Contents

Methodological papers

ANNALES MATHEMATICAE ET INFORMATICAE 32. (2005)

ANNALES MATHEMATICAE ET INFORMATICAE

TOMUS 32. (2005)

HUNGARIA, EGER

ANNALES MATHEMATICAE ET INFORMATICAE

VOLUME 32. (2005)

INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE
ESZTERHÁZY KÁROLY COLLEGE
HUNGARY, EGER

ISSN 1787-5021

A kiadásért felelős:
az Eszterházy Károly Főiskola rektora
Megjelent az EKF Líceum Kiadó gondozásában
Kiadóvezető: Kis-Tóth Lajos
Felelős szerkesztő: Tömösközi Péter
Műszaki szerkesztő: Tómács Tibor
Megjelent: 2005. december Példányszám: 50

Készítette: Diamond Digitális Nyomda, Eger
Ügyvezető: Hangácsi József

Note from the Editor

As you may observed, starting with the present volume our journal changes its name and editors. Formerly known as *Acta Academiae Paedagogicae Agriensis, Sectio Mathematicae*, established in 1974, the journal mainly served as a scientific appearance for the local mathematical community of the Eszterházy Károly College. From this issue, with extended multinational Editorial Board and peer-reviewing process we try to reach a higher international standard of scientific publication. Despite these developments we would like to preserve the traditional values of our journal: consistency and commitment for high quality. The scope of the journal also remains unchanged: original manuscripts from all fields of mathematics and computer science are welcome.

Miklós Hoffmann
Managing Editor

Annales Mathematicae et Informaticae
***32** (2005) pp. 5–44.*

Quadrature rules for periodic integrands. Bi-orthogonality and para-orthogonality*

Ruymán Cruz-Barroso[a], Leyla Daruis[a], Pablo González-Vera[a], Olav Njåstad[b]

[a]La Laguna University
ldaruis@ull.es, pglez@ull.es

[b]Norwegian University of Science and Technology

Abstract

In this paper, the algebraic construction of quadrature formulas for weighted periodic integrals is revised. For this purpose, two classical papers ([10] and [14]) in the literature are revisited and certain relations and connections are brought to light. In this respect, the concepts of "bi-orthogonality" and "para-orthogonality" are shown to play a fundamental role.

Key Words: Trigonometric polynomials, Szegő polynomials, quadratures, bi-orthogonality, para-orthogonality.

AMS Classification Number: 41A55, 33C45

1. Introduction

Let the integral $I_n(f) = \int_\Gamma f(z)d\mu(z)$ be given with Γ a certain curve in the complex plane and $d\mu$ a positive measure on Γ. By an n-point quadrature rule for this integral we mean an expression like $I_n(f) = \sum_{j=1}^n A_j f(z_j)$ with $z_j \neq z_k$ if $j \neq k$ and $\{z_j\}_{j=1}^n \subset \Gamma$ so that the weights or coefficients $\{A_j\}_{j=1}^n$ and nodes $\{z_j\}_{j=1}^n$ are to be determined by imposing that $I_n(f)$ exactly integrates i.e. $I_n(f)$ coincides with $I_\mu(f)$ for as many basis functions as possible in an appropriate function space S where the above integral exists. Two situations have been most widely considered in the literature. Namely, on the one hand, the case when Γ coincides with a subinterval of the real line, that is, $\Gamma = [a, b]$, $-\infty \leqslant a < b \leqslant \infty$ and on the other hand when Γ is the unit circle, i.e. $\Gamma = \mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$. Observe that the second case is equivalent to dealing with real integrals of the form $\int_{-\pi}^{\pi} f(\theta)d\mu(\theta)$, f being a 2π-periodic function (here by a slight abuse of notation

*This work is partially supported by the research project PI-2002-023 of Gobierno de Canarias.

we write $f(z) = f(\theta)$, $d\mu(z) = d\mu(\theta)$ for $z = e^{i\theta}$). As for the first case, it is well known that the construction of quadrature formulas to approximate integrals like $\int_a^b f(x)d\mu(x)$ represents an interesting research topic which has been exhaustively considered in the last decades and where orthogonal polynomials find one of their most direct and natural applications. Indeed, if $\{Q_k\}_{k=0}^{\infty}$ denotes the sequence of orthonormal polynomials for the measure μ, then $I_n(f) = \sum_{j=1}^n A_j f(x_j)$ with $\{x_j\}_{j=1}^n$ the zeros of $Q_n(x)$ and $\lambda_j = \left(\sum_{k=0}^n Q_k^2(x_j)\right)^{-1}$ for $j = 1, \ldots, n$ (Christoffel numbers) satisfies $I_n(P) = \int_a^b P(x)d\mu(x)$ for any polynomial P of degree $2n-1$. In this case, $\{I_n(f) \ : \ n = 1, 2, \ldots\}$ represent the well known sequence of Gaussian or Gauss–Christoffel quadrature formulas (see e.g. [8] for a survey). On the other hand, although quadratures on the unit circle and other related topics such as Szegő polynomials and the trigonometric moment problem have been receiving much recent attention because of their applications in rapidly growing fields of pure and applied mathematics (Digital Signal Processing, Operator Theory, Probability Theory, ...), there do not exist so many results about quadratures on the unit circle as in the real case. In this respect, the main aim of this paper is to emphasize the role played by certain sequences of orthogonal trigonometric polynomials introduced by Szegő [14] in the construction of quadrature rules on the unit circle by carrying out a comparision with the approach given by Jones et. al in [10]. In both approaches, a fundamental tool will be the so-called Szegő polynomials or polynomials which are orthogonal on the unit circle in the following sense: given $n \geqslant 1$, it is known (see e.g. [13]) that a unique monic polynomial $\rho_n(z)$ exists such that $\int_{-\pi}^{\pi} \rho_n(e^{i\theta})e^{-ik\theta}d\mu(\theta) = 0$ for $k = 0, 1, \ldots, n-1$. Furthermore, if we assume that the support of μ has infinitely many points, then $\int_{-\pi}^{\pi} \rho_n^2(e^{i\theta})d\mu(\theta) = \| \rho_n \|_\mu^2 > 0$. Setting $\rho_0 \equiv 1$, then $\{\rho_n\}_{n=0}^{\infty}$ is called the orthogonal sequence of monic Szegő polynomials. On the other hand, the sequence $\{\varphi_n\}_{n=0}^{\infty}$ with $\varphi_n(z) = \frac{\rho_n(z)}{\|\rho_n\|_\mu}$ represents an orthonormal sequence of Szegő polynomials (observe that such a sequence is uniquely determined by assuming that the leading coefficient of $\varphi_n(z)$ for $n = 0, 1, \ldots$ is positive). Setting $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ (sometimes we will use $\mathbb{E} = \{z \in \mathbb{C} \ : \ |z| > 1\}$, $\mathbb{C} = \mathbb{T} \cup \mathbb{D} \cup \mathbb{E}$) a fundamental property concerning the zeros of $\rho_n(z)$ for $n \geqslant 1$ (and apparently rather negative for our purposes) is the following (see e.g. [1]): "For each $n \geqslant 1$, all the zeros of $\rho_n(z)$ lie in $\mathbb{D}$". Thus, unlike the Gauss–Christoffel formulas, now the zeros of Szegő polynomials can not be directly used as nodes in our quadratures. Following two initially different paths, throughout the paper we will see how this drawback can be overcome. The paper is organized as follows. In Section 2, some preliminary results concerning trigonometric polynomials, Laurent polynomials and algebraic polynomials are presented. Then, in Section 3 the problem of the interpolation by trigonometric polynomials is analyzed whereas in Section 4 the so-called bi-orthogonal systems of trigonometric polynomials are introduced and their most relevant properties studied. The construction of quadrature rules exactly integrating trigonometric polynomials with degree as large as possible is considered in Section 5 and a connection with the unit circle presented in Section 6. Finally some illustrative numerical experiments are shown in Section 7.

2. Preliminary results

We will start by fixing some definitions and notations. Thus, for a nonnegative integer n, Π_n will denote the space of (in general complex) algebraic polynomials of degree n at most and Π the space of all polynomials. On the other hand, a real trigonometric polynomial of degree n is a function of the form

$$T_n(\theta) = a_0 + \sum_{k=1}^{n} (a_k \cos k\theta + b_k \sin k\theta), \quad a_k, b_k \in \mathbb{R}, \quad |a_n| + |b_n| > 0.$$

Clearly, when a_0, a_k and b_k are in general complex numbers for $k = 1, \dots, n$, we shall be dealing with trigonometric polynomials with complex coefficients. Thus, when we refer to a trigonometric polynomial we are implicitly meaning with real coefficients. We also denote by $\mathcal{T}_n$ the space of trigonometric polynomials of degree n at most, i.e.

$$\mathcal{T}_n = span\{1, \cos\theta, \sin\theta, \dots, \cos n\theta, \sin n\theta\}$$

and hence, $dim\,(\mathcal{T}_n) = 2n + 1$. We occasionally deal with complex trigonometric polynomials, where a_0, a_k and b_k are arbitrary complex numbers. By using the transformation $z = e^{i\theta}$ and Euler's formulas, for any complex trigonometric polynomial one can write $T_n(\theta) = L_n(e^{i\theta})$ where

$$L_n(z) = \sum_{k=-n}^{n} c_k z^k. \tag{2.1}$$

Then

$$c_0 = a_0, \quad c_k = \frac{1}{2}(a_k - ib_k), \quad k = 1, \dots, n,$$

and when the trigonometric polynomial T_n is real, a_0, a_k, b_k are real and $c_{-k} = \overline{c_k}$. Functions $L_n(z)$ as given above are called Laurent polynomials, or more generally, given p and q integers such that $p \leqslant q$, a Laurent polynomial is a function of the form

$$L_n(z) = \sum_{j=p}^{q} \alpha_j z^j, \quad \alpha_j \in \mathbb{C}. \tag{2.2}$$

We also denote by $\Lambda_{p,q}$ the space of Laurent polynomials (2.2). Observe that

$$\Lambda_{p,q} = span\left\{z^k : p \leqslant k \leqslant q\right\}.$$

Hence, $dim\,(\Lambda_{p,q}) = q - p + 1$. Thus, $L_n(z)$ given by (2.1) belongs to $\Lambda_{-n,n}$.

Now, by recalling that a double sequence $\{\mu_k\}_{k=-\infty}^{\infty}$ of complex numbers is said to be "Hermitian" if $\mu_{-k} = \overline{\mu_k}$, a Laurent polynomial $L \in \Lambda_{-n,n}$ is called Hermitian if the sequence of its coefficients is Hermitian. That is, with $L_n(z)$ in (2.1) we have $c_k = \overline{c_k}$ for $k = 0, 1, \dots, n$ and the following trivially holds,

Theorem 2.1. *Let $T_n(\theta)$ be a complex trigonometric polynomial, and set $L_n(e^{i\theta}) = T_n(\theta)$. Then T_n is real if and only if L_n is Hermitian.*

Remark 2.2. If we define $\Lambda_n^H = \{L \in \Lambda_{-n,n} \,:\, L\ Hermitian\}$ then Λ_n^H is a real vector space of dimension $2n+1$ and one can write

$$\mathcal{T}_n = \left\{T(\theta) \,:\, T(\theta) = L(e^{i\theta})\ with\ L \in \Lambda_n^H\right\}.$$

Let us next consider the connection between trigonometric polynomials and certain algebraic polynomials. For this purpose, let $P(z)$ be an algebraic polynomial of degree n, i.e.,

$$P(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C}, \quad a_n \neq 0.$$

Then, the reciprocal $P^*(z)$ of $P(z)$ is a polynomial defined by $P^*(z) = z^n P_*(z)$ where $P_*(z)$ represents the "sub-star" conjugate of $P(z)$, i.e., $P_*(z) = \overline{P\,(1/\bar{z})}$. Thus,

$$P^*(z) = z^n \overline{P\,(1/\bar{z})} = z^n \overline{P}\,(1/\bar{z}) = \sum_{j=0}^{n} \overline{a_{n-j}} z^j$$

where $\overline{P}(z) = \sum_{j=0}^{n} \overline{a_j} z^j$. Now, a usefull property of the polynomials that we shall work with is the following: for $k \in \mathbb{C}\backslash\{0\}$, a polynomial $P(z)$ is called "invariant" or more precisely, "k-invariant" if

$$P^*(z) = kP(z) \ \ \forall z \in \mathbb{C}.$$

Some direct consequences of this definition are:

1. If $P(z)$ is invariant, then $P(0) \neq 0$.

2. Let α be a zero of the invariant polynomial $P(z)$. Then, $1/\bar{\alpha}$ is also a zero of $P(z)$.

3. Let $P(z)$ be an invariant polynomial of odd degree n. Then, the number of zeros of $P(z)$ on $\mathbb{T}$ (counting multiplicities) is also odd. On the other hand, if $P(z)$ is an invariant polynomial with even degree n, it has an even number of zeros on $\mathbb{T}$.

4. Let $P(z)$ be invariant and set $P(z) = \sum_{j=0}^{n} c_j z^j = c_n \prod_{k=1}^{n}(z - z_k)$, then $|P(0)| = |c_0| = |c_n| \prod_{k=1}^{n} |z_k|$ and taking into account that $\prod_{k=1}^{n} |z_k| = 1$ it follows that $|c_0| = |c_n|$. Consequently, $c_n = k\overline{c_0}$ with $|k| = 1$. Set $k = e^{i\omega}$, $\omega \in \mathbb{R}$, and define $Q(z) = \lambda P(z)$, $\lambda \neq 0$. Then, $Q^*(z) = \overline{\lambda} k P(z) = \frac{\overline{\lambda}}{\lambda} k Q(z)$, that is, $Q(z)$ is $\frac{\overline{\lambda}}{\lambda}k$-invariant. Set now $\lambda = Re^{i\gamma}$, then $\frac{\overline{\lambda}}{\lambda}k = e^{i(\omega - 2\gamma)}$. Thus, by taking $\gamma \in \mathbb{R}$ such that $\gamma = \frac{\omega}{2} + m\pi$, with $m \in \mathbb{Z}$, then $\frac{\overline{\lambda}}{\lambda}k = 1$ and $Q(z)$ is 1-invariant.

Remark 2.3. The term "k-invariant" was introduced by Jones et. al. in [10], whereas Szegő in [14] says that a polynomial $P(z)$ is "autoreciprocal" if $P^*(z) = P(z)$ (1-invariant). Hence, we see that "invariant" polynomials are essentially "autoreciprocal".

Let $P_{2n}(z)$ be an invariant polynomial of degree $2n$. Then, there exists $\lambda_{2n} \in \mathbb{C}\backslash\{0\}$ such that $Q_{2n}(z) = \lambda_{2n} P_{2n}(z)$ is 1-invariant and we can write:

$$L_n(z) = \frac{Q_{2n}(z)}{z^n} = \sum_{j=-n}^{n} c_j z^j, \;\; c_{-j} = \overline{c_j}, \; j = 0, 1, \ldots, n$$

that is, $L_n \in \Lambda_n^H$ and by Theorem 2.1, $L_n(e^{i\theta}) = T_n(\theta)$ with $T_n \in \mathcal{T}_n$. Thus, we have

$$e^{-in\theta} P_{2n}(e^{i\theta}) = \lambda_{2n}^{-1} T_n(\theta).$$

Conversely, let $T_n \in \mathcal{T}_n$. Then

$$T_n(\theta) = L_n(e^{i\theta}), \; L_n \in \Lambda_n^H.$$

Again, $L_n(z) = \frac{P_{2n}(z)}{z^n}$, where $P_{2n}(z) \in \Pi_{2n}$ and 1-invariant. Indeed, $P_{2n}(z) = z^n L_n(z)$. Hence,

$$\begin{aligned} P_{2n}^*(z) &= z^{2n}\overline{P_{2n}\left(1/\bar{z}\right)} = z^{2n} z^{-n}\overline{L_n\left(1/\bar{z}\right)} \\ &= z^n \textstyle\sum_{j=-n}^{n} \overline{c_j} z^{-j} = z^n \sum_{j=-n}^{n} c_{-j} z^{-j} = z^n L_n(z) = P_{2n}(z). \end{aligned}$$

Next, we will see how the connection between trigonometric polynomials and invariant algebraic polynomials allows us to recover some classical results about zeros of trigonometric polynomials. Thus, let α and β be arbitrary constants, then $\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta-\beta}{2}\right)$ represents a trigonometric polynomial of degree one. Furthermore, it can be easily proved by induction that the function

$$T(\theta) = C\prod_{j=1}^{n} \sin\left(\frac{\theta-\theta_{2j-1}}{2}\right)\sin\left(\frac{\theta-\theta_{2j}}{2}\right), \;\; C \neq 0 \tag{2.3}$$

where $\{\theta_j\}_{j=1}^{2n}$ are given constants, represents a trigonometric polynomial of degree n. We will now show that a converse result also holds, i.e. any trigonometric polynomial can be factorized as (2.3). Indeed, let $T_n \in \mathcal{T}_n$, then $T_n(\theta) = L_n(e^{i\theta})$, $L_n \in \Delta_n^H$ and one can write $L_n(z) = \frac{P_{2n}(z)}{z^n}$ with $P_{2n}(z)$ an 1-invariant polynomial of degree $2n$. Therefore, $P_{2n}(z) = c_n \prod_{k=1}^{2n}(z - z_k)$, $c_n \neq 0$ (counting multiplicities) with $z_j \neq 0$ and if $z_j \notin \mathbb{T}$, then $1/\bar{z}_j$ is also a root of $P_{2n}(z)$. Let $2m$ denote the number of zeros of $P_{2n}(z)$ on $\mathbb{T}$ ($0 \leqslant m \leqslant n$). Then

$$P_{2n}(z) = c_n \prod_{j=1}^{2m}(z - z_j)\prod_{k=1}^{n-m}\left(z - \tilde{z}_k\right)\left(z - \frac{1}{\overline{\tilde{z}_k}}\right), \; c_n \neq 0 \tag{2.4}$$

where $z_j = e^{i\theta_j}$, $\theta_j \in \mathbb{R}$ for $1 \leqslant j \leqslant 2m$, are the zeros of $P_{2m}(z)$ on $\mathbb{T}$ and $\tilde{z}_k$ and $1/\overline{\tilde{z}_k}$ for $1 \leqslant k \leqslant n-m$ are the zeros not on $\mathbb{T}$, so that $\tilde{z}_k = e^{i\omega_k}$ with $\omega_k \in \mathbb{C}$, wich implies that $1/\overline{\tilde{z}_k} = e^{i\overline{\omega_k}}$. Furthermore, it can be easily checked that $e^{i\theta} - e^{i\omega} = 2i\sin\left(\frac{\theta-\omega}{2}\right)e^{i\left(\frac{\theta+\omega}{2}\right)}$. Therefore,

$$\begin{aligned} P_{2n}(e^{i\theta}) &= c_n \prod_{j=1}^{2m}\left(e^{i\theta}-e^{i\theta_j}\right)\prod_{k=1}^{n-m}\left(e^{i\theta}-e^{i\omega_k}\right)\left(e^{i\theta}-e^{i\overline{\omega_k}}\right) \\ &= c_n(-1)^n 2^{2n}\prod_{j=1}^{2m}\sin\left(\frac{\theta-\theta_j}{2}\right)e^{i\left[\frac{\theta-\theta_j}{2}\right]}\times \\ &\times \prod_{k=1}^{n-m}\sin\left(\frac{\theta-\omega_k}{2}\right)\sin\left(\frac{\theta-\overline{\omega_k}}{2}\right)e^{i\left[\theta+\frac{\omega_k+\overline{\omega_k}}{2}\right]}. \end{aligned}$$

Then, it follows that,

$$P_{2n}(e^{i\theta}) = \lambda_n e^{in\theta}\prod_{j=1}^{2m}\sin\left(\frac{\theta-\theta_j}{2}\right)\prod_{k=1}^{n-m}\sin\left(\frac{\theta-\omega_k}{2}\right)\sin\left(\frac{\theta-\overline{\omega_k}}{2}\right),\quad \lambda_n \neq 0.$$

Consequently,

$$\begin{aligned} T_n(\theta) &= L_n(e^{i\theta}) = \frac{P_{2n}(e^{i\theta})}{e^{in\theta}} \\ &= \lambda_n \prod_{j=1}^{2m}\sin\left(\frac{\theta-\theta_j}{2}\right)\prod_{k=1}^{n-m}\sin\left(\frac{\theta-\omega_k}{2}\right)\sin\left(\frac{\theta-\overline{\omega_k}}{2}\right), \end{aligned} \tag{2.5}$$

where $\lambda_n \neq 0$, $\theta_j \in \mathbb{R}$ and $\omega_k \in \mathbb{C}$ such that $\Re(\omega_k) = \psi_k + 2t\pi$, $\psi_k \in (-\pi, \pi]$, $t \in \mathbb{Z}$ and $k = 1, \ldots, n-m$. Then, we have proved the following

Theorem 2.4. *A real trigonometric polynomial $T_n(\theta)$ of the precise degree n has exactly $2n$ real or complex zeros provided that we count them as usual with their multiplicity and we restrict ourselves to the strip $-\pi < \Re(\theta) \leqslant \pi$. Furthermore, the non-real zeros appear in conjugate pairs.*

Remark 2.5. The representation (2.3) is of course not unique.

Furthermore, from (2.3) and (2.5) it can be also proved

Theorem 2.6 (L.Fejér and F.Riesz). *A real trigonometric polynomial $T(\theta)$ is nonnegative for all real θ, if and only if, it can be written in the form*

$$T(\theta) = |g(z)|^2,\quad z = e^{i\theta}$$

where $g(z)$ is an algebraic polynomial of the same degree as $T(\theta)$.

Proof. Assume $T(\theta)$ a trigonometric polynomial of degree n such that $T(\theta) = \frac{P(e^{i\theta})}{e^{in\theta}}$ with $P(z)$ a polynomial of degree $2n$. Since $T(\theta) \geqslant 0$ for all $\theta \in \mathbb{R}$, then possible real zeros of $T(\theta)$ must have even multiplicity. Furthermore, if $\theta = \alpha$ is a real zero of $T(\theta)$ then $z = e^{i\alpha}$ is a zero of $P(z)$ on $\mathbb{T}$. Hence, from (2.4), $P(z)$ can be expressed as:

$$P(z) = \lambda_n p_m^2(z) q_{n-m}(z) q_{n-m}^*(z),\quad \lambda_n \neq 0$$

where $p_m(z) \in \Pi_m$ for $0 \leqslant m \leqslant n$ and $q_{n-m}(z) \in \Pi_{n-m}$. Since $T(\theta) \geqslant 0$, for any $\theta \in \mathbb{R}$,

$$\begin{aligned} T(\theta) &= |T(\theta)| = \left|\tfrac{P(e^{i\theta})}{e^{in\theta}}\right| = |\lambda_n| \left|p_m^2(e^{i\theta})\right| \left|q_{n-m}(e^{i\theta})\right| \left|\overline{q_{n-m}(e^{i\theta})}\right| \\ &= |\lambda_n| \left|p_m^2(e^{i\theta})\right| \left|q_{n-m}(e^{i\theta})\right|^2 = \left|g(e^{i\theta})\right|^2 \end{aligned}$$

where $g(z) = \sqrt{|\lambda_n|}p_m(z)q_{n-m}(z) \in \Pi_n$.

Conversely, let $g(z)$ be an algebraic polynomial of degree n, then by setting $z = e^{i\theta}$ it follows that

$$|g(z)|^2 = g(z)\overline{g(z)} = g(z)g_*(z) = \frac{g(z)g^*(z)}{z^n} = \frac{P_{2n}(z)}{z^n}$$

where $P_{2n}(z) = g(z)g^*(z)$ is clearly an 1-invariant polynomial of degree $2n$ so that $|g(z)|^2 = L_n(z) \in \Lambda_n^H$, and by Theorem 2.1, $|g(z)|^2$ represents a trigonometric polynomial of degree n which is clearly nonnegative for any $\theta \in \mathbb{R}$. □

3. Interpolation by Trigonometric Polynomials

As it is well known, polynomial interpolation finds in the construction of quadrature formulas one of its most immediate applications. On the other hand, when considering quadrature rules based on trigonometric polynomials, similar results on interpolation will be needed. In this respect, some of the already known results will now be proved by means of the close connection between trigonometric polynomials and Hermitian Laurent polynomials shown in the preceding section. First we have,

Theorem 3.1. *Given* $(2n+1)$ *distinct nodes* $\{\theta_j\} \subset (-\pi, \pi]$*, there exists a unique* $T_n \in \mathcal{T}_n$ *such that*

$$T_n(\theta_j) = y_j, \quad j = 1, \ldots, 2n+1, \tag{3.1}$$

$\{y_j\}_{j=1}^{2n+1}$ *being a given set of real numbers.*

Proof. Set $T(\theta) = a_0 + \sum_{k=1}^n a_k \cos k\theta + b_k \sin k\theta$. We first show that the constants $\{a_k\}_{k=0}^n \cup \{b_k\}_{k=1}^n$ are uniquely determined from conditions (3.1). Now, $T(\theta) = L(e^{i\theta})$ with $L \in \Lambda_{-n,n}$ so that (3.1) is equivalent to

$$L(z_j) = y_j, \quad z_j = e^{i\theta_k}, \quad j = 1, \ldots, 2n+1. \tag{3.2}$$

Now $L(z) \in \Lambda_{-n,n}$ implies that $L(z) = \frac{P(z)}{z^n}$, $P(z) \in \Pi_{2n}$ so that (3.2) yields

$$P(z_j) = z_j^n y_j, \quad j = 1, \ldots, 2n+1. \tag{3.3}$$

Since $z_j \neq z_k$, $P(z)$ is uniquely determined by (3.3) and hence $T(\theta)$ has the desired interpolation properties. It remains to show that $T(\theta)$ has real coefficients. This will be proved by showing that $P(z)$ is 1-invariant. To see this we will show that also $P^*(z)$ satisfies the interpolation conditions (3.3). Indeed,

$$P^*(z_j) = z_j^{2n}\overline{P\left(1/\bar{z}_j\right)} = z_j^{2n}\overline{P(z_j)} = z_j^{2n}\overline{z_j^n y_j} = z_j^n y_j, \quad y_j \in \mathbb{R}.$$

Hence, by virtue of the uniqueness of polynomial $P(z)$, it follows that $P^*(z) = P(z)$ and the proof is completed. □

As for an explicit representation of $T_n \in \mathcal{T}_n$ satisfying (3.1), because of uniqueness, one can write

$$T_n(\theta) = \sum_{j=1}^{2n+1} l_j(\theta) y_j \tag{3.4}$$

where $l_j(\theta) = l_{j,n}(\theta) \in \mathcal{T}_n$ such that $l_j(\theta_k) = \delta_{j,k} = \begin{cases} 1 & if \quad j = k \\ 0 & if \quad j \neq k \end{cases}$. Since $l_j(\theta_k) = 0$ for $k = 1, \ldots, 2n+1$, $k \neq j$, clearly by (2.5),

$$l_j(\theta) = \lambda_j \prod_{k=1, k\neq j}^{2n+1} \sin\left(\frac{\theta - \theta_k}{2}\right), \quad \lambda_j \neq 0,$$

λ_j being a normalization constant such that $l_j(\theta_j) = 1$. More precisely, setting

$$W_n(\theta) = \prod_{k=1}^{2n+1} \sin\left(\frac{\theta - \theta_k}{2}\right)$$

then, it follows that

$$l_j(\theta) = \lambda_j \frac{W_n(\theta)}{\sin\left(\frac{\theta - \theta_j}{2}\right)}, \quad j = 1, \ldots, 2n+1.$$

Thus,

$$l_j(\theta_j) = \lambda_j \lim_{\theta \to \theta_j} \frac{W_n(\theta)}{\sin\left(\frac{\theta - \theta_j}{2}\right)} = \lambda_j \lim_{\theta \to \theta_j} \frac{W_n(\theta)}{\frac{\theta - \theta_j}{2}} = 2\lambda_j W_n^{'}(\theta_j).$$

Hence, taking $\lambda_j = \frac{1}{2W_n'(\theta_j)}$ one has $l_j(\theta_j) = 1$ and we can write

$$l_j(\theta) = \frac{W_n(\theta)}{2W_n'(\theta_j)\sin\left(\frac{\theta - \theta_j}{2}\right)} \quad , \; j = 1, \ldots, 2n+1.$$

Furthermore, when dealing with the construction of certain quadrature formulas exactly integrating trigonometric polynomials of degree as high as possible, the following result will be required:

Theorem 3.2. *Let $\theta_1 \dots \theta_{n+1}$ be $(n+1)$ distinct nodes on $(-\pi, \pi]$. Then there exists a unique trigonometric polynomial $H_n \in \mathcal{T}_n$ satisfying*

$$\left.\begin{array}{ll} H_n(\theta_j) = H_n^{(k)}(\theta_j) = y_j & j = 1, \dots, n+1 \\ \\ H_n^{'}(\theta_j) = H_n^{(k)\,'}(\theta_j) = y_j^{'} & j = 1, \dots, n+1,\ j \neq k \end{array}\right\} \tag{3.5}$$

where $k \in \{1, \dots, n+1\}$ is previously fixed and $\{y_j\}_{j=1}^{n+1} \cup \{y_j^{'}\}_{j=1, j\neq k}^{n+1}$ is a set of $(2n+1)$ real numbers.

Proof. Set $H_n(\theta) = L_n(e^{i\theta}) \in \Lambda_{-n,n}$. Then (3.5) becomes $H_n(\theta_j) = L_n(e^{i\theta_j}) = L_n(z_j) = y_j$ with $z_j = e^{i\theta_j} \in \mathbb{T}$ for all $j = 1, \dots, n+1$ and $z_j \neq z_k$ if $j \neq k$. On the other hand, $H_n^{'}(\theta) = L_n^{'}(e^{i\theta}) i e^{i\theta}$. Hence, $L_n^{'}(z_j) = -i\overline{z_j} H_n^{'}(\theta_j) = -i\overline{z_j} y_j^{'}$ for $j = 1, \dots, n+1$ and $j \neq k$. Since $L_n \in \Lambda_{-n,n}$, then $L_n(z) = \frac{P_{2n}(z)}{z^n}$ with $P_{2n}(z) \in \Pi_{2n}$ such that $P_{2n}(z_j) = z_j^n L_n(z_j) = z_j^n y_j$, $y_j \in \mathbb{R}$ and $z_j \in \mathbb{T}$. Furthermore, $P_{2n}^{'}(z) = n z^{n-1} L_n(z) + z^n L_n^{'}(z)$, hence

$$P_{2n}^{'}(z_j) = n z_j^{n-1} L_n(z_j) + z_j^n L_n^{'}(z_j) = z_j^{n-1}\left(n y_j - i y_j^{'}\right),\ j = 1, \dots, n+1,\ j \neq k.$$

Thus our Hermite-type trigonometric interpolation problem reduces to finding $P_{2n}(z) \in \Pi_{2n}$ such that

$$\left.\begin{array}{ll} P_{2n}(z_j) = z_j^n y_j & j = 1, \dots, n+1 \\ \\ P_{2n}^{'}(z_j) = z_j^{n-1}\left(n y_j - i y_j^{'}\right) & j = 1, \dots, n+1,\ j \neq k \end{array}\right\} \tag{3.6}$$

Now, since $z_j \neq z_l$ for $j \neq l$, it is known that the interpolation problem (3.6) has a unique solution $P_{2n}(z)$ and $T_n(\theta) = L_n(e^{i\theta}) = \frac{P_{2n}(e^{i\theta})}{e^{in\theta}}$ will be the unique solution to (3.5). As in Theorem 3.1, it remains to prove that $T_n(\theta)$ is a real trigonometric polynomial. To do this, we will show that $P_n^*(z)$ is also a solution to (3.6), hence because of uniqueness we have $P_{2n}(z) = P_{2n}^*(z)$ and the conclusion follows. Indeed,

$$\begin{aligned} P_{2n}^*(z_j) &= z_j^{2n}\overline{P_{2n}\left(1/\overline{z_j}\right)} = z_j^{2n}\overline{P_{2n}(z_j)} \\ &= z_j^{2n}\overline{z_j^n y_j} = z_j^n y_j = P_{2n}(z_j),\ j = 1, \dots, n+1. \end{aligned}$$

Furthermore, $\left(P_{2n}^*\right)^{'}(z) = 2n z^{2n-1}\overline{P_{2n}}(1/z) + z^{2n}\left(\overline{P_{2n}}\right)^{'}(1/z)\left(\frac{-1}{z^2}\right)$, yielding:

$$\left(P_{2n}^*\right)^{'}(z_j) = z_j^{2n-2}\left[2n z_j \overline{P_{2n}(z_j)} - \overline{P_{2n}^{'}(z_j)}\right].$$

(Here we are making use of the fact $\left(\overline{P}\right)^{'}(z) = \overline{(P^{\,'})}(z)$). Therefore, for $j = 1, \dots, n+1,\ j \neq k$:

$$\begin{aligned} \left(P_{2n}^*\right)^{'}(z_j) &= z_j^{2n-2}\left[2n z_j \overline{z_j^n} y_j - z_j^{-(n-1)}(n y_j + i y_j^{'})\right] \\ &= z_j^{n-1}\left[2n y_j - n y_j - i y_j^{'}\right] = z_j^{n-1}[n y_j - i y_j^{'}] = P_{2n}^{'}(z_j). \end{aligned}$$

□

As for an explicit representation of the interpolating trigonometric polynomial $H_n(\theta)$ satisfying (3.5), by virtue of uniqueness we can write for any $k \in \{1, \dots, n+1\}$,

$$H_n(\theta) = H_n^{(k)}(\theta) = t_k^{(k)}(\theta) y_k + \sum_{j=1, j\neq k}^{n+1} \left[t_j^{(k)}(\theta) y_j + s_j^{(k)}(\theta) y_j' \right] \tag{3.7}$$

where $t_j^{(k)}(\theta)$ and $s_j^{(k)}(\theta)$ are trigonometric polynomials in $\mathcal{T}_n$, such that

$$\begin{aligned}
&t_j^{(k)}(\theta_r) = \delta_{j,r} && 1 \leqslant j, r \leqslant n+1 \\
&\left(t_j^{(k)}\right)'(\theta_r) = 0 && 1 \leqslant j, r \leqslant n+1,\ r \neq k \\
&s_j^{(k)}(\theta_r) = 0 && 1 \leqslant r \leqslant n+1,\ j \neq k \\
&\left(s_j^{(k)}\right)'(\theta_r) = \delta_{j,r} && 1 \leqslant j, r \leqslant n+1,\ r \neq k,\ j \neq k.
\end{aligned} \tag{3.8}$$

Define now $W_n(\theta) = \prod_{j=1}^{n+1} \sin\left(\frac{\theta-\theta_j}{2}\right)$. If we proceed as in the previous case, after some elementary calculations we deduce the following expressions for such trigonometric polynomials for $1 \leqslant j \leqslant n+1,\ j \neq k$:

$$s_j^{(k)}(\theta) = \frac{W_n^2(\theta) \sin\left(\frac{\theta_j - \theta_k}{2}\right)}{2 \sin\left(\frac{\theta-\theta_j}{2}\right) \sin\left(\frac{\theta-\theta_k}{2}\right) \left[W_n'(\theta_j)\right]^2} \in \mathcal{T}_n, \tag{3.9}$$

$$\begin{aligned}
t_j^{(k)}(\theta) \;&= \frac{W_n^2(\theta)}{\sin^2 \frac{\theta-\theta_j}{2} \ \sin \frac{\theta-\theta_k}{2} \ \left[2W_n'(\theta_j)\right]^2} \times \\
&\times \left[\sin\left(\frac{\theta_j-\theta_k}{2}\right) + \cos\left(\frac{\theta_j-\theta_k}{2}\right) \sin\left(\frac{\theta-\theta_j}{2}\right) \right] \in \mathcal{T}_n
\end{aligned} \tag{3.10}$$

and

$$t_k^{(k)}(\theta) = \left[\frac{W_n(\theta)}{2W_n'(\theta_k) \sin\left(\frac{\theta-\theta_k}{2}\right)} \right]^2 \in \mathcal{T}_n. \tag{3.11}$$

In the rest of the section we shall be concerned with certain interpolation problems using an even number of nodes, say $2n$, in subspaces $\tilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ of dimension $2n$. For instance, $\tilde{\mathcal{T}}_n = \mathcal{T}_n \backslash span\{\cos n\theta\}$ or $\tilde{\mathcal{T}}_n = \mathcal{T}_n \backslash span\{\sin n\theta\}$. In this respect, it should be recalled that a system of continuous functions $\{f_0, \dots, f_m\}$ on an interval $[a, b]$ represents a Haar system on $[a, b]$ if and only if for any k, $1 \leqslant k \leqslant m$, $\{f_0, \dots, f_k\}$ is a Chebyshev system on $[a, b]$. Clearly,

$$\{1, \cos\theta, \sin\theta, \dots, \cos n\theta, \sin n\theta\},$$

can not be a Haar system on $[-\pi, \pi]$ (check simply that $\{1, \cos\theta\}$ is not a Chebyshev system). Hence, we can not initially assume that given $2n$ nodes $\{\theta_j\}_{j=1}^{2n}$ on $(-\pi, \pi]$ there exists $T_n \in \mathcal{T}_n \backslash span\{\cos n\theta\}$ or in $\mathcal{T}_n \backslash span\{\sin n\theta\}$ such that $T_n(\theta_j) = y_j$ for all $j = 1, \dots, 2n$. However, we can prove the following

Theorem 3.3. *Let $\{\theta_j\}_{j=1}^{2n} \subset (-\pi, \pi]$ be $2n$ distinct nodes, let $\{y_j\}_{j=1}^{2n}$ be arbitrary real numbers, and consider the interpolation problem:*

$$\tilde{T}_n(\theta_j) = y_j, \quad j = 1, \ldots, 2n. \tag{3.12}$$

Then the following hold:

1. *If $\sum_{j=1}^{2n} \theta_j \neq k\pi$ for all $k \in \mathbb{Z}$, then there is a unique solution of (3.12) in $\mathcal{T}_n \backslash span\{\cos n\theta\}$ and a unique solution of (3.12) in $\mathcal{T}_n \backslash span\{\sin n\theta\}$.*
2. *If $\sum_{j=1}^{2n} \theta_j = k\pi$ for an odd integer k, then there is a unique solution of (3.12) in $\mathcal{T}_n \backslash span\{\cos n\theta\}$.*
3. *If $\sum_{j=1}^{2n} \theta_j = k\pi$ for an even integer k, then there is a unique solution of (3.12) in $\mathcal{T}_n \backslash span\{\sin n\theta\}$.*

Proof. Assume first that we are trying to find $\tilde{T}_n(\theta) \in \mathcal{T}_n \backslash span\{\sin n\theta\}$ satisfying (3.12). Thus, we can write:

$$\tilde{T}_n(\theta) = a_0 + \sum_{j=1}^{n-1} (a_j \cos j\theta + b_j \sin j\theta) + a_n \cos n\theta = L_n(e^{i\theta}) \in \Lambda_{-n,n}$$

with $L_n(z) = \sum_{j=-n}^{n} c_j z^j$, where

$$c_j = \frac{a_j - ib_j}{2}, \; c_{-j} = \overline{c_j}, \; 1 \leqslant j \leqslant n-1, \; c_0 = a_0.$$

Thus, $c_{-j} = \overline{c_j}$ for all $0 \leqslant j \leqslant n$. Setting as usual $z_j = e^{i\theta_j}$ for all $j = 1, \ldots, 2n$, ($z_j \neq z_k$ if $j \neq k$), (3.12) becomes

$$\tilde{T}_n(\theta_j) = L_n(e^{i\theta_j}) = L_n(z_j) = y_j, \; j = 1, \ldots, 2n$$

giving rise to the linear system

$$\sum_{k=-(n-1)}^{n-1} c_k z_j^k + c_n(z_j^n + z_j^{-n}) = y_j, \; j = 1, \ldots, 2n. \tag{3.13}$$

Now, the system (3.13) has a unique solution if and only if $\Delta_n \neq 0$, where

$$\Delta_n = \begin{vmatrix} z_1^{-(n-1)} & z_1^{-(n-2)} & \cdots & 1 & \cdots & z_1^{n-1} & \left(z_1^n + z_1^{-n}\right) \\ z_2^{-(n-1)} & z_2^{-(n-2)} & \cdots & 1 & \cdots & z_2^{n-1} & \left(z_2^n + z_2^{-n}\right) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ z_{2n}^{-(n-1)} & z_{2n}^{-(n-2)} & \cdots & 1 & \cdots & z_{2n}^{n-1} & \left(z_{2n}^n + z_{2n}^{-n}\right) \end{vmatrix}.$$

By introducing the Vandermonde determinant associated with $z_1, \dots, z_{2n}$, i.e.,

$$\gamma_n = \begin{vmatrix} 1 & z_1 & \cdots & z_1^{2n-1} \\ 1 & z_2 & \cdots & z_2^{2n-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_{2n} & \cdots & z_{2n}^{2n-1} \end{vmatrix}$$

it can be easily checked that

$$\Delta_n = (z_1 \cdots z_{2n})^{n-1}(1 - z_1 \cdots z_{2n})\gamma_n. \tag{3.14}$$

On the other hand, if we consider our interpolation problem in $\tilde{T}_n(\theta) \in \mathcal{T}_n \backslash span\{\cos n\theta\}$, the associated determinant $\tilde{\Delta}_n$ of the corresponding system satisfies

$$\tilde{\Delta}_n = (z_1 \cdots z_{2n})^{n-1}(1 + z_1 \cdots z_{2n})\gamma_n. \tag{3.15}$$

Since $z_j = e^{i\theta_j}$, then $z_1 \cdots z_{2n} = e^{i\lambda_n}$ with $\lambda_n = \sum_{j=1}^{2n} \theta_j$. If $\lambda_n \neq k\pi$ for any integer k, then clearly $z_1 \cdots z_{2n} \neq \pm 1$ and from (3.14) and (3.15), both determinants Δ_n and $\tilde{\Delta}_n$ are nonzero since $\gamma_n \neq 0$, which means that the interpolation problem (3.12) has a unique solution both in $\mathcal{T}_n \backslash span\{\sin n\theta\}$ and $\mathcal{T}_n \backslash span\{\cos n\theta\}$. Next, assume that $\lambda_n = k\pi$ for some integer k. Thus, if k is even, then $e^{i\lambda_n} = 1$ and (3.15) is different from zero, whereas if k is odd, then $e^{i\lambda_n} = -1$ and (3.14) does not vanish. Thus, for instance, if $\Delta_n \neq 0$, we have found a unique $L_n \in \Lambda_{-n,n}$, $L_n(z) = \sum_{j=-n}^{n} c_j z^j$ such that $c_{-n} = c_n$ and satisfying $L_n(z_j) = y_j$ for $j = 1, \dots, 2n$. Therefore, $\tilde{T}_n(\theta) = L_n(e^{i\theta}) \in \mathcal{T}_n \backslash span\{\sin n\theta\}$ and $\tilde{T}_n(\theta_j) = y_j$ for $j = 1, \dots, 2n$. To check that $\tilde{T}_n(\theta)$ is actually a real trigonometric polynomial we proceed as in Theorem 3.1. □

Next, a Lagrange-type representation for the trigonometric polynomial $\tilde{T}_n(\theta)$ satisfying the conditions of Theorem 3.3 will be given. Indeed, set

$$\eta_n = \frac{1}{2} \sum_{j=1}^{2n} \theta_j = \frac{1}{2} \lambda_n$$

and assume that $\eta_n \neq k\pi$ for any integer k so that $\Delta_n \neq 0$. Thus, $\tilde{T}_n(\theta) \in \mathcal{T}_n \backslash span\{\sin n\theta\}$ and by virtue of uniqueness, one has $\tilde{T}_n(\theta) = \sum_{j=1}^{2n} \tilde{t}_j(\theta) y_j$ where $\tilde{t}_j \in \mathcal{T}_n \backslash span\{\sin n\theta\}$ and $\tilde{t}_j(\theta_k) = \delta_{j,k}$ for $1 \leqslant j, k \leqslant 2n$. Fix $j \in \{1, \dots, 2n\}$ and define $\alpha_j = \sum_{k=1, k\neq j}^{2n} \theta_j$. Now, we can write $\tilde{s}_j(\theta) = \frac{\tilde{l}_j(e^{i\theta})}{e^{in\theta}}$ where $\tilde{l}_j(z) \in \Pi_{2n}$ such that $\tilde{l}_j(z_k) = z_j^n \delta_{j,k}$ where, as usual, $z_k = e^{i\theta_k}$ for $k = 1, \dots, 2n$. Since $\tilde{t}_j \in \mathcal{T}_n \backslash span\{\sin n\theta\}$, the leading coefficient of $\tilde{l}_j(z)$ must coincide with $\tilde{l}_j(0)$, and one has $\tilde{l}_j(z) = c_j(z - w_j) \prod_{k=1, k\neq j}^{2n} (z - z_j) = c_j z^{2n} + \cdots + \tilde{l}_j(0)$. But $\tilde{l}_j(0) = c_j w_j \prod_{j=1, j\neq k}^{2n} z_j$, hence

$$w_j = \frac{1}{\prod_{j=1, j\neq k}^{2n} z_j} = \prod_{j=1}^{2n} \overline{z_j} = e^{-\sum_{j=1, j\neq k}^{2n} \theta_j} = e^{-i\alpha_j}.$$

Therefore, by (2.5) it follows that

$$\tilde{s}_j(\theta) = \tilde{c}_j \sin\left(\frac{\theta+\alpha_j}{2}\right) \prod_{j=1, j\neq k}^{2n} \sin\left(\frac{\theta-\theta_j}{2}\right)$$

where $\tilde{c}_j$ is to be determined such that $\tilde{s}_j(\theta_j) = 1$. Setting

$$W_n(\theta) = \prod_{j=1}^{2n} \sin\left(\frac{\theta-\theta_j}{2}\right) \in \mathcal{T}_n,$$

we have

$$\tilde{s}_j(\theta) = \tilde{c}_j \sin\left(\frac{\theta+\alpha_j}{2}\right) \frac{W_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)}.$$

Now,

$$1 = \lim_{\theta\to\theta_j} \tilde{c}_j \sin\left(\frac{\theta+\alpha_j}{2}\right) \frac{W_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)} = 2\tilde{c}_j \sin\left(\frac{\theta_j+\alpha_j}{2}\right) W_n^{'}(\theta_j).$$

Observe that $\frac{1}{2}(\theta_j+\alpha_j) = \frac{1}{2}\sum_{j=1}^{2n}\theta_j = \eta_n \neq k\pi$ for any integer k, so that $\sin\left(\frac{\theta_j+\alpha_j}{2}\right) = \sin\eta_n \neq 0$ and hence

$$\tilde{s}_j(\theta) = \frac{1}{2W_n^{'}(\theta_j)\sin\eta_n} \sin\left(\frac{\theta+\alpha_j}{2}\right) \frac{W_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)}, \quad j = 1, \ldots, 2n. \tag{3.16}$$

When dealing with the interpolant $\tilde{T}_n(\theta) \in \mathcal{T}_n \backslash span\{\cos n\theta\}$ it can be easily verified that the fundamental Lagrange-type trigonometric polynomials $\tilde{s}_j(\theta)$ are now given by

$$\tilde{s}_j(\theta) = \frac{1}{2W_n^{'}(\theta_j)\cos\eta_n} \cos\left(\frac{\theta+\alpha_j}{2}\right) \frac{W_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)}, \quad j = 1, \ldots, 2n. \tag{3.17}$$

with α_j and η_n as previously given.

4. Bi-orthogonal systems

Let $\omega(\theta)$ be a weight function on $(-\pi,\pi]$, i.e., $\omega(\theta) \geqslant 0$ on $(-\pi,\pi]$ and $0 < \int_{-\pi}^{\pi}\omega(\theta)d\theta < \infty$. The main aim of this section is briefly collecting some results by Szegő (see [14]) concerning properties of an orthogonal basis for the space $\mathcal{T}$ of real trigonometric polynomials with respect to the inner product on $\mathcal{T}$ induced by $\omega(\theta)$, namely,

$$\langle f, g\rangle_\omega = \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\omega(\theta)d\theta, \;\; \forall\, f, g \in \mathcal{T} \tag{4.1}$$

As indicated in [14], we might consider an arbitrary measure $d\mu(\theta)$ on the unit circle; in what follows, however we restrict ourselves for the sake of simplicity, to the previously defined case, i.e. to the case when $\mu(\theta)$ is absolutely continuous. Furthermore, when only real functions are considered, complex conjugation in (4.1) can be omitted. For this purpose, let us first consider the basis of $\mathcal{T}_n$ given by $\{1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta\}$ which is clearly orthogonal for $\omega(\theta) \equiv 1$ on $[-\pi, \pi]$ and let us see how this property can be extended to an arbitrary weight function $\omega(\theta)$. Certainly, this can be done by orthogonalizing the elementary functions

$$1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta$$

arranged in a linear order, according to Gram-Schmidt process. Thus, a set

$$\{f_0, f_1, g_1, \ldots, f_n, g_n\}$$

of trigonometric polynomials is generated such that f_0 is a nonzero constant,

$$f_1 \in span\{1, \cos\theta\},\ g_1 \in span\{1, \cos\theta, \sin\theta\},\ f_2 \in span\{1, \cos\theta, \sin\theta, \cos 2\theta\}$$
$$g_2 \in span\{1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta\} \quad \ldots \quad f_n \in \mathcal{T}_n \backslash span\{\sin n\theta\},\ g_n \in \mathcal{T}_n$$

and it holds that

$$\begin{array}{lll} \langle f_j, f_k\rangle_\omega = \kappa_j \delta_{j,k} & , & \kappa_j > 0 \\ \langle g_j, g_k\rangle_\omega = \kappa_j^{'} \delta_{j,k} & , & \kappa_j^{'} > 0 \\ \langle f_j, g_k\rangle_\omega = 0,\ j = 0, 1, \ldots, n & , & k = 1, \ldots, n. \end{array} \tag{4.2}$$

When the process is repeated for each $n \in \mathbb{N}$, then $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ represents an orthogonal basis for $\mathcal{T}$ with respect to $\omega(\theta)$. Now, if we set

$$\begin{aligned} f_0 &= a_{0,0} \neq 0 \\ f_j &= a_{j,0} + \textstyle\sum_{k=1}^{j} (a_{j,k} \cos k\theta + b_{j,k} \sin k\theta) \\ g_j &= c_{j,0} + \textstyle\sum_{k=1}^{j} (c_{j,k} \cos k\theta + d_{j,k} \sin k\theta) \end{aligned} \tag{4.3}$$

then, because of the linear independence it clearly follows that

$$\begin{vmatrix} a_{n,n} & b_{n,n} \\ c_{n,n} & d_{n,n} \end{vmatrix} \neq 0, \quad n \geqslant 1.$$

Conversely, we also have (see [14])

Theorem 4.1. *Let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ be a system of trigonometric polynomials such that $f_0(\theta) \equiv c \neq 0$ and for $n \geqslant 1$:*

$$\begin{aligned} f_n(\theta) &= a_{n,0} + \textstyle\sum_{k=1}^{n} (a_{n,k} \cos k\theta + b_{n,k} \sin k\theta), \\ g_n(\theta) &= c_{n,0} + \textstyle\sum_{k=1}^{n} (c_{n,k} \cos k\theta + d_{n,k} \sin k\theta). \end{aligned}$$

Assume that for $n \geqslant 1$,

$$\begin{vmatrix} a_{n,n} & b_{n,n} \\ c_{n,n} & d_{n,n} \end{vmatrix} \neq 0.$$

Then, $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ is a basis for $\mathcal{T}$.

Now, according to Szegő (see [14]) we are in a position to state the following definitions:

Definition 4.2. Two trigonometric polynomials of degree n, of the form

$$f(\theta) = a\cos n\theta + b\sin n\theta + \cdots , \quad g(\theta) = c\cos n\theta + d\sin n\theta + \cdots$$

are said to be linearly independent if and only if

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

Definition 4.3. Given the weight function $\omega(\theta)$ on $[\pi, \pi]$, a system $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ of real trigonometric polynomials with f_0 a nonzero constant will be called a bi-orthogonal system for $\omega(\theta)$ if the following holds:

1. For each $n \geqslant 1$, $f_n(\theta)$ and $g_n(\theta)$ are linearly independent.
2. The system is orthogonal with respect to the inner produc generated by $\omega(\theta)$, i.e., (4.2) is satisfied.

Next, let us see how a bi-orthogonal system can be constructed from a sequence of orthogonal polynomials on the unit circle (Szegő polynomials) for $\omega(\theta)$. To fix ideas, let $\{\rho_n(z)\}_{n=0}^{\infty}$ be the sequence of monic Szegő polynomials: $\rho_n(z) = z^n + \cdots + \delta_n$ for $n = 0, 1, \ldots$. Here, $\delta_n = \rho_n(0)$ ($\delta_0 \neq 0$; $|\delta_n| < 1$ for $n = 1, 2, \ldots$) represents the n-th reflection coefficient or Schur parameter. Let $\{\omega_n\}_{n=0}^{\infty}$ be a given sequence of nonzero complex numbers and consider $\frac{\omega_n \rho_{2n+1}(z)}{z^n} \in \Lambda_{-(n+1),n+1}$. Here, one can write

$$\omega_n e^{-in\theta} \rho_{2n+1}(e^{i\theta}) = f_{n+1}(\theta) + i g_{n+1}(\theta) \tag{4.4}$$

where $f_{n+1}(\theta)$ and $g_{n+1}(\theta)$ are real trigonometric polynomials of degree $n + 1$ ($n = 0, 1, \ldots$), and we have (see [3])

Theorem 4.4. *Let $\{\omega_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that for any $n \geqslant 0$, $\omega_n \neq 0$ and $\omega_n^2 \int_{-\pi}^{\pi} e^{i\theta} \rho_{2n+1}(e^{i\theta})\omega(\theta)d\theta$ is a real number. Then the real trigonometric polynomials $f_0 \cup \{f_{n+1}, g_{n+1}\}_{n=0}^{\infty}$ given by (4.4) with $f_0(\theta) = f_0 \neq 0$ is a bi-orthogonal system for $\omega(\theta)$.*

Remark 4.5. For an alternative construction of a bi-orthogonal system making use of orthonormal Szegő polynomials of even instead of odd degree, see [14].

Example 4.6. Take $\omega(\theta) \equiv 1$ on $[-\pi, \pi]$ (Lebesgue measure). It is known that $\rho_n(z) = z^n$ for $n = 0, 1, \ldots$ so that, for any $\omega_n \in \mathbb{C} \setminus \{0\}$:

$$\omega_n^2 \int_{-\pi}^{\pi} e^{i\theta} \rho_{2n+1}(e^{i\theta})\omega(\theta)d\theta = \omega_n^2 \int_{-\pi}^{\pi} e^{i(2n+2)\theta}d\theta = 0.$$

Hence, we can take any nonzero complex number ω_n. Set $\omega_n = \alpha_n + i\beta_n$, $\alpha_n, \beta_n \in \mathbb{R}$ and $|\alpha_n| + |\beta_n| > 0$. Then,

$$\begin{aligned} f_{n+1}(\theta) &= \alpha_n \cos(n+1)\theta - \beta_n \sin(n+1)\theta \\ g_{n+1}(\theta) &= \beta_n \cos(n+1)\theta + \alpha_n \sin(n+1)\theta. \end{aligned} \tag{4.5}$$

Furthermore, by taking $\omega_n = 1$, for $n = 0, 1, \ldots$, we obtain

$$\tilde{f}_{n+1}(\theta) = \cos(n+1)\theta, \;\; \tilde{g}_{n+1}(\theta) = \sin(n+1)\theta \tag{4.6}$$

and the well known orthogonal properties of the functions

$$\{1, \cos\theta, \sin\theta, \ldots, \cos n\theta, \sin n\theta, \ldots\}$$

with respect to the weight function $\omega(\theta) \equiv 1$ are now recovered.

Remark 4.7. It should be noted that the relations (4.5) and (4.6) between two bi-orthogonal systems for $\omega(\theta) \equiv 1$ hold for any arbitrary $\omega(\theta)$. Indeed, let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ and $\tilde{f}_0 \cup \{\tilde{f}_k, \tilde{g}_k\}_{k=1}^{\infty}$ be two bi-orthogonal systems for a given weight function $\omega(\theta)$. Since $\tilde{f}_n \in \mathcal{T}_n$ and $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ is a basis for $\mathcal{T}_n$, one has

$$\tilde{f}_n(\theta) = \alpha_0 f_0 + \sum_{j=1}^{n} \left(\alpha_j f_j(\theta) + \beta_j g_j(\theta)\right).$$

On the other hand, because of the bi-orthogonality, $\langle \tilde{f}, T \rangle_\omega = 0$ for all $T \in \mathcal{T}_{n-1}$, yielding $\tilde{f}_n(\theta) = \alpha_n f_n(\theta) + \beta g_n(\theta)$. Similarly, $\tilde{g}_n(\theta) = \gamma_n f_n(\theta) + \delta g_n(\theta)$. Both relations can be expressed in a matrix form as,

$$\begin{pmatrix} \tilde{f}_n \\ \tilde{g}_n \end{pmatrix} = M_n \begin{pmatrix} f_n \\ g_n \end{pmatrix}, \quad M_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}.$$

with

$$\begin{aligned} \alpha_n &= \tfrac{\langle \tilde{f}_n, f_n \rangle_\omega}{\|f_n\|_\omega^2}, & \beta_n &= \tfrac{\langle \tilde{f}_n, g_n \rangle_\omega}{\|g_n\|_\omega^2} \\ \gamma_n &= \tfrac{\langle \tilde{g}_n, f_n \rangle_\omega}{\|\tilde{f}_n\|_\omega^2}, & \delta_n &= \tfrac{\langle \tilde{g}_n, g_n \rangle_\omega}{\|\tilde{g}_n\|_\omega^2}. \end{aligned}$$

By changing the roles of both systems, it follows that

$$\begin{pmatrix} \tilde{f}_n \\ \tilde{g}_n \end{pmatrix} = \tilde{M}_n \begin{pmatrix} f_n \\ g_n \end{pmatrix}, \quad \tilde{M}_n = M_n^{-1}.$$

Furthermore, when dealing with bi-orthonormal systems i.e., $\| f_n \|_\omega = \| g_n \|_\omega = \| \tilde{f}_n \|_\omega = \| \tilde{g}_n \|_\omega = 1$, then it can be verified that $M_n = M_n^T$ i.e., M_n is an orthogonal matrix, as remarked in [14].

Example 4.8. Consider the weight function $\omega(\theta) = \frac{1}{T(\theta)}$, $\theta \in [-\pi, \pi]$, $T(\theta)$ being a positive trigonometric polynomial of degree m (i.e., a rational modification of the Lebesgue measure). From 2.6 we can write $T(\theta) = |h(z)|^2$, $z = e^{i\theta}$, where

$h(z) \in \Pi_m$ without zeros on $\mathbb{T}$. Moreover, we can assume without loss of generality that $h(z)$ is a monic polynomial. Hence, from [15] the monic Szegő polynomials are given by $\rho_n(z) = z^{n-m}h(z)$ for $n \geqslant m$. Hence, as in Example 4.6 it holds that $\omega_n^2 \int_{-\pi}^{\pi} e^{i\theta}\rho_{2n+1}(e^{i\theta})\omega(\theta)d\theta = 0$, and any nonzero complex number ω_n can be used, provided that $n \geqslant E\left[\frac{m-1}{2}\right] + 1$ where $E[x]$ denotes as usual the integer part of x. Thus, if we set $h(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$ and take $\omega_n = 1$, then

$$\begin{aligned} \omega_n e^{-in\theta}\rho_{2n+1}(e^{i\theta}) &= e^{i(n+1-m)\theta}h(\theta) = e^{i(n+1-m)\theta}\left(e^{im\theta} + \cdots + a_0\right) \\ &= e^{i(n+1)\theta} + \cdots + a_0 e^{i(n+1-m)\theta} = f_{n+1}(\theta) + ig_{n+1}(\theta). \end{aligned}$$

Thus, for $n \geqslant E\left[\frac{m-1}{2}\right] + 1$ a bi-orthogonal system is given by

$$\begin{aligned} f_{n+1}(\theta) &= \cos(n+1)\theta + \cdots + a_0\cos(n+1-m)\theta, \\ g_{n+1}(\theta) &= \sin(n+1)\theta + \cdots + a_0\sin(n+1-m)\theta. \end{aligned}$$

Certainly, to have a bi-orthogonal system $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ completely constructed, we must compute the Szegő polynomials $\rho_{2k+1}(z)$, $0 \leqslant k \leqslant E\left[\frac{m-1}{2}\right]$ which can be recursively done by Levinson's algorithm (see [7] or [12]).

In the rest of the section we shall be concerned with the zeros of a given bi-orthogonal system. We observe from Example 1 that $f_0 \equiv c \neq 0$, $f_n(\theta) = \cos n\theta$, $g_n(\theta) = \sin n\theta$, $n = 1, 2, \ldots$ represent a bi-orthogonal system for $\omega(\theta) \equiv 1$. Now, $f_n(\theta) = 0$ means $\theta = \frac{(2k+1)\pi}{2n}$, $k \in \mathbb{Z}$. Thus, taking $-(n-1) \leqslant k \leqslant n-1$ we see that $f_n(\theta)$ has exactly $2n$ distinct zeros on $(-\pi, \pi]$. Similarly, if a and b are two real numbers, not both zero, it can be seen that $af_n(\theta) + bg_n(\theta)$ has also $2n$ distinct zeros on $(-\pi, \pi]$. This property can be generalized to any arbitrary weight function $\omega(\theta)$.

Theorem 4.9. *Let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ be a bi-orthogonal system for $\omega(\theta)$ and let a and b be real numbers not both zero. Then the trigonometric polynomial $T(\theta) = af(\theta) + bg(\theta)$ has $2n$ real and distinct zeros on any interval of length 2π.*

Proof. To fix ideas we shall restrict ourselves to $(-\pi, \pi]$. By Theorem (2.4) we know that $T_n(\theta)$ has $2n$ real or complex zeros in the strip $-\pi < \Re(\theta) \leqslant \pi$. Furthermore, the non-real zeros appear in conjugate pairs. Le p be the number of zeros of $T_n(\theta)$ on $(-\pi, \pi]$ with odd multiplicity $(0 \leqslant p \leqslant 2n)$. Since p should be even we can set $p = 2k$, $0 \leqslant k \leqslant n$. Assume that $k < n$ and define

$$U_k(\theta) = \prod_{j=1}^{k} \sin\left(\frac{\theta - \theta_{2j}}{2}\right) \sin\left(\frac{\theta - \theta_{2j-1}}{2}\right),$$

$\{\theta_j\}_{j=1}^{2k}$ being the zeros of $T_n(\theta)$ on $(-\pi, \pi]$ with odd multiplicity (obviously, if $k = 0$ we take $U_k(\theta) \equiv 1$). Then we can write $T_n(\theta) = af_n(\theta) + bg_n(\theta) = U_k(\theta)V_{n-k}(\theta)$, where $V_{n-k}(\theta) \in \mathcal{T}_{n-k}$ and $V_{n-k}(\theta)$ has a constant sign on $(-\pi, \pi]$. Since $k < n$, by virtue of orthogonality it follows on the one hand that

$$\begin{aligned} I &= \int_{-\pi}^{\pi} T_n(\theta) U_k(\theta) \omega(\theta) d\theta \\ &= a \int_{-\pi}^{\pi} f_n(\theta) U_k(\theta) \omega(\theta) d\theta + b \int_{-\pi}^{\pi} g_n(\theta) U_k(\theta) \omega(\theta) d\theta = 0, \end{aligned}$$

whereas on the other hand

$$I = \int_{-\pi}^{\pi} U_k^2(\theta) V_{n-k}(\theta) \omega(\theta) d\theta \neq 0$$

because $\omega(\theta)$ is a weight function on $(-\pi, \pi]$. >From this contradiction it follows that $k = n$. □

Furthermore, the following interlacing property of zeros holds:

Theorem 4.10. *Under the same assumptions as in Theorem 4.9, the zeros of* $af_n(\theta) + bg_n(\theta)$ *and* $-bf_n(\theta) + ag_n(\theta)$ *interlace.*

Proof. Since we are dealing with properties of zeros, we can assume, without loss of generality that the system $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ is bi-orthonormal. We introduce the function

$$\mathcal{K}_n(\alpha, \theta) = f_0(\alpha) f_0(\theta) + \sum_{k=1}^{n} \left(f_k(\alpha) f_k(\theta) + g_k(\alpha) g_k(\theta) \right)$$

which satisfies the following reproducing property:

$$T(\alpha) = \int_{-\pi}^{\pi} \mathcal{K}_n(\alpha, \theta) T(\theta) \omega(\theta) d\theta, \ \ \forall T \in \mathcal{T}_n.$$

On the other hand, from the paper by Szegő [14], the following Christoffel-Darboux identity can be established,

$$\begin{aligned} \mathcal{K}_{n-1}(\alpha, \theta) &= \tfrac{1}{2} \tfrac{k_{2n-1}}{k_{2n}} \cot\left(\tfrac{\theta-\alpha}{2}\right) \left(f_n(\alpha) g_n(\theta) - f_n(\theta) g_n(\alpha) \right) - \\ &- \left(r_n f_n(\alpha) f_n(\theta) + s_n g_n(\alpha) g_n(\theta) \right) \end{aligned} \tag{4.7}$$

where the coefficients k_n, r_n and s_n are related to the orthonormal sequence $\{\varphi_n(z)\}_{n=0}^{\infty}$ of Szegő polynomials as follows: Set $\varphi_n(z) = k_n z^n + \cdots + l_n$ $(k_n > 0)$, then $2s_n = 1 + \frac{|l_{2n}|}{k_{2n}} > 0$ and $2r_n = 1 - \frac{|l_{2n}|}{k_{2n}}$. Furthermore, since $\rho_n(z) = \frac{\varphi_n(z)}{k_n} = z^n + \cdots + \frac{l_n}{k_n}$, then $\frac{|l_{2n}|}{k_{2n}} < 1$ and r_n is also positive. Thus

$$\begin{aligned} \mathcal{K}_{n-1}(\alpha, \alpha) &= lim_{\theta \to \alpha} \mathcal{K}_{n-1}(\alpha, \theta) \\ &= \tfrac{k_{2n-1}}{k_{2n}} \left(f_n(\alpha) g_n^{'}(\alpha) - f_n^{'}(\alpha) g_n(\alpha) \right) - \left(r_n f_n^2(\alpha) + s_n g_n^2(\alpha) \right). \end{aligned}$$

Setting $M_n(\alpha) = \left(r_n f_n^2(\alpha) + s_n g_n^2(\alpha) \right)$ we obtain for all $\alpha \in \mathbb{R}$:

$$f_n(\alpha) g_n^{'}(\alpha) - f_n^{'}(\alpha) g_n(\alpha) = \frac{k_{2n-1}}{k_{2n}} \left(M_n(\alpha) + \mathcal{K}_{n-1}(\alpha, \alpha) \right) > 0,$$

since clearly $M_n(\alpha) > 0$ and $\mathcal{K}_{n-1}(\alpha, \alpha) > 0$. >From here it can be easily seen that the zeros of $f_n(\theta)$ and $g_n(\theta)$ interlace. Finally, let us consider

$$C_n(\theta) = af_n(\theta) + bg_n(\theta), \;\; D_n(\theta) = -bf_n(\theta) + ag_n(\theta), \;\; |a| + |b| > 0.$$

Then

$$C_n(\alpha)D_n^{'}(\alpha) - C_n^{'}(\alpha)D_n(\alpha) = (a^2 + b^2)\left(f_n(\alpha)g_n^{'}(\alpha) - f_n^{'}(\alpha)g_n(\alpha)\right) > 0$$

and the proof follows. □

Remark 4.11. The two previous theorems were earlier proved by Szegő in [14] making use of the fundamental property that the zeros of any Szegő polynomial $\rho_n(z)$ lie in $\mathbb{D}$. Here, we have given alternative proofs involving only bi-orthogonality properties.

As an immediate consequence of Theorems 4.9 and 4.10, we have

Corollary 4.12. *Let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ be an orthogonal system for $\omega(\theta)$. Then,*

1. *Both f_n and g_n have $2n$ distinct zeros on any interval of length 2π.*
2. *On any interval of length 2π, the zeros of f_n and g_n interlace.*

5. Quadratures

In this section we start to properly deal with the main topic of the paper, i.e., the approximate calculation of integrals

$$I_\omega(f) = \int_{-\pi}^{\pi} f(\theta)\omega(\theta)d\theta \tag{5.1}$$

with $\omega(\theta)$ a weight function on $(-\pi, \pi]$ and f a 2π-periodic function such that $f\omega \in L_1(-\pi, \pi]$. $I_\omega(f)$ is going to be approximated by means of an n-point quadrature rule like:

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(\theta_j), \;\theta_j \neq \theta_k, \;\theta_j \in (-\pi, \pi]. \tag{5.2}$$

Here, the nodes $\{\theta_j\}_{j=1}^n$ and weights $\{\lambda_j\}_{j=1}^n$ are to be determined so that $I_n(f)$ is exact in certain subspaces of $\mathcal{T}$ with dimension as large as possible, i.e. it should hold that $I_\omega(T) = I_n(T)$ for any $T \in \mathcal{T}_{m(n)} \subset \mathcal{T}$ with $m(n)$ as large as possible. For this purpose the following results should first be taken into account:

Theorem 5.1. *There can not exist an n-point quadrature rule $I_n(f)$ like (5.2) which is exact in $\mathcal{T}_n$, i.e., $m(n) < n$.*

Proof. Proceed as in [11, pp. 73-74] for the case $\omega(\theta) \equiv 1$. □

Now, making use of the interpolation results in Section 3 the following can be proved:

Theorem 5.2. *Given n distinct nodes $\{\theta_j\}_{j=1}^n \subset (-\pi, \pi]$, there exists a certain subspace $\tilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ with dimension n such that weights $\{\lambda_j\}_{j=1}^n$ satisfying*

$$I_n(T) = \sum_{j=1}^{n} \lambda_j T(\theta_j) I_\omega(T), \ \ \forall\, T \in \tilde{\mathcal{T}}_n.$$

are uniquely determined.

Theorem 5.3. *If there exists an n-point quadrature rule $I_n(f) = \sum_{j=1}^n \lambda_j f(\theta_j)$ which is exact in $\mathcal{T}_{n-1}$, then $\lambda_j > 0$ for all $j = 1, \dots, n$ (see [11]).*

Proof. Take $t_j(\theta) = \prod_{k=1,k\neq j}^n \sin^2\left(\frac{\theta-\theta_k}{2}\right)$. Thus, $t_j(\theta) \in \mathcal{T}_{n-1}$ and $t_j(\theta) \geqslant 0$. Hence, $0 < I_\omega(t_j) = I_n(t_j) = \lambda_j t_j(\theta_j)$. Since $t_j(\theta_j) > 0$, the proof follows. □

After these preliminary considerations, we are now in a position to investigate the following problem, namely: "For $n \in \mathbb{N}$, $n \geqslant 1$, find $\theta_1, \dots, \theta_n$ with $\theta_j \neq \theta_k$ if $j \neq k$ on $(-\pi, \pi]$ and real numbers $\lambda_1, \dots, \lambda_n$ such that

$$I_n(f) = \sum_{j=1}^{n} \lambda_j f(\theta_j) = I_\omega(f), \ \ \forall f \in \mathcal{T}_{n-1}.'' \tag{5.3}$$

Since $dim\,(\mathcal{T}_{n-1}) = 2n-1$, (5.3) leads to a nonlinear system with $2n-1$ equations and $2n$ unknowns: $\theta_1, \dots, \theta_n; \lambda_1, \dots, \lambda_n$. Now, proceeding as in the polynomial situation (see e.g. [6]), instead of directly attacking the system coming from (5.3) we will try to analyze the properties of the real trigonometric polynomial whose zeros are the nodes of $I_n(f)$. For this reason we are forced to assume that the number of nodes in our quadrature rules should be even. To fix ideas, assume that this number is $2n$. Then, in the sequel our rule will be of the form

$$I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j), \ \ \{\theta_j\}_{j=1}^{2n} \subset (-\pi, \pi].$$

Set $T_n(\theta) = \prod_{j=1}^{2n} \sin\left(\frac{\theta-\theta_j}{2}\right) \in \mathcal{T}_n$. Then the following holds:

Theorem 5.4. *Let $I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j)$ be a quadrature rule such that $I_{2n}(T) = I_\omega(T)$ for all $T \in \mathcal{T}_{2n-1}$ and let $f_0 \cup \{f_k, g_k\}_{k=1}^\infty$ be a bi-orthogonal system for $\omega(\theta)$. Set $T_n(\theta) = \prod_{j=1}^{2n} \sin\left(\frac{\theta-\theta_j}{2}\right)$. Then there exist real numbers a_n and b_n not both zero such that $T_n(\theta) = a_n f_n(\theta) + b_n g_n(\theta)$.*

Proof. Set $S \in \mathcal{T}_{n-1}$, then $T_n(\theta)S(\theta) \in \mathcal{T}_{2n-1}$. Hence

$$\begin{aligned} \langle T_n, S\rangle_\omega &= I_\omega(T_n \cdot S) = \textstyle\int_{-\pi}^{\pi} T_n(\theta)S(\theta)\omega(\theta)d\theta \\ &= I_n(T_n \cdot S) = \textstyle\sum_{j=1}^{2n} \lambda_j T_n(\theta_j)S(\theta_j) = 0. \end{aligned} \tag{5.4}$$

On the other hand, since $f_0 \cup \{f_k, g_k\}_{k=1}^n$ is a basis for $\mathcal{T}_n$, one can write

$$T_n(\theta) = a_0 + \sum_{k=1}^{n} \left(a_k f_k(\theta) + b_k g_k(\theta)\right), \quad a_k = \frac{\langle T_n, f_k\rangle_\omega}{\| f_k \|_\omega^2}, \quad b_k = \frac{\langle T_n, g_k\rangle_\omega}{\| f_k \|_\omega^2}.$$

By (5.4), $a_k = 0$ for $k = 0, 1, \ldots, n-1$ and $b_k = 0$ for $k = 1, \ldots, n-1$ and the proof follows. □

Conversely, we can prove the following

Theorem 5.5. *Let $f_0 \cup \{f_k, g_k\}_{k=1}^\infty$ be a bi-orthogonal system for the weight function $\omega(\theta)$. Let a and b be real numbers not both zero and let $\{\theta_j\}_{j=1}^{2n}$ be the $2n$ zeros of $T_n(\theta) = af_n(\theta) + bg_n(\theta)$ on $(-\pi, \pi]$. Then, there exist positive numbers $\lambda_1, \ldots, \lambda_{2n}$ such that*

$$I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j) = I_\omega(f), \quad \forall\, f \in \mathcal{T}_{2n-1}.$$

Proof. Throughout the proof, $\tilde{\mathcal{T}_n}$ will denote a subspace of trigonometric polynomials coinciding either with $\mathcal{T}_n \backslash span\{\cos n\theta\}$ or $\mathcal{T}_n \backslash span\{\sin n\theta\}$, so that $dim\left(\tilde{\mathcal{T}_n}\right) = 2n$. Let $\theta_1, \ldots, \theta_{2n}$ be the $2n$ distinct zeros of $T_n(\theta) = af_n(\theta) + bg_n(\theta)$, $(|a| + |b| > 0)$. Then, by Theorem 5.2, there exist weights $\lambda_1, \ldots, \lambda_{2n}$, uniquely determined, such that

$$I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j) = I_\omega(f), \quad \forall\, f \in \tilde{\mathcal{T}_n}.$$

Let us next see that $I_{2n}(f)$ is also exact in $\mathcal{T}_{2n-1}$ (observe that $\tilde{\mathcal{T}_n} \subset \mathcal{T}_{2n-1}$). To do that, we will follow the classical pattern. Indeed, take $T \in \mathcal{T}_{2n-1}$ and let $L_n \in \tilde{\mathcal{T}_n}$ such that

$$T(\theta_j) = L_n(\theta_j), \quad j = 1, \ldots, 2n.$$

Then $T - L_n \in \mathcal{T}_{2n-1}$ and $(T - L_n)(\theta_j) = 0$ for all $j = 1, \ldots, 2n$. Hence we can write $T(\theta) - L_n(\theta) = T_n(\theta)V(\theta)$, with $V \in \mathcal{T}_{n-1}$ i.e., $T(\theta) = L_n(\theta) + T_n(\theta)V(\theta)$. Consequently

$$\begin{aligned} I_\omega(T) &= \int_{-\pi}^{\pi} T(\theta)\omega(\theta)d\theta = \int_{-\pi}^{\pi} \left(L_n(\theta) + T_n(\theta)V(\theta)\right)\omega(\theta)d\theta \\ &= \int_{-\pi}^{\pi} L_n(\theta)\omega(\theta)d\theta = I_\omega(L_n), \end{aligned}$$

since $I_\omega(T_nV) = 0$ (by definition, $T_n(\theta)$ is orthogonal to any function in $\mathcal{T}_{n-1}$). Therefore,

$$I_\omega(T) = I_\omega(L_n) = \sum_{j=1}^{2n} \lambda_j L_n(\theta_j) = \sum_{j=1}^{2n} \lambda_j T(\theta_j) = I_n(T).$$

Finally, the positive character of the weights $\{\lambda_j\}_{j=1}^{2n}$ follows from Theorem 5.3. However, we can also give an explicit integral representation. Thus, for $j = 1,\ldots,2n$, set

$$l_j(\theta) = \frac{T_n(\theta)}{2T_n'(\theta_j)\sin\left(\frac{\theta-\theta_j}{2}\right)}$$

so that $l_j(\theta_k) = \delta_{j,k}$ and $l_j^2(\theta) \in \mathcal{T}_{2n-1}$ for $j = 1,\ldots,2n$. Thus

$$I_\omega\left(l_j^2(\theta)\right) = I_{2n}\left(l_j^2(\theta)\right) = \sum_{k=1}^{2n} \lambda_k l_j^2(\theta_k) = \lambda_j$$

yielding

$$\lambda_j = \int_{-\pi}^{\pi} \left[\frac{T_n(\theta)}{2T_n'(\theta_j)\sin\left(\frac{\theta-\theta_j}{2}\right)}\right]^2 \omega(\theta)d\theta, \quad j = 1,\ldots,2n. \tag{5.5}$$

□

Theorems 5.4 and 5.5 may be summarized in the following characterization result,

Corollary 5.6. *Let $I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j)$ so that $\theta_j \neq \theta_k$ if $j \neq k$, and $\{\theta_j\} \subset (-\pi,\pi]$. Then, $I_{2n}(f) = I_\omega(f)$ for all $f \in \mathcal{T}_{2n-1}$, if and only if,*

1. *$I_{2n}(f)$ is exact in a certain subspace $\tilde{\mathcal{T}_n}$ of $\mathcal{T}_{2n-1}$ whith dimension $2n$.*
2. *There exist real numbers a and b not both zero such that $\{\theta_j\}_{j=1}^{2n}$ are the zeros of $T_n(\theta) = af_n(\theta) + bg_n(\theta)$, $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ being a bi-orthogonal system for the weight function $\omega(\theta)$.*

Furthermore, when these conditions are satisfied the weights $\{\lambda_j\}_{j=1}^{2n}$ are positive.

