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On ν-curvature tensor of C3-like conformal Finsler space

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Abstract. The purpose of the present paper is to study the properties of the ν-curvature tensor in C-reducible Finsler space and conformal ν-curvature tensor in C3-like Finsler space F^n of dimension $(n \geq 4)$, in which the conformal Cartan torsion tensor $\overline{C}_{ijk}$ is said to be a conformal C3-like Finsler space.

1 Preliminaries

Let $F^n = (M^n, L)$ be a Finsler space on a differential manifold M endowed with a fundamental function L(x,y).

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We use the following notations [2, 6]:

$$
\begin{aligned}
&a) \quad g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad g^{ij} = (g_{ij})^{-1}, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i},\\
&b) \quad C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}, \quad C^k_{ij} = \frac{1}{2}g^{km}(\dot{\partial}_m g_{ij}),\\
&c) \quad h_{ij} = g_{ij} - l_i l_j, \quad h^i_k = \delta^i_k - l^i l_k,\\
&d) \quad m_i = b_i - \beta L^{-1} l^i,\\
&e) \quad C^h_{ij} l_h = 0,\\
&f) \quad h^i_k m_i = m_k,\\
&g) \quad l^i m_i = 0,
\end{aligned} \tag{1}
$$

where l_i, m_i and n_i are the unit vectors, and h_{ij} is a angular metric tensor.

Definition 1 *Let* $F^n = (M^n, L(x, y))$ *and* $\overline{F}^n = (M^n, L(x, y))$ *be two Finsler spaces on the same underlying manifold* M^n*. If the angle in* F^n *is equal to that in* $\overline{F}^n$ *for any tangent vectors, then* F^n *is called conformal to* $\overline{F}^n$ *and the change* $L \to \overline{L} = e^\sigma L$ *of the metric is called a conformal change and* $\sigma(x)$ *is a conformal factor.*

Example 1 *We consider a Finsler space* $F^n = (M^n, L(\alpha, \beta))$*, where* α *is a Riemannian metric,* β *is a 1-form and a conformal change* $L(\alpha, \beta) \to \overline{L} = e^{\sigma(x)} L(\alpha, \beta)$*. Since* $L(\alpha, \beta)$ *is assumed to be (1)p-homogeneous in* α *and* β*, we get* $\overline{L} = L(\overline{\alpha}, \overline{\beta})$*, where* $\overline{\alpha} = e^{\sigma(x)}\alpha$ *and* $\overline{\beta} = e^{\sigma(x)}\beta$*. Thus the conformal change gives rise to the change* $(\alpha, \beta) \longrightarrow (\overline{\alpha}, \overline{\beta}) = (e^{\sigma(x)}\alpha, e^{\sigma(x)}\beta)$ *of the pair* (α, β) *independently of the form of the function* $L(\alpha, \beta)$*. Thus we also get the conformal change* $\alpha \to \overline{\alpha} = e^{\sigma(x)}\alpha$ *of the associated Riemannian space* $R^n = (M^n, \alpha)$.

Under the conformal change, we get the following relations [3, 4]:

$$
\begin{aligned}
&a) \quad \overline{g}_{ij} = e^{2\sigma} g_{ij}, \quad \overline{g}^{ij} = e^{-2\sigma} g^{ij},\\
&b) \quad \overline{C}_{ijk} = e^{2\sigma} C_{ijk}, \quad \overline{C}^i_{jk} = C^i_{jk}, \quad \overline{C}^i_{ik} = \overline{C}_k = C^i_{ik} = C_k,\\
&c) \quad \overline{l}^i = e^{-\sigma} l^i, \quad \overline{l}_i = e^\sigma l_i, \quad \overline{y}_i = e^{2\sigma} y_i,\\
&d) \quad \overline{h}_{ij} = e^{2\sigma} h_{ij}, \quad \overline{h}^i_j = h^i_j,\\
&e) \quad \overline{L} = e^\sigma L,\\
&f) \quad \overline{m}_k = e^\sigma m_k.
\end{aligned} \tag{2}
$$

Definition 2 ([1]) *A Finsler space is said to be C-reducible if it satisfies the equation*

$$C_{ijk} = (C_i h_{jk} + C_j h_{ki} + C_k h_{ij})/(n+1). \tag{3}$$

Definition 3 *A conformal Finsler space* $\overline{F}^n$ *is said to be a semi-C-reducible conformal Finsler space if* $\overline{C}_{ijk}$ *is of the form,*

$$\overline{C}_{ijk} = e^{2\sigma}\left[\frac{p}{(n+1)}\{h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\} + \frac{q}{C^2}C_iC_jC_k\right].$$

There are three kinds of torsion tensors in Cartan theory of Finsler space F^n. Two of them are $(h)h\nu$-torsion tensor C_{ijk} and (ν)-torsion tensor P_{ijk}, which are symmetric in all its indices. It is obvious that F^n is Riemannian if the tensor C_{ijk} vanishes. For a three dimensional Finsler space F^3, C_{ijk} is always written in the form [5]

$$LC_{ijk} = Hm_im_jm_k - J\mathfrak{U}_{(ijk)}\{m_im_jn_k\} + I\mathfrak{U}_{(ijk)}\{m_in_jn_k\} + Jn_in_jn_k, \tag{4}$$

where $\mathfrak{U}_{(ijk)}\{\}$ denotes the cyclic permutation of the indices i, j, k and addition H, I and J are main scalars; we assume that they are invariant under conformal change and (l_i, m_i, n_i) is Moor's frame. Here $l_i = \dot{\partial}_i L$ is the unit vector along the element of support, m_i is the unit vector along C_i, i.e., $m_i = C_i/C$, where $C^2 = g^{ij}C_iC_j$ and n_i is a unit vector orthogonal to the vector l_i and m_i.

Example 2 *The C-reducible Finsler space (3) it is written for a three dimensional case as*

$$4C_{ijk} = h_{ij}C_k + h_{jk}C_i + h_{ki}C_j. \tag{5}$$

The unit vector $m_i = \frac{C_i}{C}$ *is orthogonal to* l_i, *because* $C_iy^i = 0$. *Therefore equation (5) can be written as*

$$4LC_{ijk} = LC\left[3m_im_jm_k + \{m_im_jn_k + m_in_jm_k + n_im_jm_k\}\right], \tag{6}$$

comparing equations (4) and (6) we have, $4H = 3LC$, $LC = (H+I)$ *and* $J = 0$. *So we get* $H = 3I$. *Conversely,* $H = 3I$ *and* $LC = H + I$ *lead to the above. Therefore the necessary and sufficient condition for C-reducible is* $H = 3I$, $LC = (H+I)$ *and* $J = 0$.

Under conformal change, equation (4) can be written as,

$$\overline{LC}_{ijk} = \overline{H}\overline{m}_i\overline{m}_j\overline{m}_k - \overline{J}\mathfrak{U}_{(ijk)}\{\overline{m}_i\overline{m}_j\overline{n}_k\} + \overline{I}\mathfrak{U}_{(ijk)}\{\overline{m}_i\overline{n}_j\overline{n}_k\} + \overline{J}\overline{n}_i\overline{n}_j\overline{n}_k \tag{7}$$

Suppose H, I and J are conformal invariants, then equation (7) reduces to

$$\overline{LC}_{ijk} = e^{3\sigma} LC_{ijk}.$$

The angular metric tensor h_{ij} in F^3 can be written as [5]

$$h_{ij} = m_i m_j + n_i n_j. \tag{8}$$

Under conformal change, equation (8) can be written as

$$\begin{aligned} \overline{h}_{ij} &= \overline{m}_i \overline{m}_j + \overline{n}_i \overline{n}_j, \\ \overline{h}_{ij} &= e^{2\sigma}(m_i m_j + n_i n_j). \end{aligned}$$

After simplification, equation (7) can be written as

$$\overline{C}_{ijk} = \mathfrak{U}_{(ijk)}(\overline{h}_{ij}\overline{a}_k + \overline{C}_i \overline{C}_j \overline{b}_k), \tag{9}$$

where

$$\begin{aligned} \overline{a}_k &= \frac{1}{\overline{L}}\left\{\overline{I}\,\overline{m}_k + \frac{\overline{J}}{3}\overline{n}_k\right\}, \\ \overline{b}_k &= \frac{1}{\overline{LC}^2}\left\{\left(\frac{\overline{H}}{3} - \overline{I}\right)\overline{m}_k - \frac{4\overline{J}}{3}\overline{n}_k\right\}. \end{aligned}$$

Then – by using (2(e),(f)) – the above equation becomes

$$\begin{aligned} \overline{a}_k &= \frac{1}{L}\left\{I m_k + \frac{J}{3} n_k\right\}, \\ \overline{b}_k &= \frac{e^{2\sigma}}{LC^2}\left[\left(\frac{H}{3} - I\right) m_k - \frac{4J}{3} n_k\right]. \end{aligned}$$

Substitute $\overline{a}_k$ and $\overline{b}_k$ in (9), we get,

$$\begin{aligned} \overline{C}_{ijk} &= \mathfrak{U}_{(ijk)}(e^{2\sigma} h_{ij} a_k + e^{2\sigma} C_i C_j b_k), \\ \overline{C}_{ijk} &= e^{2\sigma}\mathfrak{U}_{(ijk)}(h_{ij} a_k + C_i C_j b_k). \end{aligned} \tag{10}$$

The equation (10) can also be written in the form of

$$\overline{C}^i_{jk} = e^{2\sigma}\mathfrak{U}_{(ijk)}(h_{jk} a^i + C_j C_k b^i). \tag{11}$$

A Finsler space F^n ($n \geq 4$) is called a C3-like conformal Finsler space if there exist two vector fields a_k and b_k, which are positively homogenous of degree -1 and +1, respectively.
The purpose of the present paper is to find the ν-curvature tensor of the conformal Finsler space $\overline{F}^n$ when it satisfies (10).

2 Properties of C3-like conformal Finsler space

Let C_{ijk} be the indicatory tensor and contract equation (10) with g^{jk}, we get

$$\begin{aligned}\overline{C}_{ijk}g^{jk} &= e^{2\sigma}(h_{ij}a_k + C_iC_jb_k + h_{jk}a_i + C_jC_kb_i + h_{ki}a_j + C_kC_ib_j)g^{jk},\\ C_i &= (C_iC_b + (n+1)a_i + C^2b_i + C^iC_b),\\ C_i - 2C_iC_b &= ((n+1)a_i + C^2b_i),\\ C_i(1-2C_b) &= (n+1)a_i + C^2b_i, \end{aligned} \tag{12}$$

where $C_b = C_ib^i$.

Lemma 1 *The three vectors* a_i, b_i, C_i *are linearly dependent vectors.*

Contracting (12) with C^i, we get

$$\begin{aligned}(1-2C_b)C_iC^i &= (n+1)a_iC^i + C^2b_iC^i,\\ (1-2C_b)C^2 &= (n+1)C_a + C^2C_b,\\ (1-2C_b)C^2 - C^2C_b &= (n+1)C_a,\\ (1-3C_b)C^2 &= (n+1)C_a.\end{aligned}$$

Lemma 2 *If* C_i *is perpendicular to* b_i, *then* $C_a = \frac{C^2}{(n+1)}$, *and if* C_i *is perpendicular to* a_i, *then* $C_b = \frac{1}{3}$.

Now equation (12) can be written as

$$b_i = \frac{(1-2C_b)C_i}{C^2} - \frac{(n+1)a_i}{C^2}. \tag{13}$$

Substitute (13) in equation (10), we get

$$\begin{aligned}\overline{C}_{ijk} &= e^{2\sigma}\mathfrak{U}_{(ijk)}\{h_{ij}a_k + C_iC_jb_k\},\\ \overline{C}_{ijk} &= e^{2\sigma}\mathfrak{U}_{(ijk)}\left[h_{ij}a_k + C_iC_j\left\{\frac{(1-2C_b)C_k}{C^2} - \frac{(n+1)a_k}{C^2}\right\}\right],\\ \overline{C}_{ijk} &= e^{2\sigma}\mathfrak{U}_{(ijk)}\left\{h_{ij}a_k - \frac{(n+1)}{C^2}C_iC_ja_k\right\} + e^{2\sigma}\frac{3(1-2C_b)}{C^2}C_iC_jC_k.\end{aligned}$$

If a_i is parallel to C_i, i.e. $a_i = \frac{p}{n+1}C_i$, where p is some scalar, then $\overline{C}_{ijk}$ reduces to

$$\begin{aligned}\overline{C}_{ijk} &= e^{2\sigma}\frac{p}{(n+1)}\{h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\} - \frac{e^{2\sigma}3P}{C^2}C_iC_jC_k +\\ &+ \frac{e^{2\sigma}3(1-2C_b)}{C^2}C_iC_jC_k,\end{aligned}$$

$$\overline{C}_{ijk} = e^{2\sigma}\frac{p}{(n+1)}\{h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\} + \frac{3e^{2\sigma}C_iC_jC_k}{C^2}(1-2C_b-P),$$
$$\overline{C}_{ijk} = e^{2\sigma}\left[\frac{p}{(n+1)}\{h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\} + \frac{q}{C^2}C_iC_jC_k\right],$$

where $q = 3(1-2C_b-p)$. Hence we state:

Theorem 1 *A* $C3$*-like conformal Finsler space reduce to a semi-C-reducible conformal Finsler space if the vectors* a_i *and* b_i *are parallel to* C_i.

3 The ν-curvature tensor of C-redusible Finsler space

The ν-curvature tensor S_{hijk} of F^n is given by

$$S_{hijk} = C_{hkr}C^r_{ij} - C_{hjr}C^r_{ik}. \tag{14}$$

Using (3), the above equation can be written as

$$S_{hijk} = [C_iC_jh_{hk} + C_hC_kh_{ij} - C_iC_kh_{hj} - C_hC_jh_{ik}]/(n+1)^2. \tag{15}$$

Therefore equation (15) reduces to,

$$S_{hijk} = C_i[C_jh_{hk} - C_kh_{hj}]/(n+1)^2 + C_h[C_kh_{ij} - C_jh_{ik}]/(n+1)^2. \tag{16}$$

Contracting (16) with respect to y^i, and after some simplification, we get

$$\begin{aligned} 0 &= C_h[C_ky_j - C_jy_k]/(n+1)^2, \\ C_ky_j &= C_jy_k. \end{aligned} \tag{17}$$

Theorem 2 *The C-reducible Finsler space and* ν*-curvature tensor* S_{hijk} *satisfies the symmetric property and it holds (17).*

Corollary 1 *Under conformal change, the C-reducible condition and* υ*-curvature tensor also satisfy property (17).*

Example 3 *Let* T_{ij} *be a tensor of (0,2)-type of a two dimensional Finsler space and* $T_{\alpha\beta}$ *be scalar components of* T_{ij} *with respect to the Berwald frame:*

$$T_{ij} = T_{11}l_il_j + T_{12}l_im_j + T_{21}m_il_j + T_{22}m_im_j.$$

If T_{ij} *is symmetric, we have* $T_{12} = T_{21}$, *and if* T_{ij} *is skew-symmetric, then* $T_{0j} = 0$, $T_{ij} = 0$; *therefore, by this condition, the* υ*-curvature tensor* S_{hijk} *of* $C\Gamma$ *of any two dimensional Finsler space vanishes identically.*

4 The ν-curvature tensor of C3-like conformal Finsler space

Under conformal change, equation (14) can be written as,

$$\overline{S}_{hijk} = e^{2\sigma}[C_{hkr}C^r_{ij} - C_{hjr}C^r_{ik}],$$

using (10), (11), the above equation becomes

$$\begin{aligned}\overline{S}_{hijk} = {} & e^{2\sigma}[\{h_{hk}a_r + C_hC_kb_r + h_{kr}a_h + C_kC_rb_h + h_{rh}a_k + C_rC_hb_k\} \times \\ & \{h_{ij}a^r + C_iC_jb^r + h^r_ja_i + C_jC^rb_i + h^r_ia_j + C^rC_ib_j\} - \\ & \{h_{hj}a_r + C_hC_jb_r + h_{jr}a_h + C_jC_rb_h + h_{rh}a_j + C_rC_hb_j\} \times \\ & \{h_{ik}a^r + C_iC_kb^r + h^r_ka_i + C_kC^rb_i + h^r_ia_k + C^rC_ib_k\}] .\end{aligned}$$

After some simplification and rearrangement, we get the following equation:

$$\begin{aligned}\overline{S}_{hijk} = {} & e^{2\sigma}\left[\left\{h_{hk}\left(\frac{a^2}{2}h_{ij} + a_ia_j + C_iC_ja_rb^r + (C_ib_j + C_jb_i)C_a\right)\right\} \times\right. \\ & \left\{h_{ij}\left(\frac{a^2}{2}h_{hk} + a_ha_k + C_hC_ka_rb^r + (C_kb_h + C_hb_k)C_a\right)\right\} - \\ & \left\{h_{hj}\left(\frac{a^2}{2}h_{ik} + a_ia_k + C_iC_ka_rb^r + (C_kb_i + C_ib_k)C_a\right)\right\} \times \\ & \left\{h_{ik}\left(\frac{a^2}{2}h_{hj} + a_ha_j + C_hC_ja_rb^r + (C_jb_h + C_hb_j)C_a\right)\right\} + \\ & \left. C^2(C_ib_h - C_hb_i)(C_kb_j - C_jb_k)\right] .\end{aligned}$$

The above equation can be rewritten in the form:

$$\begin{aligned}\overline{S}_{hijk} = {} & e^{2\sigma}[h_{hk}B_{ij} + h_{ij}B_{hk}l - h_{hj}B_{ik} - h_{ik}B_{hj} \\ & +C^2(C_ib_h - C_hb_i)(C_kb_j - C_jb_k)],\end{aligned} \tag{18}$$

where $B_{ij} = \frac{a^2}{2}h_{ij} + a_ia_j + C_iC_ja_rb^r + (C_ib_j + C_jb_i)C_a$.

Theorem 3 *The conformal ν-curvature tensor* $\overline{S}_{hijk}$ *on a C3-like conformal Finsler space reduces to equation (18).*

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Multiple radially symmetric solutions for a quasilinear eigenvalue problem

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Abstract. In this paper we study an eigenvalue problem in $\mathbb{R}^N$, which involves the p-Laplacian $(1 < p < N)$, and the nonlinear term has a global $(p-1)$-sublinear growth. We guarantee an open interval of eigenvalues, for which the eigenvalue problem has three distinct radially symmetric solutions in a weighted Sobolev space. We use a compact embedding result of Su, Wang and Willem ([6]) and a Ricceri-type three critical points theorem of Bonanno ([1]).

1 Main result

Let $V, Q : (0, \infty) \to (0, \infty)$ be two continuous functions satisfying the following hypotheses

(V) there exist real numbers a and a_0 such that

$$\liminf_{r\to\infty} \frac{V(r)}{r^a} > 0, \liminf_{r\to 0} \frac{V(r)}{r^{a_0}} > 0.$$

(Q) there exist real numbers b and b_0 such that

$$\liminf_{r\to\infty} \frac{Q(r)}{r^b} < \infty, \liminf_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty.$$

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Key words and phrases: quasilinear problem, eigenvalue problem, variational methods, weighted Sobolev space

Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support and

$$C_{0,r}^\infty(\mathbb{R}^N) = \{u \in C_0^\infty(\mathbb{R}^N) | u \text{ is radially symmetric}\}.$$

We recall that a function $u \in C_0^\infty(\mathbb{R}^N)$ is radially symmetric, if $u(|x|) = u(x)$, for any $x \in \mathbb{R}^N$.
Let $D_r^{1,p}(\mathbb{R}^N)$ be the completion of $C_{0,r}^\infty(\mathbb{R}^N)$ under

$$\|u\|^p = \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

Define the Lebesgue spaces for $p \geq 1$ and $q \geq 1$:

$$L^p(\mathbb{R}^N; V) = \{u : \mathbb{R}^N \to \mathbb{R} | u \text{ is measurable}, \int_{\mathbb{R}^N} V(|x|)|u|^p dx < \infty\}$$

$$L^q(\mathbb{R}^N; Q) = \{u : \mathbb{R}^N \to \mathbb{R} | u \text{ is measurable}, \int_{\mathbb{R}^N} Q(|x|)|u|^q dx < \infty\}$$

with the corresponding norms

$$\|u\|_{L^p(\mathbb{R}^N;V)} = \left(\int_{\mathbb{R}^N} V(|x|)|u|^p dx\right)^{1/p},$$

$$\|u\|_{L^q(\mathbb{R}^N;Q)} = \left(\int_{\mathbb{R}^N} Q(|x|)|u|^q dx\right)^{1/q}.$$

For these norms, we use the abbreviations: $\|u\|_{L^p(\mathbb{R}^N;V)} = \|u\|_{p,V}$ and $\|u\|_{L^q(\mathbb{R}^N;Q)} = \|u\|_{q,Q}$.
Then define $W_r^{1,p}(\mathbb{R}^N; V) = D_r^{1,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N; V)$, which is a Banach space under

$$\|u\|_W^p = \int_{\mathbb{R}^N} (|\nabla u|^p + V(|x|)|u|^p)\, dx.$$

In order to state the embedding theorem used in our proofs, we need to introduce the following notations:

$$q_* = \begin{cases} \dfrac{p^2(N-1+b) - ap}{p(N-1) + a(p-1)}, & b \geq a > -p, \\ \dfrac{p(N+b)}{N-p}, & b \geq -p \geq a, \\ p, & b \leq \max\{a, -p\} \end{cases}$$

$$q^* = \begin{cases} \dfrac{p(N+b_0)}{N-p}, & b_0 \geq -p, a_0 \geq -p, \\ \dfrac{p^2(N-1+b_0)-a_0p}{p(N-1)+a_0(p-1)}, & -p \geq a_0 > -\frac{N-1}{p-1}p, b_0 \geq a_0, \\ \infty, & a_0 \leq -\frac{N-1}{p-1}p, b_0 \geq a_0. \end{cases}$$

We shall use the following embedding theorem.

Theorem 1 [6, Theorem 1.] *Let* $1 < p < N$. *Assume* (V) *and* (Q). *Then we have the embedding*

$$W_r^{1,p}(\mathbb{R}^N; V) \hookrightarrow L^q(\mathbb{R}^N; Q) \tag{1}$$

for $q_* \leq q \leq q^*$, *when* $q^* < \infty$ *and for* $q_* \leq q < \infty$, *when* $q^* = \infty$.

Furthermore, the embedding is compact for $q_* < q < q^*$. *And if* $b < \max\{a, -p\}$ *and* $b_0 > \min\{-p, a_0\}$, *the embedding is also compact for* $q = p$.

Therefore, supposing besides (V) and (Q) the condition

(ab) $\qquad b < \max\{a, -p\}$ and $b_0 > \min\{-p, a_0\}$,

the embedding

$$W_r^{1,p}(\mathbb{R}^N; V) \hookrightarrow L^p(\mathbb{R}^N; Q) \tag{2}$$

is also compact.

The collection of those functions, which satisfy the conditions $(V), (Q)$ and (ab) is not empty. For example, taking

$$a = p, b = -p-1, a_0 = -p, b_0 = -p+1,$$

the functions V and Q defined by

$$V(r) = \max\left\{1, \frac{1}{r^p}\right\},$$

$$Q(x) = \min\left\{\frac{1}{r^{p+1}}, \frac{1}{r^{p-1}}\right\}$$

satisfy all three assumptions for every $1 < p < N$.

For $\lambda > 0$, we consider the following problem:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u + V(|x|)|u|^{p-2}u = \lambda Q(|x|)f(u) \text{ in } \mathbb{R}^N \\ |u(x)| \to 0, \text{as } |x| \to \infty \end{cases},$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

We say that $u \in W_r^{1,p}(\mathbb{R}^N; V)$ is a *weak radial solution* of the problem (P_λ) if

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + V(|x|)|u|^{p-2}uv)dx - \lambda \int_{\mathbb{R}^N} Q(|x|)f(u(x))v(x)dx = 0,$$

for every $v \in W_r^{1,p}(\mathbb{R}^N; V)$.

