$$

□

Particular cases

1. If $g(n) = \varphi(n)$ then

$$a_{ij} = \begin{cases} \sum_{k=1}^{n} \varphi(k) - i - j + (i,j), & i,j > 1 \\ \sum_{k=1}^{n} \varphi(k) - i - j + (i,j) + 1, & i = 1 \text{ or } j = 1 \end{cases}.$$

2. If $g(n) = 1$ then

$$a_{ij} = \begin{cases} n - \tau(i) - \tau(j) + \tau(i,j), & i,j > 1 \\ n - \tau(i) - \tau(j) + \tau(i,j) + 1, & i = 1 \text{ or } j = 1 \end{cases}.$$

The following problems remain open:

Problem 1 *Let* $F(n,m)$ *be an arithmetical function with two vriables. Determine the structure and the determinant of modified GCD matrices* $A = [a(i,j)]_{n\times n}$*, where*

$$a_{ij} = F(i,(i,j)).$$

Problem 2 *Determine the structure and the determinant of modified GCD matrices* $A = [a(i,j)]_{n\times n}$*, where*

$$a_{ij} = F(n,i,j,(i,j)).$$

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Starlike harmonic functions in parabolic region associated with a convolution structure

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Abstract. Making use of a convolution structure, we introduce a new class of complex valued harmonic functions which are orientation preserving and univalent in the open unit disc. The results presented in this paper include the coefficient bounds, distortion inequality and covering property, extreme points and certain inclusion results for this generalized class of functions

1 Introduction and preliminaries

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [3]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1}$$

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which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = f'(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, the functions h and g analytic $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \;\; g(z) = \sum_{n=1}^{\infty} b_n z^n \;\; (0 \leq b_1 < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \quad (0 \leq b_1 < 1). \tag{2}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well-known class S of normalized univalent functions if the co-analytic part of f is identically zero, i.e. $g \equiv 0$.

For functions $f \in \mathcal{H}$ given by (1) and $F \in \mathcal{H}$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}, \tag{3}$$

we recall the Hadamard product (or convolution) of f and F by

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n} \;\; (z \in \mathcal{U}). \tag{4}$$

In terms of the Hadamard product (or convolution), we choose F as a fixed function in $\mathcal{H}$ such that $(f * F)(z)$ exists for any $f \in \mathcal{H}$, and for various choices of F we get different linear operators which have been studied in recent past. To illustrate some of these cases which arise from the convolution structure (4), we consider the following examples.

(1) If

$$F(z) = z + \sum_{n=2}^{\infty} \sigma_n(\alpha_1)\, z^n + \sum_{n=1}^{\infty} \sigma_n(\alpha_1)\, \overline{z}^n \tag{5}$$

and $\sigma_n(\alpha_1)$ is defined by

$$\sigma_n(\alpha_1) = \frac{\Theta\Gamma(\alpha_1 + A_1(n-1)) \ldots \Gamma(\alpha_p + A_p(n-1))}{(n-1)!\Gamma(\beta_1 + B_1(n-1)) \ldots \Gamma(\beta_q + B_q(n-1))}\,, \tag{6}$$

where Θ is given by

$$\Theta = \left(\prod_{m=0}^{p} \Gamma(\alpha_m)\right)^{-1} \left(\prod_{m=0}^{q} \Gamma(\beta_m)\right) \tag{7}$$

and then the convolution(4) gives the Wright's generalized hypergeometric function (see [17])

$${}_p\Psi_q[(\alpha_1, A_1), \ldots; (\beta_1, B_1), \ldots; z] =_p \Psi_q[(\alpha_n, A_n)_{1,p}(\beta_n, B_n)_{1,q}; z]$$

defined by

$${}_p\Psi_q[(\alpha_n, A_n)_{1,p}(\beta_n, B_n)_{1,q}; z] = \sum_{n=0}^{\infty}\{\prod_{m=1}^{p} \Gamma(\alpha_m+nA_m)\}\{\prod_{m=1}^{q} \Gamma(\beta_m+nB_m)\}^{-1}\frac{z^m}{n!}$$

which was initially studied by Murugusundaramoorthy and Vijaya (see [10]).

(2) If $A_m = 1$ $(m = 1, \ldots, p)$ and $B_m = 1$ $(m = 1, \ldots, q)$, then we have the following relationship

$$F(z) = z + \sum_{n=2}^{\infty} \Gamma_n z^n + \sum_{n=1}^{\infty} \Gamma_n \bar{z}^n, \tag{8}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_p)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_q)_{n-1}} \frac{1}{(n-1)!},$$

and the convolution (4) gives the Dziok–Srivastava operator (see [5]):

$$\Lambda(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z)f(z) \equiv \mathcal{H}_q^p(\alpha_1, \beta_1)f(z),$$

where $\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q$ are positive real numbers, $p \leq q+1; p, q \in \mathbb{N} \cup \{0\}$, and $(\alpha)_n$ denotes the familiar Pochhammer symbol (or shifted factorial).

Remark 1 *When* $p = 1$, $q = 1$; $\alpha_1 = a$, $\alpha_2 = 1$; $\beta_1 = c$, *then (8) corresponds to the operator due to Carlson-Shaffer (see [2]) given by*

$$\mathcal{L}(a, c)f(z) := (f * F)(z),$$

where

$$F(z) := z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n + \sum_{n=1}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} \bar{z}^n \quad (c \neq 0, -1, -2, \ldots). \tag{9}$$

Remark 2 *When* $p = 1$, $q = 0$; $\alpha_1 = k+1$ $(k > -1)$, $\alpha_2 = 1$; $\beta_1 = 1$, *then (8) yields the Ruscheweyh derivative operator (see [8]) given by* $D^k f(z) := (f*F)(z)$ *where*

$$F(z) = z + \sum_{n=2}^{\infty} \binom{k+n-1}{n-1} z^n + \sum_{n=1}^{\infty} \binom{k+n-1}{n-1} \bar{z}^n, \tag{10}$$

which was initially studied by Jahangiri et al. (see [8]).

(3) If $\mathcal{D}^l f(z) = f * F$ where

$$F(z) = z + \sum_{n=2}^{\infty} n^l z^n + (-1)^l \sum_{n=1}^{\infty} n^l \bar{z}^n \quad (\ l \geq 0)\,, \tag{11}$$

was initially studied by Jahangiri et al. (see [9]).

(4) Lastly, if $\mathcal{S}_\alpha f(z) = f * F$ we have

$$F(z) = z + \sum_{m=2}^{\infty} |C_n(\alpha)| z^n + \sum_{n=1}^{\infty} |C_n(\alpha)| \bar{z}^n, \tag{12}$$

and

$$C_n(\alpha) = \frac{\prod_{j=2}^{n}(j-2\alpha)}{(n-1)!} \quad (n \in \mathbb{N} \setminus \{1\}, \mathbb{N} := \{1, 2, 3, \ldots\}) \tag{13}$$

which is decreasing in α and satisfies

$$\lim_{n\to\infty} C_n(\alpha) = \begin{cases} \infty & \text{if } \alpha < \frac{1}{2} \\ 1 & \text{if } \alpha = \frac{1}{2} \\ 0 & \text{if } \alpha > \frac{1}{2} \end{cases}. \tag{14}$$

For the purpose of this paper, we introduce here a subclass of $\mathcal{H}$ denoted by $\mathcal{R}_H(F; \lambda, \gamma)$ which involves the convolution (3) and consist of all functions of the form (1) satisfying the inequality:

$$\text{Re}\left\{(1+e^{i\psi})\frac{z(f(z)*F(z))'}{(1-\lambda)z+\lambda(f(z)*F(z))} - e^{i\psi}\right\} \geq \gamma. \tag{15}$$

Equivalently

$$\text{Re}\left\{(1+e^{i\psi})\frac{z(h(z)*H(z))' - \overline{z(g(z)*G(z))'}}{(1-\lambda)z+\lambda[h(z)*H(z)+\overline{g(z)*G(z)}]} - e^{i\psi}\right\} \geq \gamma \tag{16}$$

where $z \in \mathcal{U},\ 0 \leq \lambda \leq 1$.

Also denote $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma) = \mathcal{R}_H(F;\lambda,\gamma) \bigcap \mathcal{T}_{\mathcal{H}}$ where $\mathcal{T}_{\mathcal{H}}$ is the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \ (0 \leq b_1 < 1). \tag{17}$$

called the class of harmonic functions with negative coefficients (see [14]).

It is of special interest to note that for suitable choices of $\lambda = 0$ and $\lambda = 1$ the classes USD [13] and S_p [11] to include the following harmonic functions

$$\text{Re}\ \left\{(1 + e^{i\psi})(f(z) * F(z))' - e^{i\psi}\right\} \geq \gamma,$$

$$\text{Re}\ \left\{(1 + e^{i\psi})\frac{z(f(z) * F(z))'}{(f(z) * F(z))} - e^{i\psi}\right\} \geq \gamma.$$

We mention below some of the function classes which emerge from the function class $\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$ defined above. Indeed, we observe that if we specialize the function F by (5) to (11), and denote the corresponding reducible classes of functions of $\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$, respectively, by $\mathcal{W}_q^p(\lambda,\gamma)$, $\mathcal{G}_q^p(\lambda,\gamma)$ $\mathcal{L}_c^a(\lambda,\gamma)$, $\mathcal{R}(k,\lambda,\gamma)$, $\Omega(\lambda,\gamma)$ and $\mathcal{S}(l,\lambda,\gamma)$.

It is of special interest because for suitable choices of F from (15) we can define the following subclasses:

(i) If F is given by (5) we have $(f * F)(z) = W_q^p[\alpha_1]f(z)$ hence we define a class $\mathcal{W}_q^p(\lambda,\gamma)$ satisfying the criteria

$$\text{Re}\ \left\{(1 + e^{i\psi})\frac{z(W_q^p[\alpha_1]f(z))'}{(1-\lambda)z + \lambda W_q^p[\alpha_1]f(z)} - e^{i\psi}\right\} \geq \gamma$$

where $W_q^p[\alpha_1]$ is the Wright's generalized operator on harmonic functions (see [10]) .

(ii) If F is given by (8) we have $(f * F)(z) = H_q^p[\alpha_1]f(z)$ hence we define a class $\mathcal{G}_q^p(\lambda,\gamma)$ satisfying the criteria

$$\text{Re}\ \left\{(1 + e^{i\psi})\frac{z(H_q^p[\alpha_1]f(z))'}{(1-\lambda)z + \lambda H_q^p[\alpha_1]f(z)} - e^{i\psi}\right\} \geq \gamma$$

where $H_q^p[\alpha_1]$ is the Dziok - Srivastava operator (see [5]).

(iii) $H_1^2([a,1;c]) = \mathcal{L}(a,c)f(z)$, hence we define a class $\mathcal{L}_c^a(\lambda,\gamma)$satisfying the criteria

$$\text{Re}\ \left\{(1 + e^{i\psi})\frac{z\mathcal{L}(a,c)f(z))'}{(1-\lambda)z + \lambda\mathcal{L}(a,c)f(z)} - e^{i\psi}\right\} \geq \gamma$$

where $\mathcal{L}(a,c)$ is the Carlson - Shaffer operator (see [2]).
(iv) $H_1^2([k+1,1;1]) = D^k f(z)$, hence we define a class $\mathcal{R}(k,\lambda,\gamma)$ satisfying the criteria

$$\text{Re}\left\{(1+e^{i\psi})\frac{z(D^k f(z))'}{(1-\lambda)z+\lambda D^k f(z)} - e^{i\psi}\right\} \geq \gamma$$

where $D^k f(z)(k > -1)$ is the Ruscheweyh derivative operator (see [12]) (also see [8]).
(v) $H_1^2([2,1;2-\mu]) = \Omega_z^\mu f(z)$ we define another class $\Omega(\lambda,\gamma)$ satisfying the condition

$$\text{Re}\left\{(1+e^{i\psi})\frac{z(\Omega_z^\mu f(z))'}{(1-\lambda)z+\lambda \Omega_z^\mu f(z)} - e^{i\psi}\right\} \geq \gamma$$

given by

$$\Omega_z^\mu f(z) = \Gamma(2-\mu)z^\mu D_z^\mu f(z); (0 \leq \mu < 1) \ ,$$

where Ω_z^μ is the Srivastava-Owa fractional derivative operator (see [15]).
(vi) If F is given by (12), we have $S_\alpha(z) * f(z) = (f * F)(z)$, hence we define a class $\mathcal{PG}_{\mathcal{H}}(\alpha,\gamma)$ satisfying the criteria

$$\text{Re}\left\{(1+e^{i\psi})\frac{z(S_\alpha(z) * f(z))'}{(1-\lambda)z+\lambda(S_\alpha(z) * f(z))} - e^{i\psi}\right\} \geq \gamma, \tag{18}$$

this class was introduced and studied by Vijaya [16] for $\lambda = 1$.
(vii) If F is given by (11), we have $D^l f(z) = (f * F)(z)$, hence we define a class $S(l,\lambda,\gamma)$ satisfying the criteria

$$\text{Re}\left\{(1+e^{i\psi})\frac{z(D^l f(z))'}{(1-\lambda)z+\lambda D^l f(z)} - e^{i\psi}\right\} \geq \gamma$$

where $D^l f(z); (l \in \mathbb{N})$ is the Sălăgean derivative operator for harmonic functions (see [9]) $\lambda = 1$.