Remark 5.7. The quadrature rules characterized in Corollary 5.6 were earlier introduced by Szegő in [14] and they are sometimes refered as "quadratures with the highest degree of trigonometric precision".

Next, we will see how we can also give an explicit representation of the weights $\{\lambda_j\}_{j=1}^{2n}$ in Corollary 5.6, in terms of a bi-orthonormal system similar to the well known Christoffel numbers for the Gaussian formulas (see e.g. [8]). Indeed, we have

Theorem 5.8. *Let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ be a bi-orthonormal system for $\omega(\theta)$ and let $I_{2n}(f) = \sum_{j=1}^{2n} \lambda_j f(\theta_j)$ be a $2n$-point quadrature rule with the highest degree of trigonometric precision. Then, for $j = 1,\ldots,2n$,*

$$\lambda_j = \frac{1}{f_0^2 + \sum_{k=1}^{n-1}\left(f_k^2(\theta_j) + g_k^2(\theta_j)\right) + \left(\frac{1-|\delta_{2n}|}{2}\right) f_n^2(\theta_j) + \left(\frac{1+|\delta_{2n}|}{2}\right) g_n^2(\theta_j)} \tag{5.6}$$

where, as usual, $\delta_{2n} = \rho_{2n}(0)$, $\rho_{2n}(z)$ being the monic Szegő polynomial of degree $2n$ and $\{\theta_j\}_{j=1}^{2n}$ being the zeros of $T_n(\theta) = af_n(\theta) + bg_n(\theta)$, $|a| + |b| > 0$.

Proof. Set $T_n(\theta) = \prod_{k=1}^{2n} \sin\left(\frac{\theta-\theta_k}{2}\right) = af_n(\theta)+bg_n(\theta) \in \mathcal{T}_n$, $|a|+|b|>0$. Suppose without loss of generality that $a \neq 0$ so that $f_n(\theta_j) = \frac{-b}{a} g_n(\theta_j)$. Then, from the Christoffel-Darboux identity (4.7) it follows

$$\begin{aligned}\mathcal{K}_{n-1}(\theta,\theta_j) &= \tfrac{1}{2}\tfrac{k_{2n-1}}{k_{2n}} ctg\left(\tfrac{\theta_j-\theta}{2}\right)\left[f_n(\theta)g_n(\theta_j) - f_n(\theta_j)g_n(\theta)\right] - \\ &- \left(r_n f_n(\theta) f_n(\theta_j) + s_n g_n(\theta) g_n(\theta_j)\right) \\ &= \tfrac{1}{2a}\tfrac{k_{2n-1}}{k_{2n}} g_n(\theta_j) ctg\left(\tfrac{\theta_j-\theta}{2}\right) T_n(\theta) - \\ &- \left[\tfrac{1-|\delta_{2n}|}{2} f_n(\theta)f_n(\theta_j) + \tfrac{1+|\delta_{2n}|}{2} g_n(\theta)g_n(\theta_j)\right]\end{aligned}$$

and hence

$$\begin{aligned}\mathcal{K}_{n-1}(\theta,\theta_j) + \left[\tfrac{1-|\delta_{2n}|}{2} f_n(\theta)f_n(\theta_j) + \tfrac{1+|\delta_{2n}|}{2} g_n(\theta)g_n(\theta_j)\right] \quad = \\ \tfrac{-1}{2a}\tfrac{k_{2n-1}}{k_{2n}} g_n(\theta_j)\cos\left(\tfrac{\theta-\theta_j}{2}\right) \tfrac{T_n(\theta)}{\sin\left[\frac{\theta-\theta_j}{2}\right]}.\end{aligned} \tag{5.7}$$

As θ tends to θ_j, we get

$$\begin{aligned}f_0^2 + \textstyle\sum_{k=1}^{n-1}\left(f_k^2(\theta_j)+g_k^2(\theta_j)\right) + \left(\tfrac{1-|\delta_{2n}|}{2}\right) f_n^2(\theta_j) + \left(\tfrac{1+|\delta_{2n}|}{2}\right) g_n^2(\theta_j) = \\ \tfrac{-1}{a}\tfrac{k_{2n-1}}{k_{2n}} g_n(\theta_j) T_n^{'}(\theta_j).\end{aligned} \tag{5.8}$$

Now, due to the orthogonality conditions it follows from (5.7) that

$$1 = \frac{-1}{2a}\frac{k_{2n-1}}{k_{2n}} g_n(\theta_j) \int_{-\pi}^{\pi} \cos\left(\frac{\theta-\theta_j}{2}\right) \frac{T_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)} \omega(\theta) d\theta. \tag{5.9}$$

The combination of expressions (5.8) and (5.9) implies

$$\begin{aligned}\tfrac{1}{f_0^2+\sum_{k=1}^{n-1}\left(f_k^2(\theta_j)+g_k^2(\theta_j)\right)+\ \frac{1-|\delta_{2n}|}{2}\ f_n^2(\theta_j)+\ \frac{1+|\delta_{2n}|}{2}\ g_n^2(\theta_j)} = \\ \tfrac{1}{2T_n^{'}(\theta_j)} \textstyle\int_{-\pi}^{\pi} \cos\left(\tfrac{\theta-\theta_j}{2}\right) \tfrac{T_n(\theta)}{\sin\left[\frac{\theta-\theta_j}{2}\right]} \omega(\theta) d\theta\end{aligned} \tag{5.10}$$

On the other hand, from Corollary 5.6 one knows that the weights λ_j can be expressed as

$$\lambda_j = \int_{-\pi}^{\pi} \tilde{s}_j(\theta)\omega(\theta)d\theta, \ \ j = 1,\ldots,2n$$

where $\tilde{s}_j(\theta)$ are trigonometric polynomials of degree n at most given by (3.16) or (3.17). Thus, from (3.16) it follows

$$\tilde{s}_j(\theta) = \frac{1}{2T_n^{'}(\theta_j)\sin\eta_n} \sin\left(\frac{\theta+\alpha_j}{2}\right) \frac{T_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)}, \ \ j = 1,\ldots,2n,$$

with $\eta_n = \frac{1}{2}\sum_{j=1}^{2n}\theta_j$ and $\alpha_j = \eta_n - \frac{\theta_j}{2}$ for $j = 1,\dots,2n$. Hence, $\sin\left(\frac{\theta+\alpha_j}{2}\right) = \sin\left(\frac{\theta-\theta_j}{2}+\eta_n\right) = \sin\left(\frac{\theta-\theta_j}{2}\right)\cos\eta_n + \cos\left(\frac{\theta-\theta_j}{2}\right)\sin\eta_n$ and one can write

$$\begin{aligned}\lambda_j &= \frac{1}{2T_n'(\theta_j)\sin\eta_n}\Big[\cos\eta_n\int_{-\pi}^{\pi}T_n(\theta)\omega(\theta)d\theta + \\ &+ \sin\eta_n\int_{-\pi}^{\pi}\cos\left(\frac{\theta-\theta_j}{2}\right)\frac{T_n(\theta)}{\sin\frac{\theta-\theta_j}{2}}\omega(\theta)d\theta\Big] \\ &= \frac{1}{2T_n'(\theta_j)}\int_{-\pi}^{\pi}\cos\left(\frac{\theta-\theta_j}{2}\right)\frac{T_n(\theta)}{\sin\left[\frac{\theta-\theta_j}{2}\right]}\omega(\theta)d\theta.\end{aligned} \tag{5.11}$$

Clearly, if we now start from (3.17) the same representation (5.11) is achieved. Thus, from (5.10) and (5.11) the proof follows. □

Example 5.9. As a simple illustration of formula (5.6), let us consider $\omega(\theta) \equiv 1$. As we have already seen, a bi-orthogonal system is given by $\{1\}\cup\{\cos n\theta, \sin n\theta\}_{n=1}^{\infty}$. Thus, we have the following bi-orthonormal system:

$$f_0 = \frac{1}{\sqrt{2\pi}},\ \ f_n(\theta) = \frac{\cos n\theta}{\sqrt{\pi}},\ \ g_n(\theta) = \frac{\sin n\theta}{\sqrt{\pi}},\ \ n = 1,2,\dots.$$

Taking $a, b \in \mathbb{R}$, $|a|+|b| > 0$, the nodes of the corresponding $(2n)$-th quadrature rule are the zeros of $T_n(\theta) = af_n(\theta)+bg_n(\theta)$. Thus, when $a = 0$ and $b = 1$, i.e., $\sin n\theta = 0$, the zeros are $\theta_k = \frac{k\pi}{n}$ for all $k \in \mathbb{Z}$, i.e., the $2n$ zeros $\theta_j = \frac{(j-n)\pi}{n} = -\pi + \frac{2\pi j}{2n}$, $j = 0,1,\dots,2n-1$, are equally spaced on the interval $[-\pi,\pi]$ with step size, $h = \frac{\pi}{n}$. Moreover, since now $\rho_n(z) = z^n$ for all $n = 0,1,\dots$, then $\delta_{2n} = \rho_{2n}(0) = 0$ and formula (5.6) becomes, for all $j = 1,\dots,2n$:

$$\lambda_j = \frac{1}{\frac{1}{2\pi}+\sum_{k=1}^{n-1}\left(\frac{\cos^2(k\theta_j)}{\pi}+\frac{\sin^2(k\theta_j)}{\pi}\right)+\frac{1}{2}\left(\frac{\cos^2(n\theta_j)}{\pi}+\frac{\sin^2(n\theta_j)}{\pi}\right)} = \frac{\pi}{n}. \tag{5.12}$$

Furthermore, from (5.12) we see that independently of the expression of the nodes $\{\theta_j\}_{j=1}^{2n}$ all the weights $\{\lambda_j\}_{j=1}^{2n}$ are equal to $\frac{\pi}{n}$. This result was deduced in a different manner in [11].

Paralleling rather closely Gaussian quadrature formulas, we will give a final result involving the Hermite-type interpolation problem stated in Theorem 3.2 which could be used to give an estimation of the error for $I_{2n}(f)$. Indeed, one has

Theorem 5.10. *Let a and b real numbers not both zero and let $\{\theta_j\}_{j=1}^{2n}$ the zeros of $T_n(\theta) = af_n(\theta) + bg_n(\theta)$, $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ being a bi-orthogonal system. Let $H_{2n-1}(f,\cdot) \in \mathcal{T}_{2n-1}$ such that:*

$$H_{2n-1}(f;\theta_j) = f(\theta_j) \quad j = 1,\dots,2n$$

$$H_{2n-1}'(f;\theta_j) = f'(\theta_j) \quad j = 1,\dots,2n,\ j \neq k \in \{1,\dots,2n\}.$$

Then, $I_\omega(H_{2n-1}(f,\cdot))$ coincides with the $(2n)$-th quadrature rule with the highest degree of trigonometric precision with nodes $\{\theta_j\}_{j=1}^{2n}$. Furthermore, this formula does not depend on the parameter $k \in \{1,\dots,2n\}$ previously fixed.

Proof. The existence and unicity of the Hermite trigonometric interpolant $H_{2n-1}(f,\theta)$ is guaranteed by Theorem 3.2. Furthermore, by (3.7) we can write,

$$H_{2n-1}(f,\theta) = \sum_{j=1}^{2n} t_j(\theta) f(\theta_j) + \sum_{j=1, j\neq k}^{2n} s_j(\theta) f^{'}(\theta_j) \tag{5.13}$$

where $t_j(\theta)$ and $s_j(\theta)$ are trigonometric polynomials in $\mathcal{T}_{2n-1}$ satisfying the interpolation condition (3.8). Hence,

$$I_\omega\left(H_{2n-1}(f,\cdot)\right) = \sum_{j=1}^{2n} A_j f(\theta_j) + \sum_{j=1, j\neq k}^{2n} B_j f^{'}(\theta_j) \tag{5.14}$$

where $A_j = I_\omega(t_j)$ for $j = 1, \ldots, 2n$ and $B_j = I_\omega(s_j)$, $j = 1, \ldots, 2n$, $j \neq k$. Now, taking into account that $T_n(\theta) = af_n(\theta) + bg_n(\theta)$ is orthogonal to $\mathcal{T}_{n-1}$, it can be deduced from (3.9) that

$$B_j = \frac{\sin\left(\frac{\theta_j-\theta_k}{2}\right)}{2\left[T_n'(\theta_j)\right]^2} I_\omega \left(T_n(\theta) \frac{T_n(\theta)}{\sin\left(\frac{\theta-\theta_j}{2}\right)\sin\left(\frac{\theta-\theta_k}{2}\right)} \right) = 0, \;\; j = 1, \ldots, 2n, \; j \neq k.$$

Thus, $I_\omega\left(H_{2n-1}(f,\cdot)\right) = \sum_{j=1}^{2n} A_j f(\theta_j) = \tilde{I}_{2n}(f)$ and since for any $T \in \mathcal{T}_{2n-1}$, $H_{2n-1}(T,\theta) = T(\theta)$, we have

$$\tilde{I}_{2n}(T) = I_\omega\left(H_{2n-1}(T,\cdot)\right) = I_\omega(T), \;\; \forall\, T \in \mathcal{T}_{2n-1}.$$

Now the proof follows by Corollary 5.6. □

Remark 5.11. Quadrature rules of the form $I_n(f) = \sum_{j=1}^{n} \lambda_n f(\theta_j)$ to estimate weighted 2π-periodic integrals $I_\omega(f)$ have been constructed making use of the zeros of certain trigonometric polynomials associated to a bi-orthogonal system. For this reason we have been forced to deal with an even number of nodes and weights. Now, we might wonder if a quadrature $I_n(f)$ with n an arbitrary natural number and with the highest degree of trigonometric precision $(n-1)$ could be also constructed. It seems clear that we can not use zeros of real trigonometric polynomials anymore, since the number of these is always even. Actually, this question does not appear in the paper by Szegő [14]. In a forthcoming paper a positive answer will be given by introducing convenient technical modifications of Szegő's paper [14]. However we can also find an answer in the paper by Jones et. al. [10] which, for the sake of completeness, will be surveyed in the next Section. As a consequence, a connection between the concepts of bi-orthogonality and para-orthogonality introduced in [14] and [10] respectively will be also made.

6. A connection with the unit circle. Para-orthogonal polynomials

In this Section we shall be concerned with the approximation of integrals on the unit circle, i.e., integrals of the form $\int_{\mathbb{T}} f(z)d\mu(z)$, μ being a positive measure on $\mathbb{T}$, by means of an n-point quadrature rule:

$$I_n(f) = \sum_{j=1}^{n} A_j f(z_j), \quad z_j \neq z_k, \ j \neq k, \quad \{z_j\}_{j=1}^n \subset \mathbb{T}. \tag{6.1}$$

By a slight abuse of notation we shall set $\mu(z) = \mu(\theta)$ for $z = e^{i\theta}$. As before, and for the sake of simplicity, we will also assume that μ is an absolutely continuous measure i.e., $d\mu(\theta) = \omega(\theta)d\theta$ so that we consider integrals of the form

$$I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta \tag{6.2}$$

where $f(e^{i\theta})$ is in general a complex function. Thus $f(e^{i\theta}) = f_1(\theta) + if_2(\theta)$ with $f_j(\theta)$ for $j = 1, 2$, both real 2π-periodic functions. Here, taking into account the basic fact that any continuous function on $\mathbb{T}$ can be uniformly approximated on $\mathbb{T}$ by Laurent polynomials, the nodes $\{z_j\}_{j=1}^n$ and weights $\{A_j\}_{j=1}^n$ are to be determined by requiring that $I_n(f)$ is exact in $\Lambda_{-p,q}$ (domain of validity) with p and q as large as possible (clearly this means that $I_\omega(L) = I_n(L)$, for all $L \in \Lambda_{-p,q}$). Now, assume that for the weight function $\omega(\theta)$ and an even integer n we have found an n-point quadrature rule $I_n(f) = \sum_{j=1}^n \lambda_j f(\theta_j)$ with the highest degree of trigonometric precision (recall that $\lambda_j > 0$ and $\theta_j \neq \theta_k$ if $j \neq k$, $\{\theta_j\}_{j=1}^n \subset (-\pi, \pi]$). Take $L \in \Lambda_{-(n-1),n-1}$ so that $L(e^{i\theta}) = L_1(\theta) + iL_2(\theta)$ with $L_1, L_2 \in \mathcal{T}_n$. Then

$$\begin{aligned} I_\omega(L) &= \textstyle\int_{-\pi}^{\pi} L(e^{i\theta})\omega(\theta)d\theta = \int_{-\pi}^{\pi} L_1(\theta)\omega(\theta)d\theta + i\int_{-\pi}^{\pi} L_2(\theta)\omega(\theta)d\theta \\ &= \textstyle\sum_{j=1}^n \lambda_j L_1(\theta_j) + i\sum_{j=1}^n \lambda_j L_2(\theta_j) \\ &= \textstyle\sum_{j=1}^n \lambda_j \left(L_1(\theta_j) + iL_2(\theta_j)\right) = \sum_{j=1}^n \lambda_j L(e^{i\theta_j}) \\ &= \textstyle\sum_{j=1}^n \lambda_j L(z_j), \quad z_j = e^{i\theta_j}, \ j = 1, \ldots, n. \end{aligned}$$

(observe that $z_j \neq z_k$ if $j \neq k$). Thus, provided that n is even a quadrature rule with domain of validity $\Lambda_{-(n-1),n-1}$ for $I_\omega(f)$ has been constructed.

Conversely, let $I_n(f) = \sum_{j=1}^n A_j f(z_j)$, $z_j \neq z_k$ if $j \neq k$, be exact in $\Lambda_{-(n-1),n-1}$ and set $z_j = e^{i\theta_j}$, $\theta_j \in (-\pi, \pi]$, $\theta_j \neq \theta_k$ if $j \neq k$. Set $T \in \mathcal{T}_{n-1}$, then $T(\theta) = L(e^{i\theta})$ with $L \in \Lambda_{n-1}^H$ so that

$$\int_{-\pi}^{\pi} T(\theta)d\theta = \int_{-\pi}^{\pi} L(e^{i\theta})\omega(\theta)d\theta = \sum_{j=1}^{n} A_j L(e^{i\theta_j}) = \sum_{j=1}^{n} A_j T(\theta_j) = I_n(T)$$

with $I_n(f) = \sum_{j=1}^n A_j f(\theta_j)$. Thus, we see that the problem of constructing an n-point quadrature formula for $\omega(\theta)$ with the highest degree of trigonometric precision with n arbitrary would be solved. As immediate consequences we would also have:

1. Any quadrature rule $I_n(f) = \sum_{j=1}^n A_j f(z_j)$ with distinct nodes on $\mathbb{T}$ which is exact in $\Lambda_{-(n-1),n-1}$ has positive weights A_j, $j = 1, \ldots, n$.
2. There can not exist an n-point quadrature rule as before which is exact in $\Lambda_{-n,n}$.

Thus, in the sequel, given the integral $I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta$ we shall be concentrated on the construction of $I_n(f) = \sum_{j=1}^{n} A_j f(z_j)$ such that $z_j \neq z_k$ if $j \neq k$, $z_j \in \mathbb{T}$ for $j = 1, \ldots, n$ by imposing

$$I_n(L) = I_\omega(L), \ \forall L \in \Lambda_{-(n-1),n-1}. \tag{6.3}$$

According to [10], $\Lambda_{(n-1),n-1}$ will be called "the maximun domain of validity" for $I_n(f)$, provided that (6.3) holds. Now, set $\mu_k = \int_{-\pi}^{\pi} e^{-ik\theta}\omega(\theta)d\theta$ for any $k \in \mathbb{Z}$ (trigonometric moments) so that (6.3) gives rise to the equality

$$\sum_{k=1}^{n} A_k z_k^j = \mu_{-j}, \quad -(n-1) \leqslant j \leqslant n-1. \tag{6.4}$$

This leads to a study of the solutions of (6.4) which represents a nonlinear system with $2n$ unknowns and $2n-1$ equations. We will proceed as in the preceding section by analyzing the properties of the nodal polynomial for $I_n(f)$, $B_n(z) = \prod_{j=1}^{n}(z - z_j)$. First, take into account that in case the zeros $\{z_j\}_{j=1}^{n}$ of $B_n(z)$ satisfy $z_j \neq 0$ and $z_j \neq z_k$ if $j \neq k$, then by taking n consecutive equations in (6.4), the weights $\{A_j\}_{j=1}^{n}$ are to be uniquely determined in terms of the nodes $\{z_j\}_{j=1}^{n}$. Indeed, let p and q be nonnegative integers suche that $p + q = n - 1$ and take in (6.4) the n equations

$$\sum_{k=1}^{n} A_k z_k^j = \mu_{-j}, \quad -p \leqslant j \leqslant q. \tag{6.5}$$

Clearly, (6.5) is a linear system for the unknowns $A_1, \ldots, A_n$ admitting a unique solution because the determinant of the matrix of the system satisfies

$$\begin{vmatrix} z_1^{-p} & z_2^{-p} & \cdots & z_n^{-p} \\ z_1^{-p+1} & z_2^{-p+1} & \cdots & z_n^{-p+1} \\ \vdots & \vdots & & \vdots \\ z_1^{q} & z_2^{q} & \cdots & z_n^{q} \end{vmatrix} = (z_1 \cdots z_n)^p \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{vmatrix} \neq 0.$$

(Recall that we are assuming $z_j \neq 0$ and $z_j \neq z_k$ if $j \neq k$). Secondly, we can also deduce the following necessary conditions for the polynomials $B_n(z)$:

Theorem 6.1. *Let $I_n(f) = \sum_{j=1}^{n} A_j f(z_j)$ such that $z_j \in \mathbb{T}$ and $z_j \neq z_k$ if $j \neq k$ satisfying $I_n(L) = I_\omega(L)$, for all $L \in \Lambda_{-(n-1),n-1}$. Set $B_n(z) = \prod_{j=1}^{n}(z - z_j)$. Then,*

1. *$B_n(z)$ is invariant.*
2. $$\langle B_n(z), z^k\rangle_\omega = 0, \ 1 \leqslant k \leqslant n-1, \ \langle B_n(z), 1\rangle_\omega \neq 0, \ \langle B_n(z), z^n\rangle_\omega \neq 0. \tag{6.6}$$

Proof. 1. It trivially follows since by hypothesis the zeros of $B_n(z)$ lie on $\mathbb{T}$.

2. Set $1 \leqslant k \leqslant n-1$. Then

$$\langle B_n(z), z^k \rangle_\omega = \int_{-\pi}^{\pi} B_n(e^{i\theta})\overline{e^{ik\theta}}\omega(\theta)d\theta = \int_{-\pi}^{\pi} L(e^{i\theta})\omega(\theta)d\theta$$

where $L(z) = z^{-k}B_n(z) \in \Lambda_{-k,n-k} \subset \Lambda_{-(n-1),n-1}$ and $L(z_j) = 0$. Then, because of the exactness of $I_n(f)$ in $\Lambda_{-(n-1),n-1}$ we have

$$\begin{aligned} \langle B_(z), z^k \rangle_\omega &= \int_{-\pi}^{\pi} L(e^{i\theta})\omega(\theta)d\theta = I_n(L) \\ &= \sum_{j=1}^{n} A_j L(z_j) = 0,\ 1 \leqslant k \leqslant n-1. \end{aligned}$$

If $\langle B_n(z), 1 \rangle_\omega = 0$, then $\langle B_n(z), z^k \rangle_\omega = 0$ for $0 \leqslant k \leqslant n$, yielding $B_n(z) = \rho_n(z)$, and hence the zeros lie in $\mathbb{D}$, contrary to assumption. Similarly, if $\langle B_n(z), z^n \rangle_\omega = 0$ then $B_n(z) = \rho_n^*(z)$ and hence the zeros lie in $\mathbb{E}$, contrary to assumption. Thus $\langle B_n(z), 1 \rangle_\omega \neq 0$ and $\langle B_n(z), z^n \rangle_\omega \neq 0$.

□

Remark 6.2. From the above considerations including the fact that

$$\langle B_n(z), 1 \rangle_\omega \neq 0,\ \langle B_n(z), z^n \rangle_\omega \neq 0$$

when I_n is exact in $\Lambda_{-(n-1),n-1}$ and that the zeros of the n-th Szegő polynomial lie in $\mathbb{D}$, it follows that there can not exist an n-point quadrature formula with nodes on $\mathbb{T}$ to be exact either in $\Delta_{-(n-1),n}$ or in $\Delta_{-n,n-1}$.

Polynomials $B_n(z)$ satisfying (6.5) will play a crucial role in the construction of our quadratures $I_n(f)$ with the maximun domain of validity. This caused (see [10]) the following

Definition 6.3. A polynomial $B_n(z)$ of exact degree n, $n \geqslant 1$, is said to be para-orthogonal with respect to $\omega(\theta)$ if and only if the orthogonality conditions (6.6) are satisfied.

Now, several questions immediately arise. Indeed, for a given weight function $\omega(\theta)$ and a natural number n, does a para-orthogonal polynomial of exact degree n exist? If so, how can it be characterized? What about its zeros? The two first questions are answered in [10] where the concepts of "para-orthogonality" and "invariancy" were earlier introduced. Thus, in [10] one can find the following

Theorem 6.4. *A polynomial $B_n(z)$ of exact degree n, $n \geqslant 1$, is para-orthogonal and invariant if and only if*

$$B_n(z) = C_n\left[\rho_n(z) + \tau\rho_n^*(z)\right],\ \ C_n \neq 0,\ |\tau| = 1. \tag{6.7}$$

Now, by recalling that the sequences $\{\rho_n(z)\}_{n=0}^{\infty}$ and $\{\rho_n^*(z)\}_{n=0}^{\infty}$ satisfy the recurrence relations

$$\begin{aligned} &\rho_0(z) = \rho_0^*(z) = 1 \\ &\rho_n(z) = z\rho_{n-1}(z) + \delta_n\rho_{n-1}^*(z) \quad n = 1,2,3,\ldots \\ &\rho_n^*(z) = \overline{\delta_n}z\rho_{n-1}(z) + \rho_{n-1}^*(z) \quad n = 1,2,3,\ldots \end{aligned} \tag{6.8}$$

where, as usual, $\delta_n = \rho_n(0)$ for all $n = 1, 2, \ldots$ ($|\delta_n| < 1$), then we have

$$B_n(z) = C_n\left[\rho_n(z) + \tau\rho_n^*(z)\right] = \left(1 + \tau\overline{\delta_n}\right) C_n \left[z\rho_{n-1}(z) + \left(\frac{\tau + \delta_n}{1 + \tau\overline{\delta_n}}\right)\rho_{n-1}^*(z)\right],$$

yielding (observe that $|1 + \tau\overline{\delta_n}| \neq 0$)

$$B_n(z) = \tilde{C}_n\left[z\rho_{n-1}(z) + \lambda_n\rho_{n-1}^*(z)\right], \ \tilde{C}_n \neq 0, \ |\lambda_n| = 1 \tag{6.9}$$

(here, $\lambda_n = \frac{\tau+\delta_n}{1+\tau\overline{\delta_n}} \in \mathbb{T}$). Conversely, any polynomial $B_n(z)$ satisfying (6.9) can be expressed as in (6.7), were now $\tau = \frac{\delta_n - \lambda_n}{\overline{\delta_n}\lambda_n - 1} \in \mathbb{T}$. In short, we have obtained an alternative characterization of the para-orthogonal and invariant polynomials as shown in the following

Theorem 6.5. *A polynomial $B_n(z)$ of exact degree n, $n \geqslant 1$, is para-orthogonal and invariant if and only if*

$$B_n(z) = C_n\left[z\rho_{n-1}(z) + \tau\rho_{n-1}^*(z)\right], \ \ C_n \neq 0, \ |\tau| = 1.$$

Remark 6.6. >From this theorem we see that to compute a para-orthogonal polynomial of degree n, only the Szegő polynomial of degree $n-1$ is required.

Next, we will make a connection between certain sequences of para-orthogonal polynomials and bi-orthogonal systems of trigonometric polynomials for the same weight function $\omega(\theta)$. For this purpose, let $B_{2n}(z)$ be a polynomial of degree $2n$, para-orthogonal and invariant. Then, from the begining of Section 2, one can write (by virtue of invariance)

$$B_{2n}(e^{i\theta}) = a_n e^{in\theta} f_n(\theta), \ \ a_n \neq 0 \tag{6.10}$$

$f_n(\theta)$ being a real trigonometric polynomial of precise degree n.

Theorem 6.7. *Let $f_n(\theta) \in \mathcal{T}_n$ as given by (6.10). Then $\langle f_n(\theta), T(\theta)\rangle_\omega = 0$ for all $T \in \mathcal{T}_{n-1}$.*

Proof. Clearly, it will be enought to show that, $\langle \rho_n(z), z^j\rangle_\omega = 0$ for $-(n-1) < j \leqslant n-1$ ($z = e^{i\theta}$). By (6.10) and since $a_n \neq 0$, the above becomes

$$\langle e^{-in\theta} B_{2n}(e^{i\theta}), e^{ij\theta}\rangle_\omega = 0 \ \ , \ \ -(n-1) \leqslant j \leqslant n-1. \tag{6.11}$$

Now, by Theorem 6.4, $B_{2n}(z) = \rho_{2n}(z) + \tau\rho_{2n}^*(z)$ (observe that the constant $C_{2n} \neq 0$ is now irrelevant) so that (6.11) can be written as

$$\begin{gathered}\langle e^{-in\theta}\left(\rho_{2n}(e^{i\theta}) + \tau\rho_{2n}^*(e^{i\theta})\right), e^{ij\theta}\rangle_\omega = \\ \langle \rho_{2n}(z), z^{n+j}\rangle_\omega + \tau\langle \rho_{2n}^*(z), z^{n+j}\rangle_\omega = 0,\end{gathered}$$

because both inner products are zero by the orthogonality properties of $\rho_{2n}(z)$ and $\rho_{2n}^*(z)$. □

Now, as a direct consequence of Theorem 4.9, we can establish the fundamental property concerning the localization of the zeros of $B_n(z)$. Indeed, one has

Theorem 6.8. *Let $B_n(z)$ be a para-orthogonal and invariant polynomial of degree n. Then $B_n(z)$ has exactly n distinct zeros on the unit circle $\mathbb{T}$.*

Proof. Assume first that n is even, say $n = 2m$ so that by (6.10)

$$e^{-im\theta} B_{2m}(e^{i\theta}) = a_m h_m(\theta), \;\; a_m \neq 0, \;\; h_m \in \mathcal{T}_m.$$

Let $f_0 \cup \{f_k, g_k\}_{k=1}^{\infty}$ be a bi-orthogonal system of trigonometric polynomials. Then, by Theorem 6.7, $h_m(\theta) = \alpha_m f_m(\theta) + \beta_m g_m(\theta)$, $|\alpha_m| + |\beta_m| > 0$ and the proof follows by Theorem 4.9. Suppose now that n is odd, i.e. $n = 2m + 1$. Since $B_{2m+1}(z)$ is invariant, one knows that $B_{2m+1}(z)$ has at least one zero λ on $\mathbb{T}$ of odd multiplicity. Thus, $B_{2m+1}(z) = (z - \lambda)\tilde{B}_{2m}(z)$ with $\tilde{B}_{2m}(z)$ a polynomial of degree $2m$. Furthermore, it can be easily checked that $\tilde{B}_{2m}(z)$ is also invariant and para-orthogonal for the weight function $\tilde{\omega}(\theta) = |e^{i\theta} - \lambda|^2 \omega(\theta)$. Hence, $\tilde{B}_{2m}(z)$ has $2m$ distinct zeros on $\mathbb{T}$. Furthermore, any zero of $\tilde{B}_{2m}(z)$ is different from λ, otherwise its multiplicity would be two. This concludes the proof. □

Remark 6.9. In [10] another different and longer proof of Theorem 6.8 is presented. Here we have taken advantage of the properties of bi-orthogonal systems introduced in Section 4 to give a simpler proof.

Let $\{B_{2n}(z)\}_{n=0}^{\infty}$ be a sequence of para-orthogonal and invariant polynomials such that for each $n \geqslant 1$, $B_{2n}(z)$ has exactly degree $2n$. Because of invariance again, it can be written

$$B_{2n}(z) = a_n e^{in\theta} f_n(\theta), \;\; a_n \neq 0, \;\; f_n \in \mathcal{T}_n.$$

Then, by Theorem 6.7, $\{f_n(\theta)\}_{n=0}^{\infty}$ $(f_0(\theta) = f_0 \neq 0)$ represents a nontrivial orthogonal system of trigonometric polynomials, in the sense that for each n, $f_n(\theta)$ has the precise degree n and $\langle f_n(\theta), f_m(\theta)\rangle_\omega = K_n \delta_{n,m}$, $K_n > 0$. Now, we could ask if it is possible to find another orthogonal system $\{g_n(\theta)\}_{n=1}^{\infty}$ so that $f_0 \cup \{f_n(\theta), g_n(\theta)\}_{n=1}^{\infty}$ constitutes a bi-orthogonal system of trigonometric polynomials. To fix ideas, set

$$B_{2n}(z) = B_{2n}(z, \tau_n) = \rho_{2n}(z) + \tau_n \rho_{2n}^*(z)$$

where $\{\tau_n\}_{n=1}^{\infty}$ is a sequence of complex numbers on $\mathbb{T}$. Certainly, we can write $\tau_n = \frac{\overline{\gamma_n}}{\gamma_n}$, $\gamma_n \in \mathbb{C}$, $\gamma_n \neq 0$ so that if $\tau_n = e^{i\eta_n}$, then $\gamma_n = r_n e^{-i\eta_n/2}$, $\eta_n \in \mathbb{R}$, $r_n > 0$. On the other hand, setting $z = e^{i\theta}$:

$$\begin{aligned} z^{-n} B_{2n}(z) &= \frac{\rho_{2n}(z) + \tau_n \rho_{2n}^*(z)}{z^n} = \frac{1}{\gamma_n}\left[\frac{\gamma_n \rho_{2n}(z) + \overline{\gamma_n}\rho_{2n}^*(z)}{z^n}\right] \\ &= \frac{1}{\gamma_n}\left[\frac{\gamma_n \rho_{2n}(z) + \overline{\gamma_n} z^{2n} \rho_{(2n)*}(z)}{z^n}\right] \\ &= \frac{1}{\gamma_n}\left[\gamma_n z^{-n}\rho_{2n}(z) + \overline{\gamma_n z^{-n}\rho_{2n}(z)}\right] \\ &= \frac{2}{\gamma_n}\Re\left(\gamma_n z^{-n}\rho_{2n}(z)\right). \end{aligned}$$

Consider now $B_{2n}(z,-\tau_n) = \rho_{2n}(z) - \tau_n\rho_{2n}^*(z)$. Then, again by Theorem 6.7, one has

$$e^{-in\theta}B_{2n}(e^{i\theta},-\tau_n) = \tilde{\lambda}_n g_n(\theta), \;\; \tilde{\lambda}_n \neq 0, \; g_n \in \mathcal{T}_n$$

and $\{g_n(\theta)\}_{n=1}^{\infty}$ is an orthogonal system of trigonometric polynomials. Therefore it holds that

$$\langle g_n(\theta), g_m(\theta)\rangle_\omega = \tilde{K}_n\delta_{n,m}, \; \tilde{K}_n > 0 \quad ; \quad \langle g_n(\theta), f_m(\theta)\rangle_\omega = 0, \; n \neq m.$$

Let us also see that $\langle f_n(\theta), g_n(\theta)\rangle_\omega = 0$ for $n = 0, 1, \ldots$. As above, it can be easily shown that

$$g_n(\theta) = \tilde{C}_n \Im\left(\gamma_n z^{-n}\rho_{2n}(z)\right) = \tilde{C}_n\tilde{g}_n(\theta)$$

with $\tilde{C}_n \neq 0$ and $\tilde{g}_n \in \mathcal{T}_n$. Hence,

$$\langle f_n(\theta), g_n(\theta)\rangle_\omega = 0 \;\Leftrightarrow\; \langle \tilde{f}_n(\theta), \tilde{g}_n(\theta)\rangle_\omega = 0.$$

Now, for $z = e^{i\theta}$,

$$\begin{aligned}
\int_{-\pi}^{\pi}\left(\gamma_n z^{-n}\rho_{2n}(z)\right)^2\omega(\theta)d\theta &= \int_{-\pi}^{\pi}\left[\tilde{f}_n(\theta) + i\tilde{g}_n(\theta)\right]^2\omega(\theta)d\theta \\
&= \int_{-\pi}^{\pi}\tilde{f}_n^2(\theta)\omega(\theta)d\theta - \int_{-\pi}^{\pi}\tilde{g}_n^2(\theta)\omega(\theta)d\theta + \\
&+ \; 2i\int_{-\pi}^{\pi}\tilde{f}_n(\theta)\tilde{g}_n(\theta)\omega(\theta)d\theta.
\end{aligned}$$

Thus, by assuming that $\gamma_n^2\int_{-\pi}^{\pi} z^{-2n}\rho_{2n}^2(z)\omega(\theta)d\theta$ $(z = e^{i\theta})$ is a real number it follows that

$$\int_{-\pi}^{\pi}\tilde{f}_n(\theta)\tilde{g}_n(\theta)\omega(\theta)d\theta = \langle \tilde{f}_n(\theta), \tilde{g}_n(\theta)\rangle_\omega = 0.$$

But

$$\begin{aligned}
\gamma_n^2\int_{-\pi}^{\pi} z^{-2n}\rho_{2n}^2(z)\omega(\theta)d\theta &= \gamma_n^2\int_{-\pi}^{\pi}\rho_{2n}(z)\frac{z^{2n}+\cdots+\delta_{2n}}{z^{2n}}\omega(\theta)d\theta \\
&= \gamma_n^2\int_{-\pi}^{\pi}\rho_{2n}(z)\frac{\delta_{2n}}{z^{2n}}\omega(\theta)d\theta \\
&= \gamma_n^2\delta_{2n}\langle\rho_{2n}(z), z^{2n}\rangle_\omega.
\end{aligned}$$

Since $\langle\rho_{2n}(z), z^{2n}\rangle_\omega = \langle\rho_{2n}(z), \rho_{2n}(z)\rangle_\omega = \|\rho_{2n}(z)\|_\omega^2 > 0$, then the positivity of

$$\gamma_n^2\int_{-\pi}^{\pi} z^{-2n}\rho_{2n}^2(z)\omega(\theta)d\theta$$

reduces to $\gamma_n^2\delta_{2n} \in \mathbb{R}$, or equivalently $\overline{\gamma_n^2\delta_{2n}} \in \mathbb{R}$. In terms of the parameter $\tau_n = \frac{\overline{\gamma_n}}{\gamma_n} \in \mathbb{T}$, this condition implies $\tau_n\overline{\delta_{2n}} \in \mathbb{R}$. In other words, we have proved the following

Theorem 6.10. *Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of complex numbers in $\mathbb{T}$ such that $\tau_n\overline{\delta_{2n}} \in \mathbb{R}$ and consider the sequences of polynomials $\{B_{2n}(z,\tau_n)\}_{n=1}^{\infty}$ and $\{B_{2n}(z,-\tau_n)\}_{n=1}^{\infty}$ so that for each $n = 1, 2, \ldots$, $B_{2n}(z,\pm\tau_n)$ is a para-orthogonal and invariant polynomial of degree $2n$. Then*

1. *$e^{-in\theta}B_{2n}(e^{i\theta},\tau_n)=\lambda_n f_n(\theta)$ and $e^{-in\theta}B_{2n}(e^{i\theta},-\tau_n)=\tilde{\lambda}_n g_n(\theta)$ with λ_n and $\tilde{\lambda}_n$ nonzero complex numbers and $f_n(\theta)$ and $g_n(\theta)$ being trigonometric polynomials of the precise degree n.*
2. *Choose $f_0\neq 0$, then $f_0\cup\{f_n(\theta),g_n(\theta)\}_{n=1}^{\infty}$ represents a bi-orthogonal system for $\omega(\theta)$.*

Now, from Theorem 4.10 or Corollary 4.12 one immediately gets

Corollary 6.11. *Under the same assumptions as in Theorem 6.10, the zeros of the para-orthogonal polynomials $B_{2n}(z,\tau_n)$ and $B_{2n}(z,-\tau_n)$ interlace.*

On the other hand, a converse to Theorem 6.10 can be also given. Indeed, we have:

Theorem 6.12. *Let $f_0\cup\{f_k,g_k\}_{k=1}^{\infty}$ be a bi-orthogonal system for $\omega(\theta)$ and take a and b real numbers not both zero. Then, for $n\geqslant 1$*

$$H_n(\theta)=af_n(\theta)+bg_n(\theta)=e^{-in\theta}B_{2n}(e^{i\theta})$$

and $B_{2n}(z)$ is a para-orthogonal and 1-invariant polynomial of degree $2n$.

Proof. We can write

$$f_n(\theta)=a_0+\sum_{j=1}^{n}(a_j\cos j\theta+b_j\sin j\theta)\,,\quad g_n(\theta)=\alpha_0+\sum_{j=1}^{n}(\alpha_j\cos j\theta+\beta_j\sin j\theta)$$

with $|a_n|+|b_n|>0$, $|\alpha_n|+|\beta_n|>0$ and

$$f_n(\theta)=\sum_{k=-n}^{n}c_kz^k\in\Lambda_{-n,n},\quad g_n(\theta)=\sum_{k=-n}^{n}d_kz^k\in\Lambda_{-n,n},\quad z=e^{i\theta}$$

where for $k=1,\dots,n$,

$$\begin{array}{lll} c_0=a_0, & c_k=\frac{a_k-ib_k}{2} & c_{-k}=\frac{a_k+ib_k}{2} \\ d_0=\alpha_0, & d_k=\frac{\alpha_k-i\beta_k}{2}, & d_{-k}=\frac{\alpha_k+i\beta_k}{2}. \end{array} \tag{6.12}$$

Hence, by the transformation $z=e^{i\theta}$ it follows that

$$\begin{aligned} B_{2n}(\theta) &= z^n\left[a\sum_{k=-n}^{n}c_kz^k+b\sum_{k=-n}^{n}d_kz^k\right]=\sum_{j=0}^{2n}\left(ac_{j-n}+bd_{j-n}\right)z^j \\ &= \sum_{j=0}^{2n}e_jz^j\in\Pi_{2n}, \end{aligned}$$

and it is clear from (6.12) that $\overline{e_{2n-j}}=e_j$ for $j=0,\dots,2n$. This proves the 1-invariance property. Now, from the orthogonality conditions satisfied by $f_n(\theta)$ and $g_n(\theta)$ it follows for $j=1,\dots,2n-1$ that

$$\begin{aligned} \langle B_{2n}(\theta),e^{ij\theta}\rangle_\omega &= \langle e^{in\theta}\left[af_n(\theta)+bg_n(\theta)\right],e^{ij\theta}\rangle_\omega = \\ &= a\langle f_n(\theta),e^{i(j-n)\theta}\rangle_\omega+b\langle g_n(\theta),e^{i(j-n)\theta}\rangle_\omega=0, \end{aligned}$$

i.e., $\langle B_{2n}(z), z^j\rangle_\omega = 0$ for all $j = 1, \dots, 2n-1$. We will prove next that $\langle B_{2n}(z), 1\rangle_\omega \neq 0$ and $\langle B_{2n}(z), z^{2n}\rangle_\omega \neq 0$. Firstly observe that

$$\begin{aligned} \langle B_{2n}(z), 1\rangle_\omega &= a\langle f_n(\theta), e^{-in\theta}\rangle_\omega + b\langle g_n(\theta), e^{-in\theta}\rangle_\omega, \\ \langle B_{2n}(z), z^{2n}\rangle_\omega &= a\langle f_n(\theta), e^{in\theta}\rangle_\omega + b\langle g_n(\theta), e^{in\theta}\rangle_\omega \,. \end{aligned} \tag{6.13}$$

Writing $\cos n\theta = \frac{e^{in\theta}+e^{-in\theta}}{2}$, $\sin n\theta = \frac{e^{in\theta}-e^{-in\theta}}{2i}$, $f_n(\theta) = a_n \cos n\theta + b_n \sin n\theta + H_{n-1}(\theta)$ and $g_n(\theta) = \alpha_n \cos n\theta + \beta_n \sin n\theta + \tilde{H}_{n-1}(\theta)$, where $H_{n-1}(\theta), \tilde{H}_{n-1}(\theta) \in \mathcal{T}_{n-1}$, we deduce that

$$\begin{aligned} \langle f_n(\theta), f_n(\theta)\rangle_\omega &= \langle f_n(\theta), a_n\cos n\theta + b_n \sin n\theta + H_{n-1}(\theta)\rangle_\omega = \\ &= \tfrac{b_n+ia_n}{2i}\langle f_n(\theta), e^{in\theta}\rangle_\omega + \tfrac{-b_n+ia_n}{2i}\langle f_n(\theta), e^{-in\theta}\rangle_\omega = \\ &= h_n > 0, \end{aligned}$$

$$\begin{aligned} \langle g_n(\theta), g_n(\theta)\rangle_\omega &= \tfrac{\beta_n+i\alpha_n}{2i}\langle g_n(\theta), e^{in\theta}\rangle_\omega + \tfrac{-\beta_n+i\alpha_n}{2i}\langle g_n(\theta), e^{-in\theta}\rangle_\omega = \\ &= h_n^{'} > 0, \end{aligned}$$

$$\begin{aligned} \langle f_n(\theta), g_n(\theta)\rangle_\omega &= \langle f_n(\theta), \alpha_n\cos n\theta + \beta_n \sin n\theta + \tilde{H}_{n-1}(\theta)\rangle_\omega = \\ &= \tfrac{\beta_n+i\alpha_n}{2i}\langle f_n(\theta), e^{in\theta}\rangle_\omega + \tfrac{-\beta_n+i\alpha_n}{2i}\langle f_n(\theta), e^{-in\theta}\rangle_\omega = 0, \\ \langle g_n(\theta), f_n(\theta)\rangle_\omega &= \tfrac{b_n+ia_n}{2i}\langle g_n(\theta), e^{in\theta}\langle + \tfrac{-b_n+ia_n}{2i}\langle g_n(\theta), e^{-in\theta}\rangle_\omega = 0 \,. \end{aligned}$$

These relations can be summarized as

$$A\begin{pmatrix} \langle f_n(\theta), e^{in\theta}\rangle_\omega \\ \\ \langle f_n(\theta), e^{-in\theta}\rangle_\omega \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 2ih_n \end{pmatrix}, \; A\begin{pmatrix} \langle g_n(\theta), e^{in\theta}\rangle_\omega \\ \\ \langle g_n(\theta), e^{-in\theta}\rangle_\omega \end{pmatrix} = \begin{pmatrix} 2ih_n^{'} \\ \\ 0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} \beta_n + i\alpha_n & -\beta_n + i\alpha_n \\ \\ b_n + ia_n & -b_n + ia_n \end{pmatrix}, \quad det(A) = 2i[a_n\beta_n - \alpha_n b_n] \neq 0$$

since $f_n(\theta), g_n(\theta)$ are linearly independent trigonometric polynomials. The solutions of these systems are given by

$$\begin{aligned} \langle f_n(\theta), e^{in\theta}\rangle_\omega &= \frac{\beta_n - i\alpha_n}{a_n\beta_n - \alpha_n b_n} h_n \neq 0, \;\; \langle f_n(\theta), e^{-in\theta}\rangle_\omega = \overline{\langle f_n(\theta), e^{in\theta}\rangle_\omega}, \\ \langle g_n(\theta), e^{in\theta}\rangle_\omega &= \frac{-b_n + ia_n}{a_n\beta_n - \alpha_n b_n} h_n^{'} \neq 0, \;\; \langle g_n(\theta), e^{-in\theta}\rangle_\omega = \overline{\langle g_n(\theta), e^{in\theta}\rangle_\omega} \,. \end{aligned}$$

Now, from (6.13) it follows that

$$\langle B_{2n}(z), 1\rangle_\omega = \frac{1}{a_n\beta_n - \alpha_n b_n}\left[(a\beta_n h_n - b\beta_n h_n^{'}) + i(a\alpha_n h_n - b a_n h_n^{'})\right]$$

and $\langle B_{2n}(z), z^{2n}\rangle_\omega = \overline{\langle B_{2n}(z), 1\rangle_\omega}$. Again, since $f_n(\theta), g_n(\theta)$ are linearly independent it is easy to observe that $\langle B_{2n}(z), 1\rangle_\omega \neq 0$ and hence $\langle B_{2n}(z), z^{2n}\rangle_\omega \neq 0$. This completes the proof. □

After having established certain connections between para-orthogonal polynomials and bi-orthogonal trigonometric polynomials we are now in a position to construct an n-point quadrature rule for $I_\omega(f)$ with nodes on $\mathbb{T}$ and having the "maximum domain of validity", $\Lambda_{-(n-1),n-1}$. Indeed, we have (see [10])

Theorem 6.13. *Let $z_1, \ldots, z_n$ be the n distinct zeros of $B_n(z)$ a given polynomial of degree n, para-orthogonal and invariant. Then, there exist positive numbers $A_1, \ldots, A_n$ such that*

$$I_n(f) = \sum_{j=1}^{n} A_j f(z_j) = I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta, \ \ \forall\, f \in \Lambda_{-(n-1),n-1}.$$

Now, by considering Theorems 6.1 and 6.13 together we obtain the following characterization (see [2]):

Corollary 6.14. *Let $I_\omega(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta$ and let $I_n(f) = \sum_{j=1}^{n} A_j f(z_j)$ such that $z_j \in \mathbb{T}$, $j = 1, \ldots, n$ with $z_j \neq z_k$ if $j \neq k$ and set $B_n(z) = \prod_{j=1}^{n}(z - z_j)$. Then $I_n(L) = I_\omega(L)$ for all $L \in \Lambda_{-(n-1),n-1}$ if and only if*

1. *$I_n(L) = I_\omega(L)$, for all $L \in \Lambda_{-p,q}$, p and q being nonnegative arbitrary integers such that $p + q = n - 1$.*
2. *$B_n(z)$ is para-orthogonal and invariant.*

Furthermore, when the conditions are satisfied the weights $\{A_j\}_{j=1}^{n}$ are positive and independent of p and q.

Remark 6.15. The quadrature rules $I_n(f)$, $n = 1, 2, \ldots$ as given above are called "Szegő quadrature formulas" and were earlier introduced in [10]. They represent the analogue on the unit circle of the Gauss-Christoffel formulas. For an alternative approach of Szegő quadratures making use of the so-called orthogonal Laurent polynomials on the unit circle, see the recent paper by the authors [3]. For further details concerning these quadratures see also [4], [5] and [9].

To conclude, it should be remarked that given the integral $\int_{-\pi}^{\pi} f(\theta)\omega(\theta)d\theta$, f being a 2π-periodic function, it clearly follows from Corollary 6.14 how to construct an n-point quadrature rule with distinct nodes on $[-\pi, \pi]$ which is exact in $\mathcal{T}_{n-1}$, n being an arbitrary natural number. As a simple illustration, let us consider again the weight function $\omega(\theta) \equiv 1$. Then, $B_n(z) = z^n - \tau$, $|\tau| = 1$ and the nodes of the n-th Szegő formula are the n-th roots of τ, that is $z_j = \sqrt[n]{\tau}$, $j = 1, \ldots, n$. Thus,

$$A_j = \frac{1}{B_n'(z_j)} \int_{-\pi}^{\pi} \frac{B_n(z)}{z - z_j} d\theta = \frac{1}{n z_j^{n-1}} \frac{1}{i} \int_{\mathbb{T}} \frac{z^n - \tau}{z(z - z_j)} dz = \frac{2\pi\tau}{n z_j^n},$$

by the Residue Theorem. Since
$z_j^n = \tau$, we obtain $A_j = \frac{2\pi}{n}$, $j = 1, \ldots, n$ as previously deduced in Example 5.9.

7. Numerical examples

In order to illustrate the numerical effectiveness of the quadrature rules considered through the paper, in this section we are going to be concerned with the computation of the two-parameter integral,

$$I(m,\alpha) = \int_{-\pi}^{\pi} \frac{\cos m\theta}{\alpha + \sin^2\theta} d\theta, \ m \geq 0, \ m \in \mathbb{N}, \ \alpha > 0. \tag{7.1}$$

Observe that for $\alpha = 0$, the integral diverges. Thus, for values of α close to zero, the denominator of the integrand is also close to zero as θ tends to $\pm\pi$. Certainly, this could generate some kind of unstability when undertaking the approximation of $I(m,\alpha)$ by means of a certain quadrature rule with nodes close to $\pm\pi$.

On the other hand, for m large enough, the integral is highly oscillating on $[-\pi,\pi]$. Indeed, setting $f(\theta) = \frac{\cos m\theta}{\alpha+\sin^2\theta}$, then $f(\theta)$ clearly changes sign at the points for which $f(\theta) = 0$, i. e., at $\theta_k = \frac{(2k+1)\pi}{2m}$, $-m \leq k \leq m-1$.

Under these considerations, we propose the following in order to compute approximately the integral $I(m,\alpha)$. Note that because of simmetry, one can write

$$I(m,\alpha) = 2\int_{0}^{\pi} \frac{\cos m\theta}{\alpha + \sin^2 m\theta} d\theta. \tag{7.2}$$

First, we have approximated (7.2) by means of the $n-$point Gauss-Legendre formula for the interval $[0,\pi]$ and the Trapezoidal rule for $n = 10, 12, 14, 16$. Here n denotes both the number of nodes in the Gauss-Legendre formulas and the number of subintervals in $[0,\pi]$. The results are displayed in the following tables.

Quadrature rules	n=10	n=12	n=14	n=16
Gauss-Legendre	2.26414	0.300761	0.00937743	0.000154023
Trapezoidal	0.0224394	0.000660554	0.000194449	5.72404E-7

Table 1: $(m = 14, \ \alpha = 1)$

Quadrature rules	n=10	n=12	n=14	n=16
Gauss-Legendre	8.93136E-5	7.12412E-7	1.17708E-8	4.65022E-10
Trapezoidal	4.20833E-8	1.30695E-10	4.05799E-12	1.26807E-15

Table 2: $(m = 8, \ \alpha = 4)$

Take into account that the trapezoidal rule coincides with the quadrature formula with the highest degree of trigonometric precision (Szegő formula). This fact might explain why the results provided by the Trapezoidal rule are better than those given by Gauss-Legendre formula. However, when α is closer to zero, the results of both quadrature rules, as it could be expected, are rather poor. This is shown in Table 3 corresponding to $m = 12$ and $\alpha = 0.25$.

Quadrature rules	n=6	n=8	n=10	n=12
Gauss-Legendre	5.05696	5.60122	0.516198	0.0190433
Trapezoidal	11.2748	1.64061	0.239269	0.0349069

Table 3: $(m = 12,\ \alpha = 0.25)$

In order to overcome this drawback, we are going to take the factor $\frac{1}{\alpha+\sin^2\theta}$ as a weight function. For this purpose, set $T(\theta) = \alpha+\sin^2\theta$, so that $T(\theta)$ is a positive trigonometric polynomial of degree two. Then, by Theorem 2.6, one can write,

$$T(\theta) = \left|g\left(e^{i\theta}\right)\right|^2,\ g \in \Pi_2.$$

Since $T(\theta) = \alpha + \sin^2\theta = \alpha + \frac{1}{2}(1 - \cos 2\theta)$, then by setting $\beta = 2\alpha + 1 > 1$ and $z = e^{i\theta}$,

$$2T(\theta) = \beta - \frac{1}{2}(z^2 + z^{-2}),$$

yielding,

$$4T(\theta) = \frac{-z^4 + 2\beta z^2 - 1}{z^2}.$$

Furthermore, since $T(\theta) > 0$ and $z \in \mathbb{T}$, then

$$4T(\theta) = |4T(\theta)| = |z^4 - 2\beta z^2 + 1|. \tag{7.3}$$

If we set $z^4 - 2\beta z^2 + 1 = 0$, then $z^2 = \beta \pm \sqrt{\beta^2 - 1}$. Let $\gamma = \beta + \sqrt{\beta^2 - 1}$, then, it is easy to check that $\frac{1}{\gamma} = \beta - \sqrt{\beta^2 - 1}$. Therefore, one has

$$z^4 - 2\beta z^2 + 1 = (z^2 - \gamma)(z^2 - \gamma^{-1}). \tag{7.4}$$

On the other hand, since $z = e^{i\theta}$ and $\gamma \in \mathbb{R}$, we have:

$$\begin{aligned} |z^2 - \gamma^{-1}|^2 &= (z^2 - \gamma^{-1})\overline{(z^2 - \gamma^{-1})} = (z^2 - \gamma^{-1})(z^{-2} - \gamma^{-1}) \\ &= (z^2 - \gamma^{-1})\left(\tfrac{\gamma - z^2}{\gamma z^2}\right) = -\tfrac{1}{\gamma z^2}(z^2 - \gamma)(z^2 - \gamma^{-1}) \end{aligned}$$

$>$From (7.3) and (7.4), one has:

$$0 < |z^2 - \gamma^{-1}|^2 = -\frac{1}{\gamma z^2}4T(\theta) = \left|-\frac{1}{\gamma z^2}4T(\theta)\right| = \frac{4}{\gamma}T(\theta).$$

Thus,

$$T(\theta) = \frac{\gamma}{4}|z^2 - \gamma^{-1}|^2 = \frac{\gamma}{4}|g(z)|^2,\ g(z) = z^2 - \gamma^{-1},\ z = e^{i\theta}.$$

Now, taking into account that for integrals of the form: $\int_{-\pi}^{\pi} f(e^{i\theta}) \frac{d\theta}{2\pi|h(e^{\theta})|^2}$, with h a monic polynomial with all its zeros in $\mathbb{D}$, the coefficients of the $n-$point Szegő quadrature formulas are explicitly known ([9]), we will transform our integral $I(m,\alpha)$ as follows: $(z = e^{i\theta})$

$$\begin{aligned} I(m,\alpha) &= \int_{-\pi}^{\pi} \frac{\cos m\theta}{\alpha+\sin^2\theta} d\theta = \int_{-\pi}^{\pi} \cos m\theta \frac{d\theta}{\frac{\gamma}{4}|g(z)|^2} \\ &= \int_{-\pi}^{\pi} \left(\frac{4\pi}{\gamma}(z^m + z^{-m}) \right) \left(\frac{d\theta}{2\pi|g(z)|^2} \right). \end{aligned}$$

Therefore, we can write:

$$I(m,\alpha) = \int_{-\pi}^{\pi} f(e^{i\theta})\omega(\theta)d\theta, \tag{7.5}$$

where $f(z) = \frac{4\pi}{\gamma}(z^m + z^{-m})$ and the weight function is given by $\omega(\theta) = \frac{1}{2\pi|g(z)|^2}$, with $g(z) = z^2 - \gamma^{-1}$ and $z = e^{i\theta}$.

In this case, from Corollary 6.14, one knows that the nodes $\{z_j\}_{j=1}^n$ of the $n-$point Szegő quadrature formula are the zeros of the para-orthogonal polynomial $B_n(z) = \rho_n(z) + \tau\rho_n^*(z)$, $\rho_n(z)$ being the $n-th$ monic Szegő polynomial for $\omega(\theta) = \frac{1}{2\pi|g(z)|^2}$, with $|\tau| = 1$. Thus, from Example 4.8, $B_n(z,\tau) = z^{n-2}g(z) + \tau g^*(z) = z^{n-2}(z^2 - \gamma^{-1}) + \tau(1 - \gamma^{-1}z^2)$. On the other hand, the coefficients $\{\lambda_j\}_{j=1}^n$ of an $n-$point Szegő's formula are given by [9]:

$$\begin{aligned} \lambda_j^{-1} &= |g(z_j)|^2 \left(n - 2 + \frac{1- \frac{1}{\sqrt{\gamma}}^{\,2}}{z_j - \frac{1}{\sqrt{\gamma}}^{\,2}} + \frac{1- -\frac{1}{\sqrt{\gamma}}^{\,2}}{z_j + \frac{1}{\sqrt{\gamma}}^{\,2}} \right) \\ &= |g(z_j)|^2 \left(n - 2 + (1-\gamma^{-1}) \left(\frac{1}{z_j - \frac{1}{\sqrt{\gamma}}^{\,2}} + \frac{1}{z_j + \frac{1}{\sqrt{\gamma}}^{\,2}} \right) \right) \\ &= |g(z_j)|^2 \left(n - 2 + (1-\gamma^{-1}) \left(\frac{z_j - \frac{1}{\sqrt{\gamma}}^{\,2} + z_j + \frac{1}{\sqrt{\gamma}}^{\,2}}{|g(z_j)|^2} \right) \right) \\ &= |g(z_j)|^2 \left(n - 2 + (1-\gamma^{-1}) \left(\frac{1 - \frac{2}{\sqrt{\gamma}}\Re(z_j) + \frac{1}{\gamma} \;+\; 1 + \frac{2}{\sqrt{\gamma}}\Re(z_j) + \frac{1}{\gamma}}{|g(z_j)|^2} \right) \right) \\ &= |g(z_j)|^2 \left(n - 2 + 2(1-\gamma^{-1})(1+\gamma^{-1}) \frac{1}{|g(z_j)|^2} \right) \\ &= (n-2)|g(z_j)|^2 + 2(1-\gamma^{-2}),\ j = 1,\ldots,n. \end{aligned}$$

Note that, if $m \le n-1$, then the $n-$point Szegő quadrature formula is exact since the integrand $f \in \Delta_{-m,m}$.

Now, by (7.5), $I(m,\alpha)$ is going to be approximated by an $n-$point Szegő formula $I_n(f) = \sum_{j=1}^n \lambda_j f(z_j)$ so that the absolute errors can be exactly computed since $I(m,\alpha)$ can be calculated by the Residue's Theorem.

Indeed, since $I = \int_{-\pi}^{\pi} \frac{\sin m\theta}{\alpha+\sin^2\theta} d\theta = 0$, then

$$\begin{aligned} I(m,\alpha) &= \int_{-\pi}^{\pi} \frac{\cos m\theta}{\alpha+\sin^2\theta} d\theta + i\int_{-\pi}^{\pi} \frac{\sin m\theta}{\alpha+\sin^2\theta} d\theta = \int_{-\pi}^{\pi} \frac{\cos m\theta + i\sin m\theta}{T(\theta)} d\theta \\ &= \int_{-\pi}^{\pi} \frac{z^m}{\frac{\gamma}{4}|g(z)|^2} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{8\pi}{\gamma} z^m\right) \frac{d\theta}{(z^2-\frac{1}{\gamma})(\frac{1}{z^2}-\frac{1}{\gamma})} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (-8\pi) \frac{z^{m+2}}{(z^2-\frac{1}{\gamma})(z^2-\gamma)} d\theta = \frac{1}{2\pi i}\int_{\mathbb{T}} (-8\pi) \frac{z^{m+1}}{(z^2-\frac{1}{\gamma})(z^2-\gamma)} dz \\ &= Res\left(h, \frac{1}{\sqrt{\gamma}}\right) + Res\left(h, \frac{-1}{\sqrt{\gamma}}\right), \end{aligned}$$

where $h(z) = (-8\pi)\frac{z^{m+1}}{(z^2-\frac{1}{\gamma})(z^2-\gamma)}$.

Now,

$$Res\left(h, \frac{1}{\sqrt{\gamma}}\right) = -8\pi \frac{\frac{1}{(\sqrt{\gamma})^{m+1}}}{\frac{2}{\sqrt{\gamma}}\left(\frac{1}{\gamma}-\gamma\right)} = \frac{4\pi\gamma}{(\sqrt{\gamma})^m(\gamma^2-1)},$$

and

$$Res\left(h, \frac{-1}{\sqrt{\gamma}}\right) = -8\pi \frac{\frac{(-1)^{m+1}}{(\sqrt{\gamma})^{m+1}}}{\frac{-2}{\sqrt{\gamma}}\left(\frac{1}{\gamma}-\gamma\right)} = (-1)^{m+1}\frac{-4\pi\gamma}{(\sqrt{\gamma})^m(\gamma^2-1)}.$$

Hence,

$$I(m,\alpha) = \frac{4\pi\gamma(1-(-1)^{m+1})}{(\sqrt{\gamma})^m(\gamma^2-1)} = \begin{cases} \frac{8\pi\gamma}{(\sqrt{\gamma})^m(\gamma^2-1)}, & \text{if m is even,} \\ 0, & \text{if m is odd.} \end{cases}$$

Taking now $m = 12$ and $\alpha = 0.25$, the absolute errors for the corresponding $n-$point Szegő formula are displayed in Table 4 (Compare with Table 3).

n	Error- Szegő formula
n=4	3.18008
n=8	1.8473911237281646E-15
n=12	6.949821829035384E-15

Table 4: $(m = 12,\ \alpha = 0.25)$

The excellent behaviour of Segő formulas can be explained from [9, Theorem 3.3] taking into account that the integrand $f(z)$ in (7.5) has one only pole at the origin.

References

[1] N. I. Akhiezer The classical moment problem and some related questions in analysis, Hafner, New York, (1965).

[2] A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad Orthogonality and quadrature on the unit circle, In C. Brezinski, L. Gori, and A. Ronveaux, editors, *Orthogonal Polynomials and their Applications* Vol. 9 of IMACS Annals on Computing and Applied Mathematics (1991), 205–210. Basel, J.C. Baltzer AG.

[3] R. Cruz-Barroso, L. Daruis, P. González-Vera and O. Njåstad Sequences of orthogonal Laurent polynomials, bi-orthogonality and quadrature formulas on the unit circle. *Submitted.*

[4] L. Daruis, P. González-Vera, A. Bultheel A conecction between quadrature formulas on the unit circle and the interval $[-1, 1]$, *J. Comp. Appl. Math* Vol. 132 (1) (2000), 1–14.

[5] L.q Daruis, P. González-Vera and O. Njåstad Szegő quadrature formulas for certain Jacobi-type weight functions, *Mathematics of Computation* Vol. 71, Num. 238 (2001), 683–701.

[6] P. J. Davis, P. Rabinowitz Methods of Numerical Integration, *Academic Press* New York, (1984).

[7] P. Delsarte, Y. Genin On the role of Orthogonal Polynomials on the Unit Circle in digital signal processing applications, Orthogonal polynomials, *(Columbus, OH, 1989), NATO Adv. Study Inst. Ser. C Math. Phys. Sci* Vol. 294 (1990), 115–133. Kluwer Acad. Punl., Dordrecht.

[8] W. Gautschi A survey of Gauss-Christoffel quadrature formulae *in E.B. Christoffel; The Influence of his Work in Mathematics and the Physical Sciences, eds. P. L. Butzer and F. Fehér, Birkhäuser, Basel* (1981), 72–147.

[9] P. González-Vera, J. C. Santos-León, O. Njåstad Some results about numerical quadrature on the unit circle, *Advances in Computational Mathematics* Vol. 5 (1996), 297–328.

[10] W. B. Jones, O. Njåstad and W. J. Thron Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle, *Bull. London Math. Soc.* Vol. 21 (1989), 113-152.

[11] V. I. Krylov Approximate Calculation of Integrals, *The MacMillan Company* New York (1962).

[12] N. Levinson The Wiener RMS (root mean square) error criterion in filter design and prediction, *J.Math. Phys.* Vol. 25 (1947), 261–278.

[13] B. Simon Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, *AMS Colloquium Publications. American Mathematical Society, Providence, RI* Vol. 54.1 (2005).

[14] G. Szegő On bi-orthogonal systems of trigonometric polynomials, *Magyar Tud. Alcad. Kutató Int. Közl* Vol. 8 (1963), 255–273.

[15] G. Szegő Orthogonal polynomials, *Amer. Math. Soc. Coll. Publ.* Amer. Math. Soc. Providence, R.I. Vol. 23 (1975).

Ruymán Cruz-Barroso
Leyla Daruis
Pablo González-Vera
Department of Mathematical Analysis
38271 La Laguna. Tenerife. Spain

Olav Njåstad
Faculty of Physics, Informatics and Mathematics
N-7491 Trondheim. Norway

Annales Mathematicae et Informaticae
32 *(2005) pp. 45–51.*

Burkholder's inequality for multiindex martingales*

István Fazekas

Faculty of Informatics, University of Debrecen
e-mail: fazekasi@inf.unideb.hu

Dedicated to the memory of my teacher Péter Kiss

Abstract

Multiindex versions of Khintchine's and Burkholder's inequalities are presented.

Key Words: Khintchine's inequality, Burkholder's inequality, random field, martingale.

AMS Classification Number: 60G42, 60G60, 60E15.

1. Introduction and notation

Burkholder's inequality is a powerful tool of martingale theory. Let $(Z_n, \mathcal{F}_n)$, $n = 1, 2, \dots$, be a martingale with difference $X_n = Z_n - Z_{n-1}$. Let $p > 1$. There exist finite and positive constants C_p and D_p depending only on p such that

$$C_p \left[\mathbb{E}\left(\sum\nolimits_{k=1}^{n} X_k^2\right)^{p/2}\right]^{1/p} \leqslant (\mathbb{E}|Z_n|^p)^{1/p} \leqslant D_p \left[\mathbb{E}\left(\sum\nolimits_{k=1}^{n} X_k^2\right)^{p/2}\right]^{1/p}, \quad (1.1)$$

see Burkholder's classical paper [1] and the textbook [2]. When the random variables $X_1, X_2, \dots$ are independent (1.1) is called the Marcinkiewicz-Zygmund inequality (and in this particular case it is valid also for $p = 1$).

Let $\varepsilon_i(t)$, $i = 1, 2, \dots$, be the Rademacher system on $[0, 1]$. If $X_k = \varepsilon_k a_k$, then we obtain Khintchine's inequality. There exist finite and positive constants A_p and B_p depending only on p such that for any real sequence a_k, $k = 1, 2, \dots$,

$$A_p \left(\sum\nolimits_{k=1}^{n} a_k^2\right)^{1/2} \leqslant \left[\int_0^1 \left|\sum\nolimits_{k=1}^{n} \varepsilon_k(t) a_k\right|^p \, dt\right]^{1/p} \leqslant B_p \left(\sum\nolimits_{k=1}^{n} a_k^2\right)^{1/2}. \quad (1.2)$$

*Supported by the Hungarian Foundation of Scientific Researches under Grant No. OTKA T047067/2004 and Grant No. OTKA T048544/2005.

This inequality is valid for $p > 0$. Actually, the standard proof of (1.1) is based on (1.2), see [1]).

The two-index version of (1.1) is obtained in [8], see also [7].

The aim of this paper is to prove a multiindex version of Burkholder's inequality. The proof is based on the transform of a single parameter martingale. We also use the multiindex version of Khintchine's inequality (for the sake of completeness, we prove it).

In [9] the second inequality of (3.2) was presented (without proof) for $p > 2$. It was applied to obtain a Brunk-Prokhorov type strong law of large numbers for martingale fields (see [9], Proposition 14). For a recent overview of multiindex random processes see [6]. In [6] a certain version of the Burkholder inequality was presented for continuous parameter random fields without the details of the proof (p. 257, Theorem 4.1.2). We do not use that theorem, we give a simple proof based on well-known one-parameter results.

Our Burkholder type inequality can be used to prove convergence results for multiindex autoregressive type martingales (see [5], for the two-index case see [4]).

We use the following notation. Let d be a fixed positive integer. Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0$ the set of non-negative integers. The multidimensional indices will be denoted by $\mathbf{k} = (k_1, \dots, k_d), \mathbf{n} = (n_1, \dots, n_d), \dots \in \mathbb{N}_0^d$. Relations $\leqslant$, min are defined coordinatewise. I.e. $\mathbf{k} \leqslant \mathbf{n}$ means $k_1 \leqslant n_1, \dots, k_d \leqslant n_d$. Relation $\mathbf{k} < \mathbf{n}$ means $\mathbf{k} \leqslant \mathbf{n}$ but $\mathbf{k} \neq \mathbf{n}$.

Let $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$ for $p > 0$. Then $\|X\|_{p_1} \leqslant \|X\|_{p_2}$ for $0 < p_1 \leqslant p_2$.

2. Khintchine's inequality

Theorem 2.1. *Let $\varepsilon_i(t)$, $i = 1, 2, \dots,$ be the Rademacher system on $[0,1]$. Let $p > 0$. There exist finite and positive constants $A_{p,d}$ and $B_{p,d}$ depending only on p and d such that for any d-index sequence $a_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^d$,*

$$\begin{aligned} A_{p,d} \left(\sum\nolimits_{\mathbf{k} \leqslant \mathbf{n}} a_{\mathbf{k}}^2 \right)^{1/2} &\leqslant \left[\int_0^1 \cdots \int_0^1 \left| \sum\nolimits_{\mathbf{k} \leqslant \mathbf{n}} \varepsilon_{k_1}(t_1) \cdots \varepsilon_{k_d}(t_d) a_{\mathbf{k}} \right|^p dt_1 \dots dt_d \right]^{1/p} \\ &\leqslant B_{p,d} \left(\sum\nolimits_{\mathbf{k} \leqslant \mathbf{n}} a_{\mathbf{k}}^2 \right)^{1/2}. \end{aligned} \tag{2.1}$$

Proof. First we remark that for $d = 1$ inequality (2.1) is the original Khintchine's inequality.

Denote by ε_{i,n_i}, $n_i = 1, 2, \dots,$ $i = 1, 2, \dots, d$, independent sequences of independent Bernoulli random variables with $\mathbb{P}(\varepsilon_{i,n_i} = 1) = \mathbb{P}(\varepsilon_{i,n_i} = -1) = 1/2$ for each i and n_i. Let $s_{\mathbf{n}} = \left(\sum_{\mathbf{k} \leqslant \mathbf{n}} a_{\mathbf{k}}^2 \right)^{1/2}$ and $S_{\mathbf{n}} = \sum_{\mathbf{k} \leqslant \mathbf{n}} \varepsilon_{1,k_1} \cdots \varepsilon_{d,k_d} a_{\mathbf{k}}$. Then, by the Fubini theorem, inequality (2.1) is equivalent to

$$A_{p,d}\, s_{\mathbf{n}} \leqslant \|S_{\mathbf{n}}\|_p \leqslant B_{p,d}\, s_{\mathbf{n}}. \tag{2.2}$$

Now we prove that the products $\varepsilon_{1,k_1} \cdots \varepsilon_{d,k_d}$, $(k_1, \dots, k_d) \in \mathbb{N}^d$, are pairwise independent Bernoulli variables. By induction, it is enough to prove that $\varepsilon_{1,k_1}\varepsilon_{2,k_2}$,

$(k_1, k_2) \in \mathbb{N}^2$, are pairwise independent Bernoulli variables if ε_{1,k_1}, $k_1 \in \mathbb{N}$, and ε_{2,k_2}, $k_2 \in \mathbb{N}$, are independent sequences of pairwise independent Bernoulli variables. Indeed, if ε_1 and ε_2 are independent Bernoulli variables then their product is Bernoulli: $\mathbb{P}(\varepsilon_1\varepsilon_2 = \pm 1) = 1/2$. Now turn to the independence. It is obvious that the independence of ε_1, ε_2, ε_3, and ε_4 implies the independence of $\varepsilon_1\varepsilon_2$ and $\varepsilon_3\varepsilon_4$. Moreover, the independence of ε_1, ε_2 and ε_3 implies the independence of $\varepsilon_1\varepsilon_3$ and $\varepsilon_2\varepsilon_3$:

$$\mathbb{P}\left(\varepsilon_1\varepsilon_3 = \pm 1, \varepsilon_2\varepsilon_3 = \pm 1\right) = \frac{1}{4} = \mathbb{P}\left(\varepsilon_1\varepsilon_3 = \pm 1\right)\mathbb{P}\left(\varepsilon_2\varepsilon_3 = \pm 1\right).$$

Therefore $\|S_{\mathbf{n}}\|_2^2$ is the variance of the sum of pairwise indepenent random variables, so we have $s_{\mathbf{n}} = \|S_{\mathbf{n}}\|_2$. In particular, (2.2) is true for $p = 2$.