We assume the following conditions on f :

(f_1) there exists $C > 0$ such that $|f(s)| \leq C(1 + |s|^{p-1})$, for every $s \in \mathbb{R}$;

(f_2) $\lim_{s\to 0} \frac{f(s)}{|s|^{p-1}} = 0$;

(f_3) there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$, where $F(s) = \int_0^s f(t)dt$.

Our main result is the following

Theorem 2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies $(f_1), (f_2)$, (f_3), and assume that (V), (Q) and (ab) are verified. Then, there exists an open interval $\Lambda \subset (0, \infty)$ and a constant $\mu > 0$ such that for every $\lambda \in \Lambda$ problem (P_λ) there are at least three distinct weak radial solutions in $W_r^{1,p}(\mathbb{R}^N; V)$, whose $W_r^{1,p}(\mathbb{R}^N; V)$–norms are less than μ.*

2 Auxiliary results

In this section we give a few preliminary results. These will be used in the proof of the main result in the next section.

We denote the best embedding constant of the embedding (1) by C_q, i.e. we have the inequality:

$$\|u\|_{q,Q} \leq C_q \|u\|_W.$$

We define the energy functional corresponding to (P_λ) as

$$\mathcal{E}_\lambda : W_r^{1,p}(\mathbb{R}^N; V) \to \mathbb{R}$$

$$\mathcal{E}_\lambda(u) = \frac{1}{p}\|u\|_W^p - \lambda J(u),$$

where $J : W_r^{1,p}(\mathbb{R}^N; V) \to \mathbb{R}$ is the functional defined by

$$J(u) = \int_{\mathbb{R}^N} Q(|x|)F(u(x))dx.$$

The functional $\mathcal{E}_\lambda$ is of class C^1 (see for instance [3, Lemma 4]), and its derivative is given by

$$\langle \mathcal{E}_\lambda'(u), v\rangle = \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + V(|x|)|u|^{p-2}uv)\,dx - \lambda \int_{\mathbb{R}^N} Q(|x|)f(u(x))v(x)\,dx,$$

for every $v \in W_r^{1,p}(\mathbb{R}^N; V)$. Therefore, the critical points of the energy functional are exactly the weak radial solutions of the problem (P_λ).

Lemma 1 *For every* $\lambda > 0$*, the functional* $\mathcal{E}_\lambda : W_r^{1,p}(\mathbb{R}^N; V) \to \mathbb{R}$ *is sequentially weakly lower semicontinuous.*

Proof. Due to (f_2), for arbitrary small $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(s)| \le \varepsilon p |s|^{p-1}, \text{ for every } s \le |\delta|. \tag{3}$$

Combining this inequality with condition (f_1), we obtain

$$|f(s)| \le \varepsilon p |s|^{p-1} + K(\delta) r |s|^{r-1}, \text{ for every } s \in \mathbb{R}, \tag{4}$$

where $r \in]q_*, q^*[$ is fixed and $K(\delta) > 0$ does not depend on s.
Let $\{u_n\}$ be a sequence from $W_r^{1,p}(\mathbb{R}^N; V)$, which is weakly convergent to some $u \in W_r^{1,p}(\mathbb{R}^N; V)$. Then there exists a positive constant $M > 0$ such that

$$\|u_n\|_W \le M, \|u_n - u\|_W \le M, \forall n \in \mathbb{N}. \tag{5}$$

We claim that $|J(u_n) - J(u)| \to 0$ as $n \to \infty$. Using inequality (4), the standard mean value theorem for F and the Hölder's inequality, we obtain:

$$\begin{aligned}
|J(u_n) - J(u)| &\le \int_{\mathbb{R}^N} Q(|x|)|F(u_n(x)) - F(u(x))|dx \le \\
&\le \int_{\mathbb{R}^N} Q(|x|)|f(\theta u_n(x) - (1-\theta)u(x))||u_n(x) - u(x)|dx \le \\
&\le \varepsilon p \int_{\mathbb{R}^N} Q(|x|)|\theta u_n(x) - (1-\theta)u(x)|^{p-1}|u_n(x) - u(x)|dx + \\
&+ K(\delta) r \int_{\mathbb{R}^N} Q(|x|)|\theta u_n(x) - (1-\theta)u(x)|^{r-1}|u_n(x) - u(x)|dx \le \\
&\le \varepsilon p \int_{\mathbb{R}^N} Q(|x|)(|u_n(x)|^{p-1} + |u(x)|^{p-1})|u_n(x) - u(x)|dx + \\
&+ K(\delta) r \int_{\mathbb{R}^N} Q(|x|)|(|u_n(x)|^{r-1} + |u(x)|^{r-1})|u_n(x) - u(x)|dx \le \\
&\le \varepsilon p (\|u_n\|_{p,Q}^{p-1} + \|u\|_{p,Q}^{p-1})\|u_n - u\|_{p,Q} + \\
&+ K(\delta) r (\|u_n\|_{r,Q}^{r-1} + \|u\|_{r,Q}^{r-1})\|u_n - u\|_{r,Q}.
\end{aligned}$$

Now, using the embeddings (1), (2) and the inequalities (5) we have

$$\begin{aligned} |J(u_n)-J(u)| &\le \varepsilon p C_p^p(\|u_n\|_W^{p-1}+\|u\|_W^{p-1})\|u_n-u\|_W + \\ &+ K(\delta) r C_r^{r-1}(\|u_n\|_W^{r-1}+\|u\|_W^{r-1})\|u_n-u\|_{r,Q} \le \\ &\le 2\varepsilon p C_p^p M^p + 2K(\delta) r C_r^{r-1} M^{r-1}\|u_n-u\|_{r,Q}. \end{aligned}$$

Since the embedding $W_r^{1,p}(\mathbb{R}^N;V) \hookrightarrow L^r(\mathbb{R}^N;Q)$ is compact for $r \in]q_*, q^*[$, we have that $\|u_n-u\|_{r,Q} \to 0$, whenever $n \to \infty$. Besides that, ε is chosen arbitrarily, so the claim follows from the last inequality. ■

Lemma 2 *For every* $\lambda > 0$*, the functional* $\mathcal{E}_\lambda : W_r^{1,p}(\mathbb{R}^N;V) \to \mathbb{R}$ *is coercive.*

Proof. Let η be a constant such that

$$0 < \eta < \frac{1}{pC_p^p}, \tag{6}$$

where C_p is the best embedding constant of the embedding $W_r^{1,p}(\mathbb{R}^N;V) \hookrightarrow L^p(\mathbb{R}^N;V)$. Due to conditions (f_1) and (f_2), there is a function $k \in L^1(\mathbb{R}^N;Q)$ such that

$$|F(s)| \le \eta|s|^p + k(x), \forall s \in \mathbb{R}, \forall x \in \mathbb{R}^N. \tag{7}$$

Then, we obtain

$$\begin{aligned} \mathcal{E}_\lambda(u) &\ge \frac{1}{p}\|u\|_W^p - \eta\int_{\mathbb{R}^N} Q(|x|)|u(x)|^p dx - \int_{\mathbb{R}^N} Q(|x|)k(x)dx \ge \\ &\ge \frac{1}{p}\|u\|_W^p - \eta C_p^p\|u\|_W^p - \|k\|_{1,Q} = \\ &= \left(\frac{1}{p} - \eta C_p^p\right)\|u\|_W^p - \|k\|_{1,Q} \end{aligned}$$

By the choice of the function k, we have that $\|k\|_{1,Q}$ is bounded. Therefore, using the inequality (6), we obtain that $\mathcal{E}_\lambda(u) \to \infty$, as $\|u\|_W \to \infty$, concluding the proof. ■

Lemma 3 *For every* $\lambda > 0$*, the functional* $\mathcal{E}_\lambda : W_r^{1,p}(\mathbb{R}^N;V) \to \mathbb{R}$ *satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N;V)$ be a (PS)-sequence for the function $\mathcal{E}_\lambda$, i.e.

(1) $\{\mathcal{E}_\lambda(u_n)\}$ is bounded;

(2) $\mathcal{E}'_\lambda(u_n) \to 0$.
Since $\mathcal{E}_\lambda$ is coercive, we have that $\{u_n\}$ is bounded. The reflexivity of the Banach space $W_r^{1,p}(\mathbb{R}^N; V)$ implies the existence of a subsequence (notated also by $\{u_n\}$), such that $\{u_n\}$ is weakly convergent to an element $u \in W_r^{1,p}(\mathbb{R}^N; V)$. Therefore, we have

$$\langle \mathcal{E}'_\lambda(u), u_n - u\rangle \to 0 \text{ as } n \to \infty. \tag{8}$$

Because the inclusion $W_r^{1,p}(\mathbb{R}^N; V) \hookrightarrow L^p(\mathbb{R}^N; V)$ is compact, we have that $u_n \to u$ strongly in $L^p(\mathbb{R}^N; V)$. We would like to prove that u_n converges strongly to u in $W_r^{1,p}(\mathbb{R}^N; V)$. For this, we will use the following estimates from [2, Lemma 4.10]

$$|\xi - \zeta|^p \leq M_1(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta), \quad \text{for } p \geq 2 \tag{9}$$

$$|\xi - \zeta|^2 \leq M_2(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p}, \quad \text{for } p \in]1, 2[, \tag{10}$$

where M_1 and M_2 are some positive constants. We separate two cases. In the first case let $p \geq 2$. Then we have:

$$\begin{aligned}
||u_n - u||_W^p &= \int_{\mathbb{R}^N} |\nabla u_n(x) - \nabla u(x)|^p dx + \int_{\mathbb{R}^N} V(|x|)|u_n(x) - u(x)|^p dx \\
&\leq M_1 \int_{\mathbb{R}^N} \left[|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x)\right] (\nabla u_n(x) - \nabla u(x)) dx \\
&+ \quad M_1 \int_{\mathbb{R}^N} V(|x|) \left[|u_n(x)|^{p-2}u_n(x) - |u(x)|^{p-2}u(x)\right] (u_n(x) - u(x)) dx \\
&= \quad M_1(\langle \mathcal{E}'_\lambda(u_n), u_n - u\rangle - \langle \mathcal{E}'_\lambda(u), u_n - u\rangle + \lambda\langle J'(u_n) - J'(u), u_n - u\rangle) \\
&\leq \quad M_1 \left(||\mathcal{E}'_\lambda(u_n)||_{W_r^{1,p}(\mathbb{R}^N;V)^*} + \lambda||J'(u_n) - J'(u)||_{W_r^{1,p}(\mathbb{R}^N;V)^*}\right) ||u_n - u||_W \\
&- \quad M_1\langle \mathcal{E}'_\lambda(u), u_n - u\rangle.
\end{aligned}$$

Since $u_n \to u$ weakly in $W_r^{1,p}(\mathbb{R}^N; V)$ and J' are compact (see [3, Lemma 4]), we have that $||J'(u_n) - J'(u)||_{W_r^{1,p}(\mathbb{R}^N;V)^*} \to 0$. Moreover $||\mathcal{E}'_\lambda(u_n)|| \to 0$, hence using (8), we have that $||u_n - u||_W \to 0$, as $n \to \infty$.
In the second case, when $1 < p < 2$, we recall the following result: for all $s \in (0, \infty)$ there is a constant $c_s > 0$ such that

$$(x + y)^s \leq c_s(x^s + y^s), \quad \text{for any } x, y \in (0, \infty). \tag{11}$$

Then we obtain

$$||u_n - u||_W^2 = \left(\int_{\mathbb{R}^N} |\nabla u_n(x) - \nabla u(x)|^p dx + \int_{\mathbb{R}^N} V(|x|)|u_n(x) - u(x)|^p dx\right)^{\frac{2}{p}} \tag{12}$$

$$\leq c_p \left[\left(\int_{\mathbb{R}^N} |\nabla u_n(x) - \nabla u(x)|^p dx \right)^{\frac{2}{p}} + \left(\int_{\mathbb{R}^N} V(|x|)|u_n(x) - u(x)|^p dx \right)^{\frac{2}{p}} \right].$$

Now, using (10) and the Hölder inequalities, we get:

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n(x) - \nabla u(x)|^p dx = \int_{\mathbb{R}^N} (|\nabla u_n(x) - \nabla u(x)|^2)^{\frac{p}{2}} dx \leq \\
\leq\ & M_2 \cdot \int_{\mathbb{R}^N} \Big((|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x) - \nabla u(x)) \Big)^{\frac{p}{2}} \cdot \\
& \cdot (|\nabla u_n(x)| + |\nabla u(x)|)^{\frac{p(2-p)}{2}} dx = \\
=\ & M_2 \cdot \int_{\Omega} \Big[(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x) - \nabla u(x)) \Big]^{\frac{p}{2}} \cdot \\
& [(|\nabla u_n(x)| + |\nabla u(x)|)^p]^{\frac{2-p}{2}} dx = \\
\leq\ & \widetilde{M_2} \left(\int_{\mathbb{R}^N} |\nabla u_n(x)|^p dx + \int_{\mathbb{R}^N} |\nabla u(x)|^p dx \right)^{\frac{2-p}{2}} \cdot \\
& \left(\int_{\mathbb{R}^N} (|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x) - \nabla u(x)) dx \right)^{\frac{p}{2}} \\
\leq\ & \overline{M}_2 \left[\left(\int_{\mathbb{R}^N} |\nabla u_n(x)|^p dx \right)^{\frac{2-p}{2}} + \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx \right)^{\frac{2-p}{2}} \right] \cdot \\
& \left(\int_{\mathbb{R}^N} (|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x) - \nabla u(x)) dx \right)^{\frac{p}{2}} \\
\leq\ & \widehat{M_2} \cdot \left(\int_{\mathbb{R}^N} (|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x) - \nabla u(x)) dx \right)^{\frac{p}{2}} \\
& \left(||u_n||_W^{\frac{(2-p)p}{2}} + ||u||_W^{\frac{(2-p)p}{2}} \right).
\end{aligned}$$

Then, using again relation (11) and the above inequality, we have the estimate:

$$\left(\int_{\mathbb{R}^N} |\nabla u_n(x) - \nabla u(x)|^p dx \right)^{\frac{2}{p}} \leq \tag{13}$$

$$\begin{aligned}
\leq\ & M_2' \cdot \left(\int_{\mathbb{R}^N} (|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u(x)|^{p-2}\nabla u(x))(\nabla u_n(x) - \nabla u(x)) dx \right) \cdot \\
& \left(||u_n||_W^{2-p} + ||u||_W^{2-p} \right).
\end{aligned}$$

We introduce the following notation: $I(u) = \frac{1}{p}||u||_W^p$. As we used before, the directional derivative of I, in the direction $v \in E$ is

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \int_{\mathbb{R}^N} V(|x|)|u(x)|^{p-2}u(x)v(x)dx.$$

Using the inequalities (12), (13) we have

$$||u_n - u||_W^2 < M_2' \cdot \langle I'(u_n) - I'(u), u_n - u\rangle \cdot (||u_n||_W^{p-2} + ||u||_W^{2-p}).$$

Since u_n is bounded, the same argument as in the first case (when $p \geq 2$) shows that u_n converges to u strongly in $W_r^{1,p}(\mathbb{R}^N; V)$.

Thus $\mathcal{E}_\lambda$ satisfies the (PS) condition for all $\lambda > 0$. ■

Lemma 4

$$\lim_{t\to 0^+} \frac{\sup\{J(u) : ||u||_W^p < pt\}}{t} = 0.$$

Proof. From inequality (4) we obtain:

$$|F(s)| \leq \varepsilon|s|^p + K(\delta)|s|^r, \text{ for every } s \in \mathbb{R}, \quad (14)$$

where $r \in]q_*, q^*[$ is fixed and $K(\delta)$ does not depend on s. Then

$$J(u) \leq \varepsilon||u||_{p,Q}^p + K(\delta)||u||_{r,Q}^r.$$

Now, using emeddings (1), (2), we get:

$$J(u) \leq \varepsilon C_p^p||u||_W^p + K(\delta)C_r^r||u||_W^r.$$

Therefore,

$$\sup\{J(u) : ||u||_W^p < pt\} \leq \varepsilon C_p^p pt + K(\delta)C_r^r(pt)^{\frac{r}{p}}.$$

Since ε is chosen arbitrarily and $r > p$, by dividing this last inequality with t and taking the limit, whenever $t \to 0^+$, we conclude the proof. ■

3 Proof of theorem 2

The main tool in the proof of Theorem 2 is a Ricceri-type critical points theorem (see [4], [5]) refined by Bonanno in [1].

Theorem 3 (G. Bonanno [1]) *Let X be a separable and reflexive real Banach space, and let $\Phi, J : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that*

(i) *there exists* $x_0 \in X$, *such that* $\Phi(x_0) = J(x_0) = 0$;

(ii) $\Phi(x) \geq 0$ *for every* $x \in X$;

(iii) *there exists* $x_1 \in X$, $\rho > 0$, *such that* $\rho < \Phi(x_1)$ *and* $\sup\{J(x) : \Phi(x) < \rho\} < \rho\frac{J(x_1)}{\Phi(x_1)}$.

(iv) *the functional* $\Phi - \lambda J$ *is sequentially weakly lower semicontinuous, satisfies the Palais–Smale condition for every* $\lambda > 0$ *and it is coercive, for every* $\lambda \in [0, \bar{a}]$, *where* $\bar{a} = \dfrac{\zeta\rho}{\rho\dfrac{J(x_1)}{\Phi(x_1)} - \sup\limits_{\Phi(x)<\rho} J(x)}$, *with* $\zeta > 1$.

Then there is an open interval $\Lambda \subset [0, \bar{a}]$ *and a number* $\sigma > 0$, *such that for each* $\lambda \in \Lambda$, *the equation* $\Phi'(x) - \lambda J'(x) = 0$ *admits at least three distinct solutions in X, having norm less than* σ.

We also need the following result of Su, Wang, Willem.

Lemma 5 [6, Lemma 4] *Assuming* (V) *with* $a > -\frac{N-1}{p-1}p$, *there exists* $C > 0$, *such that for all* $u \in W_r^{1,p}(\mathbb{R}^N; V)$

$$|u(x)| \leq C|x|^{-\frac{p(N-1)+a(p-1)}{p^2}} \|u\|_W, \quad |x| \gg 1. \tag{15}$$

Proof of Theorem 2. Let $s_0 \in \mathbb{R}$ be from (f_3), i.e. $F(s_0) > 0$. We denote by B_r the N-dimensional closed ball with center 0 and radius $r > 0$.
Since Q and V are positive continuous functions, for an $R > 0$ there exist the positive constants m_Q, M_Q, M_V such that:

$$m_Q = \min_{|x|\leq R} Q(|x|), M_Q = \max_{x\leq R} Q(|x|);$$

$$M_V = \max_{|x|\leq R} V(|x|).$$

For a $\sigma \in]0, 1[$ we define the function $u_\sigma : \mathbb{R}^N \to \mathbb{R}$ by

$$u_\sigma(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}^N \setminus B_R \\ s_0, & \text{if } x \in B_{\sigma R} \\ \dfrac{s_0}{R(1-\sigma)}(R - |x|), & \text{if } x \in B_R \setminus B_{\sigma R}. \end{cases}$$

It is clear that u_σ belongs to $W_r^{1,p}(\mathbb{R}^N;V)$. Denoting the volume of the ball B_1 by ω_N, we obtain:

$$\begin{aligned}
\|u_\sigma\|_W^p &= \int_{B_{\sigma R}} V(|x|)|s_0|^p dx + \int_{B_R\setminus B_{\sigma R}} \left|\frac{s_0}{R(1-\sigma)}\right|^p dx + \\
&+ \int_{B_R\setminus B_{\sigma R}} V(|x|)\left|\frac{s_0}{R(1-\sigma)}\right|^p (R-|x|)^p dx \leq \\
&\leq |s_0|^p\omega_N R^N(\sigma^N M_V + R^{-p}(1-\sigma)^{-p}(1-\sigma^N)) + \\
&+ |s_0|^p R^{-p}(1-\sigma)^{-p} M_V \int_{B_R\setminus B_{\sigma R}} (R-|x|)^p dx \leq \\
&\leq |s_0|^p\omega_N R^N(M_V + R^{-p}(1-\sigma)^{-p}(1-\sigma^N))
\end{aligned}$$

and

$$J(u_\sigma) \geq \omega_N R^N(m_Q F(s_0)\sigma^N - M_Q \max_{|t|\leq|s_0|} F(t)(1-\sigma^N)). \tag{16}$$

By the choice of m_Q and M_Q, we have that $0 < \frac{m_Q F(s_0)}{M_Q \max_{|t|\leq|s_0|} F(t)} < 1$. Therefore, we can choose a $\sigma_0 \in \left]\left(1+\frac{m_Q F(s_0)}{M_Q \max_{|t|\leq|s_0|} F(t)}\right)^{-\frac{1}{N}}, 1\right[\subseteq]0,1[$, such that

$$J(u_{\sigma_0}) > 0. \tag{17}$$

By Lemma 4 and inequality (16) it follows the existence of a positive constant $\rho_0 > 0$ so small that

$$\rho_0 < \frac{\|u_{\sigma_0}\|_W^p}{p} \tag{18}$$

$$\frac{\sup\{J(u) : \|u\|_W^p < \rho_0\}}{\rho_0} < \frac{pJ(u_{\sigma_0})}{\|u_{\sigma_0}\|_W^p}. \tag{19}$$

Using the Lemmas from the previous section and inequalities (18), (19), all the assumptions of Theorem 3 are satisfied with the choices: $E = W_r^{1,p}(\mathbb{R}^N;V)$, $\Phi = \frac{1}{p}\|u\|_W^p$, $x_1 = u_{\sigma_0}$, $x_0 = 0$ and $\zeta = 1+\rho_0$ and

$$a = \frac{1+\rho_0}{pJ(u_{\sigma_0})\|u_{\sigma_0}\|_W^{-p} - \sup\{J(u) : \|u\|_W^p < r\}\rho_0^{-1}}.$$

Then, there exists an open interval $\Lambda \subset (0,\infty)$ and a constant $\mu > 0$ such that for every $\lambda \in \Lambda$ the equation $\mathcal{E}_\lambda = \Phi - \lambda J$ admits at least three distinct critical points: $u_\lambda^1, u_\lambda^2, u_\lambda^3 \in W_r^{1,p}(\mathbb{R}^N; V)$ such that

$$\max\{\|u_\lambda^1\|_W, \|u_\lambda^2\|_W, \|u_\lambda^3\|_W\} < \mu. \tag{20}$$

It remains to show that $|u_\lambda^i(x)| \to 0$ as $|x| \to \infty$, for $i \in \{1, 2, 3\}$. Using Lemma 5 and taking into account the estimate (20), the claim follows immediately. ■

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Density of safe matrices

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Abstract. A binary matrix A of size $m \times n$ is called *r-good* if it contains in each column at most r 1's; the matrix is called *r-schedulable* if, by deleting some zeros, the matrix becomes *r-good*; A is called *r-safe* if the first k $(1 \leq k \leq n)$ columns of the matrix contain at most kr 1's.

Let $Z = [z_{ij}]_{m \times n}$ be a matrix of independent random variables, having the common distribution $P(z_{ij} = 1) = p$ and $P(z_{ij} = 0) = 1 - p$, where $0 \leq p \leq 1$. For $m \geq 1$, lower and upper bounds are presented for the asymptotic probability of the event that a concrete realization of Z is 1-schedulable: the lower bound is connected with good, and the upper bound with safe matrices. Further exact formula is given for the critical probabilities $s_{crit}(m)$ defined as the supremum of probabilities, guaranteeing that the matrix Z is 1-safe with positive probability for arbitrary value of n and m.

1 Introduction

Percolation is a very popular research area of combinatorics [2, 3, 5, 6, 9, 10, 11, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 52] and physics [15, 28, 36, 37, 38, 39, 42, 46].

In this paper we use and extend a mathematical model proposed by Peter Winkler [53] and studied later among others in [13, 14, 19, 20, 33, 35, 46]. This model is also useful for the investigation of some scheduling problems of parallel processes [40, 51] using resources requiring mutual exclusion [1, 7, 16, 21, 34, 45, 50].

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According to the Winkler model, two processes share one unit of a resource. We extend this model for $m \geq 2$ processes and $r > 0$ units of the resource requiring mutual exclusion. The rise of the number of processes results a model describing the percolation in three or more dimensions.
Estimations of the probability of schedulability of processes are derived using different methods, first of all by investigating of asymmetric random walks across the x axis.