Motivated by the earlier works of (see [6, 9, 17]) on the subject of harmonic functions, in this paper we obtain a sufficient coefficient condition for functions f given by (2) to be in the class $\mathcal{S}_{\mathcal{H}}(F;\lambda,\gamma)$. It is shown that this coefficient condition is necessary also for functions belonging to the class $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. Further, distortion results and extreme points for functions in $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ are also obtained.

For the sake of brevity we denote the corresponding coefficient of F as C_n throughout our study unless otherwise stated.

2 Coefficient bounds

In our first theorem, we obtain a sufficient coefficient condition for harmonic functions in $\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$.

Theorem 1 *Let* $f = h + \overline{g}$ *be given by (2). If*

$$\sum_{n=1}^{\infty}\left[\frac{2n-(1+\gamma)\lambda}{1-\gamma}|a_n| + \frac{2n+(1+\gamma)\lambda}{1-\gamma}|b_n|\right]C_n \tag{19}$$

where $a_1 = 1$ *and* $0 \le \gamma < 1$, *then* $f \in \mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$.

Proof. We first show that if (19) holds for the coefficients of $f = h + \overline{g}$, the required condition (19) is satisfied. From (16) we can write

$$\begin{aligned}
&\operatorname{Re}\left\{(1+e^{i\psi})\frac{z(h(z)*H(z))' - \overline{z(g(z)*G(z))'}}{(1-\lambda)z+\lambda[h(z)*H(z)+\overline{g(z)*G(z)}]} - e^{i\psi}\right\} \ge \gamma \\
= \ &\operatorname{Re}\left\{\frac{(1+e^{i\psi})[z(h(z)*H(z))' - \overline{z(g(z)*G(z))'}]}{(1-\lambda)z+\lambda[h(z)*H(z)+\overline{g(z)*G(z)}]} - \right. \\
&\left. - \frac{e^{i\psi}[(1-\lambda)z+\lambda(h(z)*H(z)+\overline{g(z)*G(z)})]}{(1-\lambda)z+\lambda[h(z)*H(z)+\overline{g(z)*G(z)}]}\right\} = \\
= \ &\operatorname{Re}\frac{A(z)}{B(z)} \ge \gamma
\end{aligned}$$

where

$$\begin{aligned}
A(z) &= (1+e^{i\psi})[z(h(z)*H(z))' - \overline{z(g(z)*G(z))'}] - \\
&\quad - e^{i\psi}[(1-\lambda)z+\lambda(h(z)*H(z)+\overline{g(z)*G(z)})] = \\
&= z + \sum_{n=2}^{\infty}[n+(n-\lambda)e^{i\psi}]C_n a_n z^n - \sum_{n=1}^{\infty}[n+(n-\lambda)e^{i\psi}]C_n \overline{b}_n \overline{z}^n \\
\text{and } B(z) &= (1-\lambda)z+\lambda[h(z)*H(z)+\overline{g(z)*G(z)}] \\
&= z + \sum_{n=2}^{\infty}\lambda C_n a_n z^n + \sum_{n=1}^{\infty}\lambda C_n \overline{b}_n \overline{z}^n.
\end{aligned}$$

Using the fact that $\operatorname{Re}\{w\} \ge \gamma$ if and only if $|1-\gamma+w| \ge |1+\gamma-w|$, it suffices to show that

$$|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \ge 0. \tag{20}$$

Substituting for $A(z)$ and $B(z)$ in (20), we get

$$\begin{aligned}
& |A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|- \\
= & \Bigg|(2-\gamma)z+\sum_{n=2}^{\infty}[n+(n-\lambda)e^{i\psi}+(1-\gamma)\lambda]C_n a_n z^n- \\
& -\sum_{n=1}^{\infty}[n+(n-\lambda)e^{i\psi}-(1-\gamma)\lambda]C_n\overline{b}_n\ \overline{z}^n\Bigg|- \\
& -\Bigg|-\gamma z+\sum_{n=2}^{\infty}[n+(n-\lambda)e^{i\psi}-(1+\gamma)\lambda]C_n a_n z^n- \\
& -\sum_{n=1}^{\infty}[n+(n-\lambda)e^{i\psi}+(1+\gamma)\lambda]C_n\overline{b}_n\overline{z}^n\Bigg| \geq \\
\geq & (2-\gamma)|z|-\sum_{n=2}^{\infty}[n+(n-\lambda)+(1-\gamma)\lambda]C_n|a_n||z|^n- \\
& -\sum_{n=1}^{\infty}[n+(n-\lambda)-(1-\gamma)\lambda]C_n|b_n|\ |z|^n- \\
& -\gamma|z|-\sum_{n=2}^{\infty}[n+(n-\lambda)-(1+\gamma)\lambda]C_n|a_n|\ |z|^n- \\
& -\sum_{n=1}^{\infty}[n+(n-\lambda)+(1+\gamma)\lambda]C_n|b_n|\ |z|^n \geq \\
\geq & 2(1-\gamma)|z|\left\{2-\sum_{n=1}^{\infty}\left[\frac{2n-(1+\gamma)\lambda}{1-\gamma}|a_n|+\frac{2n+(1+\gamma)\lambda}{1-\gamma}|b_n|\right]C_n|z|^{n-1}\right\} \\
\geq & 2(1-\gamma)\left\{2-\sum_{n=1}^{\infty}\left[\frac{2n-(1+\gamma)\lambda}{1-\gamma}|a_n|+\frac{2n-(1+\gamma)\lambda}{1-\gamma}|b_n|\right]C_n\right\}.
\end{aligned}$$

The above expression is non negative by (19), and so $f \in \mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$. □

The harmonic function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n}x_n z^n+\sum_{n=1}^{\infty}\frac{1-\gamma}{[2n+(1+\gamma)\lambda]C_n}\overline{y}_n(\overline{z})^n \quad (21)$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1$ shows that the coefficient bound given by (19) is sharp.

The functions of the form (21) are in $\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$ because

$$\sum_{n=1}^{\infty}\left(\frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|a_n|+\frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|b_n|\right)=$$
$$=1+\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=2.$$

Next theorem establishes that such coefficient bounds cannot be improved further.

Theorem 2 *For* $a_1=1$ *and* $0\le\gamma<1$, $f=h+\overline{g}\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ *if and only if*

$$\sum_{n=1}^{\infty}\left[\frac{2n-(1+\gamma)\lambda}{1-\gamma}|a_n|+\frac{2n+(1+\gamma)\lambda}{1-\gamma}|b_n|\right]C_n\le 2. \tag{22}$$

Proof. Since $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)\subset\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (17), we notice that the condition

$$\mathrm{Re}\left\{(1+e^{i\psi})\frac{z(h(z)*H(z))'-\overline{z(g(z)*G(z))'}}{(1-\lambda)z+\lambda[h(z)*H(z)+\overline{g(z)*G(z)}]}-(e^{i\psi}+\gamma)\right\}\ge 0$$

The above inequality is equivalent to

$$\mathrm{Re}\left\{\frac{(1-\gamma)z-\sum\limits_{n=2}^{\infty}[n(1+e^{i\psi})-(1+\gamma+e^{i\psi})\lambda]C_na_nz^n}{z-\sum\limits_{n=2}^{\infty}\lambda C_na_nz^n+\sum\limits_{n=1}^{\infty}\lambda C_n\overline{b}_n\overline{z}^n}-\right.$$
$$\left.-\frac{\sum\limits_{n=1}^{\infty}[n(1+e^{i\psi})+(1+\gamma+e^{i\psi})\lambda]C_n\overline{b}_n\overline{z}^n}{z-\sum\limits_{n=2}^{\infty}\lambda C_na_nz^n+\sum\limits_{n=1}^{\infty}\lambda C_n\overline{b}_n\overline{z}^n}\right\}\ge 0.$$

The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\le z=r<1$, and noting that $\mathrm{Re}(-e^{i\psi})\ge-|e^{i\psi}|=-1$, we must have

$$\frac{(1-\gamma)-\sum\limits_{n=2}^{\infty}[2n-(1+\gamma)\lambda]C_na_nr^{n-1}-\sum\limits_{n=1}^{\infty}[2n-(1+\gamma)\lambda]C_nb_nr^{n-1}}{1-\sum\limits_{n=2}^{\infty}\lambda C_na_nr^{n-1}+\sum\limits_{n=1}^{\infty}\lambda C_nb_nr^{n-1}}\ge 0. \tag{23}$$

If the condition (22) does not hold, then the numerator in (23) is negative for r sufficiently close to 1. Hence, there exist $z_0 = r_0$ in (0,1) for which the quotient of (23) is negative. This contradicts the required condition for $f \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. This completes the proof of the theorem. □

3 Distortion bounds and extreme points

The following theorem gives the distortion bounds for functions in $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ which yields a covering result for the class $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$.

Theorem 3 *Let* $f \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. *Then for* $|z| = r < 1$, *we have*

$$(1-b_1)r - \frac{1}{C_2}\left(\frac{1-\gamma}{4-(1+\gamma)\lambda} - \frac{1+\gamma}{4-(1+\gamma)\lambda}b_1\right)r^2 \le |f(z)|$$
$$\le (1+b_1)r + \frac{1}{C_2}\left(\frac{1-\gamma}{4-(1+\gamma)\lambda} - \frac{1+\gamma}{4-(1+\gamma)\lambda}b_1\right)r^2.$$

Proof. We only prove the right hand inequality. Taking the absolute value of $f(z)$, we obtain

$$\begin{aligned}
|f(z)| &= \left|z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n\right| \le (1+b_1)|z| + \sum_{n=2}^{\infty}(a_n+b_n)|z|^n \le \\
&\le (1+b_1)r + \sum_{n=2}^{\infty}(a_n+b_n)r^2 \le (1+b_1)r + \frac{(1-\gamma)}{[4-(1+\gamma)\lambda]C_2} \\
&\quad \sum_{n=2}^{\infty}\left(\frac{[4-(1+\gamma)\lambda]C_2}{(1-\gamma)}a_n + \frac{[4-(1+\gamma)\lambda]C_2}{(1-\gamma)}b_n\right)r^2 \le \\
&\le (1+b_1)r + \frac{(1-\gamma)1}{[4-(1+\gamma)\lambda]C_2}\left(1 - \frac{1+\gamma}{1-\gamma}b_1\right)r^2 \le \\
&\le (1+b_1)r + \frac{1}{C_2}\left(\frac{1-\gamma}{4-(1+\gamma)\lambda} - \frac{1+\gamma}{4-(1+\gamma)\lambda}b_1\right)r^2.
\end{aligned}$$

The proof of the left hand inequality follows on lines similar to that of the right hand side inequality. □

The covering result follows from the left hand inequality given in Theorem 3.