Now we show that the products $\varepsilon_{1,k_1} \cdots \varepsilon_{d,k_d}$, $(k_1, \dots, k_d) \in \mathbb{N}^d$, are not (completely) independent. Indeed, if ε_1, ε_2, ε_3, and ε_4 are independent Bernoulli variables, then $\varepsilon_1\varepsilon_3$, $\varepsilon_2\varepsilon_3$, $\varepsilon_1\varepsilon_4$, and $\varepsilon_2\varepsilon_4$ are not independent:

$$\mathbb{P}\left(\varepsilon_1\varepsilon_3 = 1, \varepsilon_2\varepsilon_3 = 1, \varepsilon_1\varepsilon_4 = 1, \varepsilon_2\varepsilon_4 = 1\right) = 1/8 \neq$$

$$\neq 1/16 = \mathbb{P}\left(\varepsilon_1\varepsilon_3 = 1\right)\mathbb{P}\left(\varepsilon_2\varepsilon_3 = 1\right)\mathbb{P}\left(\varepsilon_1\varepsilon_4 = 1\right)\mathbb{P}\left(\varepsilon_2\varepsilon_4 = 1\right).$$

So relation (2.2) is really different from its one-index version.

Now we prove the second part of (2.2). We start with the case of $p \geq 2$. We use induction. For $d = 1$ it is the original Khintchine's inequality. Assume (2.2) for $d - 1$. Let

$$I_{k_1,n_2,\dots,n_d}(t_2, \dots, t_d) = \sum\nolimits_{k_2=1}^{n_2} \cdots \sum\nolimits_{k_d=1}^{n_d} \varepsilon_{k_2}(t_2) \cdots \varepsilon_{k_d}(t_d) a_{k_1,k_2,\dots,k_d}.$$

Then, by the original Khintchine's inequality,

$$\int_0^1 \left|\sum\nolimits_{k_1=1}^{n_1} \varepsilon_{k_1}(t_1) I_{k_1,n_2,\dots,n_d}(t_2, \dots, t_d)\right|^p \, dt_1 \leqslant$$

$$\leqslant B_{p,1}^p \left(\sum\nolimits_{k_1=1}^{n_1} I_{k_1,n_2,\dots,n_d}^2(t_2, \dots, t_d)\right)^{p/2}.$$

From here

$$\begin{aligned}
\|S_{\mathbf{n}}\|_p^p &\leqslant B_{p,1}^p \int_0^1 \cdots \int_0^1 \left(\sum_{k_1=1}^{n_1} I_{k_1,n_2,\dots,n_d}^2(t_2, \dots, t_d)\right)^{p/2} dt_2 \dots dt_d \\
&\leqslant B_{p,1}^p \left\{\sum_{k_1=1}^{n_1} \left[\int_0^1 \cdots \int_0^1 \left(I_{k_1,n_2,\dots,n_d}^2(t_2, \dots, t_d)\right)^{p/2} \, dt_2 \dots dt_d\right]^{2/p}\right\}^{p/2} \\
&\leqslant B_{p,1}^p \left\{\sum_{k_1=1}^{n_1} \left[B_{p,d-1}\left(\sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} a_{k_1,k_2,\dots,k_d}^2\right)^{1/2}\right]^2\right\}^{p/2}
\end{aligned}$$

$$= \quad (B_{p,d}\, s_{\mathbf{n}})^p\,,$$

where we used the triangle inequality in $L_{p/2}$ and (2.2) for $d-1$. So we proved the second part of (2.2) for $p \geq 2$.

As $\|S_{\mathbf{n}}\|_p \leqslant \|S_{\mathbf{n}}\|_2$ for $0 < p \leqslant 2$, the second part of (2.2) is true for $0 < p$.

Now turn to the first part of (2.2). We see that $s_{\mathbf{n}} = \|S_{\mathbf{n}}\|_2 \leqslant \|S_{\mathbf{n}}\|_p$ for $p \geq 2$. Therefore it is enough to prove the inequality for $0 < p < 2$. We follow the lines of [2], p. 367.

Let $0 < p < 2$. Choose $r_1, r_2 > 0$, $r_1 + r_2 = 1$, $pr_1 + 4r_2 = 2$. By Holder's inequality and the second part of (2.2), we have

$$s_{\mathbf{n}}^2 = \|S_{\mathbf{n}}\|_2^2 \leqslant \|S_{\mathbf{n}}\|_p^{pr_1} \|S_{\mathbf{n}}\|_4^{4r_2} \leqslant \|S_{\mathbf{n}}\|_p^{pr_1} B s_{\mathbf{n}}^{4r_2}.$$

From here

$$\|S_{\mathbf{n}}\|_p^{pr_1} \geqslant (1/B) s_{\mathbf{n}}^{2-4r_2} = (1/B) s_{\mathbf{n}}^{pr_1}.$$

Therefore the first part of (2.2) is true for $0 < p < 2$. □

3. Burkholder's inequality

Let $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, be a martingale difference. It means that $\mathcal{F}_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^d$, is an increasing sequence of σ-algebras, i.e. $\mathcal{F}_{\mathbf{k}} \subseteq \mathcal{F}_{\mathbf{n}}$ if $\mathbf{k} \leqslant \mathbf{n}$; $X_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$-measurable and integrable; $\mathbb{E}(X_{\mathbf{n}}|\mathcal{F}_{\mathbf{k}}) = 0$ if $\mathbf{k} < \mathbf{n}$.

To obtain Burkholder's inequality, we shall assume the so called condition (F4). I. e.

$$\mathbb{E}\left\{\mathbb{E}(\eta|\mathcal{F}_{\mathbf{m}})|\mathcal{F}_{\mathbf{n}}\right\} = \mathbb{E}\left\{\eta|\mathcal{F}_{\min\{\mathbf{m},\mathbf{n}\}}\right\} \tag{3.1}$$

for each integrable random variable η and for each $\mathbf{m}, \mathbf{n} \in \mathbb{N}^d$ (see, e.g., [6] and [3]).

Denote by $(Z_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, the martingale corresponding to the difference $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$. More precisely, let $Z_{\mathbf{n}} = 0$ and $\mathcal{F}_{\mathbf{n}} = \{\emptyset, \Omega\}$ if $\mathbf{n} \in \mathbb{N}_0^d \setminus \mathbb{N}^d$ and $Z_{\mathbf{n}} = \sum_{\mathbf{k} \leqslant \mathbf{n}} X_{\mathbf{k}}$, $\mathbf{n} \in \mathbb{N}^d$.

Theorem 3.1. *Let $(Z_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, be a martingale and $(X_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, the martingale difference corresponding to it. Assume that (3.1) is satisfied. Let $p > 1$. There exist finite and positive constants $C_{p,d}$ and $D_{p,d}$ depending only on p and d such that*

$$C_{p,d} \left[\mathbb{E}\left(\sum_{\mathbf{k} \leqslant \mathbf{n}} X_{\mathbf{k}}^2\right)^{p/2}\right]^{1/p} \leqslant (\mathbb{E}|Z_{\mathbf{n}}|^p)^{1/p} \leqslant D_{p,d} \left[\mathbb{E}\left(\sum_{\mathbf{k} \leqslant \mathbf{n}} X_{\mathbf{k}}^2\right)^{p/2}\right]^{1/p}. \tag{3.2}$$

Proof. We follow the lines of [8]. Let $u_{i,n_i} \in \{0,1\}$, $n_i = 1, 2, \ldots$, $i = 1, 2, \ldots, d$. Let

$$T_{\mathbf{n}} = \sum_{\mathbf{k} \leqslant \mathbf{n}} u_{1,k_1} \cdots u_{d,k_d} X_{\mathbf{k}} = \sum_{k_1=1}^{n_1} u_{1,k_1} Y_{k_1},$$

where

$$Y_{k_1} = Y_{k_1,n_2,\ldots,n_d} = \sum\nolimits_{k_2=1}^{n_2} \cdots \sum\nolimits_{k_d=1}^{n_d} u_{2,k_2} \cdots u_{d,k_d} X_{k_1,k_2,\ldots,k_d} .$$

First we show that

$$\mathbb{E}|Z_{\mathbf{n}}|^p \leqslant M_d \, \mathbb{E}|T_{\mathbf{n}}|^p. \tag{3.3}$$

We use induction. For $d = 1$ (3.3) is included in [1], p. 1502 (because T_n is a transform of the martingale Z_n and vice versa). Now we assume that (3.3) is true for $d - 1$. Let $n_2, \ldots, n_d$ be fixed, $\mathcal{F}_{k_1} = \mathcal{F}_{k_1,n_2,\ldots,n_d}$. Then, using (3.1), we can show that $(Y_{k_1}, \mathcal{F}_{k_1})$, $k_1 = 1, 2, \ldots,$ is a martingale difference. As the martingale $\sum_{k_1=1}^{n_1} Y_{k_1} = \sum_{k_1=1}^{n_1} u_{1,k_1}(u_{1,k_1} Y_{k_1})$ is a transform of the martingale $\sum_{k_1=1}^{n_1} (u_{1,k_1} Y_{k_1})$, by [1], p. 1502,

$$\mathbb{E}\left|\sum\nolimits_{k_1=1}^{n_1} Y_{k_1}\right|^p \leqslant M_1 \mathbb{E}\left|\sum\nolimits_{k_1=1}^{n_1} (u_{1,k_1} Y_{k_1})\right|^p . \tag{3.4}$$

Now, using (3.1), we can show that for any fixed n_1 the $(d-1)$-index sequence $\left\{\sum_{k_1=1}^{n_1} X_{k_1,k_2,\ldots,k_d}, \mathcal{F}_{n_1,k_2,\ldots,k_d}\right\}$, $(k_2, \ldots, k_d) \in \mathbb{N}^{d-1}$, is a martingale difference. Therefore, using (3.3) for $d - 1$, we obtain

$$\begin{aligned} \mathbb{E}|Z_{\mathbf{n}}|^p &= \mathbb{E}\left|\sum\nolimits_{k_2=1}^{n_2} \cdots \sum\nolimits_{k_d=1}^{n_d} \left[\sum\nolimits_{k_1=1}^{n_1} X_{k_1,\ldots,k_d}\right]\right|^p \leqslant \\ &\leqslant M_{d-1} \, \mathbb{E}\left|\sum\nolimits_{k_2=1}^{n_2} \cdots \sum\nolimits_{k_d=1}^{n_d} u_{2,k_2} \cdots u_{d,k_d} \left[\sum\nolimits_{k_1=1}^{n_1} X_{k_1,\ldots,k_d}\right]\right|^p = \\ &= M_{d-1} \, \mathbb{E}\left|\sum\nolimits_{k_1=1}^{n_1} Y_{k_1}\right|^p \leqslant M_{d-1} \, M_1 \, \mathbb{E}\left|\sum\nolimits_{k_1=1}^{n_1} (u_{1,k_1} Y_{k_1})\right|^p = \\ &= M_d \, \mathbb{E}|T_{\mathbf{n}}|^p. \end{aligned} \tag{3.5}$$

In (3.5) we applied (3.4). So we proved (3.3).

Because $Z_{\mathbf{n}}$ and $T_{\mathbf{n}}$ are each other's transforms, (3.3) implies

$$N_d \, \mathbb{E}|T_{\mathbf{n}}|^p \leqslant \mathbb{E}|Z_{\mathbf{n}}|^p \leqslant M_d \, \mathbb{E}|T_{\mathbf{n}}|^p. \tag{3.6}$$

Now we prove the first part of (3.2). By (2.1),

$$\begin{aligned} \mathbb{E}\left(\sum_{\mathbf{k}\leqslant\mathbf{n}} X_{\mathbf{k}}^2\right)^{p/2} &\leqslant \frac{1}{A_{p,d}^p} \mathbb{E}\left[\int_0^1 \cdots \int_0^1 \left|\sum_{\mathbf{k}\leqslant\mathbf{n}} \varepsilon_{k_1}(t_1) \cdots \varepsilon_{k_d}(t_d) X_{\mathbf{k}}\right|^p \, dt_1 \ldots dt_d\right] \\ &= \frac{1}{A_{p,d}^p} \int_0^1 \cdots \int_0^1 \mathbb{E}\left[\left|\sum_{\mathbf{k}\leqslant\mathbf{n}} \varepsilon_{k_1}(t_1) \cdots \varepsilon_{k_d}(t_d) X_{\mathbf{k}}\right|^p\right] \, dt_1 \ldots dt_d \\ &\leqslant \frac{1}{A_{p,d}^p} \int_0^1 \cdots \int_0^1 \frac{1}{N_d} \mathbb{E}\left[\left|\sum_{\mathbf{k}\leqslant\mathbf{n}} X_{\mathbf{k}}\right|^p\right] \, dt_1 \ldots dt_d \\ &= \frac{1}{A_{p,d}^p} \frac{1}{N_d} \mathbb{E}|Z_{\mathbf{n}}|^p. \end{aligned}$$

In the third step we applied (3.6).

We turn to the second part of (3.2). By (3.6),

$$\mathbb{E}|Z_{\mathbf{n}}|^p \leqslant M_d \, \mathbb{E}\left[\left|\sum\nolimits_{\mathbf{k}\leqslant\mathbf{n}} \varepsilon_{k_1}(t_1)\cdots\varepsilon_{k_d}(t_d)X_{\mathbf{k}}\right|^p\right].$$

From here, using (2.1),

$$\begin{aligned}\mathbb{E}|Z_{\mathbf{n}}|^p &\leqslant M_d \int_0^1 \cdots \int_0^1 \mathbb{E}\left[\left|\sum\nolimits_{\mathbf{k}\leqslant\mathbf{n}} \varepsilon_{k_1}(t_1)\cdots\varepsilon_{k_d}(t_d)X_{\mathbf{k}}\right|^p\right] \, dt_1 \ldots dt_d \\ &\leqslant M_d \, B_{p,d}^p \, \mathbb{E}\left(\sum\nolimits_{\mathbf{k}\leqslant\mathbf{n}} X_{\mathbf{k}}^2\right)^{p/2}.\end{aligned}$$

The proof is complete. □

4. Final comments

Burkholder's inequality is valid for martingales with values in $\mathbb{R}^t$ (t is a fixed positive integer). For $p > 0$ and $\mathbf{x} = (x_1, \ldots, x_t) \in \mathbb{R}^t$ let $|||\mathbf{x}|||_p = \left(\sum_{i=1}^t |x_i|^p\right)^{1/p}$.

Let $(\mathbf{X}_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, be a martingale difference with values in $\mathbb{R}^t$. Assume that condition (F4) is satisfied. Let $(\mathbf{Z}_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, be the martingale corresponding to the difference $(\mathbf{X}_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$.

Theorem 4.1. *Let $(\mathbf{Z}_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, be a martingale with values in $\mathbb{R}^t$ and $(\mathbf{X}_{\mathbf{n}}, \mathcal{F}_{\mathbf{n}})$, $\mathbf{n} \in \mathbb{N}^d$, the martingale difference corresponding to it. Assume that (3.1) is satisfied. Let $p > 1$. There exist finite and positive constants C and D depending only on t, p and d such that*

$$C\left[\mathbb{E}\left(\sum_{\mathbf{k}\leqslant\mathbf{n}} |||\mathbf{X}_{\mathbf{k}}|||_2^2\right)^{p/2}\right]^{1/p} \leqslant \left(\mathbb{E}|||\mathbf{Z}_{\mathbf{n}}|||_2^p\right)^{1/p} \leqslant D\left[\mathbb{E}\left(\sum_{\mathbf{k}\leqslant\mathbf{n}} |||\mathbf{X}_{\mathbf{k}}|||_2^2\right)^{p/2}\right]^{1/p}. \quad (4.1)$$

Proof. It is known that for any $p, q > 0$ there exist $0 < c, d < \infty$ such that $c|||\mathbf{x}|||_p \leqslant |||\mathbf{x}|||_q \leqslant d|||\mathbf{x}|||_p$ for all $\mathbf{x} \in \mathbb{R}^t$. Applying this observation and (3.2) we obtain (4.1). □

Using this theorem we can prove limit theorems for autoregressive type martingale fields. For details see [5] and [4] including the d-index case and the two-index case, respectively.

References

[1] Burkholder, D. L., Martingale transforms. *Ann. Math. Stat.* Vol. 37 (1966), 1494–1505.

[2] Chow, Y. S., Teicher, H., *Probability Theory.* Springer, New York, 1988.

[3] Fazekas, I., Convergence of vector valued martingales with multidimensional indices. *Publ. Math. Debrecen*, Vol. 30 (1983), 157–164.

[4] Fazekas, I., On the convergence of regression type martingal fields. *Proc. of the 5th Pannonian Symp.*, 43–52, Reidel, Dordrecht, 1988.

[5] Karácsony, Zs., Autoregressive type martingale fields. Accepted by *Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis.*

[6] Khosnevisan, D., *Multiparameter Processes.* Springer, New York, 2002.

[7] Ledoux, M., Inégalités de Burkholder pour martingales indexées par $N \times N$. *Two-index random processes*, Lecture Notes in Math. Vol. 863, 122–127, Springer, Berlin, 1981.

[8] Métraux, Ch., Quelques inégalités pour martingales à paramètre bidimensionnel. *Séminaire de Probabilités XII*, Université de Strasbourg, Lecture Notes in Math. Vol. 649, 170–179, Springer, Berlin, 1978.

[9] Noszály, Cs., Tómács, T., A general approach to strong laws of large numbers for fields of random variables. *Annales Univ. Sci. Budapest Eötvös Sect. Math.* Vol. 43 (2000), 61–78.

István Fazekas
Faculty of Informatics
University of Debrecen
P.O. Box 12
4010 Debrecen
Hungary

Annales Mathematicae et Informaticae
32 *(2005) pp. 53–60.*

A limit theorem for one-parameter alteration of two knots of B-spline curves*

Miklós Hoffmann[a], Imre Juhász[b]

[a]Institute of Mathematics and Computer Science
Károly Eszterházy College
e-mail: hofi@ektf.hu

[b]Department of Descriptive Geometry
University of Miskolc
e-mail: agtji@gold.uni-miskolc.hu

Abstract

Knot modification of B-spline curves is extensively studied in the past few years. Altering one knot value, curve points move on well-defined paths, the limit of which can be computed if the knot value tends to infinity. Symmetric alteration of two knot values can also be studied in a similar way. The extension of these limit theorems for general synchronized modification of two knots is discussed in this paper.

Key Words: B-spline curve, knots, paths

AMS Classification Number: 68U05

1. Introduction

B-spline curves and surfaces are well-known geometric modeling tools in computer aided design. The definition of the k^{th} order B-spline curve is as follows (c.f.[13]):

Definition 1.1. The curve $\mathbf{s}(u)$ defined by

$$\mathbf{s}(u) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u), \quad u \in [u_{k-1}, u_{n+1}]$$

*This research was supported by the Hungarian National Research Fund (OTKA) grant No. T 048523.

is called B-spline curve of order $k \leqslant n$ (degree $k-1$), where the points $\mathbf{d}_l$ are called control points or de Boor-points, while $N_l^k(u)$ is the k^{th} normalized B-spline basis function, given by the following recursive functions:

$$N_j^1(u) = \begin{cases} 1 & \text{if } u \in [u_j, u_{j+1}), \\ 0 & \text{otherwise} \end{cases}$$
$$N_j^k(u) = \frac{u-u_j}{u_{j+k-1}-u_j} N_j^{k-1}(u) + \frac{u_{j+k}-u}{u_{j+k}-u_{j+1}} N_{j+1}^{k-1}(u).$$

The numbers $u_j \leqslant u_{j+1} \in \mathbb{R}$ are called knot values or simply knots, and $0/0 \doteq 0$ by definition.

In the last few years several papers dealt with knot modification of B-spline curves. From a practical point of view, optimization techniques by changing the entire knot vector have been studied in [1] and [3], while shape control algorithms for cubic B-spline curves by changing three consecutive knots have been described in [10].

Basic theoretical results of alteration of a single knot value can be found in [8] and [9], where the notion of path has been introduced for curves $\mathbf{s}(u, u_i)$ obtained by fixing the parameter value u and modifying the knot u_i. In [8] the authors proved that these paths are rational curves. In [5] these paths are extended in a way that monotonicity of knot values was not fulfilled, i.e. we let $u_i < u_{i-1}$ and $u_i > u_{i+1}$. Here we emphasize that this extension is a pure mathematical construction that is, the functions $N_l^k(u)$ obtained by this substitution are not basis functions any more. This extension, however can help us to examine the limit properties of paths. These extended paths have been studied in [5] where the following theorem has been proved:

Theorem 1.2. *Modifying the single multiplicity knot u_i of the B-spline curve $\mathbf{s}(u)$, points of the extended paths of $\mathbf{s}(u), u \in [u_{i-1}, u_{i+1})$ tend to the control points $\mathbf{d}_i$ and $\mathbf{d}_{i-k}$ as u_i tends to $-\infty$ and ∞, respectively, i.e.,*

$$\lim_{u_i \to -\infty} \mathbf{s}(u, u_i) = \mathbf{d}_i, \lim_{u_i \to \infty} \mathbf{s}(u, u_i) = \mathbf{d}_{i-k}, \forall u \in [u_{i-1}, u_{i+1}).$$

Some of the results of knot modification have been successfully extended for B-spline surfaces as well (c.f. [4], [11]).

2. Alteration of two knots

Similarly to the previous section, one can modify two (not necessarily neighboring) knots of $\mathbf{s}(u)$ as well. Let us denote the two altered knots by u_i and u_{i+z}, $((k-1) < i < i+z < (n+1))$. If their modification is independent of each other, the possible positions of each fixed point of the curve can be described as a planar region. However if the modification of the two knots is synchronized in a way that their movement depend on a single parameter, the points of the curve will move on paths. In [6] the modification of type $u_i + \lambda$ and $u_{i+z} - \lambda$ has been discussed and the following result has been proved.

Theorem 2.1. *Symmetrically altering the knots u_i and u_{i+z} ($z \in \{1, 2, \dots, k\}$, where k is the order of the original B-spline curve), extended paths of points of the arcs $\mathbf{s}_j$, ($j = i, i+1, \dots, i+z-1$) converge to the midpoint of the segment bounded by the control points $\mathbf{d}_i$ and $\mathbf{d}_{i+z-k}$ when $\lambda \to -\infty$, i.e.*

$$\lim_{\lambda \to -\infty} \mathbf{s}(u, \lambda) = \frac{\mathbf{d}_i + \mathbf{d}_{i+z-k}}{2}, \quad u \in [u_i, u_{i+z}). \tag{2.1}$$

In this paper we extend this result for a more general movement of knots. Let the modification of the two knots be described by the following way:

$$\begin{aligned} u_i &= u_i + t\lambda \\ u_j &= u_j - (1-t)\lambda, \end{aligned}$$

where $t \in [0, 1]$ is fixed and $\lambda \in \mathbb{R}$ is a running parameter. If one intend to preserve the monotonicity of the knot values, only $\lambda \in [-c, c]$, $c = \min\{u_i - u_{i-1}, u_{i+1} - u_i, u_j - u_{j-1}, u_{j+1} - u_j\}$ is allowed, but in case of extended paths the parameter can be any real number.

3. The limit theorem

For the synchronized motion described in the previous section the following statement holds.

Theorem 3.1. *Modifying the knots*

$$u_i = u_i + t\lambda, \quad u_{i+z} = u_{i+z} - (1-t)\lambda, \quad (z = 1, 2, \dots, k) \tag{3.1}$$

the points of the extended paths of $\mathbf{s}(u)$, $u \in [u_i, u_{i+z})$ tend to a point of the line segment $\mathbf{d}_i\mathbf{d}_{i+z-k}$ the barycentric coordinates of which are t and $(1-t)$, i.e.

$$\lim_{\lambda \to -\infty} \mathbf{s}(u, t, \lambda) = t\mathbf{d}_i + (1-t)\mathbf{d}_{i+z-k}, \quad u \in [u_i, u_{i+z}), t \in [0, 1].$$

Proof. At first we prove that if $u \in [u_i, u_{i+z})$, then for $z = 1, 2, ..., k-1$

$$\begin{aligned} \lim_{\lambda \to -\infty} N^k_{i+z-k}(u, t, \lambda) &= (1-t) \\ \lim_{\lambda \to -\infty} N^k_i(u, t, \lambda) &= t \\ \lim_{\lambda \to -\infty} N^k_j(u, t, \lambda) &= 0, \quad (j \neq i, i+z-k), \end{aligned}$$

and for $z = k$

$$\begin{aligned} \lim_{\lambda \to -\infty} N^k_i(u, t, \lambda) &= 1 \\ \lim_{\lambda \to -\infty} N^k_j(u, t, \lambda) &= 0, \quad (j \neq i). \end{aligned}$$

We prove the statement by induction on k. For the sake of simplicity the variables of the basis functions are omitted.

i) $k = 3$

On the interval $[u_i, u_{i+1})$ the basis function is of the following form

$$N_i^3 = \frac{u - u_i}{u_{i+2} - u_i} \frac{u - u_i}{u_{i+1} - u_i}.$$

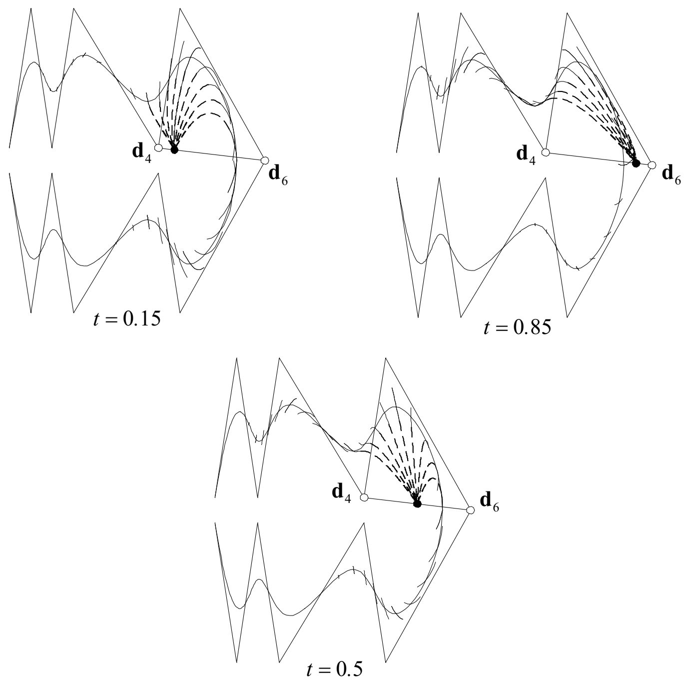

Figure 1: A cubic ($k = 4$) B-spline curve and its paths for various values of t, ($i = 6, z = 2$).

Substituting equations (3.1) into this function the numerator as well as the denominator will be quadratic in λ. The main coefficient of the numerator is t^2 independently of z. For $z = 1$ the main coefficient in the denominator can be calculated by applying

$$\begin{aligned} u_{i+2} - u_i &= u_{i+2} - (u_i + t\lambda) = u_{i+2} - u_i - t\lambda \\ u_{i+1} - u_i &= u_{i+1} - (1-t)\lambda - (u_i + t\lambda) = u_{i+1} - u_i - \lambda \end{aligned}$$

which turn to be t, while for $z = 2$ applying

$$u_{i+2} - u_i = u_{i+2} - (1-t)\lambda - (u_i + t\lambda) = u_{i+2} - u_i - \lambda$$

$$u_{i+1} - u_i = u_{i+1} - (u_i + t\lambda) = u_{i+1} - u_i - t\lambda$$

the main coefficient is t again. If $z = 3$, then due to $u_i = u_i + t\lambda$ the main coefficient is t^2. Thus we obtain, that

$$\lim_{\lambda\to-\infty} N_i^3 = \begin{cases} t, \text{ if } z = 1, 2 \\ 1, \text{ if } z = 3 \end{cases} \qquad u \in [u_i, u_{i+1}).$$

On the interval $[u_i, u_{i+1})$ the other two basis functions are of the form

$$\begin{aligned} N_{i-2}^3 &= \frac{u_{i+1} - u}{u_{i+1} - u_{i-1}} \frac{u_{i+1} - u}{u_{i+1} - u_i} \\ N_{i-1}^3 &= \frac{u - u_{i-1}}{u_{i+1} - u_{i-1}} \frac{u_{i+1} - u}{u_{i+1} - u_i} + \frac{u_{i+2} - u}{u_{i+2} - u_i} \frac{u - u_i}{u_{i+1} - u_i}. \end{aligned}$$

Similar calculation leads to the main coefficients and to the limits of these two functions, which are $(1 - t)$ and 0 for $z = 1$, 0 and $(1 - t)$ for $z = 2$ while 0 in both cases for $z = 3$. For the rest of the indices $(j \neq i - 2, i - 1, i)$ $N_j^3 \equiv 0$ always holds. Thus the proof is ready for $k = 3$ on the interval $[u_i, u_{i+1})$. For the intervals $[u_{i+1}, u_{i+2})$ and $[u_{i+2}, u_{i+3})$ the statement can be proved in an analogous way.

ii) Suppose that for $\forall u \in [u_i, u_{i+z})$

$$\lim_{\lambda\to-\infty} N_i^{k-1} = \begin{cases} t, \text{ if } z = 1, ..., k-2 \\ 1, \text{ if } z = k-1 \end{cases} \tag{3.2a}$$

$$\lim_{\lambda\to-\infty} N_{i+z-k+1}^{k-1} = \begin{cases} (1-t), \text{ if } z = 1, ..., k-2 \\ 1, \text{ if } z = k-1 \end{cases} \tag{3.2b}$$

$$\lim_{\lambda\to-\infty} N_j^{k-1} = 0, (j \neq i, i + z - k + 1), \text{ if } z = 1, ..., k-1. \tag{3.2c}$$

holds. At first we prove that the assumptions (3.2a)-(3.2c) yield

$$\lim_{\lambda\to-\infty} N_i^k = \begin{cases} t, \text{ if } z = 1, ..., k-1 \\ 1, \text{ if } z = k \end{cases} \qquad u \in [u_i, u_{i+z}). \tag{3.3}$$

By definition

$$N_i^k(u) = \frac{u - u_i}{u_{i+k-1} - u_i} N_i^{k-1}(u) + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} N_{i+1}^{k-1}(u).$$

Due to (3.2a) the limit of the first term is t if $z \leq k-2$. If $z = 1$ then $N_{i+1}^{k-1}(u) \equiv 0$, thus the limit of the second term equals 0, while for $z = 2, ..., k-2$ the limit also equals 0 due to (3.2c). For $z = k - 1$ the limit of the fraction in the first term equals t since

$$\begin{aligned} u - u_i &= u - u_i - t\lambda \\ u_{i+k-1} - u_i &= u_{i+k-1} - \lambda + t\lambda - u_i - t\lambda = u_{i+k-1} - u_i - \lambda. \end{aligned}$$

But (3.2a) yields $\lim\limits_{\lambda\to-\infty} N_i^{k-1}=1$, thus the limit of the first term is t again. Taking into account equation (3.2c) the limit of the second term is 0.

Finally, for $z=k$ the proof is analogous to that of Theorem 1.2, thus we proved (3.3).

Now applying (3.2a)-(3.2c) we verify

$$\lim_{\lambda\to-\infty} N_{i+z-k}^k = \begin{cases} (1-t), \text{ if } z=1,...,k-1 \\ 1, \text{ if } z=k \end{cases} \qquad u\in[u_i,u_{i+z}). \tag{3.4}$$

By definition

$$N_{i+z-k}^k(u)=\frac{u-u_{i+z-k}}{u_{i+z-1}-u_{i+z-k}}N_{i+z-k}^{k-1}(u)+\frac{u_{i+z}-u}{u_{i+z}-u_{i+z-k+1}}N_{i+z-k+1}^{k-1}(u).$$

Due to (3.2c) the limit of the first term equals 0 for $z\le k-1$. The limit of the fraction in the second term is 1 for $z=1,...,k-2$, while (3.2b) yields $\lim\limits_{\lambda\to-\infty} N_{i+z-k+1}^{k-1}(u)=(1-t)$. For $z=k-1$ applying

$$\begin{aligned} u_{i+k-1}-u &= u_{i+z}-(1-t)\lambda-u \\ u_{i+k-1}-u_i &= u_{i+k-1}-\lambda+t\lambda-u_i-t\lambda=u_{i+k-1}-u_i-\lambda \end{aligned}$$

the limit of the fraction in the second term equals $(1-t)$, while due to (3.2b) $\lim\limits_{\lambda\to-\infty} N_{i+z-k+1}^{k-1}(u)=1$. Thus the limit of the second term is always equal to $(1-t)$. The case $z=k$ is analogous to the proof of Theorem 1.2 again, thus (3.4) is verified.

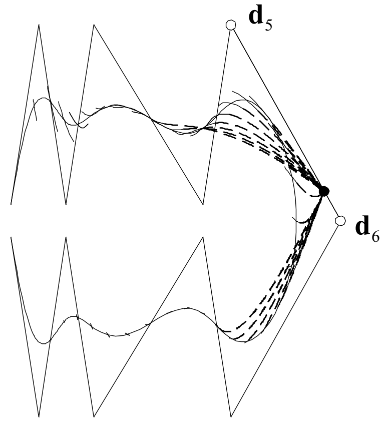

Figure 2: A quadric ($k=5$) B-spline curve and its paths for $t=0.85$, ($i=6, z=4$).

Finally, we prove that assuming (3.2a)-(3.2c)

$$\lim_{\lambda\to-\infty} N_j^k=0,(j\neq i,i+z-k),\forall z,\ u\in[u_i,u_{i+z}) \tag{3.5}$$

holds. By definition

$$N_j^k(u) = \frac{u - u_j}{u_{j+k-1} - u_j} N_j^{k-1}(u) + \frac{u_{j+k} - u}{u_{j+k} - u_{j+1}} N_{j+1}^{k-1}(u) .$$

The limit of the first term equals 0 (if $j = i + z - k + 1$ then $j + k - 1 = i + z$, thus the limit of the fraction is 0, while for the other cases the limit of the basis function in the first term is 0 due to (3.2c)). The limit of the second term equals 0 as well, (for $j + 1 = i$ the limit of the fraction equals 0, while for the rest of the cases (3.2c) yields $\lim\limits_{\lambda \to -\infty} N_{j+1}^{k-1}(u) = 0$). Thus (3.5) has also been verified and this completes the proof. $\square$

Figure 1 demonstrates the result for cubic curves, while Figure 2 shows an example for a higher order curve.

References

[1] Aszódi, B., Czuczor, Sz., Szirmay-Kalos, L., NURBS fairing by knot vector optimization, *Journal of WSCG*, Vol. 12 (2004), 19–26.

[2] Boehm, W., Inserting new knots into B–spline curves, *Computer–Aided Design*, Vol. 12 (1980), 199–201.

[3] Goldenthal, R., Bercovier, M., Spline curve approximation and design by optimal control over the knots, *Computing*, Vol. 72 (2004), 53–64.

[4] Hoffmann, M., Juhász, I., Geometric aspects of knot modification of B-spline surfaces, *Journal for Geometry and Graphics*, Vol. 6 (2002), 141–149.

[5] Hoffmann, M., Juhász, I., On the knot modification of a B-spline curve, *Publicationes Mathematicae Debrecen*, Vol. 65 (2004), 193–203.

[6] Hoffmann, M., Juhász, I., Symmetric alteration of two knots of B-spline curves, *Journal for Geometry and Graphics*, Vol. 9 (2005), 43–49.

[7] Juhász, I., A shape modification of B-spline curves by symmetric translation of two knots, *Acta Acad. Paed. Agriensis, Sect. Math.*, Vol. 28 (2001), 69–77.

[8] Juhász, I., Hoffmann, M., The effect of knot modifications on the shape of B-spline curves, *Journal for Geometry and Graphics*, Vol. 5 (2001), 111–119.

[9] Juhász, I., Hoffmann, M., Modifying a knot of B-spline curves, *Computer Aided Geometric Design*, Vol. 20 (2003), 243–245.

[10] Juhász, I., Hoffmann, M., Constrained shape modification of cubic B-spline curves by means of knots, *Computer-Aided Design*, Vol. 36 (2004), 437–445.

[11] Li Yajuan, Wang Guozhao, On Knot Modification of B-Spline and NURBS Surface, *Journal of Computer-Aided Design & Computer Graphics*, Vol. 17 (2005), 986–989.

[12] Piegl, L., Modifying the shape of rational B-splines. Part 1: curves, *Computer–Aided Design*, Vol. 21 (1989), 509–518.

[13] Piegl, L., Tiller, W., *The NURBS book*, Springer–Verlag, (1995).

Miklós Hoffmann
Institute of Mathematics and Computer Science
Károly Eszterházy College
Leányka str. 4.
H-3300 Eger
Hungary

Imre Juhász
Department of Descriptive Geometry
University of Miskolc
H-3515 Miskolc-Egyetemváros
Hungary

Annales Mathematicae et Informaticae
32 *(2005) pp. 61–78.*

Solution of a sum form equation in the two dimensional closed domain case*

Imre Kocsis

Faculty of Engineering University of Debrecen
e-mail:kocsisi@mfk.unideb.hu

Abstract

In this note we give the solution of the sum form functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i \bullet q_j) = \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j)$$

arising in information theory (in characterization of so-called entropy of degree α), where $f : [0,1]^2 \to \mathbb{R}$ is an unknown function and the equation holds for all two dimensional complete probability distributions.

Key Words: Sum form equation, additive function, multiplicative function.

AMS Classification Number: 39B22

1. Introduction

In the following we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Throughout the paper we shall use the following notations: $\underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^k$, $\underline{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^k$. For all $3 \leqslant n \in \mathbb{N}$ and for all $k \in \mathbb{N}$ we define the sets $\Gamma_n^c[k]$ and $\Gamma_n^0[k]$ by

$$\Gamma_n^c[k] = \left\{ (p_1, \ldots, p_n) : p_i \in [0,1]^k, i = 1, \ldots, n, \sum_{i=1}^{n} p_i = \underline{1} \right\}$$

*This research has been supported by the Hungarian National Foundation for Scientific Research (OTKA), grant No. T-030082.

and

$$\Gamma_n^0[k] = \left\{(p_1,\dots,p_n) : p_i \in]0,1[^k, i=1,\dots,n, \sum_{i=1}^n p_i = \underline{1}\right\},$$

respectively.

If $x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} \in \mathbb{R}^k$ then $x \bullet y = \begin{pmatrix} x_1 y_1 \\ \vdots \\ x_k y_k \end{pmatrix} \in \mathbb{R}^k$.

If we do not say else we denote the components of an element P of $\Gamma_n^c[2]$ or $\Gamma_n^0[2]$ by

$$P = (p_1,\dots,p_n) = \begin{pmatrix} p_{11} & \dots & p_{n1} \\ p_{12} & \dots & p_{n2} \end{pmatrix}.$$

A function $A : \mathbb{R}^k \to \mathbb{R}$ is additive if $A(x+y) = A(x) + A(y)$, $x, y \in \mathbb{R}^k$, a function $M :]0,1[^k \to \mathbb{R}$ is multiplicative if $M(x \bullet y) = M(x)M(y)$, $x, y \in]0,1[^k$, a function $M : [0,1]^k \to \mathbb{R}$ is multiplicative if $M(\underline{0}) = 0$, $M(\underline{1}) = 1$, and $M(x \bullet y) = M(x)M(y)$, $x, y \in [0,1]^k$.

The functional equation

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i \bullet q_j) = \sum_{i=1}^n f(p_i) \sum_{j=1}^m f(q_j) \tag{E[k]}$$

will be denoted by $(E^c[k])$ if (E[k]) holds for all $(p_1,\dots,p_n) \in \Gamma_n^c[k]$ and $(q_1,\dots,q_m) \in \Gamma_m^c[k]$, and the function f is defined on $[0,1]^k$ (closed domain case), and by $(E^0[k])$ if (E[k]) holds for all $(p_1,\dots,p_n) \in \Gamma_n^0[k]$ and $(q_1,\dots,q_m) \in \Gamma_m^0[k]$, and f is defined on $]0,1[^k$ (open domain case). The solution of equation $(E^c[1])$ is given by Losonczi and Maksa in [3], while equation $(E^0[k])$ $(k \in \mathbb{N})$ is solved by Ebanks, Sahoo, and Sander in [2].

Theorem 1.1 (Losonczi, Maksa [3]). *Let $n \geqslant 3$ and $m \geqslant 3$ be fixed integers. A function $f : [0,1] \to \mathbb{R}$ satisfies $(E^c[1])$ if, and only if, there exist additive functions $A : \mathbb{R} \to \mathbb{R}$ and $D : \mathbb{R} \to \mathbb{R}$, a multiplicative function $M : [0,1] \to \mathbb{R}$, and $b \in \mathbb{R}$ such that $D(1) = 0$, $A(1) + nmb = (A(1) + nb)(A(1) + mb)$ and*

$$f(p) = A(p) + b, \quad p \in [0,1]$$

or

$$f(p) = D(p) + M(p), \quad p \in [0,1].$$

Theorem 1.2 (Ebanks, Sahoo, Sander [2]). *Let $k \geqslant 1$, $n \geqslant 3$, and $m \geqslant 3$ be fixed integers. A function $f :]0,1[^k \to \mathbb{R}$ satisfies $(E^0[k])$ if, and only if, there exist additive functions $A : \mathbb{R}^k \to \mathbb{R}$ and $D : \mathbb{R}^k \to \mathbb{R}$, a multiplicative function $M :]0,1[^k \to \mathbb{R}$ and $b \in \mathbb{R}$ such that $D(\underline{1}) = 0$, $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$ and*

$$f(p) = A(p) + b, \quad p \in]0,1[^k$$

or

$$f(p) = D(p) + M(p), \quad p \in]0,1[^k.$$

The solution of equation ($E^c[k]$) is not known if $k \in \mathbb{N}$, $k \geq 2$. Our purpose is to solve equation ($E^c[2]$).

2. Preliminary results

Lemma 2.1. *Let $k \geqslant 1$, $n \geqslant 3$, and $m \geqslant 3$ be fixed integers. If the function $f : [0,1]^k \to \mathbb{R}$ satisfies ($E^c[k]$) and $A : \mathbb{R}^k \to \mathbb{R}$ is an additive function such that $A(\underline{1}) = 0$ then the function $g = f - A$ satisfies ($E^c[k]$), too.*

Proof.

$$\sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i \bullet q_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i \bullet q_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} A(p_i \bullet q_j) =$$

$$\Big(\sum_{i=1}^{n} f(p_i) - \sum_{i=1}^{n} A(p_i)\Big)\Big(\sum_{i=1}^{n} f(q_j) - \sum_{i=1}^{n} A(q_j)\Big) = \sum_{i=1}^{n} g(p_i)\sum_{j=1}^{m} g(q_j).$$

□

Lemma 2.2. *If $A : \mathbb{R}^2 \to \mathbb{R}$ is additive, $M :]0,1[^2 \to \mathbb{R}$ is multiplicative, $H :]0,1[\to \mathbb{R}$, and $M\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + H(x)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$ then*

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \mu(x), \qquad \begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2,$$

where $\mu :]0,1[\to \mathbb{R}$ is a multiplicative function or

$$M\begin{pmatrix} x \\ y \end{pmatrix} = y, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2.$$

Proof. Let $x, y, z \in]0,1[$. Then $A\begin{pmatrix} x \\ yz \end{pmatrix} + H(x) = M\begin{pmatrix} x \\ yz \end{pmatrix} =$ $M\begin{pmatrix} \sqrt{x} \\ y \end{pmatrix} M\begin{pmatrix} \sqrt{x} \\ z \end{pmatrix} = \left(A\begin{pmatrix} \sqrt{x} \\ y \end{pmatrix} + H(\sqrt{x})\right)\left(A\begin{pmatrix} \sqrt{x} \\ z \end{pmatrix} + H(\sqrt{x})\right)$. With fixed x and the notations $a_1(t) = A\begin{pmatrix} x \\ t \end{pmatrix}$, $t \in]0,1[$, $a_2(t) = A\begin{pmatrix} \sqrt{x} \\ t \end{pmatrix}$, $t \in]0,1[$ this implies that $a_1(yz) + H(x) = (a_2(y) + H(\sqrt{x}))(a_2(z) + H(\sqrt{x}))$, while with the substitutions $y = z = \sqrt{t}$, $a_1(t) + H(x) = (a_2(t) + H(\sqrt{x}))^2$, that is, $A\begin{pmatrix} 0 \\ t \end{pmatrix} = (a_2(t) + H(\sqrt{x}))^2 - A\begin{pmatrix} x \\ 0 \end{pmatrix} - H(x)$, $t \in]0,1[$. Since the function $t \to A\begin{pmatrix} 0 \\ t \end{pmatrix}$ is additive and $A\begin{pmatrix} 0 \\ t \end{pmatrix} \geqslant -A\begin{pmatrix} x \\ 0 \end{pmatrix} - H(x), t \in]0,1[$, there exists $c \in \mathbb{R}$ such that $A\begin{pmatrix} 0 \\ t \end{pmatrix} = ct$ (see Aczél [1]), thus $A\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ 0 \end{pmatrix} +$

cy, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$, furthermore $M\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ 0 \end{pmatrix} + H(x) + cy$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$. Let $\mu(x) = A\begin{pmatrix} x \\ 0 \end{pmatrix} + H(x)$, $x \in]0,1[$ and let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in]0,1[^2$. Then $cy_1y_2 + \mu(x_1x_2) = M\begin{pmatrix} x_1x_2 \\ y_1y_2 \end{pmatrix} = M\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} M\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (cy_1 + \mu(x_1))(cy_2 + \mu(x_2))$. Thus $(c - c^2)y_1y_2 = \mu(x_1)\mu(x_2) - \mu(x_1x_2) + c(y_1\mu(x_2) + y_2\mu(x_1))$. Taking here the limit $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we have that μ is multiplicative and

$$c(1-c)y_1y_2 = c(y_1\mu(x_2) + y_2\mu(x_1)).$$

This implies that either $c = 0$ and

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \mu(x), \quad \begin{pmatrix} x \\ y \end{pmatrix} x \in]0,1[^2$$

or $(1-c)y_1y_2 = y_1\mu(x_2) + y_2\mu(x_1)$, $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in]0,1[^2$. Since μ is multiplicative, in this case we get that $c = 1$ and $A\begin{pmatrix} x \\ 0 \end{pmatrix} + H(x) = \mu(x) = 0$, $x \in]0,1[$. Thus

$$M\begin{pmatrix} x \\ y \end{pmatrix} = y, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2.$$

□

Lemma 2.3. *Suppose that* $3 \leqslant n \in \mathbb{N}$, $3 \leqslant m \in \mathbb{N}$, $f : [0,1]^2 \to \mathbb{R}$ *satisfies equation* $(E^c[2])$ *and*

$$K = (m-1)f(\underline{0}) + f(\underline{1}) = 1. \tag{2.1}$$

Then $f(\underline{0}) = 0$ *and* $f(\underline{1}) = 1$.

Proof. Substituting $P = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma^c_m[2]$,$Q = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma^c_m[2]$ in $(E^c[2])$, by (2.1), we have $(nm-1)f(\underline{0}) + f(\underline{1}) = (n-1)f(\underline{0}) + f(\underline{1})$ and, after some calculation, we get that $n(m-1)f(\underline{0}) = 0$. This and (2.1) imply that $f(\underline{0}) = 0$ and $f(\underline{1}) = 1$. □

3. The main result

Theorem 3.1. *Let* $n \geqslant 3$ *and* $m \geqslant 3$ *be fixed integers. A function* $f : [0,1]^2 \to \mathbb{R}$ *satisfies* $(E^c[2])$ *if, and only if, there exist additive functions* $A, D : \mathbb{R}^2 \to \mathbb{R}$, *a multiplicative function* $M : [0,1]^2 \to \mathbb{R}$, *and* $b \in \mathbb{R}$ *such that* $D(\underline{1}) = 0$, $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$ *and*

$$f(p) = A(p) + b, \quad p \in [0,1]^2$$

or

$$f(p) = D(p) + M(p), \quad p \in [0,1]^2.$$

Proof. By Theorem 1.2, with $k = 2$ we have that there exist additive functions $A, D : \mathbb{R}^2 \to \mathbb{R}$, a multiplicative function $M :]0,1[^2 \to \mathbb{R}$ and $b \in \mathbb{R}$ such that $D(\underline{1}) = 0$, $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$ and

$$f(p) = A(p) + b, \quad p \in]0,1[^2$$

or

$$f(p) = D(p) + M(p), \quad p \in]0,1[^2.$$

We prove that, beside the conditions of Theorem 3.1, f has similar form with the same $b \in \mathbb{R}$ and with the additive and multiplicative extensions of the functions A,D, and M onto the whole square $[0,1]^2$, respectively. To have this result we will apply special substitutions in equation ($E^c[2]$) to get information about the behavior of f on the boundary of $[0,1]^2$.

Case 1. $f(p) = A(p) + b, \quad p \in]0,1[^2$ and $A(\underline{1}) \neq 0$.

Subcase 1.A. $K \neq 1$ (see (2.1))

Substituting $P = \begin{pmatrix} x & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, and $Q = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma_m^c[2]$ in ($E^c[2]$) we get that

$$n(m-1)f(\underline{0}) + f\begin{pmatrix} x \\ 0 \end{pmatrix} + A\begin{pmatrix} 1-x \\ 1 \end{pmatrix} + (n-1)b = \left(f\begin{pmatrix} x \\ 0 \end{pmatrix} + A\begin{pmatrix} 1-x \\ 1 \end{pmatrix} + (n-1)b\right)K.$$

Hence

$$f\begin{pmatrix} x \\ 0 \end{pmatrix} = A\begin{pmatrix} x \\ 0 \end{pmatrix} - A(\underline{1}) - (n-1)b + \frac{n(m-1)f(\underline{0})}{K-1} = A\begin{pmatrix} x \\ 0 \end{pmatrix} + b_{10}, \quad (3.1)$$

$x \in]0,1[$ for some $b_{10} \in \mathbb{R}$. A similar calculation shows that there exists $b_{20} \in \mathbb{R}$ such that

$$f\begin{pmatrix} 0 \\ y \end{pmatrix} = A\begin{pmatrix} 0 \\ y \end{pmatrix} + b_{20}, \quad y \in]0,1[. \quad (3.2)$$

Substituting $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, and $Q = (\underline{0}, \dots, \underline{0}, \underline{1}) \in \Gamma_m^c[2]$ in ($E^c[2]$) we get that

$$n(m-1)f(\underline{0}) + f\begin{pmatrix} x \\ 1 \end{pmatrix} + A\begin{pmatrix} 1-x \\ 0 \end{pmatrix} + (n-1)b_{10} = \left(f\begin{pmatrix} x \\ 1 \end{pmatrix} + A\begin{pmatrix} 1-x \\ 0 \end{pmatrix} + (n-1)b_{10}\right)K.$$

Thus

$$f\begin{pmatrix} x \\ 1 \end{pmatrix} = A\begin{pmatrix} x \\ 1 \end{pmatrix} - A(\underline{1}) - (n-1)b_{10} + \frac{n(m-1)f(\underline{0})}{K-1} = A\begin{pmatrix} x \\ 1 \end{pmatrix} + b_{11}, \quad (3.3)$$

$x \in]0,1[$ for some $b_{11} \in \mathbb{R}$. A similar calculation shows that there exists $b_{21} \in \mathbb{R}$ such that

$$f\begin{pmatrix} 1 \\ y \end{pmatrix} = A\begin{pmatrix} 1 \\ y \end{pmatrix} + b_{21}, \quad y \in]0,1[. \tag{3.4}$$

Now we show that $b = b_{10} = b_{11} = b_{20} = b_{21}$. Define the function $g : [0,1]^2 \to \mathbb{R}$ by $g\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix} - \left(A\begin{pmatrix} x \\ y \end{pmatrix} - A(\underline{1})x\right)$. Then, by (3.1),(3.2),(3.3), and (3.4), $g\begin{pmatrix} x \\ y \end{pmatrix} = A(\underline{1})x + \delta$, $\begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^2 \setminus \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$, where $\delta \in \{b, b_{10}, b_{11}, b_{20}, b_{21}\}$, respectively. It follows from Lemma 2.1 that g satisfies equation $(E^c[2])$:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i \bullet q_j) = \sum_{i=1}^{n} g(p_i)\sum_{j=1}^{m} g(q_j) \tag{3.5}$$

Thus, with the substitutions, $P = \begin{pmatrix} x_1 & \dots & x_n \\ r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$,

$Q = \begin{pmatrix} y_1 & \dots & y_m \\ s & \dots & s \end{pmatrix} \in \Gamma_m^c[2]$ in (3.5) we get that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} g\begin{pmatrix} x_i y_j \\ rs \end{pmatrix} = \sum_{i=1}^{n} g\begin{pmatrix} x_i \\ r \end{pmatrix}\sum_{j=1}^{m} g\begin{pmatrix} y_j \\ s \end{pmatrix},$$

$(x_1,\dots,x_n) \in \Gamma_n^c[1], (y_1,\dots,y_n) \in \Gamma_m^c[1]$. Let $\zeta \in]0,1[$ be fixed and $G_\zeta(x) = g(x,\zeta)$, $x \in [0,1]$. Since g does not depend on its second variable if it is from $]0,1[$, G_ζ satisfies equation $(E^c[1])$. Concerning $G_\zeta(x) = A(\underline{1})x + b$, $x \in]0,1[$ and $A(\underline{1}) \neq 0$, by Theorem 1.1, we have that $G_\zeta(x) = A(\underline{1})x + b$, $x \in [0,1]$, that is, $b = b_{20} = b_{21}$. In a similar way we can get that $b = b_{10} = b_{11}$, that is,

$$g\begin{pmatrix} x \\ y \end{pmatrix} = A(\underline{1})x + b, \ \begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^2 \setminus \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}. \tag{3.6}$$

Now we prove that (3.6) holds on $[0,1]^2$. Let $G_0(x) = g\begin{pmatrix} x \\ 0 \end{pmatrix}$, $x \in [0,1]$. $G_0(x) = A(\underline{1})x + b$, $x \in]0,1[$. Thus G_0 satisfies $(E^0[2])$. We show that G_0 satisfies $(E^c[2])$, too. Let $(p_1,\dots,p_n) = \begin{pmatrix} x_1 & \dots & x_{n-1} & x_n \\ 0 & \dots & 0 & 1 \end{pmatrix} \in \Gamma_n^c[2]$,

$(q_1,\dots,q_m) = \begin{pmatrix} y_1 & \dots & y_{m-1} & y_m \\ 0 & \dots & 0 & 1 \end{pmatrix} \in \Gamma_m^c[2]$, $x_1,\dots x_n, y_1 \dots y_m \in [0,1[$. Since $g\begin{pmatrix} t \\ 0 \end{pmatrix} = g\begin{pmatrix} t \\ 1 \end{pmatrix}$, $t \in]0,1[$ we have that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} G_0(x_i y_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i \bullet q_j) =$$

$$\sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} g(q_j) = \sum_{i=1}^{n} G_0(x_i) \sum_{j=1}^{m} G_0(q_j). \tag{3.7}$$

Substituting $x_1 = \cdots = x_{n-2} = 0, x_{n-1} = x_n = \frac{1}{2}$, $y_1 = \cdots = y_m = \frac{1}{m}$ in (3.7) and using the equalities $G_0(x) = A(\underline{1})x + b$, $x \in]0,1[$ and $A(\underline{1}) + nmb = (A(\underline{1}) + nb)(A(\underline{1}) + mb)$ we get that

$$(G_0(0) - b)(nm - 2m - nA(\underline{1}) - nmb + 2A(\underline{1}) + 2mb) = 0.$$

An easy calculation shows that the condition $A(\underline{1}) \neq 0$ implies that $(nm - 2m - nA(\underline{1}) - nmb + 2A(\underline{1}) + 2mb) \neq 0$, that is $g(\underline{0}) = G_0(0) = b$.
The substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & s & \dots & s \end{pmatrix} \in \Gamma_m^c[2]$ and $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} y_1 & \dots & y_m \\ v & \dots & v \end{pmatrix} \in \Gamma_m^c[2]$ in (3.5), using $G_0(0) = b$, imply that the function G_0 satisfies equation $(E^c[1])$ also in the remaining cases $x_1 = 1, x_2 = \cdots = x_n = 0, y_1 = 1, y_2 = \cdots = y_n = 0$ and $x_1 = 1, x_2 = \cdots = x_n = 0, (y_1, \dots, y_m) \in \Gamma_m^c[1]$. Thus, by Theorem 1.1, $G_0(x) = A(\underline{1})x + b$, $x \in [0,1]$, that is, $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = G_0(1) = A(\underline{1}) + b$. In a similar way we can get that $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = A(\underline{1}) + b$. Finally the following calculation proves that $g(\underline{1}) = A(\underline{1}) + b$. Substituting $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 1 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = (\underline{1}, \underline{0}, \dots, \underline{0}) \in \Gamma_m^c[2]$ in (3.5) we have that $(A(\underline{1}) + nb)(g(\underline{1}) - A(\underline{1}) - b) = 0$. It is easy to see that the condition $A(\underline{1}) \neq 0$ implies that $A(\underline{1}) + nb \neq 0$ thus $g(\underline{1}) = A(\underline{1}) + b$.

SUBCASE 1.B. $K = 1$ (see (2.1))

In this case, by Lemma 2.3, $f(\underline{0}) = 0$ and $f(\underline{1}) = 1$. Substituting
$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,
$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,
$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in $(E^c[2])$ we get the following system of equations.

$$\begin{aligned} I. \quad & A(\underline{1}) + 4b = (A(\underline{1}) + 2b)^2 \\ II. \quad & A(\underline{1}) + 6b = (A(\underline{1}) + 2b)(A(\underline{1}) + 3b) \\ III. \quad & A(\underline{1}) + 9b = (A(\underline{1}) + 3b)^2. \end{aligned}$$

This and the condition $A(\underline{1}) \neq 0$ imply that $b = 0$, furthermore $A(\underline{1}) = 1$, that is, $f(\underline{0}) = 0$ and $f(\underline{1}) = 1$. Substituting $P = \begin{pmatrix} 1 & 0 & 0 \dots & 0 \\ 0 & 1 & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0 & 0 \dots & 0 \\ 0 & 1 & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in $(E^c[2])$ we get that $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + f\begin{pmatrix} 0 \\ 1 \end{pmatrix} =$

$\left(f\begin{pmatrix}1\\0\end{pmatrix}+f\begin{pmatrix}0\\1\end{pmatrix}\right)^2$ thus $f\begin{pmatrix}1\\0\end{pmatrix}+f\begin{pmatrix}0\\1\end{pmatrix}\in\{0,1\}$, while with the substitutions $P=\begin{pmatrix}1&0&0\dots&0\\0&1&0\dots&0\end{pmatrix}\in\Gamma_n^0[2]$, $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in $(E^c[2])$ we get that

$$\sum_{j=1}^m f\begin{pmatrix}q_{j1}\\0\end{pmatrix}+\sum_{j=1}^m f\begin{pmatrix}0\\q_{j2}\end{pmatrix}=\left(f\begin{pmatrix}1\\0\end{pmatrix}+f\begin{pmatrix}0\\1\end{pmatrix}\right)\sum_{j=1}^m f(q_j). \tag{3.8}$$

If $f\begin{pmatrix}1\\0\end{pmatrix}+f\begin{pmatrix}0\\1\end{pmatrix}=0$ then, with fixed $Q=(q_{12},\dots,q_{m2})$, (3.8) goes over into $\sum_{j=1}^m f\begin{pmatrix}q_{j1}\\0\end{pmatrix}=c$, $(q_{11},\dots,q_{m1})\in\Gamma_m^0[1]$ with some $c\in\mathbb{R}$, so, by Theorem 1.2, there exist additive function $a_{10}:\mathbb{R}\to\mathbb{R}$ and $b_{10}\in\mathbb{R}$ such that

$$f\begin{pmatrix}x\\0\end{pmatrix}=a_{10}(x)+b_{10},\quad x\in]0,1[. \tag{3.9}$$

In a similar way we can prove that there exist an additive function $a_{20}:\mathbb{R}\to\mathbb{R}$ and $b_{20}\in\mathbb{R}$ such that

$$f\begin{pmatrix}0\\y\end{pmatrix}=a_{20}(y)+b_{20},\quad y\in]0,1[. \tag{3.10}$$

If $f\begin{pmatrix}1\\0\end{pmatrix}+f\begin{pmatrix}0\\1\end{pmatrix}=1$ then (3.8) goes over into $\sum_{j=1}^m\left[f\begin{pmatrix}q_{j1}\\q_{j2}\end{pmatrix}-f\begin{pmatrix}q_{j1}\\0\end{pmatrix}-f\begin{pmatrix}0\\q_{j2}\end{pmatrix}\right]=0$, $(q_1,\dots,q_m)\in\Gamma_m^0[2]$. Thus there exist an additive function $A_0:\mathbb{R}^2\to\mathbb{R}$ and $b_0\in\mathbb{R}$ such that

$$f\begin{pmatrix}x\\y\end{pmatrix}-f\begin{pmatrix}x\\0\end{pmatrix}-f\begin{pmatrix}0\\y\end{pmatrix}=A_0\begin{pmatrix}x\\y\end{pmatrix}+b_0,\quad\begin{pmatrix}x\\y\end{pmatrix}\in]0,1[^2. \tag{3.11}$$

With the functions $a_{10}(x)=(A-A_0)\begin{pmatrix}x\\0\end{pmatrix}$, $x\in]0,1[$ and $a_{20}(y)=(A-A_0)\begin{pmatrix}0\\y\end{pmatrix}$, $y\in]0,1[$ we have that

$$f\begin{pmatrix}x\\0\end{pmatrix}=a_{10}(x)+\left(a_{20}(y)-f\begin{pmatrix}0\\y\end{pmatrix}+b_0\right),\ x\in]0,1[$$

and

$$f\begin{pmatrix}0\\y\end{pmatrix}=a_{20}(y)+\left(a_{10}(x)-f\begin{pmatrix}x\\0\end{pmatrix}+b_0\right),\ y\in]0,1[.$$

With fixed x and y, we obtain again that (3.9) and (3.10) hold with some $b_{10}\in\mathbb{R}$ and $b_{20}\in\mathbb{R}$, respectively.

Substituting $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in $(E^c[2])$, after some calculation, we get that

$$f\begin{pmatrix} x \\ 1 \end{pmatrix} = A\begin{pmatrix} x \\ 1 \end{pmatrix}, \quad x \in]0,1[. \tag{3.12}$$

In a similar way we have that

$$f\begin{pmatrix} 1 \\ y \end{pmatrix} = A\begin{pmatrix} 1 \\ y \end{pmatrix}, \quad y \in]0,1[. \tag{3.13}$$

Substituting $P = \begin{pmatrix} x & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_m^0[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} s & \dots & s \\ s & \dots & s \end{pmatrix}$ in $(E^c[2])$, after some calculation, we have that $b_{10} = 0$ and, in a similar way, we get that $b_{20} = 0$. Substituting $P = \begin{pmatrix} x & 1-x & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$, $x \in]0,1[$ $Q = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ y & 1-y & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$, $y \in]0,1[$ in $(E^c[2])$, after some calculation, we have that

$$\left(a_{10}(x) - A\begin{pmatrix} x \\ 0 \end{pmatrix}\right) + \left(a_{20}(y) - A\begin{pmatrix} 0 \\ y \end{pmatrix} - 1\right) = a_{20}(y) - A\begin{pmatrix} 0 \\ y \end{pmatrix}.$$

This implies that either

$$a_{10}(x) = A\begin{pmatrix} x \\ 0 \end{pmatrix}, \quad x \in]0,1[\tag{3.14}$$

and

$$a_{20}(y) = A\begin{pmatrix} 0 \\ y \end{pmatrix}, \quad y \in]0,1[, \tag{3.15}$$

or none of these equations holds. It is easy to see that the later case is not possible. Thus (3.14) and (3.15) hold. Finally with the substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & r & \dots & r \end{pmatrix} \in \Gamma_m^c[2]$, $Q = \begin{pmatrix} s & \dots & s \\ s & \dots & s \end{pmatrix}$ in $(E^c[2])$, after some calculation, we have that $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In a similar way we get that $f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = A\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Case 2.

$$f(x) = A(x) + b, \quad x \in]0,1[^2, \quad A(\underline{1}) = 0 \tag{3.16}$$

or

$$f(x) = D(x) + M(x), \quad x \in]0,1[^2, \quad D(\underline{1}) = 0. \tag{3.17}$$

Define the function g by $f - A$ if (3.16) holds and by $f - D$ if (3.17) holds. It is easy to see that we have to investigate the following three subcases.

Subcase 2.A. $g(x) = 0$, $x \in]0,1[^2$, when

$$f(x) = A(x) + b, \quad b = 0, \quad x \in]0,1[^2$$

or

$$f(x) = D(x) + M(x), \quad M(x) = 0, \quad x \in]0,1[^2,$$

SUBCASE 2.B. $g(x) = 1$, $x \in]0,1[^2$, when

$$f(x) = A(x) + b, \quad b = 1, \quad x \in]0,1[^2$$

or

$$f(x) = D(x) + M(x) \quad M(x) = 1, \quad x \in]0,1[^2,$$

SUBCASE 2.C. $g(x) = 0$, $x \in]0,1[^2$, $\quad M \neq 0, M \neq 1$, when

$$f(x) = D(x) + M(x), \quad x \in]0,1[^2, \quad M \neq 0, M \neq 1.$$

By Lemma 2.1, the function g satisfies $(E^c[2])$:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} g(p_i \bullet q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} g(q_j) \tag{3.18}$$

SUBCASE 2.A. $g(x) = 0$, $x \in]0,1[^2$. With the substitutions
$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,
$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$
in (3.18), after some calculation, we have that $g(\underline{0}) = 0$. With the substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) we get that

$$g\begin{pmatrix} x \\ 0 \end{pmatrix} = 0, \quad x \in \left]0, \frac{1}{2}\right[, \tag{3.19}$$

while with the substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in (3.18) we have that

$$\sum_{j=1}^{n} g\begin{pmatrix} xq_{j1} \\ 0 \end{pmatrix} = 0, \quad (q_{11}, \dots, q_{m1}) \in \Gamma_m^0[1].$$

Hence there exists additive function $a_x : \mathbb{R} \to \mathbb{R}$ such that

$$g\begin{pmatrix} q \\ 0 \end{pmatrix} = a_x\left(\frac{x}{q}\right) - \frac{a_x(1)}{n}, \quad q \in]0,x[, \tag{3.20}$$

where x is an arbitrary fixed element of $]0,1[$. It follows from (3.19) and (3.20) that

$$g\begin{pmatrix} x \\ 0 \end{pmatrix} = 0, \quad x \in]0,1[. \tag{3.21}$$

In a similar way we get that

$$g\begin{pmatrix} 0 \\ y \end{pmatrix} = 0, \quad y \in]0,1[. \tag{3.22}$$

It is easy to see that

$$\left(g\begin{pmatrix} 1 \\ 0 \end{pmatrix}, g\begin{pmatrix} 0 \\ 1 \end{pmatrix}, g\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \in \{(0,0,0),(0,0,1),(1,0,1),(0,1,1)\}. \tag{3.23}$$

Indeed, the substitutions

$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,

$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,

$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0\dots & 0 \\ 1 & 0\dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$, and

$P = \begin{pmatrix} 1 & 0\dots & 0 \\ 1 & 0\dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0\dots & 0 \\ 1 & 0\dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$

in (3.18) imply that

$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(g\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)^2$ thus $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \{0,1\}$,

$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$,

$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = g\begin{pmatrix} 1 \\ 0 \end{pmatrix} g\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ thus if $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$ then $g\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$, and

$g\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \left(g\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)^2$ thus $g\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \{0,1\}$,

respectively. In a similar way we get that $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \{0,1\}$, and if $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$ then $g\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$, respectively, that is, (3.23) holds.

Now we show that the statement of our theorem holds in each case given by (3.23). The substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} 0 & s & \dots & s \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) imply that $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = g\begin{pmatrix} x \\ 1 \end{pmatrix} g\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ thus, if $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$, then $g\begin{pmatrix} x \\ 1 \end{pmatrix} = 1$, $x \in [0,1]$. In a similar way we have that, if $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$, then $g\begin{pmatrix} 1 \\ y \end{pmatrix} = 1$, $y \in [0,1]$. The substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ y & s & \dots & s \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) imply that $g\begin{pmatrix} x \\ 1 \end{pmatrix} g\begin{pmatrix} 1 \\ y \end{pmatrix} = 0$. Thus $g\begin{pmatrix} x \\ 1 \end{pmatrix} = 0$, $x \in [0,1]$ or $g\begin{pmatrix} 1 \\ y \end{pmatrix} = 0$, $y \in [0,1]$. In the remaining case $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$, substitute $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} y & s & \dots & s \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$, $y \in]0,1[$ in (3.18). Then we have that $g\begin{pmatrix} xy \\ 1 \end{pmatrix} = g\begin{pmatrix} x \\ 1 \end{pmatrix} g\begin{pmatrix} y \\ 1 \end{pmatrix}$, $x, y \in]0,1[$, that is, the function $\mu_1(x) = g\begin{pmatrix} x \\ 1 \end{pmatrix}$, $x \in]0,1[$ is multiplicative. In a similar way we can see that the function $\mu_2(y) = g\begin{pmatrix} 1 \\ y \end{pmatrix}$, $y \in]0,1[$ is multiplicative, too.

Subcase 2.B. $g(x) = 1, \quad x \in]0,1[^2$. The substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in (3.18), imply that

$$\sum_{j=1}^{m} \left[g\begin{pmatrix} xq_{j1} \\ 0 \end{pmatrix} - g\begin{pmatrix} x \\ 0 \end{pmatrix} \right] = 0, \quad (q_{11}, \dots, q_{1m}) \in \Gamma_m^0[1].$$

Thus there exists an additive function $a_x : \mathbb{R} \to \mathbb{R}$ such that

$$g\begin{pmatrix} q \\ 0 \end{pmatrix} = a_x\left(\frac{x}{q}\right) + g\begin{pmatrix} x \\ 0 \end{pmatrix} - \frac{a_x(1)}{n}, \quad q \in]0,x[,$$

where x is an arbitrary fixed element of $]0,1[$. This implies that there exist additive function $a_1 : \mathbb{R} \to \mathbb{R}$ and $c_1 \in \mathbb{R}$ such that

$$g\begin{pmatrix} x \\ 0 \end{pmatrix} = a_1(x) + c_1, \quad x \in]0,1[.$$

In a similar way we get that there exist additive function $a_2 : \mathbb{R} \to \mathbb{R}$ and $c_2 \in \mathbb{R}$ such that

$$g\begin{pmatrix} 0 \\ y \end{pmatrix} = a_2(y) + c_2, \quad y \in]0,1[.$$

With the substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in (3.18) we get that

$$g\begin{pmatrix} x \\ 1 \end{pmatrix} = \frac{m-1}{m} a_1(x-1) + 1, \quad x \in]0,1[.$$

Similarly we have that

$$g\begin{pmatrix} 1 \\ y \end{pmatrix} = \frac{m-1}{m} a_2(y-1) + 1, \quad y \in]0,1[.$$

With the substitutions $P = \begin{pmatrix} 0 & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 0 & s & \dots & s \\ 0 & v & \dots & v \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18), after some calculation, we get that $(g(\underline{0}))^2 = g(\underline{0})$, so $g(\underline{0}) \in \{0,1\}$.

If $g(\underline{0}) = 0$ then, with the substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18), we get that $(g(\underline{1}))^2 = g(\underline{1})$, so $g(\underline{1}) \in \{0, 1\}$. Furthermore, with the substitutions $P = \begin{pmatrix} x & 1-x & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18), we get that

$$a_1(x) = 0, \quad x \in]0,1[.$$

In a similar way we obtain that

$$a_2(y) = 0, \quad y \in]0,1[.$$

With the substitutions
$P = \begin{pmatrix} x & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} y & s & \dots & s \\ 0 & v & \dots & v \end{pmatrix} \in \Gamma_m^c[2]$, $x, y \in]0,1[$,
$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$,
$P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$ $Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & s & \dots & s \end{pmatrix} \in \Gamma_m^c[2]$, and
$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$
in (3.18), after some calculation, we get that
$c_1 = 0$ (a similar calculation shows that $c_2 = 0$),
$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} + g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(g\begin{pmatrix} 1 \\ 0 \end{pmatrix} + g\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)^2$, that is, $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} + g\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \{0, 1\}$,
$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$, and
$g(\underline{1}) = 1$, respectively.
If $g(\underline{0}) = 1$ then, with the substitutions $P = \begin{pmatrix} x & 1-x & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18), after some calculation, we get that $c_1 = 1$. In a similar way we have that $c_2 = 1$. The substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in (3.18) imply that $g(\underline{1}) = 1$. With the substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} y & s & \dots & s \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$, $y \in]0,1[$, in (3.18) we get that

$$\frac{1}{m^2} a_1(x) a_1(y) = a_1(x)\left(1 + \frac{a_1(1)}{m}\right) +$$

$$a_1(y)\left(\frac{n}{m}+\frac{a_1(1)}{m}\right)-\frac{a_1(xy)}{m}+a_1(1)(1-n-m-a_1(1))$$

From this, with $y=\frac{1}{2}$, after some calculation, we get that

$$a_1(x)=\frac{ma_1(1)}{a_1(1)+m}\left(n+a_1(1)+\frac{2m^2-1}{2m-1}\right). \tag{3.24}$$

Since a_1 is additive and the right hand side of (3.24) does not depend on x we have that

$$a_1(x)=0, \quad x\in]0,1[.$$

In a similar way, we have that

$$a_2(y)=0, \quad y\in]0,1[.$$

With the substitutions $P=\begin{pmatrix} 0 & r & \dots & r\\ 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_n^c[2]$, $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in (3.18) we get that $g\begin{pmatrix}0\\1\end{pmatrix}=1$. In a similar way, we get that $g\begin{pmatrix}1\\0\end{pmatrix}$. Thus

$$g(x)=1, \quad x\in[0,1]^2.$$

Subcase 2.C. $g(x)=M(x)$, $x\in]0,1[^2$, where $M:]0,1[^2\to\mathbb{R}$ is a multiplicative function which is different from the following four functions: $\begin{pmatrix}x\\y\end{pmatrix}\to 0,\begin{pmatrix}x\\y\end{pmatrix}\to 1,\begin{pmatrix}x\\y\end{pmatrix}\to x,\begin{pmatrix}x\\y\end{pmatrix}\to y,\begin{pmatrix}x\\y\end{pmatrix}\in]0,1[^2$. It is easy to check that this condition implies that there does not exist $c\in\mathbb{R}$ such that $\sum_{j=1}^n M(q_j)=c$ for all $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$.

With the substitutions $P=\begin{pmatrix} 0 & r & \dots & r\\ 0 & u & \dots & u\end{pmatrix}\in\Gamma_n^c[2]$, $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in (3.18) we get that

$$g(\underline{0})\left(\sum_{j=1}^n M(q_j)-m\right)=0.$$

Since there exists $Q^0\in\Gamma_m^0[2]$ such that $\sum_{j=1}^n M(q_j^0)\neq m$ thus $g(\underline{0})=0$. With the substitutions $P=\begin{pmatrix} 1 & 0 & \dots & 0\\ 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_n^c[2]$, $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in (3.18) we get that $(g(\underline{1})-1)\sum_{j=1}^n M(q_j)=0$. Since there exists $Q^0\in\Gamma_m^0[2]$ such that $\sum_{j=1}^n M(q_j^0)\neq 0$ thus $g(\underline{1})=1$. The substitutions $P=\begin{pmatrix} 1 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_n^c[2]$, $Q=\begin{pmatrix} 1 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_m^c[2]$ in (3.18) imply that $g\begin{pmatrix}1\\0\end{pmatrix}+g\begin{pmatrix}0\\1\end{pmatrix}=\left(g\begin{pmatrix}1\\0\end{pmatrix}+g\begin{pmatrix}0\\1\end{pmatrix}\right)^2$, that is, $g\begin{pmatrix}1\\0\end{pmatrix}+g\begin{pmatrix}0\\1\end{pmatrix}\in\{0,1\}$. The following calculation shows that, if there exists $x_0 \in]0,1[$ such that $g\begin{pmatrix} x_0 \\ 0 \end{pmatrix} \neq 0$, then there exists a multiplicative function $\mu :]0,1[\to \mathbb{R}$ such that $M\begin{pmatrix} x \\ y \end{pmatrix} = \mu(x)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$. The substitutions $P = \begin{pmatrix} x_0 & r & \dots & r \\ 0 & u & \dots & u \end{pmatrix} \in \Gamma_n^c[2]$, $x_0 \in]0,1[$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in (3.18), imply that

$$\sum_{j=1}^{m} \left[g\begin{pmatrix} x_0 q_{j1} \\ 0 \end{pmatrix} - g\begin{pmatrix} x_0 \\ 0 \end{pmatrix} M(q_j) \right] = 0, \quad Q = (q_1, \dots, q_m) \in \Gamma_m^0[2].$$

Thus there exists an additive function $A_1 : \mathbb{R}^2 \to \mathbb{R}$ such that

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-1}{g\begin{pmatrix} x_0 \\ 0 \end{pmatrix}} A_1\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{g\begin{pmatrix} x_0 \\ 0 \end{pmatrix}} \left[g\begin{pmatrix} x_0 x \\ 0 \end{pmatrix} - \frac{A(\underline{1})}{m} \right].$$

Hence there exist an additive function $A : \mathbb{R}^2 \to \mathbb{R}$ and a function $H :]0,1[\to \mathbb{R}$ such that

$$M\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + H(x), \quad \begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2.$$

Since the case $M\begin{pmatrix} x \\ y \end{pmatrix} = y$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$ is excluded, by Lemma 2.2, there exists multiplicative function $\mu :]0,1[\to \mathbb{R}$ such that $M\begin{pmatrix} x \\ y \end{pmatrix} = \mu(x)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$. In a similar way we can prove that, if there exists $y_0 \in]0,1[$ such that $g\begin{pmatrix} 0 \\ y_0 \end{pmatrix} \neq 0$, then there exists a multiplicative function $\mu :]0,1[\to \mathbb{R}$ such that $M\begin{pmatrix} x \\ y \end{pmatrix} = \mu(y)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$.