2 Formulation of the problem

Let m and n be positive integers, let r $(0 \leq r \leq m)$ and p $(0 \leq p \leq 1)$ be real numbers and let

$$Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & z_{2n} \\ \dots & & & \\ z_{m1} & z_{m2} & \dots & z_{mn} \end{pmatrix}$$

be a matrix of independent random variables with the common distribution

$$P(z_{ij} = k) = \begin{cases} p, & \text{if } k = 1 \text{ and } 1 \leq i \leq m,\ 1 \leq j \leq n, \\ q = 1 - p, & \text{if } k = 0 \text{ and } 1 \leq i \leq m,\ 1 \leq j \leq n. \end{cases}$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

be a concrete realization of Z.
The good, safe and schedulable matrices are defined as follows.
Matrix A is called ***r*-good** if the number of the 1's is at most r in each column. The number of different r-good matrices of size $m \times n$ is denoted by $G_r(m, n)$, and the probability that Z is good is denoted by $g_r(m, n, p)$.
Matrix A is called ***r*-safe** if

$$\sum_{i=1}^{m} \sum_{j=1}^{k} a_{ij} \leq kr \quad (k = 1, 2, \dots, n).$$

The number of different r-safe matrices of size $m \times n$ is denoted by $S_r(m, n)$ and the probability that Z is safe, is denoted by $s_r(m, n, p)$.

If $a_{ij} = 0$, then it can be deleted from A. Deletion of a_{ij} means that we decrease the second indices of $a_{i,j+1}, \ldots, a_{im}$ and add $a_{im} = 0$ to the i^{th} row of A.

Matrix A is called **Winkler r-schedulable** (shortly **r-schedulable** or r-compatible) if it can be transformed into an r-good matrix B using deletions. The number of different r-schedulable matrices of size $m \times n$ is denoted by $W_r(m, n)$, and the probability that Z is r-schedulable is denoted by $w_r(m, n, p)$. The function $w_r(m, n, p)$ is called **r-schedulability function.** The functions $g_r(m, n, r)$, $w_r(m, n, r)$ and $s_r(m, n, r)$ are called the density of the corresponding matrices. The **asymptotic density** of the good, safe and schedulable matrices are defined as:

$$g_r(m, p) = \lim_{n \to \infty} g_r(m, n, p),$$

$$s_r(m, p) = \lim_{n \to \infty} s_r(m, n, p),$$

$$w_r(m, p) = \lim_{n \to \infty} w_r(m, n, p).$$

The critical probabilities defined as

$$w_{crit,r}(m) = \sup\{p \mid w_r(m, p) > 0\},$$

$$g_{crit,r}(m) = \sup\{p \mid g_r(m, p) > 0\},$$

and

$$s_{crit,r}(m) = \sup\{p \mid s_r(m, p) > 0\}$$

represent special interest for some applications.

The aim of this paper is to characterise the density, asymptotic density and critical probability of good, schedulable and safe matrices.

The starting point of our research is due to Péter Gács [20], proving that $w_1(2, p)$ is positive for p small enough. His proof implies that $w_{crit,1}(2) \geq 10^{-400}$.

2.1 Interpretation of the problem

Although the Winkler model was proposed to study the percolation, we describe a possible interpretation as a model of parallel processes. Let m processes use r units of some resource R. The requirements of the process P_i are modelled by the sequence $a_{i1}, a_{i2}, \ldots, a_{im}$. $a_{ij} = 1$ means that the process P_i needs a unit of the given resource in the j^{th} time unit. $a_{ij} = 0$ means that the process P_i executes some background work in the j^{th} time unit which can be

delayed and executed after the last usage of R.
The special case $m = 1$ and $r = 1$ is the well-known ticket problem [52] or ballot problem [17], while the special case $m = 2$ and $r = 1$ is the Winkler model of percolation [20, 53].
The good matrices are schedulable without deletion of zeros. But some not good matrices are schedulable, since they can be transformed into good matrices using the permitted deletion operation. Safeness is a necessary condition of schedulability. Therefore, the number of good matrices gives a lower bound and the number of safe matrices results an upper bound for the number of schedulable matrices.
Since we handle the model as a model of informatics, in the sequel we follow the terminology used by Feller [17] in queueing theory.

3 Analysis

In this section first of all we investigate – using different methods – the function of the asymptotic density of 1's as the function of the probability p of the appearance of 1's and of the number of sequences m.

Some basic properties of the investigated functions ($g_r(m, n, p)$, $w_r(m, n, p)$ and $s_r(m, n, p)$) are the following:

- $n \in \mathbb{N}^+$, $r \in \mathbb{R}$ and $r \in [0, m]$, $p \in \mathbb{R}$ and $p \in [0, 1]$;
- as the functions of n they are monotonically decreasing;
- as the functions of p they are monotonically decreasing;
- as the functions of m they are monotonically decreasing;
- as the functions of r they are monotonically increasing;

In the following we suppose that $r = 1$, that is in the column of the good matrices at most one 1, and in the first k columns of the safe matrices at most k 1's are permitted. Since r everywhere equals 1, it is omitted as an index.

3.1 Preliminary results

In the further sections we need the following assertions.
Let C_n ($n \in \mathbb{N}^+$) denote the number of binary sequences $a_1, a_2, \ldots, a_{2n}$, containing n ones and n zeros in such a manner that each prefix $a_1, a_2, \ldots, a_k$ ($1 \leq k \leq 2n$) contains at most so many ones as zeros.

Lemma 1 *If* $n \geq 0$, *then*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

It is worth remark that C_n is the n^{th} Catalan number, whose explicit form appears in numerous books and papers [8, 30, 31, 32, 48, 52].

Lemma 2 *If* $0 \leq x \leq 1$, *then*

$$f(x) = x\sum_{k=0}^{\infty}\frac{1}{k+1}\binom{2k}{k}(x(1-x))^k = \begin{cases} \dfrac{x}{1-x}, & \text{if } 0 \leq x < \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Proof. See [47]. ■

If $m \geq 2$, then the columns containing only 0's are called **white** (W), the columns containing only 1's are called **black** (B) and the remaining columns are called **gray** (G).

If $m \geq 2$, then each column of the matrix A is white or gray with probability $q^m + mpq^{m-1}$, therefore $g(m,n,p) = \left(q^m + mpq^{m-1}\right)^n$. If $p > 0$, then

$$g(m,p) = \lim_{n\to\infty}\left(q^m + pq^{m-1}m\right)^n = 0,$$

so the density of the good matrices tends to zero, when the number of the columns tends to infinity.

If in the case $m = 2$ we delete the white columns from a good matrix, then only gray columns remain in the matrix, that is, each row of the matrix is *the complementer* of the other row.

The following simple assertion plays an important role in the following.

Lemma 3 *If* $m \geq 2$, *then the good matrices are schedulable, and the schedulable matrices are safe.*

Proof. If in every column of matrix A is at most one 1, then the first k columns contain at most k 1's.

If there is a k ($1 \leq k \leq n$), that the first k columns of matrix A contains more 1's than k, then – according to the pigeonhole principle – there is at least one column containing two 1's. If we delete a zero from A, then the number of the 1's in the first k columns does not decrease, therefore A is not schedulable. ■

A useful consequence of this assertion is the following corollary.

Corollary 1 *If* $m \geq 2$, *then*

$$g(m,n,p) \leq w(m,n,p) \leq s(m,n,p),$$

$$g(m,p) \leq w(m,p) \leq s(m,p),$$

$$g_{crit}(m) \leq w_{crit}(m) \leq s_{crit}(m).$$

3.2 Matrices with two rows

For the simplicity of the notations we analyse the function $u(2,n,p) = 1 - s(2,n,p)$ instead of $s(2,n,p)$. At first we derive a closed formula for $u(2,n,0.5)$.

Lemma 4 *If* $n \geq 1$, *then*

$$u(2,n,0.5) = \sum_{i=1}^{n} \sum_{j=0}^{\lfloor (i-1)/2 \rfloor} 2^{i-1-2j} C_j \binom{i-1}{2j} 4^{n-i}. \qquad (1)$$

Proof. Let's classify the possible matrices of size $2 \times n$ according to their first such column, in which the cumulated number of 1's became greater than the number of 0's. This column is called **the deciding column** of the matrix.
The index of the deciding column is $1, 2, \ldots, n-1$ or n. The matrices of the received classes can be further classified according to the number of black columns before the deciding column: the possible values of this number are $0,\ 1, \ldots,\ \lfloor (n-1)/2 \rfloor$.
The outer summing takes into account the deciding columns, while the inner summing does the black columns before the deciding column. The binomial coefficient mirrors the number of possibilities for the placement of the $2j$ black and white columns in the $i-1$ columns preceding the deciding column. The j^{th} Catalan number C_j gives the number of corresponding sequence of the black and white columns. The power of base 2 gives the number of possible arrangements of the gray columns. Finally the power of base 4 takes into account the fact, that the columns after the deciding one can be filled in arbitrary manner – the matrix will be unsafe in any case. ■

It seems that it would be hard to handle the formula (1) for $u(2,n,0.5)$. Therefore, we present a combinatorial method and three further ones based on random walks to get the explicit form of $s(2,p)$.

Lemma 5 *If* $0 \leq p \leq 1$, *then*

$$1 - s(2,p) = u(2,p) = \begin{cases} \dfrac{p^2}{q^2}, & \textit{if } 0 \leq p < \frac{1}{2}, \\ 1, & \textit{if } \frac{1}{2} \leq p \leq 1. \end{cases}$$

Proof. Some part of the unsafe matrices is unsafe due to the first black column. The general form of such matrices is G^aBA^b, where $a + b + 1 = n$, further G means a gray, B means a black and A means an arbitrary column. The asymptotic fraction of such columns is

$$\sum_{a=0}^{\infty} C_0(2pq)^a p^2 = \frac{p^2}{1-2pq} C_0.$$

The general form of the following group of the unsafe matrices is G^aBG^bW G^cBA^d, where $a+b+c+d+3 = n$. The fraction of such matrices asymptotically equals to

$$\sum_{a=0}^{\infty}(2pq)^a p^2 \sum_{b=0}^{\infty}(2pq)^b q^2 \sum_{c=0}^{\infty}(2pq)^c p^2 = \frac{p^2}{1-2pq} C_1 \frac{p^2}{1-2pq} \frac{q^2}{1-2pq} .$$

Generally, if the $(i+1)^{th}$ black column is deciding, then the asymptotic contribution of such matrices to the probability of the unsafe matrices equals to

$$\frac{p^2}{1-2pq} C_i \left(\frac{p^2}{1-2pq} \frac{q^2}{1-2pq} \right)^2 ,$$

and so

$$u(2,p) = \sum_{i=0}^{\infty} \frac{p^2}{1-2pq} C_i \left(\frac{p^2}{1-2pq} \frac{q^2}{1-2pq} \right)^i .$$

Lemma 2, gives the required formula with the substitutions $p^2/(p^2+q^2) = x$ and $q^2/(p^2+q^2) = 1-x$. ■

We get a useful method for the investigation of our matrices assigning to each matrix a random walk [17, 43] on the real axis containing a sink at the point -1.

Another proof of Lemma 5 is as follows. In the following proofs of Lemma 5 we consider only the case $0 \le p \le 1/2$, since if $1/2 \le p \le 1$, then the following famous result of György Pólya [41, 43] implies our assertion.

Lemma 6 *The probability that the moving point performing a random walk over the real axis returns infinitely often to its initial position is equal to one.*

Second proof of Lemma 5. Let's assign a random walk to matrix A so that a black column implies a step to left, a white column implies a step to right

and a gray column results that the moving point preserves its position.
Let $b_k(A)$ denote the number of 1's in the first k columns of matrix A. Then

$$b_k = \sum_{i=1}^{k} (a_{1i+2i}).$$

If $b_i \leq k$ for $i = 1, 2, \ldots, k$, then after k time units the moving point is in the point $(k - b_k, 0)$ of the real axis, otherwise the point is absorbed by the sink at -1.
We wish to determine the probability of the absorption of the moving point.
The probability of a step to left is p^2, the probability of a step to right is q^2 and $2pq$ is the probability of the event that the point does not change its position.
Using the notation $u(2, p) = x$ we have

$$x = p^2 + 2pqx + q^2x^2.$$

The roots of this equation are

$$x_{1,2} = \frac{1 - 2pq \pm \sqrt{(1 - 2pq)^2 - 4p^2q^2}}{2q^2} = \frac{p^2 + q^2 \pm \sqrt{(p^2 - q^2)^2}}{2q^2} ,$$

from where we get

$$x_1 = \frac{p^2}{q^2} \text{ and } x_2 = 1. \tag{2}$$

This formula and $s(2, p) = 1 - u(2, p)$ result the required formula. ■
Since first of all we are interested in the probability of the absorption, we can assign a random walk to matrix Z neglecting the gray columns, as the gray columns have no influence on the limit probability of the absorption (they only make the convergence slower).
Another proof of Lemma 5 is the following.
Third proof of Lemma 5. Dividing the probability of the gray columns among the black and white columns in the corresponding ratio we get for the probability a of the step to left and for the probability of the step to right that

$$a = \frac{p^2}{p^2 + q^2} \quad \text{és} \quad b = \frac{q^2}{p^2 + q^2}. \tag{3}$$

Using these probabilities, we get the equation

$$x = a + bx^3.$$

Substituting a and b into the roots of this equation, according to (3), we also get here the roots corresponding to (2). ■

Finally we present such a method, which later can be extended to arbitrary $m \geq 2$ sequences.

Fourth proof of Lemma 5. Let x_k $(k = -1, 0, 1, 2, \ldots)$ denote the probability of the event that the point starting at point k will be absorbed by the sink at $x = -1$. Let's assign again a step to left to the columns containing two 1's, a step to right to the columns containing two 0's and preserve of the position to the mixed columns.

Then we can write the following system of equations.

$$\begin{array}{lclclcl} x_0 &=& q^2x_1 &+& 2qpx_0 &+& p^2, \\ x_1 &=& q^2x_2 &+& 2qpx_1 &+& p^2x_0, \\ x_2 &=& q^2x_3 &+& 2qpx_2 &+& p^2x_1, \\ x_3 &=& q^2x_4 &+& 2qpx_3 &+& p^2x_2, \\ \ldots \end{array} \tag{4}$$

Let

$$G(z) = \sum_{i=1}^{\infty} x_i z^i$$

be the generator function of sequence $x_0, x_1, x_2, \ldots$. Multiplying the equation beginning with x_i I $= 1, 2, \ldots$ of the system of equations (4) by z^i and summing up the new equations, we get the equation:

$$G(z) = q\frac{G(z) - x_0}{z} + 2pqG(z) + p^2(1 + zG(z)).$$

From this equation $G(z)$ can be expressed in the form

$$G(z) = \frac{P(z)}{Q(z)},$$

where

$$P(z) = q^2x_0 - p^2z$$

and

$$Q(z) = p^2z^2 + 2pqz + q^2 - z.$$

In the zero places z_o with $|z_0| \leq 1$ of the polynomial $Q(z)$, according to Cauchy-Hadamard theorem [44, page 69] it must hold $P(z) = 0$. Writing the equation $Q(z) = 0$ in the form

$$(pz + q)^2 = z$$

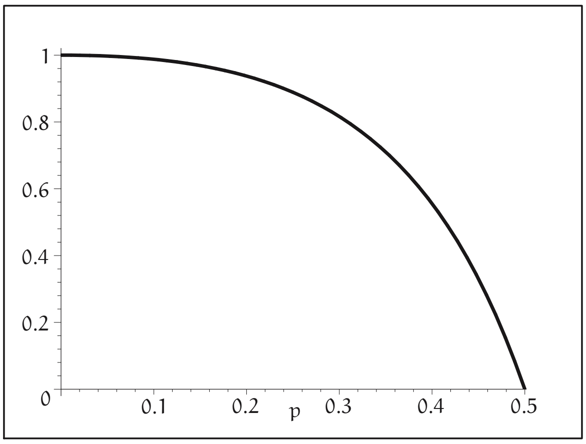

Figure 1: The curve of the schedulability function $s(2, p, 1)$ in the interval $p \in [0, 0.5]$.

we directly get that $z = 1$ is a root of the polynomial $Q(z)$. From the equation $P(1) = 0$ we get the root

$$x_0 = \frac{p^2}{q^2},$$

implying

$$s(2, p) = 1 - \frac{p^2}{q^2}, \qquad \blacksquare$$

Figure 1 shows the part belonging to the interval $p \in [0, 0.5]$ of the curve of the function $s(2, p, 1)$ defined in the interval $[0, 1]$.

According to the properties of the functions $g(2, p)$ and $s(2, p)$, the critical probabilities satisfy the following inequalities:

$$0 = g_{crit}(2) \leq w_{crit}(2) \leq s_{crit}(2) = \frac{1}{2}.$$

Let's remind that Gács proved $w_{crit}(2) \geq 10^{-400}$ [20].

Let $T(m, n)$ denote the number of binary matrices of size $m \times n$. Then $T(m, n) = 2^{mn}$.

Figure 2 contains the number and fraction of the good, schedulable and safe matrices for the case $m = 2$, $p = 0.5$, and $n = 1, 2, \ldots, 15$. In this case

n	$G(2,n)$	$\frac{G(2,n)}{T(2,n)}$	$W(2,n)$	$\frac{W(2,n)}{T(2,n)}$	$S(2,n)$	$\frac{S(2,n)}{T(2,n)}$	$\frac{W(2,n)}{S(2,n)}$
1	3	0.750	3	0.750	3	0.750	1
2	9	0.562	10	0.625	10	0.625	1
3	27	0.452	35	0.547	35	0.547	1
4	81	0.316	124	0.484	126	0.492	0.984
5	243	0.237	444	0.434	462	0.451	0.961
6	729	0.178	1592	0.389	1716	0.419	0.927
7	2187	0.133	5731	0.350	6435	0.393	0.890
8	6561	0.100	20671	0.315	24310	0.371	0.850
9	19683	0.075	74722	0.285	92378	0.352	0.808
10	59049	0.056	270521	0.258	352716	0.336	0.767
11	177147	0.042	980751	0.234	1352078	0.322	0.725
12	531441	0.032	3559538	0.212	5200300	0.310	0.684
13	1594323	0.022	12931155	0.193	20058300	0.299	0.646
14	4782969	0.018	47013033	0.175	77558760	0.289	0.606
15	14348907	0.013	171036244	0.159	300540195	0.280	0.568

Figure 2: Rounded data belonging to the parameters $m = 2$ and $p = 0.5$.

the fractions equal to the probability of the corresponding matrices. According to Lemma 5 in this case $G(m,n)/T(m,n)$, $W(m,n)/T(m,n)$, and $S(m,n)/T(m,n)$ tend to zero when n tends to infinity.

Figure 3 contains the fractions of the good, schedulable and safe matrices for the case $m = 2$, $p = 0.4$, and $n = 1, 2, \ldots, 16$. In column $s(2,n,0.4)$ of Table 3 the computed limit is $5/9 \sim 0.555$.

Figure 4 contains the fractions of the good, schedulable and safe matrices for the case $m = 2$, $p = 0.35$, and $n = 1,\ 2,\ \ldots,\ 17$. For the column $s(2,n,0.35)$ of Table 4 the computed limit is $120/169 \sim 0.710$.

3.3 Matrices with three rows

If $m = 3$, then the possible ratios of the 1's and 0's are 3:0, 2:1, 1:2 or 0:3. We assign such random walk to the investigated matrix, in which the walking point jumps by two to left with the probability p^3 of the column containing three 1's; the point makes a step to left with the probability $3p^2q$; the position is preserved with the probability q^3 of the column containing only zeros.

Using the notation x_k introduced in the fourth proof of Lemma 5, we get the

n	$T(2,n)$	$g(2,n,0.4)$	$w(2,n,0.4)$	$s(2,n,0.4)$	$\frac{w(2,n,0.4)}{s(2,n,0.4)}$
1	4	0.8400	0.8400	0.8400	1
2	16	0.7056	0.7632	0.7632	1
3	64	0.5927	0.7171	0.7171	1
4	256	0.4979	0.6795	0.6862	0.9902
5	1024	0.4182	0.6487	0.6639	0.9771
6	4096	0.3513	0.6206	0.6470	0.9592
7	16384	0.2951	0.5957	0.6339	0.9397
8	65536	0.2479	0.5731	0.6234	0.9193
9	262144	0.2082	0.5524	0.6149	0.8984
10	1048576	0.1749	0.5332	0.6078	0.8773
11	4194304	0.1469	0.5155	0.6019	0.8565
12	16777216	0.1234	0.4988	0.5967	0.8359
13	67108864	0.1037	0.4832	0.5924	0.8157
14	268435456	0.0871	0.4685	0.5886	0.7960
15	1073741824	0.0731	0.4545	0.5854	0.7764
16	4294967296	0.0644	0.4412	0.5825	0.7574
17	169779869184	0.0516	0.4286	0.5800	0.7390

Figure 3: Rounded data belonging to the parameters $m = 2$ and $p = 0.4$.

following equations:

$$\begin{array}{lllllllll} x_0 & = & q^3x_1 & + & 3q^2px_0 & + & 3qp^2 & + & p^3, \\ x_1 & = & q^3x_2 & + & 3q^2px_1 & + & 3qp^2x_0 & + & p^3, \\ x_2 & = & q^3x_3 & + & 3q^2px_2 & + & 3qp^2x_1 & + & p^3x_0, \\ x_3 & = & q^3x_4 & + & 3q^2px_3 & + & 3qp^2x_2 & + & p^3x_1, \\ \ldots \end{array} \tag{5}$$

Let

$$G(z) = \sum_{i=0}^{\infty} x_n z^n$$

be the generator function of the sequence $x_0, x_1, x_2, \ldots$. Then multiplying the equations of the system (5) with the corresponding powers of z and summing up the received equations, we get:

$$G(z) = q^3 \frac{G(z) - x_0}{z} + 3q^2pG(z) + 3qp^2(1 + zG(z)) + p^3 \left(1 + z + z^2G(z)\right),$$

n	$T(2,n)$	$g(2,0.35)$	$w(2,0.35)$	$s(2,n,0.35)$	$\frac{w(2,0.35)}{s(2,0.35)}$
1	4	0.8775	0.8775	0.8775	1
2	16	0.7700	0.8218	0.8218	1
3	64	0.6757	0.7901	0.7901	1
4	256	0.5929	0.7645	0.7699	0.9930
5	1024	0.5203	0.7441	0.7561	0.9841
6	4096	0.4565	0.7255	0.7462	0.9723
7	16384	0.4006	0.7094	0.7389	0.9601
8	65536	0.3515	0.6949	0.7334	0.9475
9	262144	0.3085	0.6817	0.7291	0.9350
10	1048576	0.2707	0.6696	0.7258	0.9226
11	4194304	0.2375	0.6585	0.7231	0.9107
12	16777216	0.2084	0.6481	0.7210	0.8989
13	67108864	0.1839	0.6383	0.7192	0.8875
14	268435456	0.1605	0.6291	0.7178	0.8764
15	1073741824	0.1401	0.6204	0.7166	0.8658
16	4294967296	0.1236	0.6122	0.7156	0.8555

Figure 4: Rounded data belonging to the parameters $m = 2$ and $p = 0.35$.

from where $G(z)$ can be expressed as the fraction of two polynomials:

$$G(z) = \frac{P(z)}{Q(z)},$$

where

$$P(z) = q^3x_0 - 3qp^2z - p^3(z + z^2)$$

and

$$Q(z) = p^3z^3 + 3p^2qz^2 + 3pq^2z + q^3 - z.$$

The equation $Q(z) = 0$ can be transformed into the form

$$(q + pz)^3 = z,$$

from where the root $z_1 = 1$ follows immediately. Expressing x_0 from the equation $P(1) = 0$, we get:

$$x_0 = \frac{3p^2}{q^2} + \frac{2p^3}{q^3}, \tag{6}$$

n	$T(3,n)$	$g(3,n,0.5)$	$w(3,n,0.5)$	$s(3,n,0.5)$	$\frac{w(3,n,0.5)}{s(3,n,0.5)}$
1	8	0.5000	0.5000	0.5000	1.0000
2	64	0.2500	0.2969	0.2969	1.0000
3	512	0.1250	0.1914	0.1914	1.0000
4	4 096	0.0625	0.1282	0.1296	0.9892
5	32 768	0.0312	0.0880	0.0907	0.9702
6	262 144	0.0156	0.0612	0.0651	0.9401
7	2 097 152	0.0078	0.0429	0.0475	0.9032
8	16 777 216	0.0039	0.0303	0.0352	0.8594

Figure 5: Rounded data belonging to the parameters $m = 3$ and $p = 0.5$.

implying

$$s(3,p) = 1 - \frac{3p^2}{q^2} - \frac{2p^3}{q^3}, \tag{7}$$

The value of the function $1 - x_0 = x_0(p/q)$ is 1 at $p/q = 0$ and it is decreasing if $0 \le p/q \le 1/2$. With the multiplication by $q = (1-p)^3$ we get the equation

$$\frac{3p^2}{(1-p)^2} + \frac{2p^3}{(1-p)^3} = 1,$$

which – by algebraic manipulations – results the value $p = 1/3$, that is $s_{crit}(3) = 1/3$.