Corollary 1 *If* $f(z) \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$, *then*

$$\left\{w : |w| < \frac{[4-(1+\gamma)\lambda]C_2 - (1-\gamma)}{[4-(1+\gamma)\lambda]C_2} - \frac{[4-(1+\gamma)\lambda]C_2 - (1+\gamma)}{[4-(1+\gamma)\lambda]C_2}|b_1|\right\} \subset f(U).$$

Proof. Using the left hand inequality of Theorem 3 and letting $r \to 1$, we prove that

$$\begin{aligned}
&(1-b_1) - \frac{1}{C_2}\left(\frac{1-\gamma}{4-(1+\gamma)\lambda} - \frac{1+\gamma}{4-(1+\gamma)\lambda}b_1\right) = \\
&= (1-b_1) - \frac{1}{C_2[4-(1+\gamma)\lambda]}[1-\gamma-(1+\gamma)b_1] = \\
&= \frac{(1-b_1)C_2[4-(1+\gamma)\lambda] - (1-\gamma) + (1+\gamma)b_1}{C_2[4-(1+\gamma)\lambda]} = \\
&= \left\{\frac{[4-(1+\gamma)\lambda]C_2 - (1-\gamma)}{[4-(1+\gamma)\lambda]C_2} - \frac{[4-(1+\gamma)\lambda]C_2 - (1+\gamma)}{[4-(1+\gamma)\lambda]C_2}|b_1|\right\} \subset f(U).
\end{aligned}$$

□

Next we determine the extreme points of closed convex hulls of $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ denoted by $\mathrm{clco}\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$.

Theorem 4 *A function* $f(z) \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ *if and only if*

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_n(z))$$

where

$$\begin{aligned}
&h_1(z) = z, h_n(z) = z - \frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n}z^n; \quad (n \geq 2), \\
&g_n(z) = z + \frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n}\overline{z}^n; \quad (n \geq 2), \\
&\sum_{n=1}^{\infty}(X_n + Y_n) = 1, \quad X_n \geq 0 \text{ and } Y_n \geq 0.
\end{aligned}$$

In particular, the extreme points of $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. First, we note that for f as in the theorem above, we may write

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_n(z)) = \\
&= \sum_{n=1}^{\infty}(X_n + Y_n)z - \sum_{n=2}^{\infty}\frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n}X_n z^n + \\
&\quad + \sum_{n=1}^{\infty}\frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n}Y_n\overline{z}^n
\end{aligned}$$

Then

$$\sum_{n=2}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|a_n| + \sum_{n=1}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|b_n| =$$
$$= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1,$$

and so $f(z) \in \mathrm{clco}\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$.

Conversely, suppose that $f(z) \in \mathrm{clco}\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. Setting

$$X_n = \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|a_n|, \quad (0 \leq X_n \leq 1, n \geq 2)$$

$$Y_n = \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma}|b_n|, \quad (0 \leq Y_n \leq 1, n \geq 1)$$

and $X_1 = 1 - \sum\limits_{n=2}^{\infty} X_n - \sum\limits_{n=1}^{\infty} Y_n$. Therefore, $f(z)$ can be rewritten as

$$\begin{aligned} f(z) &= z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n = \\ &= z - \sum_{n=2}^{\infty} \frac{1-\gamma}{[2n-(1+\gamma)\lambda]C_n} X_n z^n + \sum_{n=1}^{\infty} \frac{1-\gamma}{[2n+(1+\gamma)\lambda]C_n} Y_n \overline{z}^n = \\ &= z + \sum_{n=2}^{\infty} (h_n(z) - z) X_n + \sum_{n=1}^{\infty} (g_n(z) - z) Y_n = \\ &= z\{1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n\} + \sum_{n=2}^{\infty} h_n(z) X_n + \sum_{n=1}^{\infty} g_n(z) Y_n = \\ &= \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_n(z)) \text{ as required.} \end{aligned}$$

□

4 Inclusion results

Now we show that $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ is closed under convex combinations of its member and also closed under the convolution product.

Theorem 5 *The family $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ is closed under convex combinations.*

Proof. For $i = 1, 2, \ldots$, suppose that $f_i \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ where

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{i,n} z^n + \sum_{n=2}^{\infty} \overline{b}_{i,n} \overline{z}^n.$$

Then, by Theorem 2

$$\sum_{n=2}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{(1-\gamma)} a_{i,n} + \sum_{n=1}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{(1-\gamma)} b_{i,n} \leq 1. \tag{24}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i,n} \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \overline{b}_{i,n} \right) \overline{z}^n.$$

Using the inequality (22), we obtain

$$\sum_{n=2}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i a_{i,n} \right) + \sum_{n=1}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i b_{i,n} \right) =$$
$$= \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma} a_{i,n} + \sum_{n=1}^{\infty} \frac{[2n-(1+\gamma)\lambda]C_n}{1-\gamma} b_{i,n} \right) \leq \sum_{i=1}^{\infty} t_i = 1,$$

and therefore $\sum\limits_{i=1}^{\infty} t_i f_i \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. □

Now, we will examine the closure properties of the class $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ under the generalized Bernardi-Libera -Livingston integral operator$\mathcal{L}_c(f)$ which is defined by

$$\mathcal{L}_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, c > -1.$$

Theorem 6 *Let $f(z) \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. Then $\mathcal{L}_c(f(z)) \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$*

Proof. From the representation of $\mathcal{L}_c(f(z))$, it follows that

$$\begin{aligned}
\mathcal{L}_c(f) &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(t)+\overline{g(t)}\right]dt = \\
&= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{n=2}^{\infty}a_n t^n\right)dt+\overline{\int_0^z t^{c-1}\left(\sum_{n=1}^{\infty}b_n t^n\right)dt}\right) = \\
&= z-\sum_{n=2}^{\infty}\frac{c+1}{c+n}a_n z^n+\sum_{n=1}^{\infty}\frac{c+1}{c+n}\,b_n z^n.
\end{aligned}$$

Using the inequality (22), we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\frac{[2n-(1+\gamma)\lambda]}{1-\gamma}\left(\frac{c+1}{c+n}|a_n|\right)+\frac{[2n+(1+\gamma)\lambda]}{1-\gamma}\left(\frac{c+1}{c+n}|b_n|\right)\right)C_n \le \\
&\le \sum_{n=1}^{\infty}\left(\frac{[2n-(1+\gamma)\lambda]}{1-\gamma}|a_n|+\frac{[2n+(1+\gamma)\lambda]}{1-\gamma}|b_n|\right)C_n \le \\
&\le 2(1-\gamma), \text{ since } f(z)\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma).
\end{aligned}$$

Hence by Theorem 2, $\mathcal{L}_c(f(z))\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. □

Concluding remarks

For suitable choices of $F(z)$, as we pointed out the $\mathcal{R}_{\mathcal{H}}(F;\lambda,\gamma)$ contains, various function class defined by linear operators such as the Carlson-Shaffer operator, the Ruscheweyh derivative operator, the Sălăgean operator, the fractional derivative operator, and so on. When $\lambda = 0$ and $\lambda = 1$ the various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler harmonic function classes[1] and [8, 9, 10] respectively. The details involved in the derivations of such specializations of the results presented in this paper are fairly straight- forward, hence omitted.

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Differential subordination results for some classes of the family $\zeta(\varphi, \vartheta)$ associated with linear operator

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Abstract. For some classes of family of real valued functions defined in a unit disk, we use a linear operator to obtain some interesting differential subordination results.

1 Introduction and preliminaries

Let E_α^+ denote the family of all functions $F(z)$, in the unit disk U, of the form

$$F(z) = 1 + \sum_{n=1}^{\infty} a_n z^{n-n/\alpha}, \ \alpha = \{2, 3, 4 \dots\} \tag{1}$$

satisfying $F(0) = 1$.

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Let E_{α}^{-} denote the family of all functions $F(z)$, in the unit disk U, of the form

$$F(z) = 1 - \sum_{n=1}^{\infty} a_n z^{n-n/\alpha}, \ \alpha = \{2, 3, 4 \dots\} \tag{2}$$

which satisfy the condition $F(0) = 1$.

We know that if functions f and g are analytic in U, then f is called subordinate to g if there exists a Schwarz function $w(z)$, analytic in U such that $f(z) = g(w(z))$, and $z \in U = \{z : z \in C, \ |z| < 1\}$.

Then we denote this subordination by $f(z) \prec g(z)$ or simply $f \prec g$, but in a special case if g is univalent in U then above subordination is equivalent to $f(0) = g(0)$, and $f(U) \subset g(U)$.

Let $\phi : C^3 \times U \to C$ and let h analytic in U. Assume that p, ϕ are analytic and univalent in U and p satisfies the differential superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z). \tag{3}$$

Then p is called a solution of the differential superordination.

An analytic function q is called a subordinant if $q \prec p$, for all p satisfying equation (3). A univalent function q such that $p \prec q$ for all subordinants p of equation (3) is said to be the best subordinant.

Let E^{+} be the class of analytic functions of the form

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad z \in \mathcal{U}, \ a_n, b_n \geq 0.$$

Let $f, g \in E^{+}$ where

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

then their convolution or Hadamard product $f(z) * g(z)$ is defined by

$$f(z) * g(z) = 1 + \sum_{n=1}^{\infty} a_n b_n z^n, \ z \in \mathcal{U}.$$

Juneja et al. [1] define the family $\varepsilon(\phi, \psi)$ so that

$$\mathrm{Re}\left(\frac{f(z) * \phi(z)}{f(z) * \psi(z)}\right) > 0, z \in \mathcal{U}$$

where

$$\phi(z) = 1 + \sum_{n=1}^{\infty} \phi_n z^n$$

and

$$\psi(z) = 1 + \sum_{n=1}^{\infty} \psi_n z^n$$

are analytic in U with the conditions ϕ_n, $psi_n \geq 0$, $\phi_n \geq \psi_n$ and $\phi(z) * \psi(z) \neq 0$.

Definition 1 *Let $\zeta_\alpha^+(\varphi, \vartheta)$ be the class of family of all F(z) $\in E_\alpha^+$ such that*

$$\mathrm{Re}\left(\frac{F(z) * \varphi(z)}{F(z) * \vartheta(z)}\right) > 0, z \in \mathcal{U}$$

where

$$\varphi(z) = 1 + \sum_{n=2}^{\infty} \varphi_n z^{n-n/\alpha} \text{ and } \vartheta(z) = 1 + \sum_{n=2}^{\infty} \vartheta_n z^{n-n/\alpha}$$

are analytic in U with specific conditions, $\varphi_n, \vartheta_n \geq 0$, $\varphi_n \geq \vartheta_n$ and $F(z) * \vartheta(z) \neq 0$ and for all $n \geq 0$.

Definition 2 *Let $\zeta_\alpha^-(\varphi, \vartheta)$ be the class of family of all $F(z) \in E_\alpha^-$ such that*

$$\mathrm{Re}\left(\frac{F(z) * \varphi(z)}{F(z) * \vartheta(z)}\right) > 0, \ z \in \mathcal{U}$$

where

$$\varphi(z) = 1 - \sum_{n=2}^{\infty} \varphi_n z^{n-n/\alpha} \text{ and } \vartheta(z) = 1 - \sum_{n=2}^{\infty} \vartheta_n z^{n-n/\alpha}$$

are analytic in U with specific conditions, $\varphi_n, \vartheta_n \geq 0$, $\varphi_n \geq \vartheta_n$ and $F(z) * \vartheta(z) \neq 0$ and for all $n \geq 0$.

The aim of the present paper is to propose some sufficient conditions for all functions $F(z)$ belongs to the new classes E_α^+ and E_α^- to satisfy

$$\frac{F(z) * \varphi(z)}{F(z) * \vartheta(z)} \prec q(z), \ z \in \mathsf{U}.$$

Where $q(z)$ is a given univalent function in U such that $q(0) = 1$.