Now we show that $g\begin{pmatrix} x \\ 0 \end{pmatrix} = 0$, $x \in]0,1[$ or $g\begin{pmatrix} 0 \\ y \end{pmatrix} = 0$, $y \in]0,1[$. Indeed, suppose that there exist $x_0 \in]0,1[$ and $y_0 \in]0,1[$ such that $g\begin{pmatrix} x_0 \\ 0 \end{pmatrix} \neq 0$ and $g\begin{pmatrix} 0 \\ y_0 \end{pmatrix} \neq 0$. Then there exist multiplicative functions $\mu_1 :]0,1[\to \mathbb{R}$ and $\mu_2 :]0,1[\to \mathbb{R}$ such that $M\begin{pmatrix} x \\ y \end{pmatrix} = \mu_1(x) = \mu_2(y)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$. This implies that $M\begin{pmatrix} x \\ y \end{pmatrix} = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$ or $M\begin{pmatrix} x \\ y \end{pmatrix} = 1$, $\begin{pmatrix} x \\ y \end{pmatrix} \in]0,1[^2$, which are excluded in this case.

If $g\begin{pmatrix} x \\ 0 \end{pmatrix} = 0$, $x \in]0,1[$ and $g\begin{pmatrix} 0 \\ y \end{pmatrix} = 0$, $y \in]0,1[$ then substitute $P = \begin{pmatrix} 0 & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $Q = (q_1, \dots, q_m) \in \Gamma_m^0[2]$ in (3.18). Thus we get that

$$g\begin{pmatrix}0\\1\end{pmatrix}\sum_{j=1}^{m}M(q_j)=0.$$

Since there exists $Q^0\in\Gamma_m^0[2]$ such that $\sum_{j=1}^{m}M(q_j^0)\neq 0$ therefore $g\begin{pmatrix}0\\1\end{pmatrix}=0$. In a similar way we have that $g\begin{pmatrix}1\\0\end{pmatrix}=0$. Substituting $P=\begin{pmatrix}x & r & \dots & r\\ 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_n^c[2]$, $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in (3.18) we get that

$$\left(g\begin{pmatrix}x\\1\end{pmatrix}-M\begin{pmatrix}x\\1\end{pmatrix}\right)\sum_{j=1}^{m}M(q_j)=0.$$

Since there exists $Q^0\in\Gamma_m^0[2]$ such that $\sum_{j=1}^{m}M(q_j^0)\neq 0$ therefore

$$g\begin{pmatrix}x\\1\end{pmatrix}=M\begin{pmatrix}x\\1\end{pmatrix},\quad x\in]0,1[.$$

In a similar way we have that

$$g\begin{pmatrix}1\\y\end{pmatrix}=M\begin{pmatrix}1\\y\end{pmatrix},\quad y\in]0,1[.$$

If there exists $x_0\in]0,1[$ such that $g\begin{pmatrix}x_0\\0\end{pmatrix}\neq 0$ and $g\begin{pmatrix}0\\y\end{pmatrix}=0$, $y\in]0,1[$ then, by Lemma 2.2, there exists a multiplicative function $\mu:]0,1[\to\mathbb{R}$ such that $M\begin{pmatrix}x\\y\end{pmatrix}=\mu(x)$, $\begin{pmatrix}x\\y\end{pmatrix}\in]0,1[^2$. Substituting $P=\begin{pmatrix}x & r & \dots & r\\ 1 & 0 & \dots & 0\end{pmatrix}\in\Gamma_n^c[2]$, $x\in]0,1[$ and $Q=(q_1,\dots,q_m)\in\Gamma_m^0[2]$ in (3.18), we get that

$$\left(g\begin{pmatrix}x\\1\end{pmatrix}-\mu(x)\right)\sum_{j=1}^{m}\mu(q_{j1})=0.$$

Since there exists $(q_{11}^0,\dots,q_{m1}^0)\in\Gamma_m^0[1]$ such that $\sum_{j=1}^{m}\mu(q_{j1}^0)\neq 0$ thus $g\begin{pmatrix}x\\1\end{pmatrix}=\mu(x)$, $x\in]0,1[$.

The substitutions $P=\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & r & \dots & r\end{pmatrix}\in\Gamma_n^c[2]$, $Q=\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & s & \dots & s\end{pmatrix}\in\Gamma_m^c[2]$ in (3.18) imply that $g\begin{pmatrix}1\\0\end{pmatrix}=\left(g\begin{pmatrix}1\\0\end{pmatrix}\right)^2$, that is, $g\begin{pmatrix}1\\0\end{pmatrix}\in\{0,1\}$.

If $g\begin{pmatrix}1\\0\end{pmatrix}=1$ then, with the substitutions $P=\begin{pmatrix}1 & 0 & \dots & 0\\ x & r & \dots & r\end{pmatrix}\in\Gamma_n^c[2]$, $x\in]0,1[$, $Q=\begin{pmatrix}1 & 0 & \dots & 0\\ 0 & s & \dots & s\end{pmatrix}\in\Gamma_m^c[2]$ in (3.18), we get that $g\begin{pmatrix}1\\x\end{pmatrix}=1$, $x\in]0,1[$.

With the substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$, $Q = \begin{pmatrix} 0 & s & \dots & s \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) we get that $g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$.

With the substitutions $P = \begin{pmatrix} x & r & \dots & r \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) we get that

$$g\begin{pmatrix} x \\ 0 \end{pmatrix} = g\begin{pmatrix} x \\ 1 \end{pmatrix} = \mu(x), \quad x \in]0,1[.$$

If $g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ then, with the substitutions $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$, $Q = (q_1, \dots, q_m) \in \Gamma_m^c[2]$ in (3.18), we get that $\sum_{j=1}^m g\begin{pmatrix} q_{j1} \\ 0 \end{pmatrix} = 0$, $(q_{11}, \dots, q_{1m}) \in \Gamma_m^c[1]$. Thus there exists an additive function $a : \mathbb{R} \to \mathbb{R}$ such that $g\begin{pmatrix} x \\ 0 \end{pmatrix} = a(x) - \frac{a(1)}{m}$, $x \in [0,1]$. Since $0 = g(\underline{0}) = -\frac{a(1)}{m}$ we have that $a(1) = 0$ and

$$g\begin{pmatrix} x \\ 0 \end{pmatrix} = a(x), \quad x \in [0,1].$$

With the substitutions $P = \begin{pmatrix} x & 1-x & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$ in (3.18) we get that

$$g\begin{pmatrix} 0 \\ 1 \end{pmatrix}(a(x) + \mu(1-x) - 1) = 0.$$

Since the function a is additive, the function μ is multiplicative and different from the functions $x \to 0$, $x \to 1$, and $x \to x$, there exists $x_0 \in]0,1[$ such that $a(x_0) + \mu(1-x_0) \neq 0$ thus

$$g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

With the substitutions $P = \begin{pmatrix} x & 1-x & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ y & 1-y & 0 & \dots & 0 \end{pmatrix} \in \Gamma_m^c[2]$, $y \in]0,1[$ in (3.18) we get that $a(x) = 0$, $x \in]0,1[$. Substituting $P = \begin{pmatrix} 1 & 0 & \dots & 0 \\ x & r & \dots & r \end{pmatrix} \in \Gamma_n^c[2]$, $x \in]0,1[$, $Q = \begin{pmatrix} y_1 & y_2 & \dots & y_m \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \Gamma_n^c[2]$, $y_1, \dots, y_m \in]0,1[$, in (3.18) we get that

$$\left(g\begin{pmatrix} 1 \\ x \end{pmatrix} - 1\right)\sum_{j=1}^m \mu(y_j) = 0.$$

Since there exists $(y_1^0, \dots, y_m^0) \in \Gamma_m^0[1]$ such that $\sum_{j=1}^m \mu(y_j^0) \neq 0$ therefore $g\begin{pmatrix} 1 \\ x \end{pmatrix} = 1$, $x \in]0,1[$. $\square$

References

[1] Aczél, J. Lectures on Functional Equations and Their Applications, *Academic Press*, New York – London, 1966.

[2] Ebanks, B. R., Sahoo, P. K., Sander, W., Characterizations of information measures, *World Scientific*, Singapore – New Jersey – London – Hong Kong, 1998.

[3] Losonczi, L., Maksa, Gy., On some functional equation of information theory, *Acta Math. Acad. Sci. Hungar.*, **39** (1982), 73-82.

Imre Kocsis
H-4028, Debrecen
Ótemető u. 2-4.
Hungary

Annales Mathematicae et Informaticae
32 *(2005) pp. 79–86.*

Pointwise very strong approximation as a generalization of Fejér's summation theorem

Włodzimierz Łenski

University of Zielona Góra
Faculty of Mathematics, Informatics and Econometry
W.Lenski@wmie.uz.zgora.pl

Abstract

We will present an estimation of the $H^q_{k_r} f$ mean as a approximation versions of the Totik type generalization(see [6]) of the result of G. H. Hardy, J. E. Littlewood. Some results on the norm approximation will also given.

Key Words: very strong approximation, rate of pointwise strong summability

AMS Classification Number: 42A24

1. Introduction

Let L^p $(1 < p < \infty)$ $[resp.C]$ be the class of all 2π–periodic real–valued functions integrable in the Lebesgue sense with p–th power [continuous] over $Q = [-\pi, \pi]$ and let $X = X^p$ where $X^p = L^p$ when $1 < p < \infty$ or $X^p = C$ when $p = \infty$. Let us define the norm of $f \in X^p$ as

$$\|f\|_{X^p} = \|f(x)\|_{X^p} = \begin{cases} \left(\int_Q \mid f(x) \mid^p dx\right)^{1/p} & when\ 1 < p < \infty\ , \\ \sup_{x \in Q} \mid f(x) \mid & when\ \ p = \infty. \end{cases}$$

Consider the trigonometric Fourier series

$$Sf(x) = \frac{a_o(f)}{2} + \sum_{k=0}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) = \sum_{k=0}^{\infty} C_k f(x)$$

and denote by $S_k f$, the partial sums of Sf. Let

$$H_{k_r}^q f(x) := \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| S_{k_\nu} f(x) - f(x) \right|^q \right\}^{\frac{1}{q}}, \qquad (q > 0)$$

where $0 \leqslant k_0 < k_1 < k_2 < ... < k_r \ (\geqslant r)$.

The pointwise characteristic

$$\begin{aligned} \overline{w}_x f(\delta)_p &:= \sup_{0<h\leqslant\delta} \left\{ \frac{1}{h} \int_0^h |\varphi_x(t)|^p \, dt \right\}^{1/p}, \\ \text{where} \quad \varphi_x(t) &:= f(x+t) + f(x-t) - 2f(x) \end{aligned}$$

constructed on the base of definition of Lebesgue points ($L^1 - points$) was firstly used as a measure of approximation, by S.Aljančič, R.Bojanic and M.Tomić [1]. This characteristic was very often used, but it appears that such approximation cannot be comparable with the norm approximation beside when $X = C$. In [5] there was introduced the slight modified function:

$$w_x f(\delta)_p := \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p \, dt \right\}^{1/p}.$$

We can observe that for $p \in [1, \infty)$ and $f \in C$

$$w_x f(\delta)_p \leqslant \overline{w}_x f(\delta)_p \leqslant \omega_C f(\delta)$$

and also, with $\widetilde{p} > p$ for $f \in X^{\widetilde{p}}$, by the Minkowski inequality

$$\|w_{\cdot} f(\delta)_p\|_{X^{\widetilde{p}}} \leqslant \omega_{X^{\widetilde{p}}} f(\delta),$$

where $\omega_X f$ is the modulus of continuity of f in the space $X = X^{\widetilde{p}}$ defined by the formula

$$\omega_X f(\delta) := \sup_{0<|h|\leqslant\delta} \|f(\cdot + h) - f(\cdot)\|_X.$$

It is well-known that $H_n^q f(x) -$ means tend to 0 at the $L^p - points$ of $f \in L^p$ $(1 < p \leqslant \infty)$. In [3] this fact was by G. H. Hardy, J. E. Littlewood proved as a generalization of the Fejér classical result on the convergence of the $(C, 1)$ - means of Fourier series. Here we present an estimation of the $H_{k_r}^q f(x)$ means as a approximation version of the Totik type (see [6]) generalization of the result of G. H. Hardy, J. E. Littlewood. We also give some corollaries on norm approximation.

By K we shall designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same of each occurrence.

2. Statement of the results

Theorem 2.1. *If $f \in L^p$ $(1 < p \leqslant 2)$, then, for indices $0 \leqslant k_0 < k_1 < k_2 < ... < k_r$ $(\geqslant r)$,*

$$H_{k_r}^q f(x) \leqslant 2\left\{\sum_{k=r}^{k_r} \frac{w_x f(\frac{\pi}{k+1})_1}{k+1}\right\} + 6\left\{\frac{1}{(r+1)^{p-1}} \sum_{k=0}^{r} \frac{\left(w_x f(\frac{\pi}{k+1})_p\right)^p}{(k+1)^{2-p}}\right\}^{1/p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Applying the inequality for the norm of the modulus of continuity of f we can immediately derive from the above theorem the next one.

Theorem 2.2. *If $f \in L^p$ $(1 < p \leqslant 2)$, then for indices $0 \leqslant k_0 < k_1 < k_2 < ... < k_r$ $(\geqslant r)$,*

$$\left\|H_{k_r}^q f(\cdot)\right\|_{L^p} \leqslant 2\left\{\sum_{k=r}^{k_r} \frac{\omega_{L^p} f(\frac{\pi}{k+1})}{k+1}\right\} + 6\left\{\frac{1}{(r+1)^{p-1}} \sum_{k=0}^{r} \frac{\left(\omega_{L^p} f(\frac{\pi}{k+1})\right)^p}{(k+1)^{2-p}}\right\}^{1/p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2.3. In the special case $k_\nu = \nu$ for $\nu = 0, 1, 2, ..., r$, the first term in the above estimates is superfluous.

Next, we consider a function w_x of modulus of continuity type on the interval $[0, +\infty)$, i.e. a nondecreasing continuous function having the following properties: $w_x(0) = 0$, $w_x(\delta_1 + \delta_2) \leqslant w_x(\delta_1) + w_x(\delta_2)$ for any $0 \leqslant \delta_1 \leqslant \delta_2 \leqslant \delta_1 + \delta_2$ and let

$$L^p(w_x) = \left\{g \in L^p : w_x g(\delta)_p \leqslant w_x(\delta)\right\}.$$

In this class we can derive the following

Theorem 2.4. *Let $f \in L^p(w_x)$ $(1 < p \leqslant 2)$ and $0 \leqslant k_0 < k_1 < k_2 < ... < k_r$ $(\geqslant r)$. If w_x satisfy, for some $A > 1$ the condition $\limsup_{\delta \to 0+} \left(\frac{w_x(A\delta)}{w_x(\delta)}\right)^p < A^{p-1}$, then*

$$H_{k_r}^q f(x) \leqslant K w_x\left(\frac{\pi}{r+1}\right) \log \frac{k_r + 1}{r+1}.$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In the same way for subclass

$$L^p(\omega) = \{g \in L^p : \omega_{L^p} f(\delta) \leqslant \omega(\delta), \quad \text{with modulus of continuity} \quad \omega\}$$

we can obtain

Theorem 2.5. *Let* $f \in L^p(\omega)$ $(1 < p \leqslant 2)$ *and* $0 \leqslant k_0 < k_1 < k_2 < ... < k_r$ $(\geqslant r)$. *If* ω *satisfy , for some* $A > 1$ *and an integer* $s \geqslant 1$, *the condition* $\limsup_{\delta \to 0+} \frac{\omega(A\delta)}{\omega(\delta)} < A^s$, *then*

$$\left\| H_{k_r}^q f(\cdot) \right\|_{L^p} \leqslant K\omega\left(\frac{\pi}{r+1}\right) \log \frac{k_r+1}{r+1},$$

where $\frac{1}{p} + \frac{1}{q} = 1$

For the proof of Theorem 2.2 we will need the following lemma of N. K. Bari and S. B. Stechkin [2].

Lemma 2.6. *If a continuous and non-decreasing on* $[0, \infty)$ *function* w *satisfies conditions:* $w(0) = 0$ *and* $\limsup_{\delta \to 0+} \frac{w(A\delta)}{w(\delta)} < A^s$ *for some* $A > 1$ *and an integer* $s \geqslant 1$, *then*

$$u^s \int_u^\pi \frac{w(t)}{t^{s+1}} dt \leqslant Kw(u) \quad \text{for} \quad u \in (0, \pi],$$

where the constant K *depend only on* w *and in other way the fulfillment of the above inequality for all* $u \in (0, \pi]$ *imply the existence of a constant* $A > 1$ *for which* $\limsup_{\delta \to 0+} \frac{w(A\delta)}{w(\delta)} < A^s$ *with some integer* $s \geqslant 1$.

3. Proofs of the results

We only prove Theorems 2.1 and 2.4.

Proof of Theorem 2.1. Let as usually

$$\begin{aligned} H_{k_r}^q f(x) &= \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_0^\pi \varphi_x(t) D_{k_\nu}(t) \, dt \right|^q \right\}^{1/q} \\ &\leqslant A_{k_r} + B_{k_r} + C_{k_r}, \end{aligned}$$

where $D_{k_\nu}(t) = \frac{\sin \frac{(2k_\nu+1)t}{2}}{2 \sin \frac{t}{2}}$,

$$A_{k_r}(\delta) = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_0^\delta \varphi_x(t) D_{k_\nu}(t) \, dt \right|^q \right\}^{1/q},$$

$$B_{k_r}(\gamma, \delta) = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_\delta^\gamma \varphi_x(t) D_{k_\nu}(t) \, dt \right|^q \right\}^{1/q}$$

and

$$C_{k_r}(\gamma) = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_\gamma^\pi \varphi_x(t) D_{k_\nu}(t) \, dt \right|^q \right\}^{1/q},$$

with $\delta = \frac{\pi}{k_r+1}$ and $\gamma = \frac{\pi}{r+1}$.

Since $k_\nu \leqslant k_r$, for $\nu = 0,1,2,...,r$,we conclude that $|D_{k_\nu}(t)| \leqslant k_r + 1$ and $|D_{k_\nu}(t)| \leqslant \frac{\pi}{2|t|}$. Hence

$$A_{k_r}(\delta) \leqslant \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left[\frac{k_r+1}{\pi} \int_0^\delta |\varphi_x(t)|\, dt \right]^q \right\}^{1/q} = w_x f(\delta)_1$$

and

$$B_{k_r}(\gamma,\delta) = \left\{ \frac{1}{r+1} \sum_{\nu=0}^{r} \left[\frac{1}{2} \int_\delta^\gamma \frac{|\varphi_x(t)|}{t} dt \right]^q \right\}^{1/q} = \frac{1}{2} \int_\delta^\gamma \frac{|\varphi_x(t)|}{t} dt.$$

Integrating by parts, we obtain

$$\begin{aligned} B_{k_r}(\gamma,\delta) &= \frac{1}{2} \left\{ w_x f(t)_1 \,|_{t=\delta}^{\gamma} + \int_\delta^\gamma \frac{w_x f(t)_1}{t} dt \right\} \\ &= \frac{1}{2} w_x f(\gamma)_1 - \frac{1}{2} w_x f(\delta)_1 + \frac{1}{2} \int_{r+1}^{k_r+1} \frac{w_x f(\pi/u)_1}{u} du \end{aligned}$$

and by simple calculation we have

$$\begin{aligned} B_{k_r}(\gamma,\delta) &\leqslant \frac{1}{2} w_x f(\gamma)_1 - \frac{1}{2} w_x f(\delta)_1 + \frac{1}{2} \sum_{k=r+1}^{k_r} \int_k^{k+1} \frac{w_x f(\pi/u)_1}{u} du \\ &\leqslant \frac{1}{2} w_x f(\gamma)_1 - \frac{1}{2} w_x f(\delta)_1 + \frac{1}{2} \sum_{k=r+1}^{k_r} \frac{k+1}{k} \frac{w_x f(\pi/k)_1}{k} \end{aligned}$$

$$\begin{aligned} &\leqslant \frac{1}{2} w_x f(\gamma)_1 - \frac{1}{2} w_x f(\delta)_1 + \frac{1}{2} \left(1 + \frac{1}{r+1} \right) \sum_{k=r}^{k_r-1} \frac{w_x f(\pi/k)_1}{k} \\ &\leqslant w_x f(\gamma)_1 + 2 \sum_{k=r}^{k_r-1} \frac{w_x f(\pi/k)_1}{k}. \end{aligned}$$

Putting $D_{k_\nu}(t) = \frac{1}{2} \sin(k_\nu t) \cot \frac{t}{2} + \frac{1}{2} \cos(k_\nu t)$, by the Hausdorff–Young inequality,

$$\begin{aligned} & C_{k_r}(\gamma) \\ \leqslant\ & \frac{1}{2(r+1)^{1/q}} \left\{ \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_\gamma^\pi \varphi_x(t) \cot \frac{t}{2} \sin(k_\nu t)\, dt \right|^q \right\}^{1/q} \\ & + \frac{1}{2(r+1)^{1/q}} \left\{ \sum_{\nu=0}^{r} \left| \frac{1}{\pi} \int_\gamma^\pi \varphi_x(t) \cos(k_\nu t)\, dt \right|^q \right\}^{1/q} \end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\frac{1}{\pi}\int_{\gamma}^{\pi}\left|\varphi_{x}\left(t\right)\cot\frac{t}{2}\right|^{p}dt\right\}^{1/p}\\
&\quad+\frac{1}{2\left(r+1\right)^{1/q}}\left\{\frac{1}{\pi}\int_{\gamma}^{\pi}\left|\varphi_{x}\left(t\right)\right|^{p}dt\right\}^{1/p}\\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\int_{\gamma}^{\pi}\left|\frac{\varphi_{x}\left(t\right)}{t/\pi}\right|^{p}dt\right]^{1/p}+\pi^{1/p}w_{x}f(\pi)_{p}\right\}
\end{aligned}
$$

and by partial integration,

$$
\begin{aligned}
&C_{k_r}\left(\gamma\right)\\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\frac{\left[w_{x}f(t)_{p}\right]^{p}}{t^{p-1}}\Big|_{t=\gamma}^{\pi}+p\int_{\gamma}^{\pi}\left|\frac{w_{x}f(t)_{p}}{t}\right|^{p}dt\right]^{1/p}\right.\\
&\quad\left.+\pi^{1/p}w_{x}f(\pi)_{p}\right\}\\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\pi^{1-p}\left[w_{x}f(\pi)_{p}\right]^{p}+p\int_{1}^{r+1}\left|\frac{w_{x}f(\pi/u)_{p}}{\pi/u}\right|^{p}\frac{\pi}{u}du\right]^{1/p}\right.\\
&\quad\left.+\pi^{1/p}w_{x}f(\pi)_{p}\right\}.
\end{aligned}
$$

Therefore, analogously as before,

$$
\begin{aligned}
&C_{k_r}\left(\gamma\right)\\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\pi^{1-p}\left[w_{x}f(\pi)_{p}\right]^{p}+p\pi^{1-p}\sum_{k=1}^{r}\int_{k}^{k+1}\frac{\left[w_{x}f(\pi/u)_{p}\right]^{p}}{u^{2-p}}du\right]^{1/p}\right.\\
&\quad\left.+\pi^{1/p}w_{x}f(\pi)_{p}\right\}\\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\pi^{1-p}\left[w_{x}f(\pi)_{p}\right]^{p}+p\pi^{1-p}\sum_{k=1}^{r}\frac{k+1}{k}\frac{\left[w_{x}f(\pi/k)_{p}\right]^{p}}{k^{2-p}}\right]^{1/p}\right.\\
&\quad\left.+\pi^{1/p}w_{x}f(\pi)_{p}\right\}\\
&\leqslant \frac{1}{2\left(r+1\right)^{1/q}}\left\{\left[\left(1+p\right)\pi^{1-p}\sum_{k=1}^{r}\frac{\left[w_{x}f(\pi/k)_{p}\right]^{p}}{k^{2-p}}\right]^{1/p}+\pi^{1/p}w_{x}f(\pi)_{p}\right\}\\
&\leqslant K\left\{\frac{1}{\left(r+1\right)^{p-1}}\sum_{k=1}^{r}\frac{\left[w_{x}f(\pi/\left(k+1\right))_{p}\right]^{p}}{\left(k+1\right)^{2-p}}\right\}^{1/p}.
\end{aligned}
$$

Finally, since

$$
w_{x}f(\gamma)_{1} \quad\leqslant\quad w_{x}f(\gamma)_{p}\left\{\frac{p}{\left(r+1\right)^{p}}\sum_{k=0}^{r}\frac{1}{\left(k+1\right)^{1-p}}\right\}^{1/p}
$$

$$\leqslant \quad \left\{ \frac{p}{(r+1)^{p-1}} \sum_{k=1}^{r} \frac{[w_x f(\pi/(k+1))_p]^p}{(k+1)^{2-p}} \right\}^{1/p},$$

our result follows. □

Proof of Theorem 2.4. It is clear that if $f \in L^p(w_x)$ $(1 < p \leqslant 2)$ then $w_x f(\delta)_1 \leqslant w_x f(\delta)_p \leqslant w_x(\delta)$. Thus, by Theorem 2.1,

$$H_{k_r}^q f(x) \leqslant 2 \left\{ \sum_{k=r}^{k_r} \frac{w_x(\frac{\pi}{k+1})}{k+1} \right\} + 6 \left\{ \frac{1}{(r+1)^{p-1}} \sum_{k=0}^{r} \frac{\left(w_x(\frac{\pi}{k+1})\right)^p}{(k+1)^{2-p}} \right\}^{1/p}$$

and, by the monotonicity of w_x and simple inequality $w_x(\pi) \leqslant 2w_x(\frac{\pi}{2})$, we obtain

$$\begin{aligned}
H_{k_r}^q f(x) \quad &\leqslant \quad 2 \left\{ \sum_{k=r}^{k_r} \frac{w_x(\frac{\pi}{k+1})}{k+1} \right\} \\
&\quad + 6 \left\{ \frac{1}{(r+1)^{p-1}} \left((w_x(\pi))^p + \sum_{k=1}^{r} \frac{\left(w_x(\frac{\pi}{k+1})\right)^p}{(k+1)^{2-p}} \right) \right\}^{1/p} \\
&\leqslant \quad 2 \left\{ \sum_{k=r}^{k_r} \frac{w_x(\frac{\pi}{k+1})}{k+1} \right\} + 6 \left\{ \frac{5}{(r+1)^{p-1}} \sum_{k=1}^{r} \frac{\left(w_x(\frac{\pi}{k+1})\right)^p}{(k+1)^{2-p}} \right\}^{1/p} \\
&\leqslant \quad 2 \left\{ w_x(\frac{\pi}{r+1}) \sum_{k=r}^{k_r} \frac{1}{k+1} \right\} \\
&\quad + 6 \left\{ \frac{5}{(r+1)^{p-1}} \sum_{k=1}^{r} \int_k^{k+1} \frac{\left(w_x(\frac{\pi}{t})\right)^p}{t^{2-p}} dt \right\}^{1/p} \\
&\leqslant \quad 2 \left\{ w_x(\frac{\pi}{r+1}) \int_r^{k_r+1} \frac{1}{t} dt \right\} \\
&\quad + 6 \left\{ \frac{5}{(r+1)^{p-1}} \pi^{p-1} \int_{\frac{\pi}{r+1}}^{\pi} \frac{(w_x(u))^p}{u^{p-2}} \frac{du}{u^2} \right\}^{1/p} \\
&\leqslant \quad 2w_x(\frac{\pi}{r+1}) \log \frac{k_r+1}{r} \\
&\quad + 6 \left\{ 5 \left(\frac{\pi}{r+1} \right)^{p-2} \frac{\pi}{r+1} \int_{\frac{\pi}{r+1}}^{\pi} \frac{(w_x(u))^p \, u^{2-p}}{u^2} du \right\}^{1/p}
\end{aligned}$$

Now, we observe that, by our assumption , the function $(w_x(u))^p u^{2-p}$ satisfy the condition

$$\limsup_{\delta\to 0+} \frac{(w_x(A\delta))^p (A\delta)^{2-p}}{(w_x(\delta))^p (\delta)^{2-p}} = A^{2-p} \limsup_{\delta\to 0+} \frac{(w_x(A\delta))^p}{(w_x(\delta))^p} < A^{2-p}A^{p-1} = A$$

i.e. the condition of Lemma 2.6 with $s = 1$. Therefore

$$\frac{\pi}{r+1}\int_{\frac{\pi}{r+1}}^{\pi} \frac{(w_x(u))^p u^{2-p}}{u^2} du \leqslant \left(w_x(\frac{\pi}{r+1})\right)^p \left(\frac{\pi}{r+1}\right)^{2-p}.$$

Hence

$$\begin{aligned} H_{k_r}^q f(x) &\leqslant 2w_x(\frac{\pi}{r+1}) \log \frac{k_r+1}{r} \\ &\quad +6\left\{5\left(\frac{\pi}{r+1}\right)^{p-2}\left(w_x(\frac{\pi}{r+1})\right)^p\left(\frac{\pi}{r+1}\right)^{2-p}\right\}^{1/p} \\ &\leqslant \left(2+6\ 5^{1/p}\right) w_x(\frac{\pi}{r+1}) \log \frac{k_r+1}{r}, \end{aligned}$$

and our result is proved. □

References

[1] S. Aljančič, R. Bojanic and M. Tomić, On the degree of convergence of Fejér–Lebesgue sums, L'Enseignement Mathematique, Geneve, Tome XV (1969) 21–28.

[2] N. K. Bari, S. B. Stečkin, Best approximation and differential properties of two conjugate functions, (in Russian) ,Trudy Moscovsco Mat. o-va, 1956, T.5, 483–522.

[3] G. H. Hardy, J. E. Littlewood, Sur la série de Fourier d'une function a caré sommable, Comptes Rendus, Vol.28,.(1913), 1307–1309.

[4] L. Leindler, Strong approximation by Fourier series, Akadémiai Kiadó, Budapest, 1985.

[5] W. Łenski, On the rate of pointwise strong (C, α) summability of Fourier series, Colloquia Math. Soc. János Bolyai, 58 Approx. Theory, Kecskemét (Hungary),(1990), 453–486.

[6] V. Totik, On the strong approximation of Fourier series, Acta Math. Acad. Hungar.35 (1-2), (1980), 151–172.

Włodzimierz Łenski
University of Zielona Góra
Faculty of Mathematics, Informatics and Econometry
65-516 Zielona Góra, ul. Szafrana 4a
Poland

Annales Mathematicae et Informaticae
***32** (2005) pp. 87–94.*

Towards a non-selfadjoint version of Kadison's theorem

Martin Mathieu

Queen's University Belfast
e-mail: m.m@qub.ac.uk

Abstract

Kadison's theorem of 1951 describes the unital surjective isometries between unital C^*-algebras as the Jordan *-isomorphisms. We propose a non-selfadjoint version of his theorem and discuss the cases in which this is known to be true.

Key Words: Spectral isometry, Jordan isomorphism, C^*-algebra.

AMS Classification Number: Primary 47A65; Secondary 46L10, 47A10, 47B48.

1. Introduction

Among the most important linear mappings between Banach spaces are the isometries; no wonder therefore that they have been given a lot of attention. One of the best-known results is the classical Banach–Stone theorem, proved by Banach in 1932 under the assumption of separability and by Stone in 1937 in the general case.

Theorem 1.1 (Banach–Stone). *Let X, Y be compact Hausdorff spaces. Let $T\colon C(X) \to C(Y)$ be a surjective linear isometry between the associated Banach spaces of complex-valued continuous functions. Then there exist a uniquely determined function $h \in C(Y)$ with $|h| = 1$ and a uniquely determined homeomorphism $\varphi\colon Y \to X$ such that $Tf = h\,(f \circ \varphi)$ for all $f \in C(X)$.*

In particular, if T is *unital*, that is, $T1 = 1$, then T is multiplicative, hence an algebra isomorphism. Note that we get for free that T preserves the canonical involution on the spaces of continuous functions: $T\bar{f} = \overline{Tf}$, where $\bar{f}$ denotes the complex-conjugate function.

In 1951 Kadison obtained the following generalisation of the above result to arbitrary C^*-algebras [8].

Theorem 1.2 (Kadison). *Let A, B be unital C^*-algebras. Let $T\colon A \to B$ be a surjective linear isometry between A and B. Then there exist a uniquely determined unitary $u \in B$ and a uniquely determined Jordan *-isomorphism $\Phi\colon A \to B$ such that $Tx = u\,\Phi x$ for all $x \in A$.*

Here, a Jordan *-isomorphism Φ is a bijective linear mapping with the property that $\Phi(x^2) = (\Phi x)^2$ for all $x \in A$ and which is *selfadjoint*, i.e. preserves selfadjoint elements. It follows easily that Φ indeed preserves the Jordan product $x \cdot y = \frac{1}{2}(xy+yx)$, $x, y \in A$. Sometimes such mappings are referred to as *C^*-isomorphisms* and in a certain sense they can be built from isomorphisms and anti-isomorphisms, see [1, Section 6.3] for example.

Every Jordan *-isomorphism Φ is an isometry. Indeed, let $x \in A$ be positive; then $x = y^2$ for some $y \in A_{sa}$. Hence, $\Phi x = \Phi(y^2) = (\Phi y)^2$ is positive. By the Russo–Dye theorem [15, Corollary 2.9], $\|\Phi\| = \|\Phi 1\|$ and since it is easily seen that every Jordan *-isomorphism is unital, it follows that Φ is a contraction. Applying the same argument to Φ^{-1} yields the claim.

Part of Kadison's argument establishes the fact that $u = \Phi 1$ is a unitary in B, whenever Φ is a surjective isometry. Therefore one can reduce to the case of a *unital* isometry and thus Kadison's theorem is in fact a characterisation of the unital surjective isometries between unital C^*-algebras as the Jordan *-isomorphisms. A similar reduction applies to the more general mappings discussed below, hence we will from this point on deal exclusively with unital mappings. On the other hand, it is well known that the assumption of surjectivity is inevitable.

Suppose that $T\colon A \to B$ is a unital surjective isometry; then T is selfadjoint. Indeed, let $a \in A_{sa}$, $\|a\| = 1$ and write $Ta = b + ic$, where b, $c \in B_{sa}$. If $c \neq 0$ then there is $\gamma \in \sigma(c)$, the spectrum of c, which is non-zero; we can assume that $\gamma > 0$. Since T is an isometry, for large $n \in \mathbb{N}$, we find

$$\|a + in\|^2 = 1 + n^2 < (\gamma + n)^2 \leqslant \|c + n\|^2 \leqslant \|T(a + in)\|^2.$$

This entails that $c = 0$ and so $Ta \in B_{sa}$.

We thus observe that unital isometries are *intrinsically* selfadjoint. In the sequel we wish to discuss a concept of 'non-selfadjoint' isometries that is capable of characterising not necessarily selfadjoint Jordan isomorphisms in a way analogous to Kadison's theorem.

2. Spectral isometries

Suppose that $T\colon A \to B$ is a Jordan isomorphism between the unital C^*-algebras A and B. It is well known that $a \in A$ is invertible if and only if $Ta \in B$ is invertible; see, e.g., [7, Lemma 4.1]. Consequently T preserves the spectrum of every element, that is, $\sigma(Ta) = \sigma(a)$ for every $a \in A$. A fortiori T preserves the spectral radius $r(a)$, and it turns out that this is the decisive property.

Definition 2.1. A linear mapping $T\colon A \to B$ between two unital C^*-algebras is called a *spectral isometry* if $r(Ta) = r(a)$ for every $a \in A$.

Since norm and spectral radius coincide in the commutative case, it is evident that we only obtain a new notion if at least A or B is not commutative. However, it turns out that in fact *both* have to be non-commutative, otherwise we are back to the notion of an isometry.

Proposition 2.2. *Let $T\colon A \to B$ be a unital surjective spectral isometry between the unital C^*-algebras A and B. If A or B is commutative then T is a multiplicative isomorphism.*

This result follows easily from the results in [10] and the Banach–Stone theorem. More generally every surjective spectral isometry restricts to an isomorphism of the centres of general C^*-algebras. To see this, let us first note two properties.

(1) Every spectral isometry is injective.

(2) Every surjective spectral isometry preserves central elements.

Suppose $T\colon A \to B$ is a spectral isometry, and let $a \in A$ be such that $Ta = 0$. For $x \in A$ we obtain $r(a + x) = r(Ta + Tx) = r(Tx) = r(x)$; hence, by Zemánek's characterisation of the radical [2, Theorem 5.3.1], a belongs to the radical of A which is zero. Thus $a = 0$ and (1) holds.

Now assume in addition that T is surjective. Let $z \in Z(A)$, the centre of A. For $b \in B$ take $a \in A$ such that $Ta = b$. Then

$$r(Tz + b) = r(T(z + a)) = r(z + a) \leqslant r(z) + r(a) = r(Tz) + r(b).$$

By Pták's characterisation of the centre [16, Proposition 2.1] it follows that $Tz \in Z(B)$. This shows (2).

Combining these properties with the Banach–Stone theorem and applying (2) to the spectral isometry $T^{-1}\colon B \to A$, we obtain the stated result.

Proposition 2.3. *Let $T\colon A \to B$ be a unital surjective spectral isometry between the unital C^*-algebras A and B. Then $T_{|Z(A)}$ induces a *-isomorphism between $Z(A)$ and $Z(B)$.*

We shall soon make good use of this result. But let us first compare the two concepts of isometry and spectral isometry more closely. Every unital surjective isometry between unital C^*-algebras is a Jordan *-isomorphism by Theorem 1.2, hence a spectral isometry. We remark in passing that we do not know of a direct argument proving this without using Kadison's theorem. Conversely, every selfadjoint unital surjective spectral isometry is an isometry. To see this, let $a \in A_+$, $\|a\| = 1$. Then $\|a-1\| \leqslant 1$ and therefore $\|Ta\| = 1$ and $\|Ta-1\| \leqslant 1$, since $Ta \in B_{sa}$ and norm and spectral radius coincide for selfadjoint elements. Consequently Ta is positive which shows that T is a positive map. Applying the Russo–Dye theorem once again we deduce that $\|T\| = \|T1\| = 1$ so T is a contraction. The same argument for T^{-1} yields the result.

Proposition 2.4. *Let $T\colon A \to B$ be a unital surjective linear map. Then T is an isometry if and only if T is a selfadjoint spectral isometry.*

A 35-year old problem of Kaplansky [9] asks whether every surjective spectrum-preserving linear mapping between unital C^*-algebras has to be a Jordan isomorphism. An important step forward was made by Aupetit [4] by establishing the result for von Neumann algebras. However, to-date no answer appears to be known if neither of the C^*-algebras is real rank zero. Nevertheless, Kaplansky's question together with the above evidence made us surmise the following in [12].

Conjecture 2.5. *Every unital surjective spectral isometry between unital C^*-algebras is a Jordan isomorphism.*

Evidently this conjecture is harder than Kaplansky's; the point we wish to make here is that the statement provides a non-selfadjoint analogue of Kadison's theorem.

In the remainder of this note we shall explain what by now is known on Conjecture 2.5 and discuss some of the techniques involved in proving our results.

3. The theorem

Before stating the main result and discussing the ingredients of its proof we need two more properties of spectral isometries.

(3) Every surjective spectral isometry is bounded (and hence open).

(4) Every surjective spectral isometry preserves nilpotent elements.

Both properties in fact hold for the wider class of spectrally bounded operators. A linear mapping $T\colon A \to B$ is said to be *spectrally bounded* if there is a constant $M > 0$ such that $r(Tx) \leqslant M\, r(x)$ for all $x \in A$. The surjectivity and the *semisimplicity of B* then yield the boundedness of T; see [2, Theorem 5.5.1] and, slightly more general, [5]. This gives (3). Property (4) was obtained in [13, Lemma 3.1], once again for surjective spectrally bounded maps. It follows that, if T is a surjective spectral isometry and $a \in A$, then $a^n = 0$ if and only if $(Ta)^n = 0$. Spectrally bounded maps originally were introduced in connection with the non-commutative Singer–Wermer conjecture, see [6] for more details. A number of their basic properties are discussed in [12].

Apart from the commutative situation, which is somewhat special, an important technique employed by many authors to show that a spectral isometry (or, more generally, a spectrally bounded operator) has the Jordan property has been to evaluate it on projections. This, of course, only works if the domain is well supplied with projections. In fact, we do not know of any result that goes beyond the scope of C^*-algebras with real rank zero at present. Indeed, Aupetit's theorem [4] does not rely on the structure of von Neumann algebras but extends to C^*-algebras of real rank zero, see, e.g., [11, Theorem 1.1].

The reason for this approach is the following result, by now standard and being used by many authors.

Proposition 3.1. *Let $T\colon A \to B$ be a bounded linear operator between the C*-algebras A and B. Suppose that A has real rank zero. If T maps projections in A onto idempotents in B then T is a Jordan homomorphism, that is, $T(x^2) = (Tx)^2$ for all $x \in A$.*

The idea of the argument is as follows. If $p \in A$ is a projection then, by assumption, $Tp \in B$ is an idempotent. If $q \in A$ is a projection orthogonal to p, then an easy argument shows that the idempotent Tq is orthogonal to Tp. Hence, if $a \in A$ is of the form $a = \sum_{j=1}^{n} \lambda_j p_j$ for some scalars λ_j and finitely many mutually orthogonal projections p_j, then

$$T(a^2) = T\big(\sum_{j=1}^{n} \lambda_j^2 p_j\big) = \sum_{j=1}^{n} \lambda_j^2 Tp_j = (Ta)^2.$$

The assumption on A to have real rank zero amounts to the fact that every selfadjoint element can be approximated by elements of the above form; hence the continuity of T entails the Jordan property on A_{sa}. Finally, the cartesian decomposition $x = a + ib$, $a, b \in A_{sa}$ completes the argument. For more details see [13, Lemma 2.1].

Combining Proposition 3.1 with property (3) above opens up the way to deal with spectral isometries.

Corollary 3.2. *Let $T\colon A \to B$ be a unital surjective spectral isometry between the unital C*-algebras A and B. If A has real rank zero and T maps projections in A onto idempotents in B then T is a Jordan isomorphism.*

We now state the result which, to our knowledge, is the most general so far.

Theorem 3.3. *Let $T\colon A \to B$ be a unital surjective spectral isometry between the unital C*-algebras A and B. If either*

(i) *A is a von Neumann algebra without direct summand of type* II_1

or

(ii) *A is a simple C*-algebra with real rank zero and without tracial states*

then T is a Jordan isomorphism.

Outline of proof. In view of Corollary 3.2 our aim is to show that, whenever $p \in A$ is a projection, then Tp is an idempotent. Let $q = 1 - p$ and suppose, without loss of generality, that $p \neq 0 \neq q$. If A satisfies the assumptions in (ii), then every element in the subalgebras pAp and qAq is a finite sum of elements of square zero. This follows from results by Marcoux, Pop and Zhang, see [11]. If A is a properly infinite von Neumann algebra, then we can reduce to the case where both pAp and qAq are properly infinite and, by using results due to Pearcy and Topping, obtain the same statement; for details see [13].

Hence, there are finitely many $a_i \in pAp$, $b_j \in qAq$ such that $p = \sum_i a_i$, $q = \sum_j b_j$, and $a_i^2 = b_j^2 = 0$ for all i, j. We claim that

$$(Tp)(Tq) + (Tq)(Tp) = 0 \tag{3.1}$$

which implies that

$$2\,(Tp - (Tp)^2) = (Tp)(1 - Tp) + (1 - Tp)(Tp) = 0,$$

as $T1 = 1$. Consequently, Tp is idempotent.

Since $(a_i + b_j)^2 = 0$ for all i, j, property (4) above entails that $(T(a_i + b_j))^2 = 0$ for all i, j. On the other hand,

$$(T(a_i{+}b_j))^2 = (Ta_i)^2{+}(Ta_i)(Tb_j){+}(Tb_j)(Ta_i){+}(Tb_j)^2 = (Ta_i)(Tb_j){+}(Tb_j)(Ta_i),$$

wherefore $(Ta_i)(Tb_j) + (Tb_j)(Ta_i) = 0$ for all i, j. Summing over all indices yields the claim (3.1).

If A is a general von Neumann algebra we write it in its type decomposition, but under the hypothesis (i), we can assume that the type II_1 part is absent:

$$A = A_{\mathrm{I}_{fin}} \oplus A_{\mathrm{I}_\infty} \oplus A_{\mathrm{II}_\infty} \oplus A_{\mathrm{III}}.$$

Now comes an important step. Each of the direct summands above is of the form eA for some central projection e in A. By Proposition 2.4 we know that $f = Te$ is a central projection in B. But, in addition, $T(ex) = (Te)(Tx)$ for all $x \in A$ and so T maps the C*-subalgebra eA onto the C*-subalgebra fB. It follows that T restricts to a unital surjective spectral isometry from eA onto fB. This is obtained in [14]. As a result, we can treat each of the parts separately, since T will be a Jordan isomorphism if and only if each of the restrictions is.

The last three summands we already dealt with as they are properly infinite; so it remains to cover the finite type I case. In other words, we can assume that A is of the form $A = \prod_{n\in\mathbb{N}} C(X_n, M_n)$, where each X_n is a hyperstonean space and M_n denotes the complex $n \times n$ matrices. Since each of the von Neumann subalgebras $C(X_n, M_n)$ once again is of the form eA for a central projection $e \in A$, we can employ the same reduction argument as above in order to assume that, in fact, $A = C(X, M_n)$ for some hyperstonean space X and some $n \in \mathbb{N}$.

Since the centre $Z(A)$ is isomorphic to $C(X)$ and is generated by its projections, an argument as in Proposition 3.1 gives us the identity $T(zx) = (Tz)(Tx)$ for all $z \in Z(A)$, $x \in A$ from the analogous identity for central projections e noted above. Let I be a Glimm ideal of A, that is, an ideal of the form $I = MA$ for a (unique) maximal ideal M of $Z(A)$. It follows that $J = TI$ is a Glimm ideal of B, since $TI = T(MA) = NB$, where $N = TM$ is a maximal ideal in $Z(B)$ by Proposition 2.3. Every Glimm ideal of A is in fact a maximal ideal, as it is of the form

$$I = \{f \in C(X, M_n) \mid f(t) = 0 \text{ for some } t \in X\},$$

and the quotient A/I is isomorphic to M_n. The induced unital mapping $\hat{T}\colon A/I \to B/J$ turns out to be a spectral isometry onto B/J, by [14, Proposition 9]. Since $\dim A/I = n^2$ and T is a linear isomorphism, B/J is a finite-dimensional C^*-algebra of dimension n^2 with trivial centre (which is isomorphic to $Z(A/I) = \mathbb{C}$). Consequently, $\hat{T}$ in fact is a unital surjective spectral isometry from M_n to M_n. Each such spectral isometry has been shown to be a Jordan isomorphism in [3, Proposition 2]. Since the Glimm ideals separate the points, it finally follows that T is a Jordan isomorphism, and the proof is complete. □

Slight extensions beyond the situation of C^*-algebras covered by condition (ii) in Theorem 3.3 are possible, but do not give insight into the open unknown cases. These are, on the one hand, C^*-algebras not of real rank zero; here, even the case $C([0,1], M_n)$ appears to be open at the time of this writing, and on the other hand, finite von Neumann algebras; e.g., the case of the hyperfinite II_1 factor is still unsettled. It is intriguing that the non-selfadjoint version of Kadison's theorem needs, at least at present, different techniques for different types of algebras whereas the characterisation of onto isometries allows for such an elegant and comprehensive proof.

Acknowledgements. A talk on the above topic with the title *On 'non-selfadjoint' isometries between C^*-algebras* was given at the Fejér–Riesz Conference in Eger on 10 June 2005. The author gratefully acknowledges the hospitality of the Eszterházy Károly College.

References

[1] P. Ara, M. Mathieu, Local multipliers of C^*-algebras, Springer-Verlag, London, 2003.

[2] B. Aupetit, A primer on spectral theory, Springer-Verlag, New York, 1991.

[3] B. Aupetit, Spectral characterisation of the radical in Banach and Jordan–Banach algebras, *Math. Proc. Camb. Phil. Soc.*, Vol. 114 (1993), 31–35.

[4] B. Aupetit, Spectrum-preserving linear mappings between Banach algebras or Jordan–Banach algebras, *J. London Math. Soc.*, Vol. 62 (2000), 917–924.

[5] B. Aupetit, M. Mathieu, The continuity of Lie homomorphisms, *Studia Math.*, Vol. 138 (2000), 193–199.

[6] M. Brešar, M. Mathieu, Derivations mapping into the radical, III, *J. Funct. Anal.*, Vol. 133 (1995), 21–29.

[7] P. Civin, B. Yood, Lie and Jordan structures in Banach algebras, *Pac. J. Math.*, Vol. 15 (1965), 775–797.

[8] R. V. Kadison, Isometries of operator algebras, *Annals of Math.*, Vol. 54 (1951), 325–338.

[9] I. Kaplansky, Algebraic and analytic aspects of operator algebras, CBMS Series Vol. 1, Amer. Math. Soc., Providence, RI, 1970.

[10] M. Mathieu, Spectrally bounded traces on C^*-algebras, *Bull. Austral. Math. Soc.*, Vol. 68 (2003), 169–173.

[11] M. Mathieu, Spectrally bounded operators on simple C^*-algebras, II, *Irish Math. Soc. Bull.*, Vol. 54 (2004), 33–40.

[12] M. Mathieu, G. J. Schick, First results on spectrally bounded operators, *Studia Math.*, Vol. 152 (2002), 187–199.

[13] M. Mathieu, G. J. Schick, Spectrally bounded operators from von Neumann algebras, *J. Operator Theory*, Vol. 49 (2003), 285–293.

[14] M. Mathieu, A. R. Sourour, Hereditary properties of spectral isometries, *Arch. Math.*, Vol. 82 (2004), 222–229.

[15] V. I. Paulsen, Completely bounded maps and operator algebras, Cambridge Univ. Press, Cambridge, 2002.

[16] V. Pták, Derivations, commutators and the radical, *Manuscripta math.*, Vol. 23 (1978), 355–362.

Martin Mathieu
Department of Pure Mathematics
Queen's University Belfast
Belfast BT7 1NN
Northern Ireland

Annales Mathematicae et Informaticae
32 *(2005) pp. 95–123.*

Construction of ECT-B-splines, a survey*

Günter W. Mühlbach[a] **and Yuehong Tang**[b]

[a]Institut für Angewandte Mathematik, Universität Hannover, Germany
e-mail: mb@ifam.uni-hannover.de
[b]Department of Mathematics, Nanjing University of Aeronautics and Astronautics
P. R. China

Abstract

s-dimensional *generalized polynomials* are linear combinations of functions forming an ECT-system on a compact interval with coefficients from $\mathbb{R}^s$. *ECT-spline curves in* $\mathbb{R}^s$ are constructed by glueing together at interval endpoints generalized polynomials generated from different local ECT-systems via connection matrices. If they are nonsingular, lower triangular and totally positive there is a basis of the space of 1-dimensional ECT-splines consisting of functions having minimal compact supports normalized to form a nonnegative partition of unity. Its functions are called *ECT-B-splines*. One way (which is semiconstructional) to prove existence of such a basis is based upon zero bounds for ECT-splines. A constructional proof is based upon a definition of ECT-B-splines by generalized divided differences extending Schoenberg's classical construction of ordinary polynomial B-splines. This fact eplains why ECT-B-splines share many properties with ordinary polynomial B-splines. In this paper we survey such constructional aspects of ECT-splines which in particular situations reduce to classical results.

Key Words: ECT-systems, ECT-B-splines, ECT-spline curves, de-Boor algorithm

AMS Classification Number: 41A15, 41A05

1. ECT-systems and their duals, rET- and lET-systems

Let J be a nontrivial compact subinterval of the real line $\mathbb{R}$. A system of functions $U = (u_1, \dots, u_n)$ in $C^{n-1}(J; \mathbb{R})$ is called an *extended Tchebycheff system*

*supported in part by INTAS 03-51-6637

(*ET-system*, for short) *of order* n *on* J provided for all $T = (t_1, \ldots, t_n)$, $t_1 \le \ldots \le t_n$, $t_j \in J$,

$$V \left| \begin{matrix} u_1, \ldots, u_n \\ t_1, \ldots, t_n \end{matrix} \right|_r := \det \ (D_r^{\nu_j} u_i(t_j))\,|_{i,j=1,\ldots,n} > 0$$

with

$$\nu_j := \ \max\{l : t_j = t_{j-1} = \ldots = t_{j-l} \ge t_1\}, \quad j = 1, \ldots, n, \tag{1.1}$$

where $Df(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ denotes the operator of differentiation appropriately one sided at an endpoint of J. Then span U will be called an *ET-space of dimension* n *on* J. An ET-system $U = (u_1, \ldots, u_n)$ is called *complete* or an *ECT-system* provided $(u_1, \ldots, u_k)$ is an ET-system of order k on J for $k = 1, \ldots, n$.

The following characterization of ECT-systems is well known [8] p. 376f, [30] p. 364:

Theorem 1.1. *Let* $u_1, \ldots, u_n$ *be of class* $C^{n-1}(J;\mathbb{R})$. *Then the following assertions are equivalent:*

(i) $(u_1, \ldots, u_n)$ *is an ECT-system of order* n *on* J.

(ii) All Wronskian determinants

$$W(u_1, \ldots, u_k)(x) = det \ \left(D^{j-1}u_i(x)\right)_{i=1,\ldots,k}^{j=1,\ldots,k} > 0 \quad k = 1, \ldots, n;\ x \in J$$

are positive on J.

(iii) There exist positive weight functions $w_j \in C^{n-j}(J;\mathbb{R})$, $j = 1, \ldots, n$, *and for every* $c \in J$ *coefficients* $c_{j,i} \in \mathbb{R}$ *such that*

$$u_j(x) = w_1(x) \cdot \int_c^x w_2(t_2) \int_c^{t_2} w_3(t_3) \int_c^{t_3} \ldots \int_c^{t_{j-1}} w_j(t_j)\, dt_j \ldots dt_2 \tag{1.2}$$
$$+ \sum_{i=1}^{j-1} c_{j,i} \cdot u_i(x), \quad j = 1, \ldots, n; \quad x \in J.$$

Clearly, the functions $s_j(x,c) := u_j(x) - \sum_{i=1}^{j-1} c_{j,i} \cdot u_i(x) \quad j = 1, \ldots, n$ satisfy

$$s_j(x,c) = w_1(x) \cdot h_{j-1}(x, c; w_2, \ldots, w_j) \quad j = 1, \ldots, n \tag{1.3}$$

where $h_0(x,c) := 1$ and for $1 \le m \le n$

$$h_m(x, c; w_1, \ldots, w_m) := \int_c^x w_1(t) \cdot h_{m-1}(t, c; w_2, \ldots, w_m) dt.$$

The system (1.3) $(s_1, \ldots, s_n)$ forms a special basis of span$\{u_1, \ldots, u_n)$ which we call an ECT-system in *canonical form with respect to c*.

Example 1.2. If $w_j = \mathbf{1}$ for $j = 1, \dots, n$ where $\mathbf{1}$ denotes the constant function equal to one then

$$h_m(x, c; \mathbf{1}, \dots, \mathbf{1}) = \frac{(x-c)^m}{m!} \qquad m = 0, \dots, n \quad \text{and}$$
$$s_j(x, c) = \frac{(x-c)^{j-1}}{(j-1)!} \qquad j = 1, \dots, n$$

and $\operatorname{span}\{s_1, \dots, s_n\} = \pi_{n-1}$, the space of ordinary polynomials of degree $n-1$ or of order n at most.

Example 1.3. (cf. also [3], [31]) If $n \geq 3$ and $w_j = \mathbf{1}$ for $j = 1, \dots, n-2$,

$$w_{n-1}(x) = \frac{(n-2)!}{(x-a+\varepsilon)^{n-1}}, \quad w_n(x) = \frac{(n-1)(b-a+2\varepsilon)(x-a+\varepsilon)^{n-2}}{(b+\varepsilon-x)^n}$$

with $\varepsilon > 0$ a parameter, then for any $c \in [a, b]$

$$s_j(x, c) = \frac{(x-c)^{j-1}}{(j-1)!}, \qquad j = 1, \dots, n-2 \tag{1.4}$$
$$s_{n-1}(x, c) = \frac{(x-c)^{n-2}}{(x-a+\varepsilon)(c-a+\varepsilon)^{n-2}} \tag{1.5}$$
$$s_n(x, c) = \frac{(x-c)^{n-1}(b-a+2\varepsilon)}{(x-a+\varepsilon)(b+\varepsilon-x)(b+\varepsilon-c)^{n-1}} \tag{1.6}$$

is a *Cauchy-Vandermonde-system* in canonical form with respet to c whose first $n-2$ functions are polynomials and the last two are proper rational functions, s_{n-1} having a pole of order 1 at $x = a - \varepsilon$ and s_n having poles of order 1 at $x = a - \varepsilon$ and at $x = b + \varepsilon$.

Associated with an ECT-system (1.2) or (1.3) are the linear differential operators

$$D_0 u = u, \quad D_j u = D\Big(\frac{u}{w_j}\Big) \qquad j = 1, \dots, n$$
$$\hat{L}_j u = D_j \cdots D_0 u \qquad j = 0, \dots, n$$
$$L_j u = \frac{1}{w_{j+1}} \hat{L}_j u \qquad j = 0, \dots, n-1.$$

For $\mu \in \mathbb{N}$ by $\boldsymbol{L}[f](t) := (L_0 f(t), \dots, L_{\mu-1} f(t))^T$ we denote *the ECT-derivative vector of dimension* μ of a sufficiently smooth function f. Also, we will use the limits

$$\boldsymbol{L}^\mu[f](t-) := \lim_{\tau \to t-0} \boldsymbol{L}^\mu[f](\tau), \qquad \boldsymbol{L}^\mu[f](t+) := \lim_{\tau \to t+0} \boldsymbol{L}^\mu[f](\tau).$$

Obviously, $\ker \hat{L}_j = \operatorname{span}\{u_1, \dots, u_j\}, \quad j = 1, \dots, n, \quad$ and

$$L_j s_{j+1}(x, c) = 1 \qquad j = 0, \dots, n-1,$$

$$L_j s_{l+1}(c,c) = \delta_{j,l} \qquad j,l = 0,\dots,n-1.$$

There is a Taylor's Theorem with respect to ECT-systems. The initial value problem

$$\begin{aligned} \hat{L}_n u(x) &= f(x), & x &\in J \\ L_j u(c) &= c_j, & j &= 0,\dots,n-1, \end{aligned}$$

with $f \in C(J;\mathbb{R})$ and $c_j \in \mathbb{R}$ given, has the solution

$$u(x) = \sum_{j=0}^{n-1} c_j s_{j+1}(x,c) + \int_c^x f(t) s_n(x,t)dt. \tag{1.7}$$

Associated with any ECT-system $U = (s_j)_{j=1}^n$ of order n on J in canonical form with respect to $c \in J$ with weights $w_1,\dots,w_n$ its *dual canonical system* $U^* = (s_i^*)_{i=1}^n$ *with respect to* $c \in J$ is defined by

$$s^*_{j,n}(x,c) := h_{j-1}(x,c;w_n,\dots,w_{n+2-j}) \qquad j = 1,\dots,n. \tag{1.8}$$

It is again an ECT-system of order n on J with weights $(w_1^*,\dots,w_n^*) = (\mathbf{1}, w_n,\dots,w_2)$ provided

$$w_j \in C^{\max\{n-j,j-2\}}(J;\mathbb{R}), \qquad j = 2,\dots,n. \tag{1.9}$$

Assuming this, with the dual canonical ECT-system with respect to c associated are the linear differential operators

$$\begin{aligned} &D_0^* f = f, \quad D_1^* f = Df, \quad D_j^* f = D\left(\frac{f}{w_{n+2-j}}\right), & j &= 2,\dots,n \\ &\hat{L}_j^* f = D_j^* \cdots D_0^* f, & j &= 0,\dots,n, \\ &L_0^* f = f, \quad L_j^* f = \frac{1}{w_{n+1-j}} \hat{L}_j^* f, & j &= 1,\dots,n. \end{aligned}$$

The function

$$g(x,y) := \begin{cases} w_1(x) h_{n-1}(x,y;w_2,\dots,w_n) & x \ge y \\ 0 & \text{otherwise} \end{cases}$$

has the characteristic behaviour of a Green's function for the differential operator L_{n-1} acting on the varable x, i.e.

$$\begin{aligned} &L_{n-1} g_n(x,y)|_{x=y-} = 0 \\ &L_{n-1} g_n(x,y)|_{x=y+} = L_{n-1} s_n(x,y)|_{x=y} = 1. \end{aligned}$$

In particular, for $x,y,c \in J$

$$h(x,y) := s_n(x,y) = w_1(x) h_{n-1}(x,y;w_2,\dots,w_n)$$

$$= \sum_{k=1}^{n} (-1)^{n-k} s_k(x,c) s^*_{n+1-k,n}(y,c) \tag{1.10}$$
$$= (-1)^{n-1} w_1(x) h_{n-1}(y,x;w_n,\dots,w_2)$$
$$= (-1)^{n-1} w_1(x) s^*_{n,n}(y,x)$$

where the right hand side of (1.10) is independent of c [10].

Example 1.1. (continued) If $w_1 = \ldots = w_n = \mathbf{1}$, then $s^*_j(x,c) = \frac{(x-c)^{j-1}}{(j-1)!}$, $j = 1,\dots,n$, and (1.10) reduces to the Binomial Theorem

$$s_n(x,y) = h(x,y) = \frac{(x-y)^{n-1}}{(n-1)!} = \sum_{k=1}^{n} (-1)^{n-k} \frac{(x-c)^{k-1}}{(k-1)!} \cdot \frac{(y-c)^{n-k}}{(n-k)!}.$$

Example 1.2. (continued, cf. [31]) If $n \geq 3$ and the weight functions are taken as in Example 1.2 then $(w^*_1,\dots,w^*_n) = (\mathbf{1}, w_n, w_{n-1},\dots,w_2)$, and if for any $c \in [a,b]$

$$\gamma(k,n,c) := \frac{(n-2)!}{(k-3)!} (c-a+\varepsilon)^{k-1-n} \sum_{\kappa=0}^{k-3} \binom{k-3}{\kappa} \frac{(-1)^{k-3-\kappa}}{n-2-\kappa}$$
$$\delta(k,n) := (b-a+2\varepsilon) \frac{(n-1)!}{(k-3)!}$$
$$\lambda(\nu,n) := (-1)^{n-1-\nu} \binom{n-1}{\nu} (b-a+2\varepsilon)^{\nu}, \qquad 1 \leq \nu \leq n-1$$
$$\mu(k,n,\nu,c) := \frac{1}{\nu} \binom{k-3}{n-\nu-1} \sum_{i=0}^{\nu+k-n-2} (-1)^{k-i} \binom{\nu+k-n-2}{i} \cdot$$
$$\cdot (b-a+2\varepsilon)^i \frac{(c-a+\varepsilon)^{\nu+k-n-2-i}}{\nu-1-i}, \qquad n-k-2 \leq \nu \leq n-1$$
$$\psi_\nu := \psi_\nu(x,b,c,\varepsilon) := \frac{1}{(b+\varepsilon-x)^{\nu}} - \frac{1}{(b+\varepsilon-c)^{\nu}}, \quad 1 \leq \nu \leq n-1$$

then

$$s^*_{1,n}(x,c) = \mathbf{1}$$
$$s^*_{2,n}(x,c) = \sum_{\nu=1}^{n-1} \psi_\nu \cdot \alpha(2,n,\nu,c), \qquad \alpha(2,n,\nu,c) = \lambda(\nu,n) \tag{1.11}$$

and for $3 \leq k \leq n$

$$s^*_{k,n}(x,c) = \sum_{\nu=1}^{n-1} \psi_\nu \cdot \alpha(k,n,\nu,c) \tag{1.12}$$

where

$$\alpha(k,n,\nu,c)=\begin{cases}\gamma(k,n,c)\lambda(\nu,n) & 1\le\nu\le n-k+1\\ \gamma(k,n,c)\lambda(\nu,n)+\delta(k,n)\mu(k,n,\nu,c) & n-k+2\le\nu\le n-1.\end{cases}$$

The representations (1.11) and (1.12) are proved by calculating the integrals according to the definition of the dual system in its canonical form with respect to c. In example 1.2 according to (1.10)

$$h(x,y)=(-1)^{n-1}\cdot s^*_{n,n}(y,x).$$

Let J be a subinterval of the real line $\mathbb{R}$ that is open to the right. For $n\in\mathbb{N}_0$ let

$C_r^n(J;\mathbb{R}) := \{f\in C(J;\mathbb{R})$: for every $x\in J$ and for $\nu=1,\dots,n$ there exists the right derivative of f of order ν at x and $J\ni x\mapsto D_r^\nu f(x)$ is right continuous$\}$.

A system of functions $U=(u_1,\dots,u_n)$ in $C_r^{n-1}(J;\mathbb{R})$ is called a *right-sided extended Tchebycheff system* (*rET-system*, for short) *of order* n *on* J provided for all $T=(t_1,\dots,t_n)$, $t_1\le\dots\le t_n$, $t_j\in J$,

$$V\left|\begin{matrix}u_1,\dots,u_n\\ t_1,\dots,t_n\end{matrix}\right|_r := \det\ (D_r^{\nu_j}u_i(t_j))\,|_{i,j=1,\dots,n}>0$$

with ν_j defined by (1.1) where $D_rf(x):=\lim_{h\to 0+}\frac{f(x+h)-f(x)}{h}$ denotes the operator of ordinary right differentiation. Then span U will be called an *rET-space of dimension* n *on* J.

If $q\in$ span U where U is an rET-system of order n on J, a point $x_0\in J$ is called a *zero of* q *of right multiplicity* ν_0 iff $q(x_0)=0, D_r^1q(x_0)=0,\dots,D_r^{\nu_0-1}q(x_0)=0, D_r^{\nu_0}q(x_0)\neq 0$.

The following characterization of rET-spaces is an immediate consequence of the Alternative Theorem of Linear Algebra, as is the corresponding well known characterization for ET-spaces (cf. [8], p. 376).

Theorem 1.4. *(i)* $(u_1,\dots,u_{n-1},u_n)$ *or* $(u_1,\dots,u_{n-1},-u_n)$ *is an rET-sytem of order* n *on* J.

(ii) Every nontrivial element of span$(u_1,\dots,u_n)$ *has at most* $n-1$ *zeros in* J *counting right multiplicities.*

(iii) Every problem of right sided Hermite interpolation

$$H(U,T+,f):\quad\left\{\begin{array}{l}\text{given points } t_1\le\dots\le t_n \text{ in } J,\\ \text{given } f\in C_r^{n-1}(J;\mathbb{R}),\\ \text{find } q\in \text{ span } U \text{ such that}\\ D_r^{\nu_j}q(t_j)=D_r^{\nu_j}f(t_j)\qquad j=1,\dots,n\end{array}\right.\tag{1.13}$$

has a unique solution.

Analogously, *left sided ET-systems* and *lET-spaces* and related concepts as the problem of *left sided Hermite interpolation* $H(U, T-, f)$ are defined. In the analysis of dual functionals to ECT-B-splines naturally certain rET-and lET-spaces arise (see (5.1) and (5.2) below) that are no ET-spaces.

If $U = (u_1, \ldots, u_n)$ is an rET-system on J then the leading coefficient (that before u_n) of the unique $q \in$ span U that solves $H(U, T+, f)$ is called the *right sided generalized divided difference of* f with respect to $u_1, \ldots, u_n$ and with nodes $t_1, \ldots, t_n$. By Cramer's rule it is

$$\begin{bmatrix} u_1, \ldots, u_n \\ t_1, \ldots, t_n \end{bmatrix}_r f = \frac{V \begin{vmatrix} u_1, \ldots, u_{n-1}, f \\ t_1, \ldots, t_{n-1}, t_n \end{vmatrix}_r}{V \begin{vmatrix} u_1, \ldots, u_{n-1}, u_n \\ t_1, \ldots, t_{n-1}, t_n \end{vmatrix}_r}.$$

Developing the numerator determinant along its last column one sees

$$\begin{bmatrix} u_1, \ldots, u_n \\ t_1, \ldots, t_n \end{bmatrix}_r f = \sum_{j=1}^{n} c_j \cdot D_r^{\nu_j} f(t_j), \quad c_n = \frac{V \begin{vmatrix} u_1, \ldots, u_{n-1} \\ t_1, \ldots, t_{n-1} \end{vmatrix}_r}{V \begin{vmatrix} u_1, \ldots, u_{n-1}, u_n \\ t_1, \ldots, t_{n-1}, t_n \end{vmatrix}_r} \tag{1.14}$$

with coefficients c_j that do not depend on f.

For lET- or ET-systems we use similar notations with the suffix r replaced by l or omitted, respectively.

It is known [17] that if $(u_1, \ldots, u_{n+1})$, $(u_1, \ldots, u_n)$ are ECT-systems, and, if $n \geq 2$, also $(u_1, \ldots, u_{n-1})$ is an ECT-system, then if $t_1 \neq t_{n+1}$

$$\begin{bmatrix} u_1, \ldots, u_{n+1} \\ t_1, \ldots, t_{n+1} \end{bmatrix} f = \frac{\begin{bmatrix} u_1, \ldots, u_n \\ t_2, \ldots, t_{n+1} \end{bmatrix} f - \begin{bmatrix} u_1, \ldots, u_n \\ t_1, \ldots, t_n \end{bmatrix} f}{\begin{bmatrix} u_1, \ldots, u_n \\ t_2, \ldots, t_{n+1} \end{bmatrix} u_{n+1} - \begin{bmatrix} u_1, \ldots, u_n \\ t_1, \ldots, t_n \end{bmatrix} u_{n+1}}.$$

This formula holds for the right or left sided generalized divided differences as well [25].

2. rECT-splines; the spaces $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ and $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{\mathbf{ext}})$

Assume that x is a real number and that in nontrivial closed intervals $J_0 = [a, x]$ and $J_1 = [x, b]$ left and right to x there are given two ECT-systems of order n

$$U^{[0]} := U_n^{[0]} := (u_1^{[0]}, \ldots, u_n^{[0]}), \quad U^{[1]} := U_n^{[1]} := (u_1^{[1]}, \ldots, u_n^{[1]}),$$

with weights $w_j^{[i]}$ $(j = 1, \ldots, n;\ i = 0, 1)$ and associated linear differential operators $L_j^{[i]}$ $(j = 0, \ldots, n-1;\ i = 0, 1)$, correspondingly. Suppose that μ is an integer, $0 \leq \mu \leq n$, and that A is a square $(n-\mu)-$dimensional real matrix which is nonsingular. A function $s : [a, b] \mapsto \mathbb{R}$ such that $s|_{[a,x)} \in$ span $U^{[0]}$ and $s|_{[x,b]} \in$ span $U^{[1]}$ and

$$\boldsymbol{L}^{[1]n-\mu}[s](x+) = A \cdot \boldsymbol{L}^{[0]n-\mu}[s](x-) \tag{2.1}$$

where $\boldsymbol{L}^{[i]n-\mu}[s](t)$ $(i=0,1)$ denote the ECT-derivative vectors of s at t of dimension $n-\mu$ is called $(U^{[0]},U^{[1]},A)-$*smooth of order* $n-\mu$ *at* x. The equations (2.1) are called the *connection equations of* s *at the knot* x and A is called a *connection matrix* at x. We allow $0\le\mu\le n$ where in case $\mu=n$ there is no condition on s at x. In case $\mu=0$ the knot x is a knot with no freedom. If $1\le\mu\le n$ at x, given s on $[a,x)$, there are μ degrees of freedom in extending s to $[x,b]$ as a function belonging to span $U^{[1]}$ such that $s\in C_r^{n-1}([a,b];\mathbb{R})$. Symmetrically, if $1\le\mu\le n$ at x, given s on $(x,b]$, there are μ degrees of freedom in extending s to $[a,x]$ as a function belonging to span $U^{[0]}$ such that $s\in C_l^{n-1}([a,b];\mathbb{R})$.

It should be observed that $(U^{[0]},U^{[1]},A)-$smoothness in general does not imply smoothness in the ordinary sense. But it is not hard to give conditions that a function being $(U^{[0]},U^{[1]},A)-$smooth at x of order $n-\mu$ is smooth at x of order m in the usual sense [31].

Let $[a,b]\subset\mathbb{R}$ be either a nontrivial compact interval or the real line. By X we denote a finite or a bi-infinite partition of $[a,b]$ respectively, i.e.

$$\begin{aligned}&X=\{x_0,\dots,x_{k+1}\}\quad\text{with}\quad a=x_0<x_1<\dots<x_{k+1}=b\quad\text{or}\\&X=(x_i)_{i\in\mathbb{Z}}\ \text{with}\ \dots<x_{-1}<x_0<x_1<\dots\ \text{and}\ \lim_{i\to\pm\infty}x_i=\pm\infty.\end{aligned}$$

The points of X which are not endpoints are called *inner knots* and endpoints are called *auxiliary knots*. The index sets for inner knots are

$$K_X:=\begin{cases}\{1,\dots,k\} & \text{if } X=\{x_0,\dots,x_{k+1}\}\\ \mathbb{Z} & \text{if } X=(x_i)_{i\in\mathbb{Z}}.\end{cases}$$

In any case by $\Delta=(J_i)$, $\;J_i:=[x_i,x_{i+1})$ and $\check\Delta=(\check J_i)$, $\;\check J_i:=(x_i,x_{i+1}]$ for all i except the last resp. first we denote the corresponding partition of $[a,b]$ into subintervals called *r- resp. l-knot intervals* where in case of a finite partition of a compact interval the last $r-$ resp. first $l-$knot interval is $J_k:=[x_k,x_{k+1}]$ resp. $\check J_0=[x_0,x_1]$.

Assume that on each closed interval $\bar J_i=[x_i,x_{i+1}]$ the system

$$U_n^{[i]}=\left(u_1^{[i]},\dots,u_n^{[i]}\right)\tag{2.2}$$

is an ECT-system of order n with associated weight functions

$$w_j^{[i]}\in C^{n-j}(\bar J_i;(0,\infty)),\quad j=1,\dots,n\tag{2.3}$$

and associated linear differential operators $L_j^{[i]}$ and ECT-derivative vectors $\boldsymbol{L}^{[i]\mu}[f](t)=\left(L_0^{[i]}f(t),\dots,L_{\mu-1}^{[i]}f(t)\right)^T$ of dimension μ.

By $\mathcal{U}=\mathcal{U}_n=\left(U^{[i]}\right)_i$ we denote the sequence of ECT-systems. Assume that corresponding to the inner knots we are given a sequence of integers $M=(\mu_i)$, $\;0\le\mu_i\le n$, and a sequence of nonsingular matrices

$$\mathcal{A}=\mathcal{A}_n=(A^{[i]}),\qquad A^{[i]}\in\mathbb{R}^{(n-\mu_i)\times(n-\mu_i)}.$$

A function $s : [a,b] \mapsto \mathbb{R}$ is called an *rECT-* resp. *lECT-spline function on* $[a,b]$ *with respect to the generating sequences* $\mathcal{U}, \mathcal{A}, M, X$ provided

$$\begin{aligned} &s|_{J_i} \in \text{ span } U^{[i]} \text{ resp. } s|_{\check{J}_i} \in \text{ span } U^{[i]} \text{ for all } i \text{ and} \\ &s \text{ is } (U^{[i-1]}, U^{[i]}, A^{[i]})\text{–smooth at } x_i \text{ for all inner knots.} \end{aligned} \tag{2.4}$$

The sets of all such functions will be denoted by $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ and $\check{\mathcal{S}}_n(\mathcal{U}, \mathcal{A}, M, X)$, respectively.