Figure 5 contains the fraction of the good, schedulable and safe matrices for the case $m = 3$, $p = 0.5$, and $n = 1, 2, \ldots, 8$.

In this table $g(3,n,0.5)$, $w(3,n,0.5)$, and $s(3,n,0.5)$ all have to tend to zero when n tends to infinity.

Figure 6 contains fraction of the good, schedulable and safe matrices for the case $m = 3$, $p = 0.25$, and $n = 1, 2, \ldots, 5$.

In this table $g(3,n,0.25)$ has to tend to zero, if n tends to infinity, but according to formula (7) $16/23 \sim 0.593$ is the computed limit for $s(3,n,0.25)$ when n tends to infinity.

We remark that the master thesis of Rudolf Szendrei [49] contains further simulation results.

n	$T(3,m)$	$g(3,n,0.25)$	$w(3,n,0.25)$	$s(3,n,0.25)$	$\frac{w(3,n,0.25)}{s(3,n,0.25)}$
1	8	0.8437	0.8437	0.8437	1.0000
2	64	0.7119	0.7712	0.7712	1.0000
3	512	0.6007	0.7286	0.7286	1.0000
4	4 096	0.5068	0.6981	0.7004	0.9967
5	32 768	0.4276	0.6748	0.6804	0.9917

Figure 6: Rounded data belonging to the parameters $m = 3$ and $p = 0.25$.

4 Main result

The analysis of the safe matrices of size $m \times n$ in the case of $m \geq 4$ is similar. If a column of matrix A contains at least $b \geq 3$ 1's, then the walking point jumps $(b-2)$ positions to left; if the column contains two 1's then the point makes a step to left; in the case of one 1 the point preserves its position and if the column contains only 0's, then the point makes a step to right. The corresponding probabilities are $\binom{m}{b}p^{b-2}q^{n-b+2}$, $\binom{m}{2}p^{m-2}q^2$, $\binom{m}{1}pq^{m-1}$ and $\binom{m}{0}q^m$,. So we get the following equations:

$$\begin{array}{lclclcl} x_0 & = & \binom{m}{0}q^m x_1 & + & \binom{m}{1}pq^{m-1}x_0 & + & \binom{m}{2}p^2q^{m-2} \\ & + & \binom{m}{3}p^3q^{m-3} & + & \dots & + & \binom{m}{m}p^m, \\ x_1 & = & \binom{m}{0}q^m x_2 & + & \binom{m}{1}pq^{m-1}x_1 & + & \binom{m}{2}p^2q^{m-2}x_0 \\ & + & \binom{m}{3}p^3q^{m-3} & + & \dots & + & \binom{m}{m}p^m, \\ x_2 & = & \binom{m}{0}q^m x_3 x_3 & + & \binom{m}{1}pq^{m-1}x_2 & + & \binom{m}{2}p^2q^{m-2}x_1 \\ & + & \binom{m}{3}p^3q^{m-3}x_0 & + & \dots & + & \binom{m}{m}p^m, \\ \dots & & & & & & \end{array} \tag{8}$$

Let

$$G(z) = \sum_{i=0}^{\infty} x_n z^n$$

be the generator function of the sequence $x_0, x_1, x_2, \dots$. Then multiplying the equations in (8) with the corresponding powers of z and summing up them, we get:

$$G(z) = \binom{m}{0}q^m \frac{G(z) - x_0}{z} + \binom{m}{1}pq^{m-1}G(z) + \binom{m}{2}p^2q^{m-2}(1 + zG(z))$$

$$+\binom{m}{3}p^3q^{m-3}(1+z+z^2G(z))+\dots+\binom{m}{m}p^m\left(1 + z + \dots + z^{m-2} + z^{m-1}G(z)\right),$$

from where one can express $G(z)$ as the fraction of two polynomials:

$$G(z) = \frac{P(z)}{Q(z)},$$

where

$$P(z) = \binom{m}{0} q^m x_0 - \sum_{i=2}^{m} \left(\binom{m}{i} p^i q^{m-i} \sum_{j=0}^{i-2} z^j \right).$$

If the denominator has a root x with $|x| \leq 1$, then the value of the nominator at x must be zero.
Reordering the equation $Q(z) = 0$ to the form

$$(q + pz)^m = 1$$

we get the root $z_1 = 1$. Division of the equation $P(1) = 0$ by q^m results the equation

$$x_0 = \sum_{i=2}^{m} \binom{m}{i} \left(\frac{p}{1-p} \right)^i (i-1).$$

The value of the function $x_0 = x_0(p)$ is zero at $p = 0$, and the function is increasing, if p is positive. From the equation $x_0 = 1$ we get $p = 1/m$.
Taking into account the results received above for cases $m = 2$ and $m = 3$, we received the following result.

Theorem 1 *If* $m \geq 2$ *and* $0 \leq p \leq m$, *then*

$$s_{crit}(m) = \frac{1}{m}. \tag{9}$$

and

$$s(m,p) = \begin{cases} 1 - \sum_{i=2}^{m} \binom{m}{i} \left(\frac{p}{1-p} \right)^i (i-1), & \text{if } 0 \leq p < \frac{1}{m}, \\ 0, & \text{if } \frac{1}{m} \leq p \leq 1. \end{cases} \tag{10}$$

Proof. a) The special case $m = 2$ is equivalent with Lemma 5.
b) The special case $m = 3$ is equivalent with formula (7).
c) For the case $m \geq 4$, see the proof before the theorem. ■

5 Summary

We determined the explicit form of the asymptotic density $s(m, p)$ for every number of the rows $m \geq 2$ and probability of 1's p. Furthermore we gave the exact values of the critical probabilities $s_{crit}(m)$ for $m \geq 2$. The value of $s_{crit}(2)$ is 0.5, which is characteristic to several other two dimensional critical probabilities. The further critical probabilities are decresse when m grows.
According to the simulation experiments the critical probabilities are near to the received upper bounds: Table 2 shows the data belonging to $m = 2$ and $p = 0.5$, Table 3 the data belonging to $m = 2$ and $p = 0.4$, Table 4 the data for $m = 2$ and $p = 0.35$, Table 5 the data belonging to $m = 3$ and $p = 0.5$, and Table 6 presents the data belonging to $m = 3$ and $p = 0.25$.
On the base of the data of the figures we suppose that the bound $p \geq 10^{-400}$ in [20] can be improved, but the analysis of the behaviour of fraction $w(m, p)/s(m, p)$ requires further work.
We are able to give a bit better lower and upper bounds of the investigated $w_r(m, n, p)$ probabilities, but the more precise characterisation of the critical probabilities requires more useful matrices than the good and safe ones.

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Generalized Möbius-type functions and special set of k-free numbers

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Abstract. In [3] Bege introduced the generalized Apostol's Möbius functions $\mu_{k,m}(n)$. In this paper we present new properties of these functions. By introducing the special set of k-free numbers, we have obtained some asymptotic formulas for the partial sums of these functions.

1 Introduction

Möbius function of order k, introduced by T. M. Apostol [1], is defined by the following formula:

$$\mu_k(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^{k+1} \mid n \text{ for some prime } p, \\ (-1)^r & \text{if } n = p_1^k \cdots p_r^k \prod_{i>r} p_i^{\alpha_i}, \quad \text{with } 0 \le \alpha_i < k, \\ 1 & \text{otherwise.} \end{cases}$$

The generalized function is denoted by $\mu_{k,m}(n)$, where $1 < k \le m$.
If $m = k$, $\mu_{k,k}(n)$ is defined to be $\mu_k(n)$, and if $m > k$ the function is defined as follows:

$$\mu_{k,m}(n) = \begin{cases} 1 & \text{if } n = 1, \\ 1 & \text{if } p^k \nmid n \text{ for each prime } p, \\ (-1)^r & \text{if } n = p_1^m \cdots p_r^m \prod_{i>r} p_i^{\alpha_i}, \quad \text{with } 0 \le \alpha_i < k, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

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In this paper we show some relations that hold among the functions $\mu_{k,m}(n)$. We introduce the new type of k-free integers and we make a connection between generalized Möbius function and the characteristic function $q^*_{k,m}(n)$ of these. We use these to derive an asymptotic formula for the summatory function of $q^*_{k,m}(n)$.

2 Basic lemmas

The generalization $\mu_{k,m}$, like Apostol's $\mu_k(n)$, is a multiplicative function of n, so it is determined by its values at the prime powers. We have

$$\mu_k(p^\alpha) = \begin{cases} 1 & \text{if} \quad 0 \le \alpha < k, \\ -1 & \text{if} \quad \alpha = k, \\ 0 & \text{if} \quad \alpha > k, \end{cases}$$

whereas

$$\mu_{k,m}(p^\alpha) = \begin{cases} 1 & \text{if} \quad 0 \le \alpha < k, \\ 0 & \text{if} \quad k \le \alpha < m, \\ -1 & \text{if} \quad \alpha = m, \\ 0 & \text{if} \quad \alpha > m. \end{cases} \tag{2}$$

In [1] Apostol obtained the asymptotic formula

$$\sum_{n \le x} \mu_k(n) = A_k x + O(x^{\frac{1}{k}} \log x), \tag{3}$$

where

$$A_k = \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right).$$

Later, Suryanarayana [5] showed that, on the assumption of the Riemann hypothesis, the error term in (3) can be improved to

$$O\left(x^{\frac{4k}{4k^2+1}} \omega(x)\right), \tag{4}$$

where

$$\omega(x) = \exp\{A \log x (\log\log x)^{-1}\}$$

for some positive constant k.
In 2001 A. Bege [3] proved the following asymptotic formulas.

Lemma 1 ([3], Theorem 3.1.) *For* $x \geq 3$ *and* $m > k \geq 2$*, we have*

$$\sum_{\substack{r \leq x \\ (r,n)=1}} \mu_{k,m}(r) = \frac{xn^2\,\alpha_{k,m}}{\zeta(k)\psi_k(n)\alpha_{k,m}(n)} + O\left(\theta(n)x^{\frac{1}{k}}\delta(x)\right) \tag{5}$$

uniformly in x*,* n *and* k*, where* $\theta(n)$ *is the number of square-free divisors of* n*,*

$$\alpha_{k,m} = \prod_p \left(1 - \frac{1}{p^{m-k+1} + p^{m-k+2} + \cdots + p^m}\right),$$

$$\alpha_{k,m}(n) = n\prod_{p|n} \left(1 - \frac{1}{p^{m-k+1} + p^{m-k+2} + \cdots + p^m}\right),$$

$$\psi_k(n) = n\prod_{p|n}\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{k-1}}\right),$$

and

$$\delta_k(x) = \ exp\,\{-A\ k^{-\frac{8}{5}}\log^{\frac{3}{5}} x\ (\log\log x)^{-\frac{1}{5}}\}, \quad A > 0.$$

Lemma 2 ([3], Theorem 3.2.) *If the Riemann hypothesis is true, then for* $x \geq 3$ *and* $m > k \geq 2$ *we have*

$$\sum_{\substack{r \leq x \\ (r,n)=1}} \mu_{k,m}(r) = \frac{xn^2\,\alpha_{k,m}}{\zeta(k)\psi_k(n)\alpha_{k,m}(n)} + O\left(\theta(n)x^{\frac{2}{2k+1}}\omega(x)\right) \tag{6}$$

uniformly in x*,* n *and* k*.*

Lemma 3 ([2]) *If* $s > 0$*,* $s \neq 1$*,* $x \geq 1$*, then*

$$\sum_{n \leq x} \frac{1}{n^s} = \zeta(s) - \frac{1}{(s-1)x^{s-1}} + O\left(\frac{1}{x^s}\right).$$

3 Generalized k-free numbers

Let Q_k denote the set of k-free numbers and let $q_k(n)$ to be the characteristic function of this set. Cohen [4] introduced the Q_k^* set, the set of positive integers n with the property that the multiplicity of each prime divisor of n is not a multiple of k. Let $q_k^*(n)$ be the characteristic function of these integers.

$$q_k^*(n) = \begin{cases} 1, & \text{if } n = 1 \\ 1, & \text{if } n = p_1^{\alpha_1} \dots p_k^{\alpha_k},\ \alpha_i \not\equiv 0 \pmod{k} \\ 0, & \text{otherwise.} \end{cases}$$

We introduce the following special set of integers

$$\begin{aligned} Q_{k,m} : \ = \ & \{n \mid n = n_1 \cdot n_2,\ (n_1, n_2) = 1,\ n_1 \in Q_k, \\ & n_2 = 1 \text{ or } n_2 = (p_1 \dots p_i)^m,\ p_i \in \mathbb{P}\}, \end{aligned}$$

with the characteristic function

$$q_{k,m}(n) = \begin{cases} 1, & \text{if } n \in Q_{k,m} \\ 0, & \text{if } n \notin Q_{k,m}. \end{cases}$$

The function $q_{k,m}(n)$ is multiplicative and

$$q_{k,m}(n) = |\mu_{k,m}(n)|. \tag{7}$$

We introduce the following set $Q_{k,m}^*$ which, in the generalization of Q_k^*. The integer n is in the set $Q_{k,m}^*$, $1 < k < m$ iff the power of each prime divisor of n divided by m has the remainder between 1 and $k-1$. The characteristic functions of these numbers is

$$q_{k,m}^*(n) = \begin{cases} 1, & \text{if } n = p_1^{\alpha_1} \dots p_k^{\alpha_k},\ \exists \ell :\ \ell m < \alpha_i < \ell m + k \\ 0, & \text{otherwise.} \end{cases}$$

If we write the generating functions for this functions, we have the following result.

Theorem 1 *If* $m \geq k$ *and the series converges absolutely, we have*

$$\sum_{n=1}^{\infty} \frac{\mu_{k,m}(n)}{n^s} = \zeta(s) \prod_p \left(1 - \frac{1}{p^{ks}} - \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}}\right), \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{q_{k,m}^*(n)}{n^s} = \zeta(s)\zeta(ms) \prod_p \left(1 - \frac{1}{p^{ks}} - \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}}\right), \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{q_{k,m}(n)}{n^s} = \zeta(s) \prod_p \left(1 - \frac{1}{p^{ks}} + \frac{1}{p^{ms}} - \frac{1}{p^{(m+1)s}}\right). \tag{10}$$

Proof. Because the function $\mu_{k,m}(n)$ is multiplicative, when the series converges absolutely ($s > 1$), we have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mu_{k,m}(n)}{n^s} &= \prod_p \left(1 + \frac{\mu_{k,m}(p)}{p^s} + \ldots + \frac{\mu_{k,m}(p^\alpha)}{p^{\alpha s}} + \ldots\right) = \\
&= \prod_p \left(1 + \frac{1}{p^s} + \ldots + \frac{1}{p^{(k-1)s}} - \frac{1}{p^{ms}}\right) = \\
&= \prod_p \frac{1}{1 - \frac{1}{p^s}} \prod_p \left(1 - \frac{1}{p^{ks}} - \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}}\right) = \\
&= \zeta(s) \prod_p \left(1 - \frac{1}{p^{ks}} - \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}}\right).
\end{aligned}$$

In a similar way, because $q^*_{k,m}(n)$ is multiplicative, we have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{q^*_{k,m}(n)}{n^s} &= \prod_p \left(1 + \frac{q^*_{k,m}(p)}{p^s} + \ldots + \frac{q^*_{k,m}(p^\alpha)}{p^{\alpha s}} + \ldots\right) = \\
&= \prod_p \left(1 + \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots + \frac{1}{p^{(k-1)s}}\right) + \right. \\
&+ \left. \left(\frac{1}{p^{(m+1)s}} + \frac{1}{p^{(m+2)s}} \ldots + \frac{1}{p^{(m+k-1)s}}\right) + \ldots\right) = \\
&= \prod_p \left(1 + \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots + \frac{1}{p^{(k-1)s}}\right)\left(1 + \frac{1}{p^{ms}} + \frac{1}{p^{2ms}} + \ldots\right)\right) \\
&= \prod_p \left(1 + \frac{\frac{1}{p^s} - \frac{1}{p^{ks}}}{1 - \frac{1}{p^s}} \frac{1}{1 - \frac{1}{p^{ms}}}\right) = \\
&= \zeta(s)\zeta(ms) \prod_p \left(1 - \frac{1}{p^{ks}} - \frac{1}{p^{ms}} + \frac{1}{p^{(m+1)s}}\right).
\end{aligned}$$

Because $q_{k,m}(n)$ is multiplicative and $q_{k,m}(n) = |\mu_{k,m}(n)|$, we have:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{q_{k,m}(n)}{n^s} &= \prod_p \left(1 + \frac{q_{k,m}(p)}{p^s} + \ldots + \frac{q_{k,m}(p^\alpha)}{p^{\alpha s}} + \ldots\right) = \\
&= \prod_p \left(1 + \frac{1}{p^s} + \ldots + \frac{1}{p^{(k-1)s}} + \frac{1}{p^{ms}}\right) = \\
&= \prod_p \frac{1}{1 - \frac{1}{p^s}} \prod_p \left(1 - \frac{1}{p^{ks}} + \frac{1}{p^{ms}} - \frac{1}{p^{(m+1)s}}\right) = \\
&= \zeta(s) \prod_p \left(1 - \frac{1}{p^{ks}} + \frac{1}{p^{ms}} - \frac{1}{p^{(m+1)s}}\right).
\end{aligned}$$

■

In the particular case when $m = k$, we have $\mu_{k,m}(n) = \mu_k(n)$, $q_{k,m}(n) = q_{k+1}(n)$ and

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s} &= \zeta(s) \prod_p \left(1 - \frac{2}{p^{ks}} + \frac{1}{p^{(k+1)s}}\right), \\
\sum_{n=1}^{\infty} \frac{q_{k+1}(n)}{n^s} &= \frac{\zeta(s)}{\zeta\big((k+1)s\big)}.
\end{aligned}$$

We have the following convolution type formulas.

Theorem 2 *If* $m \geq k$

$$q^*_{k,m}(n) = \sum_{d^m \delta = n} \mu_{k,m}(\delta), \tag{11}$$

$$\mu_{k,m}(n) = \sum_{d^m \delta = n} \mu(d) q^*_{k,m}(\delta). \tag{12}$$

Proof. Because $q_{k,m}(n)$ and $\mu_{k,m}(n)$ are multiplicative, it results that both sides of (11) are multiplicative functions. Hence it is enough if we verify the identity for $n = p^\alpha$, a prime power.
If $\alpha = \ell m + i$ and $0 < i < k$

$$\begin{aligned}
\sum_{d^m \delta = p^\alpha} \mu_{k,m}(\delta) &= \mu_{k,m}(p^{\ell m+i}) + \mu_{k,m}(p^{(\ell-1)m+i}) + \ldots + \mu_{k,m}(p^{m+i}) + \\
&+ \mu_{k,m}(p^i) = 1 = q_{k,m}(p^\alpha).
\end{aligned}$$

If $\alpha = \ell m + i$ and $k < i < m$, then

$$\begin{aligned}\sum_{d^m\delta=p^\alpha} \mu_{k,m}(\delta) &= \mu_{k,m}(p^{\ell m+i}) + \mu_{k,m}(p^{(\ell-1)m+i}) + \ldots + \mu_{k,m}(p^{m+i}) + \\ &+ \mu_{k,m}(p^i) = 0 = q_{k,m}(p^\alpha).\end{aligned}$$

If $\alpha = \ell m$

$$\begin{aligned}\sum_{d^m\delta=p^\alpha} \mu_{k,m}(\delta) &= \mu_{k,m}(p^{\ell m}) + \mu_{k,m}(p^{(\ell-1)m}) + \ldots + \mu_{k,m}(p^m) + \mu_{k,m}(1) = \\ &= -1 + 1 = 0 = q_{k,m}(p^\alpha).\end{aligned}$$

(12) results from the Möbius inversion formula.
■

4 Asymptotic formulas

Theorem 3 *For* $x \geq 3$ *and* $m > k \geq 2$, *we have*

$$\sum_{r \leq x} q^*_{k,m}(r) = \frac{x\alpha_{k,m}\zeta(m)}{\zeta(k))} + O\left(x^{\frac{1}{k}}\delta(x)\right) \tag{13}$$

uniformly in x, n *and* k, *where*

$$\alpha_{k,m} = \prod_p \left(1 - \frac{1}{p^{m-k+1} + p^{m-k+2} + \cdots + p^m}\right)$$

$$\delta(x) = \mathit{exp}\,\{-A\ \log^{\frac{3}{5}} x\ (\log\log x)^{-\frac{1}{5}}\},$$

for some absolute constant $A > 0$.

Proof. Based on (11) and (5) with $n = 1$, we have

$$\sum_{r \leq x} q^*_{k,m}(n) = \sum_{\delta d^m \leq x} \mu_{k,m}(\delta) = \sum_{d \leq x^{\frac{1}{m}}} \sum_{\delta \leq \frac{x}{d^m}} \mu_{k,m}(\delta) =$$

$$= \sum_{d \leq x^{\frac{1}{m}}} \left\{ \frac{\left(\frac{x}{d^m}\right)\alpha_{k,m}}{\zeta(k)} + O\left(\frac{x^{\frac{1}{k}}}{d^{\frac{m}{k}}}\delta\left(\frac{x}{d^m}\right)\right)\right\} =$$

$$= \frac{x\alpha_{k,m}}{\zeta(k)} \sum_{d \le x^{\frac{1}{m}}} \frac{1}{d^m} + O\left(\delta(x)x^{\epsilon}x^{\frac{1}{k}-\epsilon} \sum_{d \le x^{\frac{1}{m}}} \frac{1}{d^{\frac{m}{k}-\epsilon m}}\right).$$

Now we use (3), and the fact that $\delta(x)x^{\epsilon}$ is increasing for all $\epsilon > 0$, we choose $\epsilon > 0$, so that $\frac{m}{k} - \epsilon m > 1 + \epsilon'$ and we obtain (13). ■

Applying the method used to prove Theorem 1, and making use of Lemma 2, we get

Theorem 4 *If the Riemann hypothesis is true, then for* $x \ge 3$ *and* $m > k \ge 2$ *we have*

$$\sum_{r \le x} q^*_{k,m}(r) = \frac{x\alpha_{k,m}\zeta(m)}{\zeta(k)} + O\left(x^{\frac{2}{2k+1}}\omega(x)\right) \tag{14}$$

uniformly in x, n *and* k.

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Existence and data dependence for multivalued weakly Ćirić-contractive operators

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Abstract. In this paper we define the concept of weakly Ćirić-contractive operator and give a fixed point result for this type of operators. Then we study the data dependence for the fixed point set.

1 Introduction

Let (X, d) be a metric space. A singlevalued operator T from X into itself is called contractive if there exists a real number $r \in [0, 1)$ such that $d(T(x), T(y)) \leq rd(x, y)$ for every $x, y \in X$. It is well known that if X is a complete metric space, then a contractive operator from x into itself has a unique fixed point in X.

In 1996, Japanese mathematicians O. Kada, T. Suzuki and W. Takahashi introduced the w-distance (see [4]) and discussed some properties of this new distance. Later, T. Suzuki and W. Takahashi, starting by the definition above, gave some fixed points result for a new class of operators, weakly contractive operators (see [8]).