Define the function $\varphi_\alpha(a,c;z)$ by

$$\varphi_\alpha(a,c;z) = 1 + \sum_{1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-n/\alpha},\ z \in U,\ c \in \Re \setminus \{0,-1,-2\ldots\}$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(n+a)}{\Gamma(a)} = \begin{cases} 1 & \text{if } n = 0 \\ a(a+1)(a+2)\cdots(a+n-1) & \text{if } n \in N \end{cases}$$

Corresponding to the function $\varphi_\alpha(a,c;z)$, define a linear operator $I_\alpha(a,c)$, by

$$I_\alpha(a,c)F(z) = I_\alpha(a,c;z) * F(z),\ \ F(z) \in E_\alpha^+,$$

or equivalently by

$$I_\alpha(a,c)F(z) = 1 + \sum_{1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-n/\alpha}, z \in U, c \in \Re \setminus \{0,-1,-2\ldots\}$$

Different authors have used this linear operator for various types of classes of univalent functions namely, Uralgaddi and Somanatha [4], Cho, Kwon and Srivastava [5], Saitoh [6], and Sokol and Spelina [7], respectively.

The classes E_α^+ and E_α^- defined above exhibit some interesting properties. We need the following lemmas.

Lemma 1 [3]. *Let $q(z)$ be univalent in the unit U disk and $\theta(z)$ be analytic in a domain D containing q(U). If $zq'(z)\theta(q)$ is starlike in U , and*

$$zp'(z)\theta(p(z)) \prec zq'(z)\theta(q(z))$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 1 *Let the function $q(z)$ be univalent in the unit disk U such that $q'(z) \neq (0)$ and $\dfrac{zq'(z)}{q(z)} \neq 0$ is starlike in U, if $F(z) \in E_\alpha^+$ satisfies the subordination*

$$b\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \prec \frac{bzq'(z)}{q(z)}$$

then,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec q(z)$$

Then is $q(z)$ the best dominant.

Proof. First we defined the function $p(z)$,

$$p(z) = \left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]$$

then,

$$\frac{bzp'(z)}{p(z)} = b\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \tag{4}$$

By setting, $\theta(\omega) = \dfrac{b}{\omega}$, it can easily observed that $\theta(\omega)$ is analytic in $C \setminus \{0\}$. Then we obtain that,

$$\theta(p(z)) = \frac{b}{p(z)} \text{ and } \theta(q(z)) = \frac{b}{q(z)}.$$

So from equation (4), we have

$$zp'(z)\theta(p(z)) \preceq b\frac{q'(z)}{q(z)} = zq'(z)\theta(q(z)),$$

this implies,

$$zp'(z)\theta(p(z)) \prec zq'(z)\theta(q(z))$$

from lemma (1), we have

$$p(z) \prec q(z)$$

this implies,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec q(z)$$

□

Corollary 1 *If* $F(z)$ *satisfies the subordination*

$$b\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \prec \left[\frac{b(A-B)z}{(1+Az)(1+BZ)}\right]$$

then,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec \left[\frac{1+Az}{1+Bz}\right], \quad -1 \le A \le B \le 1,$$

and $\dfrac{(1+Az)}{(1+Bz)}$ *is the best dominant.*

Corollary 2 *If* $F(z)$ *satisfies the subordination*

$$b\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)}-\frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right]\prec\left[\frac{2bz}{(1+z)(1+z)}\right]$$

then,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]\prec\left[\frac{1+z}{1-z}\right],\quad -1\le A\le B\le 1,$$

and $\dfrac{(1+z)}{(1+z)}$ *is the best dominant.*

Lemma 2 [2]. *Let* $q(z)$ *be convex in the unit disk* U *with* $q(0)=1$ *and* $\Re(q)>1/2$, $z\in U$. *If* $0\le U<1$, p *is analytic function in with* $p(0)=1$ *and if*

$$(1-\mu)p^2(z)+(2\mu-1)p(z)-\mu+(1-\mu)zp'(z)$$

$$\prec(1-\mu)q^2(z)+(2\mu-1)q(z)-\mu+(1-\mu)zq'(z)$$

then $p(z)\prec q(z)$ *and* $q(z)$ *is the best dominant.*

Theorem 2 *Let* $q(z)$ *be convex in the unit disk* U *with* $q(0)=1$ *and* $\Re(q)>1/2$. *If* $F(z)\in E_\alpha^+$ *and* $\left[\dfrac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]$ *is an analytic function in* U *satisfies the subordination*

$$\begin{aligned}&(1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]^2+(2\mu-1)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]-\mu+\\ +\quad&(1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)}-\frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right]\prec\\ \prec\quad&(1-\mu)q^2(z)+(2\mu-1)q(z)-\mu+(1-\mu)zq'(z)\end{aligned}$$

Then,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]\prec q(z)$$

and $q(z)$ *is the best dominant.*

Proof. Let the function $p(z)$ be defined by

$$p(z)=\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right],\ z\in U$$

since $p(0) = 1$, therefore

$$\begin{aligned}
&(1-\mu)p^2(z) + (2\mu-1)p(z) - \mu + (1-\mu)zp'(z) = \\
&= (1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]^2 + (2\mu-1)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] - \mu + \\
&\quad + (1-\mu)z\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]' = \\
&= [1-\mu]\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]^2 + [2\mu-1]\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] - [\mu] + \\
&\quad + (1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \prec \\
&\prec (1-\mu)q^2(z) + (2\mu-1)q(z) - \mu + (1-\mu)zq'(z)
\end{aligned}$$

now by using the Lemma 2, we have

$$p(z) \prec q(z)$$

implies that,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec q(z)$$

and $q(z)$ is the best dominant. □

Corollary 3 *If* $F(z) \in E_\alpha^+$ *and* $\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]$ *is an analytic function in* U *satisfying the subordination*

$$\begin{aligned}
&(1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]^2 + (2\mu-1)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] - \mu + \\
&\quad + (1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \prec \\
&\prec (1-\mu)\left[\frac{1+Az}{1+Bz}\right]^2 + (2\mu-1)\left[\frac{1+Az}{1+Bz}\right] - \mu + \\
&\quad + (1-\mu)\left[\frac{1+Az}{1+Bz}\right]\left[\frac{(A-B)z}{(1+Az)(1+Bz)}\right]
\end{aligned}$$

Then,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec \left[\frac{(1+Az)}{(1+Bz)}\right]$$

and $\left[\dfrac{1+Az}{1+Bz}\right]$ *is the best dominant.*

Proof. Let us define $q(z)$ by

$$q(z) = \left[\frac{1+Az}{1+Bz}\right], \ z \in U$$

this implies that $q(0) = 1$ and $\Re(q) > 1/2$ for arbitrary $A, B, z \in U$ where

$$\frac{zq'(z)}{q(z)} = \frac{(A-B)z}{(1+Az)(1+Bz)}$$

Then applying the Theorem 2, we obtain the result. □

Corollary 4 *If* $F(z) \in E_\alpha^+$ *and* $\left[\dfrac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]$ *is an analytic function in* U *satisfying the subordination*

$$(1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]^2 + (2\mu-1)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] - \mu +$$
$$+ (1-\mu)\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right]\left[\frac{z(I_\alpha(a,c)\phi(z))'}{I_\alpha(a,c)\phi(z)} - \frac{z(I_\alpha(a,c)\psi(z))'}{I_\alpha(a,c)\psi(z)}\right] \prec$$
$$\prec (1-\mu)\left[\frac{1+z}{1-z}\right]^2 + (2\mu-1)\left[\frac{1+z}{1-z}\right] - \mu + (1-\mu)\left[\frac{1+z}{1-z}\right]\left[\frac{2z}{(1+z)(1-z)}\right]$$

Then,

$$\left[\frac{I_\alpha(a,c)\phi(z)}{I_\alpha(a,c)\psi(z)}\right] \prec \frac{(1+z)}{(1-z)}$$

and $\dfrac{1+z}{1-z}$ *is the best dominant.*

Proof. Let the function $q(z)$ be defined by

$$q(z) = \left[\frac{1+z}{1-z}\right], \ z \in U,$$

then in view of Theorem 2 we obtain the result. □

Definition 3 *The fractional integral of order* α *is defined, for a function* $f(z)$ *by*

$$I_z^{\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z f(z)(z-\zeta)^{\alpha-1} d\zeta, \quad 0 \leq \alpha < 1$$

where, the function $f(z)$ *is analytic in simply-connected region of the complex z-plane containing the origin and the multiplicity of* $(z-\zeta)^{\alpha-1}$ *is removed by requiring* $\log(z-\zeta)$ *to be real when* $(z-\zeta) > 0$. *Note that* $I_z^{\alpha} f(z) = f(z) \times z^{\alpha-1}/\Gamma(\alpha)$ *for* $z > 0$ *and 0 (see [8, 9, 10, 11]). Let*

$$f(z) = \sum_0^{\infty} \phi_n z^{n-n/\beta+1-\alpha},$$

this implies that,

$$\begin{aligned} I_z^{\alpha} f(z) &= f(z) \times z^{\alpha-1}/\Gamma(\alpha) = z^{\alpha-1}/\Gamma(\alpha) \sum_0^{\infty} \phi_n z^{n-n/\beta+1-\alpha} \quad \textit{for } z > 0 \\ &= \sum_o^{\infty} a_n z^{n-n/\beta}, \quad \textit{where} \quad a_n = \phi_n/\Gamma(\alpha), \end{aligned}$$

thus,

$$1 \pm I_z^{\alpha} f(z) \in M_{\alpha}^{+}(M_{\alpha}^{-})$$

then we have the following results.

Theorem 3 *Let* $q(z)$ *be convex in the unit disk* U *with* $q(0) = 1$ *and* $R(q(z)) > 1/2$. *If* $F(z) \in \mathcal{E}_{\alpha}^{+}$ *and* $\dfrac{(1 + I_z^{\alpha} f(z)) * \varphi(z)}{(1 + I_z^{\alpha} f(z)) * \vartheta(z)}$ *is an analytic function in* U *satisfies the subordination*

$$\begin{aligned} &(1-u)\left[\frac{(1+I_z^{\alpha}f(z)) * \varphi(z)}{(1+I_z^{\alpha}f(z)) * \vartheta(z)}\right]^2 (z) + (2u-1)\left[\frac{(1+I_z^{\alpha}f(z)) * \varphi(z)}{(1+I_z^{\alpha}f(z)) * \vartheta(z)}\right] - u + \\ &+ (1-u)\left[\frac{(1+I_z^{\alpha}f(z)) * \varphi(z)}{(1+I_z^{\alpha}f(z)) * \vartheta(z)}\right]\left[\frac{z(1+I_z^{\alpha}f(z)) * \varphi(z))'}{(1+I_z^{\alpha}f(z)) * \varphi(z))} - \frac{z(1+I_z^{\alpha}f(z)) * \vartheta(z))'}{(1+I_z^{\alpha}f(z)) * \vartheta(z))}\right] \\ &\prec (1-u)q^2(z) + (2u-1)q(z) - u + (1-u)zq'(z) \end{aligned}$$

then,

$$\left[\frac{(1+I_z^{\alpha}f(z)) * \varphi(z)}{(1+I_z^{\alpha}f(z)) * \vartheta(z)}\right] \prec q(z).$$

Proof. Let the function $p(z)$ be defined by

$$F(z) = \frac{(1 + I_z^\alpha f(z)) * \varphi(z)}{(1 + I_z^\alpha f(z)) * \vartheta(z)}, \quad z \in U$$

then in view of Theorem 2 we obtain the result. □

Theorem 4 *Let the function* $q(z)$ *be univalent in the unit disk* U *such that* $q'(z) \neq 0$ *and* $\dfrac{zq'(z)}{q(z)} \neq 0$ *is starlike in* U, *if* $(1 - I_z^\alpha f(z)) \in \mathcal{E}_\alpha^-$ *satisfies the subordination*

$$b\left[\frac{(1 - I_z^\alpha f(z)) * \varphi(z))'}{(1 - I_z^\alpha f(z)) * \varphi(z))} - \frac{(1 - I_z^\alpha f(z)) * \vartheta(z))'}{(1 - I_z^\alpha f(z)) * \vartheta(z))}\right] \prec \frac{bzq'(z)}{q(z)}$$

then,

$$b\left[\frac{(1 - I_z^\alpha f(z)) * \varphi(z)}{(1 - I_z^\alpha f(z)) * \vartheta(z)}\right] \prec q(z)$$

then $q(z)$ *is the best dominant.*

Proof. Let the function $p(z)$ be defined by

$$\frac{(1 - I_z^\alpha f(z)) * \varphi(z)}{(1 - I_z^\alpha f(z)) * \vartheta(z)}, \quad z \in U$$

then in view of Theorem 2 we obtain the result. □

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Subordination results for certain classes of analytic functions defined by a generalized differential operator