Clearly, every rECT-spline function is right continous everywhere and jumps may occur only at the knots. If all ECT-systems $U^{[i]}$ have the first weight function

$$w_1^{[i]}(x) = \mathbf{1} \qquad x \in \bar{J}_i, \quad \text{for all } i \tag{2.5}$$

and all connection matrices $A^{[i]}$ have the form

$$A^{[i]} = \text{ diag } (1, \bar{A}^{[i]}) \tag{2.6}$$

where $\mu_i \leq n-1$ and $\bar{A}^{[i]} \in \mathbb{R}^{(n-1-\mu_i)\times(n-1-\mu_i)}$ is nonsingular for all i then $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ and $\check{\mathcal{S}}_n(\mathcal{U}, \mathcal{A}, M, X) \subset C([a,b];\mathbb{R})$ and both spaces contain the constant functions.

In the sequel we shall treat rECT-spaces only. Clearly, every result for rECT-splines has an analogue for lECT-splines.

Under the assumptions (2.5), (2.6) and that $\mathcal{A} = \mathcal{A}^+ := (A^{[i]})_i$ where for every i the connection matrix

$$A^{[i]} \text{ is nonsingular, lower triangular, totally positive} \tag{2.7}$$

it is possible to construct for the space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ a local support basis (N_j) that is normalized to form a nonnegative partition of unity. In order to give the definitions the following notation is usefull. For any partition $X = (x_i)$ of $[a,b]$, finite or biinfinite, with corresponding sequence of multiplicities of inner knots $M = (\mu_i)$ such that $1 \leq \mu_i \leq n$ for all i, we denote by $\boldsymbol{\xi}$ resp. by $\boldsymbol{\xi}_{\text{ext}}$ the weakly increasing sequence of inner resp. of all knots where auxiliary knots by definition have multiplicity n, each repeated according to its multiplicity, the enumeration being fixed by the convention $\xi_1 = \xi_2 = \ldots = \xi_{\mu_1} = x_1$. In this case we will also use the notation $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) = \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{\text{ext}})$.

By $\varphi : j \mapsto i_j$ we denote the mapping which assigns to each ξ_j the unique knot x_{i_j} such that $\xi_j = x_{i_j}$. Then

$$X = \varphi(\boldsymbol{\xi}_{\text{ext}}),\ M_{\text{ext}} := (\mu_i) \text{ with } \mu_i = \text{card } \varphi^{-1}(\{x_i\}).$$

It will be convenient to use the index set

$$J_\varphi = J_\varphi^n := \begin{cases} \{-n+1, \ldots, \mu\} & \text{if } [a,b] \text{ is compact} \\ \mathbb{Z} & \text{if } [a,b] = \mathbb{R}. \end{cases}$$

Observe that the sequences $\boldsymbol{\xi}$ or $\boldsymbol{\xi}_{\text{ext}}$ are well defined as nonvoid sequences of $\mu := \sum \mu_i$ terms also in case $0 \le \mu_i \le n$ for all i provided $1 \le \mu_1 \le n$. Only in case that all inner knots have multiplicities zero, $M = (0)_i$, we have $\boldsymbol{\xi} = (\ )$, a void sequence.

Remark 2.1. The space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ was introduced by Barry [1], p. 396. Barry has constructed de Boor-Fix functionals first and used them to derive existence of a local support basis for this space. ECT-splines are studied from a blossom point of view by Mazure [13],[14],[16] and Pottmann [15],[27] and more recently by Prautzsch [28], and from a constructive point of view by Mühlbach [23],[24]. Cardinal ECT-splines with simple knots are discussed in [31].

Remark 2.2. If $U^{[i]} = U_n|_{\bar{J}_i}$ where U_n is a fixed global ECT-system of order n on $[a, b]$ and $A^{[i]}$ is the $(n - \mu_i)-$dimensional identity matrix then $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ is the space of *Tchebycheff splines* of order n on $[a, b]$ with knots $x_1, \ldots, x_k$ of multiplicities $\mu_1, \ldots, \mu_k$, respectively.

Remark 2.3. If $U^{[i]} = (1, x, \ldots, x^{n-1})|_{\bar{J}_i}$ for $i = 0, \ldots, k$ then $\mathcal{S}_n = \mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X) = \mathcal{S}_n(x_1, \ldots, x_k | A^{[1]}, \ldots, A^{[k]})$ is the space of piecewise ordinary polynomials of order n generated by connection matrices $A^{[i]}$ considered by Dyn and Micchelli [4], p. 321, and by Barry et al [2]. If moreover each $A^{[i]}$ is an identity matrix then $\mathcal{S}_n$ is the well known Schoenberg space of ordinary polynomial spline functions of order n with knots x_i of multilicity μ_i, $i = 1, \ldots, k$.

According to the definitions given an rECT-spline $s \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ may be represented by

$$s = \sum_i \sum_{j=1}^{n} c_j^{[i]} \cdot u_j^{[i]}$$

meaning that

$$s|_{J_i} = \sum_{j=1}^{n} c_j^{[i]} \cdot u_j^{[i]}, \qquad \text{all } i$$

with coefficients $c_j^{[i]}$ $(j = 1, \ldots, n - \mu_i)$ that are related by the connection equations (2.4). There remain μ_i degrees of freedom for s right to x_i.

From this it is easily seen that with the usual pointwise defined algebraic operations $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ is a linear space over the reals whose dimension is

$$d = \dim\, \mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X) = n + \mu, \qquad \mu = \sum \mu_i \tag{2.8}$$

where the sum is extended over all inner knots. A basis generalizing the *truncated powers* is constructed as follows. It will be suficient to consider the case of a compact interval $[a, b]$. For $i = 0$ let $b_j(x)|_{J_0} := s_j^{[0]}(x, c)$, $j = 1, \ldots, n$, where

$s_1^{[0]}, \ldots, s_n^{[0]}$ is the prescribed ECT-system on $\bar{J}_0$ in its canonical form with respect to a fixed point c with $x_0 \le c \le x_1$. Then extend b_j to J_1 such that the extension satisfies the connection equations (2.4) at x_1. Since $A^{[1]}$ is nonsingular there is a μ_1-parameter family of such extensions. Actually, in extending the basic functions b_j to the right for every knot x_i we choose in the connection equations (2.4) the connection matrix of the form

$$C^{[i]} := \text{diag }(A^{[i]}, I_{\mu_i}) \in \mathbb{R}^{n\times n} \tag{2.9}$$

where I_ν denotes the identity matrix of dimension ν requiring

$$L_l^{[i]} b_j(x_i+) = L_l^{[i-1]} b_j(x_i-), \quad l = n-\mu_i, \ldots, n-1,\ i = 1, \ldots, k.$$

If $1 \le i \le k$ and $j = n + \sum_{l=1}^{i-1} \mu_l + m,\ \ m = 1, \ldots, \mu_i$, take $b_j(x)|_{J_i} = s^{[i]}_{n-\mu_i+m}(x, x_i)$ where $s_1^{[i]}, \ldots, s_n^{[i]}$ is the ECT-system on $\bar{J}_i$ in its canonical form with respect to $c = x_i$, extend b_j to the left by zero and to the right across each knot $x_p,\ i+1 \le p \le k$, via the connection equations (2.4) with the connection matrices (2.9). By construction, the functions $b_1, \ldots, b_d$ belong to $\mathcal{S}_n(\mathcal{U}, \mathcal{A}, M, X)$ and they are linearly independent on $[a, b]$. Since their cardinality equals the dimension of $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ we have constructed a basis of this space.

3. A zero bound for splines in $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$

We use the zero counting convention due to Goodman [6]). In this section and in the rest of the paper we make the basic assumptions (2.5),(2.6) and (2.7) which ensure, in particular, $\mathbf{1} \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) \subset C([a, b]; \mathbb{R})$.

Definition 3.1. Let $f \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ and $t \in (a, b)$. We set

$$f(t)^+ := \begin{cases} 1 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ is positive on } (t, t+\varepsilon) \\ 0 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ vanishes identically on } (t, t+\varepsilon) \\ -1 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ is negative on } (t, t+\varepsilon) \end{cases}$$

$$f(t)^- := \begin{cases} 1 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ is positive on } (t-\varepsilon, t) \\ 0 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ vanishes identically on } (t-\varepsilon, t) \\ -1 & \text{there exists } \varepsilon > 0 \text{ such that } f \text{ is negative on } (t-\varepsilon, t). \end{cases}$$

If f is not identically zero in some neighborhood of t then $f(t)^+ f(t)^- \neq 0$. In this case there exist nonnegative integers $l, r \le n-1$ such that

$$f(t-) = f'(t-) = \ldots = f^{(l-1)}(t-) = f(t+) = f'(t+) = \ldots = f^{(r-1)}(t+) = 0$$

and $f^{(l)}(t-) f^{(r)}(t+) \neq 0$. Let $q^* :=$ max (l, r). We say that f has a *point zero of multiplicity m at the point t* where

$$m = \begin{cases} q^* & \text{if } f(t)^- f(t)^+ (-1)^{q^*} > 0 \\ q^* + 1 & \text{if } f(t)^- f(t)^+ (-1)^{q^*} < 0 \end{cases}$$

As a consequence, $f(t)^+ f(t)^- = (-1)^m$. If $x_0 \le \alpha < \beta \le x_{k+1}$ we set $k(\alpha,\beta) := \sum_{\alpha < x_l < \beta} \mu_l$.

Definition 3.2. Let $f \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ and $a \le \alpha < \beta \le b$.

(i) If $f(\alpha)^- f(\beta)^+ \ne 0$ and $f(x) = 0$ for $\alpha < x < \beta$, then α and β are knots, $\alpha = x_p$ and $\beta = x_q$ with $0 < p < q < k+1$, and we say that f *has an interval zero* $[\alpha, \beta]$ *of multiplicity*

$$Z(f|[\alpha,\beta]) = n + 1 + k(\alpha,\beta).$$

(ii) If $f(x) = 0$ for all $a \le x < \beta$ while $f(\beta)^+ \ne 0$, then β is a knot, $\beta = x_q$ with $0 < q < k+1$, and we say that f *has an interval zero* $[a,\beta]$ *of multiplicity*

$$Z(f|[a,\beta]) = n + k(a,\beta).$$

(iii) If $f(x) = 0$ for all $\alpha < x \le b$ while $f(\alpha)^- \ne 0$, then α is a knot, $\alpha = x_p$ with $0 < p < k+1$, and we say that f *has an interval zero* $[\alpha, b]$ *of multiplicity*

$$Z(f|[\alpha,b]) = n + k(\alpha,b).$$

The total number of zeros of f in an interval J will be denoted by $Z(f|J)$.

Dyn and Micchelli [4], p. 324-327 have established a zero bound for polynomial splines via connection matrices as in remark 2.3 under the basic assumptions (2.6) and (2.7). A carefull examination of their proof shows that it can be adapted to the space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$. The reason is that also for ECT-spaces there holds a Budan-Fourier-Theorem [30], p. 371. From this as in [4] a Boudan-Fourier-Theorem for $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ can be derived (see theorem 3.3 of [25]), and this in turn yields the following

Theorem 3.3. *Let $f \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ with $X = (x_0, \ldots, x_{k+1})$ being a partition of a compact interval $[a,b]$. Under the basic assumptions* (2.5),(2.6) *and* (2.7) *if f is not identically zero then*

$$Z(f|[x_0, x_{k+1}]) \le n - 1 + \mu, \qquad \mu = \sum_{i=1}^{k} \mu_i.$$

For the particular case that all multiplicities are zero also Barry [1] has given this bound.

Corollary 3.4. *If $f \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ is not the zero function and vanishes identically on* $([a, x_1)$ *and on* $(x_k, b])$ *then*

$$Z(f|(x_1, x_k)) \le \max\{\mu - n - 1, 0\}.$$

It is the situation of corollary 3.4 that is needed for constructing a B-spline basis for the space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ or rECT-splines. For the space of piecewise ordinary polynomials of order n via totally positive connection matrices, Dyn and Micchelli [4] have constructed such a basis. Again, a careful inspection of their proof shows that it carries over to rECT-splines yielding the following theorem.

Theorem 3.5. *Suppose that $n \geq 2$ and $[a, b] \subset \mathbb{R}$ is compact. Under the basic assumptions (2.5),(2.6) and (2.7) with $1 \leq \mu_i \leq n - 1$ for $i = 1, \ldots, k$, then there is a basis $(N_j^n)_{j=-n+1}^{\mu}$ of the space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{ext})$ having the properties*

$$\begin{aligned}
&N_j(x) := N_j^n(x) := N_j(x|\xi_j, \ldots, \xi_{j+n}) && \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{ext})\\
&N_j(x) > 0 && x \in (\xi_j, \xi_{j+n})\\
&N_j(x) = 0 && x \notin [\xi_j, \xi_{j+n}]\\
&N_j^{(l)}(\xi_j+) = 0 \quad \text{for } l = 0, \ldots, n-1-\mu_j^+, && D_+^{n-\mu_j^+} N_j(\xi_j) > 0,\\
&N_j^{(l)}(\xi_{j+n}-) = 0 \quad \text{for } l = 0, \ldots, n-1-\mu_{j+n}^-, && D_-^{n-\mu_{j+n}^-} N_j(\xi_{j+n}) < 0,\\
&\sum_{j=-n+1}^{\mu} N_j(x) = 1 && x \in [a, b].
\end{aligned}$$

Here $\mu_j^{\pm} := \#\{l \geq 0 : \xi_j = \xi_{j \pm l}\}$ denote the right and left multiplicities of a knot ξ_j in the sequence $(\xi_l)_{l=-n+1}^{\mu+n}$.

Another proof of theorem 3.5 based upon right sided generalized divided differences can be found in [24]. It should be remarked that for arbitrary knot sequences total positivity of the connection matrices is a sufficient condition to ensure existence of a local support basis forming a nonnegative partition of unity. As shown by Mazure [14] it is not necessary. It is an open problem to give conditions which are necessary and sufficient for existence of such a basis. Given arbitrary nonsingular connection matrices, for Chebycheff splines local support bases forming a partition of unity exist for knot sequences which are dense in the set of all possible knot sequences, as is shown recently by Prautzsch [28].

4. Interpolation properties of the spline spaces $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$

Consider the spline space $\mathcal{S}_n = \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ with $X = (x_i)_{i=0}^{k+1}$ a partition of a compact interval $[a, b]$. Assume that there are given d *nodes* or *interpolation points* y_j,

$$Y = (y_1, \ldots, y_d) \quad \text{where} \quad x_0 \leq y_1 \leq y_2 \leq \ldots \leq y_d \leq x_{k+1}. \tag{4.1}$$

Here d denotes the dimension (2.8) of the space $\mathcal{S}_n$. Since its elements are continuous functions that are piecewise generalized polynomials of order n we assume that

$$\nu_j := \max\{l \geq 0 : y_j = y_{j-1} = \ldots = y_{j-l}\} \leq n-1, \quad j = 1, \ldots, d, \tag{4.2}$$

i.e. each node has multiplicity not greater than n.

For every node y_j there is a unique integer h such that

$$y_j = x_h, \quad h \in \{0, \ldots, k+1\} \quad \text{or} \quad y_j \in \mathrm{int} J_h, \quad h \in \{0, \ldots, k\}.$$

When $y_j = x_h$ with $h \in \{1, \ldots, k\}$ we suppose that

$$\nu_j + \mu_h \leq n-1, \qquad j = 1, \ldots, d. \tag{4.3}$$

This condition is called the *accumulation condition.* It allows that nodes are knots. Only finite endpoints or knots of multiplicity 0 may be nodes of multiplicity n. If a node y_j equals an inner knot x_h whose multiplicity μ_h is not zero then the accumulation condition guarantees that for every $f \in \mathcal{S}_n$ the rECT-derivative of highest order $L^{[h]}_{\nu_j} f(y_j+)$ does exist. Then also $D^{\nu_j}_+ f(y_j+)$ exists.

We consider the problem $H(\mathcal{S}_n, Y_+, f)$ of *right sided Hermite interpolation* (cf. (1.13))

$$H(\mathcal{S}_n, Y_+, f) : \quad \begin{cases} \text{given } y_1 \leq \ldots \leq y_d, \quad y_j \in [a,b], \\ \text{given } f \in C^N_+([a,b];\mathbb{R}) \text{ with } N = \max_{j=1,\ldots,d} \nu_j, \\ \text{find } s \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) \text{ such that} \\ D^{\nu_j}_+ s(y_j) = D^{\nu_j}_+ f(y_j), \; j = 1, \ldots, d. \end{cases}$$

The following theorem gives conditions which are necessary and sufficient for this problem to have a unique solution.

Theorem 4.1. *We make the assumptions* (2.5),(2.6),(2.7),(4.1),(4.2) *and* (4.3). *Then the following assertions are equivalent*

(i) $H(\mathcal{S}_n, Y+, f)$ *has a unique solution for every admissible function* f.

(ii)

$$y_i < \xi_i < y_{i+n} \qquad i = 1, \ldots, \mu, \quad \mu := \sum_{l=1}^{k} \mu_l.$$

(iii)

$$y_i \in M_i \qquad i = 1, \ldots, d,$$

where

$$M_i := \begin{cases} [x_0, \xi_i) & i = 1, \ldots, n \\ (\xi_{i-n}, \xi_i) & i = n+1, \ldots, d-n \\ (\xi_{i-n}, x_{k+1}] & i = d-n+1, \ldots, d. \end{cases}$$

The conditions (ii) and (iii) are called the *mixing conditions of the first* resp. *second kind.*

Theorem 4.1 generalizes in part the interpolation theorems of Schoenberg and Whitney [29] for ordinary polynomial splines with simple knots, of Karlin and Ziegler [9] for Chebycheffian splines with multiple knots and an interpolation theorem of Dyn and Micchelli [4] for polynomial splines via totally positive connection matrices. It is consistent with theorems 4.67 and 9.33 of Schumaker [30] on right sided Hermite interpolation by ordinary polynomial splines or by Tchebycheffian splines, respectively, since all our interpolation functions are continuous and the nodes satisfy the conditions (4.2) and (4.3). For the same reasons it is also consistent with the particular case $q = d$ of the more general result of Lyche and Schumaker [11] on modified Hermite interpolation by LB-splines.

In case $M = (0)$ when all inner knots have multiplicity zero the mixing conditions of both kinds are void. We then have

Corollary 4.2. *Under the assumptions of theorem 4.1 the space $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, (0), X)$ is an rET-space of order n consisting of continuous functions. It has a basis $p_1, \ldots, p_n$ such that*

$$V \begin{vmatrix} p_1, \ldots, p_n \\ y_1, \ldots, y_n \end{vmatrix}_+ := det \ \left(D_+^{\nu_j} p_i(y_j) \right) > 0$$

for all $y_1 \leq \ldots \leq y_n$ in $[a, b]$.

Corollary 4.3. *Under the assumptions of theorem 4.1 for all systems of nodes $y_1 \leq \cdots \leq y_d$ in $[a, b]$*

$$V := V \begin{vmatrix} b_1, \ldots, b_d \\ y_1, \ldots, y_d \end{vmatrix}_+ \geq 0$$

with strict inequality iff the mixing conditions hold. Here $b_1, \ldots, b_d$ is the basis of generalized truncated powers constructed in section 2.

Corollary 4.4. *Under the assumptions of theorem 4.1 and of theorem 3.2 we have*

$$V \begin{vmatrix} N_{-n+1}, \ldots, N_{d-n} \\ y_1, \ldots, y_d \end{vmatrix}_+ \geq 0 \tag{4.4}$$

with strict inequality iff the mixing conditions hold. Here $(N_j)_{j=-n+1}^{d-n}$ is the basis of theorem 3.5.

It is an open problem if the generalized Vandermonde matrix of (4.4) for simple nodes is totally positive as in the particular case of ordinary polynomial B-splines.

Corollary 4.5. *Under the assumptions of theorem 4.1 and assuming that each connection matrix can be partitioned according to $A^{[i]} = diag(1, A_1^{[i]}, A_2^{[i]})$ where $A_1^{[i]}$*

and $A_2^{[i]}$ *are square matrices of dimensions* $n-m-1$ *and* m*, respectively, that both satisfy* (2.7)*, then the space* $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, (0), X)$ *is an rET-space of order* n *that has an rET-subspace of order* $n-m$*. There is a basis* $p_1, \ldots, p_n$ *of* $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, (0), X)$ *such that*

$$V \begin{vmatrix} p_1, \ldots, p_{n-m} \\ y_1, \ldots, y_{n-m} \end{vmatrix}_r := det \ (D_r^{\nu_j} p_l(y_j)) > 0$$

for all $y_1 \leq \ldots \leq y_{n-m}$ *in* $[a, b]$*.*

Corollary 4.6. *Under the assumptions of corollary 4.5 every nontrivial* $f \in$ *span* $\{p_1, ..., p_{n-m}\} \subset \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, (0), X)$ *has at most* $n-m-1$ *zeros in* $[a, b]$*.*

Corollary 4.7. *Assuming* (2.5) *and that each connection matrix* $A^{[i]}$ *is a nonsingular positive diagonal matrix with* $a_{11}^{[i]} = 1$*, then the space* $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, (0), X)$ *is an rECT-space of order* n*, i.e. this space has a basis* $p_1, \ldots, p_n$ *such that for* $m = 1, \ldots, n$

$$V \begin{vmatrix} p_1, \ldots, p_m \\ y_1, \ldots, y_m \end{vmatrix}_r := det \ (D_r^{\nu_j} p_l(y_j)) > 0$$

for all $y_1 \leq \ldots \leq y_m$ *in* $[a, b]$*.*

In the situation of corollary 4.7 every interpolation problem $H(\mathcal{S}_n, Y+, f)$ can be solved recursively either using Newton's method via generalized divided differences [17], [18], [21], or using the generalized Neville-Aitken algorithm [19], [20]. This proves particular usefull in computing the spline weights recursively that occur in the recurrence relation for rECT-B-splines (see (7.5) below).

5. Pólya-polynomials and Marsden's identity generalized to ECT-splines

As in the preceding sections we adopt the general assumptions (2.5), (2.6) and (2.7). Let $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{\text{ext}}) = \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$ be an rECT-spline space as in section 2. Assuming for the weights of every local ECT-system (1.9) we set

$$\mathcal{C}_{\mathcal{A}^+} := (C^{[i]})_{i \in K_X} \qquad \text{with} \qquad C^{[i]} = \text{diag}(A^{[i]}, I_{\mu_i})$$
$$\mathcal{E}_{\mathcal{A}^+} := (E^{[i]})_{i \in K_X} \qquad \text{with} \qquad E^{[i]T} := R^{-1} \left(C^{[i]}\right)^{-1} R$$

where $R = R_n$ is the $n-$dimensional orthogonal matrix defined by

$$R^T := \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & (-1) & 0 \\ 0 & 0 & \ldots & (-1)^2 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ (-1)^{n-1} & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}.$$

We use the spaces

$$\mathcal{P}_n := \mathcal{P}_n(\mathcal{U}, \mathcal{C}_{\mathcal{A}^+}, X) := \mathcal{S}_n(\mathcal{U}, \mathcal{C}_{\mathcal{A}^+}, (0), X) := \{f : f \in C_r^{n-1}([a,b];\mathbb{R}), \tag{5.1}$$
$$f|_{J_i} \in \text{ span } U^{[i]},\ f \text{ is } (U^{[i-1]}, U^{[i]}, C^{[i]}) - \text{smooth at } x_i,\ i \in K_X\} \quad \text{and}$$
$$\mathcal{P}_n^* := \mathcal{P}_n(\mathcal{U}^*, \mathcal{E}_{\mathcal{A}^+}, X) := \check{\mathcal{S}}_n(\mathcal{U}^*, \mathcal{E}_{\mathcal{A}^+}, (0), X) := \{f : f \in C_l^{n-1}([a,b];\mathbb{R}), \tag{5.2}$$
$$f|_{\check{J}_i} \in \text{ span } U^{[i]^*},\ f \text{ is } (U^{[i-1]^*}, U^{[i]^*}, E^{[i]}) - \text{smooth at } x_i,\ i \in K_X\}.$$

Clearly, $\mathcal{P}_n \subset \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$. From the assumption (2.6) it follows that

$$E^{[i]} = \text{ diag } (I_{\mu_i}, \bar{E}^{[i]}_{n-1-\mu_i}, 1), \quad \bar{E}^{[i]}_{n-1-\mu_i} = R^{-1}_{n-1-\mu_i}\left(\bar{A}^{[i]}\right)^{-T} R_{n-1-\mu_i}. \tag{5.3}$$

is a n-dimensional square matrix which satisfies (2.7).

Later we will also use spaces

$$\mathcal{P}_{n+1} := \mathcal{S}_{n+1}(\hat{\mathcal{U}}, \hat{\mathcal{C}}_{\mathcal{A}^+}, (0), X)$$

having $\mathcal{P}_n$ as a subspace and

$$\mathcal{P}^*_{n+1} := \check{\mathcal{S}}_{n+1}(\hat{\mathcal{U}}^*, \hat{\mathcal{E}}_{\mathcal{A}^+}, (0), X)$$

having $\mathcal{P}_n^*$ as a subspace. They are defined by the extensions

$$\hat{\mathcal{U}} = \left(\hat{U}^{[i]}\right)_{i\in K_X},\ \hat{\mathcal{C}}_{\mathcal{A}^+} = \left(\hat{C}^{[i]}\right)_{i\in K_X},\ \hat{\mathcal{U}}^* = \left(\hat{U}^{[i]*}\right)_{i\in K_X},\ \hat{\mathcal{E}}_{\mathcal{A}^+} = \left(\hat{E}^{[i]}\right)_{i\in K_X}$$

where $\hat{U}^{[i]} = (u_1^{[i]}, \ldots, u_n^{[i]}, u_{n+1}^{[i]})$ is an ECT-system generated by the weights $(w_1^{[i]}, \ldots, w_{n+1}^{[i]}) = (\mathbf{1}, w_2^{[i]}, \ldots, w_n^{[i]}, \hat{w}_{n+1}^{[i]})$ and $\hat{U}^{[i]*} = (u_1^{[i]*}, \ldots, u_n^{[i]*}, u_{n+1}^{[i]*})$ is an ECT-system generated by the weights $(w_1^{[i]*}, \ldots\ldots, w_{n+1}^{[i]*}) = (\mathbf{1}, w_n^{[i]}, \ldots, w_2^{[i]}$ $, w_{n+1}^{[i]*})$. Here $0 < \hat{w}_{n+1}^{[i]} \in C^0(\bar{J}_i;\mathbb{R})$ and $0 < w_{n+1}^{[i]*} \in C^0(\bar{J}_i;\mathbb{R})$ may be chosen arbitrarily, where now we assume that

$$w_j^{[i]} \in C^{\max\{n+1-j, j-1\}}(J_i;\mathbb{R}), \qquad j = 2, \ldots, n.$$

The connection matrices for $\mathcal{P}_n$ resp. for $\mathcal{P}^*_{n+1}$ for every $i \in K_X$ are defined by $\hat{C}^{[i]} := \text{diag } (C^{[i]}, 1)$ resp. $\hat{E}^{[i]} := \text{diag}(E^{[i]}, 1)$. Here $\hat{E}^{[i]} = \text{diag}(I_{\mu_i}, \overline{E}^{[i]}_{n-1-\mu_i}, 1, 1)$ if $A^{[i]}$ may be partitioned as in (2.6).

According to corollary 4.2 $\mathcal{P}_n$ resp. $\mathcal{P}_n^*$ is an rET- resp. lET-space of order n each and $\mathcal{P}^*_{n+1}$ is an lET-space of order $n+1$. By corollary 4.5 the space $\mathcal{P}^*_{n+1}$ has a basis

$$q_1, \ldots, q_{n+1} \tag{5.4}$$

such that for $\nu = 0, 1, 2$ the system $q_1, \ldots, q_{n-1+\nu}$ is an lET-sytem of order $n-1+\nu$. Such a basis is obtained by fixing in any knot interval $\bar{J}_i$ the local ECT-system (1.8) in canonical form with respect to any $c \in [x_i, x_{i+1}]$ $(s^{[i]*}_{1,n+1}(x,c), \ldots, s^{[i]*}_{n+1,n+1}(x,c))$,

$x \in (x_i, x_{i+1}]$, and extending these functions by the connection equations of $\mathcal{P}^*_{n+1}$ to the left and right of J_i. Since $s^{[i]*}_{j,n} = s^{[i]*}_{j,n+1}, \quad j = 1, \ldots, n$, the basis $q_1, \ldots, q_{n+1}$ of $\mathcal{P}^*_{n+1}$ constructed this way under the hypothesis (2.6) indeed yields in the sections $q_1, \ldots, q_{n-1}$ and $q_1, \ldots, q_n$ lET-systems of orders $n-1$ and n, respectively.

Generalized Pólya polynomials are defined for $j \in J_\varphi$ by

$$\begin{aligned} &M_j(y) = M^n_j(x) = \\ &= M_j(y|\xi_{j+1}, \ldots, \xi_{j+n-1}) = (-1)^{n-1} r q_n \begin{bmatrix} q_1, \ldots, q_{n-1} \\ \xi_{j+1}, \ldots, \xi_{j+n-1} \end{bmatrix}_l (y), \end{aligned}$$

denoting by $rf \begin{bmatrix} u_1,\ldots,u_n \\ t_1,\ldots,t_n \end{bmatrix}_l (y) := f(y) - pf \begin{bmatrix} u_1,\ldots,u_n \\ t_1,\ldots,t_n \end{bmatrix}_l (y)$ the interpolation remainder where $pf \begin{bmatrix} u_1,\ldots,u_n \\ t_1,\ldots,t_n \end{bmatrix}_l (y)$ is the solution of the Hermite interpolation problem $H(U, T-, f)$.

$M_j \in \mathcal{P}^*_n$ has exactly $n-1$ zeros $\xi_{j+1}, \ldots, \xi_{j+n-1}$, counting left multiplicities, and no other zeros, and M_j has leading coefficient $(-1)^{n-1}$ in every interval $\check{J}_i$. Therefore M_j is positive for $x < \xi_{j+1}$. It is not hard to show that every n consecutive generalized Pólya polynomials $(M_j)^{l+n-1}_{j=l}$, $l \in J_\varphi$, form a basis of $\operatorname{span}\{q_1, \ldots, q_n\}$.

Barry [1] has constructed *de Boor-Fix functionals*

$$\Lambda_j(x)[f] := \sum_{p=0}^{n-1} (-1)^{n-1-p} L^{[r]}_p f(x) \cdot L^{[r]*}_{n-1-p} M_j(x),\ x \in J_r,\ \xi_j < x < \xi_{j+n}.$$

Actually, it is easily derived from (1.10) that under our general assumptions (2.5), (2.6) and (2.7) for every $j \in J_\varphi$ and $f \in \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{\text{ext}})$ the function

$$x \mapsto \Lambda_j(x)[f]$$

is a constant function of $x \in (\xi_j, \xi_{j+n})$. As a consequence,

$$\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X) \ni f \mapsto \Lambda_j(f) := \Lambda_j(x)[f], \quad x \in (\xi_j, \xi_{j+n})$$

is a well defined linear functional for every $j \in J_\varphi$. It follows that they are dual functionals for the rECT-B-spline basis (N_j) of order n. This is due to Barry [1], cf. also [24].

Theorem 5.1.

$$\Lambda_j(N_i) = \delta_{i,j} \qquad i, j \in J_\varphi.$$

The following theorem will be basic both for the definition of the rECT-B-splines in terms of generalized divided differences and for their general recursion relation. It's proof is a straightforward extension of the proof due to Dyn and Micchelli [4] of the similar result for polynomial splines via connection matrices.

Theorem 5.2. *Under the assumptions* (2.5), (2.6), (2.7) *there exists a unique function* $[a,b] \times [a,b] \ni (x,y) \mapsto h(x,y)$ *such that*

(i) for each $y \in [a,b]$ $\quad h(\,\cdot\,,y) \in \mathcal{P}_n$,

(ii) for each $x \in [a,b]$ $\quad h(x,\,\cdot\,) \in \mathcal{P}_n^*$.

(iii) Whenever for some $l \in K_X \quad x \in J_l$ *and* $y \in \check{J}_l$ *then*

$$\begin{aligned} h(x,y) &= w_1^{[l]}(x) h_{n-1}(x,y;w_2^{[l]},\ldots,w_n^{[l]}) = s_n^{[l]}(x,y) \\ &= \sum_{k=1}^{n} (-1)^{n-k} s_k^{[l]}(x,c) \cdot s_{n+1-k,n}^{[l]*}(y,c) \\ &= w_1^{[l]}(x) \sum_{k=1}^{n} h_{k-1}(x,c;w_2^{[l]},\ldots,w_k^{[l]}) h_{n-k}(y,c;w_{k+1}^{[l]},\ldots,w_n^{[l]}) \end{aligned}$$

with $c \in \bar{J}_l$ *arbitrary.*

(iv) For $i \in K_X$ *fixed,* $c \in \bar{J}_i$ *and* $j = 1,\ldots,n$ *let* $p_j(\,\cdot\,,c) \in \mathcal{P}_n$ *be defined by*

$$p_j(x,c) = w_1^{[i]}(x) h_{j-1}(x,c;w_2^{[i]},\ldots,w_j^{[i]}) = s_j^{[i]}(x,c),\ x \in J_i,$$

and for $i \in K_X$ *fixed,* $c \in \bar{J}_i$ *and* $j = 1,\ldots,n$ *let* $q_j(\,\cdot\,,c) \in \mathcal{P}_n^*$ *be defined by*

$$q_j(y,c) = h_{j-1}(y,c;w_n^{[i]},\ldots,w_{n+2-j}^{[i]}) = s_{j,n}^{[i]*}(y,c),\ y \in \check{J}_i. \tag{5.5}$$

Then the function h *has the representation*

$$h(x,y) = \sum_{k=1}^{n} (-1)^{n-k} p_k(x,c) q_{n+1-k}(y,c), \quad (x,y) \in [a,b] \times [a,b],\ c \in \bar{J}_i \tag{5.6}$$

where the right hand side is independent of i *and of* $c \in \bar{J}_i$.

The function h will be called *generating function* of rECT-B-splines for reasons that will become clear soon.

Remark 5.3. If $U^{[i]} = (\mathbf{1}, x, \ldots, x^{n-1})|_{J_i}$ for all i then $h(x,y) = \frac{(x-y)^{n-1}}{(n-1)!}$ whenever $x \in J_l$ and $y \in \check{J}_l$ for some l and (5.6) reduces to formula (3.67) of [4]. If moreover $A^{[i]} = I_{n-\mu_i}$ for all i then (5.6) reduces to the binomial theorem

$$\frac{(x-y)^{n-1}}{(n-1)!} = \sum_{k=1}^{n} (-1)^{n-k} \frac{(x-c)^{k-1}}{(k-1)!} \frac{(y-c)^{n-k}}{(n-k)!}.$$

When $U^{[i]} = U|_{J_i}$ where $U = (u_1, \ldots, u_n)$ is an ECT-system on $[a,b]$ and for all $i \in \mathbb{Z}$ $A^{[i]} = I_{n-\mu_i}$ is an identity matrix then (5.6) reduces to Marsden's identity for Tchebycheff splines [30] p. 382.

The next theorem is a generalization of Marsden's identity to rECT-B-splines.

Theorem 5.4. *Under the assumptions* (2.5), (2.6) *and* (2.7) *for the function* h *of theorem 5.2 there holds*

$$h(x,y) = \sum_{i \in J_\varphi} N_i(x) M_i(y) \qquad \text{for all} \quad (x,y) \in [a,b] \times [a,b].$$

Remark 5.5. Theorems 5.1 and 5.4 are equivalent in the sense that each is a consequence of the other [24].

6. rECT-B-splines defined by generalized divided differences

Let

$$g(x,y) := \begin{cases} h(x,y) & x \geq y \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

where h is the function of theorem 3.5 By the properties of h, with y fixed, as a function of x, $g(x,y)$ belongs piecewise, for $x \geq y$ and for $x < y$, to $\mathcal{P}_n \subset \mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, M, X)$. If x is fixed, as a function of y, $g(x,y)$ belongs piecewise, for $x \geq y$ and for $x < y$, to $\mathcal{P}_n^*$. The function g is separately, with respect to x, in $C_r^{n-1}([a,b];\mathbb{R})$, and with respect to y, in $C_l^{n-1}([a,b];\mathbb{R})$, since h has this property, and the $(n-1)$st ECT-derivative of g with respect to x at $x = y$ has the characteristic jump discontinuity of a Green's function: for every i

$$\lim_{x \to y-} L_\nu^{[i]} g(x,y)| = 0 \qquad \nu = 0, \ldots, n-1, \quad \text{if } x_i < y \leq x_{i+1}$$

$$\lim_{x \to y+} L_\nu^{[i]} g(x,y)| = \begin{cases} 0 & \nu = 0, \ldots, n-2 \\ 1 & \nu = n-1 \quad \text{if } x_i \leq y < x_{i+1}. \end{cases}$$

We call (6.1) the *Green's function with respect to the spaces* $\mathcal{P}_n(\mathcal{U}, \mathcal{C}_{\mathcal{A}^+}, X)$ and $\mathcal{P}_n(\mathcal{U}^*, \mathcal{E}_{\mathcal{A}^+}, X)$.

Definition 6.1. For $j \in J_\varphi$ and $x \in [a,b]$

$$\begin{aligned} \tilde{N}_j^n(x) &:= \tilde{N}(x|\xi_j, \ldots, \xi_{j+n}) \\ &:= (-1)^n \left(\begin{bmatrix} q_1, \ldots, q_n \\ \xi_{j+1}, \ldots, \xi_{j+n} \end{bmatrix}_l g(x, \cdot\,) - \begin{bmatrix} q_1, \ldots, q_n \\ \xi_j, \ldots, \xi_{j+n-1} \end{bmatrix}_l g(x, \cdot\,) \right). \end{aligned} \tag{6.2}$$

Here we have made use of the notation (1.14) for the left sided generalized divided differences of a function, and the functions $q_1, \ldots, q_n$ are those defined in Theorem 5.2.

Theorem 6.2. *For $j \in J_\varphi$ and $x \in [a,b]$*

$$\tilde{N}_j^n(x) = (-1)^n f_{n,j} \cdot \begin{bmatrix} q_1, \dots, q_{n+1} \\ \xi_j, \dots, \xi_{j+n} \end{bmatrix}_l g(x,\,\cdot\,), \tag{6.3}$$

$$f_{n,j} = \begin{bmatrix} q_1, \dots, q_n \\ \xi_{j+1}, \dots, \xi_{j+n} \end{bmatrix}_l q_{n+1} - \begin{bmatrix} q_1, \dots, q_n \\ \xi_j, \dots, \xi_{j+n-1} \end{bmatrix}_l q_{n+1}.$$

Here $q_1, \dots, q_{n+1} \in \mathcal{P}_{n+1}^*$ are the functions defined by (5.4) *and $q_1, \dots, q_n$ are those of Theorem 5.2.*

Remark 6.3. In case of ordinary polynomial splines of order n where all connection matrices are identity matrices (6.3) simplifies to

$$\tilde{N}_j^n(x) = (-1)^n(\xi_{j+n} - \xi_j)[\xi_j, \dots, \xi_{j+n}]_l(x - \,\cdot\,)_+^{n-1}$$

where $[\xi_j, \dots, \xi_{j+n}]_l f$ denotes the ordinary left sided divided difference of the function $f \in C_l^{n-1}(J;\mathbb{R})$ with respect to the polynomials of degree n at most. In case of Tchebycheffian splines of order n where all connection matrices are identity matrices (6.3) extends Lyche's definition (6.2) of Tchebycheffian B-splines [10].

In [24] it is proved

Theorem 6.4. *For $j \in J_\varphi$* $\quad \tilde{N}_j^n = N_j^n$.
Here N_j^n are the rECT-B-splines of theorem 3.5.

Remark 6.5. It is definition 6.1 which, under suitable assumptions, leads to a recursive method for computing ECT-B-splines and ECT-spline curves developed in section 7 [25]. In [31] cardinal ECT-B-splines with simple knots defined by connection matrices are computed directly according to theorem 6.2 Actually, there the left sided generalized divided differences are computed directly via a certain characteristic polynomial wherein also the Taylor's expansion (1.7) with respect to an ECT-system and that with respect to its dual are involved.

7. Computing rECT-B-splines recursively

Recursive methods for computing B-splines (normalized to form a nonnegative partition of unity) for splines of particular classes are well known. For ordinary polynomial splines (with connection matrices that all are identity matrices) best known is the deBoor-Mansion-Cox recurrence relation (7.3),(7.4) which is a two-term recursion. Using a contour integral approach Walz [32] has proved more general more-term recursions. For Tchebycheff splines there is a two-term recursion due to Lyche [10] where the spline weights are expressed as quotients of determinants. For a wide class of Tchebycheff splines, LB-splines and complex splines Dyn and Ron [5] have given four-term recursions.

It should be noticed that our constructive approach does not cover trigonometric B-splines. Stable two-term recursions for ordinary trigonometric B-splines (with all connection matrices equal to identity matrices) are due to Lyche and Winther [12] and more general ones due to Walz [33].

It is Lyche's approach to Tchebycheff B-splines [10] that can be extended to rECT-B-splines (and similarly to lECT-B-splines). Two ideas are basic in establishing the recurrence relation. One is due to Lyche defining auxiliary B-splines of lower orders which are used as intermediate results in computing the B-splines N_l^n of order n. As before we adopt the assumptions (2.5),(2.6) and (2.7).

Definition 7.1. For $n \in \mathbb{N}$, $k = 1, \ldots, n$ and $x \in \mathbb{R}$ and $j \in \mathbb{Z}$ let

$$N_j^{k,n}(x) := \tag{7.1}$$
$$\begin{cases} (-1)^n \left(\left[\begin{smallmatrix} q_1(\cdot,c),\ldots,q_n(\cdot,c) \\ \xi_{j+1},\ldots,\underbrace{x,\ldots,x}_{n-k},\ldots,\xi_{j+k} \end{smallmatrix} \right]_l g(x,\cdot) \right. & \\ \left. - \left[\begin{smallmatrix} q_1(\cdot,c),\ldots,q_n(\cdot,c) \\ \xi_j,\ldots,\underbrace{x,\ldots,x}_{n-k},\ldots,\xi_{j+k-1} \end{smallmatrix} \right]_l g(x,\cdot) \right) & \text{if } \xi_j < \xi_{j+k} \text{ and } \xi_j \le x < \xi_{j+k} \\ 0 & \text{otherwise} \end{cases}$$

Here g is defined by (6.1) and the $q_l(y,c)$ are defined by (5.5). It can be shown [25] that $N_j^{k,n}(x)$ is independent of $c \in \bar{J}_i$ and of $i \in \mathbb{Z}$. We call these functions *auxiliary B-splines of suborder* k of the B-spline N_l^n of order n such that $\xi_l \le \xi_j < \xi_{j+k} \le \xi_{l+n}$.

The B-splines of lowest orders are

$$N_j^{1,1}(x) = N_j^1(x) = N(x|\xi_j, \xi_{j+1}) = \begin{cases} 1 & \text{if } \xi_j \le x < \xi_{j+1} \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to show that the auxiliary B-splines have similar properties as the B-splines themselves (see theorem 3.2) though, at least in general, they do not belong to $\mathcal{S}_n(\mathcal{U}, \mathcal{A}^+, \boldsymbol{\xi}_{\text{ext}})$. The second basic idea dates back to Popoviciu [26]. He has modified generalized divided differences by introducing further nodes as is done in definition 7.1. This idea was elaborated for ECT-systems by Lyche [10] and for lET-systems by Mühlbach and Tang in lemmata 4.2 and 4.3 of [25].

Remark 7.2. In case of ordinary polynomial splines of order n the $N_j^{k,n}$ are the polynomial B-splines of order k:

$$N_j^{k,n}(x) = (-1)^k (\xi_{j+k} - \xi_j)[\xi_j, \ldots, \xi_{j+k}]_l (x - \cdot)_+^{k-1} = N_j^{k,k}(x), \qquad k = 1, \ldots, n. \tag{7.2}$$

It is well known that the polynomial B-splines (7.2) can be computed by the de Boor-Mansion-Cox recursion

$$N_j^{k+1,n}(x) = \lambda_j^{k,n}(x) N_j^{k,n}(x) + \mu_{j+1}^{k,n}(x) N_{j+1}^{k,n}(x), \quad k = 1, \ldots, n-1 \tag{7.3}$$

starting with

$$N_j^{1,n}(x) = N_j^{1,1}(x) = \begin{cases} 1 & \text{if } \xi_j \le x < \xi_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

where the coefficients are the Neville-Aitken weights of polynomial interpolation that are independent of n

$$\lambda_j^{k,n}(x) = \frac{x - \xi_j}{\xi_{j+k} - \xi_j}, \quad \mu_{j+1}^{k,n}(x) = 1 - \lambda_{j+1}^{k,n}(x) = \frac{\xi_{j+k+1} - x}{\xi_{j+k+1} - \xi_{j+1}}. \tag{7.4}$$

They occur in the Neville-Aitken interpolation formula

$$\begin{aligned} p_{k+1} f[\xi_j, \ldots, \xi_{j+k}](x) = &\frac{x - \xi_j}{\xi_{j+k} - \xi_j} p_k f[\xi_{j+1}, \ldots, \xi_{j+k}](x) \\ &+ \frac{\xi_{j+k} - x}{\xi_{j+k} - \xi_j} p_k f[\xi_j, \ldots, \xi_{j+k-1}](x) \end{aligned}$$

where $p_l f[z_1, \ldots, z_l]$ is that polynomial of order l interpolating the function f at the nodes $z_1, \ldots, z_l$ in the sense of Hermite. The equality (7.2) relies on the factorization of algebraic polynomials. For arbitrary ECT-systems $\boldsymbol{s}^{[i]}$ there is no similar interpretation of the auxiliary ECT-B-splines of suborder k as in (7.2). Only for particular ECT-systems, for instance of rational functions with prescribed poles there is a similar interpretation (see section 8). However, under suitable assumptions also in the general case the auxiliary ECT-B-splines of suborder k can be computed recursively by a de Boor-like recursion. It turns out that the spline weight factors also in the general case can be interpreted in terms of interpolation theory with respect to an lET-system $q_1, \ldots, q_n$. They are again certain generalized Neville-Aitken weights.

Theorem 7.3. *Suppose that $N_j^{k,n}(x)$ for $k = 1, \ldots, n$ and all $j \in \mathbb{Z}$ is defined by* (7.1)*. Assume that the basis* (5.5) *$q_1(\,\cdot\,, c), \ldots, q_n(\,\cdot\,, c)$ of $\mathcal{P}_n(\mathcal{U}^*, \mathcal{E}_{\mathcal{A}^+}, X)$ is an lET-system on $[a, b]$ of order n that has the property that also $q_1(\,\cdot\,, c), \ldots, q_{n-1}(\,\cdot\,, c)$ is an lET-systems on $[a, b]$. Here $c \in \bar{J}_i$ and $i \in \mathbb{Z}$ are arbitrary. In view of* (5.3) *according to corollary 4.5 this holds true due to the basic assumption* (2.6)*. Then for $k = 1, \ldots, n-1$, $x \in \mathbb{R}$ and $j \in \mathbb{Z}$*

$$N_j^{k+1,n}(x) = \lambda_j^{k,n}(x) \cdot N_j^{k,n}(x) + \mu_{j+1}^{k,n}(x) \cdot N_{j+1}^{k,n}(x) \tag{7.5}$$

with the initialization

$$N_j^{1,n}(x) = \begin{cases} 1 & \text{if } \xi_j \le x < \xi_{j+1} \\ 0 & \text{for all other } x, \end{cases}$$

and $N_j^{1,n}(x) \equiv 0$ iff $\xi_j = \xi_{j+1}$, where the spline weights can be computed by the following formulas:

$$\lambda_j^{k,n}(x) \equiv 0 \qquad\qquad \text{if } \xi_j = \ldots = \xi_{j+k}$$

$$\lambda_j^{k,n}(x) := \begin{cases} 0 & \text{if } x = \xi_j \\ \dfrac{D_l^{\mu^+(\xi_j)-1} rq_n[\xi_{j+1},\ldots,\underbrace{x,\ldots,x}_{n-k},\ldots,\xi_{j+k-1}]_l(\xi_j)}{D_l^{\mu^+(\xi_j)-1} rq_n[\xi_{j+1},\ldots,\underbrace{x,\ldots,x}_{n-k-1},\ldots,\xi_{j+k}]_l(\xi_j)} & \text{if } \xi_j < x \le \xi_{j+k} \end{cases}$$

where $\mu^+(\xi_j)$ is the multiplicity of ξ_j in $(\xi_j, \xi_{j+1}, \ldots, \xi_{j+k})$ and the interpolation remainders are with respect to the lET-system $q_1(\,\cdot\,, c), \ldots, q_{n-1}(\,\cdot\,, c)$ with $c \in \mathbb{R}$ arbitrary. Moreover

$$\mu_{j+1}^{k,n}(x) \equiv 0 \qquad \text{if } \xi_{j+1} = \ldots = \xi_{j+k+1}$$

$$\mu_{j+1}^{k,n}(x) = \begin{cases} 1 & \text{if } x = \xi_{j+1} \\ \dfrac{D_l^{\mu^-(\xi_{j+k+1})-1} rq_n[\xi_{j+2},\ldots,\underbrace{x,\ldots,x}_{n-k},\ldots,\xi_{j+k}]_l(\xi_{j+k+1})}{D_l^{\mu^-(\xi_{j+k+1})-1} rq_n[\xi_{j+1},\ldots,\underbrace{x,\ldots,x}_{n-k-1},\ldots,\xi_{j+k}]_l(\xi_{j+k+1})} & \\ & \text{if } \xi_{j+1} < x \le \xi_{j+k+1} \end{cases}$$

where $\mu^-(\xi_{j+k+1})$ is the multiplicity of ξ_{j+k+1} in $(\xi_{j+1}, \ldots, \xi_{j+k+1})$.

Moreover, we have

$$0 < \lambda_j^{k,n}(x) < 1 \qquad \text{if } \xi_j < x < \xi_{j+k},$$

and $x \mapsto \lambda_j^{k,n}(x)$ is left continuous everywhere and strictly increasing from 0 to 1 for $\xi_j < x < \xi_{j+k}$. Similarly, we have

$$0 < \mu_{j+1}^{k,n}(x) < 1 \qquad \text{if } \xi_{j+1} < x < \xi_{j+k+1},$$

and $x \mapsto \mu_{j+1}^{k,n}(x)$ is left continuous everywhere and strictly decreasing from 1 to 0 for $\xi_{j+1} < x < \xi_{j+k+1}$.

Remark 7.4. The spline weight factors can be interpreted in terms of interpolation theory, cf. [25] remark 4.2.

Remark 7.5. If for every i the connection matrix $A^{[i]}$ is a diagonal matrix with only positive diagonal elements then the basis $q_1(\,\cdot\,, c), \ldots, q_n(\,\cdot\,, c)$ is an lECT-system, and all weights $\lambda_j^{k,n}(x), \mu_{j+1}^{k,n}(x)$ can be computed recursively since the remainders $rq_n[y_1, \ldots, y_n]_l(y)$ can be computed recursively (cf. [19]).

Remark 7.6. It should be observed that theorem 7.3 also covers the case of *Bézier-ECT-splines.* This type of splines arises if we consider a compact interval $[a, b]$, a finite partition $X = (x_i)_{i=0}^{k+1}$ of $[a, b]$, $a = x_0 < x_1 < \ldots < x_k < x_{k+1} = b$, into knot intervals $J_i = [x_i, x_{i+1})$, $i = 0, \ldots, k-1$, with the last knot interval

$J_k = [x_k, x_{k+1}]$ resp. $\check{J}_i = (x_i, x_{i+1}]$, $i = 1, \ldots, k$, with the first knot interval $\check{J}_0 = [x_0, x_1]$, with multiplicities $\mu_0 = \mu_{k+1} = n$, $\mu_i = 0$ for $i = 1, \ldots, k$, with local ECT-systems (2.2) on $\bar{J}_i$ generated by weights (2.3) satisfying (2.5), with full connection matrices $A^{[i]} \in \mathbb{R}^{n \times n}$ that satisfy (2.6) and (2.7). Notice that the ECT-system $U^{[0]}$ on $\bar{J}_0 = [x_0, x_1]$ may be extended as an ECT-system to a larger interval $\hat{J}_0 = [x_0 - \delta, x_1]$ for $\delta > 0$ simply by extending the weights $w_j^{[0]}$ to $\hat{J}_0$ maintaining the smoothness properties (2.3).

According to theorem 7.3 for $\xi_j \leq x < \xi_{j+1}$ the B-spline curve of order n

$$s(x) = \sum_{l=j-n+1}^{j} c_l \cdot N_l^n(x)$$

where the *control points* $c_{j-n+1}, \ldots, c_j \in \mathbb{R}^s$ $(s \in \mathbb{N})$ are given can be computed recursively by the following de Boor-like algorithm.

Algorithm 7.7
initialisation:

$$c_l^{1,n}(x) := c_l \qquad l = j - n + 1, \ldots, j$$

algorithm:

$$\begin{aligned} c_i^{k+1,n}(x) = \lambda_i^{n-k,n}(x) \cdot c_i^{k,n}(x) + (1 - \lambda_i^{n-k,n}(x)) \cdot c_{i-1}^{k,n}(x), \\ \xi_j \leq x < \xi_{j+1},\ i = j - n + k + 1, \ldots, j,\ k = 1, \ldots, n-1. \end{aligned}$$

output:

$$c_j^{n,n}(x) = s(x), \qquad \xi_j \leq x < \xi_{j+1}.$$

Moreover, at level k $(k = 1, \ldots, n)$ there holds

$$s(x) = \sum_{l=j-n+k}^{j} c_l^{k,n}(x) \cdot N_l^{n+1-k,n}(x), \qquad \xi_j \leq x < \xi_{j+1}.$$

8. Examples

Example 8.1. In case of ordinary polynomial splines of order n with all connection matrices equal to identity matrices we have

$$q_j(y, c) = \frac{(y - c)^{j-1}}{(j-1)!} \qquad j = 1, \ldots, n$$

and from theorem 7.3 for $\xi_j \le x < \xi_{j+1}$ and $l = j-k,\dots,j$ the spline weights

$$\lambda_l^{k,n}(x) = \frac{D_l^{\mu^+(\xi_l)-1} rq_n[\xi_{l+1},\dots,\underbrace{x,\dots,x}_{n-k},\dots,\xi_{l+k-1}]_l(\xi_l)}{D_l^{\mu^+(\xi_l)-1} rq_n[\xi_{l+1},\dots,\underbrace{x,\dots,x}_{n-k-1},\dots,\xi_{l+k}]_l(\xi_l)} = \frac{x-\xi_l}{\xi_{l+k}-\xi_l},$$

$$\mu_l^{k,n}(x) = 1-\lambda_l^{k,n}(x) = \frac{\xi_{l+k}-x}{\xi_{l+k}-\xi_l}.$$

In this example algorithm 7.7 reduces to the de Boor algorithm computing points of B-spline curves in case of ordinary polynomial splines of order n when all connection matrices are identity matrices.

Example 8.2. In case of Tchebycheff splines of order n with all connection matrices equal to identity matrices we have

$$q_j(y,c) = s_{j,n}^*(y,c) \qquad j=1,\dots,n$$

and from theorem 7.1 for $\xi_j \le x < \xi_{j+1}$ and $l = j-k,\dots,j$ the spline weights

$$\lambda_j^{k,n}(x) = \frac{D_l^{\mu^+(\xi_j)-1} rq_n[\xi_{j+1},\dots,\underbrace{x,\dots,x}_{n-k},\dots,\xi_{j+k-1}]_l(\xi_j)}{D_l^{\mu^+(\xi_j)-1} rq_n[\xi_{j+1},\dots,\underbrace{x,\dots,x}_{n-k-1},\dots,\xi_{j+k}]_l(\xi_j)}$$

$$\mu_j^{k,n}(x) = 1-\lambda_j^{k,n}(x)$$

giving new interpretations to the weights due to Lyche [10].

Example 8.3. For the global ECT-system $s_{1,n}^*(y,c),\dots,s_{n,n}^*(y,c)$ (1.11), (1.12) on $[a,b]$ of example 1.2 where all connection matrices are identity matrices the N_j^n are are Chebycheff-B-splines with respect to this ECT-system. In this case the functions $q_1(\,\cdot\,,x),\dots,q_n(\,\cdot\,,x)$ of theorem 5.2 are known as the rational functions (1.4), (1.5), (1.6). This ECT-system being also a Cauchy-Vandermonde system with respect to the poles $b_1=\infty,\dots,b_{n-2}=\infty$, $b_{n-1}=a-\varepsilon$, $b_n=b+\varepsilon$ allows to compute the spline weights of theorem 5.2 and of algorithm 7.7 explicitly using the explicit expression (42) of [22] of the interpolation remainder in terms of the nodes and the poles. If $\xi_j \le x < \xi_{j+1},\ j\in\{-n+1,\dots,\mu\}$ according to theorem 5.2 for $n\in\mathbb{N}$ at level $k=1,\dots,n$ for $m=j-k+1,\dots,j$

$$\lambda_m^{k,n}(x) := \begin{cases} \lim\limits_{\tilde{\xi}_m\to\xi_m-0} \dfrac{rq_n[\xi_{m+1},\dots,\underbrace{x,\dots,x}_{n-k},\dots,\xi_{m+k-1}]_l(\tilde{\xi}_m)}{rq_n[\xi_{m+1},\dots,\underbrace{x,\dots,x}_{n-k-1},\dots,\xi_{m+k}]_l(\tilde{\xi}_m)} & \text{if } \xi_j < x < \xi_{j+1} \end{cases} \tag{8.1}$$

$$= \frac{\xi_m - x}{\xi_m-\xi_{m+k}}\,\frac{b+\varepsilon-\xi_{m+k}}{b+\varepsilon-x} \qquad \text{if } \xi_j \le x < \xi_{j+1}$$

and

$$\mu_m^{k,n}(x) = 1 - \lambda_m^{k,n}(x) = \frac{\xi_{m+k} - x}{\xi_{m+k} - \xi_m} \frac{b + \varepsilon - \xi_{m+k}}{b + \varepsilon - x}. \tag{8.2}$$

The spline weights (8.1) and (8.2) agree with the weights given by Gresbrand [7] where splines with the ordinary smoothness conditions constructed from Cauchy-Vandermonde systems with respect to arbitrary given poles $b_1, b_2, \ldots, b_n$ outside $[a, b]$ are considered. This shows that also in the simple case of Chebycheff ECT-B-splines with one pole of order $n-1$ at $b+\varepsilon$ the auxiliary ECT-B-splines are the ECT-B-splines of lower orders.

More analytical and some numerical examples can be found in [25]. Of course, it will depend on the applications what kind of splines a designer will choose. The family of ECT-B-splines provides a real alternative to the classical polynomial B-splines. In fact, they allow more freedoms without increasing computational costs too much.

References

[1] Barry, P.J., de Boor-Fix dual functionals and algorithms for Tchebycheffian B-splines curves, *Constructive Approximation* 12 (1996), 385-408

[2] Barry, P.J. Dyn, N. Goldman, R.N. Micchelli, C.A., Identities for piecewise polynomial spaces determined by connection matrices, *Aequationes Math.* 42 (1991), 123-136

[3] Buchwald, B., Mühlbach, G., Rational splines with prescribed poles, *JCAM* 167 (2004), 271-291

[4] Dyn, N., Micchelli, C.A., Piecewise polynomial spaces and geometric continuity of curves, *Numerische Mathematik* 54, 319-337

[5] Dyn, N., Ron, A., Recurrence Relations for Tchebycheffian B-Splines, *J. Analyse Math.* 51 (1988), 118-138

[6] Goodman, T.N.T., Properties of beta-splines, *J. Approx. Theory* 44 (1985), 132-153

[7] Gresbrand, A., Rational B-splines with prescribed poles,*Num. Algorithms* 12 (1996), 151-158

[8] Karlin, S. & Studden, W.J. , Tchebycheff Systems: With Applications in Analysis and Statistics, *Interscience Publishers* (1966).

[9] Karlin, S., Ziegler, Z., Tchebysheffian spline functions, *SIAM Num.* 3 (1966), 514-543

[10] Lyche, T.,A recurrence relation for Chebychevian B-splines,*Constr. Approx.* 1 (1985), 155 - 173

[11] Lyche, T., Schumaker, L.L., Total Positivity Properties of LB-splines, in: *Total Positivity and its Applications, M. Gasca and C. Micchelli (eds.), Kluwer Academic Publishers* (1996), 35-46

[12] Lyche, T., Winther, R., A Stable Recurrence Relation for Trigonometric B-splines, *J. Approx. Theory* 25 (1979), 266-279

[13] Mazure, M.-L., Blossoming: a geometric approach, *Constructive Approximation* 15 (1999), 33-68

[14] Mazure, M.-L., Chebyshev splines beyond total positivity, *Advances in Computational Mathematics* 14 (2001), 129-156

[15] Mazure, M.-L., Pottmann H., Tchebycheff Curves, in: *Total Positivity and its Applications, M. Gasca and C.A. Micchelli eds, Kluwer Academic Publ.* (1996), 187-218

[16] Mazure, M.-L., Laurent, P.-J., Piecewise smooth spaces in duality: applications to blossoming, *J. Approx. Thory* 98 (1999), 316-353

[17] Mühlbach, G., A Recurrence Formula for Generalized Divided Differences and Some Applications, *J. Approx. Theory* 9 (1973), 165-172

[18] Mühlbach, G., Newton- und Hermite Interpolation mit Čebyšev-Systemen, *ZAMM* 54 (1974), 541-550

[19] Mühlbach, G., Neville-Aitken Algorithms for Interpolation by Functions of Čebyšev-Systems in the Sense of Newton and in a Generalized Sense of Hermite, in: *Approximation Theory with Applications, A.G. Law and B.N. Sahney eds, Academic Press* (1976), 200-212

[20] Mühlbach, G., Recursive triangles, in: *Proceedings of the 3d Int. Coll. on Numerical Analysis, D. Bainov and V. Covachev eds, VSP,* (1995), 129-134

[21] Mühlbach, G., A recurrence relation for generalized divided differences with respect to ECT-systems, *Numerical Algorithms* 22 (1999), 319-326

[22] Mühlbach, G., Interpolation by Cauchy-Vandermonde systems and applications, *JCAM* 122 (2000), 203-222

[23] Mühlbach, G., One Sided Hermite Interpolation by Piecewise Different Generalized Polynomials, *JCAM* (2005), in print

[24] Mühlbach, G., ECT-B-splines defined by generalized divided differences, *JCAM* (2005), in print

[25] Mühlbach, G., Tang, Y., Calculating ECT-B-splines recursively, *Numerical Algorithms* (2006), in print

[26] Popoviciu, T., Sur le reste dans certaines formules linéaires d'approximation de l'analyse, *Mathematica (Cluj)* 1, (24) (1959), 95-142

[27] Pottmann, H., The geometry of Tchebycheffian splines, *CAGD* 10 (1993), 181-210

[28] Prautzsch, H., B-Splines with Arbitrary Connection Matrices, *Constr. Approx.* 20 (2004), 191-205

[29] Schoenberg, I.J., Whitney, A., On Pólya frequency functions III. The positivity of translation determinants with applications to the interpolation problem by spline curves, *TAMS* 74 (1953), 246-259

[30] Schumaker, L.L., Spline functions. Basic Theory, *Wiley Interscience* (1981), New York

[31] Tang, Y., Mühlbach, G., Cardinal ECT-splines, *Num. Alg.* 38 (2005), 259-283

[32] Walz, G., A Unified Approach to B-Spline Recursions and Knot Insertion, with Applications to New Recursion Formulas, *Advances Comp. Math.* 3 (1995), 89-100

[33] Walz, G.; Identities for Trigonometric B-splines, with Applications to Curve Design, *BIT* 37 (1997), 189-201

Günter W. Mühlbach
Institut für Angewandte Mathematik
Universität Hannover
Welfengarten 1
D 30167 Hannover
Germany

Yuehong Tang
Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Yu Dao Street 29
Nanjing 210016
Jiangsu, P. R. China

Annales Mathematicae et Informaticae
32 *(2005) pp. 125–127.*

A remark on Rainwater's theorem

Olav Nygaard

Agder University College
olav.nygaard@hia.no

Abstract

We define Rainwater sets as subsets of the dual of a Banach space for which Rainwater's theorem holds and show that (I)-generating subsets have this property. We apply this observation to give a proof of James' theorem when the dual unit ball is sequentially compact in its weak-star topology.

Key Words: Rainwater set, James boundary, (I)-generation

AMS Classification Number: 46B20

1. (I)-generating sets and Rainwater's theorem

For a subset B in the dual unit ball B_{X^*} of a Banach space X, in [2] a property was localized between the properties $\overline{\text{conv}}(B) = B_{X^*}$ and $\overline{\text{conv}}^{w^*}(B) = B_{X^*}$:

Definition 1.1. B is said to (I)-generate B_{X^*} if whenever B is written as a countable union, $B = \bigcup B_i$, then $B_{X^*} = \overline{\text{conv}}\,\big(\bigcup_i \overline{\text{conv}}^{w^*}\,(B_i)\big)$.

Note the following equivalent definition: Whenever B is written as an increasing countable union $B = \bigcup B_i\ \uparrow$, then $\bigcup_i \overline{\text{conv}}^{w^*}\,(B_i)$ is norm-dense in B_{X^*}. (I)-generation of course makes sense in any w^*-compact convex subset of X^*.

Recall that a set $B \subset B_{X^*}$ is called a *James boundary* if, for every $x \in X$, the maximum over B_{X^*} is attained on B. As a standard example, for any Banach space X the extreme points of B_{X^*} is a James boundary. The fundamental result from [2] is the following:

Theorem 1.2 ([2, Thm. 2.3]). *If B is a James boundary, then B (I)-generates B_{X^*}. The same is true for a James boundary in any w^*-compact convex subset of X^*.*

Note how this theorem both generalizes and sharpens the Krein-Milman theorem in this situation. It generalizes because it works for any James boundary and sharpens because (I)-generation is a stronger property than $\overline{\mathrm{conv}}^{w^*}(B) = B_{X^*}$ as a simple example in [2] shows. If B is separable and (I)-generates, then we already have $\overline{\mathrm{conv}}(B) = B_{X^*}$.

In 1963 (see [3] or [1, p. 155]) the following theorem was published under the pseudomym J. Rainwater: *For a bounded sequence in a Banach space X to converge weakly it is enough that it converges pointwise on the extreme points of the unit ball in the dual, B_{X^*}.* The proof is an application of Choquet's theorem. Later on S. Simons (see [4] or [5]) gave a completely different argument to show that Rainwater's theorem is true with any James boundary.

Definition 1.3. Let X be a Banach space. A subset B of B_{X^*} is called a Rainwater set if every bounded sequence that converges pointwise on B converges weakly.

Rainwater's original theorem then reads: The extreme points of B_{X^*} is a Rainwater set. Simons' more general version reads: Any James boundary is a Rainwater set. We want in this little note just to remark the simple but general fact that (I)-generating sets are Rainwater sets and give an application of this observation to a proof of James' theorem in a rather wide class of Banach spaces.

Theorem 1.4. *Let X be a Banach space. Suppose B (I)-generates B_{X^*}. Then B is a Rainwater set.*

Proof. Let (x_i) be a bounded sequence in X. Let M be such that $\|x_i\|, \|x\| \leqslant M$ for all i. Pick an arbitrary $x^* \in B_{X^*}$ and let $\varepsilon > 0$. Define

$$B_i = \{y^* \in B\ : \forall j \geqslant i, |y^*(x_j - x)| < \varepsilon\}.$$

Then, since $y^*(x_i) \to y^*(x)$ for every $y^* \in B$, (B_i) is an increasing covering of B.

Since B (I)-generates, there is a y^* in some $\overline{\mathrm{conv}}^{w^*}(B_N)$ such that $\|x^* - y^*\| < \varepsilon$. Note that for every $y^* \in \overline{\mathrm{conv}}^{w^*}(B_N)$, $j \geqslant N$ implies that $|y^*(x_j - x)| \leqslant \varepsilon$. Now, the triangle inequality show that for $j \geqslant N$

$$\begin{aligned} |x^*(x_j - x)| &\leqslant |x^*(x_j) - y^*(x_j)| + |y^*(x_j) - y^*(x)| + |y^*(x) - x^*(x)| \\ &\leqslant (1 + 2M)\varepsilon, \end{aligned}$$

and hence (x_i) converges weakly to x. □

Note that Simons' version of Rainwater's theorem follows from Theorem 1.4 and Theorem 1.2. Remark also that completeness is not needed in Definition 1.3 and also not in Theorem 1.4.

2. A proof of James' theorem when the dual unit ball is weak-star sequentially compact

Recall James famous characterization of reflexive spaces: *If every $x^* \in X^*$ attains its supremum over B_X, then X is reflexive.* In other words, if S_X is a

James boundary for $B_{X^{**}}$, then $B_X = B_{X^{**}}$. We now prove this result when B_{X^*} is sequentially compact in its weak-star topology. Such spaces are discussed in [1, Chapter XIII], the basic result being the Amir-Lindenstrauss theorem telling us that any subspace of a weakly compactly generated space is of this type.

Here is the argument: Suppose every $x^* \in X^*$ attains its supremum over S_X. Then S_X is a James boundary of $B_{X^{**}}$. Thus, from Theorem 1.2 and 1.4, X is a Grothendieck space, that is, weak and w^*-convergence of (bounded) sequences coincide in X^*. Since B_{X^*} is w^*-sequentially compact it is weakly sequentially compact and hence, by Eberlein's theorem, weakly compact. Hence X^*, and thus X, is reflexive.

Whether it is true in general that X is reflexive whenever S_X (I)-generates $B_{X^{**}}$ is to my best knowledge an open question. Let us end this little note by analyzing this problem a little more:

Definition 2.1. A Banach space X where S_X (I)-generates $B_{X^{**}}$ is called an (I)-space.

By Theorem 1.4 it is clear that (I)-spaces are Grothendieck spaces. The point in the proof of James' theorem in the sequentially weak-star compact dual unit ball case is that Grothendieck together with weak-star compact dual unit ball imply reflexivity, by Eberleins theorem. The standard example of a non-reflexive Grothendieck space is ℓ_∞. A starting point in characterizing (I)-spaces should be to decide whether ℓ_∞ is an (I)-space or not. But even this is a hard task since we have no description of ℓ_∞^{**}.

References

[1] J. Diestel, *Sequences and Series in Banach Spaces,* Graduate Texts in Mathematics, vol. 92, Springer, 1984.

[2] V. P. Fonf and J. Lindenstrauss, *Boundaries and generation of convex sets,* Israel. J. Math. **136** (2003) 157-172.

[3] J. Rainwater, *Weak convergence of bounded sequences,* Proc. Amer. Math. Soc. **14** (1963) 999.

[4] S. Simons *A convergence theory with boundary.* Pacific J. Math. **40** (1972) 703-708

[5] S. Simons *An Eigenvector Proof of Fatou's Lemma for Continuous Functions.* The Math. Int. **17** (No 3) (1995) 67-70

Olav Nygaard
Faculty of Mathematics and Science
Agder University College
Gimlemoen 25J
Servicebox 422
4604 Kristiansand
Norway

Annales Mathematicae et Informaticae
32 *(2005) pp. 129–139.*

Means of positive matrices: Geometry and a conjecture*

Dénes Petz

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
H–1053 Budapest, Reáltanoda u. 13–15, Hungary
e-mail: petz@math.bme.hu

Abstract

Means of positive numbers are well-know but the theory of matrix means due to Kubo and Ando is less known. The lecture gives a short introduction to means, the emphasis is on matrices. It is shown that any two-variable-mean of matrices can be extended to more variables. The n-variable-mean $\mathbf{M}_n(A_1, A_2, \ldots, A_n)$ is defined by a symmetrization procedure when the n-tuple $(A_1, A_2, \ldots, A_n)$ is ordered, it is continuous and monotone in each variable. The geometric mean of matrices has a nice interpretation in terms of an information geometry and the ordering of the n-tuple is not necessary for the definition. It is conjectured that this strong condition might be weakened for some other means, too.

Key Words: operator means, information geometry, logarithmic mean, geometric mean, positive matrices.

AMS Classification Number: 47A64 (15A48, 47A63)

1. Introduction

The **geometric mean** $\sqrt{xy}$, the **arithmetic mean** $(x+y)/2$ and the inequality between them go back to the ancient Greeks. That time the means of the positive numbers x and y were treated as geometric proportions.

The arithmetic mean can be extended to more variables as

$$\mathbf{A}(x_1, x_2, \ldots, x_n) := \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

*Written form of the lecture delivered in the Riesz–Fejér Conference, Eger, June, 2005. The work was partially supported by the Hungarian grant OTKA T032662.

and the formula makes sense not only for positive numbers but in any vectorspace. The quantity appears under various names depending on the context of the application, for example, average, barycenter, or the center of mass.

Suppose we have a device which can compute the mean of two variables. How to compute the mean of three? Assume that we aim to obtain the mean of x, y and z. We can make a new device

$$W : (a, b, c) \mapsto (\mathbf{A}(a, b), \mathbf{A}(a, c), \mathbf{A}(b, c)) \tag{1.1}$$

which applied to (x, y, z) many times gives the mean of x, y and z. More mathematically,

$$W^n(x, y, z) \to \mathbf{A}(x, y, z) \quad \text{as} \quad n \to \infty. \tag{1.2}$$

To show the relation (1.2), we have various possibilities. The simplest might be to observe that $W^n(x, y, z)$ is a convex combination of x, y and z,

$$W^n(x, y, z) = \lambda_1^{(n)} x + \lambda_2^{(n)} y + \lambda_3^{(n)} z.$$

One can compute the coefficients $\lambda_i^{(n)}$ explicitely and show that $\lambda_i^{(n)} \to 1/3$.

Another possibility is to compute the eigenvalues of the linear transformation W. It turns out that 1 is the only eigenvalue and the only peripheral eigenvalue, so W^n converges to the corresponding eigenprojection according to ergodic theory. In other words,

$$W^n(x, y, z) \to \frac{1}{3}(x + y + z)\,.$$

Assume that x, y and z are linearly independent vectors in a vectorspace. Their convex hull is a triangle Δ_0. Let the convex hull of the three vectors $W^n(x, y, z)$ be the triangle Δ_n. Since

$$\bigcap_{n=0}^{\infty} \Delta_n = \left\{ \frac{x + y + z}{3} \right\}$$

we can visualize the convergence (1.2) and we can observe its exponential speed, since the diameter of Δ_{n+1} is the half of that of Δ_n.

Of course, the above approach to the three-variable arithmetic mean is a possibility. If $x_1, x_2, \ldots, x_n$ are numbers, then one can minimize the functional

$$z \mapsto \sum_{i=1}^{n} (x_i - z)^2. \tag{1.3}$$

The minimizer is

$$z = \mathbf{A}(x_1, x_2, \ldots, x_n)\,.$$

The aim of this lecture is to study the symmetrization procedure (1.1) not only for the arithmetic mean but for several other means and not for numbers but mostly for matrices.

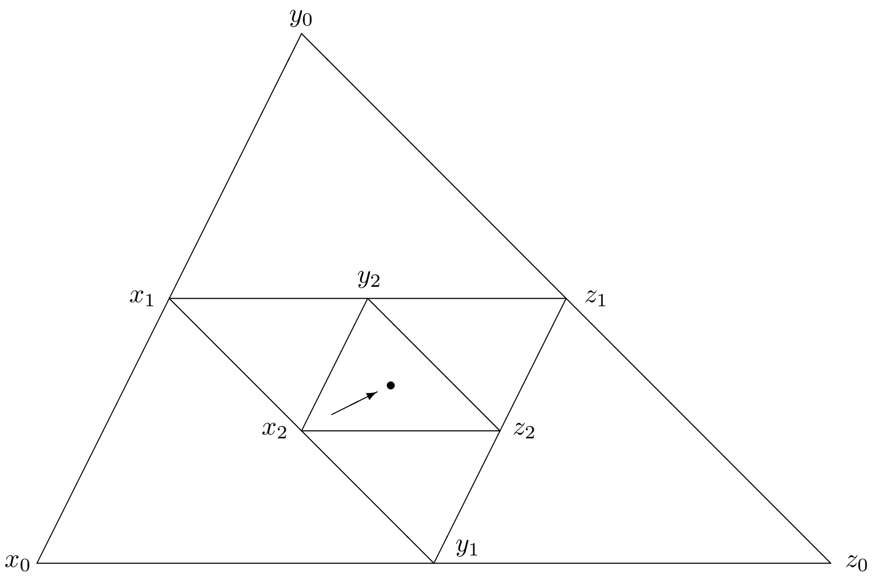

Figure 1: The triangles Δ_0, Δ_1 and Δ_2.

2. Means for three variables

The geometric mean, the arithmetic mean and the harmonic mean are extended to three (and more) variables as

$$\sqrt{xyz}, \quad (x+y+z)/3, \quad 3/(x^{-1}+y^{-1}+z^{-1}).$$

Our target is to extend an arbitrary mean $M_2(x,y)$ of two variables to three variables. In order to do this we need to specify what we mean by a mean. A function $\mathbf{M} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ may be called a *mean* of positive numbers if

(i) $\mathbf{M}(x,x) = x$ for every $x \in \mathbb{R}^+$.

(ii) $\mathbf{M}(x,y) = \mathbf{M}(y,x)$ for every $x, y \in \mathbb{R}^+$.

(iii) If $x < y$, then $x < \mathbf{M}(x,y) < y$.

(iv) If $x < x'$ and $y < y'$, then $\mathbf{M}(x,y) < \mathbf{M}(x',y')$.