The purpose of this paper is to give a fixed point theorem for a new class of operators, namely the so-called weakly Ćirić-contractive operators. Then, we present a data dependence result for the fixed point set.

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2 Preliminaries

Let (X, d) be a complete metric space. We will use the following notations:
$P(X)$ - *the set of all nonempty subsets of X*;
$\mathcal{P}(X) = P(X) \bigcup \emptyset$
$P_{cl}(X)$ - *the set of all nonempty closed subsets of X*;
$P_b(X)$ - *the set of all nonempty bounded subsets of X*;
$P_{b,cl}(X)$ - *the set of all nonempty bounded and closed, subsets of X*;
For two subsets $A, B \in P_b(X)$, we recall the following functionals:
$D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, D(Z, Y) = \inf\{d(x, y) : x \in Z \ , y \in Y\}$, $Z \subset X$ – *the gap functional.*
$\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \delta(A, B) := \sup\{d(a, b) | x \in A, b \in B\}$ – *the diameter functional*;
$\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \rho(A, B) := \sup\{D(a, B) | a \in A\}$ – *the excess functional*;
$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, H(A, B) := \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}$ – *the Pompeiu-Hausdorff functional*;
$Fix\, F := \{x \in X \mid x \in F(x)\}$ – *the set of the fixed points of* F;

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see [4]) as follows:
Let (X,d) be a metric space, $w : X \times X \to [0, \infty)$ is called w-distance on X if the following axioms are satisfied :

1. $w(x, z) \leq w(x, y) + w(y, z)$, for any $x, y, z \in X$;

2. for any $x \in X : w(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;

3. for any $\varepsilon > 0$, there exists $\delta > 0$ such that $w(z, x) \leq \delta$ and $w(z, y) \leq \delta$ implies $d(x, y) \leq \varepsilon$.

Let us give some examples of w-distances (see [4]).

Example 1 *Let* (X, d) *be a metric space . Then the metric "d" is a w-distance on X.*

Example 2 *Let X be a normed liniar space with norm* $\|\cdot\|$*. Then the function* $w : X \times X \to [0, \infty)$ *defined by* $w(x, y) = \|x\| + \|y\|$ *for every* $x, y \in X$ *is a w-distance.*

Example 3 *Let (X,d) be a metric space and let* $g : X \to X$ *a continuous mapping. Then the function* $w : X \times Y \to [0, \infty)$ *defined by:*

$$w(x, y) = max\{d(g(x), y), d(g(x), g(y))\}$$

for every $x, y \in X$ *is a w-distance.*

For the proof of the main results we need the following crucial result for w-distance (see [8]).

Lemma 1 *Let* (X, d) *be a metric space, and let* w *be a* w*-distance on* X. *Let* (x_n) *and* (y_n) *be two sequences in* X, *let* (α_n), (β_n) *be sequences in* $[0, +\infty[$ *converging to zero and let* $x, y, z \in X$. *Then the following holds:*

1. *If* $w(x_n, y) \leq \alpha_n$ *and* $w(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* $y = z$.
2. *If* $w(x_n, y_n) \leq \alpha_n$ *and* $w(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$, *then* (y_n) *converges to* z.
3. *If* $w(x_n, x_m) \leq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* (x_n) *is a Cauchy sequence.*
4. *If* $w(y, x_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* (x_n) *is a Cauchy sequence.*

3 Existence of fixed points for multivalued weakly Ćirić-contractive operators

At the beginning of this section let us define the notion of multivalued weakly Ćirić-contractive operators.

Definition 1 *Let* (X, d) *be a metric space and* $T : X \to P(X)$ *a multivalued operator. Then* T *is called weakly Ćirić-contractive if there exists a* w*-distance on* X *such that for every* $x, y \in X$ *and* $u \in T(x)$ *there is* $v \in T(y)$ *with* $w(u, v) \leq q$ *max*$\{w(x, y), D_w(x, T(x), D_w(y, T(y)), \frac{1}{2}D_w(x, T(y))\}$,
for every $q \in [0, 1)$.

Let (X, d) be a metric space, w be a w-distance on X $x_0 \in X$ and $r > 0$. Let us define:

$B_w(x_0; r) := \{x \in X | w(x_0, x) < r\}$ the open ball centered at x_0 with radius r with respect to w;

$\widetilde{B_w}(x_0; r) := \{x \in X | w(x_0, x) \leq r\}$ the closed ball centered at x_0 with radius r with respect to w;

$\widetilde{B_w}^d(x_0; r)$- the closure in (X, d) of the set $B_w(x_0; r)$.

One of the main results is the following fixed point theorem for weakly Ćirić-contractive operators.

Theorem 1 *Let* (X, d) *be a complete metric space,* $x_0 \in X$*,* $r > 0$ *and* $T : \widetilde{B_w}(x_0; r) \to P_{cl}(X)$ *a multivalued operator such that:*

(i) T *is weakly Ćirić-contractive operator;*

(ii) $D_w(x_0, T(x_0)) \leq (1-q)r$.

Then there exists $x^* \in X$ *such that* $x^* \in T(x^*)$.

Proof. Since $D_w(x_0, T(x_0)) \leq (1-q)r$, then for every $x_0 \in X$ there exists $x_1 \in T(x_0)$ such that $D_w(x_0, T(x_0)) \leq w(x_0, x_1) \leq (1-q)r < r$.
Hence $x_1 \in \widetilde{B_w}(x_0; r)$.
For $x_1 \in \widetilde{B_w}(x_0; r)$, there exists $x_2 \in T(x_1)$ such that:

i. $w(x_1, x_2) \leq qw(x_0, x_1)$

ii. $w(x_1, x_2) \leq qD_w(x_0, T(x_0)) \leq qw(x_0, x_1)$

iii. $w(x_1, x_2) \leq qD_w(x_1, T(x_1)) \leq qw(x_1, x_2)$

iv. $w(x_1, x_2) \leq \frac{q}{2}D_w(x_0, T(x_1)) \leq \frac{q}{2}w(x_0, x_2)$

$w(x_1, x_2) \leq \frac{q}{2}[w(x_0, x_1) + w(x_1, x_2)]$

$(1 - \frac{q}{2})w(x_1, x_2) \leq \frac{q}{2}w(x_0, x_1)$

$w(x_1, x_2) \leq \frac{q}{2-q}w(x_0, x_1)$.

Then $w(x_1, x_2) \leq \max\{q, \frac{q}{2-q}\}w(x_0, x_1)$
Since $q > \frac{q}{2-q}$ for every $q \in [0, 1)$, then $w(x_1, x_2) \leq qw(x_0, x_1) \leq q(1-q)r$.
Then $w(x_0, x_2) \leq w(x_0, x_1) + w(x_1, x_2) < (1-q)r + q(1-q)r = (1-q^2)r < r$.
Hence $x_2 \in \widetilde{B_w}(x_0; r)$.
For $x_1 \in \widetilde{B_w}(x_0; r)$ and $x_2 \in T(x_1)$, there exists $x_3 \in T(x_2)$ such that

i. $w(x_2, x_3) \leq qw(x_1, x_2)$

ii. $w(x_2, x_3) \leq qD_w(x_1, T(x_1)) \leq qw(x_1, x_2)$

iii. $w(x_2, x_3) \leq qD_w(x_2, T(x_2)) \leq qw(x_2, x_3)$

iv. $w(x_2, x_3) \leq \frac{q}{2}D_w(x_1, T(x_2)) \leq \frac{q}{2}w(x_1, x_3)$

$w(x_2, x_3) \leq \frac{q}{2}[w(x_1, x_2) + w(x_2, x_3)]$

$(1 - \frac{q}{2})w(x_2, x_3) \leq \frac{q}{2}w(x_1, x_2)$

$w(x_2, x_3) \leq \frac{q}{2-q}w(x_1, x_2)$.

Then $w(x_2, x_3) \leq \max\{q, \frac{q}{2-q}\}w(x_1, x_2)$.
Since $q > \frac{q}{2-q}$ for every $q \in [0, 1)$, then $w(x_2, x_3) \leq qw(x_1, x_2) \leq q^2(x_0, x_1) \leq q^2(1-q)r$.

Then $w(x_0, x_3) \leq w(x_0, x_2) + w(x_2, x_3) \leq (1-q^2)r + q^2(1-q)r =$
$= (1-q)(1+q+q^2)r = (1-q^3)r < r$. Hence $x_3 \in \widetilde{B_w}(x_0; r)$.
By this procedure we get a sequence $(x_n)_{n\in\mathbb{N}} \in X$ of successive applications for T starting from arbitrary $x_0 \in X$ and $x_1 \in T(x_0)$, such that

(1) $x_{n+1} \in T(x_n)$, for every $n \in \mathbb{N}$;

(2) $w(x_n, x_{n+1}) \leq q^n w(x_0, x_1) \leq q^n(1-q)r$, for every $n \in \mathbb{N}$.

For every $m, n \in \mathbb{N}$, with $m > n$, we have

$$w(x_n, x_m) \leq w(x_n, x_{n+1}) + w(x_{n+1}, x_{n+2}) + ... + w(x_{m-1}, x_m) \leq$$
$$\leq q^n w(x_0, x_1) + q^{n+1} w(x_0, x_1) + ... + q^{m-1} w(x_0, x_1) \leq$$
$$\leq \frac{q^n}{1-q} w(x_0, x_1) \leq q^n r.$$

By Lemma 1(3) we have that the sequence $(x_n)_{n\in\mathbb{N}} \in \widetilde{B_w}(x_0; r)$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, then there exists $x^* \in \widetilde{B^d_w}(x_0; r)$ such that $x_n \xrightarrow{d} x^*$.
Fix $n \in \mathbb{N}$. Since $(x_m)_{m\in\mathbb{N}}$ converge to x^* and $w(x_n, \cdot)$ is lower semicontinuous, we have

$$w(x_n, x^*) \leq \lim_{m\to\infty} \inf w(x_n, x_m) \leq \frac{q^n}{1-q} w(x_0, x_1) \leq q^n r.$$

For $x^* \in \widetilde{B^d_w}(x_0; r)$ and $x_n \in T(x_{n-1})$, there exists $u_n \in T(x^*)$ such that

i. $w(x_n, u_n) \leq q w(x_{n-1}, x^*) \leq \frac{q^n}{1-q} w(x_0, x_1)$

ii. $w(x_n, u_n) \leq q D_w(x_{n-1}, T(x_{n-1})) \leq q w(x_{n-1}, x_n) \leq ... \leq q^n w(x_0, x_1)$

iii. $w(x_n, u_n) \leq q D_w(x^*, T(x^*)) \leq q w(x^*, u_n) \leq \frac{q^n}{1-q} w(x_0, x_1)$

iv. $w(x_n, u_n) \leq \frac{q}{2} D_w(x_{n-1}, T(x^*)) \leq \frac{q}{2} w(x_{n-1}, u_n) \leq \frac{q}{2} \cdot \frac{q^{n-1}}{1-q} w(x_0, x_1)$
$= \frac{q^n}{2(1-q)} w(x_0, x_1)$.

Then $w(x_n, u_n) \leq max\{\frac{q^n}{1-q}, q^n, \frac{q^n}{2(1-q)}\} w(x_0, x_1)$.
Since for $q \in [0, 1)$ we have true $\frac{q^n}{1-q} > q^n$ and $\frac{q^n}{1-q} > \frac{q^n}{2(1-q)}$ we get that $w(x_n, u_n) \leq \frac{q^n}{1-q} w(x_0, x_1) \leq q^n r$.
So, for every $n \in \mathbb{N}$ we have:
$w(x_n, x^*) \leq q^n r$
$w(x_n, u_n) \leq q^n r$.

Then, from 1(2), we obtain that $u_n \xrightarrow{d} x^*$. As $u_n \in T(x^*)$ and using the closure of T result that $x^* \in T(x^*)$. ■

A global result for previous theorem is the following fixed point result for multivalued weakly Ćirić-contractive operators.

Theorem 2 *Let* (X,d) *be a complete metric space,* $T : X \to P_{cl}(X)$ *a multivalued weakly Ćirić-contractive operator. Then there exists* $x^* \in X$ *such that* $x^* \in T(x^*)$.

4 Data dependence for weakly Ćirić-contractive multivalued operators

The main result of this section is the following data dependence theorem with respect to the above global theorem 2.

Theorem 3 *Let* (X,d) *be a complete metric space,* $T_1, T_2 : X \to P_{cl}(X)$ *be two multivalued weakly Ćirić-contractive operators with* $q_i \in [0,1)$ *with* $i = \{1,2\}$. *Then the following are true:*

1. $FixT_1 \neq \emptyset \neq FixT_2$;
2. *We suppose that there exists* $\eta > 0$ *such that for every* $u \in T_1(x)$ *there exists* $v \in T_2(x)$ *such that* $w(u,v) \leq \eta$, *(respectively for every* $v \in T_2(x)$ *there exists* $u \in T_1(x)$ *such that* $w(v,u) \leq \eta$*).*

Then for every $u^* \in FixT_1$, *there exists* $v^* \in FixT_2$ *such that*

$$w(u^*, v^*) \leq \frac{\eta}{1-q}, \text{ where } q = q_i \text{ for } i = \{1,2\};$$

(respectively for every $v^* \in FixT_2$ *there exists* $u^* \in FixT_1$ *such that*

$$w(v^*, u^*) \leq \frac{\eta}{1-q}, \text{ where } q = q_i \text{ for } i = \{1,2\}).$$

Proof. From the above theorem we have that $FixT_1 \neq \emptyset \neq FixT_2$.
Let $u_0 \in FixT_1$, then $u_0 \in T_1(u_0)$. Using the hypothesis (2) we have that there exists $u_1 \in T_2(u_0)$ such that $w(u_0, u_1) \leq \eta$.
Since T_1, T_2 are weakly Ćririć-contractive with $q_i \in [0,1)$ and $i = \{1,2\}$ we have that for every $u_0, u_1 \in X$ with $u_1 \in T_2(u_0)$ there exists $u_2 \in T_2(u_1)$ such that

i. $w(u_1,u_2) \leq qw(u_0,u_1)$
ii. $w(u_1,u_2) \leq D_w(u_0, T_2(u_0)) \leq qw(u_0,u_1)$
iii. $w(u_1,u_2) \leq D_w(u_1, T_2(u_1)) \leq qw(u_1,u_2)$
iv. $w(u_1,u_2) \leq \frac{q}{2} D_w(u_0, T_2(u_1)) \leq \frac{q}{2} w(u_0,u_2)$
$w(u_1,u_2) \leq \frac{q}{2}[w(u_0,u_1) + w(u_1,u_2)]$
$w(u_1,u_2) \leq \frac{q}{2-q} w(u_0,u_1)$.

Then $w(u_1,u_2) \le \max\{q, \frac{q}{2-q}\} w(u_0,u_1)$.
Since for $q \in [0,1)$ we have true $q > \frac{q}{2-q}$, then we have

$$w(u_1,u_2) \le q w(u_0,u_1).$$

For $u_1 \in X$ and $u_2 \in T_2(u_1)$, there exists $u_3 \in T_2(u_2)$ such that

i. $w(u_2,u_3) \le q w(u_1,u_2)$

ii. $w(u_2,u_3) \le D_w(u_1,T_2(u_1)) \le q w(u_1,u_2)$

iii. $w(u_2,u_3) \le D_w(u_2,T_2(u_2)) \le q w(u_2,u_3)$

iv. $w(u_2,u_3) \le \frac{q}{2} D_w(u_1,T_2(u_2)) \le \frac{q}{2} w(u_1,u_3)$
$w(u_2,u_3) \le \frac{q}{2}[w(u_1,u_2) + w(u_2,u_3)]$
$w(u_2,u_3) \le \frac{q}{2-q} w(u_1,u_2)$

Then $w(u_2,u_3) \le \max\{q, \frac{q}{2-q}\} w(u_1,u_2)$.
Since for $q \in [0,1)$ we have true $q > \frac{q}{2-q}$, then we have

$$w(u_2,u_3) \le q w(u_1,u_2) \le q^2 w(u_0,u_1).$$

By induction we obtain a sequence $(u_n)_{n\in\mathbb{N}} \in X$ such that

(1) $u_{n+1} \in T_2(u_n)$, for every $n \in \mathbb{N}$;

(2) $w(u_n,u_{n+1}) \le q^n w(u_0,u_1)$.

For $n, m \in \mathbb{N}$, with $m > n$ we have the inequality

$$\begin{gathered} w(u_n,u_m) \le w(u_n,u_{n+1}) + w(u_{n+1},u_{n+2}) + \cdots + w(u_{m-1},u_m) \le \\ < q^n w(u_0,u_1) + q^{n+1} w(u_0,u_1) + \cdots + q^{m-1} w(u_0,u_1) \le \\ \le \frac{q^n}{1-q} w(u_0,u_1) \end{gathered}$$

By Lemma 1(3) we have that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X,d) is a complete metric space, we have that there exists $v^* \in X$ such that $u_n \xrightarrow{d} v^*$.
By the lower semicontinuity of $w(x,\cdot): X \to [0,\infty)$ we have

$$w(u_n,v^*) \le \liminf_{m\to\infty} w(u_n,u_m) \le \frac{q^n}{1-q} w(u_0,u_1).$$

For $u_{n-1}, v^* \in X$ and $u_n \in T_2(u_{n-1})$ there exists $z_n \in T_2(v^*)$ such that we have

i. $w(u_n,z_n) \le q w(u_{n-1},v^*) \le \frac{q^n}{1-q} w(u_0,u_1)$

ii. $w(u_n,z_n) \le q D_w(u_{n-1},T_2(u_{n-1})) \le q w(u_{n-1},u_n) \le \ldots \le q^n w(u_0,u_1)$

iii. $w(u_n,z_n) \le q D_w(v^*,T_2(v^*)) \le w(v^*,z_n) \le \frac{q^n}{1-q} w(u_0,u_1)$

iv. $w(u_n,z_n) \le \frac{q}{2} D_w(u_{n-1},T_2(v^*)) \le \frac{q}{2} w(u_{n-1},z_n) \le \frac{q^n}{2(1-q)} w(u_0,u_1)$.

Then $w(u_n, z_n) \le \max\{\frac{q^n}{1-q}, q^n, \frac{q^n}{2(1-q)}\} w(u_0, u_1)$.
Since $\frac{q^n}{1-q} > q^n$ and $\frac{q^n}{1-q} > \frac{q^n}{2(1-q)}$ for every $q \in [0, 1)$ we have that

$$w(u_n, z_n) \le \frac{q^n}{1-q} w(u_0, u_1).$$

So, we have:
$w(u_n, v^*) \le \frac{q^n}{1-q} w(u_0, u_1)$
$w(u_n, z_n) \le \frac{q^n}{1-q} w(u_0, u_1)$.
Applying Lemma 1(2), from the above relations we have that $z_n \xrightarrow{d} v^*$.
Then, we know that $z_n \in T_2(v^*)$ and $z_n \xrightarrow{d} v^*$. In this case, by the closure of T_2, it results that $v^* \in T_2(v^*)$. Then, by $w(u_n, v^*) \le \frac{q^n}{1-q} w(u_0, u_1)$, with $n \in \mathbb{N}$, for $n = 0$, we obtain

$$w(u_0, v^*) \le \frac{1}{1-q} w(u_0, u_1) \le \frac{\eta}{1-q},$$

which completes the proof. ∎

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Experimental results on probable primality

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Abstract. In this paper we present experimental results on probable primality. More than four billion of randomly chosen integers having 256, 512 and 1024 bits were tested. We realized more experiments than Rivest did in 1991, and can confirm his observation: Miller–Rabin test does not ameliorate the small prime divisors test followed by Fermat test with the only base 2.

1 Introduction

Prime integers play a fundamental role in mathematics. They have always been a source of interest and fascination. Since the appearance of public key cryptography at the end of 1970's (see, for instance [1,9]), they have become more and more useful. RSA, Rabin cryptosystem, elliptic curve method, discrete logarithm problem and many digital signature protocols are completely based on large prime integers. By large, we mean that the considered numbers have at least 256 binary digits, or around 77 decimal digits.

It is well-known (see, for instance, [5]) that the running time of algorithms for constructing cryptosystem keys is dominated by the running time for generating prime integers. Finding rapid procedures for this latter task has, therefore, great importance.

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It is also well known that there is no efficient and practical deterministic algorithm for quickly producing prime integers. That is why we only look for non deterministic algorithms which give us integers that are primes with a strong probability. There is a new look at the primality concept: an integer is taken as prime, not because it is really prime in an exact mathematical sense, but instead of that, it is prime because one thinks that nobody can factorize it. Recently, an integer is called industrial-grade prime (the term is due to H. Cohen) if its primality has not been proven, but it has undergone probable prime test(s).

The purpose of this work is to confirm what was concluded by Rivest in [10], as we made more experiments than Rivest. By analyzing experimental results on 4.13 billion randomly selected large integers, we show that a particular probabilistic algorithm for generating large prime integers based on three tests is likely equivalent to a similar algorithm, but based on only two tests. More precisely, our experimental results tend to indicate that using only two tests, division by small prime divisors followed by the Fermat test (see, for example [3,12]) produces the same results as using three tests: division by small primes, then the application of Fermat test, followed by Rabin-Miller test (see, for example [7,8]) with eight random bases. The Miller–Rabin test seems to be a waste of time when added as the third one to the first two aforesaid tests.

The paper is organized as follows. In section 2 we review the three tests composing the main algorithm and specify their formal parameters. In section 3 we briefly recall Rivest experimental results, and then we describe our own experiments, present and analyze the computing results. Section 4 contains conclusion and suggestion on possible forthcoming work.

In the sequel, we will adopt classical notation. In particular, $\mathbb{N}$ is the set of non-negative integers. Let $a, b, c \in \mathbb{N}$. Then $\gcd(a, b)$ denotes the great common divisor of a and b, while the remainder of a, when divided by b is denoted by $a \bmod b$. We write $a = b\ [c]$ if c divides the difference $a - b$. As usual, let $\pi(x)$ denote the number of primes less than or equal to the real number x. Finally, the bit length l_b of a positive integer $n = \sum_{i=0}^{k-1} 2^i a_i$ is $l_b = k$, where $a_{k-1} = 1$, and $a_i \in \{0, 1\}$ if $i = 0, \dots k - 2$.

2 Three known tests

In this section we review three known tests and specify their formal parameters. Let $n > 1$ be an odd integer for which we want to test primality.

2.1 Small division test T_1

This test is the trial division by small divisors, namely by primes that are less than a fixed bound B. We divide n by all primes less than B. If we find one divisor, then n is composite, otherwise n is a candidate to be prime. Eratostenes sieve is applied to generate all primes between 2 and the bound B.

2.2 Fermat test T_2

Here we use a test based on the little Fermat theorem (see, for example [3,12]). If an integer a satisfies $\mathrm{gcd}(a, n) = 1$, we calculate $a^{n-1} \bmod n$ and compare it to 1. If $a^{n-1} \neq 1$ $[n]$, then n is composite, otherwise n is a candidate to be prime.

2.3 Miller–Rabin test T_3

Miller–Rabin test (see, for example [7,8]) is more efficient than Solovay and Strassen probabilistic test (see, for instance [11,6]). Since n is odd, we can uniquely find two positive integers r and s such that $n - 1 = 2^r s$. Let a be any integer such that $\mathrm{gcd}(a, n) = 1$. If $a^s \neq 1$ $[n]$ and $\forall\ j \in \{0, 1, \ldots, r - 1\}$: $a^{2^j s} \neq -1$ $[n]$, then n is composite.

If n passes all three tests, then it is probably a prime integer. In other words, we believe in its primality.

3 Results of our experiments

In this section, first we recall Rivest experimental tests [10], and then describe our own experiments providing the main results.

3.1 Rivest experiences

In 1991, Rivest examined 718 million randomly chosen 256-bit integers. Firstly he tested them by small divisors with the upper bound $B = 10^4$. $43,741,404$ passed this first test. Of those, $4,058,000$ passed Fermat test with the base 2. Of those, no one was eliminated by Miller–Rabin test with 8 random bases.