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Abstract. In this paper, we derive some subordination results for certain classes of analytic functions defined by a generalized differential operator using the principle of subordination and a subordination theorem. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

1 Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We denote by $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$, the class of all functions in $\mathcal{A}$ which are, respectively, univalent, starlike, convex and close-to-convex in $\mathcal{U}$. For functions f given by (1) and g given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

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the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $\mathcal{T}(\gamma, \alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the inequality

$$\Re\left(\frac{zf'(z) + \gamma z^2 f''(z)}{(1-\gamma)f(z) + \gamma z f'(z)}\right) > \alpha, \qquad z \in \mathcal{U},$$

for some α $(0 \leq \alpha < 1)$ and γ $(0 \leq \gamma < 1)$, and let $\mathcal{C}(\gamma, \alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the inequality

$$\Re\left(\frac{\gamma z^3 f'''(z) + (2\gamma+1) z^2 f''(z) + z f'(z)}{\gamma z^2 f''(z) + z f'(z)}\right) > \alpha, \qquad z \in \mathcal{U},$$

for some α $(0 \leq \alpha < 1)$ and γ $(0 \leq \gamma < 1)$. We note that

$$f \in \mathcal{C}(\gamma, \alpha) \Longleftrightarrow zf' \in \mathcal{T}(\gamma, \alpha).$$

The classes $\mathcal{T}(\gamma, \alpha)$ and $\mathcal{C}(\gamma, \alpha)$ were introduced and investigated by O. Altıntaş [2], and M. Kamali and S. Akbulut [4], respectively.

Let $\mathcal{M}(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \beta, \ z \in \mathcal{U},$$

for some β $(\beta > 1)$, and let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \beta, \ z \in \mathcal{U},$$

for some β $(\beta > 1)$. The classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ were introduced and investigated by S. Owa and H. M. Srivastava [6] (see also J. Nishiwaki and S. Owa [5], S. Owa and J. Nishiwaki [7], H. M. Srivastava and A. A. Attiya [9]).

Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s$ $(q, s \in \mathbb{N} \cup \{0\}, q \leq s+1)$ be complex numbers such that $\beta_k \neq 0, -1, -2, \ldots$ for $k \in \{1, 2, \ldots, s\}$. The generalized hypergeometric function ${}_qF_s$ is given by

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_q)_n}{(\beta_1)_n (\beta_2)_n \ldots (\beta_s)_n} \frac{z^n}{n!}, \qquad (z \in \mathcal{U}),$$

where $(x)_n$ denotes the Pochhammer symbol defined by

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) \text{ for } n \in \mathbb{N} \text{ and } (x)_0 = 1.$$

Corresponding to a function $\mathcal{G}^p_{q,s}(\alpha_1;\beta_1;z)$ defined by

$$\mathcal{G}_{q,s}(\alpha_1,\beta_1;z) := z\,{}_qF_s(\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s;z),$$

we now define the following generalized differential operator:

$$\begin{aligned}
D^0_{\lambda\mu}(\alpha_1,\beta_1)f(z) &= f(z) * \mathcal{G}_{q,s}(\alpha_1,\beta_1;z),\\
D^1_{\lambda\mu}(\alpha_1,\beta_1)f(z) &= D_{\lambda\mu}(\alpha_1,\beta_1)f(z) = \lambda\mu z^2(f(z) * \mathcal{G}_{q,s}(\alpha_1,\beta_1;z))''+\\
&\quad + (\lambda-\mu)z(f(z) * \mathcal{G}_{q,s}(\alpha_1,\beta_1;z))'+\\
&\quad + (1-\lambda+\mu)(f(z) * \mathcal{G}_{q,s}(\alpha_1,\beta_1;z)), \text{ and}\\
D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z) &= D_{\lambda\mu}(D^{m-1}_{\lambda}(\alpha_1,\beta_1)f(z)),
\end{aligned}$$

where $0 \le \mu \le \lambda \le 1$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If $f(z) \in \mathcal{A}$, then we have

$$D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z) = z + \sum_{n=2}^{\infty} \vartheta_n^m\, \sigma_n\, a_n z^n, \tag{2}$$

where

$$\vartheta_n = 1 + (\lambda\mu n + \lambda - \mu)(n-1) \tag{3}$$

and

$$\sigma_n = \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\ldots(\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\ldots(\beta_s)_{n-1}\,(n-1)!}. \tag{4}$$

It can be seen that, by specializing the parameters the operator $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$ reduces to many known and new differential operators. In particular, when $m = 0$ the operator $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$ reduces to the well- known Dziok-Srivastava operator [3] and for $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, and $\alpha_2 = 1$, it reduces to the operator introduced by F. M. Al-Oboudi [1]. Further we remark that, when $\lambda = 1$, $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, and $\alpha_2 = 1$ the operator $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$ reduces to the operator introduced by G. S. Sălăgean [8].

For simplicity, in the sequel, we will write $D^m_{\lambda\mu}f(z)$ instead of $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$.

Motivated by the above mentioned function classes, we now introduce the following subclasses of $\mathcal{A}$ involving the generalized differential operator $D^m_{\lambda\mu}f(z)$.

Definition 1 *A function* $f \in \mathcal{A}$ *is said to be in the class* $\mathcal{S}^m_{\lambda\mu}(\gamma, \alpha)$ *if it satisfies the following inequality*

$$\Re\left\{\frac{(1-\gamma)D^{m+1}_{\lambda\mu}f(z) + \gamma D^{m+2}_{\lambda\mu}f(z)}{(1-\gamma)D^{m}_{\lambda\mu}f(z) + \gamma D^{m+1}_{\lambda\mu}f(z)}\right\} > \alpha, \qquad z \in \mathcal{U},$$

where

$$m \in \mathbb{N}_0, \quad 0 \le \gamma \le 1, \quad 0 \le \alpha < 1.$$

It is easy to see that the classes $\mathcal{T}(\gamma, \alpha)$ and $\mathcal{C}(\gamma, \alpha)$ are special cases of the class $\mathcal{S}^m_{\lambda\mu}(\gamma, \alpha)$.

Definition 2 *A function* $f \in \mathcal{A}$ *is said to be in the class* $\mathcal{M}^m_{\lambda\mu}(\gamma, \beta)$ *if it satisfies the following inequality*

$$\Re\left\{\frac{(1-\gamma)D^{m+1}_{\lambda\mu}f(z) + \gamma D^{m+2}_{\lambda\mu}f(z)}{(1-\gamma)D^{m}_{\lambda\mu}f(z) + \gamma D^{m+1}_{\lambda\mu}f(z)}\right\} < \beta, \qquad z \in \mathcal{U},$$

where

$$m \in \mathbb{N}_0, \quad 0 \le \gamma \le 1, \quad \beta > 1.$$

It is also easy to see that the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ are special cases of the class $\mathcal{M}^m_{\lambda\mu}(\gamma, \beta)$.

We now provide some coefficient inequalities associated with the function classes $\mathcal{S}^m_{\lambda\mu}(\gamma, \alpha)$ and $\mathcal{M}^m_{\lambda\mu}(\gamma, \beta)$.

2 Coefficient inequalities

Theorem 1 *Let* $0 \le \alpha < 1$ *and* $0 \le \gamma \le 1$. *If* $f \in \mathcal{A}$ *satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty} (1 - \gamma + \gamma\vartheta_n)(\vartheta_n - \alpha)\, \vartheta_n^m\, \sigma_n\, |a_n| \le 1 - \alpha, \tag{5}$$

where ϑ_n *and* σ_n *are given by (3) and (4) respectively, then* $f \in \mathcal{S}^m_{\lambda\mu}(\gamma, \alpha)$.

Proof. It is suffices to show that

$$\left|\frac{(1-\gamma)D^{m+1}_{\lambda\mu}f(z) + \gamma D^{m+2}_{\lambda\mu}f(z)}{(1-\gamma)D^{m}_{\lambda\mu}f(z) + \gamma D^{m+1}_{\lambda\mu}f(z)} - 1\right| < 1 - \alpha, \qquad z \in \mathcal{U}.$$

Now we note that for any $z \in \mathcal{U}$,

$$\left|\frac{(1-\gamma)D_{\lambda\mu}^{m+1}f(z)+\gamma D_{\lambda\mu}^{m+2}f(z)}{(1-\gamma)D_{\lambda\mu}^{m}f(z)+\gamma D_{\lambda\mu}^{m+1}f(z)}-1\right| = \left|\frac{\sum\limits_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)(\vartheta_n-1)\vartheta_n^m\sigma_n a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)\vartheta_n^m\sigma_n a_n\, z^{n-1}}\right|$$
$$\leq \frac{\sum\limits_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)(\vartheta_n-1)\vartheta_n^m\sigma_n|a_n|}{1-\sum\limits_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)\vartheta_n^m\sigma_n|a_n|}.$$

It follows from (5) that the last expression is bounded by $1-\alpha$. This completes the proof of the theorem. □

Theorem 2 *Let $\beta > 1$ and $0 \leq \gamma \leq 1$. If $f \in \mathcal{A}$ satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)(\vartheta_n+|\vartheta_n-2\beta|)\,\vartheta_n^m\,\sigma_n\,|a_n| \leq 2(\beta-1), \tag{6}$$

where ϑ_n and σ_n are given by (3) and (4) respectively, then $f \in \mathcal{M}_{\lambda\mu}^m(\gamma,\beta)$.

Proof. It is sufficient to show that

$$\left|\frac{(1-\gamma)D_{\lambda\mu}^{m+1}f(z)+\gamma D_{\lambda\mu}^{m+2}f(z)}{(1-\gamma)D_{\lambda\mu}^{m}f(z)+\gamma D_{\lambda\mu}^{m+1}f(z)}\right| < \left|\frac{(1-\gamma)D_{\lambda\mu}^{m+1}f(z)+\gamma D_{\lambda\mu}^{m+2}f(z)}{(1-\gamma)D_{\lambda\mu}^{m}f(z)+\gamma D_{\lambda\mu}^{m+1}f(z)}-2\beta\right|, \tag{7}$$

where $z \in \mathcal{U}$.