(v) $\mathbf{M}(x,y)$ is continuous.

(vi) $\mathbf{M}(tx,ty) = t\mathbf{M}(x,y)$ $(t,x,y \in \mathbb{R}^+)$.

A two-variable function $\mathbf{M}(x,y)$ satisfying condition (vi) can be reduced to a one-variable function $f(x) := \mathbf{M}(1,x)$. Namely, $\mathbf{M}(x,y)$ is recovered from f as

$$\mathbf{M}(x,y) = xf\left(\frac{y}{x}\right). \tag{2.1}$$

What are the properties of f which imply conditions (i)–(v)? They are as follows.

(i)$'$ $f(1) = 1$

(ii)$'$ $tf(t^{-1}) = f(t)$

(iii)$'$ $f(t) > 1$ if $t > 1$ and $f(t) < 1$ if $0 < t < 1$.

(iv)$'$ f is monotone increasing.

(v)$'$ f is continuous.

Then a homogeneous and continuous mean is uniquely described by a function f satisfying the properties (i)$'$–(v)$'$.

Assume that $0 < x \leqslant y \leqslant z$ and define a recursion

$$x_1 = x, \qquad y_1 = y, \qquad z_1 = z \tag{2.2}$$

$$x_{n+1} = M_2(x_n, y_n), \qquad y_{n+1} = M_2(x_n, z_n), \qquad z_{n+1} = M_2(y_n, z_n). \tag{2.3}$$

Below we refer to this recursion as **symmetrization procedure**.

One can show by induction that $x_n \leqslant y_n \leqslant z_n$, moreover the sequence (x_n) is increasing and (z_n) is decreasing. Therefore, the limits

$$L := \lim_{n\to\infty} x_n \qquad \text{and} \qquad U = \lim_{n\to\infty} z_n \tag{2.4}$$

exist. It is not difficult to show that $L = U$ must hold [12].

Given a mean $\mathbf{M}_2(x, y)$ of two variables, we define $\mathbf{M}_3(x, y, z)$ as the limit $\lim_n x_n = \lim_n y_n = \lim_n z_n$ in the above recursion (2.2) and (2.3) for any $x, y, z \in \mathbb{R}^+$. Note that the existence of the limit does not require the symmetry of the $\mathbf{M}_2(x, y)$.

Example 2.1. Let

$$\mathbf{M}_f(x, y) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right)$$

be a **quasi-arithmetic mean** defined by a strictly monotone function f [5]. For these means

$$\mathbf{M}_3(x, y, z) = f^{-1}\left(\frac{f(x) + f(y) + f(z)}{3}\right).$$

Note that arithmetic, geometric and harmonic means belong to this class. $\square$

When our paper [12] was written, we were not aware of the paper [6], in which the following definition was given. Assume that m is a mean of two variables. A mean $\mathbf{M}$ of three variables is said to be of type 1 invariant mean with respect to m if

$$\mathbf{M}(m(a, c), m(a, b), m(b, c)) = \mathbf{M}(a, b, c).$$

It is obtained in [6] that to each m there exists a unique $\mathbf{M}$ which is type 1 invariant with respect to m. The proof is exactly the above symmetrization procedure.

A theory of means of positive operators was developed by Kubo and Ando [7]. The key point of the theory that for positive matrices A and B formula (2.1) should be modified as

$$\mathbf{M}(A,B) = A^{1/2} f\big(A^{-1/2}BA^{-1/2}\big)A^{1/2} \tag{2.5}$$

and the function f is required to be operator monotone. This property implies that f' is decreasing and $f'(1) = 1/2$. Since the matrix case is in the center of our interest, we assume these properties below.

Assume that $x < y$.

$$\frac{\mathbf{M}(x,y) - x}{y-x} = \frac{x(f(y/x)-1)}{y-x} = \frac{x(y/x-1)}{y-x} f'(t) = f''(t),$$

where $1 < t < y/x$. When f' is decreasing, we have

$$f'(y/x) \le \frac{\mathbf{M}(x,y) - x}{y-x} \le f'(1). \tag{2.6}$$

Lemma 2.2. *Let $0 < x \leqslant y \leqslant z$ and assume that f' is decreasing and $f'(1) = 1/2$. Then*

$$\frac{\mathbf{M}(y,z) - \mathbf{M}(x,y)}{z-x} \le 1 - f'(y/x).$$

Proof. From (2.6) we have

$$\frac{y - \mathbf{M}(x,y)}{y-x} \le 1 - f'(y/x) \qquad \text{and} \qquad \frac{\mathbf{M}(y,z) - y}{z-y} \le f'(1),$$

moreover $\max\big(f'(1), 1 - f'(y/x)\big) = 1 - f'(y/x)$. □

It is an important consequence of the lemma that the limit (2.4) is exponential. Namely,

$$z_n - x_n \le (z-x)\big(1 - f'(z/x)\big)^n \tag{2.7}$$

holds. When n is large, then z_n/x_n is close to 1 and $(z_{n+1} - x_{n+1})/(z_n - x_n)$ has an upper bound close to $1/2$.

A sort of mean of three positive matrices can be obtained by a symmetrization procedure from the two-variable-means, at least under some restriction.

Theorem 2.3. *Let $A, B, C \in \mathbf{M}_n(\mathbb{C})$ be positive definite matrices and let $\mathbf{M}_2$ be an operator mean. Assume that $A \leqslant B \leqslant C$. Set a recursion as*

$$A_1 = A, \qquad B_1 = B, \qquad C_1 = C, \tag{2.8}$$

$$A_{n+1} = \mathbf{M}_2(A_n, B_n), \qquad B_{n+1} = \mathbf{M}_2(A_n, C_n), \qquad C_{n+1} = \mathbf{M}_2(B_n, C_n). \tag{2.9}$$

Then the limits

$$\mathbf{M}_3(A,B,C) := \lim_n A_n = \lim_n B_n = \lim_n C_n \tag{2.10}$$

exist.

Proof. By the monotonicity of $\mathbf{M}_2$ and mathematical induction, we see that $A_n \leqslant B_n \leqslant C_n$. It follows that the sequence (A_n) is increasing and (C_n) is decreasing. Therefore, the limits

$$L := \lim_{n\to\infty} A_n \qquad \text{and} \qquad U = \lim_{n\to\infty} C_n$$

exist. We claim that $L = U$.

Assume that $L \neq U$. By continuity, $B_n \to \mathbf{M}_2(L, U) =: M$, where $L < M < U$. Since

$$\mathbf{M}_2(B_n, C_n) = C_{n+1},$$

the limit $n \to \infty$ gives $\mathbf{M}_2(M, U) = U$, which contradicts $M < U$. □

By the symmetrization procedure any operator mean of two variables has an extension to those triplets (A, B, C) which can be ordered. In a few cases, the latter restriction can be skipped. The arithmetic and the harmonic means belong to this class.

The three-variable matrix means defined by symmetrization are continuous and monotone in each of the variables. This facts follow straightforwardly from the construction.

Example 2.4. The **logarithmic mean** of the positive numbers x and y is

$$\frac{x-y}{\log x - \log y}\,. \tag{2.11}$$

The corresponding function

$$f(x) = \frac{x-1}{\log x}$$

is operator monotone and so the mean makes sense for positive matrices as well. If A and B are positive matrices, then

$$\mathbf{G}(A, B) \leq \mathbf{L}(A, B) \leq \mathbf{A}(A, B)$$

holds for the geometric, logarithmic and arithmetic means. When $A \leq B \leq C$ are positive matrices, then the symmetrization procedure defines the three-variable means $\mathbf{G}_3, \mathbf{L}_3$ and $\mathbf{A}_3$. (Of course, $\mathbf{A}_3$ is clear without symmetrization.) It follows from the procedure, that

$$\mathbf{G}_3(A, B, C) \leq \mathbf{L}_3(A, B, C) \leq \mathbf{A}_3(A, B, C).$$

Note that the paper aimed to discuss the three-variable mean $\mathbf{L}_3$ for numbers, but only some inequalities were obtained.

3. Means of matrices and information geometry

An important non-trivial example of operator means is the **geometric mean**:

$$A\#B = A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^{1/2}A^{1/2} \tag{3.1}$$

which has the special property

$$(\lambda A)\#(\mu B) = \sqrt{\lambda\mu}(A\#B) \tag{3.2}$$

for positive numbers λ and μ. The geometric mean was found before the general theory of matrix means, see [13], in a very different context.

Theorem 3.1. *Let $A, B, C \in M_n(\mathbb{C})$ be positive definite matrices. Set a recursion as*

$$A_1 = A, \qquad B_1 = B, \qquad C_1 = C, \tag{3.3}$$

$$A_{n+1} = A_n\#B_n, \qquad B_{n+1} = A_n\#C_n, \qquad C_{n+1} = B_n\#C_n. \tag{3.4}$$

Then the limit

$$G(A,B,C) := \lim_n A_n = \lim_n B_n = \lim_n C_n \tag{3.5}$$

exists.

Proof. Choose positive numbers λ and μ such that

$$A' := A < B' := \lambda B < C' := \mu C.$$

Start the recursion with these matrices. By Theorem 2.3 the limits

$$G(A',B',C') := \lim_n A'_n = \lim_n B'_n = \lim_n C'_n$$

exist. For the numbers

$$a := 1, \qquad b := \lambda \qquad \text{and} \qquad c := \mu$$

the recursion provides a convergent sequence (a_n, b_n, c_n) of triplets.

$$(\lambda\mu)^{1/3} = \lim_n a_n = \lim_n b_n = \lim_n c_n.$$

Since

$$A_n = A'_n/a_n, \qquad B_n = B'_n/b_n \qquad \text{and} \qquad C_n = C'_n/c_n$$

due to property (3.2) of the geometric mean, the limits stated in the theorem must exist and equal $G(A',B',C')/(\lambda\mu)^{1/3}$. $\square$

The result of Theorem 3.1 was obtained in [2] but our proof is different and completely elementary. A different approach is based on Riemannian geometry [3, 9].

The positive definite matrices might be considered as the variance of multivariate normal distributions and the information geometry of Gaussians yields a natural Riemannian metric. The simplest way to construct an information geometry is to start with an **information potential function** and to introduce the Riemannian metric by the Hessian of the potential. We want a geometry on the family of non-degenerate multivariate Gaussian distributions with zero mean vector. Those distributions are given by a positive definite real matrix A in the form

$$f_A(x) := \frac{1}{\sqrt{(2\pi)^n \det A}} \exp\big(-\langle A^{-1}x, x\rangle/2\big) \qquad (x \in \mathbb{R}^n). \tag{3.6}$$

We identify the Gaussian (3.6) with the matrix A, and we can say that the Riemannian geometry is constructed on the space of positive definite real matrices. There are many reasons (originated from statistical mechanics, information theory and mathematical statistics) that the **Boltzmann entropy**

$$S(f_A) := C + \frac{1}{2}\log\det A \qquad (C \text{ is a constant}) \tag{3.7}$$

is a candidate for being an information potential.

The $n \times n$ real symmetric matrices can be identified with the Euclidean space of dimension $n(n+1)/2$ and the positive definite matrices form an open set. Therefore the set of Gaussians has a simple and natural manifold structure. The tangent space at each foot point is the set of symmetric matrices. The Riemannian metric is defined as

$$g_A(H_1, H_2) := \frac{\partial^2}{\partial s \partial t} S(f_{A+tH_1+sH_2})\Big|_{t=s=0}, \tag{3.8}$$

where H_1 and H_2 are tangents at A. The differentiation easily gives

$$g_A(H_1, H_2) = \operatorname{Tr} A^{-1}H_1A^{-1}H_2\,. \tag{3.9}$$

The corresponding information geometry of the Gaussians was discussed in [10] in details. We note here that this geometry has many symmetries, each similarity transformation of the matrices becomes a symmetry. In the statistical model of multivariate distributions (3.9) plays the role of the **Fisher-Rao metric**.

(3.9) determines a Riemannian metric on the set $\mathcal{P}$ of all positive definite complex matrices as well and below we prefer to consider the complex case. The geodesic connecting $A, B \in \mathcal{P}$ is

$$\gamma(t) = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2} \qquad (0 \le t \le 1)$$

and we observe that the midpoint $\gamma(1/2)$ is just the geometric mean $A\#B$. The geodesic distance is

$$\delta(A, B) = \|\log(A^{-1/2}BA^{-1/2})\|_2\,,$$

where $\|\cdot\|_2$ stands for the Hilbert–Schmidt norm. (It was computed in [1] that the scalar curvature of the space $\mathcal{P}$ is constant.) These observations show that the information Riemannian geometry is adequate to treat the geometric mean of positive definite matrices [3, 9].

Let A, B and C be positive definite matrices. The mean $C' := A\#B$ is the middle point of the geodesic connecting A with B, $B' := C\#A$ and $C' := A\#B$ have similar geometric description. Since

$$\delta(A\#C, B\#C) \le \tfrac{1}{2}\delta(A, B) \tag{3.10}$$

(see Prop. 6 in [3]), the diameter of the triangle $A'B'C'$ is at most the half of the diameter of ABC.

When A_n, B_n, C_n are defined by the symmetrization procedure, the sequences (A_n), (B_n) and (C_n) form Cauchy sequences with respect to the geodesic distance δ. The space is complete with respect to this metric and the three sequences have a common limit point.

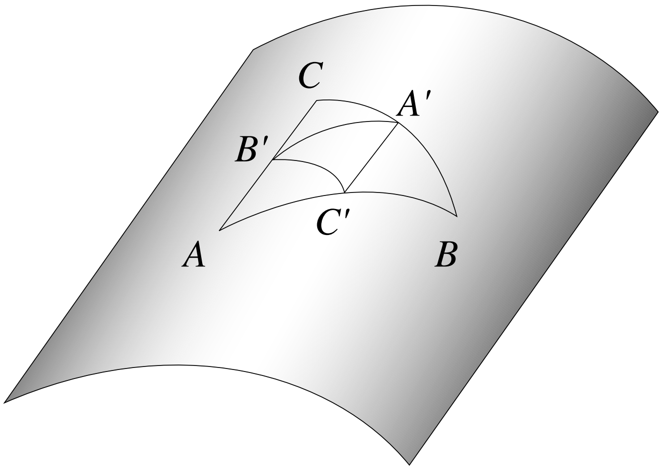

Figure 2: Geometric view of the symmetrization

The geometric view of the symmetrization procedure concerning the geometric mean in the Riemannian space of positive definite matrices resembles very much the procedure concerning the arithmetic mean in the flat space.

The arithmetic mean of matrices A_1, A_2 and A_3 is the minimizer of the functional

$$Z \mapsto \|Z - A_1\|^2 + \|Z - A_2\|^2 + \|Z - A_3\|^2,$$

where the norm is the Hilbert–Schmidt norm. Following this example, one may define the geometric mean of the positive matrices A_1, A_2 and A_3 as the minimizer of the functional

$$Z \mapsto \delta(Z, A_1)^2 + \delta(Z, A_2)^2 + \delta(Z, A_3)^2.$$

This approach is discussed in several papers [8, 9, 3]. The minimizer is unique, the mean is well-defined but it is different from the three-variable geometric mean coming out from the symmetrization procedure.

Note that there are several natural Riemannian structures on the cone of positive definite matrices. When such a matrix is considered as a quantum statistical operator (without the normalization constraint), the information geometries correspond to different Riemannian metrics, see [11].

4. Discussion

It seems that there are more 3-variable-means without the ordering constraint than the known arithmetic, geometric and harmonic means. (Computer simulation has been carried out for some other means as well.)

An operator monotone function f associated with a matrix mean has the property $f(1) = 1$ and $f'(1) = 1/2$. The latter formula follows from (ii)$'$. From the power series expansion around 1, one can deduce that there are some positive numbers ε and δ such that

$$\|A^{1/2} f(A^{-1/2} B_1 A^{-1/2}) A^{1/2} - A^{1/2} f(A^{-1/2} B_2 A^{-1/2}) A^{1/2}\| \le (1-\delta)\|B_1 - B_2\|$$

whenever $I - \varepsilon \le A, B_1, B_2 \le I + \varepsilon$. This estimate equivalently means

$$\|\mathbf{M}_f(A, B_1) - \mathbf{M}_f(A, B_2)\| \le (1-\delta)\|B_1 - B_2\|\,, \tag{4.1}$$

where $\|\cdot\|$ denotes the operator norm.

If the **diameter** of a triplet (A, B, C) is defined as

$$D(A, B, C) := \max\{\|A - B\|, \|B - C\|, \|A - C\|\}\,,$$

then we have

$$D(A_{n+1}, B_{n+1}, C_{n+1}) \le (1-\delta) D(A_n, B_n, C_n),$$

provided that the condition $I - \varepsilon \le A_1, B_1, C_1 \le I + \varepsilon$ holds. Therefore in a small neighborhood of the identity the symmetrization procedure converges exponentially fast for any triplet of matrices and for any matrix mean. We conjecture that this holds in general.

References

[1] Andai, A., Information geometry in quantum mechanics, Ph.D. dissertation (in Hungarian), BUTE, 2004.

[2] Ando, T., Li, C-K., Mathias, R., Geometric means, *Linear Algebra Appl.*, Vol. 385 (2004), 305–334.

[3] Bhatia, R., Holbrook, J., Geometry and means, preprint, 2005.

[4] Carlson, B. C., The logarithmic mean, *Amer. Math. Monthly*, Vol. 79 (1972), 615–618.

[5] Daróczy, Z., Páles, Zs., Gauss-composition of means and the solution of the Matkowski-Sutő problem, *Publ. Math. Debrecen*, **61**(2002), 157–218.

[6] Horwitz, A., Invariant means, *J. Math. Anal. Appl.*, Vol. 270 (2002), 499–518.

[7] Kubo, F., Ando, T., Means of positive linear operators, *Math. Ann.*, Vol. 246 (1980), 205–224.

[8] Lim, Y., Geometric means of symmetric connes, *Arch. Math.*, Vol. 75 (2000), 39–45.

[9] Moakher, M., A differential geometric approach to the geometric mean of symmetric positive definite matrices, *SIAM J. Matrix Anal. Appl.*, Vol. 26 (2005), 735–747.

[10] Ohara, A., Suda, N., Amari, S., Dualistic differential geometry of positive definite matrices and its applications to related problems, *Linear Algebra Appl.*, Vol. 247 (1996), 31–53.

[11] Petz, D., Monotone metrics on matrix spaces, *Linear Algebra Appl.*, Vol. 244 (1996), 81–96.

[12] Petz, D., Temesi, R., Means of positive numbers and matrices, *SIAM J. Matrix Anal. Appl.*, to appear.

[13] Pusz, W., Woronowicz, S. L., Functional calculus for sesquilinear forms and the purification map, *Rep. Math. Phys.*, Vol. 8 (1975), 159–170.

[14] Skovgaard, L. T., A Riemannian geometry of the multivariate normal model, *Scand. J. Statistics*, Vol. 11 (1984), 211–223.

Dénes Petz
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
H–1053 Budapest, Reáltanoda u. 13–15, Hungary

Annales Mathematicae et Informaticae
32 *(2005) pp. 141–152.*

Discrete spectral synthesis

László Székelyhidi

Institute of Mathematics, University of Debrecen
e-mail: szekely@math.klte.hu

Abstract

Discrete spectral analysis and synthesis study the description of translation invariant function spaces over discrete Abelian groups. The basic building bricks are the exponential monomials. A remarkable result of R. J. Elliot in 1965 claimed that spectral synthesis holds on any Abelian group, which means that the exponential monomials span a dense linear subspace in any pointwise-closed translation invariant linear space of complex valued functions over the group. Unfortunately, the proof of this theorem had several gaps. In this paper we give a short survey about the present status of discrete spectral analysis and synthesis, we show that Elliot's theorem is false, we give a necessary condition for Abelian groups to have spectral synthesis and we formulate a conjecture about a possible characterization of Abelian groups having spectral synthesis.

Key Words: spectral synthesis, torsion free rank, polynomial functions

AMS Classification Number: 43A45, 39A70, 20K15

1. Introduction

Spectral analysis and spectral synthesis deal with the description of translation invariant function spaces over locally compact Abelian groups. We consider the space $\mathcal{C}(G)$ of all complex valued continuous functions on a locally compact Abelian group G, which is a locally convex topological linear space with respect to the pointwise linear operations (addition, multiplication with scalars) and to the topology of uniform convergence on compact sets. Continuous homomorphisms of G into the additive topological group of complex numbers, and into the multiplicative topological group of nonzero complex numbers are called *additive* and *exponential functions*, respectively. A *polynomial* is a finite linear combination of products of additive functions and an *exponential monomial* is a product of a polynomial and an exponential function. Linear combinations of exponential monomials are called *exponential polynomials*.

Suppose that a closed linear subspace in the space $\mathcal{C}(G)$ is given, which is *translation invariant*, which means that if f belongs to this subspace then the *translate* $\tau_y f$ *of* f *by* y, defined by

$$\tau_y f(x) = f(x+y)$$

belongs to the subspace as well, for any x, y in G. Such subspaces are called *varieties* and these are the main objectives of spectral analysis and spectral synthesis.

It turns out that exponential functions, or more generally, exponential monomials can be considered as basic building bricks of varieties. A given variety may or may not contain any exponential function or exponential monomial of the above mentioned form. If it contains an exponential function, then we say that *spectral analysis holds* for the variety. An exponential function in a variety can be considered as a kind of *spectral value* and the set of all exponential functions in a variety is called the *spectrum* of the variety. It follows that spectral analysis for a variety means that the spectrum of the variety is nonempty. On the other hand, the set of all exponential monomials contained in a variety is called the *spectral set* of the variety. It turns out that if an exponential monomial belongs to a variety, then the exponential function appearing in the representation of this exponential monomial belongs to the variety, too. Hence if the spectral set of a variety is nonempty, then also the spectrum of the variety is nonempty and spectral analysis holds. There is, however an even stronger property of some varieties, namely, if the spectral set of the variety span a dense subspace of the variety. In this case we say that *spectral synthesis holds* for the variety. It follows, that for nonzero varieties spectral synthesis implies spectral analysis. If spectral analysis, resp. spectral synthesis holds for every variety on an Abelian group, then we say that *spectral analysis* holds, resp. *spectral synthesis* holds on the Abelian group. A famous and pioneer result of L. Schwartz [1] exhibits the situation by stating that if the group is the reals with the Euclidean topology, then spectral values do exist, that is, any nonzero variety contains an exponential function, the spectrum is nonempty, spectral analysis holds. Furthermore, spectral synthesis also holds in this situation: there are sufficiently many exponential monomials in the variety in the sense that their linear hull is dense in the subspace.

An interesting particular case is presented by discrete Abelian groups. Here the problem seems to be purely algebraic: all complex functions are continuous, and convergence is meant in the pointwise sense. The archetype is the additive group of integers: in this case the closed translation invariant function spaces can be characterized by systems of homogeneous linear difference equations with constant coefficients. It is known that these function spaces are spanned by exponential monomials corresponding to the characteristic values of the equation, together with their multiplicities. In this sense the classical theory of homogeneous linear difference equations with constant coefficients can be considered as spectral analysis and spectral synthesis on the additive group of integers.

The next simplest case is the case of systems of homogeneous linear difference equations with constant coefficients in several variables, or, in other words, spectral analysis and spectral synthesis on free Abelian groups with a finite number of generators. As in this case a structure theorem is available, namely, any group of this type is a direct product of finitely many copies of the additive group of integers, it is not very surprising to have the corresponding - nontrivial - result by M. Lefranc [2]: on finitely generated free Abelian groups spectral analysis and spectral synthesis holds for any closed translation invariant subspace.

Based on these results the natural question arises: what about other discrete Abelian groups? In his 1965 paper [4] R. J. Elliot presented a theorem on spectral synthesis for arbitrary Abelian groups. However, in 1987 Z. Gajda drew my attention to the fact that the proof of Elliot's theorem had several gaps. Since then several efforts have been made to solve the problem of discrete spectral analysis and spectral synthesis on Abelian groups. In the subsequent paragraphs we present a summary about the status of these problems. From now on we consider only discrete Abelian groups and all the above mentioned concepts are meant in the discrete setting.

2. Spectral analysis and spectral synthesis on finitely generated Abelian groups

The first general result about spectral synthesis is due to M. Lefranc on free Abelian groups of finite rank, that means, on groups of the form $\mathbb{Z}^k$ with some nonnegative integer k (see [2]).

Theorem 2.1. *Spectral synthesis holds for any free Abelian group of finite rank.*

Using the following simple lemma we can clarify the connection between spectral synthesis and spectral analysis.

Lemma 2.2. *Let G be an Abelian group, V a variety in $\mathcal{C}(G)$, $p : G \to \mathbb{C}$ a nonzero polynomial and $m : G \to \mathbb{C}$ an exponential function. If the exponential monomial $p\,m$ belongs to V, then m belongs to V, too.*

Proof. The statement is obvious if p is a nonzero constant. On the other hand, if p is a nonconstant polynomial, then $\Delta_y p$ is a nonzero polynomial for some y in G, with degree one less than that of p. Moreover, by the identity

$$\Delta_y p(x)\, m(x) = p(x+y)m(x) - p(x)m(x) = p(x+y)m(x+y)m(-y) - p(x)m(x)$$

which holds for each x, y in G it follows that the exponential monomial $(\Delta_y p)\, m$ belongs to V for each y in G. Hence our statement follows by induction on the degree of p. □

From this lemma we infer the following theorem.

Theorem 2.3. *If spectral synthesis holds for an Abelian group, then also spectral analysis holds for it.*

Using Theorem 2.1 we have the following easy consequence.

Theorem 2.4. *Spectral analysis holds for any free Abelian group of finite rank.*

The following theorem makes it possible to extend the above results.

Theorem 2.5. *If spectral synthesis holds for an Abelian group then it holds for its homomorphic images, too.*

Proof. Suppose that G is an Abelian group, H is a homomorphic image of G and let $F: G \to H$ be a surjective homomorphism. If V is a variety in $\mathcal{C}(H)$, then we let

$$V_F = \{f \circ F : \quad f \in V\}.$$

Using the surjectivity of F a routine calculation shows that V_F is a variety in $\mathcal{C}(H)$. Let Φ be an exponential monomial in V_F of the form

$$\Phi(x) = P(A_1(x), A_2(x), \ldots, A_n(x))M(x)\,, \tag{2.1}$$

where $A_1, A_2, \ldots, A_n$ are linearly independent additive functions on G, M is an exponential on G, and P is a complex polynomial in n variables. By Lemma 2.2 the exponential M is in V_F, too, hence $M = m \circ F$ holds for some m in V. If u, v are arbitrary in H, then $u = F(x)$ and $v = F(y)$ for some x, y in G, which implies

$$m(u+v) = m(F(x)+F(y)) = m(F(x+y)) = M(x+y) = M(x)M(y) =$$

$$= m(F(x))\,m(F(y)) = m(u)m(v)\,.$$

As m is never zero, hence m is an exponential in V. On the other hand, (2.1) implies that

$$q(x) = P(A_1(x), A_2(x), \ldots, A_n(x)) = p(F(x))$$

holds for each x in G with some function $p: H \to \mathbb{C}$. We show that p is a polynomial on H. Using the Newton Interpolation Formula and the Taylor Formula in several variables it follows easily that the functions $A_1, A_2, \ldots, A_n$ can be expressed as a linear combination of some translates of q. On the other hand, if $F(x) = F(y)$ for some x, y in G, then $q(x+z) = q(y+z)$ holds for each z in G, hence $A_i(x) = A_i(y)$ for $i = 1, 2, \ldots, n$. It follows that we can define the functions $a_i : H \to \mathbb{C}$ for $i = 1, 2, \ldots, n$ by the equation

$$a_i(u) = A_i(F(x))\,,$$

where x is arbitrary in G with the property $F(x) = u$. Further, we see immediately that a_i is additive for $i = 1, 2, \ldots, n$. On the other hand,

$$p(u) = p(F(x)) = P(A_1(x), A_2(x), \ldots, A_n(x)) = P(a_1(u), a_2(u), \ldots, a_n(u))$$

holds for any u in H, hence p is a polynomial on H. This means that the exponential monomial Φ above has the form $\Phi = \varphi \circ F$ with some exponential monomial φ in V. Finally, it is straightforward to verify that if the exponential monomials span a dense subspace in V_F, then the corresponding exponential monomials span a dense subspace in V, so our proof is complete. $\square$

Using the well-known fact that every finitely generated Abelian group is the homomorphic image of some free Abelian group of finite rank we have the following result.

Theorem 2.6. *Spectral synthesis and spectral analysis holds for any finitely generated Abelian group.*

At this point a simple question can be formulated: is there any non-finitely generated Abelian group, on which spectral synthesis, or spectral analysis holds?

3. Spectral analysis and spectral synthesis on arbitrary Abelian groups

In 1965 R. J. Elliot published the following result in the Proc. Cambridge Phil. Soc. (see [4]):

Theorem 3.1. *Spectral synthesis holds on any Abelian group.*

Of course a theorem of this type would have closed all open problems concerning discrete spectral analysis and spectral synthesis. Unfortunately, in 1990 the polish mathematician Zbigniew Gajda called my attention to the fact that the proof of Elliot's theorem had several gaps. After several efforts of Gajda and myself we were unable either to fill those gaps or to find a counterexample to Elliot's result. Obviously, the question about spectral analysis on arbitrary Abelian groups turned to be open again. In this respect we could prove the following result (see [9]).

Theorem 3.2. *Spectral analysis holds on every Abelian torsion group.*

Proof. We show that every nonzero variety in $\mathcal{C}(G)$ contains a character. Let V be any nonzero variety in $\mathcal{C}(G)$. Then by the Hahn–Banach theorem V is equal to the annihilator of its annihilator, that is, there exists a set Λ of finitely supported complex measures on G such that V is exactly the set of all functions in $\mathcal{C}(G)$ which are annihilated by all members of Λ:

$$V = V(\Lambda) = \{f | f \in \mathcal{C}(G), \langle \lambda, f \rangle = 0 \text{ for all } \lambda \in \Lambda\}.$$

We show that for any finite subset Γ in Λ its annihilator, $V(\Gamma)$ contains a character. Indeed, let F_Γ denote the subgroup generated by the supports of the measures belonging to Γ. Then F_Γ is a finitely generated torsion group. The measures belonging to Γ can be considered as measures on F_Γ and the annihilator of Γ in $\mathcal{C}(F_\Gamma)$ will be denoted by $V(\Gamma)_{F_\Gamma}$. This is a variety in $\mathcal{C}(F_\Gamma)$. It is also nonzero. Indeed, if f belongs to V then its restriction to F_Γ belongs to $V(\Gamma)_{F_\Gamma}$. If, in addition, we have $f(x_0) \neq 0$ and y_0 is in F_Γ, then the translate of f by $x_0 - y_0$ belongs to V, its restriction to F_Γ belongs to $V(\Gamma)_{F_\Gamma}$ and at y_0 it takes the value $f(x_0) \neq 0$. Hence $V(\Gamma)_{F_\Gamma}$ is a nonzero variety in $\mathcal{C}(F_\Gamma)$. As F_Γ is finitely generated, by Theorem 2.6 spectral analysis holds, and, in particular $V(\Gamma)_{F_\Gamma}$ contains exponential functions.

As F_Γ is a torsion group, any exponential function on F_Γ is a character. That means, $V(\Gamma)_{F_\Gamma}$ contains a character of F_Γ. It is well-known (see e.g. [3]) that any character of F_Γ can be extended to a character of G, and obviously any such extension belongs to $V(\Gamma)$.

Now we have proved that for any finite subset Γ of the set Λ the annihilator $V(\Gamma)$ contains a character. Let $char(V)$ denote the set of all characters contained in V. Obviously $char(V)$ is a compact subset of $\widehat{G}$, the dual of G, because $char(V)$ is closed and $\widehat{G}$ is compact. On the other hand, the system of nonempty compact sets $char(V(\Gamma))$, where Γ is a finite subset of Λ, has the finite intersection property:

$$char(V(\Gamma_1 \cup \Gamma_2)) \subseteq char(V(\Gamma_1)) \cap char(V(\Gamma_2)).$$

We infer that the intersection of this system is nonempty, and obviously

$$\emptyset \neq \bigcap_{\Gamma \subseteq \Lambda \ \text{finite}} char(V(\Gamma)) \subseteq char(V).$$

That means, $char(V)$ is nonempty, and the theorem is proved. $\square$

This theorem presents a partial answer to our previous question: as obviously there are Abelian torsion groups which are not finitely generated, hence there are non-finitely generated Abelian groups on which spectral analysis holds.

In 2001 G. Székelyhidi in [8] presented a different approach to the result of Lefranc, and he actually proved that spectral analysis holds on countably generated Abelian groups, further, his method strongly supported the conjecture that spectral analysis - hence also spectral synthesis - might fail to hold on free Abelian groups having no generating set with cardinality less than the continuum. At the 41st International Symposium on Functional Equations in 2003, Noszvaj, Hungary we presented a counterexample to Theorem 3.1 of Elliot in [4]. The counterexample depends on the following observation (see [10]).

Theorem 3.3. *Let G be an Abelian group. If there exists a symmetric bi-additive function $B : G \times G \to \mathbb{C}$ such that the variety V generated by the quadratic function $x \mapsto B(x,x)$ is of infinite dimension, then spectral synthesis fails to hold for V.*

Proof. Let $f(x) = B(x,x)$ for all x in G. By the equation

$$f(x+y) = B(x+y, x+y) = B(x,x) + 2B(x,y) + B(y,y) \tag{3.1}$$

we see that the translation invariant subspace generated by f is generated by the functions 1, f and all the additive functions of the form $x \mapsto B(x,y)$, where y runs through G. Hence our assumption on B is equivalent to the condition that there are infinitely many functions of the form $x \mapsto B(x,y)$ with y in G, which are linearly independent. This also implies that there is no positive integer n such that B can be represented in the form

$$B(x,y) = \sum_{k=1}^{n} a_k(x) b_k(y),$$

where $a_k, b_k : G \to \mathbb{C}$ are additive functions $(k = 1, 2, \dots, n)$. Indeed, the existence of a representation of this form would mean that the number of linearly independent additive functions of the form $x \mapsto B(x, y)$ is at most n.

It is clear that any translate of f, hence any function g in V satisfies

$$\Delta_y^3 g(x) = 0 \tag{3.2}$$

for all x, y in G: this can be checked directly for f. Hence any exponential m in V satisfies the same equation, which implies

$$m(x)\big(m(y) - 1\big)^3 = 0$$

for all x, y in G, and this means that m is identically 1. It follows that any exponential monomial in V is a polynomial. By the results in [5] (see also [6]) and by (3.2) g can be uniquely represented in the following form:

$$g(x) = A(x, x) + c(x) + d$$

for all x in G, where $A : G \times G \to \mathbb{C}$ is a symmetric bi-additive function, $c : G \to \mathbb{C}$ is additive and d is a complex number. Here "uniqueness" means that the "monomial terms" $x \mapsto A(x, x)$, $x \mapsto c(x)$ and d are uniquely determined (see [6]). In particular, any polynomial p in V has a similar representation, which means that it can be written in the form

$$p(x) = \sum_{k=1}^{n} \sum_{l=1}^{m} c_{kl}\, a_k(x) b_l(x) + c(x) + d = p_2(x) + c(x) + d$$

with some positive integers n, m, additive functions $a_k, b_l, c : G \to \mathbb{C}$ and constants c_{kl}, d. Suppose that p_2 is not identically zero. By assumption, p is the pointwise limit of a net formed by linear combinations of translates of f, that means, by functions of the form (3.1). Linear combinations of functions of the form (3.1) can be written as

$$\varphi(x) = c\, B(x, x) + A(x) + D,$$

with some additive function $A : G \to \mathbb{C}$ and constants c, D. Any net formed by these functions has the form

$$\varphi_\gamma(x) = c_\gamma\, B(x, x) + A_\gamma(x) + D_\gamma.$$

By pointwise convergence

$$\lim_\gamma \frac{1}{2} \Delta_y^2 \varphi_\gamma(x) = \frac{1}{2} \Delta_y^2 p(x) = p_2(y)$$

follows for all x, y in G. On the other hand,

$$\lim_\gamma \frac{1}{2} \Delta_y^2 \varphi_\gamma(x) = B(y, y) \lim_\gamma c_\gamma,$$

holds for all x, y in G, hence the limit $\lim_\gamma c_\gamma = c$ exists and is different from zero, which gives $B(x,x) = \frac{1}{c}\, p_2(x)$ for all x in G and this is impossible.

We infer that any exponential monomial φ in V is actually a polynomial of degree at most 1, which satisfies

$$\Delta_y^2 \varphi(x) = 0 \tag{3.3}$$

for each x, y in G, hence any function in the closed linear hull of the exponential monomials in V satisfies this equation. However f does not satisfy (3.3), hence the linear hull of the exponential monomials in V is not dense in V. □

Using this theorem we are in the position to disprove the result Theorem 3.1 of Elliot. In what follows $\mathbb{Z}^\omega$ denotes the (non-complete) direct sum of countably many copies of the additive group of integers, or, in other words, the set of all finitely supported $\mathbb{Z}$-valued functions on the nonnegative integers.

Theorem 3.4. *Spectral synthesis fails to hold on any Abelian group with torsion free rank at least ω.*

Proof. First of all we will show that there exists a symmetric bi-additive function $B : \mathbb{Z}^\omega \times \mathbb{Z}^\omega \to \mathbb{C}$ with the property that there are infinitely many linearly independent functions of the form $x \mapsto B(x, y)$, where y is in $\mathbb{Z}^\omega$. For any nonnegative integer n let p_n denote the projection of the direct sum $\mathbb{Z}^\omega$ onto the n-th copy of $\mathbb{Z}$. This means that for any x in $\mathbb{Z}^\omega$ the number $p_n(x)$ is the coefficient of the characteristic function of the singleton $\{n\}$ in the unique representation of x. It is clear that the functions p_n are additive and linearly independent for different choices of n. Let

$$B(x,y) = \sum_n p_n(x) p_n(y)$$

for each x, y in $\mathbb{Z}^\omega$. The sum is finite for any fixed x, y, and obviously B is symmetric and bi-additive. On the other hand, if χ_k is the characteristic function of the singleton $\{k\}$, then we have

$$B(x, \chi_k) = \sum_n p_n(x) p_n(\chi_k) = p_k(x),$$

hence the functions $x \mapsto B(x, \chi_k)$ are linearly independent for different nonnegative integers k.

The next step is to show that if G is an Abelian group, H is a subgroup of G and $B : H \times H \to \mathbb{C}$ is a symmetric, bi-additive function, then B extends to a symmetric bi-additive function on $G \times G$. Then the extension obviously satisfies the property given in Theorem 3.3 and our statement follows. On the other hand, the existence of the desired extension is proved in [7], Theorem 2.

The proof is complete. □

By this theorem Lefranc's result is the best possible for free Abelian groups: spectral synthesis holds exactly on the finitely generated ones. Hence the following question naturally arises: can spectral synthesis hold on non-finitely generated Abelian groups? If the answer is "yes" then we can ask: is it true that if spectral synthesis fails to hold on an Abelian group, then its torsion free rank is at least ω? In the subsequent paragraphs we shall give partial answers to these questions.

4. Spectral synthesis on Abelian torsion groups

In [11] we proved the following theorem.

Theorem 4.1. *Spectral synthesis holds on any Abelian torsion group.*

Proof. Let V be a proper variety in $\mathcal{C}(G)$ and let W denote the linear span of the set of all characters contained in V. We have to prove that W is dense in V. Supposing the contrary there exists a finitely supported measure x on G such that $\langle x, \gamma\rangle = 0$ whenever γ is a character in V, but $\langle x, f_0\rangle \neq 0$ for some f_0 in V.

Let J denote the support of x; then J is a finite subset of G. Let $\mathcal{H}$ denote the family of all finite subgroups of G containing J. For every H in $\mathcal{H}$ let V_H denote the set of the restrictions of the elements of V to H. It is easy to check that V_H is a variety in $\mathcal{C}(H)$. Whenever a function Φ is defined on J then we put $\langle x, \Phi\rangle = \sum_{g\in J} x(g)\Phi(g)$. If H is in $\mathcal{H}$ then $\langle x, f_0|H\rangle = \langle x, f_0\rangle \neq 0$. Since spectral synthesis holds on H and $f_0|H$ belongs to V_H, there is a character γ_H of H such that γ_H belongs to V_H and $\langle x, \gamma_H\rangle \neq 0$.

Hence we have a net (γ_H) along the directed set $\mathcal{H}$ in the product space $\mathbb{T}^G$ ($\mathbb{T}$ is the complex unit circle). From its compactness it follows that this net has an accumulation point, that is, there is a function $\gamma_0 : G \to \mathbb{T}$ such that for every finite subset F of G and for every $\varepsilon > 0$ there exists an H in $\mathcal{H}$ with $F \cup J$ is included in H and $|\gamma_0(g) - \gamma_H(g)| < \varepsilon$ holds for each g in F. It is clear that γ_0 is a character of G. As V is closed, we also have that γ_0 belongs to V.

Since each element g in J has a finite order, the set of values $\gamma(g)$, where γ is a character and g is in J is finite. This implies that the set $\langle x, \gamma_H\rangle$ for H in $\mathcal{H}$ is a finite set of complex numbers. As $\langle x, \gamma_0\rangle$ is one of these numbers it follows $\langle x, \gamma_0\rangle \neq 0$. This, however, contradicts the fact that γ_0 is in V. □

This theorem shows that there are non-finitely generated Abelian groups on which spectral synthesis holds. Hence we can formulate a quite reasonable conjecture: spectral synthesis holds on an Abelian group if and only if its torsion free rank is finite.

5. Characterization of Abelian groups with spectral analysis and spectral synthesis

In [12] M. Laczkovich and G. Székelyhidi proved the following result.

Theorem 5.1. *Spectral analysis holds on an Abelian group if and only if its torsion free rank is less than the continuum.*

According to Theorem 3.4 there are Abelian groups on which spectral analysis holds and spectral synthesis fails to hold: for instance, any Abelian group with torsion free rank ω, like $\mathbb{Z}^\omega$ above. On the other hand, a complete description of those Abelian groups on which spectral synthesis holds is still missing. The conjecture formulated in the previous section has neither been proved nor disproved yet. An interesting situation can be presented by the additive group of rational numbers. It is not known if spectral synthesis holds on this group. Actually, this group is not finitely generated, however, its torsion free rank is 1. If spectral synthesis does not hold on the rationals, then the above conjecture is drastically disproved: for Abelian groups with torsion free rank zero spectral synthesis holds, as these are exactly the torsion groups. The next simplest case is obviously the case of torsion free rank 1. On the other hand, if spectral synthesis holds on the rational group, then this is the first example for a torsion free group where spectral synthesis holds and the group is not finitely generated.

In addition to the above conjecture in [13] we proved the following theorem.

Theorem 5.2. *The torsion free rank of any Abelian group is equal to the dimension of the linear space consisting of all complex additive functions of the group in the sense that either both are finite and equal, or both are infinite.*

Proof. Let G be an Abelian group and let let $k = r_0(G) \leqslant +\infty$. Then G has a subgroup isomorphic to $\mathbb{Z}^k$. If k is infinite then this is equal to the non-complete direct product of k copies of $\mathbb{Z}$. We will identify this subgroup with $\mathbb{Z}^k$. Obviously $\mathbb{Z}^k$ has at least k linearly independent complex additive functions; for instance we can take the projections onto the different factors of the product group. On the other hand, by the above mentioned result in [3] any homomorphism of a subgroup of an Abelian group into a divisible Abelian group can be extended to a homomorphism of the whole group. As the additive group of complex numbers is obviously divisible, the above mentioned linearly independent complex additive functions of $\mathbb{Z}^k$ can be extended to complex homomorphisms of the whole group G, and the extensions are clearly linearly independent, too. Hence the dimension of the linear space of all complex additive functions of G is not less then the torsion free rank of G.

Now we suppose that $k < +\infty$. Let Φ denote the natural homomorphism of G onto the factor group with respect to $\mathbb{Z}^k$. As it is a torsion group, hence for each element g of G there is a positive integer n such that

$$0 = n\Phi(g) = \Phi(ng)\,,$$

thus ng belongs to the kernel of Φ, which is $\mathbb{Z}^k$. This means that there exist integers $m_1, m_2, \dots, m_k$ such that

$$ng = (m_1, m_2, \dots, m_k)\,.$$

Suppose now that there are $k+1$ linearly independent complex additive functions $a_1, a_2, \dots, a_{k+1}$ on G. Then there exist elements $g_1, g_2, \dots, g_{k+1}$ in G such that the

$(k+1) \times (k+1)$ matrix $\big(a_i(g_j)\big)$ is regular. For $l = 1, 2, \ldots, k$ we let e_l denote the vector in $\mathbb{C}^k$ whose l-th coordinate is 1, the others are 0. By our above consideration there are integers $m_l^{(j)}$, n_j for $l = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, k+1$ such that

$$n_j g_j = (m_1^{(j)}, m_2^{(j)}, \ldots, m_k^{(j)})\,.$$

Hence we have

$$a_i(n_j g_j) = a_i(m_1^{(j)}, m_2^{(j)}, \ldots, m_k^{(j)}) =$$

$$= m_1^{(j)} a_i(e_1) + m_2^{(j)} a_i(e_2) + \cdots + m_k^{(j)} a_i(e_k)\,,$$

and therefore

$$a_i(g_j) = \sum_{l=1}^{k} \frac{m_l^{(j)}}{n_j}\, a_i(e_l)$$

holds for $i, j = 1, 2, \ldots, k+1$. This means that the linearly independent columns of the matrix $\big(a_i(g_j)\big)$ are linear combinations of the columns of the matrix $\big(a_i(e_l)\big)$ for $i = 1, 2, \ldots, k+1$; $l = 1, 2, \ldots, k$. But this is impossible, because the latter matrix has only k columns, hence its rank is at most k.

We have shown that if the torsion free rank of G is the finite number k then the dimension of the linear space consisting of all complex additive functions of G is at most k, hence the theorem is proved. □

Another characterization of Abelian groups with finite torsion free rank is given by the following result (see [13]).

Theorem 5.3. *The torsion free rank of an Abelian group is finite if and only if any complex bi-additive function is a bilinear function of complex additive functions.*

Hence our conjecture has two more equivalent formulations:

- Spectral synthesis holds on an Abelian group if and only if there are only finitely many linearly independent additive functions on the group.

- Spectral synthesis holds on an Abelian group if and only if any complex bi-additive function is a bilinear function of complex additive functions.

References

[1] Schwartz, L., Théorie génerale des fonctions moyenne-périodiques, *Ann. of Math.*, Vol. 48(2) (1947), 857–929.

[2] Lefranc, M., L‘analyse harmonique dans $\mathbb{Z}^n$, *C. R. Acad. Sci. Paris*, Vol. 246 (1958), 1951–1953.

[3] Hewitt, E., Ross, K., *Abstract Harmonic Analysis I.,II.*, Springer Verlag, Berlin, 1963.

[4] Elliot, M. J., Two notes on spectral synthesis for discrete Abelian groups, *Proc. Cambridge Phil. Soc.*, Vol. 61 (1965), 617–620.

[5] Djoković, D. Z., A representation theorem for $(X_1 - 1)(X_2 - 1)\dots(X_n - 1)$ and its applications, *Ann. Polon. Math.*, Vol. 22 (1969), 189–198.

[6] Székelyhidi, L., *Convolution type functional equations on topological Abelian groups*, World Scientific Publishing Co. Pte. Ltd., Singapore, New Jersey, London, Hong Kong, 1991.

[7] Székelyhidi, L., On the extension of exponential polynomials, *Math. Bohemica*, vol. 125, no. 3, pp. 365–370, 2000.

[8] Székelyhidi, G., *Spectral Synthesis on Locally Compact Abelian Groups*, (essay), Cambridge, Trinity College, 2001.

[9] Székelyhidi, L., A Wiener Tauberian theorem on discrete abelian torsion groups, *Annales Acad. Paedag. Cracov., Studia Mathematica I.*, Vol. 4 (2001), 147–150.

[10] Székelyhidi, L., The failure of spectral synthesis on some types of discrete Abelian groups, *Jour. Math. Anal. Appl.*, Vol. 291 (2004), 757–763.

[11] Bereczky, A., Székelyhidi, L., Spectral synthesis on torsion groups, *Jour. Math. Anal. Appl.*, Vol. 304/2, (2005) 607–613.

[12] Laczkovich, M., Székelyhidi, G., Harmonic analysis on discrete Abelian groups, to appear in *Proc. Amer. Math. Soc.*

[13] Székelyhidi, L., Polynomial functions and spectral synthesis, to appear in *Aequationes Math.*

László Székelyhidi
Institute of Mathematics
University of Debrecen
4010 Debrecen, Pf. 12. Hungary

Annales Mathematicae et Informaticae
32 *(2005) pp. 153–157.*

Note on symmetric alteration of knots of B-spline curves

Ibolya Szilágyi

Institute of Mathematics and Computer Science
Károly Eszterházy College
e-mail: szibolya@ektf.hu

Abstract

The shape of a B-spline curve can be influenced by the modification of knot values. Previously the effect caused by symmetric alteration of two knots have been studied on the intervals between the altered knots. Here we show how symmetric knot alteration influences the shape of the B-spline curve over the rest of the domain of definition in the case $k = 3$.

Key Words: B-spline curve, knot modification, path

AMS Classification Number: 68U05

1. Introduction

The properties and capabilities of B-spline curves make them widely used in computer aided geometric design. B-spline curves are polynomial curves defined as linear combination of the control points by some polynomial functions called basis functions. These basis functions are defined in a piecewise way over a closed interval and the subdivision values of this interval are called knots. The basic definitions of the basis functions and the curve are the following.

Definition 1.1. The recursive function $N_j^k(u)$ given by the equations

$$\begin{aligned} N_j^1(u) &= \begin{cases} 1 & \text{if } u \in [u_j, u_{j+1}), \\ 0 & \text{otherwise} \end{cases} \\ N_j^k(u) &= \tfrac{u-u_j}{u_{j+k-1}-u_j} N_j^{k-1}(u) + \tfrac{u_{j+k}-u}{u_{j+k}-u_{j+1}} N_{j+1}^{k-1}(u) \end{aligned}$$

is called normalized B-spline basis function of order k (degree $k-1$). The numbers $u_j \leqslant u_{j+1} \in \mathbf{R}$ are called knot values or simply knots, and $0/0 \doteq 0$ by definition.

Definition 1.2. The curve $\mathbf{s}(u)$ defined by

$$\mathbf{s}(u) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u), \quad u \in [u_{k-1}, u_{n+1}]$$

is called B-spline curve of order $k \leqslant n$ (degree $k-1$), where $N_l^k(u)$ is the l^{th} normalized B-spline basis function, for the evaluation of which the knots $u_0, u_1, \ldots, u_{n+k}$ are necessary. The points $\mathbf{d}_i$ are called control points or de Boor-points, while the polygon formed by these points is called control polygon.

The j^{th} arc can be written as

$$\mathbf{s}_j(u) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(u), \quad u \in [u_j, u_{j+1}).$$

The data structure of these polynomial curves include their order, control points and knot values. Obviously, any modification of these data has some effect on the shape of the curve. In case of control point repositioning the effect is widely studied (c.f. [2], [9] or [10]). The modification of any knot value influences the given curve as well. Applications of knot modifications in computer aided geometric design can be found in [1], [3], [8].

The question, how the alteration of a single knot effects the shape of the curve was studied in [4],[6],[7]. When a knot u_i is altered, points of the curve move on special curves called paths. In [6] the authors proved that these paths are rational curves. In [4] the paths have been extended allowing $u_i < u_{i-1}$ and $u_i > u_{i+1}$.

Instead of a single knot one can modify two knots at the same time. In [5] a general theorem about the extended path obtained by the symmetric alteration of two knots has been verified. (The definition of symmetric knot alteration can be found in [5].) The following statement has been shown:

Theorem 1.3 (Hoffmann-Juhász). *Symmetrically altering the knots u_i, u_{i+z}, $(z = 1, 2, \ldots, k)$, extended paths of points of the arcs $\mathbf{s}_j$, $(j = i, i+1, \ldots, i+z-1)$ converge to the midpoint of the segment bounded by the control points $\mathbf{d}_i$ and $\mathbf{d}_{i+z-k}$ when $\lambda \to -\infty$, i.e.*

$$\lim_{\lambda \to -\infty} \mathbf{s}(u, \lambda) = \frac{\mathbf{d}_i + \mathbf{d}_{i+z-k}}{2}, \quad u \in [u_i, u_{i+z}). \tag{1.1}$$

As the above theorem shows, the authors have studied the effect of symmetric alteration of two knots on the intervals between the altered knots. As the definition of the basis functions shows, the altered knots has effect on some neighbouring intervals as well.

The purpose of the present paper is to extend the above theorem, and describe the effects of the modification of two knots on the neighbouring intervals.

2. Symmetric alteration of two knots: new results

In Theorem 1.3 the extended paths of points of the $\mathbf{s}_j$ arcs has been analyzed in the case $\lambda \to -\infty$. A closer look on the basis functions appearing in the above arcs shows, that the same statement holds for k=3 in the case $\lambda \to +\infty$ as well. This can be verified by the following observation: Substituting $u_i = u_i + \lambda$ and $u_{i+z} = u_{i+z} - \lambda$ into the basis functions we get that λ has the same sign in the numerator and in the denominator of the rational functions.

In Theorem 1.3 the effect of symmetric alteration of two knots is analyzed between the modified knots. However altered knots modify the curve over some neigbouring intervals as well. In the following we analyze the extended paths over these intervals.

We consider the case $k = 3$. The shape of the curve changes over four further inetrvals $([u_{i-2}, u_{i-1}), [u_{i-1}, u_i), [u_{i+z}, u_{i+z+1}), [u_{i+z+1}, u_{i+z+2}))$. On the interval $[u_{i+1}, u_{i+2})$ the only nonzero basis functions are

$$\begin{aligned}
N_{i-1}^3 &= \frac{u_{i+2} - u}{u_{i+2} - u_i} \frac{u_{i+2} - u}{u_{i+2} - u_{i+1}} \\
N_i^3 &= \frac{u - u_i}{u_{i+2} - u_i} \frac{u_{i+2} - u}{u_{i+2} - u_{i+1}} + \frac{u_{i+3} - u}{u_{i+3} - u_{i+1}} \frac{u - u_{i+1}}{u_{i+2} - u_{i+1}} \\
N_{i+1}^3 &= \frac{u - u_{i+1}}{u_{i+3} - u_{i+1}} \frac{u - u_{i+1}}{u_{i+2} - u_{i+1}}.
\end{aligned}$$

First we consider the case $z = 1$, i.e. we alter the u_i, u_{i+1} knots. Substituting $u_i = u_i + \lambda$ and $u_{i+1} = u_{i+1} - \lambda$ in the above functions, while the denominators of N_{i-1}^3, N_i^3 are second degree polynomials in λ the numerators are linear. In the case N_{i+1}^3 the numerator as well as the denominator is quadratic polynomial of λ, and the coefficient of λ^2 is equal to 1 both in the numerator and in the denominator. This yields

$$\lim_{\lambda\to\infty} N_{i-1}^3 = 0, \ \lim_{\lambda\to\infty} N_i^3 = 0, \ \lim_{\lambda\to\infty} N_{i+1}^3 = 1, \qquad u \in [u_{i+1}, u_{i+2}).$$

The geometric meaning of the above result is that the extended paths of arc s_{i+1} converge to the control point $\mathbf{d}_{i+1}$ when $\lambda \to \infty$. Similar calculation proofs the same statement for the case $z = 2, 3$. This yields the following lemma.

Lemma 2.1. *In the case $k = 3$ symmetrically altering the knots u_i and u_{i+z}, $(z = 1, 2, 3)$, extended paths of points of the arcs $\mathbf{s}_{i+z}, \mathbf{s}_{i-1}$ converge to the control points $\mathbf{d}_{i+z}$ and $\mathbf{d}_{i-1}$ respectively, when $\lambda \to \infty$, i.e.*

$$\lim_{\lambda\to\infty} \mathbf{s}(u, \lambda) = \mathbf{d}_{i+z}, \quad u \in [u_{i+z}, u_{i+z+1}). \tag{2.1}$$

$$\lim_{\lambda\to\infty} \mathbf{s}(u, \lambda) = \mathbf{d}_{i-1}, \quad u \in [u_{i-1}, u_i). \tag{2.2}$$

The symmetric modification of knots u_i and u_{i+z}, $(z = 1, 2, 3)$ effects the shape of the arcs s_{i+z+1}, s_{i-2} as well. On the interval $[u_{i+2}, u_{i+3})$ the only nonzero basis functions are

$$\begin{aligned} N_i^3 &= \frac{u_{i+3} - u}{u_{i+3} - u_{i+1}} \frac{u_{i+3} - u}{u_{i+3} - u_{i+2}} \\ N_{i+1}^3 &= \frac{u - u_{i+1}}{u_{i+3} - u_{i+1}} \frac{u_{i+3} - u}{u_{i+3} - u_{i+2}} + \frac{u_{i+4} - u}{u_{i+4} - u_{i+2}} \frac{u - u_{i+2}}{u_{i+3} - u_{i+2}} \\ N_{i+2}^3 &= \frac{u - u_{i+2}}{u_{i+4} - u_{i+2}} \frac{u - u_{i+2}}{u_{i+3} - u_{i+2}}. \end{aligned}$$

First we consider the case $z = 1$, i.e. we substitute $u_i = u_i + \lambda$ and $u_{i+1} = u_{i+1} - \lambda$ in the above functions. A short calculation shows that the alteration of the mentioned knots does not effect the basis function N_{i+2}^3. In the case N_i^3 while λ does not appear in the numerator, the denominator is a linear polynomial of λ. N_{i+1}^3 is the sum of two rational functions. While both the numerator and the denominator are linear polynomials of λ in one of the terms, the other one is not effected by the knot alteration.

Consequently, for $u \in [u_{i+2}, u_{i+3})$ the following equalities hold:

$$\begin{aligned} \lim_{\lambda\to\infty} N_i^3 &= 0 \\ \lim_{\lambda\to\infty} N_{i+1}^3 &= 1 + \frac{u_{i+4} - u}{u_{i+4} - u_{i+2}} \frac{u - u_{i+2}}{u_{i+3} - u_{i+2}} \\ \lim_{\lambda\to\infty} N_{i+2}^3 &= N_{i+2}^3. \end{aligned}$$

Similar calculation proofs the same statement for the case $z = 2, 3$. This yields to the following lemma.

Lemma 2.2. *In the case $k = 3$ symmetrically altering the knots u_i and u_{i+z}, $(z = 1, 2, 3)$, extended paths of points of the arcs $\mathbf{s}_{i+z+1}, \mathbf{s}_{i-2}$ converge to*

$$\begin{aligned} \lim_{\lambda\to\infty} \mathbf{s}(u, \lambda) &= \mathbf{d}_{i+z}\Big(1 + \frac{u_{i+z+3} - u}{u_{i+z+3} - u_{i+z+1}} \frac{u - u_{i+z+1}}{u_{i+z+2} - u_{i+z+1}}\Big) + \mathbf{d}_{i+z+1} N_{i+z+1} \\ &= \mathbf{d}_{i+z} + (\mathbf{d}_{i+z}(u_{i+z+3} - u) + \mathbf{d}_{i+z+1}(u - u_{i+z+1}))\cdot \\ &\quad \cdot \frac{u - u_{i+z+1}}{(u_{i+z+3} - u_{i+z+1})(u_{i+z+2} - u_{i+z+1})}, \quad u \in [u_{i+z+1}, u_{i+z+2}). \end{aligned}$$

$$\begin{aligned} \lim_{\lambda\to\infty} \mathbf{s}(u, \lambda) &= \mathbf{d}_{i-4} N_{i-4} + \mathbf{d}_{i-3}\Big(\frac{u - u_{i-3}}{u_{i-1} - u_{i-3}} \frac{u_{i-1} - u}{u_{i-1} - u_{i-2}} + 1\Big) \\ &= \mathbf{d}_{i-3} + (\mathbf{d}_{i-3}(u - u_{i-3}) + d_{i-4}(u_{i-1} - u))\cdot \\ &\quad \cdot \frac{u_{i-1} - u}{(u_{i-1} - u_{i-3})(u_{i-1} - u_{i-2})}, \quad u \in [u_{i-2}, u_{i-1}). \end{aligned}$$

The alteration of knots u_i, u_{i+z} does not change the shape of any further arcs. Therefore, the effect of symmetric alteration of two knots in the case = 3 is fully explored on the domain of definition of the B-spline curve.

Obviously the above two lemmas are valid for the case $\lambda \to -\infty$. Here we note, that the limits of the basis functions have been considered by substituting $u_i + \lambda, u_{i+z} - \lambda$ to the computed functions. Since these functions are defined recursively, the limit can be interpreted in a different way as well.

References

[1] Aszódi, B., Czuczor, Sz., Szirmay-Kalos, L.: *NURBS fairing by knot vector optimization*, Journal of WSCG, **12**, 19–26 (2004).

[2] Fowler, B., Bartels, R.: *Constraint–based curve manipulation*, IEEE Computer Graphics and Applications, **13**, 43–49 (1993).

[3] Goldenthal, R., Bercovier, M.: *Spline curve approximation and design by optimal control over the knots*, Computing, **72**, 53–64 (2004).

[4] Hoffmann, M., Juhász, I.: *On the knot modification of a B-spline curve*, Publicationes Mathematicae Debrecen, **65**, 193–203 (2004)

[5] Hoffmann, M., Juhász, I.: *Symmetric alteration of two knots of a B-spline curves*, Journal for Geometry and Graphics, **9**, 43–49 (2005).

[6] Juhász, I., Hoffmann, M.: *The effect of knot modifications on the shape of B-spline curves*, Journal for Geometry and Graphics, **5**, 111–119 (2001).

[7] Juhász, I., Hoffmann M.: *Modifying a knot of B-spline curves*, Computer Aided Geometric Design, **20**, 243–245 (2003).

[8] Juhász, I., Hoffmann M.: *Constrained shape modification of cubic B-spline curves by means of knots*, Computer-Aided Design, **36**, 437–445 (2004).

[9] Piegl, L.: *Modifying the shape of rational B-splines. Part 1: curves*, Computer–Aided Design, **21**, 509–518 (1989).

[10] Piegl, L., Tiller, W.: *The NURBS book*, Springer–Verlag, (1995).

Ibolya Szilágyi
Institute of Mathematics and Computer Science
Károly Eszterházy College
Leányka str. 4.
H-3300 Eger
Hungary

Annales Mathematicae et Informaticae
***32** (2005) pp. 159–166.*

Some special cases of a general convergence rate theorem in the law of large numbers

Tibor Tómács

Department of Applied Mathematics, Eszterházy Károly College
e-mail: tomacs@ektf.hu

Abstract

Tómács in [6] proved a general convergence rate theorem in the law of large numbers for arrays of Banach space valued random elements. We shall study this theorem in case Banach space of type φ and for two special arrays.

Key Words: Convergence rates; Arrays of Banach space valued random variables; Banach space of type φ

1. Introduction and notation

Let $\mathbb{N}$ be the set of the positive integers and $\mathbb{R}$ the set of real numbers. Let Φ_0 denote the set of functions $f\colon [0,\infty) \to [0,\infty)$, that are nondecreasing. A function $f \in \Phi_0$ is said to satisfy the Δ_2-condition ($f \sim \Delta_2$) if there exists a constant $c > 0$ such that $f(2t) \leqslant cf(t)$ for all $t > 0$.

Let B be a real separable Banach space with norm $\|.\|$ and zero element $\mathbf{0}$. If X is a B-valued random variable (r.v.) and $\boldsymbol{E}\|X\| < \infty$ then $\boldsymbol{E}X$ stands for the Bochner integral of X.

Throughout the paper let $\{k_n, n \in \mathbb{N}\}$ be a strictly increasing sequence of positive integers. Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ be an array of B-valued r.v.'s. It is rowwise independent, if $X_{n1}, \ldots, X_{nk_n}$ are independent r.v.'s for any fixed $n \in \mathbb{N}$. Let $S_{k_n} = \sum_{k=1}^{k_n} X_{nk}$. If $k_n = n$ for all n, then we denote S_{k_n} by S_n. This corresponds to the case of ordinary sequences.

The array $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ is said to be *bounded in probability* if for all $\varepsilon > 0$ there exists $A > 0$ such that $\boldsymbol{P}(\|X_{nk}\| \geqslant A) < \varepsilon$ for all $n \in \mathbb{N}$ and $k = 1, \ldots, k_n$.

The following remark give a sufficient condition for the boundedness in probability.

Remark 1.1. If there exists a constant $M > 0$ such that $\boldsymbol{E}\,\|X_{nk}\| \leqslant M$ for every $n \in \mathbb{N}, k = 1, \dots, k_n$, then the array $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ is bounded in probability. (The reader can readily verify this statement.)

Definition 1.2 (Gut [2]). We say that the array $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ is *weakly mean dominated* (w.m.d.) by the r.v. X, if for some $\gamma > 0$,

$$\frac{1}{k_n}\sum_{k=1}^{k_n} \boldsymbol{P}(\|X_{nk}\| > t) \leqslant \gamma \boldsymbol{P}(|X| > t) \quad \text{for all} \quad t \geqslant 0 \quad \text{and} \quad n \in \mathbb{N}.$$

The following theorem a general convergence rate theorem, which is proved in [6].

Theorem 1.3 (Tómács [6], Theorem 3.1). *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Assume that there exists a sequence $\{\gamma_n, n \in \mathbb{N}\}$ of positive real numbers such that $\{\|S_n\|/\gamma_n, n \in \mathbb{N}\}$ is bounded in probability. Let $\alpha, \vartheta, \varphi \in \Phi_0$, and assume that α is not bounded, $\vartheta, \varphi \sim \Delta_2$, $\vartheta \not\equiv 0$. Let $\beta(n) = \varphi(\alpha(n+1)) - \varphi(\alpha(n))$, $n = 0, 1, 2, \dots$. Assume that $\boldsymbol{E}\,\varphi(|X|) < \infty$, $\boldsymbol{E}\,\vartheta(|X|) < \infty$ and $\lim_{n\to\infty} \alpha(n)/\gamma_n = \infty$.*

Let either $\mu(n) = \beta(n-1)$ for all $n \in \mathbb{N}$ or $\mu(n) = \beta(n)$ for all $n \in \mathbb{N}$. In second case assume that there exists a constant $c > 0$ such that for $n \in \mathbb{N}$ large enough $c\beta(n) \leqslant \beta(n-1)$.

Let $n_0 \in \mathbb{N}$ be such that $\vartheta(\alpha(n)) > 0$ for all $n \geqslant n_0$. If there exist $j \in \mathbb{N}$ and $r > 0$ such that

$$\sum_{n=n_0}^{\infty} \frac{\mu(n)}{n}\left(\frac{rn + \vartheta(\gamma_n)}{\vartheta(\alpha(n))}\right)^{2^j} < \infty,$$

then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \boldsymbol{P}\big(\|S_n\| > \varepsilon\alpha(n)\big) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

In the following two corollaries of Theorem 1.3 we use some special notations: Following Gut [1], introduce the functions ψ and M_r with

$$\psi(t) = \mathrm{Card}\{n \in \mathbb{N} : k_n \leqslant t\} \quad \text{for} \quad t \geqslant 0,$$

and

$$M_r(t) = \sum_{i=1}^{[t]} k_i^{r-1} \quad \text{if} \quad t \geqslant 1 \quad \text{and} \quad M_r(t) = k_1^{r-1} \quad \text{if} \quad 0 \leqslant t < 1,$$

where $r \in \mathbb{R}$, $\mathrm{Card}A$ is the cardinality of the set A and $[.]$ denotes the integer function. Let $M = M_2$. Let $f \circ g$ be the composite function of functions f and g.

Remark 1.4. $M_r \circ \psi \in \Phi_0$ and

$$(M_r \circ \psi)(t) = M_r(\psi(t)) = \begin{cases} \sum\limits_{i=1}^{n} k_i^{r-1} = M_r(n), & \text{if} \quad k_n \leqslant t < k_{n+1}, \\ k_1^{r-1} = M_r(1), & \text{if} \quad 0 \leqslant t < k_1. \end{cases}$$

The following corollary is a generalization of Theorem 6.2 of Fazekas [5].

Corollary 1.5 (Tómács [6], Corollary 3.2). *Let* $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ *be an array of rowwise independent* B*-valued r.v.'s which is w.m.d. by the r.v.* X. *Let* $M \circ \psi \sim \Delta_2$, $r, s, t > 0$, $rs > t$. *Assume that* $\{\|S_{k_n}\| / k_n^{1/s}, n \in \mathbb{N}\}$ *is bounded in probability. Furthermore, if* $r > 2$ *we assume that* $\{M(n)/M(n-1), n \in \mathbb{N}\}$ *is bounded. If* $\boldsymbol{E} M^{r/2}(\psi(|X|^{t/r})) < \infty$ *and* $\boldsymbol{E}|X|^s < \infty$, *then*

$$\sum_{n=1}^{\infty} \big(M(n)\big)^{r/2-1} \boldsymbol{P}\left(\|S_{k_n}\| > \varepsilon k_n^{r/t}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

The following corollary is a version of Corollary 4.1 of Hu et al. [3].

Corollary 1.6 (Tómács [6], Corollary 3.3). *Let* $\{X_{nk}, n \in \mathbb{N}, k = 1, \ldots, k_n\}$ *be an array of rowwise independent* B*-valued r.v.'s which is w.m.d. by the r.v.* X. *Let* $r \in \mathbb{R}$, $0 < t < s$ *and* $M_r \circ \psi \sim \Delta_2$. *Assume that* $\{\|S_{k_n}\| / k_n^{1/s}, n \in \mathbb{N}\}$ *is bounded in probability. If* $\boldsymbol{E} M_r(\psi(|X|^t)) < \infty$ *and* $\boldsymbol{E}|X|^s < \infty$, *then*

$$\sum_{n=1}^{\infty} k_n^{r-2} \boldsymbol{P}\left(\|S_{k_n}\| > \varepsilon k_n^{1/t}\right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

In Section 2 we give a sufficient condition for the boundedness in probability and in Section 3 we study two concrete sequences k_n in Corollary 1.5 and 1.6.

2. The boundedness in probability in case Banach space of type φ

If B has an appropriate geometric property, then a moment condition can imply the boundedness of $\{\|S_{k_n}\| / \gamma_{k_n}, n \in \mathbb{N}\}$.

Definition 2.1. A function $\varphi\colon [0,\infty) \to [0,\infty)$ is said to be an *Orlicz function* if it is continuous, convex, $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. For an Orlicz function φ the *Orlicz space* $l_\varphi(B)$ consists of those B-valued sequences $\{u_n, n \in \mathbb{N}\}$ for which

$$\sum_{n=1}^{\infty} \varphi\left(\|u_n\| / a\right) < \infty \quad \text{for some} \quad a > 0.$$

Let $\varepsilon_1, \varepsilon_2, \ldots$ be independent r.v.'s with $\boldsymbol{P}(\varepsilon_n = 1) = \boldsymbol{P}(\varepsilon_n = -1) = 1/2$ for all $n \in \mathbb{N}$. B is said to be of *type* φ, if $\sum_{n=1}^{\infty} \varepsilon_n u_n$ converges in probability for all $\{u_n, n \in \mathbb{N}\} \in l_\varphi(B)$.

Definition 2.2. An Orlicz function φ is said to satisfy the Δ_2^0-condition ($\varphi \sim \Delta_2^0$) if there exist constants $c > 0$ and $t_0 > 0$ such that $\varphi(2t) \leqslant c\varphi(t)$ is satisfied for all $0 \leqslant t \leqslant t_0$.

Lemma 2.3. *Let φ be an Orlicz function and $\varphi \sim \Delta_2^0$. B is of type φ iff there exists a constant $c > 0$ such that*

$$\boldsymbol{E}\left\|\sum_{k=1}^{n} X_k\right\| \leqslant c\boldsymbol{E}\inf_{y>0}\left\{\frac{1}{y}\left(1+\sum_{k=1}^{n}\varphi(y\left\|X_k\right\|)\right)\right\}$$

for all $n \in \mathbb{N}$ and every independent B-valued r.v.' $X_1, \dots, X_n$ with $\boldsymbol{E}X_k = \mathbf{0}$, $k = 1, \dots, n$.

For the proof see Fazekas [4].

The following lemma is a generalization of Lemma 2.1 of Gut [2] and Lemma 2.7 (b) of Fazekas [5].

Lemma 2.4 (Tómács [6], Lemma 4.4). *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ be an array of B-valued r.v.'s which is w.m.d. by the r.v. X and constant γ. If $\varphi \in \Phi_0$ then*

$$\frac{1}{k_n}\sum_{k=1}^{k_n}\boldsymbol{E}\,\varphi(\|X_{nk}\|) \leqslant \max\{1,\gamma\}\boldsymbol{E}\,\varphi(|X|).$$

The following theorem show that in Theorem 1.3 we can write moment conditions instead of the boundedness of $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$ if B is of type φ.

Theorem 2.5. *Let $\varphi \in \Phi_0$ be a submultiplicative Orlicz function, $\varphi \sim \Delta_2^0$ and let B be a space of type φ. Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_n\}$ be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Assume that the sequence $\{k_n\varphi(1/\gamma_{k_n}), n \in \mathbb{N}\}$ is bounded for some sequence $\{\gamma_n, n \in \mathbb{N}\}$ of positive real numbers. If $\boldsymbol{E}X_{nk} = \mathbf{0}$ for every $n \in \mathbb{N}$, $k = 1, \dots, k_n$ and $\boldsymbol{E}\,\varphi(|X|) < \infty$, then $\{\|S_{k_n}\|/\gamma_{k_n}, n \in \mathbb{N}\}$ is bounded in probability.*

Proof. By Lemma 2.3 and 2.4 there exists a constant $c > 0$ such that

$$\begin{aligned}
\boldsymbol{E}\frac{\|S_{k_n}\|}{\gamma_{k_n}} &\leqslant \frac{c}{\gamma_{k_n}}\boldsymbol{E}\inf_{y>0}\left\{\frac{1}{y}\left(1+\sum_{k=1}^{k_n}\varphi(y\left\|X_{nk}\right\|)\right)\right\} \\
&\leqslant c\boldsymbol{E}\left(1+\sum_{k=1}^{k_n}\varphi\left(\|X_{nk}\|/\gamma_{k_n}\right)\right) \\
&\leqslant c\big(1+\varphi\left(1/\gamma_{k_n}\right)\max\{1,\gamma\}k_n\boldsymbol{E}\,\varphi(|X|)\big).
\end{aligned}$$

Thus Remark 1.1 implies the statement. $\square$

3. Convergence rate theorems for two concrete sequences k_n

Lemma 3.1. *$f \sim \Delta_2$ iff there exist constants $k > 1$ and $c > 0$ such that*

$$f(kt) \leqslant cf(t) \quad \text{for all} \quad t > 0. \tag{3.1}$$

Proof. If $f \sim \Delta_2$ then in case $k = 2$ we get (3.1). Now suppose that there exist constants $k > 1$ and $c > 0$ such that the inequality (3.1) is true for all $t > 0$. Then we can obtain with induction that

$$f(k^n t) \leqslant c^n f(t) \quad \text{for all} \quad t > 0 \quad \text{and for all} \quad n \in \mathbb{N}.$$

It follows that there exists $n_0 \in \mathbb{N}$ such that

$$f(2t) \leqslant f(k^{n_0} t) \leqslant c^{n_0} f(t) \quad \text{for all} \quad t > 0.$$

Thus we get $f \sim \Delta_2$. □

The reader can readily verify the following lemma.

Lemma 3.2. *Let $g \colon [k_1, \infty) \to \mathbb{R}$ be a nondecreasing function which has the property that $g(k_n) \geqslant M_r(n)$ for all $n \in \mathbb{N}$. Then $M_r(\psi(x)) \leqslant g(x)$ for all $x \geqslant k_1$.*

Lemma 3.3. *Let $r \in \mathbb{R}$. Assume that there exists strictly increasing sequence $\{a_n, n \in \mathbb{N}\}$ of positive integers and there exist constants $k > 1$, $c > 0$ such that*

$$\frac{k_n}{k_{a_n}} \leqslant \frac{1}{k} \quad \text{and} \quad \frac{M_r(a_n)}{M_r(n-1)} \leqslant c \quad \text{for all} \quad n \in \mathbb{N}.$$

Then $M_r \circ \psi \sim \Delta_2$.