3.2 Our own experiments

Three kinds of experiments were realized with Maple software, versions 9.5 and 10, depending on the bit length: 256, 512 or 1024. We used ordinary personal computers working with Pentium IV 3.4 GHz processor and 248 MB of RAM. The main parameters were taken as in Rivest experiments:

- the upper limit of small primes is $B = 10^4$,
- the Fermat base is $b = 2$,
- the eight bases in the Miller–Rabin test are randomly chosen from the set $\{3, 4, \dots, n-2\}$.

In our case, we used blocks of integers and the number of randomly selected integers in each block was mainly between 5 and 10 million. Sometimes we used smaller or larger blocks as well.

We summarize the results in the next table where numbers N, N_1, N_2 and N_3 are defined as follows.

- N is the number of the randomly selected integers,
- N_1 denotes the number of integers which passed the first test T_1,
- N_2 is the number of integers which passed both T_1 and T_2,
- N_3 shows the number of integers which passed all the three tests.

Moreover, for $i = 1$, 2, and 3 let $R_i = 100\dfrac{N_i}{N}$.

We began to test more than one billion of integers whose bit lenght is 256, more than what was tested by Rivest. We found that the time required by PCs to run every range of 5 million of integers is around 40 minutes and around 80 minutes for every range of 10 million. For data see Table 1.

bit length	N	N_1	N_2	N_3
256	1.13×10^9	68 781 054	6 381 145	6 381 145
512	10^9	60 875 654	2 820 804	2 820 804
512	10^9	60 893 522	2 822 109	2 822 109
1024	10^9	60 876 414	1 408 923	1 408 923

bit length	N	$R_1(\%)$	$R_2(\%)$	$R_3(\%)$
256	1.13×10^9	6.0868189	0.5647031	0.5647031
512	10^9	6.0875654	0.2820804	0.2820804
512	10^9	6.0893522	0.2822109	0.2822109
1024	10^9	6.0876414	0.1408923	0.1408923

Table 1.

Then we tested two times one billion integers with bit lenght 512. And, finally, we tested one billion of integers with $l_b = 1024$. For comparison, see again Table 1.

We emphasized that, in the three kinds of experiment, we found $N_3 = N_2$ implying $R_3 = R_2$.

4 Conclusions

I. In this work, we realized new experiments on large integers in order to determine their primality. We tested more than four billion integers having 256, 512 and 1024 bits. They were all selected randomly. The main fact is that, from those which passed the small divisor test and the Fermat test, no one was blocked by the Miller–Rabin test. This result, based on more experiments, confirms what was already observed by Rivest. With the parameters mentioned above, the Miller–Rabin test does not improve the probabilistic algorithm based on the two first tests. Hence it seems that the Miller–Rabin test is unnecessary as the third stage of the three tests.

On the other hand, for future work, we suggest to replace the Miller–Rabin test by an alternative one and to verify experimentally if this modification brings any amelioration or not.

II. It seems that the upper bound on small primes is unnecessarily high. Both Rivest and us first used $B = 10^4$, but now we suggest $B = 300$ or $B = 3000$ instead. Why? Because with $B = 10^4$ we filtered 93.91% of the attendants independently from the bit length (supposing that l_b is large enough). If we have all the primes $p_1 = 2,\ p_2 = 3,\ \ldots,\ p_m \leq B$, then in the first step of the 3 tests they exclude expectedly

$$1 - \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_m}\right) \tag{1}$$

part of the attendants. This formula gives 50% for $B = 2$, approximately 66.667% for $B = 3$, and so on, and provides $93,911\%$ for $B = 10^4$ (this was the preferred case). But for $B = 300$ we already have $90,245\%$, and going further, for $B = 3000$ we obtain $93,003\%$, which almost coincides with what we had for $B = 10^4$ before.

The following table shows the comparison of running experiences of different values B if $l_b = 256$. One can observe, that if we decrease B, then the number of random integers which failed the small prime divisor test also decreases, but the final ratio of the integers survived all the three tests is approximately constant. Furthermore, the values in the third column of Table 2 coincide with the values forecasted by (1).

B	Size of sample	Failed T_1	Passed $T_1 \wedge T_2 \wedge T_3$
10000	1.13 billion	93.913181	0.564703
3000	100 million	93.005612	0.564496
300	100 million	90.251749	0.564111

Table 2.

III. In the experiment we randomly chose a huge number of integers to classify them by three consecutive primality tests. Therefore, it is natural to compare the number of integers passing through all three tests (the number of industrial-grade primes) and the expected value of primes. Now we recall the thesis of Dusart [2], providing good approximations of the function $\pi(x)$.

Theorem 1 *(Dusart, [2], p.36.) If* $x \geq 1.332 \cdot 10^{10}$*, then*

$$\frac{x}{\ln x}\left(1+\frac{1}{\ln x}+\frac{1.8}{\ln^2 x}\right) \leq \pi(x) \leq \frac{x}{\ln x}\left(1+\frac{1.0992}{\ln x}\right).$$

Let π_n and $d_n = \frac{\pi_n}{2^{n-1}}$ denote the number of primes and the density of the primes in the interval $I_n = [2^{n-1}; 2^n - 1]$, respectively. By Theorem 1, we obtain

$$0.0056\,424 \leq d_{256} = \frac{\pi_{256}}{2^{255}} \leq 0.0056\,509\,,$$
$$0.0028\,194 \leq d_{512} = \frac{\pi_{512}}{2^{511}} \leq 0.0028\,217\,,$$
$$0.001409\,299 \leq d_{1024} = \frac{\pi_{1024}}{2^{1023}} \leq 0.001409\,875\,.$$

Note, that in the experiment we investigated 1.13 and 2 and 1 billion random integers from the interval I_{256}, I_{512} and I_{1024}, respectively. Hence, with the given cardinality of the samples, the expected values E_{256}, E_{512} and E_{1024} of primes, by Dusart's theorem, satisfy the inequalities

$$6375919 \leq E_{256} \leq 6385530\,,$$
$$5638897 \leq E_{512} \leq 5643385\,,$$
$$1409299 \leq E_{1024} \leq 1409874\,.$$

The following table recalls the statistics about the candidates for primality (the integers survived the three tests).

	256	512	1024
cardinality of sample	1.13 billion	2 billion	1 billion
number of candidates	6381145	5642913	1408923

Table 3.

When the bit length is 256, then we gained 6381145 industrial-grade primes and this number is in the interval [6375919 ; 6385530] bounding E_{256}. Similarly it is true when we choose random integers from I_{512}. In the case of longest bit length, 1408923 is not in the interval around E_{1024}, but less then its lower limit 1409299 (better case).

These data also reinforce the primality of integers passing through the three tests.

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Approximating poles of complex rational functions

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Abstract. In this paper we investigate the application of the Nelder–Mead simplex method to approximate poles of complex rational functions. To our knowledge, there isn't any algorithm which is able to find the poles of a function when only the values on the unit circle are given. We will show that this method can accurately approximate 1, 2 or even 3 poles without any preliminary knowledge of their locations. The work presented here has implications in the study of ECG signals.

1 Introduction

The research presented in this article is motivated primarily by the fact that by combining a couple of simple complex rational functions and examining the values on the unit circle, the result can be very similar to an ECG signal (see Fig. 1). These functions can be applied for analysis, compression and denoising of ECG signals. Diagnostic applications may also be possible.

Rational functions play an important role in control theory. The Malmquist–Takenaka systems are often used to identify the transfer function of a system, see [1], [2], [9], and [10]. However automatic approximation of the poles of these functions proved not trivial when only the values on the unit circle are given and we have no preliminary knowledge about the locations of the poles.

A function, such as the one in Fig. 1, can be defined by its *poles* and the corresponding coefficients. The coefficients can be expressed by means of scalar

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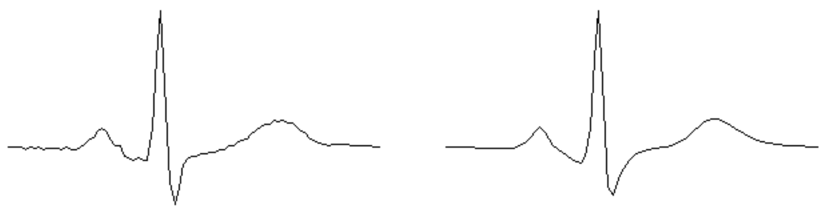


(a) An ECG signal from [3]. (Apnea-ECG Database)

(b) An approximating function.

Figure 1: An ECG signal and a very similar function produced using complex rational functions.

products when using an orthonormal system, so the main problem is to find the poles generating an appropriate system.

In this paper we investigate the application of the Nelder–Mead simplex method to approximate poles of generated complex rational functions given by their values on the unit circle. The question of H^∞ approximation (see [1] and [2]) is also to be analyzed.

2 Mathematical background

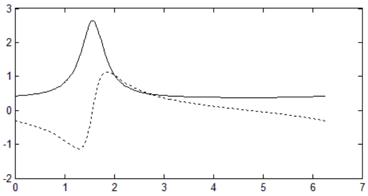
In this section we will introduce our functions of interest and recall some properties of related orthonormal systems. Then we give a summary of the Nelder–Mead simplex method, a commonly used nonlinear optimization algorithm.

2.1 Complex rational functions

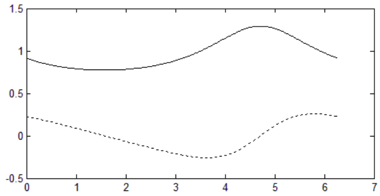
Denote by $\mathbb{C}$ the set of complex numbers and let $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ be the open unit disk, $\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$ the unit circle and $\mathbb{D}^* := \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$. The natural numbers will be considered as the set $\mathbb{N} := \{ 1, 2, 3, \dots \}$.

The disk algebra, i.e. the set of functions analytic on $\mathbb{D}$ and continuous on $\mathbb{D} \cup \mathbb{T}$, will be denoted by $\mathcal{A}$. The scalar product on $\mathbb{T}$ is defined by:

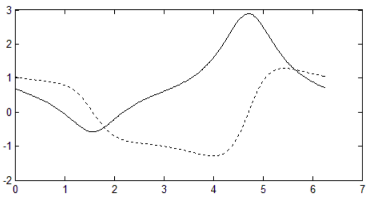
$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g}(e^{it}) \, dt.$$



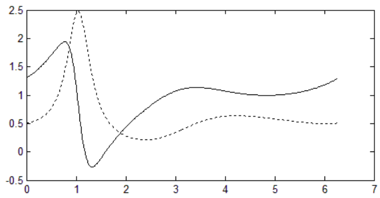
(a) $\mathfrak{a} = \left(\frac{3}{4}e^{\frac{1}{2}i\pi}\right), \mathfrak{c} = (1)$



(b) $\mathfrak{a} = \left(\frac{1}{4}e^{\frac{3}{2}i\pi}\right), \mathfrak{c} = (1)$



(c) $\mathfrak{a} = \left(\frac{1}{2}e^{\frac{1}{2}i\pi}, \frac{1}{2}e^{\frac{3}{2}i\pi}\right), \mathfrak{c} = (-1, 2)$



(d) $\mathfrak{a} = \left(\frac{1}{4}e^{i\pi}, \frac{3}{4}e^{\frac{1}{3}i\pi}\right), \mathfrak{c} = (1, i)$

Figure 2: Examples of complex rational functions.

We shall examine functions generated by the collection of

$$\varphi_n(z) := \frac{1}{(1 - \overline{a_n}z)^{m_n}} \qquad (z \in \mathbb{C};\ m \in \mathbb{N};\ n = 1, \ldots, m),$$

where $a_n \in \mathbb{D}$ $(n = 1, \ldots, m)$ and $m_n = \sum_{i \leq n, a_i = a_n} 1$ the multiplicity of the parameter a_n. We note that φ_n has a pole in $a_n^* = 1/\overline{a_n} \in \mathbb{D}^*$ and $\Phi := (\varphi_n \colon n = 1, \ldots, m) \subset \mathcal{A}$.

Fig. 2 illustrates some rational functions of the form $f = \sum_{n=1}^{m} c_n \varphi_n$ with $\mathfrak{a} = (a_1, \ldots, a_m)$, $\mathfrak{c} = (c_1, \ldots, c_m)$ and $m = 1, 2$. [1]

By applying the Gram–Schmidt orthogonalization procedure to Φ, we obtain an orthonormal system $\Psi := (\psi_n \colon n = 1, \ldots, m)$ on $\mathbb{T}$, the so-called Malmquist–Takenaka system (introduced in [7] and [11], see also [6]), which can be expressed by the Blaschke functions:

$$B_b(z) := \frac{z - b}{1 - \overline{b}z} \qquad (b \in \mathbb{D};\ z \in \mathbb{C}).$$

[1]The values on $\mathbb{T}$ are shown, i.e. for a function f we plot $f(z) = f(e^{it})$, where $t \in [0, 2\pi]$. The solid line is the real part, the dashed line is the imaginary part of $f(z)$.

Namely

$$\psi_n(z) = \frac{\sqrt{1-|a_n|^2}}{1-\overline{a_n}z}\prod_{k=1}^{n-1} B_{a_k}(z) \qquad (z\in\mathbb{C};\, n=1,\dots,m),$$

which suggests a convenient computation method for the values of ψ_n.

The orthonormality of the Malmquist–Takenaka functions is defined by

$$\langle\psi_k,\psi_l\rangle = \delta_{kl} \qquad (k,l=1,\dots,m),$$

where δ_{kl} is the Kronecker symbol. Note that span $\Phi = \operatorname{span}\Psi$, i.e. the systems Φ and Ψ generate the same m-dimensional subspace.

Given a function $f\in\mathcal{A}$ we can compute $\mathcal{P}_\Psi f = \mathcal{P}_{a_1,\dots,a_m} f$, the orthogonal projection of f on the subspace span Ψ by the formula

$$\mathcal{P}_\Psi f = \sum_{n=1}^{m}\langle f,\psi_n\rangle\,\psi_n.$$

Let $\mathcal{E}_\Psi f = \mathcal{E}_{a_1,\dots,a_m} f$ denote the best approximation of f in $\|.\|_2$, in span Ψ:

$$\mathcal{E}_\Psi f := \|f-\mathcal{P}_\Psi f\|_2 = \sqrt{\langle f-\mathcal{P}_f, f-\mathcal{P}_f\rangle}.$$

Our aim is to minimize $\mathcal{E}_\Psi f$ for a given function $f\in\mathcal{A}$ (f is given by its values on $\mathbb{T}$) and $m\in\mathbb{N}$ dimension by choosing the parameters $a_1,a_2,\dots,a_m$ of the Ψ (or Φ) system 'well'.

Naturally, in our computations we use the discrete approximation of the scalar product:

$$[f,g] := [f,g]_N = \frac{1}{2\pi N}\sum_{k=0}^{N-1} f(e^{2\pi i k/N})\,\overline{g}(e^{2\pi i k/N})$$

for a sufficiently large N. Let us choose e.g. $N=256$. Furthermore a function is given by its values on the set

$$\mathbb{T}_N := \left\{ z\in\mathbb{T} : z = e^{2\pi i k/N};\ k=0,\dots,N-1 \right\}.$$

2.2 The Nelder–Mead algorithm

The method introduced by Nelder and Mead in [8] is for the minimalization of a function of n variables, which depends only on the comparison of function values at the $n+1$ vertices of a general simplex, followed by the replacement of the vertex with the highest value by another point. The simplex adapts itself to the local landscape and contracts on to the final minimum. The method has been shown to be effective and computationally compact. Though there are very few proofs concerning its convergence properties (see [4] and [5]), it is widely used in practice in natural sciences and engineering for function optimization.

The method is described as follows. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be an arbitrary function and $x_1, x_2, \ldots, x_{n+1} \in \mathbb{R}^n$ the vertices of the current (nondegenerate) simplex in n-dimensions. Usually the vertices are defined by an x_s starting point and $w > 0$:

$$x_1 := x_s, \quad x_i := x_s + w \cdot e_{i-1} \qquad (i = 2, \ldots, n+1),$$

where e_i is the ith element of the canonical basis in $\mathbb{R}^n$.

Let $y_i := f(x_i)$ $(i = 1, \ldots, n+1)$ and define the indices h and l such that

$$y_h = \max\{\, y_i \,:\, i = 1, \ldots, n+1 \,\}, \quad y_l = \min\{\, y_i \,:\, i = 1, \ldots, n+1 \,\},$$

the *highest* and *lowest* values. Further define $\overline{x}$ the *centroid* of the points x_i with $i \neq h$.

At each stage in the process x_h is replaced by a new point; three operations are used: *reflection, expansion* and *contraction* with the following parameters: $\alpha = 1, \beta = \frac{1}{2}$ and $\gamma = 2$, since the natural ('standard') strategy given by these values proved to be the best, see [8]. These are defined as follows:

- The reflection of x_h is defined by the relation

  $$x_r := (1+\alpha)\overline{x} - \alpha x_h, \quad y_r := f(x_r).$$

  If $y_l \leq y_r < y_h$, then x_h is replaced by x_r and we start again with the new simplex.

- If $y_r < y_l$, i.e. the reflection has produced a new minimum, then we expand x_r to x_e by the relation:

  $$x_e := \gamma x_r + (1-\gamma)\overline{x}, \quad y_e := f(x_e).$$

  If $y_e < y_r$, we replace x_h by x_e and restart the process; but if $y_e \geq y_l$, then we have a failed expansion, and we replace x_h by x_r before restarting.

- If on reflecting x_h to x_r we find that $y_r \geq y_i$ for all $i \neq h$, i.e. that replacing x_h by x_r leaves y_r the maximum, then we define a new x_h to be either the old x_h or x_r, whichever has the lower y value (when $y_h = y_r$ then choose x_h), and form

$$x_c := \beta x_h + (1-\beta)\overline{x}, \quad y_c := f(x_c).$$

We then accept x_c for x_h and restart, unless $x_c > \min\{x_h, x_r\}$, i.e. the contracted point is worse than the better of x_h and x_r. For such a failed contraction we replace all x_i points by $\frac{1}{2}(x_i + x_l)$ and restart the process.[2]

We stop the iteration when the standard deviation is less then ε, a small preset value:

$$\left(\frac{1}{n}\sum_{i=1}^{n+1}(y_i - \overline{y})^2\right)^{\frac{1}{2}} < \varepsilon,$$

where

$$\overline{y} = \frac{1}{n+1}\sum_{i=1}^{n+1} y_i.$$

Let us choose e.g. $\varepsilon = 10^{-6}$.

The Nelder–Mead method is an effective and robust algorithm, but it often stops near local minima ignoring better global solutions. In these cases a reinitialization of the simplex at another starting point may prove helpful.

In Fig. 3 we illustrate the steps of a 2-dimensional simplex with starting point $x_s = (4, 6)$ and $w = 1$ optimizing the quadratic function:

$$f(x) = f(x', x'') = \frac{1}{16}(x' - 2)^2 + (x'' - 3)^2 \qquad (x', x'' \in \mathbb{R}).$$

The effects of reflection, expansion and contraction can be observed, as defined above. It is clear that in this simple case the simplex contracts on the minimum $x_{min} = (2, 3)$.[3]

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[2]This operation is called a *shrink* and a shrinking parameter δ can also be defined. The standard choice is $\delta = \frac{1}{2}$.

[3]This algorithm is also known as the *amoeba method* for the similarity of the simplex's moves to the named unicellular creature.

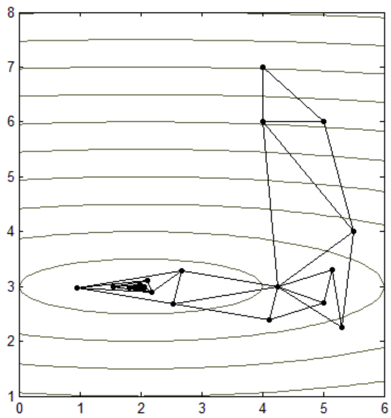


Figure 3: Moves of a Nelder–Mead simplex optimizing a quadratic function.

3 Methodology

In this section we explain how the Nelder–Mead method can be applied to find suitable parameters for the approximation. Furthermore, we describe our experiments and measurements.

Our goal is to minimize the function $\mathcal{E}_\Psi f = \mathcal{E}_{a_1,\ldots,a_m} f$ introduced in Section 2.1 for a given $f \in \mathcal{A}$ function (f is given by its values on $\mathbb{T}$) and $m \in \mathbb{N}$ dimension by choosing the parameters $a_1, a_2, \ldots, a_m$ of the Ψ system. For solving this minimization problem, we shall use the Nelder–Mead simplex algorithm described in Section 2.2.

The parameters of the Ψ system are to be chosen from $\mathbb{D}$. The simplex method requires vertices from $\mathbb{R}^n$. So in order to allow the simplex to move freely in $\mathbb{R}^n$ i.e. without any constraints to its steps, we set $n = 2m$ and use the map

$$\mathbb{R}^2 \ni (u, v) \quad \longmapsto \quad z = \frac{u}{\sqrt{1+u^2+v^2}} + \frac{v}{\sqrt{1+u^2+v^2}} i \in \mathbb{D}.$$

This map is a bijection between $\mathbb{R}^2$ and $\mathbb{D}$. Then a map from $\mathbb{R}^{2m}$ to $\mathbb{D}^m$ can be easily given by considering pairs of coordinates in $\mathbb{R}^{2m}$.

The traditional map used comes from the following idea. Imagine a half sphere on the complex unit disk $\mathbb{D}$ and then lay a plane $\mathbb{R}^2$ on the half sphere. Then the corresponding $z \in \mathbb{D}$ to an $(u, v) \in \mathbb{R}^2$ point is given by joining

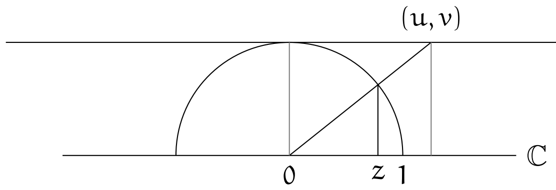


Figure 4: Mapping $(u, v) \in \mathbb{R}^2$ to $z \in \mathbb{D}$.

the complex zero with (u, v) by a straight line and projecting its intersection with the half sphere in $\mathbb{C}$ as seen in Fig. 4. The formula can be deduced from properties of the similar triangles on the figure.

So to find the $(a_1, \ldots, a_m) \in \mathbb{D}^m$ parameter values minimizing $\mathcal{E}_\Psi f$, we use a simplex in $\mathbb{R}^{2m}$. Let the starting position x_s be the zero of $\mathbb{R}^{2m}$, and $w = 0.1$. The iteration stops when the standard deviation of the $\mathcal{E}_\Psi f$ values in the vertices of the simplex descend below $\varepsilon = 10^{-6}$.

For a given $m \in \mathbb{N}$ and function $f \in \mathcal{A}$ of the form

$$f(z) = \sum_{n=1}^{m} c_n \varphi_n(z) \qquad (z \in \mathbb{T};\, c_n \in \mathbb{C})$$

with parameters $a_1, \ldots, a_m \in \mathbb{D}$ defining the functions φ_n $(n = 1, \ldots, m)$ and the approximations $b_1, \ldots, b_m \in \mathbb{D}$ of the parameters a_n define

$$\mathcal{D}f := \max\{\, |a_n - b_n| \,:\, n = 1, \ldots, m \,\},$$

the error of the approximation of the poles. (For convenience, we sometimes refer to the a_n parameters of the rational system as *poles* although these are actually not poles of the functions in focus. The $1/\overline{a_n}$ values are.) Further define

$$\mathcal{H}f := \frac{\max\{\, |f(t) - (\mathcal{P}_{b_1,\ldots,b_m} f)(t)| \,:\, t \in \mathbb{T} \,\}}{\max\{\, |f(t)| \,:\, t \in \mathbb{T} \,\}},$$

the relative error of the approximation in H^∞ norm. Naturally, in our computations we use the discrete approximation of $\mathcal{H}f$. And finally define

$$\mathcal{N}f \in \mathbb{N},$$

the number of calculations the simplex algorithm performs before terminating, where one calculation means evaluating $\mathcal{E}_{b_1,\ldots,b_m} f$ for a given function and set of parameters.