Now, we define $M \in \mathbb{R}$ by

$$\begin{aligned}
M :=& \left|(1-\gamma)D_{\lambda\mu}^{m+1}f(z)+\gamma D_{\lambda\mu}^{m+2}f(z)\right| - \\
&- \left|(1-\gamma)D_{\lambda\mu}^{m+1}f(z)+\gamma D_{\lambda\mu}^{m+2}f(z)-2\beta\big((1-\gamma)D_{\lambda\mu}^{m}f(z)+\gamma D_{\lambda\mu}^{m+1}f(z)\big)\right| = \\
=& \left|z+\sum_{n=2}^{\infty}\left[(1-\gamma)\vartheta_n^{m+1}+\gamma\vartheta_n^{m+2}\right]\sigma_n a_n z^n\right| - \\
&- \left|z+\sum_{n=2}^{\infty}\left[(1-\gamma)\vartheta_n^{m+1}+\gamma\vartheta_n^{m+2}\right]\sigma_n a_n z^n - \right. \\
&\left. - 2\beta\left\{z+\sum_{n=2}^{\infty}\left[(1-\gamma)\vartheta_n^{m}+\gamma\vartheta_n^{m+1}\right]\sigma_n a_n z^n\right\}\right|.
\end{aligned}$$

Thus, for $|z| = r < 1$, we have

$$M \leq r + \sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)\vartheta_n^{m+1}\sigma_n|a_n|r^n -$$
$$-\left[(2\beta-1)r - \sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)|\vartheta_n - 2\beta|\vartheta_n^m\sigma_n|a_n|r^n\right] <$$
$$< \left(\sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)(\vartheta_n + |\vartheta_n - 2\beta|)\vartheta_n^m\sigma_n|a_n| - 2(\beta-1)\right)r.$$

It follows from (6) that $M < 0$, which implies that (7) holds. This completes the proof of the theorem. □

In view of Theorem (1) and Theorem (2), we now introduce the subclasses

$$\widetilde{\mathcal{S}}_{\lambda\mu}^m(\gamma,\alpha) \subset \mathcal{S}_{\lambda\mu}^m(\gamma,\alpha) \qquad \text{and} \qquad \widetilde{\mathcal{M}}_{\lambda\mu}^m(\gamma,\beta) \subset \mathcal{M}_{\lambda\mu}^m(\gamma,\beta),$$

which consist of functions $f \in \mathcal{A}$ whose Taylor-Maclaurin coefficients satisfy the inequalities (5) and (6) respectively. We now derive some subordination results for the function classes $\widetilde{\mathcal{S}}_{\lambda\mu}^m(\gamma,\alpha)$ and $\widetilde{\mathcal{M}}_{\lambda\mu}^m(\gamma,\beta)$.

3 Subordination result for the class $\widetilde{\mathcal{S}}_{\lambda\mu}^m(\gamma,\beta)$

We will use of the following definitions and lemma to prove our result.

Definition 3 *(Subordination Principle) Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, and write*

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z)$$

if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$ with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathcal{U}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathcal{U}).$$

In particular, if the function $g(z)$ is univalent in $\mathcal{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathcal{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

Definition 4 *(Subordinating Factor Sequence) A sequence* $\{b_n\}_{n=1}^{\infty}$ *of complex numbers is said to be a subordinating factor sequence if, whenever* $f(z)$ *of the form (1) is analytic, univalent and convex in* $\mathcal{U}$*, we have the subordination given by*

$$\sum_{n=1}^{\infty} a_n\, b_n\, z^n \prec f(z) \qquad (z \in \mathcal{U};\ a_1 := 1).$$

Lemma 1 *(See Wilf [11]) The sequence* $\{b_n\}_{n=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$\Re\left(1 + 2\sum_{n=1}^{\infty} b_n\, z^n\right) > 0 \qquad (z \in \mathcal{U}).$$

Theorem 3 *Let the function* $f(z)$ *defined by (1) be in the class* $\widetilde{\mathcal{S}}_{\lambda\mu}^{m}(\gamma, \alpha)$*. If* $g(z) \in \mathcal{K}$*, then*

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\,\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\,\sigma_2]}\,(f*g)(z) \prec g(z) \tag{8}$$

$$(z \in \mathcal{U},\ m \in \mathbb{N}_0,\ 0 \le \gamma \le 1,\ 0 \le \alpha < 1)$$

and

$$\Re(f) > -\frac{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\,\sigma_2}{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\,\sigma_2}, \tag{9}$$

where ϑ_n *and* σ_n *are given by (3) and (4) respectively. The constant factor in the subordination result (8)*

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\,\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\,\sigma_2]}$$

cannot be replaced by a larger one.

Proof. Let $f(z) \in \widetilde{\mathcal{S}}_{\lambda\mu}^{m}(\gamma, \alpha)$ and suppose that

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}.$$

Then we readily have

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\,\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\,\sigma_2]}\,(f*g)(z) =$$

$$= \frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\,\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\,\sigma_2]}\left(z + \sum_{n=2}^{\infty} a_n\, c_n z^n\right).$$

Thus, by Definition 4, the subordination result (8) will holds if

$$\left\{\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2]}\,a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1 this is equivalent to the following inequality:

$$\Re\left\{1+\sum_{n=1}^{\infty}\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,a_n z^n\right\} > 0 \qquad (z\in\mathcal{U}). \tag{10}$$

Since $(1-\gamma+\gamma\vartheta_n)(\vartheta_n-\alpha)\,\vartheta_n^m\sigma_n \quad (n\geq 2,\ m\in\mathbb{N}_0)$ is an increasing function of n, we have

$$\begin{aligned}
&\Re\left\{1+\sum_{n=1}^{\infty}\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,a_n z^n\right\}\\
&\quad=\Re\left\{1+\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,z\right.\\
&\qquad+\frac{1}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\cdot\\
&\qquad\left.\cdot\sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2\,a_n z^n\right\}\\
&\quad\geq 1-\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,r\\
&\qquad-\frac{1}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\cdot\\
&\qquad\cdot\sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)(\vartheta_n-\alpha)\vartheta_n^m\sigma_n\,|a_n|\,r^n\\
&\quad>1-\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,r\\
&\qquad-\frac{1-\alpha}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,r\\
&\quad=1-r>0 \qquad (|z|=r<1),
\end{aligned}$$

where we have also made use of the assertion (5) of Theorem 1. This evidently proves the inequality (10), and hence also the subordination result (8) asserted

by Theorem 3. The inequality (9) follows from (8) upon setting

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in \mathcal{K}.$$

Next we consider the function

$$q(z) := z - \frac{1-\alpha}{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2} z^2, \tag{11}$$

$$(m \in \mathbb{N}_0,\ 0 \le \gamma \le 1,\ 0 \le \alpha < 1),$$

where ϑ_n and σ_n are given by (3) and (4) respectively, which is a member of the class $\widetilde{\mathcal{S}}_{\lambda\mu}^m(\gamma,\alpha)$. Then, by using (8), we have

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2]}\, q(z) \prec \frac{z}{1-z} \qquad (z \in \mathcal{U}).$$

One can easily verify for the function $q(z)$ defined by (11) that

$$\min\left\{\Re\left(\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2]}\, q(z)\right)\right\} = -\frac{1}{2} \qquad (z \in \mathcal{U}),$$

which completes the proof of Theorem 3. □

Remark 1 *Setting* $\gamma = 0$, $\lambda = 1$, $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, *and* $\alpha_2 = 1$ *in Theorem 3, we get the corresponding result obtained by S. Sümer Eker et al. [10].*

4 Subordination result for the class $\widetilde{\mathcal{M}}_{\lambda\mu}^m(\gamma,\beta)$

The proof of the following subordination result is similar to that of Theorem 3. We, therefore, omit the analogous details involved.

Theorem 4 *Let the function* $f(z)$ *defined by (1) be in the class* $\widetilde{\mathcal{M}}_{\lambda\mu}^m(\gamma,\beta)$. *If* $g(z) \in \mathcal{K}$, *then*

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m\sigma_2}{2[2(\beta-1)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m\sigma_2]}\,(f*g)(z) \prec g(z) \tag{12}$$

$$(z \in \mathcal{U},\ m \in \mathbb{N}_0,\ 0 \le \gamma \le 1,\ 0 \le \alpha < 1)$$

and

$$\Re(f) > -\frac{2(\beta-1)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m\sigma_2}{(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m\sigma_2},$$

where ϑ_n *and* σ_n *are given by (3) and (4) respectively. The constant factor*

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m\sigma_2}{2[2(\beta-1)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m\sigma_2]}$$

in the subordination result (12) cannot be replaced by a larger one.

Remark 2 *Setting* $m = 0$, *or* $\gamma = 0$, $\lambda = 1$, $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ *and* $\alpha_2 = 1$ *in Theorem 3, we get the corresponding results obtained by H. M. Srivastava and A. A. Attiya. [9].*

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Longest runs in coin tossing. Comparison of recursive formulae, asymptotic theorems, computer simulations

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Abstract. The coin tossing experiment is studied. The length of the longest head run can be studied by asymptotic theorems [3, 4], by recursive formulae [7, 11] or by computer simulations [1]. The aim of the paper is to compare numerically the asymptotic results, the recursive formulae, and the simulation results. Moreover, we consider also the longest run (i.e. the longest pure heads or pure tails). We compare the distribution of the longest head run and that of the longest run.

1 Introduction

The success-run in a sequence of Bernoulli trials has been studied in a large number of papers. Consider the well-known coin tossing experiment. Let R_n denote the length of the longest run of consecutive heads (longest head run). Moreover, let $\mathsf{R}_n^{'}$ denote the longest run of consecutive heads or consecutive tails (longest run). The asymptotic distribution of R_n is studied in several

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papers (see, e.g. [3, 4, 5, 6, 9, 10]). However, these results give approximations being accurate for large enough n. Precise values of the distributions can be calculated by certain recursive formulae (see, e.g. [2, 7, 11]). However it is difficult and slow to calculate them numerically for large n. The distributions of R_n and R_n' can be calculated by simulations, as well. Simulations can be applied both for small and large values of n, but they offer only approximations (which can be improved by using large number of repetitions). The comparison of the asymptotic theorems and the simulations are given in [1].

In this paper we compare numerically the asymptotic theorems, the recursive formulae and the simulations. As the case of a fair coin is well-known, we focus on a biased coin (i.e. when $P(\text{head}) = p \neq \frac{1}{2}$). Moreover, as our aim is to obtain precise numerical results, we emphasize the importance of the recursive formulae. We give detailed proofs for the (known) recursive formulae. Finally, we remark that most results in the literature concern the longest head run (i.e. R_n) but in practice people are interested in the longest run (i.e. R_n'). Therefore, we concentrate mainly on R_n'.

The numerical results show that the asymptotic theorems give bad results for small n (i.e. $n \leq 250$) and give practically precise results for large n (i.e. $n \geq 3000$). It can also be seen that for large n the distribution of R_n' is close to that of R_n if $p > \frac{1}{2}$ (p is the probability of a head).

We present recursion formulae offering the exact distribution of the longest run of heads (Section 2), and the distribution of the longest whatever run (Section 3). We consider the situation in which the probability of a head can take any value in $(0, 1)$.

2 The longest head run

Consider n independent tosses of a (biased) coin, and let R_n denote the length of the longest head run. The (cumulative) distribution function of R_n is the following

$$F_n(x) = P(R_n \leq x) = \sum_{k=0}^{n} C_n^{(k)}(x) p^k q^{n-k}, \tag{1}$$

where $C_n^{(k)}(x)$ is the number of strings of length n where exactly k heads occur, but not more than x heads occur consecutively. We have the following recursive formula for $C_n^{(k)}(x)$.

Proposition 1 *(See [11])*

$$C_n^{(k)}(x) = \begin{cases} \sum_{j=0}^{x} C_{n-1-j}^{(k-j)}(x), & \textit{if} \quad x < k < n, \\ \binom{n}{k}, & \textit{if} \quad 0 \le k \le x, \\ 0, & \textit{if} \quad x < k = n. \end{cases} \tag{2}$$

Proof. If $x < k = n$, then $C_n^k(x) = 0$, because in this case all elements (being more than x) are heads, so there is no series containing less than or equal to x heads consecutively.

If $0 \le k \le x$, then the value of $C_n^k(x)$ is equal to the binomial coefficient. In this case there are less than or equal to x heads among n elements and we have to count those cases when the length of the longest head run is less than or equal to x. All possible sequences have this property, therefore $C_n^{(k)}(x) = \binom{n}{k}$.

If $x < k < n$, then we need to consider the following. Our series may start with $j = 0, 1, 2, \ldots, x$ heads, then must be one tail, then a sequence follows containing $k-j$ heads among the remaining $n-j-1$ objects. In this sequence the length of the longest head run must be less than or equal to x. The number of these sequences equals exactly $C_{n-1-j}^{(k-j)}(x)$.

$$\underbrace{\mathrm{H}\ \ldots\ \mathrm{H}}_{j \text{ heads}} \quad \mathrm{T} \quad \underbrace{\ldots \quad \mathrm{H} \quad \ldots \quad \mathrm{T} \quad \ldots}_{\substack{n-j-1 \text{ elements, containing } k-j \text{ heads,} \\ \text{and the length of the longest head run is less than or equal to } x}} \qquad \square$$

The following table displays the values of $C_n^k(3)$ for $n \le 8$.