Proof. Assume that $k_n \leqslant t < k_{n+1}$. Then Remark 1.4 implies

$$M_r(\psi(kt)) \leqslant M_r(\psi(kk_{n+1})) \leqslant M_r(\psi(k_{a_{n+1}})) = M_r(a_{n+1}) \leqslant cM_r(n) = cM_r(\psi(t)).$$

Similarly if $0 < t < k_1$ then

$$M_r(\psi(kt)) \leqslant M_r(\psi(kk_1)) \leqslant M_r(\psi(k_{a_1})) = M_r(a_1) \leqslant cM_r(0) = cM_r(\psi(t)).$$

It follows that $M_r(\psi(kt)) \leqslant cM_r(\psi(t))$ for all $t > 0$. Thus, by Lemma 3.1 we get the statement. □

Lemma 3.4. *Let $l \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + (ln)^k}{1^k + 2^k + \cdots + (n-1)^k} = \begin{cases} l^{k+1}, & \text{if} \quad k > -1, \\ 1, & \text{if} \quad k \leqslant -1. \end{cases}$$

Proof. It is easy to see that

$$x^{k+1}-(x-1)^{k+1}\leqslant (k+1)x^k\leqslant (x+1)^{k+1}-x^{k+1}\quad \text{for all}\quad x\geqslant 1, k\geqslant 0$$

and

$$(x+1)^{k+1}-x^{k+1}\leqslant (k+1)x^k\leqslant x^{k+1}-(x-1)^{k+1}\quad \text{for all}\quad x\geqslant 1, -1<k<0.$$

Apply these inequalities for $x=1,2,\dots,n$. Then we have

$$\lim_{n\to\infty}\frac{1^k+2^k+\dots+n^k}{n^{k+1}}=\frac{1}{k+1}\quad \text{for all}\quad k>-1,$$

which implies the statement for $k>-1$.

It is well known that $\frac{1}{1^c}+\frac{1}{2^c}+\cdots\frac{1}{n^c}$ is convergent if $c>1$. It follows that the statement is true in case $k<-1$ as well.

Finally in case $k=-1$ the inequalities

$$1+\frac{\frac{l-1}{l}}{1+\frac{1}{2}+\dots+\frac{1}{n-1}}<\frac{1+\frac{1}{2}+\dots+\frac{1}{ln}}{1+\frac{1}{2}+\dots+\frac{1}{n-1}}<1+\frac{l}{1+\frac{1}{2}+\dots+\frac{1}{n-1}}$$

imply the statement. □

Lemma 3.5. *Let $k_1, d\in\mathbb{N}$, $q\in\mathbb{N}\setminus\{1\}$. If $k_n=k_1q^{n-1}$ or $k_n=k_1n^d$ then $M_r\circ\psi\sim\Delta_2$ for all $r\in\mathbb{R}$.*

Proof. In the first case, when $k_n=k_1q^{n-1}$, let $a_n=n+1$ and $k=q$. Then

$$\frac{k_n}{k_{a_n}}=\frac{k_1q^{n-1}}{k_1q^n}=\frac{1}{q}\leqslant\frac{1}{k}.$$

Let $Q=q^{r-1}$ and assume that $r>1$. In this case $|1/Q|<1$, thus we get

$$\frac{M_r(a_n)}{M_r(n-1)}=\frac{M_r(n+1)}{M_r(n-1)}=\frac{1+Q+\dots+Q^n}{1+Q+\dots+Q^{n-2}}=\frac{Q^2-\frac{1}{Q^{n-1}}}{1-\frac{1}{Q^{n-1}}}\to Q^2.$$

If $r<1$ then $1/Q>1$, thus

$$\frac{M_r(a_n)}{M_r(n-1)}=\frac{Q^2-\frac{1}{Q^{n-1}}}{1-\frac{1}{Q^{n-1}}}\to 1.$$

If $r=1$ then $Q=1$, so

$$\frac{M_r(a_n)}{M_r(n-1)}=\frac{n+1}{n-1}\to 1.$$

Thus we get that $\frac{M_r(a_n)}{M_r(n-1)}$ is bounded for all $r\in\mathbb{R}$. Hence conditions of Lemma 3.3 are satisfied, which implies the statement.

In the second case, when $k_n = k_1 n^d$, let $a_n = 2n$ and $k = 2^d$. Then

$$\frac{k_n}{k_{a_n}} = \frac{k_1 n^d}{k_1 (2n)^d} = \frac{1}{2^d} \leqslant \frac{1}{k}.$$

On the other hand it follows from Lemma 3.4 that $\frac{M_r(a_n)}{M_r(n-1)}$ is bounded for all $r \in \mathbb{R}$. So Lemma 3.3 implies the statement. □

Theorem 3.6. *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_1 n^d\}$ ($k_1, d \in \mathbb{N}$ are fixed) be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $t > 0$, $r \geqslant 2d/(d+1)$, $s > t/r$ and $v = \max\{s, t(d+1)/(2d)\}$. If $\{\|S_{k_1 n^d}\| / n^{d/s}, n \in \mathbb{N}\}$ is bounded in probability and $\boldsymbol{E}\, |X|^v < \infty$, then*

$$\sum_{n=1}^{\infty} n^{(d+1)(r/2-1)} \boldsymbol{P}\left(\|S_{k_1 n^d}\| > \varepsilon n^{dr/t} \right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

Proof. We shall prove that conditions of Corollary 1.5 are satisfied. Let $k_n = k_1 n^d$. Then by Lemma 3.4 $\{M(n)/M(n-1), n \in \mathbb{N}\}$ is bounded. Let $Y = M^{r/2}(\psi(|X|^{t/r}))$. Now we turn to the proof of $\boldsymbol{E}Y < \infty$. It is well known that

$$1^d + \dots + n^d = a_1 n^{d+1} + a_2 n^d + \dots + a_{d+2}$$

for some $a_1, a_2, \dots, a_{d+2} \in \mathbb{R}$. Let

$$g\colon [k_1, \infty) \to \mathbb{R}, \quad g(x) = \sum_{i=1}^{d+2} |a_i| \left(k_1^{i-2} x^{d+2-i} \right)^{1/d}.$$

Then g is nondecreasing, $g(k_n) \geqslant M(n)$ and $g(x) \leqslant \text{const.} x^{(d+1)/d}$ for all $x \geqslant k_1$. Therefore by Lemma 3.2 we have

$$M^{r/2}\left(\psi(x^{t/r})\right) \leqslant \text{const.} x^{t(d+1)/(2d)} \quad \text{for all} \quad x^{t/r} \geqslant k_1.$$

It follows that

$$Y = Y\boldsymbol{I}(|X|^{t/r} < k_1) + Y\boldsymbol{I}(|X|^{t/r} \geqslant k_1) \leqslant k_1^{r/2} + \text{const.} |X|^{t(d+1)/(2d)},$$

where $\boldsymbol{I}(A)$ denotes the indicator function of the set A. So $\boldsymbol{E}Y < \infty$. By Lemma 3.5 $M \circ \psi \sim \Delta_2$. It is easy to see that the other conditions of Corollary 1.5 hold true as well, on the other hand $M(n) \geqslant \text{const.} n^{d+1}$. So this theorem is consequence of Corollary 1.5. □

Theorem 3.7. *Let $\{X_{nk}, n \in \mathbb{N}, k = 1, \dots, k_1 q^{n-1}\}$ ($k_1 \in \mathbb{N}, q \in \mathbb{N} \setminus \{1\}$ are fixed) be an array of rowwise independent B-valued r.v.'s which is w.m.d. by the r.v. X. Let $w \geqslant 0$, $t > 0$, $s > t$ and $v = \max\{s, t(w+1)\}$. If $\{\|S_{k_1 q^{n-1}}\| / q^{n/s}, n \in \mathbb{N}\}$ is bounded in probability and $\boldsymbol{E}\, |X|^v < \infty$, then*

$$\sum_{n=1}^{\infty} q^{nw} \boldsymbol{P}\left(\|S_{k_1 q^{n-1}}\| > \varepsilon q^{n/t} \right) < \infty \quad \textit{for all} \quad \varepsilon > 0.$$

Proof. We shall prove that conditions of Corollary 1.6 are satisfied. Let $k_n = k_1 q^{n-1}$, $r = w + 2$ and $Y = M_r(\psi(|X|^t))$. Then $M_r(n) = k_1^{r-1}\frac{Q^n-1}{Q-1}$, where $Q = q^{r-1}$. Let

$$g\colon [k_1,\infty) \to \mathbb{R}, \quad g(x) = k_1^{r-1}\frac{Q^{1+\log(x/k_1)/\log q}-1}{Q-1}.$$

Then g is nondecreasing, $g(k_n) = M_r(n)$ and $g(x) \leqslant \text{const.}x^{r-1}$ for all $x \geqslant k_1$. Therefore by Lemma 3.2 we have

$$M_r\left(\psi(x^t)\right) \leqslant \text{const.}x^{t(w+1)} \quad \text{for all} \quad x^t \geqslant k_1.$$

It follows that

$$Y = Y\boldsymbol{I}(|X|^t < k_1) + Y\boldsymbol{I}(|X|^t \geqslant k_1) \leqslant k_1^{r-1} + \text{const.}|X|^{t(w+1)}.$$

So $\boldsymbol{E}Y < \infty$. By Lemma 3.5 $M_r \circ \psi \sim \Delta_2$. The other conditions of Corollary 1.6 hold true as well. Thus Corollary 1.6 implies the statement. □

References

[1] Gut, A., On complete convergence in the law of large numbers for subsequences, *Ann. Probab.,* 13 (1985) 1286–1291.

[2] Gut, A., Complete convergence for arrays, *Periodica Math. Hungar.,* 25 (1) (1992) 51–75.

[3] Hu, T.-C., Rosalsky, A., Szynal, D., Volodin, A. I., On complete convergence for arrays of rowwise independent random elements in Banach spaces, *Stoch. Anal. Appl.,* 17 (6) (1999) 963–992.

[4] Fazekas, I., A strong law of large numbers in Banach spaces of type Φ, *In Mathematical Statistics and Probability Theory,* Vol. A, 89–97, Reidel, Dordrecht, 1987.

[5] Fazekas, I., Convergence rates in the law of large numbers for arrays, *Publ. Math. Debrecen,* 41/1-2 (1992) 53–71.

[6] Tómács, T., Convergence rates in the law of large numbers for arrays of Banach space valued random elements, *Statist. Probab. Lett.,* 72 (2005) 59–69.

Tibor Tómács
Department of Applied Mathematics
Eszterházy Károly College
P.O. Box 43
H-3301 Eger
Hungary

Annales Mathematicae et Informaticae
***32** (2005) pp. 167–186.*

Wiener amalgams and summability of Fourier series*

Ferenc Weisz

Department of Numerical Analysis
Eötvös L. University
e-mail: weisz@ludens.elte.hu

Abstract

Some recent results on a general summability method, on the so-called θ-summability is summarized. New spaces, such as Wiener amalgams, Feichtinger's algebra and modulation spaces are investigated in summability theory. Sufficient and necessary conditions are given for the norm and a.e. convergence of the θ-means.

Key Words: Wiener amalgam spaces, Feichtinger's algebra, homogeneous Banach spaces, Besov-, Sobolev-, fractional Sobolev spaces, modulation spaces, Herz spaces, Hardy-Littlewood maximal function, θ-summability of Fourier series, Lebesgue points.

AMS Classification Number: Primary 42B08, 46E30, Secondary 42B30, 42A38.

1. Introduction

In this paper we consider a general method of summation, the so called θ-summation, which is generated by a single function θ. A natural choice of θ is a function from the Wiener algebra $W(C, \ell_1)(\mathbb{R}^d)$. All concrete summability methods investigated in the literature satisfy this condition.

We shall investigate some function spaces known from other topics of analysis, for example Wiener amalgam spaces, Feichtinger's Segal algebra $\mathbf{S}_0(\mathbb{R}^d)$, modulation and Herz spaces. Feichtinger's algebra and modulation spaces are very intensively investigated in Gabor analysis (see e.g. Feichtinger and Zimmermann [6] and Gröchenig [13]). $\mathbf{S}_0(\mathbb{R}^d)$ is the minimal (non-trivial) Banach space which is isometrically invariant under translation, modulation and Fourier transform.

*This research was supported by Lise Meitner fellowship No M733-N04 and the Hungarian Scientific Research Funds (OTKA) No T043769, T047128, T047132.

In Sections 4 and 5 we deal with norm convergence of the θ-means of multi-dimensional Fourier series and Fourier transforms. We will show that if θ is in the Wiener algebra then the θ-means $\sigma_n^\theta f$ of the Fourier series of $f \in L_2(\mathbb{T}^d)$ converge to f in L_2 norm as $n \to \infty$. Moreover, $\sigma_n^\theta f \to f$ uniformly (resp. at each point) for all $f \in C(\mathbb{T}^d)$ if and only if $\sigma_n^\theta f \to f$ in L_1 norm for all $f \in L_1(\mathbb{T}^d)$ if and only if $\hat{\theta} \in L_1(\mathbb{R}^d)$. If B is a homogeneous Banach space on $\mathbb{T}^d$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ then $\sigma_n^\theta f \to f$ in B norm for all $f \in B$. If θ is continuous and has compact support then the uniform convergence of the θ-means is equivalent to the L_1 norm convergence of the θ-means and this is equivalent to the condition $\theta \in \mathbf{S}_0(\mathbb{R}^d)$. In all cases we investigate convergence over the diagonal.

In Sections 7 and 8 the a.e. convergence of the θ-means is considered. We show that $\hat{\theta}$ is in the homogeneous Herz space $\dot{E}_q(\mathbb{R}^d)$ for some $1 < q \leqslant \infty$ if and only if the maximal operator of the θ-means of the Fourier transform of f can be estimated by the modified Hardy-Littlewood maximal function $M_p f$, where p is the dual index to q. Since M_p is of weak type (p,p) we obtain $\sigma_T^\theta f \to f$ a.e. as $T \to \infty$ for all $f \in L_r(\mathbb{R}^d)$, $p \leqslant r < \infty$. Under the condition $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$ this convergence holds also for functions from the Wiener amalgam space $W(L_p, \ell_\infty)(\mathbb{R}^d)$. The set of convergence is also characterized, the convergence holds at every p-Lebesgue point of f. The converse holds also, more exactly, $\sigma_T^\theta f(x) \to f(x)$ at each p-Lebesgue point of $f \in L_p(\mathbb{R}^d)$ (resp. of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$) if and only if $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$.

In Sections 6 and 9 we give some sufficient conditions for θ such that $\hat{\theta} \in L_1(\mathbb{R}^d)$, or $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ or $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$. More exactly, if θ is in a suitable Besov, Sobolev, fractional Sobolev, weighted Wiener amalgam or modulation space then all convergence results above hold.

Most of the proofs of the results of this survey paper can be found in Feichtinger and Weisz [5, 4]. This paper was the base of my talk given at the Fejér-Riesz Conference, June 2005, in Eger (Hungary).

2. Wiener amalgams and Feichtinger's algebra

Let us fix $d \geqslant 1$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$ let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \ldots \times \mathbb{Y}$ taken with itself d-times. We shall prove results for $\mathbb{R}^d$ or $\mathbb{T}^d$, therefore it is convenient to use sometimes the symbol $\mathbb{X}$ for either $\mathbb{R}$ or $\mathbb{T}$, where $\mathbb{T}$ is the torus. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ set

$$u \cdot x := \sum_{k=1}^d u_k x_k, \qquad \|x\|_p := \Big(\sum_{k=1}^d |x_k|^p\Big)^{1/p}, \qquad |x| := \|x\|_2.$$

We briefly write L_p or $L_p(\mathbb{X}^d)$ instead of $L_p(\mathbb{X}^d, \lambda)$ space equipped with the norm (or quasi-norm) $\|f\|_p := (\int_{\mathbb{X}^d} |f|^p \, d\lambda)^{1/p}$ $(0 < p \leqslant \infty)$, where $\mathbb{X} = \mathbb{R}$ or $\mathbb{T}$ and λ is the Lebesgue measure. We use the notation $|I|$ for the Lebesgue measure of the set I.

The *weak* L_p space, $L_{p,\infty}(\mathbb{X}^d)$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\rho>0} \rho\lambda(|f| > \rho)^{1/p} < \infty,$$

while we set $L_{\infty,\infty}(\mathbb{X}^d) = L_\infty(\mathbb{X}^d)$. Note that $L_{p,\infty}(\mathbb{X}^d)$ is a quasi-normed space (see Bergh and Löfström [1]). It is easy to see that for each $0 < p \leqslant \infty$,

$$L_p(\mathbb{X}^d) \subset L_{p,\infty}(\mathbb{X}^d) \qquad \text{and} \qquad \|\cdot\|_{L_{p,\infty}} \leqslant \|\cdot\|_p.$$

The space of continuous functions with the supremum norm is denoted by $C(\mathbb{X}^d)$ and we will use $C_0(\mathbb{R}^d)$ for the space of continuous functions vanishing at infinity. $C_c(\mathbb{R}^d)$ denotes the space of continuous functions having compact support.

A measurable function f belongs to the *Wiener amalgam space* $W(L_p, \ell_q^{v_s})(\mathbb{R}^d)$ $(1 \leqslant p, q \leqslant \infty)$ if

$$\|f\|_{W(L_p,\ell_q^{v_s})} := \Big(\sum_{k\in\mathbb{Z}^d} \|f(\cdot + k)\|^q_{L_p[0,1)^d} v_s(k)^q \Big)^{1/q} < \infty$$

with the obvious modification for $q = \infty$, where the weight function v_s is defined by $v_s(\omega) := (1 + |\omega|)^s$ $(\omega \in \mathbb{R}^d)$. If $s = 0$ then we write simply $W(L_p, \ell_q)(\mathbb{R}^d)$. $W(L_p, c_0)(\mathbb{R}^d)$ is defined analogously, where c_0 denotes the space of sequences of complex numbers having 0 limit, equipped with the supremum norm. If we replace the space $L_p[0,1)^d$ by $L_{p,\infty}[0,1)^d$ then we get the definition of $W(L_{p,\infty}, \ell_q)(\mathbb{R}^d)$. The closed subspace of $W(L_\infty, \ell_q)(\mathbb{R}^d)$ containing continuous functions is denoted by $W(C, \ell_q)(\mathbb{R}^d)$ $(1 \leqslant q \leqslant \infty)$. The space $W(C, \ell_1)(\mathbb{R}^d)$ is called *Wiener algebra*. It is used quite often in Gabor analysis, because it provides a convenient and general class of windows (see e.g. Walnut [33] and Gröchenig [12]). As we can see later, it plays an important rule in summability theory, too.

It is easy to see that $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and

$$W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset W(L_1, \ell_\infty)(\mathbb{R}^d) \qquad (1 \leqslant p \leqslant \infty).$$

For more about amalgam spaces see e.g. Heil [14].

Translation and *modulation* of a function f are defined, respectively, by

$$T_x f(t) := f(t - x) \quad \text{and} \quad M_\omega f(t) := e^{2\pi\imath\omega\cdot t} f(t) \qquad (x, \omega \in \mathbb{R}^d).$$

Recall that the *Fourier transform* and the *short-time Fourier transform* (STFT) with respect to a window function g are defined by

$$\mathcal{F}f(x) := \hat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi\imath x\cdot t}\, dt \qquad (x \in \mathbb{R}^d, \imath = \sqrt{-1})$$

and

$$S_g f(x, \omega) := \int_{\mathbb{R}^d} f(t)\overline{g(t - x)} e^{-2\pi\imath\omega t}\, dt = \langle f, M_\omega T_x g\rangle \qquad (x, \omega \in \mathbb{R}^d),$$

respectively, whenever the integrals do exist.

Feichtinger's algebra $\mathbf{S}_0(\mathbb{R}^d)$ and the *modulation spaces* $M_1^{v_s}(\mathbb{R}^d)$ (see e.g. Feichtinger [7] and Gröchenig [13]) are intoduced by

$$M_1^{v_s}(\mathbb{R}^d) := \Big\{ f \in L^2(\mathbb{R}^d) : \|f\|_{M_1^{v_s}} := \|S_{g_0} f \cdot v_s\|_{L_1(\mathbb{R}^{2d})} < \infty \Big\} \qquad (s \geqslant 0),$$

where $g_0(x) := e^{-\pi |x|^2}$ is the Gauss function and $v_s(x, \omega) := (1 + |\omega|)^s$ $(x, \omega \in \mathbb{R}^d)$. In case $s = 0$ we write $\mathbf{S}_0(\mathbb{R}^d) := M_1(\mathbb{R}^d)$.

It is known that $\mathbf{S}_0(\mathbb{R}^d)$ is isometrically invariant under translation, modulation and Fourier transform. Actually, $\mathbf{S}_0$ is the minimal space having this property (see Feichtinger [7]). Moreover, the embeddings $\mathcal{S}(\mathbb{R}^d) \hookrightarrow M_1^{v_s}(\mathbb{R}^d)$ $(s \geqslant 0)$ and $\mathbf{S}_0(\mathbb{R}^d) \hookrightarrow W(C, \ell_1)(\mathbb{R}^d)$ are dense and continuous (see e.g. Feichtinger and Zimmermann [6] and Gröchenig [13]), where $\mathcal{S}$ denotes the Schwartz functions.

A Banach space B consisting of Lebesgue measurable functions on $\mathbb{X}^d$ is called a *homogeneous Banach space*, if

(i) for all $f \in B$ and $x \in \mathbb{X}^d$, $T_x f \in B$ and $\|T_x f\|_B = \|f\|_B$,

(ii) the function $x \mapsto T_x f$ from $\mathbb{X}^d$ to B is continuous for all $f \in B$,

(iii) the functions in B are uniformly locally integrable, i.e. for every compact set $K \subset \mathbb{X}^d$ there exists a constant C_K such that

$$\int_K |f| \, d\lambda \leqslant C_K \|f\|_B \qquad (f \in B).$$

If furthermore B is a dense subspace of $L_1(\mathbb{X}^d)$ it is called a *Segal algebra* (cf. Reiter [20]). Note that the continuous embedding into $L_1(\mathbb{X}^d)$ is a consequence of the closed graph theorem. For an introduction to homogeneous Banach spaces see Katznelson [16] or Shapiro [24]. It is easy to see that the spaces $L_p(\mathbb{X}^d)$ $(1 \leqslant p < \infty)$, $C(\mathbb{T}^d)$, $C_0(\mathbb{R}^d)$, Lorentz spaces $L_{p,q}(\mathbb{X}^d)$ $(1 < p < \infty, 1 \leqslant q < \infty)$, Hardy spaces $H_1(\mathbb{X}^d)$ (for the definitions see e.g. Weisz [37]), Wiener amalgams $W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \leqslant p, q < \infty)$, $W(L_p, c_0)(\mathbb{R}^d)$ $(1 \leqslant p < \infty)$, $W(C, \ell_q)(\mathbb{R}^d)$ $(1 \leqslant q < \infty)$ and $\mathbf{S}_0(\mathbb{R}^d)$ are homogeneous Banach spaces. Note that if B is a homogeneous Banach space on $\mathbb{R}^d$ then $B \hookrightarrow W(L_1, \ell_\infty)(\mathbb{R}^d)$ (see Katznelson [16]).

3. θ-summability of Fourier series

The θ-summation was considered in a great number of papers and books, such as Butzer and Nessel [3], Trigub and Belinsky [32], Bokor, Schipp, Szili and Vértesi [22, 2, 23, 28, 29], Natanson and Zuk [18], Weisz [35, 36, 37, 38] and Feichtinger and Weisz [5, 4]. We assume that the function θ is from the Wiener algebra $W(C, \ell_1)(\mathbb{R}^d)$. We have seen in Feichtinger and Weisz [5, 4] that this is a natural choice of θ and all summability methods considered in Butzer and Nessel [3] and Weisz [37] satisfy this condition.

Recall that for a distribution $f \in \mathcal{S}'(\mathbb{T}^d)$ the nth *Fourier coefficient* is defined by $\hat{f}(n) := f(e^{-2\pi \imath n \cdot x})$ $(n \in \mathbb{Z}^d)$. In special case, if $f \in L_1(\mathbb{T}^d)$ then

$$\hat{f}(n) = \int_{\mathbb{T}^d} f(t) e^{-2\pi \imath n \cdot t}\, dt \qquad (n \in \mathbb{Z}^d).$$

Given a function $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ the *θ-means* of a distribution f are defined by

$$\sigma_n^\theta f(x) := \sum_{j=1}^d \sum_{k_j=-\infty}^\infty \theta\Big(\frac{-k_1}{n_1+1}, \dots, \frac{-k_d}{n_d+1}\Big) \hat{f}(k) e^{2\pi \imath k \cdot x} = \int_{\mathbb{T}^d} f(x-t) K_n^\theta(t)\, dt,$$

where $x \in \mathbb{T}^d, n \in \mathbb{N}^d$ and the *θ-kernels* K_n^θ are given by

$$K_n^\theta(t) := \sum_{j=1}^d \sum_{k_j=-\infty}^\infty \theta\Big(\frac{-k_1}{n_1+1}, \dots, \frac{-k_d}{n_d+1}\Big) e^{2\pi \imath k \cdot t} \qquad (t \in \mathbb{T}^d).$$

Under $\sum_{j=1}^d \sum_{k_j=-\infty}^\infty$ we mean the sum $\sum_{k_1=-\infty}^\infty \cdots \sum_{k_d=-\infty}^\infty$. It is easy to see that

$$\begin{aligned} \sum_{j=1}^d \sum_{k_j=-\infty}^\infty \Big|\theta\Big(\frac{k_1}{n_1+1}, \dots, \frac{k_d}{n_d+1}\Big)\Big| &\leqslant \sum_{l \in \mathbb{Z}^d} \Big(\prod_{j=1}^d (n_j+1)\Big) \sup_{x \in [0,1)^d} |\theta(x+l)| \\ &= \Big(\prod_{j=1}^d (n_j+1)\Big) \|\theta\|_{W(C,\ell_1)} < \infty, \end{aligned}$$

and hence $K_n^\theta \in L_1(\mathbb{T}^d)$. We will always suppose that $\theta(0) = 1$.

Now we present some well known one-dimensional summability methods as special cases of the θ-summation. For more examples see Feichtinger and Weisz [5, 4].

Example 3.1 (Fejér summation). Let

$$\theta(x) := \begin{cases} 1 - |x| & \text{if } 0 \leqslant |x| \leqslant 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

$$\sigma_n^\theta f(x) := \sum_{k=-n}^n (1 - \frac{|k|}{n+1}) \hat{f}(k) e^{2\pi \imath k \cdot x}.$$

Example 3.2 (Riesz summation). Let

$$\theta(x) := \begin{cases} (1 - |x|^\gamma)^\alpha & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

for some $0 \leqslant \alpha, \gamma < \infty$. The Riesz operators are given by

$$\sigma_n^\theta f(x) := \sum_{k=-n}^n \Big(1 - \Big|\frac{k}{n+1}\Big|^\gamma\Big)^\alpha \hat{f}(k) e^{2\pi \imath k \cdot x}.$$

Example 3.3 (Weierstrass summation). Let

$$\theta(x) = e^{-|x|^\gamma} \qquad (0 < \gamma < \infty),$$

$$\sigma_n^\theta f(x) := \sum_{k=-\infty}^{\infty} e^{-(\frac{|k|}{n+1})^\gamma} \hat{f}(k) e^{2\pi \imath k \cdot x}.$$

The most known form of the Weierstrass means are

$$W_r^\theta f(x) := \sum_{k=-\infty}^{\infty} r^{|k|^\gamma} \hat{f}(k) e^{2\pi \imath k \cdot x} \qquad (0 < r < 1).$$

Example 3.4 (Generalized Picar and Bessel summations). Let

$$\theta(x) = \frac{1}{(1+|x|^\gamma)^\alpha}$$

for some $0 < \alpha, \gamma < \infty$ such that $\alpha\gamma > 1$. The θ-means are given by

$$\sigma_n^\theta f(x) := \sum_{k=-\infty}^{\infty} \frac{1}{\Big(1 + (\frac{|k|}{n+1})^\gamma\Big)^\alpha} \hat{f}(k) e^{2\pi \imath k \cdot x}.$$

Example 3.5 (de La Vallée-Poussin summation). Let

$$\theta(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant 1/2, \\ -2x+2 & \text{if } 1/2 < x \leqslant 1, \\ 0 & \text{if } x > 1. \end{cases}$$

Example 3.6. Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m$ and $\beta_0, \ldots, \beta_m$ $(m \in \mathbb{N})$ be real numbers, $\beta_0 = 1$, $\beta_m = 0$. Suppose that $\theta(\alpha_j) = \beta_j$ $(j = 0, 1, \ldots, m)$, $\theta(x) = 0$ for $x \geqslant \alpha_m$, θ is a polynomial on the interval $[\alpha_{j-1}, \alpha_j]$ $(j = 1, \ldots, m)$.

4. Norm Convergence of the θ-means of Fourier series

In this section we collect some results about the norm convergence of $\sigma_{\mathbf{n}}^\theta f$ as $n \to \infty$. The proofs of the theorems can be found in Feichtinger and Weisz [5]. Note that $\mathbf{x}$ denotes the vector $(x, \ldots, x) \in \mathbb{R}^d$ $(x \in \mathbb{R})$.

Theorem 4.1. *If $\theta \in W(C, \ell_1)(\mathbb{R}^d)$ and $\theta(0) = 1$ then for all $f \in L_2(\mathbb{T}^d)$*

$$\lim_{n\to\infty} \sigma_{\mathbf{n}}^\theta f = f \qquad \textit{in } L_2(\mathbb{T}^d) \textit{ norm.}$$

If the Fourier transform of θ is integrable then the θ-means can be written as a singular integral of f and the Fourier transform of θ in the following way.

Theorem 4.2. *If* $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ *and* $\hat\theta \in L_1(\mathbb{R}^d)$ *then*

$$\sigma_{\mathbf{n}}^{\theta} f(x) = (n+1)^d \int_{\mathbb{R}^d} f(x-t)\,\hat\theta\big((n+1)t\big)\,dt$$

for all $x \in \mathbb{T}^d$, $n \in \mathbb{N}$ *and* $f \in L_1(\mathbb{T}^d)$.

For the uniform and L_1 norm convergence of $\sigma_{\mathbf{n}}^{\theta} f \to f$ a sufficient and necessary condition can be given.

Theorem 4.3. *If* $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ *and* $\theta(0) = 1$ *then the following conditions are equivalent:*

(i) $\hat\theta \in L_1(\mathbb{R}^d)$,

(ii) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iii) $\sigma_{\mathbf{n}}^{\theta} f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iv) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *in* $L_1(\mathbb{T}^d)$ *norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$.

One part of the preceding result is generalized for homogeneous Banach spaces.

Theorem 4.4. *Assume that* B *is a homogeneous Banach space on* $\mathbb{T}^d$. *If* $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0) = 1$ *and* $\hat\theta \in L_1(\mathbb{R}^d)$ *then for all* $f \in B$

$$\lim_{n\to\infty} \sigma_{\mathbf{n}}^{\theta} f = f \qquad \text{in } B \text{ norm.}$$

Since $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ implies $\theta \in W(C,\ell_1)(\mathbb{R}^d)$ and $\hat\theta \in \mathbf{S}_0(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$, the next corollary follows from Theorems 4.3 and 4.4.

Corollary 4.5. *If* $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ *and* $\theta(0) = 1$ *then*

(i) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(ii) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *in* $L_1(\mathbb{T}^d)$ *norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$,

(iii) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *in* B *norm for all* $f \in B$ *as* $n \to \infty$ *if* B *is a homogeneous Banach space.*

If θ has compact support then $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ is equivalent to the conditions $\theta, \hat\theta \in L_1(\mathbb{R}^d)$ (see Feichtinger and Zimmermann [6]). This implies

Corollary 4.6. *If* $\theta \in C(\mathbb{R}^d)$ *has compact support and* $\theta(0) = 1$ *then the following conditions are equivalent:*

(i) $\theta \in \mathbf{S}_0(\mathbb{R}^d)$,

(ii) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *uniformly for all* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iii) $\sigma_{\mathbf{n}}^{\theta} f(x) \to f(x)$ *for all* $x \in \mathbb{T}^d$ *and* $f \in C(\mathbb{T}^d)$ *as* $n \to \infty$,

(iv) $\sigma_{\mathbf{n}}^{\theta} f \to f$ *in* $L_1(\mathbb{T}^d)$ *norm for all* $f \in L_1(\mathbb{T}^d)$ *as* $n \to \infty$.

5. Norm convergence of the θ-means of Fourier transforms

All the results above can be shown for non-periodic functions $f \in L_p(\mathbb{R}^d)$. Suppose first that $f \in L_p(\mathbb{R}^d)$ for some $1 \leqslant p \leqslant 2$. The Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(u) e^{2\pi \imath x \cdot u} \, du \qquad (x \in \mathbb{R}^d)$$

holds if $\hat{f} \in L_1(\mathbb{R}^d)$.

In the investigation of Fourier transforms we can take a larger space than $W(C, \ell_1)(\mathbb{R}^d)$, we will assume that $\theta \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$. The θ-*means* of $f \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant 2)$ are defined by

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}^d} \theta\Big(\frac{-t_1}{T_1}, \dots, \frac{-t_d}{T_d}\Big) \hat{f}(t) e^{2\pi \imath x \cdot t} \, dt = \int_{\mathbb{R}^d} f(x-t) K_T^\theta(t) \, dt$$

where $x \in \mathbb{R}^d, T \in \mathbb{R}_+^d$ and

$$K_T^\theta(x) = \int_{\mathbb{R}^d} \theta\Big(\frac{-t_1}{T_1}, \dots, \frac{-t_d}{T_d}\Big) e^{2\pi \imath x \cdot t} \, dt = \Big(\prod_{j=1}^d T_j\Big) \hat{\theta}(T_1 x_1, \dots, T_d x_d),$$

$(x \in \mathbb{R}^d)$. Thus the θ-means can rewritten as

$$\sigma_T^\theta f(x) = \Big(\prod_{j=1}^d T_j\Big) \int_{\mathbb{R}^d} f(x-t) \hat{\theta}(T_1 t_1, \dots, T_d t_d) \, dt \tag{5.1}$$

which is the analogue to Theorem 4.2. Note that $\theta \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ implies $\theta \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty)$. Now we formulate Theorem 4.1 for Fourier transforms.

Theorem 5.1. *If $\theta \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ and $\theta(0) = 1$ then for all $f \in L_2(\mathbb{R}^d)$*

$$\lim_{T \to \infty} \sigma_{\mathbf{T}}^\theta f = f \qquad \text{in } L_2(\mathbb{R}^d) \text{ norm.}$$

Since σ_T^θ is defined only for $f \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant 2)$, instead of Theorem 4.3 we have

Theorem 5.2. *If $\theta \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ and $\theta(0) = 1$ then the following conditions are equivalent:*

(i) $\hat{\theta} \in L_1(\mathbb{R}^d)$,

(ii) $\sigma_{\mathbf{T}}^\theta f \to f$ *in $L_1(\mathbb{R}^d)$ norm for all $f \in L_1(\mathbb{R}^d)$ as $T \to \infty$.*

If θ has compact support then $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ is also an equivalent condition.

If $\hat{\theta} \in L_1(\mathbb{R}^d)$, the definition of the θ-*means* extends to $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$ by

$$\sigma_T^\theta f := f * K_T^\theta \qquad (T \in \mathbb{R}_+^d),$$

where $*$ denotes the convolution. Note that $\theta \in L_1(\mathbb{R}^d)$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ imply $\theta \in C_0(\mathbb{R}^d)$.

The analogue of Theorem 4.4 follows in the same way:

Theorem 5.3. *Assume that B is a homogeneous Banach space on $\mathbb{R}^d$. If $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ (e.g. $\theta \in \mathbf{S}_0(\mathbb{R}^d)$) then for all $f \in B$*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^\theta f = f \qquad \textit{in } B \textit{ norm.}$$

Since the space $C_u(\mathbb{R}^d)$ of uniformly continuous bounded functions endowed with the supremum norm is also a homogeneous Banach space, we have

Corollary 5.4. *If f is a uniformly continuous and bounded function, $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$ and $\hat{\theta} \in L_1(\mathbb{R}^d)$ then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^\theta f = f \qquad \textit{uniformly.}$$

6. Sufficient conditions

In this section we give some sufficient conditions for a function θ, which ensures that $\hat{\theta} \in L_1(\mathbb{R}^d)$, resp. $\theta \in \mathbf{S}_0(\mathbb{R}^d)$. As mentioned before $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ implies also that $\hat{\theta} \in L_1(\mathbb{R}^d)$. Recall that $\mathbf{S}_0(\mathbb{R}^d)$ contains all Schwartz functions. If $\theta \in L_1(\mathbb{R}^d)$ and $\hat{\theta}$ has compact support or if $\theta \in L_1(\mathbb{R}^d)$ has compact support and $\hat{\theta} \in L_1(\mathbb{R}^d)$ then $\theta \in \mathbf{S}_0(\mathbb{R}^d)$.

Sufficient conditions can be given with the help of Sobolev, fractional Sobolev and Besov spaces, too. For a detailed description of these spaces see Triebel [31], Runst and Sickel [21], Stein [26] and Grafakos [11].

A function $\theta \in L_p(\mathbb{R}^d)$ is in the *Sobolev space* $W_p^k(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty, k \in \mathbb{N})$ if $D^\alpha\theta \in L_p(\mathbb{R}^d)$ for all $|\alpha| \leqslant k$ and

$$\|\theta\|_{W_p^k} := \sum_{|\alpha|\leqslant k} \|D^\alpha \theta\|_p < \infty,$$

where D denotes the distributional derivative.

This definition is extended to every real s in the following way. The *fractional Sobolev space* $\mathcal{L}_p^s(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty, s \in \mathbb{R})$ consists of all tempered distribution θ for which

$$\|\theta\|_{\mathcal{L}_p^s} := \|\mathcal{F}^{-1}((1 + |\cdot|^2)^{s/2}\hat{\theta})\|_p < \infty.$$

It is known that $\mathcal{L}_p^s(\mathbb{R}^d) = W_p^k(\mathbb{R}^d)$ if $s = k \in \mathbb{N}$ and $1 < p < \infty$ with equivalent norms.

In order to define the Besov spaces take a non-negative Schwartz function $\psi \in \mathcal{S}(\mathbb{R})$ with support $[1/2, 2]$ which satisfies $\sum_{k=-\infty}^{\infty} \psi(2^{-k}s) = 1$ for all $s \in \mathbb{R} \setminus \{0\}$. For $x \in \mathbb{R}^d$ let

$$\phi_k(x) := \psi(2^{-k}|x|) \quad \text{for} \quad k \geqslant 1 \quad \text{and} \quad \phi_0(x) = 1 - \sum_{k=1}^{\infty} \phi_k(x).$$

The *Besov space* $B^s_{p,r}(\mathbb{R}^d)$ $(0 < p, r \leqslant \infty, s \in \mathbb{R})$ is the space of all tempered distributions f for which

$$\|f\|_{B^s_{p,r}} := \Big(\sum_{k=0}^{\infty} 2^{ksr} \|(\mathcal{F}^{-1}\phi_k) * f\|_p^r \Big)^{1/r} < \infty.$$

The Sobolev, fractional Sobolev and Besov spaces are all quasi Banach spaces and if $1 \leqslant p, r \leqslant \infty$ then they are Banach spaces. All these spaces contain the Schwartz functions. The following facts are known: in case $1 \leqslant p, r \leqslant \infty$ one has

$$W^m_p(\mathbb{R}^d), B^s_{p,r}(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d) \qquad \text{if} \quad s > 0, m \in \mathbb{N},$$

$$W^{m+1}_p(\mathbb{R}^d) \hookrightarrow B^s_{p,r}(\mathbb{R}^d) \hookrightarrow W^m_p(\mathbb{R}^d) \qquad \text{if} \quad m < s < m+1,$$

$$B^s_{p,r}(\mathbb{R}^d) \hookrightarrow B^s_{p,r+\epsilon}(\mathbb{R}^d), B^{s+\epsilon}_{p,\infty}(\mathbb{R}^d) \hookrightarrow B^s_{p,r}(\mathbb{R}^d) \qquad \text{if} \quad \epsilon > 0,$$

$$B^{d/p_1}_{p_1,1}(\mathbb{R}^d) \hookrightarrow B^{d/p_2}_{p_2,1}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d) \qquad \text{if} \quad 1 \leqslant p_1 \leqslant p_2 < \infty.$$

Theorem 6.1.

(i) If $1 \leqslant p \leqslant 2$ and $\theta \in B^{d/p}_{p,1}(\mathbb{R}^d)$ then $\hat{\theta} \in L_1(\mathbb{R}^d)$ and

$$\|\hat{\theta}\|_1 \leqslant C \|\theta\|_{B^{d/p}_{p,1}}.$$

(ii) If $s > d$ then $\mathcal{L}^s_1(\mathbb{R}^d) \hookrightarrow \mathbf{S}_0(\mathbb{R}^d)$.

(iii) If d' denotes the smallest even integer which is larger than d and $s > d'$ then

$$B^s_{1,\infty}(\mathbb{R}^d) \hookrightarrow W^{d'}_1(\mathbb{R}^d) \hookrightarrow \mathbf{S}_0(\mathbb{R}^d).$$

The embedding $W^2_1(\mathbb{R}) \hookrightarrow \mathbf{S}_0(\mathbb{R})$ follows from (iii). With the help of the usual derivative we give another useful sufficient condition for a function to be in $\mathbf{S}_0(\mathbb{R}^d)$.

A function θ is in $V^k_1(\mathbb{R})$ $(k \geqslant 2, k \in \mathbb{N})$, if there are numbers $-\infty = a_0 < a_1 < \ldots < a_n < a_{n+1} = \infty$ such that $n = n(\theta)$ is depending on θ and

$$\theta \in C^{k-2}(\mathbb{R}), \quad \theta \in C^k(a_i, a_{i+1}), \quad \theta^{(j)} \in L_1(\mathbb{R})$$

for all $i = 0, \ldots, n$ and $j = 0, \ldots, k$. Here C^k denotes the set of k-times continuously differentiable functions. The norm of this space is introduced by

$$\|\theta\|_{V^k_1} := \sum_{j=0}^{k} \|\theta^{(j)}\|_1 + \sum_{i=1}^{n} |\theta^{(k-1)}(a_i + 0) - \theta^{(k-1)}(a_i - 0)|$$

where $\theta^{(k-1)}(a_i \pm 0)$ denote the right and left limits of $\theta^{(k-1)}$. These limits do exist and are finite because $\theta^{(k)} \in C(a_i, a_{i+1}) \cap L_1(\mathbb{R})$ implies

$$\theta^{(k-1)}(x) = \theta^{(k-1)}(a) + \int_a^x \theta^{(k)}(t)\,dt$$

for some $a \in (a_i, a_{i+1})$. Since $\theta^{(k-1)} \in L_1(\mathbb{R})$ we establish that $\lim_{-\infty} \theta^{(k-1)} = \lim_{\infty} \theta^{(k-1)} = 0$. Similarly, $\theta^{(j)} \in C_0(\mathbb{R})$ for $j = 0, \ldots, k-2$.

Of course, $W_1^2(\mathbb{R})$ and $V_1^2(\mathbb{R})$ are not identical. For $\theta \in V_1^2(\mathbb{R})$ we have $\theta' = D\theta$, however, $\theta'' = D^2\theta$ only if $\lim_{a_i+0} \theta' = \lim_{a_i-0} \theta'$ $(i = 1, \ldots, n)$.

We generalize the previous definition for the d-dimensional case as follows. For $d > 1$ and $k \geqslant 2$ let $\theta \in V_1^k(\mathbb{R}^d)$ if θ is even in each variable and

$$\theta \in C^{k-2}(\mathbb{R}^d), \quad \theta \in C^k([0,\infty)^d \setminus \{(0,\ldots,0)\}), \quad \partial_1^{i_1} \cdots \partial_d^{i_d} \theta(t) \in L_1([0,\infty)^l)$$

for each $i_j = 0, \ldots, k$ $(j = 1, \ldots, d)$ and fixed $0 < t_{m_1}, \ldots, t_{m_{d-l}} < \infty$ $(1 \leqslant m_1 < m_2 < \cdots < m_{d-l} \leqslant d)$ and $1 \leqslant l \leqslant d$.

Theorem 6.2. *If $\theta \in V_1^2(\mathbb{R}^d)$ then $\theta \in \mathbf{S}_0(\mathbb{R}^d)$.*

The next Corollary follows from the definition of $\mathbf{S}_0(\mathbb{R}^d)$.

Corollary 6.3. *If each $\theta_j \in V_1^2(\mathbb{R})$ $(j = 1, \ldots, d)$ then $\theta := \prod_{j=1}^d \theta_j \in \mathbf{S}_0(\mathbb{R}^d)$.*

7. A.e. convergence of the θ-means of Fourier transforms

For the a.e. convergence we will investigate first Fourier transforms rather than Fourier series, because the theorems for Fourier transforms are more complicated. The proofs of the results can be found in Feichtinger and Weisz [4].

$L_p^{loc}(\mathbb{X}^d)$ $(1 \leqslant p \leqslant \infty)$ denotes the space of measurable functions f for which $|f|^p$ is locally integrable, resp. f is locally bounded if $p = \infty$. For $1 \leqslant p \leqslant \infty$ and $f \in L_p^{loc}(\mathbb{X}^d)$ let us define a generalization of the *Hardy-Littlewood maximal function* by

$$M_p f(x) := \sup_{x \in I} \Big(\frac{1}{|I|} \int_I |f|^p \, d\lambda \Big)^{1/p} \qquad (x \in \mathbb{X}^d)$$

with the usual modification for $p = \infty$, where the supremum is taken over all cubes with sides parallel to the axes. If $p = 1$, this is the usual Hardy-Littlewood maximal function. The following inequalities follow easily from the case $p = 1$, which can be found in Stein [27] or Weisz [37]:

$$\|M_p f\|_{L_{p,\infty}} \leqslant C_p \|f\|_p \qquad (f \in L_p(\mathbb{X}^d)) \tag{7.1}$$

and

$$\|M_p f\|_r \leqslant C_r \|f\|_r \qquad (f \in L_r(\mathbb{X}^d), p < r \leqslant \infty). \tag{7.2}$$

The first inequality holds also if $p=\infty$.

The space $\dot{E}_q(\mathbb{R}^d)$ contains all functions $f \in L_q^{loc}(\mathbb{R}^d)$ for which

$$\|f\|_{\dot{E}_q} := \sum_{k=-\infty}^{\infty} 2^{kd(1-1/q)} \|f \mathbf{1}_{P_k}\|_q < \infty,$$

where $P_k := \{2^{k-1} \leqslant |x| < 2^k\}$, $(k \in \mathbb{Z})$. These spaces are special cases of the Herz spaces [15] (see also Garcia-Cuerva and Herrero [9]). The non-homogeneous version of the space $\dot{E}_q(\mathbb{R}^d)$ was used by Feichtinger [8] to prove some Tauberian theorems. It is easy to see that

$$L_1(\mathbb{R}^d) = \dot{E}_1(\mathbb{R}^d) \hookleftarrow \dot{E}_q(\mathbb{R}^d) \hookleftarrow \dot{E}_{q'}(\mathbb{R}^d) \hookleftarrow \dot{E}_\infty(\mathbb{R}^d), \qquad 1 < q < q' < \infty.$$

To prove pointwise convergence of the θ-means we will investigate the *maximal operator*

$$\sigma_\square^\theta f := \sup_{T>0} |\sigma_{\mathbf{T}}^\theta f|.$$

If $\hat{\theta} \in L_1(\mathbb{R}^d)$ then (5.1) implies

$$\|\sigma_\square^\theta f\|_\infty \leqslant \|\hat{\theta}\|_1 \|f\|_\infty \qquad (f \in L_\infty(\mathbb{R}^d)).$$

In the one-dimensional case Torchinsky [30] proved that if there exists an even function η such that η is non-increasing on $\mathbb{R}_+$, $|\hat{\theta}| \leqslant \eta$, $\eta \in L_1$ then $\sigma_\square^\theta$ is of weak type $(1,1)$ and a.e. convergence holds. Under similar conditions we will generalize this result for the multi-dimensional setting. First we introduce an equivalent condition.

Theorem 7.1. *For $\theta \in L_1(\mathbb{R}^d)$ let $\eta(x) := \sup_{\|t\|_r \geqslant \|x\|_r} |\hat{\theta}(t)|$ for some $1 \leqslant r \leqslant \infty$. Then $\hat{\theta} \in \dot{E}_\infty(\mathbb{R}^d)$ if and only if $\eta \in L_1(\mathbb{R}^d)$ and*

$$C^{-1}\|\eta\|_1 \leqslant \|\hat{\theta}\|_{\dot{E}_\infty} \leqslant C\|\eta\|_1.$$

Theorem 7.2. *Let $\theta \in L_1(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$ and $1/p + 1/q = 1$. If $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$ then*

$$\|\sigma_\square^\theta f\|_{L_{p,\infty}} \leqslant C_p \|\hat{\theta}\|_{\dot{E}_q} \|f\|_p$$

for all $f \in L_p(\mathbb{R}^d)$. Moreover, for every $p < r \leqslant \infty$,

$$\|\sigma_\square^\theta f\|_r \leqslant C_r \|\hat{\theta}\|_{\dot{E}_q} \|f\|_r \qquad (f \in L_r(\mathbb{R}^d)).$$

The proof of this theorem follows from the pointwise inequality

$$\sigma_\square^\theta f(x) \leqslant C \|\hat{\theta}\|_{\dot{E}_q} M_p f(x) \tag{7.3}$$

and from (7.1) and (7.2). Inequality (7.3) is proved in Feichtinger and Weisz [4].

Theorem 7.2 and the usual density argument due to Marcinkiewicz and Zygmund [17] imply

Corollary 7.3. *If* $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$, $1 \leqslant p \leqslant \infty$, $1/p+1/q = 1$ *and* $\hat\theta \in \dot E_q(\mathbb{R}^d)$ *then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^\theta f = f \qquad a.e.$$

if $f \in L_r(\mathbb{R}^d)$ *for* $p \leqslant r < \infty$ *or* $f \in C_0(\mathbb{R}^d)$.

Note that $\dot E_q(\mathbb{R}^d) \supset \dot E_{q'}(\mathbb{R}^d)$ whenever $q < q'$. If $\hat\theta$ is in a smaller space (say in $\dot E_\infty(\mathbb{R}^d)$) then we get convergence for a wider class of functions (namely for $f \in L_r(\mathbb{R}^d)$, $1 \leqslant r \leqslant \infty$).

In order to generalize the last theorem and corollary for the larger space $W(L_1, \ell_\infty)(\mathbb{R}^d)$, we have to define the *local Hardy-Littlewood maximal function* by

$$m_p f(x) := \sup_{0<r\leqslant 1} \Big(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f|^p \, d\lambda \Big)^{1/p} \qquad (x \in \mathbb{R}^d),$$

where $f \in L_p^{loc}(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$ and $B(x,r)$ denotes the ball with center x and radius r. It is easy to see that inequalities (7.1) and (7.2) imply

$$\|m_p f\|_{W(L_{p,\infty},\ell_s)} \leqslant C_p \|f\|_{W(L_p,\ell_s)} \qquad (f \in W(L_p,\ell_s)(\mathbb{R}^d)) \tag{7.4}$$

and

$$\|m_p f\|_{W(L_r,\ell_s)} \leqslant C_r \|f\|_{W(L_r,\ell_s)} \qquad (f \in W(L_r,\ell_s)(\mathbb{R}^d)) \tag{7.5}$$

for all $p < r \leqslant \infty$ and $1 \leqslant s \leqslant \infty$. Recall that

$$\|f\|_{W(L_{p,\infty},\ell_\infty)} = \sup_{k\in\mathbb{Z}^d} \sup_{\rho>0} \rho\, \lambda(|f| > \rho, [k, k+1))^{1/p}.$$

Theorem 7.4. *Let* $\theta \in L_1(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$ *and* $1/p+1/q = 1$. *If* $\hat\theta \in \dot E_q(\mathbb{R}^d)$ *then*

$$\|\sigma_\square^\theta f\|_{W(L_{p,\infty},\ell_\infty)} \leqslant C_p \|\hat\theta\|_{\dot E_q} \|f\|_{W(L_p,\ell_\infty)}$$

for all $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$. *Moreover, for every* $p < r \leqslant \infty$,

$$\|\sigma_\square^\theta f\|_{W(L_r,\ell_\infty)} \leqslant C_r \|\hat\theta\|_{\dot E_q} \|f\|_{W(L_r,\ell_\infty)} \qquad (f \in W(L_r,\ell_\infty)(\mathbb{R}^d)).$$

It is easy to see that

$$M_p f \leqslant C m_p f + C_p \|f\|_{W(L_p,\ell_\infty)} \qquad (1 \leqslant p \leqslant \infty).$$

The proof of Theorem 7.4 follows from (7.3)–(7.5).

Corollary 7.5. *If* $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$, $1 \leqslant p < \infty$, $1/p+1/q = 1$ *and* $\hat\theta \in \dot E_q(\mathbb{R}^d)$ *then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^\theta f = f \quad a.e.$$

if $f \in W(L_p, c_0)(\mathbb{R}^d)$.

Note that $W(L_p,c_0)(\mathbb{R}^d)$ contains all $W(L_r,c_0)(\mathbb{R}^d)$ spaces for $p \leqslant r \leqslant \infty$.

We can characterize the set of convergence in the following way. Lebesgue differentiation theorem says that

$$\lim_{h\to 0} \frac{1}{|B(0,h)|} \int_{B(0,h)} f(x+u)\,du = f(x)$$

for a.e. $x \in \mathbb{X}^d$, where $f \in L_1^{loc}(\mathbb{X}^d)$, $\mathbb{X} = \mathbb{T}$ or $\mathbb{X} = \mathbb{R}$. A point $x \in \mathbb{X}^d$ is called a *p-Lebesgue point* (or a Lebesgue point of order p) of $f \in L_p^{loc}(\mathbb{X}^d)$ if

$$\lim_{h\to 0} \Big(\frac{1}{|B(0,h)|} \int_{B(0,h)} |f(x+u)-f(x)|^p\,du\Big)^{1/p} = 0 \qquad (1 \leqslant p < \infty)$$

resp.

$$\lim_{h\to 0} \sup_{u\in B(0,h)} |f(x+u)-f(x)| = 0 \qquad (p=\infty).$$

Usually the 1-Lebesgue points, called simply Lebesgue points are considered (cf. Stein and Weiss [25] or Butzer and Nessel [3]). One can show that almost every point $x \in \mathbb{X}^d$ is a p-Lebesgue point of $f \in L_p^{loc}(\mathbb{X}^d)$ if $1 \leqslant p < \infty$, which means that almost every point $x \in \mathbb{R}^d$ is a p-Lebesgue point of $f \in W(L_p,\ell_\infty)(\mathbb{R}^d)$. $x \in \mathbb{X}^d$ is an ∞-Lebesgue point of $f \in L_\infty^{loc}(\mathbb{X}^d)$ if and only if f is continuous at x. Moreover, all r-Lebesgue points are p-Lebesgue points, whenever $p < r$.

Stein and Weiss [25, p. 13] (see also Butzer and Nessel [3, pp. 132-134]) proved that if $\eta(x) := \sup_{|t|\geqslant|x|} |\hat\theta(t)|$ and $\eta \in L_1(\mathbb{R}^d)$ then one has convergence at each Lebesgue point of $f \in L_p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty)$. Using the $\dot E_q$ spaces we generalize this result.

Theorem 7.6. *Let $\theta \in L_1(\mathbb{R}^d)$, $\theta(0)=1$, $1 \leqslant p \leqslant \infty$ and $1/p+1/q=1$. If $\hat\theta \in \dot E_q(\mathbb{R}^d)$ then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^\theta f(x) = f(x)$$

for all p-Lebesgue points of $f \in W(L_p,\ell_\infty)(\mathbb{R}^d)$.

Note that $W(L_1,\ell_\infty)(\mathbb{R}^d)$ contains all $L_p(\mathbb{R}^d)$ spaces and amalgam spaces $W(L_p,\ell_q)(\mathbb{R}^d)$ for the full range $1 \leqslant p,q \leqslant \infty$.

If f is continuous at a point x then x is a p-Lebesgue point of f for every $1 \leqslant p \leqslant \infty$.

Corollary 7.7. *Let $\theta \in L_1(\mathbb{R}^d)$, $\theta(0)=1$, $1 \leqslant p \leqslant \infty$ and $1/p+1/q=1$. If $\hat\theta \in \dot E_q(\mathbb{R}^d)$ and $f \in W(L_p,\ell_\infty)(\mathbb{R}^d)$ is continuous at a point x then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^\theta f(x) = f(x).$$

Recall that $\dot E_1(\mathbb{R}^d) = L_1(\mathbb{R}^d)$ and $W(L_\infty,\ell_\infty)(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$. If f is uniformly continuous then we have uniform convergence (see Corollary 5.4).

Let us consider converse-type problems. The partial converse of Theorem 7.2 is given in the next result.

Theorem 7.8. *Let $\theta \in L_1(\mathbb{R}^d)$, $\hat{\theta} \in L_1(\mathbb{R}^d)$, $1 \leqslant p < \infty$ and $1/p + 1/q = 1$. If*

$$\sigma_{\square}^{\theta} f(x) \leqslant C M_p f(x) \tag{7.6}$$

for all $f \in L_p(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ then $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$.

The converse of Theorem 7.6 reads as follows.

Theorem 7.9. *Suppose that $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$, $\hat{\theta} \in L_1(\mathbb{R}^d)$, $1 \leqslant p < \infty$ and $1/p + 1/q = 1$. If*

$$\lim_{T \to \infty} \sigma_{\mathbf{T}}^{\theta} f(x) = f(x) \tag{7.7}$$

for all p-Lebesgue points of $f \in L_p(\mathbb{R}^d)$ then $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$.

Corollary 7.10. *Suppose that $\theta \in L_1(\mathbb{R}^d)$, $\theta(0) = 1$, $\hat{\theta} \in L_1(\mathbb{R}^d)$, $1 \leqslant p < \infty$ and $1/p + 1/q = 1$. Then*

$$\lim_{T \to \infty} \sigma_{\mathbf{T}}^{\theta} f(x) = f(x)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{R}^d)$ (resp. of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$) if and only if $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$.

If we take the supremum in the maximal θ-operator over a cone, say over $\{T \in \mathbb{R}_+^d : 2^{-\tau} \leqslant T_i/T_j \leqslant 2^\tau; i, j = 1, \ldots, d\}$ for some fixed $\tau \geqslant 0$:

$$\sigma_c^{\theta} f := \sup_{\substack{2^{-\tau} \leqslant T_i/T_j \leqslant 2^\tau \\ i,j=1,\ldots,d}} |\sigma_T^{\theta} f|,$$

then all the results above can be shown for σ_c^θ. In this case, under the conditions above we obtain the convergence $\sigma_T^\theta f \to f$ a.e. as $T \to \infty$ and $2^{-\tau} \leqslant T_i/T_j \leqslant 2^\tau$ $(i, j = 1, \ldots, d)$. This convergence has been investigated in a great number of papers (e.g. in Marcinkiewicz and Zygmund [17], Zygmund [39], Weisz [34, 36, 37]). For more details see Feichtinger and Weisz [4]. The unrestricted convergence of $\sigma_T^\theta f$, i.e. as $T_j \to \infty$ for each $j = 1, \ldots, d$, is also investigated in that paper.

8. A.e. convergence of the θ-means of Fourier series

All the results of Section 7 holds also for Fourier series. In this case we define the *maximal operator* of the θ-means by

$$\sigma_{\square}^{\theta} f := \sup_{n \in \mathbb{N}} |\sigma_{\mathbf{n}}^{\theta} f|.$$

Similarly to Theorem 7.4 we have

Theorem 8.1. *Let $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$ and $1/p+1/q=1$. If $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$ then*

$$\|\sigma_{\square}^{\theta} f\|_{L_{p,\infty}} \leqslant C_p \|\hat{\theta}\|_{\dot{E}_q} \|f\|_p$$

for all $f \in L_p(\mathbb{T}^d)$. Moreover, for every $p < r \leqslant \infty$,

$$\|\sigma_{\square}^{\theta} f\|_r \leqslant C_r \|\hat{\theta}\|_{\dot{E}_q} \|f\|_r \qquad (f \in L_r(\mathbb{T}^d)).$$

The analogue of Theorems 7.6, 7.9 and Corollary 7.10 reads as follows.

Theorem 8.2. *Let $\theta \in W(C,\ell_1)(\mathbb{R}^d)$, $\theta(0)=1$, $\hat{\theta} \in L_1(\mathbb{R}^d)$, $1 \leqslant p < \infty$ and $1/p+1/q=1$. Then*

$$\lim_{n\to\infty} \sigma_{\mathbf{n}}^{\theta} f(x) = f(x)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{T}^d)$ if and only if $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$.

Corollary 8.3. *Let $\theta \in L_1(\mathbb{R}^d)$, $\theta(0)=1$, $1 \leqslant p \leqslant \infty$ and $1/p+1/q=1$. If $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$ and $f \in L_p(\mathbb{T}^d)$ is continuous at a point $x \in \mathbb{T}^d$ then*

$$\lim_{n\to\infty} \sigma_{\mathbf{n}}^{\theta} f(x) = f(x).$$

9. Besov, modulation and Sobolev spaces

The next theorem was proved in Herz [15], Peetre [19] and Girardi and Weis [10].

Theorem 9.1. *If $1 \leqslant p \leqslant 2$, $1/p+1/q=1$ and $\theta \in B_{p,1}^{d/p}(\mathbb{R}^d)$ then $\hat{\theta} \in \dot{E}_q(\mathbb{R}^d)$ and*

$$\|\hat{\theta}\|_{\dot{E}_q} \leqslant C_p \|\theta\|_{B_{p,1}^{d/p}}.$$

Theorem 9.1 implies the following result.

Corollary 9.2. *If $\theta \in L_1(\mathbb{R}^d) \cap B_{p,1}^{d/p}(\mathbb{R}^d)$ for some $1 \leqslant p \leqslant 2$ and $\theta(0)=1$ then Theorems 7.2, 7.4, 7.6 and Corollaries 7.3, 7.5 and 7.7 hold.*

For the connection between the $\dot{E}_q(\mathbb{R}^d)$ and amalgam spaces we have proved the following result.

Theorem 9.3. *If $f \in W(L_q,\ell_1^{v_{d/p}})(\mathbb{R}^d)$ for some $1 \leqslant q \leqslant \infty$, $1/p+1/q=1$ then $f \in \dot{E}_q(\mathbb{R}^d)$ and*

$$\|f\|_{\dot{E}_q} \leqslant C_q \|f\|_{W(L_q,\ell_1^{v_{d/p}})}.$$

Corollary 9.4. *Let $\theta \in L_1(\mathbb{R}^d)$, $\theta(0)=1$, $1 \leqslant p \leqslant \infty$ and $1/p+1/q=1$. If $\hat{\theta} \in W(L_q,\ell_1^{v_{d/p}})(\mathbb{R}^d)$ then Theorems 7.2, 7.4, 7.6 and Corollaries 7.3, 7.5 and 7.7 hold.*

In particular, if $q=\infty$ then we get the condition $\hat{\theta} \in W(C, \ell_1^{v_d})(\mathbb{R}^d)$. Note that if $\theta \in L_1(\mathbb{R}^d)$ and $\hat{\theta}$ has compact support then Corollary 9.4 holds. Actually $\theta \in \mathbf{S}_0(\mathbb{R}^d)$ in this case (see e.g. Feichtinger and Zimmerman [6]).

If θ is in a suitable modulation space then the θ-means converge a.e. to f. Indeed, using Theorem 9.3 we can show that $\hat{\theta} \in W(L_\infty, \ell_1^{v_d})(\mathbb{R}^d) \subset \dot{E}_\infty(\mathbb{R}^d)$ if $\theta \in M_1^{v_d}(\mathbb{R}^d)$ and

$$C^{-1}\|\hat{\theta}\|_{\dot{E}_\infty} \leqslant \|\hat{\theta}\|_{W(L_\infty, \ell_1^{v_d})} \leqslant C\|\theta\|_{M_1^{v_d}}.$$

Theorem 9.5. *If $\theta \in M_1^{v_d}(\mathbb{R}^d)$ and $\theta(0)=1$ then*

$$\lim_{T\to\infty} \sigma_{\mathbf{T}}^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$. Moreover,

$$\sup_{\rho>0} \rho\, \lambda(\sigma_\square^\theta > \rho) \leqslant C\|\theta\|_{M_1^{v_d}} \|f\|_1 \qquad (f \in L_1(\mathbb{R}^d)),$$

$$\|\sigma_\square^\theta f\|_{W(L_{1,\infty}, \ell_\infty)} \leqslant C\|\theta\|_{M_1^{v_d}} \|f\|_{W(L_1, \ell_\infty)} \qquad (f \in W(L_1, \ell_\infty)(\mathbb{R}^d))$$

and, for every $1<p\leqslant\infty$,

$$\|\sigma_\square^\theta f\|_p \leqslant C_p\|\theta\|_{M_1^{v_d}} \|f\|_p \qquad (f \in L_p(\mathbb{R}^d)),$$

$$\|\sigma_\square^\theta f\|_{W(L_p, \ell_\infty)} \leqslant C_p\|\theta\|_{M_1^{v_d}} \|f\|_{W(L_p, \ell_\infty)} \qquad (f \in W(L_p, \ell_\infty)(\mathbb{R}^d)).$$

Similarly to Theorem 6.2 we give sufficient conditions for θ to be in the modulation space $\theta \in M_1^{v_d}(\mathbb{R}^d)$.

Theorem 9.6. *If $\theta \in V_1^k(\mathbb{R}^d)$ $(k \geqslant 2)$ then $\theta \in M_1^{v_s}(\mathbb{R}^d)$ for all $0 \leqslant s < k-1$.*

Corollary 9.7. *If each $\theta_j \in V_1^k(\mathbb{R})$ $(k \geqslant 2, j=1,\ldots,d)$ then $\theta := \prod_{j=1}^d \theta_j \in M_1^{v_s}(\mathbb{R}^d)$ for all $0 \leqslant s < k-1$.*

$V_1^2(\mathbb{R})$ is not contained in $M_1^{v_1}(\mathbb{R})$, however, the same results hold as in Theorem 9.5.

Corollary 9.8. *If $\theta \in V_1^2(\mathbb{R})$ then $\hat{\theta} \in \dot{E}_\infty(\mathbb{R})$ and Theorem 9.5 hold.*

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces, an Introduction*, Springer, Berlin, 1976.

[2] J. Bokor and F. Schipp, Approximate identification in Laguerre and Kautz bases, *Automatica*, 34:463–468, 1998.

[3] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, Birkhäuser Verlag, Basel, 1971.

[4] H. G. Feichtinger and F. Weisz, Wiener amalgams and pointwise summability of Fourier transforms and Fourier series, *Math. Proc. Camb. Phil. Soc.*, 140, 2006. (to appear).

[5] H. G. Feichtinger and F. Weisz, The Segal algebra $\mathbf{S}_0(\mathbb{R}^d)$ and norm summability of Fourier series and Fourier transforms, *Monatshefte Math.* (to appear).

[6] H. G. Feichtinger and G. Zimmermann, A Banach space of test functions for Gabor analysis, In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms. Theory and Applications.*, Applied and Numerical Harmonic Analysis, pages 123–170, Boston, MA, 1998. Birkhäuser. chap. 3.

[7] H. G. Feichtinger, On a new Segal algebra, *Monatsh. Math.*, 92:269–289, 1981.

[8] H. Feichtinger, An elementary approach to Wiener's third Tauberian theorem for the Euclidean n-space, In *Symposia Math.*, volume XXIX of *Analisa Armonica*, pages 267–301, Cortona, 1988.

[9] J. Garcia-Cuerva and M.-J. L. Herrero, A theory of Hardy spaces associated to the Herz spaces, *Proc. London Math. Soc.*, 69:605–628, 1994.

[10] M. Girardi and L. Weis, Operator-valued Fourier multiplier theorems on Besov spaces, *Math. Nachr.*, 251:34–51, 2003.

[11] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, New Jersey, 2004.

[12] K. Gröchenig and C. Heil, Gabor meets Littlewood–Paley: Gabor expansions in $L^p(\mathbf{R}^d)$, *Studia Math.*, 146:15–33, 2001.

[13] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[14] C. Heil, An introduction to weighted Wiener amalgams, In M. Krishna, R. Radha, and S. Thangavelu, editors, *Wavelets and their Applications*, pages 183–216. Allied Publishers Private Limited, 2003.

[15] C. Herz, Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms, *J. Math. Mech.*, 18:283–324, 1968.

[16] Y. Katznelson, *An Introduction to Harmonic Analysis*, Cambridge Mathematical Library. Cambridge University Press, 1976.

[17] J. Marcinkiewicz and A. Zygmund, On the summability of double Fourier series, *Fund. Math.*, 32:122–132, 1939.

[18] G. M. Natanson and V. V. Zuk, *Trigonometric Fourier Series and Approximation Theory*, Izdat. Leningrad Unta, Leningrad, 1983. (in Russian).

[19] J. Peetre, Applications de la théorie des espaces d'interpolation dans l'analyse harmonique, *Ricerche Mat.*, 15:1–37, 1966.

[20] H. Reiter, *Classical Harmonic Analysis and Locally Compact Abelian Groups*, Oxford University Press, 1968.

[21] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, Walter de Gruyter, Berlin, New York, 1996.

[22] F. Schipp and J. Bokor, L^∞ system approximation algorithms generated by φ summations, *Automatica*, 33:2019–2024, 1997.

[23] F. Schipp and L. Szili, Approximation in $\mathcal{H}_\infty$-norm, In *Approximation Theory and Function Series*, volume 5, pages 307–320. Bolyai Soc. Math. Studies, Budapest, 1996.

[24] H. S. Shapiro, *Topics in Approximation Theory*, volume 187 of *Lecture Notes in Math.* Springer, Berlin, 1971.

[25] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, N.J., 1971.

[26] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J., 1970.

[27] E. M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, N.J., 1993.

[28] L. Szili and P. Vértesi, On uniform convergence of sequences of certain linear operators, *Acta Math. Hungar.*, 91:159–186, 2001.

[29] L. Szili, On the summability of trigonometric interpolation processes, *Acta Math. Hungar.*, 91:131–158, 2001.

[30] A. Torchinsky, *Real-variable Methods in Harmonic Analysis*, Academic Press, New York, 1986.

[31] H. Triebel, *Theory of Function Spaces II*, Birkhäuser, 1992.

[32] R. M. Trigub and E. S. Belinsky, *Fourier Analysis and Approximation of Functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2004.

[33] D. Walnut, Continuity properties of the Gabor frame operator, *J. Math. Anal. Appl.*, 165(2):479–504, 1992.

[34] F. Weisz, Cesàro summability of multi-dimensional trigonometric-Fourier series, *J. Math. Anal. Appl.*, 204:419–431, 1996.

[35] F. Weisz, θ-summation and Hardy spaces, *J. Appr. Theory*, 107:121–142, 2000.

[36] F. Weisz, Several dimensional θ-summability and Hardy spaces, *Math. Nachr.*, 230:159–180, 2001.

[37] F. Weisz, *Summability of Multi-dimensional Fourier Series and Hardy Spaces*, Mathematics and Its Applications. Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[38] F. Weisz, θ-summability of Fourier series, *Acta Math. Hungar.*, 103:139–176, 2004.

[39] A. Zygmund, *Trigonometric Series*, Cambridge Press, London, 3th edition, 2002.

Ferenc Weisz
Department of Numerical Analysis
Eötvös L. University
Pázmány P. sétány 1/C.
H-1117 Budapest
Hungary

Methodological papers

Annales Mathematicae et Informaticae
32 *(2005) pp. 189–201.*

Homogeneity properties of subadditive functions

Pál Burai and Árpád Száz

Institute of Mathematics, University of Debrecen
e-mail: buraip@math.klte.hu
e-mail: szaz@math.klte.hu

Abstract

We collect, supplement and extend some well-known basic facts on various homogeneity properties of subadditive functions.

Key Words: Homogeneous and subadditive functions, seminorms and preseminorms.

AMS Classification Number: 39B72

1. Introduction

Subadditive functions, with various homogeneity properties, play important roles in many branches of mathematics. First of all, they occur in the Hahn-Banach theorems and the derivation of vector topologies. (See, for instance, [2] and [14] .)

A positively homogeneous subadditive function is usually called sublinear. While, an absolutely homogeneous subadditive function may be called a seminorm. However, some important subadditive functions are only preseminorms.

Moreover, it is also worth noticing that subbadditive functions are straightforward generalizations of the real-valued additive ones. Therefore, the study of additive functions should, in principle, be preceded by that of the subadditive ones.

Subadditive functions have been intensively studied by several authors. Their most basic algebraic and analytical properties have been established by R. Cooper [5] , E. Hille [7, pp. 130–145] , R. A. Rosenbaum [17] , E. Berz [1] , M. Kuczma [9, pp. 400–423] and J. Matkowski [11] .

In this paper, we are only interested in the most simple homogeneity properties of subadditive functions. Besides collecting some well-known basic facts, for instance, we prove the following theorem.

Theorem 1.1. *If p is a quasi-subadditive function of a vector space X over $\mathbb{Q}$, and moreover $x \in X$ and $0 \neq k \in \mathbb{Z}$, then*

(1) $\frac{1}{k}\, p(l\,x) \leq p\left(\frac{l}{k}\,x\right)$ *for all* $l \in \mathbb{Z}$;

(2) $p\left(\frac{l}{k}\,x\right) \leq l\,p\left(\frac{1}{k}\,x\right) \geq \frac{l}{k}\,p(x)$ *for all* $0 < l \in \mathbb{Z}$;

(3) $p\left(\frac{l}{k}\,x\right) \geq l\,p\left(\frac{1}{k}\,x\right) \leq \frac{l}{k}\,p(x)$ *for all* $0 > l \in \mathbb{Z}$.

Remark 1.2. If p is a subodd subadditive function of X, then p is additive. Therefore, the corresponding equalities are also true.

While, if p is an even subadditive function of X, then we can only prove that

$$\frac{1}{|k|}\, p(l\,x) \leq p\left(\frac{l}{k}\,x\right) \leq |l|\,p\left(\frac{1}{k}\,x\right)$$

for all $x \in X$, $0 \neq k \in \mathbb{Z}$ and $l \in \mathbb{Z}$.

2. Superodd and subhomogeneous functions

Definition 2.1. A real-valued function p of a group X will be called

(1) subodd if $p(-x) \leq -p(x)$ for all $x \in X$;

(2) superodd if $-p(x) \leq p(-x)$ for all $x \in X$.

Remark 2.2. Note that thus p may be called odd if it is both subodd and superodd.

Moreover, p is superodd if and only if $-p$ is subodd. Therefore, superodd functions need not be studied separately.

However, because of the forthcoming applications, it is more convenient to study superodd functions. By the above definition, we evidently have the following

Proposition 2.3. If p is a superodd function of a group X, then

(1) $0 \leq p(0)$; (2) $-p(-x) \leq p(x)$ for all $x \in X$.

Hint. Clearly, $-p(0) \leq p(-0) = p(0)$. Therefore, $0 \leq 2\,p(0)$, and thus (1) also holds.

Remark 2.4. Note that if p is subodd, then just the opposite inequalities hold. Therefore, if in particular p is odd, then the corresponding equalities are also true.

Analogously to Definition 2.1, we may also naturally introduce the following

Definition 2.5. A real-valued function p of a group X will be called

(1) $\mathbb{N}$–subhomogeneous if $p(nx) \leq n\,p(x)$ for all $n \in \mathbb{N}$ and $x \in X$;

(2) $\mathbb{N}$–superhomogeneous if $n\,p(x) \leq p(nx)$ for all $n \in \mathbb{N}$ and $x \in X$.

Remark 2.6. Note that thus p may be called $\mathbb{N}$–homogeneous if it is both $\mathbb{N}$–subhomogeneous and $\mathbb{N}$–superhomogeneous.

Moreover, p is $\mathbb{N}$–superhomogeneous if and only if $-p$ is $\mathbb{N}$–subhomogeneous. Therefore, $\mathbb{N}$–superhomogeneous functions need not be studied separately.

Concerning $\mathbb{N}$–subhomogeneous functions, we can easily establish the following

Proposition 2.7. If p is an $\mathbb{N}$–subhomogeneous function of a group X, then

$$p(kx) \leq -k\,p(-x).$$

for all $x \in X$ and $0 > k \in \mathbb{Z}$.

Proof. Under the above assumptions, we evidently have $p(kx) = p\big((-k)(-x)\big)$ $\leq (-k)\,p(-x) = -k\,p(-x)$. □

Now, as an immediate consequence of Definition 2.5 and Proposition 2.7, we can also state

Proposition 2.8. If p is an $\mathbb{N}$–subhomogeneous function of a group X and $x \in X$, then

(1) $\frac{1}{k}\,p(kx) \leq p(x)$ for all $0 < k \in \mathbb{Z}$;

(2) $-p(-x) \leq \frac{1}{k}\,p(kx)$ for all $0 > k \in \mathbb{Z}$.

Moreover, by using this proposition, we can easily prove the following

Theorem 2.9. *If p is an $\mathbb{N}$–subhomogeneous function of a vector space X over $\mathbb{Q}$, and moreover $x \in X$ and $l \in \mathbb{Z}$, then*

(1) $\frac{1}{k}\,p\big(lx\big) \leq p\left(\frac{l}{k}\,x\right)$ *for all* $0 < k \in \mathbb{Z}$*;*

(2) $-\frac{1}{k}\,p\big(-lx\big) \leq p\left(\frac{l}{k}\,x\right)$ *for all* $0 > k \in \mathbb{Z}$*.*

Proof. If $0 < k \in \mathbb{Z}$, then by Proposition 2.8 it is clear that

$$\frac{1}{k}\,p\big(lx\big) = \frac{1}{k}\,p\left(k\left(\frac{l}{k}\,x\right)\right) \leq p\left(\frac{l}{k}\,x\right).$$

While, if $0 > k \in \mathbb{Z}$, then by the above inequality it is clear that

$$-\frac{1}{k}\,p\big(-lx\big) = \frac{1}{-k}\,p\big(l(-x)\big) \leq p\left(\frac{l}{-k}\,(-x)\right) = p\left(\frac{l}{k}\,x\right).$$

□

Remark 2.10. Note that if p is $\mathbb{N}$–superhomogeneous, then just the opposite inequalities hold. Therefore, if in particular p is $\mathbb{N}$–homogeneous, then the corresponding equalities are also true.