In our first experiments we used 1024 functions with one pole a_1 randomly chosen from the uniform distribution on $\mathbb{D}$, forming $f(z) = c_1\varphi_1(z)$ with c_1 also randomly chosen from $\mathbb{D}$. We avoided extreme values of a_1 and c_1 too close to zero (less than 0.05), because these values would result in almost constant function values on $\mathbb{T}$. We also avoided a_1 values too close to $\mathbb{T}$ (greater than 0.95), because φ_1 can no longer be defined with its parameter in $\mathbb{T}$ and our discretization may prove insufficient to reflect the properties of these extreme functions. For each function f we applied the Nelder–Mead algorithm as described above to find its pole and measured the previously defined $\mathcal{D}f$, $\mathcal{H}f$ and $\mathcal{N}f$ values.

Then we generated another 1024 functions with two poles i.e. functions of the form $f(z) = c_1\varphi_1(z) + c_2\varphi_2(z)$ with a_1, a_2, c_1, c_2 chosen similarly to the previous case and measured the $\mathcal{D}f$, $\mathcal{H}f$ and $\mathcal{N}f$ values again.

This experiment has been repeated for another 1024 functions with three random poles and coefficients: $f(z) = \sum_{i=1}^{3} c_i\varphi_i(z)$.

Finally we investigated the iterated application of the simplex algorithm in the case $m = 3$. This means that if the result was not good enough (e.g. $\mathcal{D}f > 10^{-4}$), we reinitalized the simplex with $w = 0.1$ and x_s in the position reached in the previous iteration and started the optimization process again, at most 5 times.

4 Results

The statistics of our measurement results of the $\mathcal{D}f$, $\mathcal{H}f$ and $\mathcal{N}f$ values are summarized in Table 1. The histograms of these values are shown in Fig. 5, 6 and 7. Fig. 5 and 6 show the number of functions (out of 1024) with $\mathcal{D}f$ and $\mathcal{H}f$ approximation error values with an order of magnitude of 10^{-8}, 10^{-7}, etc. Fig. 7 shows the number of functions (out of 1024) with $\mathcal{N}f$ values in the intervals shown on the horizontal axis.

One can observe that in the case $m = 1$, i.e. the case of functions with one pole, the algorithm always gives a very good approximation of the pole. The order of the approximation error is better than 10^{-6} and so is the approximation error in the H^∞ norm. The algorithm is very effective and fast, it requires 90 calculations on average. We also found that the algorithm needs more steps when applied to a function with its pole closer to $\mathbb{T}$. In these cases the approximation is usually more accurate too.

In the case $m = 2$ (i.e. the case of functions with two poles) in most cases the poles can be approximated with precision at least of order 10^{-6}. The

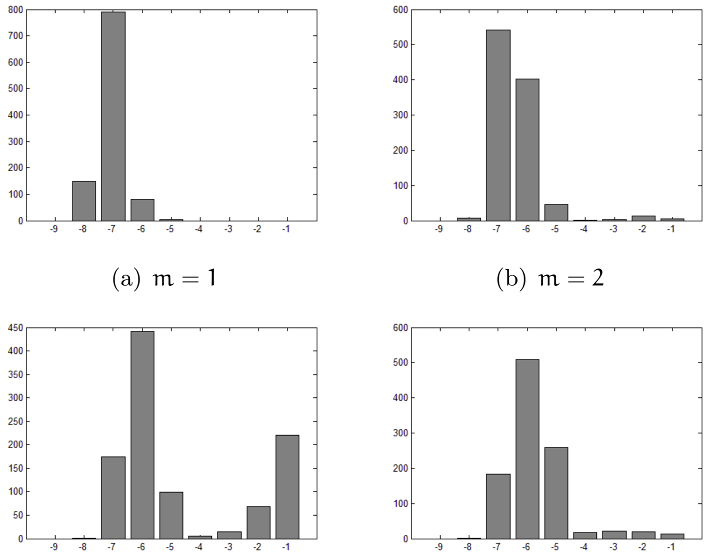
		min	max	avg	std. dev.
1 pole	$\mathcal{D}f$	$2.83 \cdot 10^{-8}$	$2.10 \cdot 10^{-5}$	$4.74 \cdot 10^{-7}$	$1.15 \cdot 10^{-6}$
	$\mathcal{H}f$	$7.20 \cdot 10^{-8}$	$2.19 \cdot 10^{-5}$	$6.52 \cdot 10^{-7}$	$1.34 \cdot 10^{-6}$
	$\mathcal{N}f$	56	116	89.92	8.30
2 poles	$\mathcal{D}f$	$5.74 \cdot 10^{-8}$	$4.27 \cdot 10^{-1}$	$1.85 \cdot 10^{-3}$	$1.95 \cdot 10^{-2}$
	$\mathcal{H}f$	$6.95 \cdot 10^{-8}$	$5.87 \cdot 10^{-3}$	$1.09 \cdot 10^{-5}$	$1.89 \cdot 10^{-4}$
	$\mathcal{N}f$	91	792	283.58	68.03
3 poles	$\mathcal{D}f$	$7.46 \cdot 10^{-8}$	$1.78 \cdot 10^{0}$	$1.13 \cdot 10^{-1}$	$2.71 \cdot 10^{-1}$
	$\mathcal{H}f$	$8.53 \cdot 10^{-8}$	$3.37 \cdot 10^{-1}$	$1.82 \cdot 10^{-3}$	$1.49 \cdot 10^{-2}$
	$\mathcal{N}f$	181	2006	712.64	272.56
3 poles iterated	$\mathcal{D}f$	$7.46 \cdot 10^{-8}$	$5.94 \cdot 10^{-1}$	$3.91 \cdot 10^{-3}$	$3.03 \cdot 10^{-2}$
	$\mathcal{H}f$	$8.53 \cdot 10^{-8}$	$2.57 \cdot 10^{-4}$	$1.91 \cdot 10^{-6}$	$1.24 \cdot 10^{-5}$
	$\mathcal{N}f$	430	2782	944.54	361.27

Table 1: The measured minimum, maximum, average and standard deviation values of $\mathcal{D}f$, $\mathcal{H}f$ and $\mathcal{N}f$ in the four investigated cases.

approximation in the H^∞ norm is also very good. The algorithm requires about 280 calculations on average. The cases when the $\mathcal{D}f$ value is in the order of 10^{-1} or 10^{-2}, are the ones when the two poles are very close to each other and there is a significant difference in the absolute values of the coefficients. In such cases the function could be almost as precisely approximated using functions with only one pole as using functions with two poles.

For functions with three poles ($m = 3$), there are lot more cases when $\mathcal{D}f$ is of the order 10^{-1}, even if the H^∞ error is small enough. We observed that in these cases the algorithm finds two poles with high precision, but the third one is far from the original. Then if we start again by initializing the simplex in the point reached (we iterate the application of the algorithm), the third pole is also find usually with an error less than 10^{-5} and the error of the H^∞ approximation also decreases. Naturally the computation cost rises with m and with the iterated application of the algorithm.

In the case of functions with even more poles, our few experiments show that this algorithm is not as powerful as in the cases detailed above (See also [4].) For instance, if the function is generated with 8 different poles, then the simplex method usually finds 4 of the poles with very small errors, but the others remain unknown.



(a) $m = 1$ (b) $m = 2$

(c) $m = 3$ (d) $m = 3$, iterated

Figure 5: Number of functions (out of 1024) vs. order of $\mathcal{D}f$.

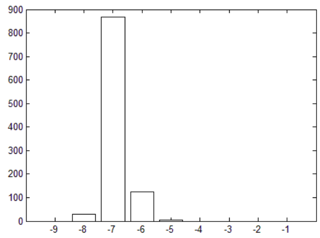
5 Conclusions

Our results show that the Nelder–Mead simplex algorithm can be applied effectively to solve the problem of approximating poles of complex rational functions with 1, 2 or even 3 poles, when the functions are given by their values on $\mathbb{T}$ and we have no preliminary knowledge about the location of the poles. We also get a satisfying approximation in H^∞ norm.

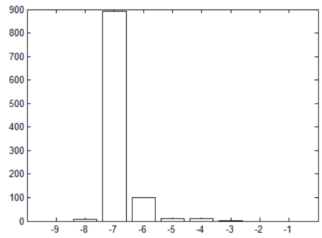
The results presented here have proven sufficient to perform promising calculations in the case of approximating ECG signals.

6 Further research

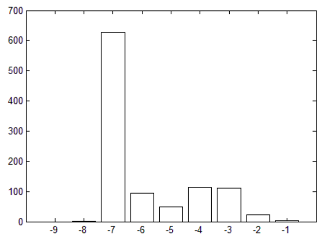
The main area of application of this research is the processing and analysis of ECG signals. The representation using complex rational functions may give an efficient way to compress and store these signals. We can gain a new method for denoising too, because of the smoothness of the functions applied. The potentials in diagnostics are also to be explored.



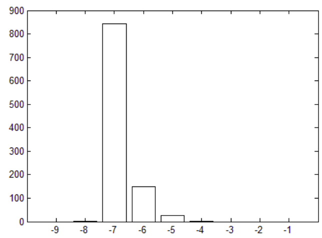
(a) $m = 1$



(b) $m = 2$

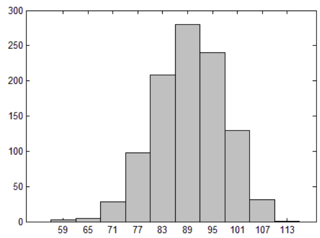


(c) $m = 3$

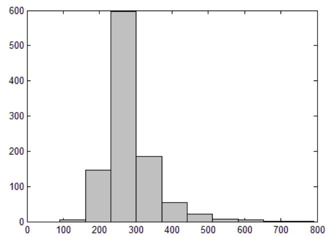


(d) $m = 3$, iterated

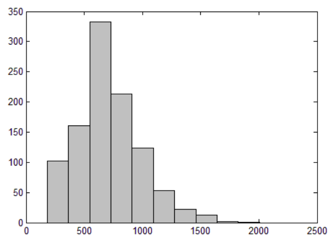
Figure 6: Number of functions (out of 1024) vs. order of $\mathcal{H}f$.



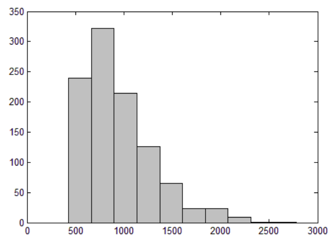
(a) $m = 1$



(b) $m = 2$



(c) $m = 3$



(d) $m = 3$, iterated

Figure 7: Histogram of $\mathcal{N}f$.

The effect of adding noise to the examined functions may also be investigated.

The direct use of $\mathbb{D}$ and hyperbolic coordinates instead of $\mathbb{R}^2$ in the implementation of the algorithm also seems to be an interesting field of research.

The design of new algorithms or possible improvement of the Nelder–Mead method for finding poles of functions with more singularities effectively is also to be studied.

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On perfect numbers connected with the composition of arithmetic functions

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Abstract. We study two extensions of notions related to perfect numbers. One is the extension of "superperfect" numbers, the other one is a new notion called "aperfect" numbers. As particular cases, many results involving the arithmetical functions σ, σ^*, σ^{**}, φ, φ^*, ψ and their compositions are presented in a unitary way.

1 Introduction

Let $\sigma(n)$ denote the sum of distinct divisors of the positive integer n. It is well-known that n is called perfect if $\sigma(n) = 2n$. Euclid and Euler have determined all even perfect numbers (see [8] for history of this theorem) by showing that they are of the form $n = 2^k \cdot q$, where $q = 2^{k+1} - 1$ is a prime $(k \geq 1)$. Prime numbers of the form $2^a - 1$ are called Mersenne primes, and it is one of the most difficult open problems of mathematics the proof of the infinitude of such primes. Up to now, only 46 Mersenne primes are known (see e.g. http://www.mersenne.org/). On the other hand, no odd perfect number is known ([3]). In 1969 D. Suryanarayana [10] defined the so-called superperfect numbers n, having the property $\sigma(\sigma(n)) = 2n$; and he and H. J. Kanold [4] obtained the general form of even superperfect numbers. All odd superperfect numbers must be perfect squares, but we do not know if there

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exists at least one such number.
In what follows we denote by $\mathbb{N}$ the non-zero positive integers: $\mathbb{N} = \{1, 2, \ldots\}$.
We call a g function *multiplicative*, if $g(ab) = g(a)g(b)$ for all $a, b \geq 1$, with $(a, b) = 1$.
In what follows, we denote by $\sigma^*(n)$ the sum of unitary divisors of n, i.e. those divisors $d|n$, with the property $(d, n/d) = 1$. A divisor d of n is called *bi-unitary* if the greatest common unitary divisor of d and n/d is 1. It is well-known that (see e.g. [2], [8]) σ^* and σ^{**} are multiplicative functions, and

$$\sigma^*(p^\alpha) = p^\alpha + 1, \tag{1}$$

$$\sigma^{**}(p^\beta) = \begin{cases} 1 + p + \ldots + p^{2\alpha} - p^\alpha, \text{ if } \beta = 2\alpha \\ 1 + p + \ldots + p^{2\alpha+1} = \sigma(p^\alpha), \text{ if } \beta = 2\alpha + 1 \end{cases}, \tag{2}$$

where p is an arbitrary prime and $\alpha \geq 1$ is a positive integer.
Clearly, σ is also a multiplicative function and

$$\sigma(p^\alpha) = 1 + p + \ldots + p^\alpha, \tag{3}$$

for any prime p and $\alpha \geq 1$.
The Euler's totient function is a multiplicative function with

$$\varphi(p^\alpha) = p^{\alpha-1} \cdot (p - 1), \tag{4}$$

while its unitary analogue is a multiplicative function with

$$\varphi^*(p^\alpha) = p^\alpha - 1, \tag{5}$$

(see e.g. [2], [9]).
Finally, Dedekind's arithmetical function ψ is a multiplicative function with the property

$$\psi(p^\alpha) = p^{\alpha-1} \cdot (p + 1), \tag{6}$$

(see e.g. [3], [7]).
In what follows, we shall call a number n "f-perfect", if

$$f(n) = 2n \tag{7}$$

Thus the classical perfect numbers are the σ-perfect numbers, while the superperfect numbers are in fact $\sigma \circ \sigma$-perfect numbers.
In 1989 the first author [6] determined all even $\psi \circ \sigma$-perfect numbers. In fact, he proved that for all even n one has

$$\psi\big(\sigma(n)\big) \geq 2n, \tag{8}$$

with equality only if $n = 2^k$, where $2^{k+1} - 1$ is a Mersenne prime. Since $\sigma(m) \geq \psi(m)$ for all m, from (8) we get:

$$\sigma\big(\sigma(n)\big) \geq \psi\big(\sigma(n)\big) \geq 2n \text{ for } n = \text{even}, \tag{9}$$

an inequality, which refines in fact the Kanold-Suryanarayana theorem.
We note the contrary to the $\sigma \circ \sigma$-perfect numbers; at least one odd solution to $\psi \circ \sigma$-perfect numbers is known, namely $n = 3$.

2 Extensions of even superperfect numbers

The main result of this section is contained in the following.

Theorem 1 *Let* $f, g : N \to N$ *be two arithmetic functions having the following properties:*

1. g *is multiplicative*
2. $f(ab) \geq af(b)$ *for all* $a, b \geq 1$
3. $g(m) \geq m$*, with equality only for* $m = 1$
4. $f\big(g(2^k)\big) \geq 2^{k+1}$*, with equality only if* $2^{k+1} - 1 \in A$*, where* A *is a set of positive integers*

Then for all even n *one has*

$$f\big(g(n)\big) \geq 2n, \tag{10}$$

and all even $f \circ g$*-perfect numbers are of the form* 2^k*, where* $2^{k+1} - 1 \in A$*.*

Proof. Let $n = 2^k \cdot m$ with $m =$ odd, be an even integer. By condition *1.* one has $g(n) = g(2^k)g(m)$, so by *2.* we can write that $f\big(g(n)\big) = f\big(g(2^k)g(m)\big) \geq g(m)f\big(g(2^k)\big)$. Since $g(m) \geq m$ (by *3.*) and $f\big(g(2^k)\big) \geq 2^{k+1}$ (by *4.*), we get that $f\big(g(m)\big) \geq 2n$, so (10) follows. For equality we must have $g(m) = 1$ and $f\big(g(2^k)\big) = 2^{k+1}$, so $m = 1$ and $2^{k+1} - 1 \in A$. This finishes the proof of Theorem 1. ■

Remark 1 *If at least one of the inequalities 2.–4. is strict, then in (10) one has strict inequality. As a consequence,* n *cannot be an even* $f \circ g$*-perfect number.*

Corollary 1 *(Sándor [6])*
All even $\psi \circ \sigma$*-perfect numbers* n *have the form* $n = 2^k$*, where* $2^{k+1} - 1$ *is prime.*

– The first $\psi \circ \sigma$*-perfect numbers are:* $2 = 2^1$, 3, $4 = 2^2$, $16 = 2^4$, $64 = 2^6$, $4096 = 2^{12}$, $65536 = 2^{16}$, $262144 = 2^{18}$, *where* $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, $2^7 - 1 = 127$, $2^{13} - 1 = 8191$, $2^{17} - 1 = 131071$, $2^{19} - 1 = 524287$ *are Mersenne primes.*

– Put $f(n) = \psi(n)$ *and* $g(n) = \sigma(n)$ *in Theorem 1. Then property 2. is known (see e.g. [7]), while 3. and 4. are well known. Since for* $t > 1$ *one has* $\psi(t) \geq t + 1$*, with equality only for* $t =$ *prime, by* $\sigma(2^k) = 2^{k+1} - 1$*, we get* $A =$ *set of primes of the form* $2^{k+1} - 1$*.*

Corollary 2 *(Sándor [6])*
The only even $\sigma \circ \psi$*-perfect number* n *is* $n = 2$*.*

– Put $f(n) = \sigma(n)$ *and* $g(n) = \psi(n)$ *in Theorem 1. Then properties 1.–3. are well-known; for 4. one has* $\psi(2^k) = 2^{k-1} \cdot 3$*; so* $\sigma(2^{k-1} \cdot 3) = ((2^k - 1) \cdot 4) \geq 2^{k+1} \Leftrightarrow 2^k \geq 2$*. Thus* $k = 1$ *and* $A = \{3\}$*.*

Corollary 3 *(Kanold-Suryanarayana [4])*
All even $\sigma \circ \sigma$*-perfect numbers* n *have the form* $n = 2^k$*, where* $2^{k+1} - 1$ *is prime.*

– The first $\sigma \circ \sigma$*-perfect numbers are:*
$2 = 2^1$, $4 = 2^2$, $16 = 2^4$, $64 = 2^6$, $4096 = 2^{12}$, $65536 = 2^{16}$, $262144 = 2^{18}$, $1073741824 = 2^{30}$, $1152921504606846976 = 2^{60}$, *where* $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, $2^7 - 1 = 127$, $2^{13} - 1 = 8191$, $2^{17} - 1 = 131071$, $2^{19} - 1 = 524287$, $2^{31} - 1 = 2147483647$, $2^{61} - 1 = 2305843009213693951$ *are Mersenne primes.*

– This also follows from inequality (10) for $f(n) = \sigma(n)$ *and* $g(n) = \psi(n)$*, but a direct proof applies for* $f(n) = g(n) = \sigma(n)$*.*

Corollary 4 *(Sándor [6])*
There is no even $\psi \circ \psi$-perfect number.
– Put $f(n) = g(n) = \psi(n)$. Since inequality 4. will be strict, inequality (10) holds true also with strict inequality.

Remark 2 *In [6] it is proved also that the only odd $\psi \circ \psi$-perfect number is $n = 3$.*

Corollary 5 *The only even $\sigma \circ \sigma^{**}$-perfect number n is $n = 2$.*
– Let $f(n) = \sigma(n)$ and $g(n) = \sigma^{**}(n)$ in Theorem 1. Clearly 3. holds true, as more generally it is known that (see e.g. [1], [8]):

$$\sigma^{**}(m) \geq m + 1 \text{ for } m > 1, \tag{11}$$

with equality only for $m = p$ or $m = p^2$ (p = prime).
Now, let k be odd. Then $\sigma^{**}(2^k) = \sigma(2^k) = 2^{k+1} - 1$ and $\sigma(\sigma^{**}(2^k)) = \sigma(2^{k+1} - 1) \geq 2^{k+1}$, with equality only if $2^{k+1} - 1$ = prime. For $k \geq 3$, as k is odd, clearly $k + 1$ is even, so it is immediate that $2^{k+1} - 1 \equiv 0 \pmod 3$. Thus we must have $k = 1$, i.e. $n = 2$ is a solution.
When k is even, put $k = 2a$. Then $\sigma^{**}(2^k) = \sigma^{**}(2^{2a}) = 1 + 2 + \ldots + 2^{a-1} + \underbrace{2^{a+1} + \ldots + 2^{2a}}_{2^{a+1}\cdot(1+2+\ldots+2^{a-1})} = (1+2+\cdots+2^{a-1}) \cdot (1+2^{a+1}) = (2^a - 1)(2^{a+1}+1)$. Thus, $\sigma(\sigma^*(2^k)) = \sigma((2^a - 1) \cdot (2^{a+1} + 1)) \geq (2^a - 1)\sigma(2^a - 1) \geq (2^{a+1} + 1) \cdot 2^a > 2^{2a+1} = 2^{k+1}$, so inequality 4) is strict for k even number.

3 Aperfect numbers

The equality $f(n) = n + 2$, for $f(n) > n$ is a kind of additive analogue of $f(n) = n \cdot 2$, i.e. of classical perfect numbers. We shall call a number n f-*plus aperfect* (aperfect = "additive perfect"), if

$$f(n) = n + 2. \tag{12}$$

This notion also extends the notion of perfect numbers. Put e.g. $f(n) = \sigma(n) - n + 2$. Then $\sigma(n) = 2n$, so we obtain again the perfect numbers.
Similary, for $f(n) < n$ we have a similar notion. We call n f-*minus aperfect*, if

$$f(n) = n - 2. \tag{13}$$

We can state the following general result:

Theorem 2 *Let* $f, g : N \to N$ *be two arithmetic functions such that* $g(n) \geq n + 1$ *for* $n > 1$*, with equality only for* $n = p^{\alpha}$ *(*p *prime,* $\alpha \geq 1$ *integer) and* $f(m) \geq m + 1$*, for* $m > 1$*, with equality only for* $m = q^{\beta}$ *(*q *prime,* $\beta \geq 1$ *integer). Then one has the inequality*

$$f\big(g(n)\big) \geq n + 2 \tag{14}$$

for all n*, and* n *is* $f \circ g$*-plus aperfect only if the prime powers* p^{α} *and* q^{β} *satisfy the equation*

$$g(p^{\alpha}) = q^{\beta}. \tag{15}$$

Proof. From the stated conditions, one can write $f\big(g(n)\big) \geq g(n) + 1 \geq (n + 1) + 1 = n + 2$. One has equality only if $n = p^{\alpha}$ and $g(n) = q^{\beta}$, i.e. $g(p^{\alpha}) = q^{\beta}$, which means equality (15). ■

Corollary 6 *(Sándor [6])*
All $\sigma \circ \sigma^{*}$*-plus aperfect numbers* n *have form* $n = 2^{s}$*, where* $2^{s} + 1$ *is a prime (i.e. Fermat prime,* $s = 2^{a}$ *).*

– The first $\sigma \circ \sigma^{*}$*-plus aperfect numbers are:* $2 = 2^1$*,* $4 = 2^2$*,* $16 = 2^4$*,* $256 = 2^8$*,* $65536 = 2^{16}$*, where* $2^1 + 1 = 3$*,* $2^2 + 1 = 5$*,* $2^4 + 1 = 17$*,* $2^8 + 1 = 257$*,* $2^{16} + 1 = 65537$ *are Fermat primes.*

– Let $f(n) = \sigma(n), g(n) = \sigma^{*}(n)$*. Then (15) may be written as* $\sigma^{*}(p^{\alpha}) = q^{\beta}$*. Since* $\sigma(m) = m + 1$ *only for* $m =$ *prime, we have* $\beta = 1$*, thus* $p^{\alpha} + 1 = q$*. For* $p \geq 3, p^{\alpha} + 1$ *is even number, so we must have* $p = 2$*, i.e.* $q = 2^{\alpha} + 1$*. Since* $n = p^{\alpha} = 2^{\alpha}$*, then result follows.*

Corollary 7 *All* $\sigma^{*} \circ \sigma$*-plus aperfect numbers are* $n = 2$*, or have the form* $n = 2^{k} - 1$*, where* $2^{k} - 1$ *is a Mersenne prime.*

– The first $\sigma^{*} \circ \sigma$*-plus aperfect numbers are:* 2*,* $2^2 - 1 = 3$*,* $2^3 - 1 = 7$*,* $2^5 - 1 = 31$*,* $2^7 - 1 = 127$*,* $2^{13} - 1 = 8191$*,* $2^{17} - 1 = 131071$*,* $2^{19} - 1 = 524287$ *(Mersenne primes).*

– Let $f(n) = \sigma^{*}(n), g(n) = \sigma(n)$ *in Theorem 2. Then (15) has the form* $\sigma(p^{\alpha}) = q^{\beta}$*. Since* $\alpha = 1$*, one has* $p + 1 = q^{\beta}$*, i.e.* $p = q^{\beta} - 1$*. For* $q \geq 3$ *this is even, so we must have* $q = 2$*, when* $p = 2^{\beta} - 1$ *is Mersenne prime. When* $q = 3$ *for* $\beta = 1$ *we get the prime 2, the first* $\sigma^{*} \circ \sigma$*-plus aperfect number.*

Corollary 8 *The only* $\sigma \circ \sigma$*-plus aperfect number* n *is* $n = 2$.
– *Let* $f(n) = g(n) = \sigma(n)$. *Then we get* $\alpha = \beta = 1$ *so* $\sigma(p) = q$, *i.e.* $p + 1 = q$ *with* p, q *This is possible only for* $p = 2, q = 3$.