8									0
7								0	0
6							0	1	10
5						0	**2**	**12**	40
4					0	**3**	12	31	65
3				1	**4**	10	20	35	56
2			1	**3**	6	10	15	21	28
1		1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	1
k/n	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*

The first four rows of the table ($k = 0, 1, 2, 3$) are part of Pascal's triangle. Entries above that four rows are computed by taking diagonal sums of four

entries from the rows and columns below and to the left. The 'hockey stick' (printed in boldface in the table) illustrates the case $C_7^{(5)}(3) = 2+3+4+3 = 12$. Tossing a biased coin 8 times, we now have the probability of obtaining not more than three consecutive heads: $F_8(3) = C_8^{(0)}(3)p^0q^8 + C_8^{(1)}(3)p^1q^7 + \ldots + C_8^{(7)}(3)p^7q^1 + C_8^{(8)}(3)p^8q^0 = 1q^8 + 8pq^7 + 28p^2q^6 + 56p^3q^5 + 65p^4q^4 + 40p^5q^3 + 10p^6q^2 + 0 + 0$. Knowing the value of p we can calculate the exact result.

The asymptotic behaviour is described by the following theorem.

Theorem 1 (See [5].) *Let* $\mu(n) = -\frac{\log n}{\log p}$, $q = 1 - p$ *and let* W *have a double exponential distribution (i.e.* $P(W \le t) = \exp(-\exp(-t))$*), then uniformly in* t:

$$P\left(R_n - \mu(qn) \le t\right) - P\left(\left[\frac{W}{-\log p} + \{\mu(qn)\}\right] - \{\mu(qn)\} \le t\right) \to 0 \qquad (3)$$

as $n \to \infty$ *where* $[a]$ *denotes the integer part of* a *and* $\{a\} = a - [a]$.

We emphasize that the above theorem does not offer a limiting law for $R_n - \mu(qn)$ but it gives a sequence of accompanying laws. The distances of the laws in the two sequences converge to 0 (as $n \to \infty$). So the above theorem is a merge theorem. Observe, the periodic property in the sequence of the accompanying laws.

3 The longest run

For a coin with $p \neq 0.5$ the (cumulative) distribution function $F'_n(x)$ is complicated. Let R'_n denote the length of the longest run in the sequence of n coin tossings. That is the maximum of the longest head run and the longest tail run. Let F'_n be the distribution function of R'_n.

$$F'_n(x) = P(R'_n \le x) = \sum_{k=0}^{n} \overline{C}_n^{(k)}(x) p^k q^{n-k} \qquad (4)$$

where $\overline{C}_n^{(k)}(x)$ is the number of strings of length n with exactly k heads, but not more than x of heads and not more than x of tails occur consecutively (p is the probability of a head and $q = 1 - p$). First consider

$$\overline{C}_{m+k}^{(k)}(x) = C_{x+1}(m, k). \qquad (5)$$

Here $C_t(m,k)$ denotes the number of strings of m indistinguishable objects of type A and k indistinguishable objects of type B in which no t-clump (run of length t) occurs. (A and B may interpret head and tail, respectively.) We have the following recursive formulas for $C_t(m,k)$.

Proposition 2 (See [2].)

$$C_t(m,k) = \sum_{i=0}^{t-1} C_t(m-1,k-i) - \sum_{i=1}^{t-1} C_t(m-t,k-i) + e_t(m,k), \qquad (6)$$

where

$$e_t(m,k) = \begin{cases} 1, & \text{if} \quad m=0 \quad \text{and} \quad 0 \le k < t, \\ -1, & \text{if} \quad m=t \quad \text{and} \quad 0 \le k < t, \\ 0, & \text{in all other cases} \end{cases},$$

moreover if $m = k = 0$*, then* $C_t(0,0) = 1$*, if* m *or* k *is negative, then* $C_t(m,k) = 0$.

We give a detailed proof which is not contained in [2].

Proof.
Case $m = 0$.

If $0 \le k < t$, then $C_t(0,k) = 1$, because this means that there is only one type of the elements but the number of objects is less than the length of the run. So there can not be any t run. As the elements are indistinguishable, this means only one order. In this case (6) means $1 = 0 - 0 + 1$. For example: $C_3(0,2) = C_3(-1,2) + C_3(-1,1) + C_3(-1,0) - [C_3(-3,1) + C_3(-3,0)] + 1 = 0 + 0 + 0 - [0+0] + 1 = 1$.

If $k \ge t$, then $C_t(0,k) = 0$, because there is only one type of the elements, but the number of objects is greater or equal to the length of the run. So there is no one sequence in which there is no t run. For example: $C_3(0,4) = C_3(-1,4) + C_3(-1,3) + C_3(-1,2) - [C_3(-3,3) + C_3(-3,2)] + 0 = 0 + 0 + 0 - [0+0] + 0 = 0$.

In case of $0 < m < t$ our formula (6) is the following.

$$C_t(m,k) = \sum_{i=0}^{t-1} C_t(m-1,k-i) - 0 + 0.$$

Because this case means the following

$$\underbrace{\text{B} \ \ldots \ \text{B}}_{i \text{ of B}} \quad \text{A} \quad \underbrace{\ldots \quad \text{B} \quad \ldots \quad \text{A} \quad \ldots}_{\substack{(k-i) \text{ of B, } (m-1) \text{ of A,} \\ \text{and there is no } t\text{-run among them}}}$$

The number of these sequences is: $C_t(m, -1, k-i)$. As $m < t$, so there can not be t run from A, so we do not need to subtract anything. For example: $C_3(2,2) = C_3(1,2) + C_3(1,1) + C_3(1,0) - [C_3(-1,1) + C_3(-1,0)] + 0 = 3 + 2 + 1 - [0+0] + 0 = 6$.

The case of $m = t$ and $0 \le k < t$.

This means that there are less than t of B elements, and the number of A elements is equal to t. In this case our formula is the following: $C_t(m,k) = \sum_{i=0}^{t-1} C_t(m-1, k-i) - \sum_{i=1}^{t-1} C_t(0, k-i) - 1$. The first sum consists of $k+1$ positive terms (not t), when the i-th term starts with i of B objects, then follows A, then follows a sequence consisting of $m-1$ A and $k-i$ B and not containing t run.

$$\underbrace{\text{B} \ \ldots \ \text{B}}_{i \text{ of B}} \quad \text{A} \quad \underbrace{\ldots \quad \text{B} \quad \ldots \quad \text{A} \quad \ldots}_{\substack{(k-i) \text{ of B, } (m-1) \text{ of A,} \\ \text{and there is no } t\text{-run}}}$$

But there is a 'bad' term in each of them, when the $m = t$ A objects are consecutive. As the second sum consists of k terms, so the above $k+1$ bad cases are subtracted. For example: $C_3(3,2) = C_3(2,2) + C_3(2,1) + C_3(2,0) - [C_3(0,1) + C_3(0,0)] - 1 = 6 + 3 + 1 - [1+1] - 1 = 7$.

In the case of $m = t$ and $k \ge t$, our formula is the following

$$C_t(m,k) = \sum_{i=0}^{t-1} C_t(m-1, k-i) - \sum_{i=1}^{t-1} C_t(0, k-i) + 0.$$

If $i = 0$ in the first sum, then our possibility is the following

$$\text{A} \quad \underbrace{\ldots \ \text{A} \ \ldots \ \text{B} \ \ldots}_{\substack{k \text{ of B, } (m-1) \text{ of A} \\ \text{and there is no } t\text{-run}}}$$

The number of these sequences is $C_t(m-1, k)$. Seemingly there is one 'bad' event among them, when in the second part starts with $m-1$ A objects and they make a t run with the very first A object. But the k B objects are in the end of the second part and they would make a t run, so the above 'bad' situation is not included in $C_t(m-1, k)$.

If $i = 1, 2, \ldots, t-1$, then we have

$$\underbrace{\text{B} \ \ldots \ \text{B}}_{i \text{ of B}} \quad \text{A} \quad \underbrace{\ldots \quad \text{B} \quad \ldots \quad \text{A} \quad \ldots}_{\substack{(k-i) \text{ of B, } (m-1) \text{ of A,} \\ \text{and there is no } t\text{-run}}}$$

The number of these sequences is $C_t(m-1, k-i)$. But there can be a 'bad' event in this situation, when all objects A ($m = t$) are next to each other, so we have to subtract $C_t(0, k-i)$ (it can be equal to 0 as well). For example: $C_3(3,4) = C_3(2,4) + C_3(2,3) + C_3(2,2) - [C_3(0,3) + C_3(0,2)] + 0 = 6 + 7 + 6 - [0 + 1] + 0 = 18$.

Case $m > 0$ and $m > t$.

Our sequence may start with i (i is less than t) same type objects (for example with B) then follows a different one (A) and ends with a string without t run.

$$\underbrace{\text{B} \ \ldots \ \text{B}}_{i \text{ of B}} \quad \text{A} \quad \underbrace{\ldots \quad \text{B} \quad \ldots \quad \text{A} \quad \ldots}_{\substack{(k-i) \text{ of B, } (m-1) \text{ of A,} \\ \text{and there is no } t\text{-run}}}$$

The number of these sequences is: $\sum_{i=1}^{t-1} C_t(m-1, k-i)$.

But among them there may be sequences when there are same A objects after the individual A, so that together there are t consecutive A objects and after them there is no t run

$$\underbrace{\text{B} \ \ldots \ \text{B}}_{i \text{ of B}} \ \underbrace{\text{A} \ \ldots \ \text{A}}_{t \text{ of A}} \underbrace{\ldots \quad \text{B} \quad \ldots \quad \text{A} \quad \ldots}_{\substack{(k-i) \text{ of B, } (m-t) \text{ of A and} \\ \text{there is no } t\text{-run}}}$$

The number of these strings is $\sum_{i=1}^{t-1} C_t(m-t, k-i)$, that we have to subtract from the previous sum. But in these there can be such sequences, when A object stands after the t run of A, so there can be another t run. The number of these can be denoted by $\sum_{i=1}^{t-1} C_t^*(m-t, k-i)$.

What happens is if $i = 0$, so our sequence starts with A? In this case the first object is A and then there is no t run

$$\text{A} \quad \underbrace{\ldots \ \text{A} \ \ldots \ \text{B} \ \ldots}_{\substack{k \text{ of B, } (m-1) \text{ of A} \\ \text{and there is no } t\text{-run}}}$$

The number of these strings is $C_t(m-1, k)$. But in these strings there can be some sequences starting with t run and then there is no t run

$$\underbrace{\text{A} \ \ldots \ \text{A}}_{t \text{ of A,}} \underbrace{\text{B} \ \ldots \ \text{B}}_{i \text{ of B}} \underbrace{\text{A} \ldots \ \text{B} \ldots \ \text{A} \ \ldots}_{\substack{(m-t) \text{ of A, } (k-i) \text{ of B} \\ \text{and there is no } t\text{-run} \\ (1 \le i \le (t-1))}}$$

The numbers of these strings is $\sum_{i=1}^{t-1} C_t^*(m-t, k-i)$, that we have to subtract from the previous sum.

Summarizing our results we get the following

$$C_t(m,k) = \sum_{i=1}^{t-1} C_t(m-1, k-i) - \left\{ \sum_{i=1}^{t-1} C_t(m-t, k-i) - \sum_{i=1}^{t-1} C_t^*(m-t, k-i) \right\} +$$

$$+ \left\{ C_t(m-1, k) - \sum_{i=1}^{t-1} C_t^*(m-t, k-i) \right\} + e_t(m,k).$$

For example: $C_3(5,2) = C_3(4,2) + C_3(4,1) + C_3(4,0) - [C_3(2,1) + C_3(2,0)] + 0 = 6 + 1 + 0 - [3+1] + 0 = 3$.