Corollary 9 *The only* $\sigma^{**} \circ \sigma^{**}$*-plus aperfect numbers* n *are* $n = 2, 3, 4$.
– *Since the equality* $\sigma^{**}(n) = n + 1$ *is satisfied only if* $n = p$ *or* $n = p^2$ *(*p *prime), we must study the equality:*

$$\sigma^{**}(p^{\alpha}) = q^{\beta} \tag{16}$$

for $\alpha, \beta \in \{1, 2\}$.
If $\alpha = 1$, *then* $\beta = 1$ *implies* $p + 1 = q$, *which is possible only for* $p = 2, q = 3$. *Now for* $\alpha = 1, \beta = 2$ *we get* $p + 1 = q^2$, *so* $p = q^2 - 1 = (q - 1)(q + 1)$, *which is possible only for* $q = 2$ *and* $p = 3$. *Thus* $p = 3$ *is acceptable too.*
If $\alpha = 1, \beta = 2$, *we get* $q^2 + 1 = p$, *i.e.* $q^2 = p - 1$. *Here* $p = 2$ *is not possible, while for* $p \geq 3, p - 1$ *is even, thus* $2 | q^2$. *This means* $q = 2$. *So* $p = 5$ *is another solution. For* $q^2 + 1 = p^2$ *we get* $q^2 = (p - 1)(p + 1)$, *which for* $p = 2$ *gives* $q^2 = 3$, *which is impossible. For* $p \geq 3$ *we get* $q = 3$, *so* $5 = p^2$, *which is again impossible. Then result follows.*

Similarly to Theorem 2, we may prove the following:

Theorem 3 *Let* $f, g : N \to N$ *be two arithmetic functions, such that* $g(n) \leq n - 1$ *for* $n > 1$, *with equality only for* $n = p^{\alpha}$ *(*p *prime,* $\alpha \geq 1$ *integer) and* $f(m) \leq m - 1$, *for* $m > 1$, *with equality only for* $m = q^{\beta}$ *(*q *prime,* $\beta \geq 1$ *integer). Then one has the inequality:*

$$f\big(g(n)\big) \leq n - 2 \tag{17}$$

for all $n > 2$, *and* n *is* $f \circ g$*-minus aperfect only if the prime powers* p^{α} *and* q^{β} *satisfy the equation*

$$g(p^{\alpha}) = q^{\beta}. \tag{18}$$

Proof. From the stated properties one can write $f\big(g(n)\big) \leq g(n) - 1 \leq (n - 1) - 1 = n - 2$, with equality only if $n = p^{\alpha}$ and $g(n) = q^{\beta}$, so (18) follows. ■

Corollary 10 *All* $\varphi \circ \varphi^*$*-minus aperfect numbers* n *are* $n = 3$*, or have the form* $n = 2^a$*, where* $2^a - 1$ *is a Mersenne prime.*

– *The first* $\varphi \circ \varphi^*$*-minus aperfect numbers are:* 3, $4 = 2^2$, $8 = 2^3$, $32 = 2^5$, $128 = 2^7$, $8192 = 2^{13}$, *where* $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, $2^7 - 1 = 127$, $2^{13} - 1 = 8191$ *are Mersenne primes.*

– *Let* $f(n) = \varphi(n), g(n) = \varphi^*(n)$ *in Theorem 3. As* $\varphi(m) = m - 1$ *only for* $m =$ *prime, we have* $\beta = 1$*, so (18) becomes* $\varphi^*(p^\alpha) = q$*, i.e.* $p^\alpha - 1 = q$*. Then* $p = 2$*, so* $q = 2^\alpha - 1$ *is a Mersenne prime. Here* $n = 2^\alpha$*, so the result follows. When* $p = 3$ *and* $\alpha = 1$*, then* $q = 1$*, and we obtain the first* $\varphi \circ \varphi^*$*-minus aperfect number:* 3*.*

Corollary 11 *All* $\varphi^* \circ \varphi$*-minus aperfect numbers* n *have the form* $n = 2^a + 1 =$ *Fermate prime.*

– *The first* $\varphi^* \circ \varphi$*-minus aperfect numbers are:* $3 = 2^1 + 1$, $5 = 2^2 + 1$, $17 = 2^4 + 1$, $257 = 2^8 + 1$ *Fermat primes.*

– *Put* $f(n) = \varphi^*(n), g(n) = \varphi(n)$ *in Theorem 3. Now* $\alpha = 1$*, so* $\varphi(p) = q^\beta$*, i.e.* $p - 1 = q^\beta$*, implying* $p = q^\beta + 1$*. Since* p, q *are primes, one must have* $q = 2$*. Thus* $p = 2^\beta + 1$ *and* $n = p$*, which implies the assertion.*

Remark 3 *At the present state of the science, there are only 5 Fermat primes known, namely* $n = 3, 5, 17, 257, 65537$ *(see [5], [3]).*

Corollary 12 *All* $\varphi^* \circ \varphi^*$*-minus aperfect numbers are* $n = 9$ *or* $n = 2^\alpha$ *with* $2^\alpha - 1$ *is Mersenne prime, or* $n = 2^\beta + 1$ *is Fermat prime.*

– *The first* $\varphi^* \circ \varphi^*$*-minus aperfect numbers are: 3, 4, 5, 8, 9, 17, 32, 128, 257, 8192.*

– *We have* $\varphi^*(n) = n - 1$ *only if* $n = p^\alpha$*, so we must solve the equation* $\varphi^*(p^\alpha) = p^\alpha - 1 = q^\beta$*.*

Case 1) *If* $q \geq 3$*, then as* $p^\alpha = q^\beta + 1 =$ *even, we get* $2 | p^\alpha$*, so* $p = 2$*. We get the equation:*

$$q^\beta = 2^\alpha - 1. \tag{19}$$

Equation (19) has been studied in [9] (Lemma 6'), so we get $\beta = 1, q = 2^\alpha - 1$ *is Mersenne prime.*

Case 2) *If* $q = 2$*, then we get the equation:*

$$p^\alpha = 2^\beta + 1, \tag{20}$$

studied in [9] (Lemma 4). Thus we have: a) $p = 3, \alpha = 2, \beta = 3$, *in which case* $n = p^{\alpha} = 3^2 = 9$*; b)* $\alpha = 1, p = 2^{\beta} + 1$ *is Fermat prime.*
This finishes the proof of Corollary 12.

Remark 4 *It is easy to see that the only* $\varphi \circ \varphi$*-minus aperfect number is* $n = 3$.

Remark 5 *Since the result of Corollary 7 is a characterisation of odd solutions, it could be used as a Mersenne prime test, too; and Corollary 11 could be used as a Fermat prime test, too.*

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Closedness of the solution mapping to parametric vector equilibrium problems

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Abstract. The goal of this paper is to study the parametric vector equilibrium problems governed by vector topologically pseudomonotone maps. The main result gives sufficient conditions for closedness of the solution map defined on the set of parameters.

1 Introduction

M. Bogdan and J. Kolumbán [3] gave sufficient conditions for closedness of the solution map defined on the set of parameters. They considered the parametric equilibrium problems governed by topological pseudomonotone maps depending on a parameter. In this paper we extend this result for parametric vector equilibrium problems.

Let X be a Hausdorff topological space and let P (the set of parameters) be another Hausdorff topological space. Let $\mathcal{Z}$ be a real topological vector space with an ordering cone C, where C is a closed convex cone in $\mathcal{Z}$ with $\operatorname{Int} C \neq \emptyset$ and $C \neq \mathcal{Z}$.

We consider the following parametric vector equilibrium problem, in short $(VEP)_p$:

Find $a_p \in D_p$, such that

$$f_p(a_p, b) \notin -C \setminus \{0\}, \ \forall b \in D_p,$$

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where D_p is a nonempty subset of X and $f_p : X \times X \to \mathcal{Z}$ is a given function.

It is well-known that VEP contains several problems as special cases, namely, vector optimization problem, vector saddle point problem, vector variational inequality problem, vector complementarity problem, etc.

Denote by $S(p)$ the set of the solutions for a fixed p. Suppose that $S(p) \neq \emptyset$, for all $p \in P$. For sufficient conditions for the existence of solutions see [8], [13].

The paper is organized as follows. In Section 2, we introduce a new notion of the vector topological pseudomonotonicity and we recall the notion of the Mosco convergence of the sets. Section 3 is devoted to the closedness of the solution map for parametric vector equilibrium problems.

2 Preliminaries

In this section, we will introduce a new definition of the vector topologically pseudomonotone bifunctions with values in $\mathcal{Z}$. First, the definition of the suprema and the infima of subsets of $\mathcal{Z}$ are given. Following [1], for a subset A of $\mathcal{Z}$ the suprema of A with respect to C is defined by:

$$\operatorname{Sup} A = \left\{ z \in \bar{A} : A \cap (z + \operatorname{Int} C) = \emptyset \right\},$$

and the infima of A with respect to C is defined by:

$$\operatorname{Inf} A = \left\{ z \in \bar{A} : A \cap (z - \operatorname{Int} C) = \emptyset \right\}.$$

Let $(z_i)_{i \in I}$ be a net in $\mathcal{Z}$. Let $A_i = \{z_j : j \geq i\}$ for every i in the index set I. The limit inferior of (z_i) is given by:

$$\operatorname{Liminf} z_i = \operatorname{Sup} \left(\bigcup_{i \in I} \operatorname{Inf} A_i \right).$$

Similarly, the limit superior of (z_i) can be defined as

$$\operatorname{Limsup} z_i = \operatorname{Inf} \left(\bigcup_{i \in I} \operatorname{Sup} A_i \right).$$

Theorem 1 ([7], Theorem 2.1) *Let $(z_i)_{i \in I}$ be a net in $\mathcal{Z}$ convergent to z, and let $A_i = \{z_j : j \geq i\}$.*

i) *If there is an i_0 such that, for every $i \geq i_0$, there exists $j \geq i$ with $\operatorname{Inf} A_j \neq \emptyset$, then $z \in \operatorname{Liminf} z_i$.*

ii) *If there is an* i_0 *such that, for every* $i \geq i_0$*, there exists* $j \geq i$ *with* $\operatorname{Sup} A_j \neq \emptyset$*, then* $z \in \operatorname{Limsup} z_i$.

We introduce the definition of vector topologically pseudomonotonicity, which plays a central role in our main results.

Definition 1 *Let* (X, σ) *be a Hausdorff topological space, and let* D *be a nonempty subset of* X*. A function* $f : D \times D \to \mathcal{Z}$ *is called vector topologically pseudomonotone if for every* $b \in D$*,* $\nu \in C$ *and for each net* $(a_i)_{i \in I}$ *in* D *satisfying* $a_i \xrightarrow{\sigma} a \in D$ *and*

$$\operatorname{Liminf} f(a_i, a) \cap (-\operatorname{Int} C) = \emptyset, \tag{1}$$

then for every i *in the index set* I

$$\overline{\{f(a_j, b) : j \geq i\}} \cap [f(a, b) + \nu - C] \neq \emptyset.$$

In Definition 1, if $\mathcal{Z} = \mathbb{R}$, and if C is the set of all non-negative real numbers, then we get back the well-known topological pseudomonotonicity introduced by Brézis [4].

Let us consider σ and τ two topologies on X. Suppose that τ is stronger than σ on X.

For the parametric domains in $(VEP)_p$, we shall use a slight generalization of Mosco's convergence [14].

Definition 2 ([3], Definition 2.2.) *Let* D_p *be subsets of* X *for all* $p \in P$*. The sets* D_p *converge to* D_{p_0} *in the Mosco sense (*$D_p \xrightarrow{M} D_{p_0}$*) as* $p \to p_0$ *if:*

a) for every subnet $(a_{p_i})_{i \in I}$ *with* $a_{p_i} \in D_{p_i}$*,* $p_i \to p_0$ *and* $a_{p_i} \xrightarrow{\sigma} a$ *implies* $a \in D_{p_0}$*;*

b) for every $a \in D_{p_0}$*, there exists* $a_p \in D_p$ *such that* $a_p \xrightarrow{\tau} a$ *as* $p \to p_0$.

3 Closedness of the solution map

This section is devoted to prove the closedness of the solution map for parametric vector equilibrium problems.

Theorem 2 *Let* X *be a Hausdorff topological space with* σ *and* τ *two topologies, where* τ *is stronger than* σ*. Let* D_p *be nonempty sets of* X*, and let* $p_0 \in P$ *be fixed. Suppose that* $S(p) \neq \emptyset$ *for each* $p \in P$ *and the following conditions hold:*

i) $D_p \xrightarrow{M} D_{p_0}$;

ii) For each net of elements $(p_i, a_{p_i}) \in \mathsf{Graph}S$*, if* $p_i \to p_0$*,* $a_{p_i} \xrightarrow{\sigma} a$*,* $b_{p_i} \in D_{p_i}$*,* $b \in D_{p_0}$*,and* $b_{p_i} \xrightarrow{\tau} b$*, then*

$$\operatorname{Liminf}\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right) \cap (-\operatorname{Int} C) \neq \emptyset.$$

iii) $f_{p_0} : X \times X \to \mathcal{Z}$ *is vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ *is closed at* p_0*, i.e. for each net of elements* $(p_i, a_{p_i}) \in \mathsf{Graph}S$*,* $p_i \to p_0$ *and* $a_{p_i} \xrightarrow{\sigma} a$ *imply* $(p_0, a) \in \mathsf{Graph}S$*.*

Proof. Let $(p_i, a_{p_i})_{i\in I}$ be a net of elements $(p_i, a_{p_i}) \in \mathsf{Graph}S$, i.e.

$$f_{p_i}(a_{p_i}, b) \notin -C \setminus \{0\}, \ \forall b \in D_{p_i}, \tag{2}$$

with $p_i \to p_0$ and $a_{p_i} \xrightarrow{\sigma} a$. By the Mosco convergence of the sets D_p, we get $a \in D_{p_0}$. Moreover, there exists a net $(b_{p_i})_{i\in I}$, $b_{p_i} \in D_{p_i}$ such that $b_{p_i} \xrightarrow{\tau} a$. From the assumption ii) we obtain that

$$\operatorname{Liminf}\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a)\right) \cap (-\operatorname{Int} C) \neq \emptyset. \tag{3}$$

Since $-\operatorname{Int} C$ is an open cone, it follows that there exists a subnet (a_{p_i}) denoted by the same indexes such that

$$f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a) \in -\operatorname{Int} C \text{ for all } i \in I. \tag{4}$$

By replacing b with b_{p_i} in (2), we get

$$f_{p_i}(a_{p_i}, b_{p_i}) \notin -C \setminus \{0\}. \tag{5}$$

From (5) and (4) we obtain that

$$f_{p_0}(a_{p_i}, a) \in (-C)^c \subset (-\operatorname{Int} C)^c, \text{ for all } i \in I,$$

since $(-\operatorname{Int} C)^c$ is closed, it follows

$$\operatorname{Liminf} f_{p_0}(a_{p_i}, a) \cap (-\operatorname{Int} C) = \emptyset.$$

Now, we can apply iii) and we obtain that for every $b \in D_{p_0}$, $\nu \in C$, and for every $i \in I$ we have

$$\overline{\left\{f_{p_0}\left(a_{p_j}, b\right) : j \geq i\right\}} \cap \left[f_{p_0}(a, b) + \nu - C\right] \neq \emptyset. \tag{6}$$

We have to prove that

$$f_{p_0}(a, b) \notin -C \setminus \{0\}, \ \forall b \in D_{p_0}.$$

Assume the contrary, that there exists $\overline{b} \in D_{p_0}$ such that

$$f_{p_0}\left(a, \overline{b}\right) \in -C \setminus \{0\}.$$

Let be $f_{p_0}\left(a, \overline{b}\right) = -v$, where $v \in C \setminus \{0\}$. From (6) we obtain that for every $i \in I$ we have

$$\overline{\left\{f_{p_0}\left(a_{p_j}, \overline{b}\right) : j \geq i\right\}} \cap (-C) \neq \emptyset, \tag{7}$$

i.e. there exists a subnet (a_{p_i}) denoted by the same indexes such that

$$f_{p_0}\left(a_{p_i}, \overline{b}\right) \in -C \text{ for all } i \in I, \tag{8}$$

or

$$f_{p_0}\left(a_{p_i}, \overline{b}\right) \text{ converges to a point in } -\partial C. \tag{9}$$

Since $\overline{b} \in D_{p_0}$ from the Mosco convergence of the sets D_p, we have that there exists $\left(\overline{b}_{p_i}\right)_{i \in I} \subset D_{p_i}$ such that $\overline{b}_{p_i} \stackrel{\tau}{\to} \overline{b}$. By using again the assumption ii), it follows that there exists a subnet (a_{p_i}) denoted by the same indexes, for which

$$f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) - f_{p_0}\left(a_{p_i}, \overline{b}\right) \in -\operatorname{Int} C, \text{ for all } i \in I. \tag{10}$$

From (8), (9) and (10) it follows that there exists an index $i_0 \in I$ such that

$$f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) \in -\operatorname{Int} C, \ i \geq i_0, \tag{11}$$

but on the other side $(p_i, a_{p_i}) \in \mathsf{Graph} S$, and

$$f_{p_i}\left(a_{p_i}, \overline{b}_{p_i}\right) \notin -C \setminus \{0\},$$

which is a contradiction. Hence $(p_0, a) \in \mathsf{Graph} S$. ■

M. Bogdan and J. Kolumbán [3] showed that the topological pseudomonotonicity and the assumption ii) are essential in scalar case.

Remark 1 *The assigment* ii) *can not be replaced by*

ii') *For each net of elements* $(p_i, a_{p_i}) \in \mathsf{Graph} S$, *if* $p_i \to p_0$, $a_{p_i} \stackrel{\sigma}{\to} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, *and* $b_{p_i} \stackrel{\tau}{\to} b$, *then*

$$\operatorname{Liminf}\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right) \cap \left(-\operatorname{Int} C \cup \{0\}\right) \neq \emptyset.$$

Therefore Theorem 2 does not imply Theorem 1 in [3].

The following example confirms this statement.

Example 1 *Let* $P = \mathbb{N} \cup \{\infty\}$, $p_0 = \infty$ *(∞ means $+\infty$ from real analysis), where we consider the topology induced by the metric given by* $d(m,n) = |1/m - 1/n|$, $d(n,\infty) = d(\infty,n) = 1/n$, *for* $m,n \in \mathbb{N}$, *and* $d(\infty,\infty) = 0$. *Let* $X = [0,1]$ *where* σ, τ *are natural topologies,* $\mathcal{Z} = \mathbb{R}^2$, $D_p = [0,1]$, $p \in P$, *the real vector functions* $f_n : [0,1] \times [0,1] \to \mathbb{R}^2$. *The ordering cone* C *is the third quadrant, i.e.* $C = \{(a,b) \in \mathbb{R}^2 : a \le 0, b \le 0\}$.

Let $f_n(a,b) = (a - b - 2/n, 1 - 2a)$, $n \in \mathbb{N}$ *and the function* f_∞ *be defined by*

$$f_\infty(a,b) = \begin{cases} (a-b, 1-a) & \text{if } a > 0 \\ (b,1) & \text{if } a = 0 \end{cases}.$$

The f_∞ *is vector topologically pseudomonotone. Indeed, for* $a > 0$, f_∞ *is continuous, therefore it is vector topologically pseudomonotone. Let us study the case when* $a = 0$.

We have to prove that for every $b \in [0,1]$, $v \in C$ *for each* $(a_n)_n$, $a_n \in [0,1]$ *with* $a_n \to 0$ *satisfying*

$$\operatorname{Liminf} f_\infty(a_n, 0) \cap (-\operatorname{Int} C) = \emptyset,$$

then for every $m \in \mathbb{N}$ *we have*

$$\overline{\{f_\infty(a_n,b) : n \ge m\}} \cap [f_\infty(a,b) + v - C] \ne \emptyset.$$

If $a_n = 0$, *for all* $n \in \mathbb{N}$, *one has the obvious relation for every* $b \in [0,1]$, $v \in C$

$$\overline{\{f_\infty(0,b) : n \ge m\}} \cap [f_\infty(0,b) + v - C] \ne \emptyset, \ \forall m \in \mathbb{N}.$$

If there exists a $k \in \mathbb{N}$ *such that* $a_k \ne 0$, *then one has that*

$$f_\infty(a_k, 0) \in \operatorname{Liminf} f_\infty(a_n, 0). \tag{12}$$

Indeed, $f_\infty(a_k,0)$ *is an inferior point, because otherwise it has to exist an* $j > k$ *such that*

$$(a_j, 1 - a_j) \in (a_k, 1 - a_k) - \operatorname{Int} C.$$

This implies that

$$\begin{cases} a_j > a_k \\ 1 - a_j > 1 - a_k, \end{cases}$$

which is a contradiction. Similarly we can prove that $f_\infty(a_k,0)$ *is a superior point.*

Since $f_\infty(a_k, 0) \in (-\operatorname{Int} C)$*, it follows from (12), that*

$$\operatorname{Liminf} f_\infty(a_n, 0) \cap (-\operatorname{Int} C) \neq \emptyset,$$

so f_∞ *is vector topologically pseudomonotone.*

If $a_n = 1/n$ *for all* $n \in \mathbb{N}$*, the assumption* $ii')$ *holds. Indeed, from Theorem 1, it follows that*

$$(0,0) \in \operatorname{Liminf}(f_n(a_n, b_n) - f_\infty(a_n, b)),$$

where $b_n \to b$*. We have* $(n, 1/n) \in \mathsf{GraphS}$ *for each* $n \in \mathbb{N}$*,* $S(\infty) = \{1\}$*, so* $0 \notin S(\infty)$*. Hence* S *is not closed at* ∞*.*

If the $(VEP)_p$ is defined on constant domains, $D_p = X$ for all $p \in P$, we can omit the Mosco convergence. In this case condition $ii)$ can be weakened.

Theorem 3 *Let* (X, σ) *be a Hausdorff topological space, and let* $p_0 \in P$ *be fixed. Suppose that* $S(p) \neq \emptyset$*, for each* $p \in P$*, and*

i) For each net of elements $(p_i, a_{p_i}) \in \mathsf{GraphS}$*, if* $p_i \to p_0$*,* $a_{p_i} \xrightarrow{\sigma} a$*, and* $b \in X$*, then*

$$\operatorname{Liminf}(f_{p_i}(a_{p_i}, b) - f_{p_0}(a_{p_i}, b)) \cap (-\operatorname{Int} C) \neq \emptyset.$$

ii) $f_{p_0} : X \times X \to \mathcal{Z}$ *is vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ *is closed at* p_0*.*

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