So recursive formula (6) is satisfied. □

Proposition 3 (See [2].) *Let* $t \geq 2$. *Then*

$$C_t(m,k) = C_t(m-1,k) + C_t(m,k-1) - C_t(m-t,k-1) - C_t(m-1,k-t) \quad (7)$$

$$+ C_t(m-t,k-t) + e_t^*(m,k),$$

where

$$e_t^*(m,k) = \begin{cases} 1, & \text{if} \quad (m,k) = (0,0) \quad \text{or} \quad (m,k) = (t,t), \\ -1, & \text{if} \quad (m,k) = (0,t) \quad \text{or} \quad (m,k) = (t,0), \\ 0, & \text{in all other cases,} \end{cases}$$

moreover if $m = k = 0$, *then* $C_t(0,0) = 1$, *if* m *or* k *is negative, then* $C_t(m,k) = 0$.

Here we give a proof being different from the one in [2].

Proof. Our sequence may start either with A or B

$$\left. \text{A} \quad \underbrace{\ldots \text{ A} \ldots \text{B} \ldots}_{(m-1) \text{ of A and } k \text{ of B}} \right\} \text{The number of these sequences is } C_t(m-1,k).$$

$$\left. \text{B} \quad \underbrace{\ldots \text{ A} \ldots \text{B} \ldots}_{m \text{ of A and } (k-1) \text{ of B}} \right\} \text{The number of these sequences is } C_t(m,k-1).$$

We have to subtract the number of those sequences in which after the first A element there are $t-1$ consecutive A's (so there is a t-clump) and then there is a different element and there is a string with no t-clump

$$\left.\underbrace{\mathrm{A}\ldots\mathrm{A}}_{t\text{ of A}}\quad \mathrm{B}\quad \underbrace{\ldots\ \mathrm{A}\ldots\mathrm{B}\ldots}_{(m-t)\text{ of A and }(k-1)\text{ of B}}\right\}\ \text{The number of these sequences is } C_t(m-t,k-1).$$

$$\left.\underbrace{\mathrm{B}\ldots\mathrm{B}}_{t\text{ of B}}\quad \mathrm{A}\quad \underbrace{\ldots\ \mathrm{A}\ldots\mathrm{B}\ldots}_{(m-1)\text{ of A and }(k-t)\text{ of B}}\right\}\ \text{The number of these sequences is } C_t(m-1,k-t).$$

But these cases contain the following sequences as well.

The sequence starts with t consecutive A's followed with t consecutive B's and ends with a string containing $m-t$ A and $k-t$ B elements and not containing t clump but starting with A. The number of these sequences is $C_t^{(A)}(m-t,k-t)$. Changing the role of A and B we get again $C_t^{(B)}(m-t,k-t)$ sequences. But for the sum of them we have $C_t^{(A)}(m-t,k-t)+C_t^{(B)}(m-t,k-t)=C_t(m-t,k-t)$.

Summarizing the above statements we can get our formula

$$C_t(m,k)=C_t(m-1,k)+C_t(m,k-1)-$$

$$-\{C_t(m-t,k-1)+C_t(m-1,k-t)-C_t(m-t,k-t)\}+e_t^*(m,k).$$

□

To see how these work, let us calculate some data in case where $t=3$ and m and k are less than 10:

$m\diagdown k$	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
0	1	1	1	0	0	0	0	0	0	0
1	1	2	3	2	1	0	0	0	0	0
2	1	3	6	7	6	3	1	0	0	0
3	0	2	7	**14**	**18**	16	10	4	1	0
4	0	1	6	18	34	45	43	30	15	5
5	0	0	3	**16**	**45**	**84**	113	114	87	50
6	0	0	1	10	43	**113**	208	285	300	246
7	0	0	0	4	30	114	285	518	720	786
8	0	0	0	1	15	87	300	720	1296	1823
9	0	0	0	0	5	50	246	786	1823	3254

For example $C_3(6,5) = (84+45+16)-(18+14) = 113$. (See the numbers in bold style in the above table.)

The number of terms on the right hand side of (6), increases with t, but in formula (7), the right hand side has only six terms no matter how large t is.

Let p denote the probability of a head. To find the asymptotic behaviour of R'_n, denote by $V_n(p)$ the probability that the longest run in n trials is formed by heads. Then, by Theorem 5 of [8],

$$\lim_{n\to\infty} V_n(p) = \begin{cases} 0, & \text{if} \quad 0 \le p < 1/2, \\ 1, & \text{if} \quad 1/2 < p \le 1. \end{cases} \tag{8}$$

Therefore, if $p > 1/2$, the asymptotic behaviour of R'_n is the same as that of R_n.

It means that "the one with lower chances" will not intervene in the formation of the longest run. When n is sufficiently large, the values that $F'_n(x)$ are well approximated by the values of $F_n(x)$ calculated for the case of $P(\text{head}) = \max\{p, 1-p\}$. The longest run will almost certainly be composed of whichever is more likely between heads and tails.

4 Numerical results, simulations

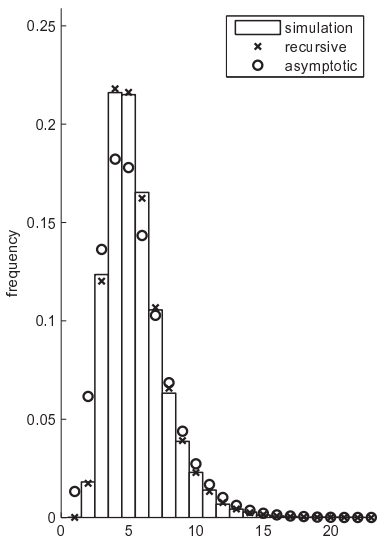
For numerical calculation we used MATLAB software. The data of the computer are INTEL Core2 Quad Q9550 processor, 4Gb, memory DDR3. The following table shows some running times

	n	repetition	running time
	3,100	20.000	172.6209 sec
$p = 0.6$	1,000	20.000	15.4258 sec
	250	20.000	2.8452 sec
	30	20.000	2.0678 sec

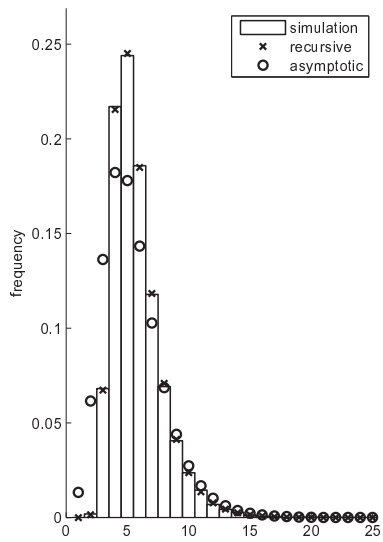
We calculated the distributions of R_n and R'_n. We considered the precise values obtained by recursion, the asymptotic values offered by asymptotic theorems, and used simulation with 20.000 repetitions. On the figures below $\times$ denotes the result of the recursion, o belongs to the asymptotic result, while the histogram shows the relative frequencies calculated by simulation. If n is small, the recursive algorithm is fast, but it slows down if n increases. For biased coin we used $p = 0.6$. We show the results for short trials ($n = 30$), medium trials ($n = 250$), and long trials ($n = 1000$ and $n = 3100$). We can see from the results that the asymptotic theorem does not give good (close to

the recursive) results for small n. But we should say that if $n > 3000$, then the results of the recursion and the results of the asymptotic theorem are almost the same. As the algorithm is slowing down, we offer to use the asymptotic theorem instead of the recursion in case of large n. The asymptotic value is a good approximation if $n \geq 1000$. The figures below show that the distribution of R'_n is far from that of R_n for small n ($n = 30$). However, they are practically the same if n is large.

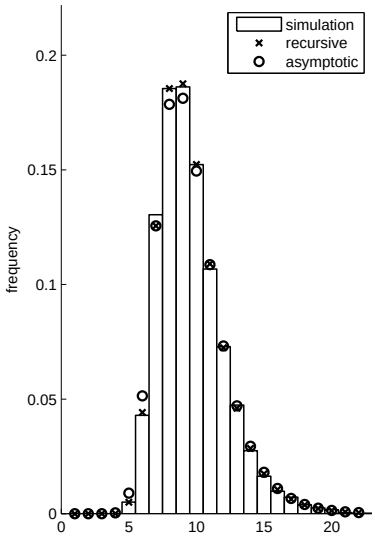
If p is much larger than $1/2$, the distribution of R'_n is quite close to R_n for moderate values of n as well. These facts give numerical evidence of (8).



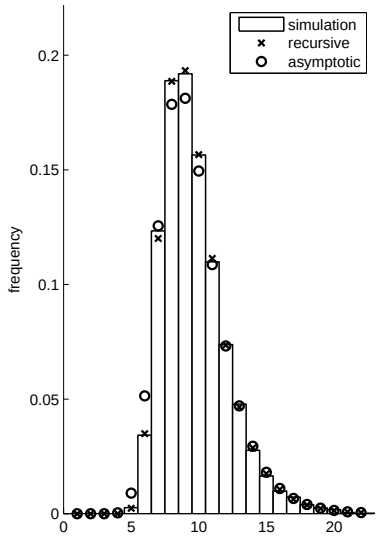
Distribution of the longest head run $p = 0.6, n = 30$.



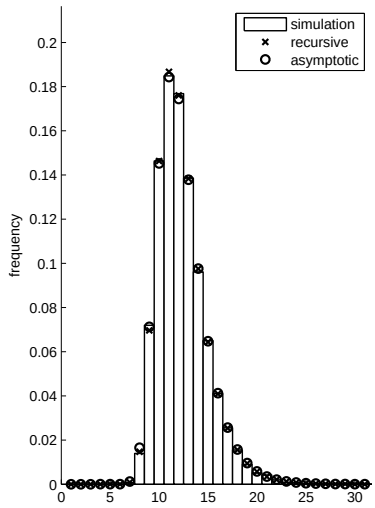
Distribution of the longest run $p = 0.6, n = 30$.



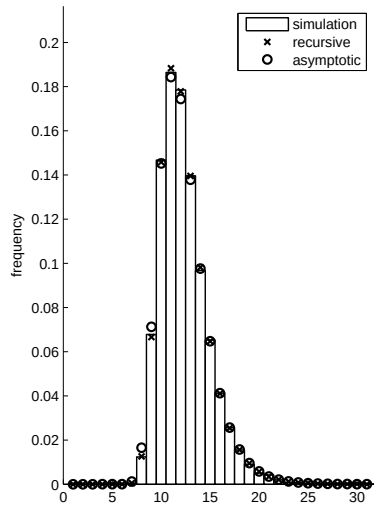
Distribution of the longest head run $p = 0.6, n = 250$.



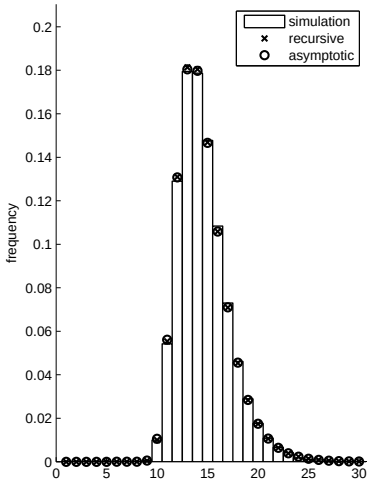
Distribution of the longest run $p = 0.6, n = 250$.



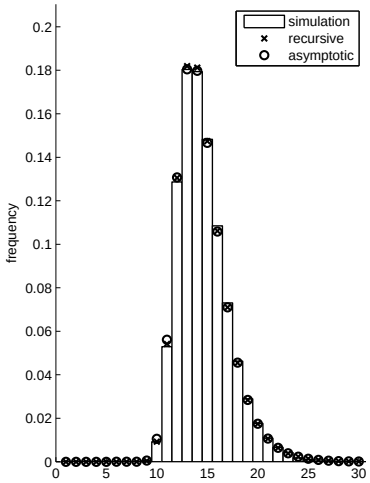
Distribution of the longest head run $p = 0.6, n = 1000$.



Distribution of the longest run $p = 0.6, n = 1000$.



Distribution of the longest head run $p = 0.6, n = 3100$.



Distribution of the longest run $p = 0.6, n = 3100$.

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