3. Subadditive and quasi-subadditive functions

Following the terminology of Hille [7, p. 131] and Rosenbaum [17, p. 227], we may also naturally have the following

Definition 3.1. A real-valued function p of a group X will be called

(1) subadditive if $p(x+y) \le p(x)+p(y)$ for all $x, y \in X$;

(2) superadditive if $p(x)+p(y) \le p(x+y)$ for all $x, y \in X$.

Definition 3.2. Note that thus p is additive if and only if it is both subadditive and superadditive.

Moreover, p is superadditive if and only if $-p$ is subadditive. Therefore, superadditive functions need not be studied separately.

The appropriateness of Definitions 2.1 and 2.5 is apparent from the following theorem whose proof can also be found in Kuczma [9, p. 401].

Theorem 3.3. *If p is a subadditive function of a group X, then p is superodd and $\mathbb{N}$–subhomogeneous.*

Proof. Clearly, $p(0) = p(0+0) \le p(0)+p(0)$, and thus $0 \le p(0)$. Moreover, if $x \in X$, then we have

$$0 \le p(0) = p(x+(-x)) \le p(x)+p(-x).$$

Therefore, $-p(x) \le p(-x)$, and thus p is superodd.

Moreover, if $p(nx) \le np(x)$ for some $n \in \mathbb{N}$, then we also have

$$p\big((n+1)x\big) = p(nx+x) \le p(nx)+p(x) \le np(x)+p(x) = (n+1)p(x).$$

Hence, by the induction principle, it is clear that $p(nx) \le np(x)$ for all $n \in \mathbb{N}$. Therefore, p is $\mathbb{N}$–subhomogeneous. □

Remark 3.4. Note that if p is superadditive, then p is subodd and $\mathbb{N}$–superhomogeneous. Therefore, if in particular p is additive then p is odd and $\mathbb{N}$–homogeneous.

Because of Theorem 3.3, we may also naturally introduce the following

Definition 3.5. A real-valued function p of a group X will be called

(1) quasi-subadditive if it is superodd and $\mathbb{N}$–subhomogeneous;

(2) quasi-superadditive if it is subodd and $\mathbb{N}$–superhomogeneous.

Remark 3.6. Note that thus p may be called quasi-additive if it is both quasi-subadditive and quasi-superadditive.

Moreover, p is quasi-superadditive if and only if $-p$ is quasi-subadditive. Therefore, quasi-superadditive functions need not be studied separately.

Now, in addition to Propositions 2.7 and 2.8, we can also prove the following

Proposition 3.7. If p is a quasi-subadditive function of a group X and $x \in X$, then

(1) $k\,p(x) \le p(k\,x)$ for all $0 > k \in \mathbb{Z}$;

(2) $\frac{1}{k}\,p(k\,x) \le p(x)$ for all $0 \ne k \in \mathbb{Z}$.

Proof. If $0 > k \in \mathbb{Z}$, then by the corresponding definitions we have

$$-p(k\,x) \le p(-(k\,x)) = p((-k)\,x) \le (-k)\,p(x) = -k\,p(x).$$

Therefore,

$$k\,p(x) \le p(k\,x), \qquad \text{and hence} \qquad \frac{1}{k}\,p(k\,x) \le p(x).$$

Moreover, from Proposition 2.8 we know that the latter inequality is also true for $0 < k \in \mathbb{Z}$. □

Now, by using the above proposition, we can easily prove the following counterpart of Theorem 2.9.

Theorem 3.8. *If p is a quasi–subadditive function of a vector space X over $\mathbb{Q}$, and moreover $x \in X$ and $0 \ne k \in \mathbb{Z}$, then*

(1) $\frac{1}{k}\,p(l\,x) \le p\left(\frac{l}{k}\,x\right)$ *for all* $l \in \mathbb{Z}$;

(2) $p\left(\frac{l}{k}\,x\right) \le l\,p\left(\frac{1}{k}\,x\right) \ge \frac{l}{k}\,p(x)$ *for all* $0 < l \in \mathbb{Z}$;

(3) $p\left(\frac{l}{k}\,x\right) \ge l\,p\left(\frac{1}{k}\,x\right) \le \frac{l}{k}\,p(x)$ *for all* $0 > l \in \mathbb{Z}$.

Proof. If $l \in \mathbb{Z}$, then by Proposition 3.7 (2), it is clear that

$$\frac{1}{k}\,p(l\,x) = \frac{1}{k}\,p\left(k\left(\frac{l}{k}\,x\right)\right) \le p\left(\frac{l}{k}\,x\right).$$

Moreover, if $0 < l \in \mathbb{Z}$, then by using the $\mathbb{N}$–subhomogeneity of p and the $l = 1$ particular case of Theorem 3.8 (1) we can see that

$$p\left(\frac{l}{k}\,x\right) = p\left(l\,\frac{1}{k}\,x\right) \le l\,p\left(\frac{1}{k}\,x\right) \quad \text{and} \quad \frac{l}{k}\,p(x) = l\,\frac{1}{k}\,p(x) \le l\,p\left(\frac{1}{k}\,x\right).$$

While, if $0 > l \in \mathbb{Z}$, then by using Proposition 3.7 (1) and the $l = 1$ particular case of Theorem 3.8 (1) we can see that

$$l\,p\left(\frac{1}{k}\,x\right) \le p\left(l\,\frac{1}{k}\,x\right) = p\left(\frac{l}{k}\,x\right) \quad \text{and} \quad l\,p\left(\frac{1}{k}\,x\right) \le l\,\frac{1}{k}\,p(x) = \frac{l}{k}\,p(x).$$

□

Remark 3.9. Note that if p is quasi-superadditive, then just the opposite inequalities hold. Therefore, if p is in particular quasi-additive, then the corresponding equalities are also true.

4. Some further results on subadditive functions

Whenever p is subadditive, then in addition to Theorem 3.8 we can also prove the following

Theorem 4.1. *If p is a subadditive function of a group X, then for any $x, y \in X$ we have*

(1) $-p(-(x-y)) \le p(x) - p(y) \le p(x-y)$;

(2) $-p(-(-y+x)) \le p(x) - p(y) \le p(-y+x)$.

Proof. We evidently have

$$p(x) = p(x - y + y) \le p(x-y) + p(y),$$

and hence also

$$p(y) \le p(y-x) + p(x) = p(-(x-y)) + p(x).$$

Therefore, (1) is true.

Moreover, quite similarly we also have

$$p(x) = p(y - y + x) \le p(y) + p(-y+x),$$

and hence also

$$p(y) \le p(x) + p(-x+y) = p(x) + p(-(-y+x)).$$

Therefore, (2) is also true. □

Now, as a useful consequence of the above theorem, we can also state

Corollary 4.2. *If* p *is a subadditive function of a group* X, *then for any* $x, y \in X$ *we have*

(1) $\big|\, p(x) - p(y) \,\big| \leq \max\big\{\, p(x-y),\ p(-(x-y)) \big\}$ *;*

(2) $\big|\, p(x) - p(y) \,\big| \leq \max\big\{\, p(-y+x),\ p(-(-y+x)) \big\}$ *.*

Proof. If
$$M = \max\{\, p(x-y),\ p(-(x-y)) \},$$
then by Theorem 4.1 we have
$$p(x)-p(y) \leq p(x-y) \leq M \qquad \text{and} \qquad -(p(x)-p(y)) \leq p\big(-(x-y)\big) \leq M .$$
Therefore, $|\, p(x) - p(y) \,| \leq M$, and thus (1) is true. The proof of (2) is quite similar. □

By using Theorem 4.1, we can easily prove the following improvement of Kuczma's [9, Lemma 9, p. 402]. (See also Cooper [5, Theorem IX, p. 430].)

Theorem 4.3. *If* p *is a real-valued function of a group* X, *then the following assertions are equivalent:*

(1) p *is additive;*

(2) p *is odd and subadditive.*

(3) p *is subodd and subadditive.*

Proof. If (1) holds, then by Remark 3.4 it is clear that p is odd, and thus (2) also holds. Therefore, since (2) trivially implies (3), we need actually show that (3) implies (1).

For this, note that if (3) holds, then by Remark 2.4 and Theorem 4.1 we have
$$p(x) + p(y) \leq p(x) + (-p(-y)) = p(x) - p(-y) \leq p(x - (-y)) = p(x+y)$$
for all $x, y \in X$. Hence, by the subadditivity of p, it is clear that (1) also holds. □

From the above theorem, by using Remark 3.9, we can immediately get

Corollary 4.4. *If* p *is a subodd subadditive function of a vector space* X *over* $\mathbb{Q}$, *then* $p(rx) = r\,p(x)$ *for all* $r \in \mathbb{Q}$ *and* $x \in X$.

Hence, it is clear that in particular we also have

Corollary 4.5. *If* p *is a subodd subadditive function of* $\mathbb{Q}$, *then* $p(r) = p(1)\,r$ *for all* $r \in \mathbb{Q}$.

5. Even superodd and subhomogeneous functions

Because of quasi-subadditive functions, it is also worth studying even superodd and $\mathbb{N}$–subhomogeneous functions.

Definition 5.1. A real-valued function p of a group X will be called even if $p(-x) = p(x)$ for all $x \in X$.

Remark 5.2. Now, in contrast to Definition 2.1, the subeven and supereven functions need not be introduced. Namely, we have evidently the following

Proposition 5.3. If p is a real-valued function of a group X, then the following assertions are equivalent:

(1) p is even;

(2) $p(-x) \leq p(x)$ for all $x \in X$;

(3) $p(x) \leq p(-x)$ for all $x \in X$.

Hint. If (3) holds, then for each $x \in X$ we also have $p(-x) \leq p(-(-x)) = p(x)$. Therefore, $p(-x) = p(x)$, and thus (1) also holds.

Remark 5.4. Note that a counterpart of the above proposition fails to hold for odd functions. Namely, if for instance $p(x) = |x|$ for all $x \in \mathbb{R}$, then p is superodd, but not odd.

By using Definition 5.1, in addition to Proposition 2.3, we can also easily establish the following extension of Cooper's [5, Theorem X, p. 430]. (See also Kuczma [9, Lemma 8, p. 402].)

Proposition 5.5. If p is an even superodd function of a group X, then $0 \leq p(x)$ for all $x \in X$.

Proof. Namely, if $x \in X$, then $-p(x) \leq p(-x) = p(x)$. Therefore, $0 \leq 2p(x)$, and thus $0 \leq p(x)$ also holds. □

Remark 5.6. Hence, it is clear that if p is an even subodd function of X, then $p(x) \leq 0$ for all $x \in X$.

Therefore, if in particular p is an even and odd function of X, then we necessarily have $p(x) = 0$ for all $x \in X$.

Moreover, by using Proposition 2.7 and Theorem 2.9, we can also easily prove the following counterparts of Proposition 3.7 and Theorem 3.8.

Proposition 5.7. If p is an even $\mathbb{N}$–subhomogeneous function of a group X, then

$$p(kx) \leq |k| p(x)$$

for all $x \in X$ and $0 \neq k \in \mathbb{Z}$.

Proof. If $0 < k \in \mathbb{Z}$, then the corresponding definitions we evidently have $p(kx) \leq kp(x) = |k|p(x)$.

While, if $0 > k \in \mathbb{Z}$, then by Proposition 2.7 and the corresponding definitions we also have $p(kx) \leq -kp(-x) = |k|p(x)$. □

Theorem 5.8. *If p is an even $\mathbb{N}$–subhomogeneous function of a vector space X over $\mathbb{Q}$, then*

$$\frac{1}{|k|}\,p(lx) \leq p\Big(\frac{l}{k}\,x\Big) \leq |l|\,p\Big(\frac{1}{k}\,x\Big)$$

for all $x \in X$ and $k,\, l \in \mathbb{Z}$ with $k,\, l \neq 0$.

Proof. If $k > 0$, then Theorem 2.9 (1) and Proposition 5.7 it is clear that

$$\frac{1}{|k|}\,p(lx) = \frac{1}{k}\,p(lx) \leq p\Big(\frac{l}{k}\,x\Big) = p\Big(l\,\frac{1}{k}\,x\Big) \leq |l|\,p\Big(\frac{1}{k}\,x\Big).$$

While, if $k < 0$, then by Theorem 2.9 (2) and Proposition 5.7, it is clear that

$$\frac{1}{|k|}\,p(lx) = -\frac{1}{k}\,p(-lx) \leq p\Big(\frac{l}{k}\,x\Big) = p\Big(l\,\frac{1}{k}\,x\Big) \leq |l|\,p\Big(\frac{1}{k}\,x\Big).$$

□

Remark 5.9. To compare the above theorem with Theorem 3.8, note that by the $l = 1$ particular case of Theorem 5.8 now we also have

$$\frac{|l|}{|k|}\,p(x) \leq |l|\,p\Big(\frac{1}{k}\,x\Big).$$

Finally, we note that by Corollary 4.2 we can also state the following

Proposition 5.10. If p is an even subadditive function of a group X, then

$$|p(x) - p(y)| \leq \min\{p(x-y),\ p(-y+x)\}.$$

for all $x,\, y \in X$.

6. Homogeneous subadditive functions

Definition 6.1. A real-valued function p of a vector space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ will be called

(1) homogeneous if $p(\lambda x) = \lambda p(x)$ for all $\lambda \in \mathbb{K}$ and $x \in X$;

(2) positively homogeneous if $p(\lambda x) = \lambda p(x)$ for all $\lambda > 0$ and $x \in X$;

(3) absolutely homogeneous if $p(\lambda x) = |\lambda|\,p(x)$ for all $\lambda \in \mathbb{K}$ and $x \in X$.

Remark 6.2. Note that if p is homogeneous (absolutely homogeneous), then p is, in particular, odd (even) and positively homogeneous.

Moreover, if p is positively homogeneous, then in particular we have $p(0) = p(2 \cdot 0) = 2p(0)$, and hence $p(0) = 0$. Therefore, $p(0x) = p(0) = 0 = 0p(x)$ is also true.

Now, as some useful characterizations of positively and absolutely homogeneous functions, we can also easily prove the following two propositions.

Proposition 6.3. If p is a real-valued function of a vector space X over $\mathbb{R}$, then the following assertions are equivalent:

(1) p is positively homogeneous;

(2) $p(\lambda x) \leq \lambda p(x)$ for all $\lambda > 0$ and $x \in X$;

(3) $\lambda p(x) \leq p(\lambda x)$ for all $\lambda > 0$ and $x \in X$.

Proposition 6.4. If p is a real-valued function of a vector space X over $\mathbb{K}$, then the following assertions are equivalent:

(1) p is absolutely homogeneous;

(2) $p(\lambda x) \leq |\lambda| p(x)$ for all $0 \neq \lambda \in \mathbb{K}$ and $x \in X$;

(3) $|\lambda| p(x) \leq p(\lambda x)$ for all $0 \neq \lambda \in \mathbb{R}$ and $x \in X$.

Hint. If (3) holds, then for any $0 \neq \lambda \in \mathbb{R}$ and $x \in X$ we also have

$$p(\lambda x) = |\lambda| \left| \frac{1}{\lambda} \right| p(\lambda x) \leq |\lambda| p\Big(\frac{1}{\lambda} \lambda x \Big) = |\lambda| p(x).$$

Therefore, the corresponding equality is also true. Moreover, from Remark 6.2, we can see that $p(0x) = p(0) = 0 = |0| p(x)$. Therefore, (1) also holds.

In addition to the above propositions, it is also worth establishing the following

Theorem 6.5. *If p is a real-valued function of a vector space X over $\mathbb{R}$, then*

(1) p is homogeneous if and only if p is odd and positively homogeneous;

(2) p is absolutely homogeneous if and only if p is even and positively homogeneous.

Hint. If p is even and positively homogeneous, then for any $\lambda < 0$ and $x \in X$ we also have $p(\lambda x) = p\big(-\lambda(-x)\big) = -\lambda p(-x) = |\lambda| p(x)$. Hence, by the second part of Remark 6.2, it is clear that p is absolutely homogeneous.

Remark 6.6. >From Remark 6.2 and Theorem 4.3, we can see that a homogeneous subadditive function is necessarily linear.

Therefore, only some non-homogeneous subbadditive functions have to be studied separately. The most important ones are the norms.

Definition 6.7. A real-valued, absolutely homogeneous, subadditive function p of a vector space X is called a seminorm on X.

In particular, the seminorm p is called a norm if $p(x) \neq 0$ for all $x \in X \setminus \{0\}$.

Remark 6.8. Note that if p is a seminorm on X, then by Remark 6.2, Theorem 3.3 and Proposition 5.5, we necessarily have $0 \leq p(x)$ for all $x \in X$.

Definition 6.9. A real-valued subadditive function p of a vector space X over $\mathbb{K}$ is called a preseminorm on X if

(1) $\lim\limits_{\lambda \to 0} p(\lambda x) = 0$ for all $x \in X$;

(2) $p(\lambda x) \leq p(x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$.

In particular, the preseminorm p is called a prenorm if $p(x) \neq 0$ for all $x \in X \setminus \{0\}$.

Remark 6.10. By Remark 6.8, it is clear that every seminorm p on X is, in particular, a preseminorm.

Moreover, if p is a preseminorm on X, then defining $p^*(x) = \min\{1, p(x)\}$ $\big($or $p^*(x) = p(x)/\big(1 + p(x)\big)\big)$ for all $x \in X$, it can be shown that p^* is a preseminorm on X such that p^* is not a seminorm.

Most of the following basic properties of preseminoms have also been established in [18] . The simple proofs are included here for the reader's convenience.

Theorem 6.11. *If p is a preseminorm on a vector space X over $\mathbb{K}$ and $x \in X$, then*

(1) $p(0) = 0$ *(2)* $0 \leq p(x)$;

(3) $p(\lambda x) = p(|\lambda| x)$ *for all* $\lambda \in \mathbb{K}$;

(4) $|p(x) - p(y)| \leq p(x - y)$ *for all* $y \in X$;

(5) $p(\lambda x) \leq p(\mu x)$ *for all* $\lambda, \mu \in \mathbb{K}$ *with* $|\lambda| \leq |\mu|$;

(6) $p(\lambda x) \leq n\, p(x)$ *for all* $\lambda \in \mathbb{K}$ *and* $n \in \mathbb{N}$ *with* $|\lambda| \leq n$;

(7) $\frac{1}{|k|} p(l x) \leq p\left(\frac{l}{k} x\right) \leq |l|\, p\left(\frac{1}{k} x\right)$ *for all* $k, l \in \mathbb{Z}$ *with* $k \neq 0$.

Proof. By Definition 6.9 (1), we evidently have

$$p(0) = \lim_{\lambda \to 0} p(0) = \lim_{\lambda \to 0} p(\lambda 0) = 0.$$

Moreover, if $\lambda, \mu \in \mathbb{K}$ such that $|\lambda| \leq |\mu|$ and $\mu \neq 0$, then by using Definition 6.9 (2) we can see that $p\big(\lambda x\big) = p\big((\lambda/\mu)\mu x\big) \leq p\big(\mu x\big)$. Hence, since $|\lambda| \leq |\mu|$ and $\mu = 0$ imply $\lambda = 0$, it is clear (5) is also true.

Now, by (5) and the inequalities $|\lambda| \leq ||\lambda|| \leq |\lambda|$, it is clear that in particular we also have $p(\lambda x) \leq p\big(|\lambda| x\big) \leq p(\lambda x)$. Therefore, (3) is also true.

Moreover, if $n \in \mathbb{N}$ such that $|\lambda| \leq n$, then by (5) and Theorem 3.3, it is clear that $p(\lambda x) \leq p(nx) \leq n p(x)$ also holds.

Finally, to complete the proof, we note that by (3) p is, in particular, even. Therefore, by Propositions 5.3 and 5.10 and Theorem 5.8, assertions (2), (4) and (7) are also true. □

Remark 6.12. From the above proof, it is clear that if p is a subadditive function of a vector space X over $\mathbb{K}$ such that in addition to Definition 6.9 (2) we only have $\inf_{\lambda \neq 0} p(\lambda x) \leq 0$ for all $x \in X$, then p is already a preseminorm.

Finally, we note that by using Theorem 6.11 (2) and (6) we can also prove

Corollary 6.13. *If p is a nonzero preseminorm on a one-dimensional vector space X over $\mathbb{K}$, then p is necessarily a prenorm on X.*

Proof. Namely, if this not the case, then there exists $x \in X$ such that $x \neq 0$ and $p(x) = 0$. Hence, by using $\dim(X) = 1$, we can see that $X = \mathbb{K}x$. Moreover, if $\lambda \in \mathbb{K}$, then by choosing $n \in \mathbb{N}$ such that $|\lambda| \leq n$ we can see that $0 \leq p(\lambda x) \leq n p(x) = 0$, and thus $p(\lambda x) = 0$. Therefore, p is identically zero, which is a contradiction. □

Remark 6.14. The importance of preseminorms lies mainly in the fact that in contrast to seminorms, a nonzero preseminorm can be bounded by Remark 6.10.

Thus, by an idea of Fréchet, any sequence $(p_n)_{n=1}^{\infty}$ preseminorms on X can be replaced by a single preseminorm $q = \sum_{n=1}^{\infty} (1/2^n)\, p_n^*$ which induces the same topology on X.

In this respect, it is also worth mentioning that, in contrast to seminorms, each vector topology on X can be derived from a family preseminorms on X. (See, for instance, [14].)

References

[1] Berz, E., Sublinear functions on $\mathbb{R}$, *Aequationes Math.*, Vol. 12 (1975), 200–206.

[2] Buskes, G., The Hahn–Banach theorem surveyed, *Dissertationes Math.*, Vol. 327, (1993) 1–49.

[3] Bruckner, A. M., Minimal superadditive extensions of superadditive functions, *Pacific J. Math.*, Vol. 10 (1960), 1155–1162.

[4] Bruckner, A. M., Some relationships between locally superadditive functions and convex functions, *Proc. Amer. Math. Soc.*, Vol. 15 (1964), 61–65.

[5] Cooper, R., The converses of the Cauchy–Hölder inequality and the solutions of the inequality $g(x+y) \leq g(x) + g(x)$, *Proc. London Math. Soc.*, Vol. 26 (1927), 415–432.

[6] Gajda, Z., Kominek, Z., On separation theorems for subadditive and superadditive functions, *Studia Math.*, Vol. 100 (1991), 25–38.

[7] Hille, E., Functional analysis and Semi-Groups, *Amer. Math. Soc. Coll. Publ.*, Vol. 31, New York 1948.

[8] Jolly, R. F., Concerning periodic subadditive functions, *Pacific J. Math.*, Vol. 15 (1965), 159–169.

[9] Kuczma, M., An Introduction to the Theory of Functional Equations and Inequalities, *Państwowe Wydawnictwo Naukowe*, Warszawa 1985.

[10] Laatsch, R. G., Extensions of subadditive functions, *Pacific J. Math.*, Vol. 14 (1964), 209–215.

[11] Matkowski, J., Subadditive functions and a relaxation of the homogeneity condition of seminorms, *Proc. Amer. Math. Soc.*, Vol. 117 (1993), 991–1001.

[12] Matkowski, J., On subadditive functions and ψ–additive mappings, *Central European J. Math.*, Vol. 4 (2003), 435–440.

[13] Matkowski, J., Światkowski, T., On subadditive functions, *Proc. Amer. Math. Soc.*, Vol. 119 (1993), 187–197.

[14] Metzler, R., Nakano, H., Quasi-norm spaces, *Trans. Amer. Math. Soc.*, Vol. 123 (1966), 1–31.

[15] Nikodem, K., Sadowska, E., Wasowicz, S., A note on separation by subadditive and sublinear functions, *Ann. Math. Sil.*, Vol. 14 (2000), 33–39.

[16] Rathore, S. P. S., On subadditive and superadditve functions, *Amer. Math. Monthly*, Vol. 72 (1965), 653–654.

[17] Rosenbaum, R. A., Sub-additive functions, *Duke Math. J.*, Vol. 17 (1950), 227–247.

[18] Száz, Á., Preseminormed spaces, *Publ. Math.*, Debrecen Vol. 30 (1983), 217–224.

[19] Száz, Á., The intersection convolution of relations and the Hahn–Banach type theorems, *Ann. Polon. Math.*, Vol. 69 (1998), 235–249.

[20] Száz, Á., Preseminorm generating relations and their Minkowski functionals, *Publ. Elektrotehn. Fak., Ser. Math., Univ. Beograd*, Vol. 12 (2001), 16–34.

[21] Wong, J. S. W., A note on subadditive functions, *Proc. Amer. Math. Soc.*, Vol. 44 (1974), 106.

Burai Pál
H-4010 Debrecen, Pf. 12, Hungary

Száz Árpád
H-4010 Debrecen, Pf. 12, Hungary

Annales Mathematicae et Informaticae
32 *(2005) pp. 203–210.*

Solving initial value problem by different numerical methods: Practical investigation

Gábor Geda

Department of Computer Science, Eszterházy Károly College
e-mail: gedag@aries.ektf.hu

Abstract

Our aim was to study what kind of bases can be provided to understand the basic terms of differential equation through teaching mathematical material in secondary school and to what extent this basis has to be expanded so that we can help the demonstration of differential equation. So if we give up the usual expansion of mathematical device we have to find another method which is easy to algorithmise and lies on approach. Such method and its practical experience are shown in this paper.

AMS Classification Number: 65L05, 65L06, 53A04, 97D99

1. Introduction

It is well-known how important the differential equation models are in the mathematical description of different processes and systems. Our aim is to find approximate methods which are based on the approach and there is no need for higher mathematical knowledge to understand and apply them. Moreover, they are easy to algorithmise even in the possession of the secondary school material. The problems, concerning this topic can be given as an explicit first-order ordinary differential equation (abbreviated as ODE in the followings).

$$y' = f(x, y(x)) \tag{1.1}$$

The solutions for these equations (if they exist at all) are y(x) functions. In most cases to give such a function is a difficult task which needs the knowledge of serious mathematical devices. Starting from the side of approach we can say that by giving (1.1) we give the slope of those tangents which can be drawn to all points of the curves of the functions providing the solution. So (1.1) correspond a given steepness to points of the place, the value of the steepness of tangent taken in a given P point concerning the solution of ODE. The only problem is that we do not know which points should be considered belonging to the same curve among the points close to one another. In certain cases there is no need to present all the possible solutions, only the *y(x)* is necessary on the curve of which a given $P_0(x_0,y_0)$ fits. In this case we can say that we solve an initial value problem. By expressing an initial value problem we chose one of the curves which are solutions for ODE.

Other times we have to be contented with the approximate solution of the problem. In a geometrical point of view the solution for an initial value problem by approximation is giving a P_0, $P_1, \ldots$, P_n point serial the elements of which fit to the chosen curve by desired accuracy. The serial of vectors $\overrightarrow{P_iP_{i+1}}$ $(0 \leqq i < n)$ determines a broken line the points of which approximate well the points of the curve. The accuracy of the approximation is influenced by several factors. The most important ones among them are the approximate algorithm and ODE itself. This way, when we select the successive elements of the point serial we should take the changes of the curve of the function into consideration.

2. Demonstration of an approximating method

Let ODE (1.1), $P_0(x_0,y_0)$ be given and minor d distance. We would like to determine the broken line running through a given point and giving the function curve in the surroundings of a given point.

$m_0 = f(x_0, y_0)$ is the steepness of a tangent belonging to point P_0. Let $\vec{a}$ be a vector parallel with line of m_0 steepness and $\|\vec{a}\| = d$. Define point $Q(x,y)$ where $\overrightarrow{P_0Q} = \vec{a}$ (see Figure 1). Steepness (m) belonging to Q can also be calculated.

If d is minor enough, then Q is close enough to the curve which is the solution for the initial value problem. This way, m well approximates the steepness of the curve in one of its points near to Q.

Let $\vec{b}$ be a vector parallel with the tangents the steepness of which is m and $\|\vec{b}\| = d$, and in addition, we must chose the direction of $\vec{b}$ in a way, that the angle of $\vec{a}$ and $\vec{b}$ must be equal with the angle of lines with m_0 and m steepness.

In the narrow surroundings of P_0 the curve of the function can be well approximated with a proper arc (c_2) which is the circle of the curvature in P_0. Similarly we can fit an arc (c_1) in the narrow surroundings of point P_1 to a curve on which Q fits. If $m_0 \neq m$ then perpendiculars of the tangents running through P_0 and Q intersect at O. This point is considered to be the common central point of the two circles (c_1 and c_2) if the d is minor enough (see Figure 3). Knowing O, we can determine the P_1 point where $\|\overrightarrow{P_0O}\| = \|\overrightarrow{OP_1}\|$ (see Figure 1).

To determine the following approximate point, the starting point will be P_1 as it was P_0 earlier. The promptness of the approximation depends on the selection of the value d. If it is too big O will not be a good approximation of the common centre of the two osculating circles. At the same time, if we find an appropriate O point then the distance of O and P_0 approximate the radiant of the circle of

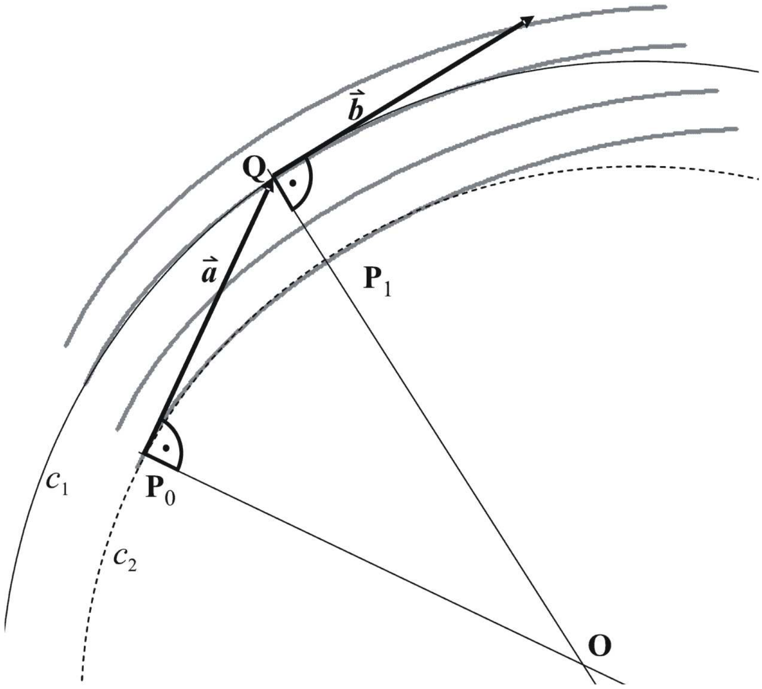

Figure 1

curvature at P_1. This can be used to get a better defining of the value of d. If we can choose the value of d according to the characteristics of the curve we can approximate the function more precisely, and the algorithm will be faster.

If m_0 equals with m then the perpendiculars placed in the P_0 and Q points of the tangents are parallel so they do not have a point of intersection. In this case knowing m_0 and m the place of P_1 has to be defined in another way.

To understand the operation of this method we only need the knowledge of graphic meaning of the differential quotient as the exact definition is not used in this case. If we regard an ODE as a function which orders value of steepness to the points of the place then the point serial giving the solution can be written by the help of vector operation based on the method mentioned above (in the followings abbreviated as **OCM** *Osculating Circle Method*) which approximates the solution of initial value problem.

$$\vec{p}_1 = \vec{p}_0 + \overrightarrow{\mathrm{P}_0\mathrm{O}} + \overrightarrow{\mathrm{OP}_1}$$

where the end-point of $\vec{p}_0$ and $\vec{p}_1$ local vectors are always P_0 and P_1 (see Figure 1). To give the algorithm we need to the know how to solve an equation system.

3. The experience of practical examination of OCM

3.1. Demonstration of family of curves through OCM

Through studying the differential equation it is common to draw direction field. Good consequences can be drawn concerning the placement of the curves. In spite of this, it would be useful to present the curves of a family of curves running through the elements of a point serial. In many cases we can gain the same information from a clearer figure.

$$F(x,y) = (x^2+y^2)^2 - 2c^2(x^2-y^2) - a^2 + c^4 = 0$$

is the equation of *Cassini curves* and by changing a and c we get the different member of the family of curves.

Let

$$\frac{dy}{dx} = -\frac{x^3+xy^2-c^2x}{y^3+x^2y+c^2y} \tag{3.1}$$

be a differential equation derived from

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

The computer programme made on the basis of OCM assign points along y=0.001 line running at the same step and approaches the points of the curves running through the assigned points (3<$|x_0|$<6).

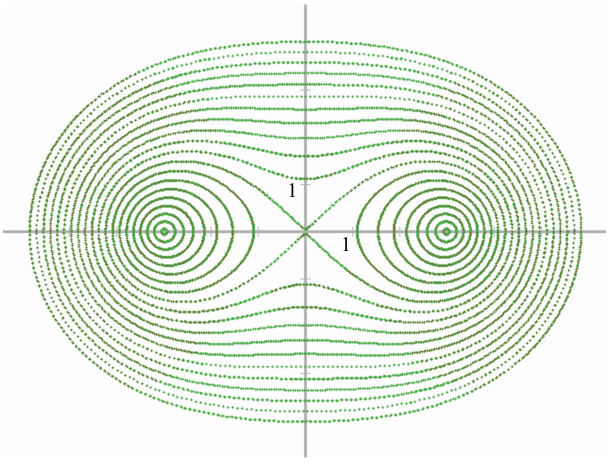

Figure 2: This figure was made by using (3.1). The algorithm drew the curves running through the members of the point serial (3<$|x_0|$<6) along y=0.001 line running at the same step (c=3).

3.2. The approximation of initial values problem by OCM

In the followings we examine how exactly the different algorithms follow the solution. (RKn means the n^{th} -order Runge-Kutta method in the followings.) To the comparison we chose an initial value problem:

$$y' = \frac{dy}{dx} = \frac{1}{2y}, \qquad y_0(0) = 0.005$$

The $y(x)$ functions giving the solution can not be differentiated in the points where the function and the x axes intersect. We examined how the applied methods can follow the changes of the curve if we give the starting point in the surroundings of such a point (see Figure 3).

Through the experiment, the initial value of d was 0.005 in the programme which was made on the basis of the OCM.

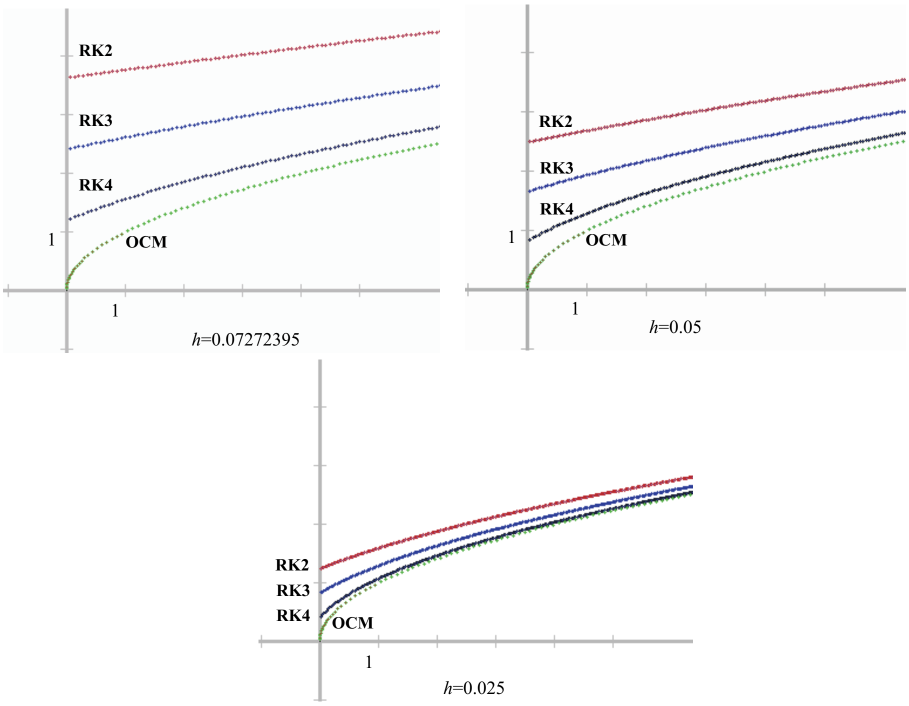

Figure 3: Where h is the step-size of Runge-Kutta algorithms.

Naturally, we can not ignore the fact, that the algorithm based on OCM changes the value of d in the function of curvature,(which depends on radius of curvature), while the step-size is a constant in RK2, RK3 and RK4.

For the next practical examination of the algorithm we chose the following differential equations

$$y' = \frac{dy}{dx} = -\frac{1}{2y}.$$

$y_0(0) = 2$ was an initial condition and we approximated the solution by three different methods. (RK2, RK3, OCM)

First we carried out OCM and in the case of d=0.05 56 approximating points were presented above $[0; 4]$. As the distance of the succeeding approximating points of x-coordinates is different we chose its mean as the step-size of RK algorithms.

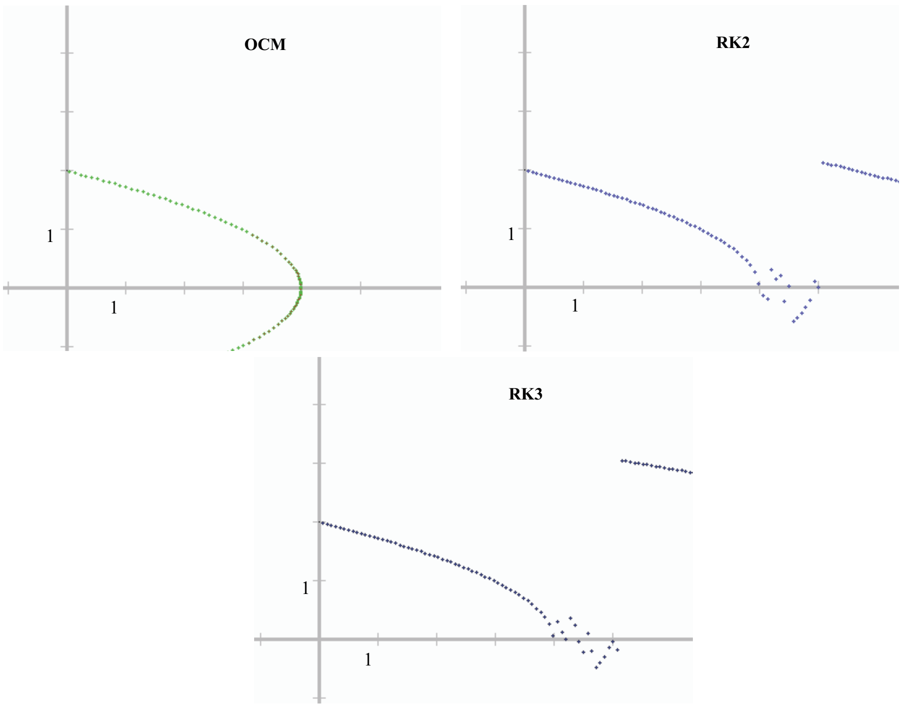

Figure 4

The promptness of approximations was characterised by the average and maximum of the series of the differences between approximated and calculated values of function (see Table 1).

The graph shows the difference between the calculated value and the values produced by the three procedures in [0;4]. We left out the last two points of 56 points for the sake of better visibility in all three diagrams.

It can be observed that the difference between the approximated values and the calculated ones is increasing in all three cases approaching the zero. It is worth observing and comparing the values belonging to the points showed. It can be seen the last values belonging to RK3 and OCM are almost the same. The figure

	RK2	**RK3**	**OCM**
$\frac{1}{n}\sum_{i=1}^{n} \lvert y_i - y(x_i) \rvert$	0.001029571	0.00084471	$9.27051\ 10^{-6}$
$\max_{i=1,\ldots,n} (\lvert y_i - y(x_i) \rvert)$	0.056674050	0.04690024	0.000151908

Table 1

shows that points belonging to RK2 and RK3 are farther from the point where the function and the X-axis cross each other.

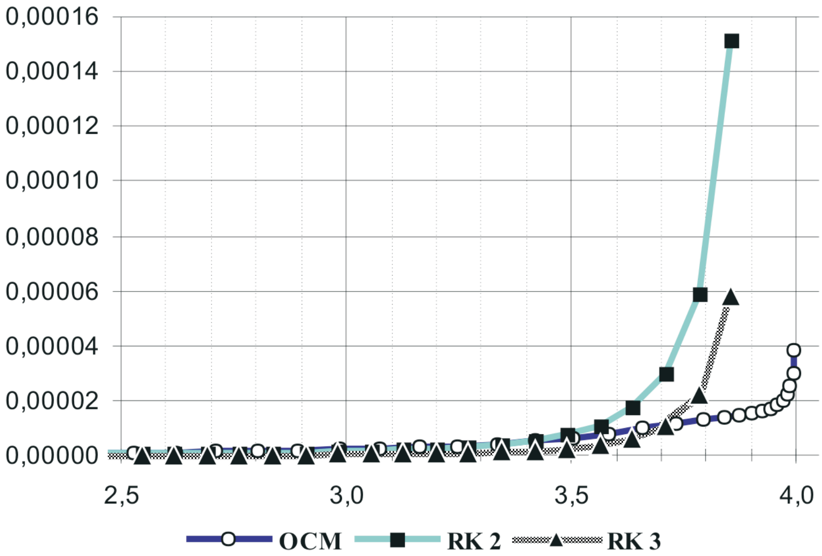

Figure 5: We can see that OCM at the same number of steps differs less and better approaches the critical point.

4. Conclusion

It could be seen that with the help of figures produced by a programme operating on the base of OCM we can receive a graphic image about the route of curves running through an assigned points of a domain. Not to mention the fact we can get an image about the route of a curve compared to other curves because this procedure follows the route of curves running through given points. We must not ignore the drawback of OCM namely it needs more calculation than calculation of (1.1) point by point. The other advantage is that it serves the presentation of an approximating method in the field of solution of ODE at the disposal of basic knowledge mentioned above.

Observations in the field of the promptness of ODE proved that it can operate in the surroundings of critical points with fewer mistakes. This difference is smaller than the mistake of RK3, although, it uses two values of steepness to produce an approximating point. At the same time it needs more calculation and it can be used efficiently in the case when the calculation need of $f(x, y)$ given in (1.1) is high enough.

To receive a clear picture of the promptness of OCM further observations are needed though which it should be compared to other methods using adaptive step-size.

At the same time we can see further possibilities to refine this method.

References

[1] Geda, G., Solving Initial Value Problem by Approximation in Different Graphic Ways, 5^{th} *Intrnational Conference of PhD Students, Miskolc* (2005).

[2] Stoyan, G., Tako, G., Numerikus módszerek, *ELTE-TypoTEX, Budapest*, (1993).

[3] Hatvani, L., Pintér, L., Differenciálegyenletes modellek a középiskolában, *POLY-GON, Szeged*, (1997).

[4] Rontó, M., Raisz, Péterné., Differenciálegyenletek műszakiaknak, *Miskolci Egyetemi Kiadó, Miskolc*, (2004).

[5] Szőkefalvi-Nagy Gy., Gehér L., Nagy P., Differenciálgeometria, *Műszaki Könyvkiadó, Budapest,*, (1979).

[6] Geda, G., Kezdeiérték-probléma közelítő megoldásának egy geometriai szemléltetése, *Tavaszi Szél, Debrecen*, (2005).

[7] Pólya, Gy., Matematikai módszerek a természettudományban, *Gondolat, Budapest*, (1984).

Gábor Geda
Department of Computer Science
Eszterházy Károly College
Leányka str. 6.
H-3300 Eger, Hungary

Annales Mathematicae et Informaticae
32 *(2005) pp. 211–224.*

On the fundamental theorem of compact and noncompact surfaces

Eszter Kónya

Ferenc Kölcsey Reformed Teacher Training College
e-mail: hke@kfrtkf.hu

Abstract

In this article an overview of the history of surface topology is given. From the Euler-formula, to the Kerékjártó Theorem we follow the development process of the fundamental theorem of compact and non-compact surfaces. We refer to the works of Riemann, Möbius, Jordan, Klein and others, but our main focus point is to show the work of the Hungarian mathematician, Béla Kerékjártó.

Key Words: history of surface topology, classification of surfaces, homeomorphism, topological invariants, Kerékjártó, compact and noncompact surfaces

AMS Classification Number: 01A50, 01A55, 01A60, 57N05, 5703

1. Introduction

We can observe many similarities between the developmental process of mathematics and the teaching process of a particular topic. Among others, Imre Lakatos was dealt with these similarities. In his work, *Proofs and Refutations* [16] he demonstrated the creative and informal nature of a real mathematical discovery, and suggested a way of teaching in which the concepts and theorems are thought by reproducing the historical steps.

"Mathematics develops, according to Lakatos, ... by a process of conjecture, followed by attempts to 'prove' the conjecture (i.e. to reduce it to other conjectures) followed by criticism via attempts to produce counter-examples both to the conjectured theorem and to the various steps in the proof."[1]

The development of topology, especially of surface topology is similar to the process of teaching mathematics in two aspects: First, the development of surface

[1] J. Worrall: Imre Lakatos (1922-1974) : philosopher of mathematics and philosopher of science, Z. Allgemeine Wissenschaftstheorie 5 (2) (1974), 211-217.

topology gives an instructive example of the difficulties of formulating an intuitive and practical problem into a precise mathematical model. Such problems in topology include: The definition of the topological transformation: From the intuitive idea of "a change in form without tearing and sticking together" researchers reached the mathematical concept of the one to one and continuous mapping. The definition of the surface: from the intuitive idea through the concept of triangulation mathematicians developed the concept of two dimensional topological manifolds. Second, studying the developmental process of surface topology we can observe definitions and theorems that precede the birth of a concrete result. In our paper we illustrate this process describing the fundamental theorem of compact and non-compact surfaces. Since the development of the theorem includes the same phases that students go through when formulating and solving a problem, the history of fundamental theorem of surface topology is a useful parallel to the process of teaching and learning mathematics.

The starting point of the theory of 2-dimensional manifolds as well as for many topological theories was the Euler-theorem (1750). Euler's famous formula is for a polyhedron: $v - e + f = 2$, where v is the number of the vertices, e is the number of edges and f is the number of faces.

The generalisation by L'Huilier (1811) led to the first known result on a topological invariant.

Another type of generalization was made by Schläfli and Poincare. The Euler-formula was extended to n-dimensional spaces by Schläfli, and proved by Poincaré (1893): $N_0 - N_1 + N_2 - \cdots + (-1)^{n-1} N_{n-1} = 1 - (-1)^n$, where $N_0 = v$, $N_1 = e$, $N_2 = f$, and N_k is analogically the number of k-dimensional figures. [6]

Riemann examined the connectivity of surfaces in 1851 and 1857. The Euler-formula for an n-connected polyhedron is the following: $v - e + f = 3 - n$, where n is the connectivity number of the polyhedron. [11]

In the 1860-s Möbius(1863, 1865) and Jordan (1866) worked independently from each other on the problem of topological equivalent surfaces. They elaborated the classification of compact orientable surfaces. Although Listing (1862) mentioned first the so called Möbius-strip as an example of a one-sided surface, Möbius (1865) described its properties in terms of non-orientability.

In the 1870-s, Schläfli and Klein discussed on the orientability of the real 2-dimensional subspaces of the projective space. Klein introduced the concept of relative and absolute properties of a manifold, and identified orientability as an absolute property. In 1882 he described the so called Klein-bottle, one of the most famous non-orientable closed surfaces.

The classification of non-orientable compact surfaces was published in the paper of van Dyck in 1888. Seeing that neither the concept of homeomorfism nor the concept of an abstract surface was completely precise, the first essentially rigorous proof of the classification theorem for compact surfaces was given only in 1907 by Dehn and Heegaard. After Brahana's exact algebraic proof in 1921 some additional proofs were made in the 20. century too.

The problem of non-compact surfaces didn't occur in works in the 19. century.

The starting point for the compactification was perhaps the concept of the projective plane. Kerékjártó in 1923 gave the classification theorem of non-compact surfaces after introducing boundary components to compactify the open surface. Richards (1962) and Goldmann (1971) proved more precisely the theorem and gave some consequences to it.

Of course there exists the generalisation of the theory of surfaces for higher dimensions, but this is not the topic of the present paper.

2. Compact surfaces

One of the earliest topological results is the Euler-formula[2] from 1750 ($v-e+f=2$). Euler was looking for a relation similar to that, which exists between the numbers of vertices and sides of a polygon ($v=s$). At that time the concept of polyhedron was an intuitive extending of the fives Platonic solids (pyramids, prisms etc.). Before the appearance of the Euler-theorem it was no reason to give a precise definition.

After trying to prove the theorem, and in connection with this, after the appearance of different counterexamples, it was necessary to define a polyhedron. Cauchy's proof in 1813[3], and L'Huilier's well-known counterexamples[4] led the examinations to the direction of topology. The novelty of Cauchy's proof was that he considered a polyhedron not a rigid body, but a surface. He omitted one of the faces of polyhedron, and embedded it in the plane admitting some deformations of edges and faces, and examined a connected graph on the plane. At that time the concepts of function and geometrical transformation was not developed to the level, that Cauchy would reach the idea of topological transformation.

Observing transparent crystals L'Huilier tried to prove the Euler-theorem for polyhedron with holes. He became the result, that the number $v-e+f$ is not always 2, there exists surfaces with other numbers too.

Works of Poinsot, Vandermonde, Cauchy, Poincare, and others led to further counterexamples, and to the generalization of the Euler-theorem, as well as to the concept of Euler-characteristics.

In the 1850-s and 1860-s the surface topology was developed through the works of Möbius, Jordan, Riemann and Listing. Möbius and Jordan gave a definition of topological transformation, and with Riemann used the concept of surfaces more generally as others before.

[2] L. Euler: Opera Omnia I, Bd. 26, 71-93.

[3] A. L. Cauchy: Recherches sur les polyédres. In: Journal de l'École Polytechnique, 1813, 68-86.

[4] S. A. J. L'Huillier: Mémoire sur la polyédrométrie, In: Annales de Mathématiques, Pures et Appliquées, 1812-13, 179, 194.

2.1. Topological transformation

Let A, B two sets of points, and $f : A \to B$ a function. We can see that f is a mapping of the set A into the set B. The distance between two points in the sets A and B is defined.

A function $f : A \to B$ is continuous at a point $x_0 \in A$, if for $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that whenever x differs from x_0 by less then δ, $f(x)$ differs from $f(x_0)$ by less then ϵ. If a mapping $f : A \to B$ is continuous at every point $x_0 \in A$, then we say that f is continuous.

A mapping $f : A \to B$ is said to be bijective, if the preimage of every point of B is exactly one point of A. For a bijective mapping $f : A \to B$ we can define the inverse mapping $f^{-1} : B \to A$.

A mapping $f : A \to B$ is said to be a homeomorphism or topological transformation, if it is both bijective, and f as well as its inverse f^{-1} are continuous. Intuitively, homeomorphism is a mapping of a set on another set that involves no tearing (the continuity condition) and no gluing together (the bijective condition).

Two figures A and B are homeomorphic or topologically equivalent if there exists a mapping $f : A \to B$, which is homeomorphism. For example the cube is homeomorphic to a sphere. (See Figure 1.)

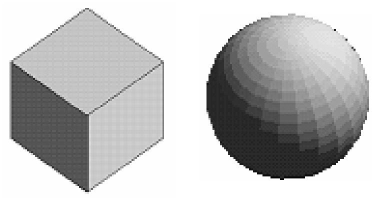

Figure 1.

Properties of figures unchanged by homeomorphisms are called topological properties, or topological invariants. One of the first known topological invariant of a surface S was the Euler-characteristic, the number $\chi(S) = v - e + f$.

2.2. Concept of surfaces

There are many definitions of surface, in geometry, in differential geometry, in topology. The modern topological definition is the following: A surface is a connected two-dimensional manifold.

Our terminology must be start from the generalization of the Euler-theorem: The relation $v-e+f = 2$ is true for any polyhedron whose surface is homeomorphic to a sphere and each of whose faces is homeomorphic to a disk. A figure is called surface without boundary, if each of its points x has a neighbourhood homeomorfic to a disk. (A neighbourhood of a point x is a set, whose points differ from x less then a given positive number.)

To our further review of history of surface topology we need the definition of triangulation. Triangulation is a method dividing a surface in set of triangles, which are in bijective relation to planar triangles. We dealt with triangulable surfaces only.

A surface is a set of triangles satisfying the following properties:
1. The inner points of a triangle belong to this triangle only.
2. Every edge of a triangle belongs to exactly two triangles, which do not have any other common points besides this edge.
3. Every vertex of a triangle belongs to a finite, cyclically ordered set of triangles, in which set every two consecutive triangles have a common edge containing this vertex.
4. For every pair of triangles there exists a not necessarily unique finite sequence of triangles, in which the first and last elements are the given triangles, and every two consecutive triangles have exactly one common edge.

If the number of triangles is finite then the surface is called closed, otherwise, it is called open.

A bordered surface is a finite set of triangles satisfying the following properties:
1'. The inner points of a triangle belong to this triangle only.
2'. Every edge of a triangle belongs to exactly one triangle, or belongs to exactly two triangles, which do not have any other common points besides this edge.
3'. Every vertex of a triangle belongs to a finite, cyclically or linearly ordered set of triangles, in which set every two consecutive triangles have a common edge containing this vertex.
4'. For every pair of triangles there exists a not necessarily unique finite sequence of triangles, in which the first and last elements are the given triangles, and every two consecutive triangles have exactly one common edge. [14]

A boundary of a surface consists of edges of triangles belonging to exactly one triangle. The boundary contains finite number of simple closed curves without common points, they are called boundary components.

2.3. Johann Benedict Listing (1808-1882)

He wrote two important works related to topology. In 1847 in his paper *Vorstudien zur Topologie* he used the term topology instead of Analysis Situs. This means that his main focus was on connection, relative position and continuity.

In 1862 in his other work *Der Census Räumlicher Complexe oder Verallgemeinerung des Eulerschen Satzes von der Polyedern* the Möbius-strip as an example appeared, and he denoted, that it has *"quite different properties"*, and it's bound by one closed curve but did not describe exactly this one-sided surface. He generalized the Euler-formula to a sphere and to surfaces homeomorphic to a sphere. [12]

2.4. Bernhard Riemann (1826-1866)

The interpretation of complex numbers and functions of a variable complex quantity in Riemann's works lead to the concept of connectivity of surfaces. In his dissertation *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*, in 1851 he defined surfaces which cover a domain in the complex plane, the *"Riemann-surfaces"*.

He recognised the importance of topological ideas and applied topological methods to his problems in complex analysis. In his study *Theorie der Abelschen Functionen* in 1857 he introduced the connectivity number as a topological invariant. He said that surface S is simply connected, if it falls in two parts by any cross cut (a line which runs through the interior of the surface without self-intersections, and joins one boundary point to another). For not simply connected surfaces he described a cutting method which gives the minimal number of cutting resulting only simply connecting surfaces. If the surface is closed, also without boundary, Riemann made a surface with boundary through a perforation.

According to this method we give a closed curve on the surface, and give the second curve so, that it joins two not necessary different points of the first curve. Drawing of curves should be continued until, till it is possible to draw a new curve without intersecting of the sequence of previous curves. For example the sphere is simply connected surface, and the connectivity number of the torus is 3.

He recognised the relation between the Euler-characteristic and the number of connectivity: $n = 3 - \chi(S)$. [12]

2.5. August Ferdinand Möbius (1790-1868)

In our point of view Möbius had two important papers. The first, *Theorie der elementaren Verwandtschaft* in 1863 described the concept of topological transformation as an *"elementary relationship"* in the following intuitive way: Two points of a figure which are infinitely near each other are corresponding to two points of the other figure which are infinitely near each other too.

"Zwei geometrische Figuren sollen einander elementar verwandt heissen, wenn jedem nach allen Dimensionen unendlich kleinen Elemente der einen Figur ein dergleichen Element in der anderen dergestalt entspricht, dass von je zwei an eiander gremzenden Elementen der einen Figur die zwei ihnen entsprechende Elemente der anderen ebenfalls zusammenstossen; oder, was dasselbe ausdrückt: wenn je einem Puncte der einen Figur ein Punct der anderen also entspricht, dass von je zwei einander unendlich nahen Puncten der einen auch die ihnen entsprechenden der anderen einander unendlich nahe sind." [17]

Möbius examined elementary relationships of closed and bordered surfaces which are without self-intersection in the Euclidien plane or space. He showed that each such surface can be constructed from two elementary equivalent surfaces each with n boundary components which are pasted together at the boundary components. He called n the class of the surface. On a surface nth class we can draw $n - 1$ closed curves not decomposing it.

He said the classification theorem the following way: Two closed surfaces are elementary equivalents if and only if they belong to the same class.

"Je zwei geschlossene Fläche φ und φ', welche zu derselben Klasse gehören, sind elementar verwandt. Dagegen sind zwei zu verschiedenen Klassen gehörige

geschlossene Flächen nicht in elementarer Verwandschaft. [17]

Möbius generalized the Euler-theorem from the direction of surface topology and not from the direction of counterexamples. He gave the normal forms of closed or bordered, orientable surfaces. The relation between the number n and the Euler-characteristic is: $\chi = 2(2 - n)$.

Since all surfaces were considered as embedded into R^3, non-orientable surfaces were found quite late. In his paper *Über die Bestimmung des Inhaltes eines Polyeders* in 1858 (printed in 1865) Möbius defined the orientation of surfaces, and described one-sided and two sided surfaces.

He defined an orientation of a surface (polyhedron) nearly the following way: Let S be, a triangulable closed or bordered surface. Give orientation to the edges of triangles so that if H_1 and H_2 are neighbouring triangles (with one common edge), then their common edge has different orientations. (See Figure 2.)

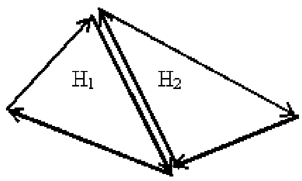

Figure 2.

Den Perimetern der ein Polyeder umgrenzenden Flächen können solche Sinne beigelegt werden, dass für jede Kante des Polyeders die zwei Richtungen, welche derselben, als der gemeinschaftlichen Kante zweier Polyederflächen, in Folge der Sinne dieser zwei Flächen zukommen, einander entgegengesetzt sind. [18]

Möbius called this construction "rule of edges" (*Der Gesetz der Kannten*). If there exists a $H_1, H_2, \ldots, H_n, H_1$ closed chain containing neighbouring triangles of S, and the orientation of H_1 triangle is changed, then the surface S is called non-orientable. If such a chain doesn't exist, then S is called an orientable surface.

Möbius found a surface, by which the "rule of edges" could not apply. This was the Möbius-strip, the first example of a non-orientable surface. It is a bordered surface which is obtained by taking a rectangular strip, twisting it once and gluing its ends together. (See Figure 3.)

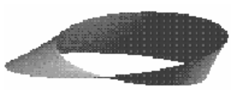

Figure 3.

This is a one-sided surface, if we go over a Möbius-strip with a "paintbrush", then we return to the starting point on the "opposite side".

"Auch hat diese Fläche nur eine Seite; denn wenn man sie von einer beliebigen Stelle aus mit einer Farbe zu überstreichen anfängt und damit fortfährt, ohne mit

dem Pinsel über die Grenzlinie hinaus auf die andere Seite überzugehen, so werden nichtsdestoweniger zuletzt an jeder Stelle die zwei daselbst einander gegenüberliegenden Seiten der Fläche gefärbt sein." [18]

2.6. Camille Jordan (1838-1921)

Independently from Möbius Jordan defined the topological transformation as mapping, and classified the orientable surfaces too. In his work *Sur la deformation des surfaces* in 1866 he wrote the following theorem: The maximal number of recurrent cuts which do not dissect the surface into disconnected pieces, and the number of boundary components are invariant properties and classifies uniquely the compact orientable surfaces. [12]

2.7. Felix Klein (1849-1925)

He distinguished absolute and relative properties of surfaces. For example non-orientability is an absolute, but one-sidedness is a relative property. The definition of one-sidedness involves not only the surface, but also its disposition in space. Orientability depends only on the surface. (*Über den Zusammenhang der Flächen*, 1875) In his paper *Über Riemanns Theorie der algebrischen Functionen und ihre Integrale*, in 1882 Klein gave the normal forms of closed surfaces and described a non-orientable closed surface, the Klein-bottle. It is impossible to embed this surface in three-dimensional Euclidean space without self-intersection. [22] (See Figure 4.)

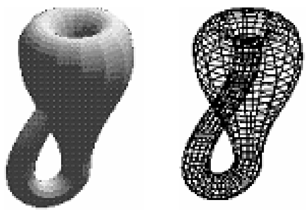

Figure 4.

2.8. Walter von Dyck (1856-1934)

In his study *Beiträge zur Analysis Situs*, in 1888 he dealt with absolute properties of compact orientable and non-orientable 2-dimensional manifolds. He defined surfaces with recurrent and non-recurrent indicatrix. We draw a small circle around a point of the surface, which is not a boundary point, and oriented it. The circle with its orientation (clockwise or anticlockwise) is called an indicatrix. If there is a closed path on the surface, whose traversal reverses the orientation of the indicatrix, the surface with recurrent indicatrix is non-orientable, and the surface with non-recurrent indicatrix is orientable.

He described the cross-cap, a surface homeomorphic to a Möbius-strip. (See Figure 5.) (We cut a square from a half-sphere, and glue together the diagonally opposite vertices of the square.) This representation of the Möbius-strip has a self intersection, but its boundary curve is homeomorphic to a circle, hence cross-caps can be glued into holes in a sphere.

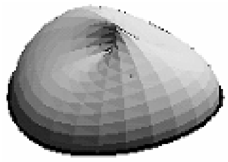

Figure 5.

He proved the following **fundamental theorem of compact surfaces**:

Two closed or bordered triangulable surfaces are topological equivalents if and only if, they have the same number of boundary curves, the same Euler-characteristic and are either both orientable or non-orientable.

Classification of closed or bordered surfaces:

Let $H(p, r)$ a surface which is derived from a sphere with r holes by adding p handles. (See Figure 6.)

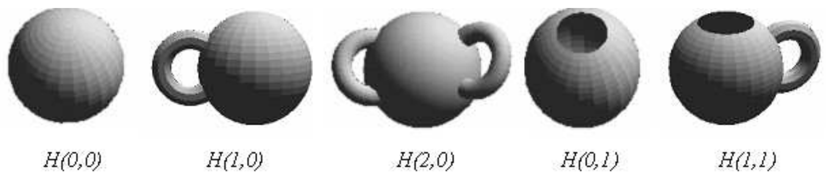

Figure 6.

Let $C(q, r)$ be a surface which is derived from a sphere with r holes by adding q cross-caps. (See Figure 7.)

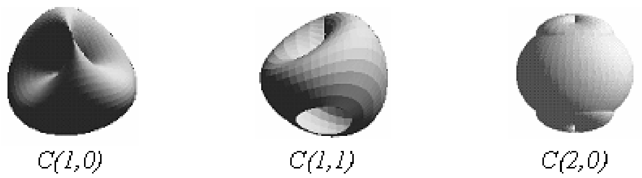

Figure 7.

It is proved, that every closed or bordered surface belongs to one and only one of these classes. [7]

Roughly speaking, the number of handles or cross-caps is called the genus of the surface. This concept appeared in Riemanns work already as the largest number of nonintersecting simple closed curves that can be drawn on the surface without separating it. The genus is related to the Euler- characteristic. More precisely, if the compact surface is orientable, the genus: $g = 1/2(2 - \chi - r)$, where χ is the Euler-characteristic, r the number of boundary components. The genus of a non-orientable surface is $g = 2 - r - \chi$. The genus of a surface is one of the oldest known topological invariants and much of topology has been created in order to generalize this concept of surface topology.

Examples of the connection between invariant properties: (See Table 1.) [15]

The surface	χ	**g**	**r**	**Orientable**	**Class**
sphere	2	0	0	yes	H(0,0)
cube	2	0	0	yes	H(0,0)
torus	0	1	0	yes	H(1,0)
Möbius-strip	0	1	1	no	C(1,1)
sphere with 1 cross-cap	1	1	0	no	C(1,1)
Klein-bottle	0	2	0	no	C(2,0)
disk	1	0	1	yes	H(0,1)
half-sphere	1	0	1	yes	H(0,1)
sphere with 2 handles	-2	2	0	yes	H(2,0)
ring	0	0	2	yes	H(0,2)
cylinder	0	0	2	yes	H(0,2)
sphere with 1 handle and 1 hole	-1	1	1	yes	H(1,1)
cross-cup	0	1	1	no	C(1,0)

Table 1.

2.9. Max Dehn (1878-1952) and Poul Heegaard (1871-1948)

In an encyclopaedia article, in 1907 they elaborated the axiomatic structure of combinatorial topology, this approach allowed them to establish a normal form of surfaces and give the first rigorous proof of the fundamental theorem of compact surfaces. (*Analysis Situs, Enzyklopädie der Mathematischen Wissenschaften*).

2.10. Henry R. Brahana

In his doctoral thesis, *Systems of circuits on two-dimensional manifolds* (1921) he gave on algebraic proof of the classification of closed two-dimensional surfaces, and gave a method of reducing any two-dimensional manifold to one of the known polygonal normal forms through a series of transformations by cutting and joining them. [4]

3. Noncompact surfaces

3.1. Béla Kerékjártó (1898-1946)

The work of the Hungarian mathematician, Béla Kerékjáró according to the non-compact surfaces was the last step in the process of development of surface topology. The process started with the Euler-theorem for polyhedron and went through the classification of compact orientable, then compact non-orientable surfaces till the classification of non-compact surfaces.

Kerékjártó was born in 1898 in Budapest, and died in 1946 in Gyöngyös. He received his Phd in 1920, and 2 years later he became a Full Professor at the University of Szeged. In 1922-23 he was a visiting professor at the University of Göttingen, where he wrote his book *Vorlesungen über Topologie.* This was the first research monograph, and the first textbook on this topic. A chapter of this book contains the theorem of open surfaces, which is known as Kerékjártó's Theorem. With this theorem the problem of topological equivalence of compact and non-compact surfaces is completely solved.

Kerékjártó's main idea was that he defined the ideal boundary of an open surface. This compactification process is a generalisation of the projective closure of the Euclidean space. With the help of these ideal points he compactified the open surface to a closed surface.

An ideal point of a surface S is a nested sequence $G_1 \supset G_2 \supset G_3 \supset \dots$ of connected, unbounded regions (open connected sets) in S satisfying the following properties:

- The boundary curves of G_k regions are simple closed curves of S, for $\forall k \in N$
- The sequence of regions doesn't have any common points.

This sequence of regions defines an ideal point, a boundary point. (See Figure 8.)

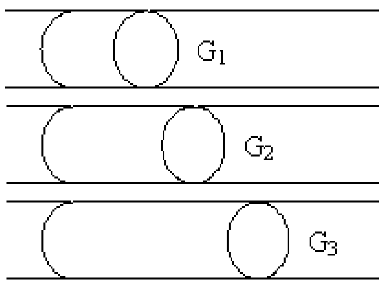

Figure 8.

For example the ideal boundary of a disk can be realized as a circle or the Euclidian plane can be compactified with one point.

Two $G_1, G_2, \dots$ and $G'_1, G'_2, \dots$ sequences of regions define the same ideal point if for $\forall k \in N$ there is a corresponding integer n such that $G'_n \supset G_k$ and $G_n \supset G'_k$.

The invariants of compact surfaces can be generalized to open surfaces.

We distinguish planar and orientable ideal points. A closed surface is planar, if every Jordan curve separates it. The ideal points of an open surface are planar or orientable, if G_k, the element of the sequence of regions are planar or orientable for k sufficiently large. [14] An open surface S is of finite genus, if there is a bounded subsurface A such that $S - A$ is homeomorphic to a subset of the Euclidean plane. Otherwise S is of infinite genus. We have four orientability classes of surfaces: If the surface S is non-orientable, then it is either finitely or infinitely non-orientable. S is infinitely non-orientable, if there is no subset A so that $S - A$ is orientable. If S is finitely non-orientable, we distinguish odd or even non-orientability according to whether every sufficiently large bounded subsurface has an odd or even genus. [20]

Kerékjártó's Theorem

Let S′ and S″ be two open triangulable surfaces of the same genus and orientability class. Then S′ and S″ are homeomorphic if and only if their ideal boundaries are homeomorphic, and the sets of planar and orientable ideal points are homeomorphic too.

Examples of open surfaces that are homeomorphic include:

I. The Euclidean plane and a sphere perforated in one point

II. Grate-surface and a sphere with infinite handles (See Figure 9.)

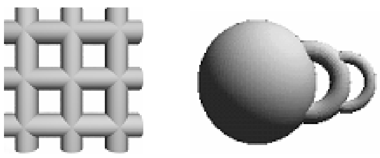

Figure 9.

III. The Euclidean plane with one cross-cap and a cross-cap-surface perforated in one point

We can construct every surface from the following five bordered surface-elements: (See Figure 10.)

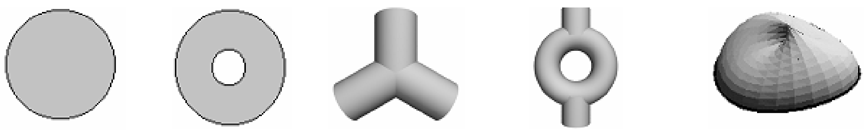

Figure 10.

The way of the construction: We take one from these surface-elements, then stick an other one to all its every boundary curves, so we get a new bordered surface. We repeat this construction so that we have different surfaces stuck to different curves. [14]

3.2. Further results

Of course the formulation of Kerékjártó's Theorem in 1922 did not correspond to the level of exactness in modern abstract mathematics. New versions and proofs of the theorem have been published. The ideal points for open surfaces are generalized as boundary components of a surface imbedded in a topological space.

- Kerékjártó Béla: *A nyílt felületek topológiájáról* (1931)

He gave a new definition of ideal points. [13]

- Ian Richards: *On the classification of noncompact 2-manifolds* (1960)

He described a complete topological classification of non-compact triangulable surfaces, and gave a concrete model for arbitrary surface, similar to the classical normal form. He reconceptualised the Kerékjártó's Theorem and made more precise the proof. [20]

- M. E. Goldmann: *An algebraic classification of noncompact 2-manifolds* (1971)

He proved a theorem, which is for an open surface an algebraic version of the Kerékjártó Classification Theorem. [9]

References

[1] Ahlfors, L. V., Sario, L., Riemann surfaces, *Princeton, New Yersey*, 1960.

[2] Aull, C. E., Lowen, R. (Ed.), Handbook of the history of general topology, Vol. 1, Kluwer A. P., Dordrecht, 1997.

[3] Blackett, D. W., Elementary topology, A combinatorial and algebraic approach, Academic Press, London, 1982.

[4] Brahana, H. R., Systems of circuits on two-dimensional manifolds, *Annals of Mathematics,* 23, (1921.), 144-168.

[5] Boltyanski, V. G., Efremovich, V. A., Intuitive combinatorial topology, Springer-Verlag, New York, 2001.

[6] Coxeter, H. S. M., Regular polytopes, Dower Publications, Inc., New York, 1973.

[7] Dyck, W., Beitrage zur Analysis Situs, *Matematische Annalen,* XXXII (1888), 457-512.

[8] Firby, P. A., Gardiner, C. F., Surface topology, Ellis Horwood Ltd, New York, 1882.

[9] Goldman, M. E., An algebraic classification of noncompact 2-manifolds, *Trans. Amer. Math. Soc.,* 156 (1971), 241-258.

[10] Griffiths, H. B., Surfaces, Cambridge University Press, Cambridge, 1981

[11] Hilbert, D., Cohn-Vossen, S., Szemléletes geometria, Gondolat, Budapest, 1982.

[12] James, I. M. (ed.), History of Topology, Elsevier, Amsterdam, 1999.

[13] Kerékjártó, B., A nyílt felületek topológiájáról, *Matematikai és fizikai lapok,* 38 (1931), 146-155.

[14] Kerékjártó B., Vorlesungen über Topologie, Verlag von Julius Springer, Berlin, 1923.

[15] Kónya, E., Kerékjártó Béla tétele a nyílt felületek topológiájáról, *Polygon,* XII/1-2 (2003), 23-34.

[16] Lakatos, I., Bizonyítások és cáfolatok, Typotex, Budapest, 1998.

[17] Möbius, A. F., Theorie der elementaren Verwandschaften, Gesammelte Werke, Bd. 2, 473-512, Verlag von S. Hirzel, Leipzig, 1886.

[18] Möbius, A. F., Über die Bestimmung des Inhaltes eines Polyeders, Gesammelte Werke, Bd. 2, 433-471, Verlag von S. Hirzel, Leipzig, 1886.

[19] Prasolov, V. V., Intuitive topology, Mathematical World, Vol. 4, American Mathematical Society, 1995.

[20] Richards, I., On the classification of noncompact surfaces, *Trans. Amer. Math. Soc.,* 106, 1963., 259-269

[21] Scholz, E., Geschichte des mannigfaltigkeitsbegriffs von Riemann bis Poincaré, Birkhäuser, Boston, 1980.

[22] Weyl, H., Die Idee der Riemsnnschen Fläche, Druck und Verlag von B. G. Teubner, Leipzig und Berlin, 1913.

Eszter Kónya
4026 Debrecen
Péterfia u. 1-7.
Hungary

Annales Mathematicae et Informaticae
32 *(2005) pp. 225–236.*

Remarks on the concepts of affine transformation and collineation in teaching geometry in teachers' training college

István Krisztin Német

Mathematical Department, University of Szeged,
Juhász Gyula Teachers' Training College
e-mail: krisztin@jgytf.u-szeged.hu

Abstract

In [11] and [12] (textbooks for teachers' training colleges written by B. Pelle) isometry and similarity are defined not in the classical way but as a product. We continue this way of definition refer to the affine transformation and collineation, and study the fundamental theorems of these mappings analogously to those of isometry and similarity.

AMS Classification Number: 00A35 (ZDM: G55, G59)

1. Introduction

In the last textbooks on geometry for teachers' training colleges, in [11] and in [12], (plane-) isometry is defined as a product of reflections in line, similarity is defined as a product of isometry and central dilatation, instead of the classical way which is based on the properties of distance- and ratio-preserving. We call this way of definition "constructive" because it provides technique to give the mapping. However, this way doesn't proceed in [11] for studying affine transformations and collineations. (These subjects are not involved in [12].)

As it is known, in the classical treatment in the Euclidean plane-geometry affine transformation is a line-preserving transformation of the plane, and axial affine transformation is defined as an affine transformation with an axis. In the classical projective plane-geometry collineation is defined as a line preserving transformation of the plane, and central (axial) collineation is a collineation with a center (or an axis).

In [11] the concept of the affine transformation is aproached from projective geometry. At first, the concept of collineation is defined in the classical way ([11] p. 329.). Affine transformation is a collineation, which leaves the line at infinity (the ideal line) invariant ([11] p. 332.). Axial affine transformation is a central collineation, whose center is a point at infinity (an ideal point), and the axis is an ordinary line ([11] p. 338.).

In our opinion building a uniform system is a very inportant didactical principle especially in teacher training. Therefore in our present paper we'll apply the "constructive" way – only in the plane geometry – for these topics, too. (In [9] we wrote our remarks on studying similarities in the constructive way.) We'll follow the reversed way for both cases mentioned above. At first, we'll define the axial affine transformation in the Euclidean plane in a metric way. Then we'll build the concept of the affine transformation on the concept of axial affine transformation and similarity. We'll deal with the concept of collineation in a similar way. So in all cases at first we define a special mapping, and then we get the general type by using this special one and the previous mapping. This way fulfills another important didactical principle, too, namely: progressing from the special case towards the general one. Beside providing a uniform definition, our further aim is to study the fundamental theorems related to the new concepts analogously to the studying of isometries and similarities, in order to form a uniform and consistent system in the "constructive" way. We'll point out several analogies which also strengthen unity and understanding. Meanwhile, we'll touch upon the connection between the classical and "constructive" ways, too.

2. Preliminaries

In this paper by transformation we mean a bijective mapping of the plane onto itself. Two transformations are said to be equal, if they transform any point into the same point. By line-preserving mapping we mean a mapping, which transforms collinear points into collinear points. If a point coincides whith its image under a mapping, then we call it fixed point. If a straight line is fixed pointwise by a mapping, then we call it axis. If a line coincides with its image under a mapping, then we call it invariant line. By center of a mapping we mean a point, through which every line passing is invariant. By plane-flag we mean the union of a halfplane and a ray on its boundary. If three lines have common point or they are parallel to each other, then we call them concurrent lines. We use directed line segments; we define the operations related to them in the usual way. We use the concepts of the affine- and the cross ratio on the Euclidean plane almost totally in the usual metric sense. We study their elementary properties and the Pappos-Steiner theorem also in the classical way (e.g. in [4]). The only difference is that by the affine ratio (ABC) of the collinear points A, B and C we mean the ratio $\frac{AC}{BC}$ as in [7]. This definition is involved in [11] (p. 141), too, but there the ratio is negative, iff C does not separate A and B, on the contrary as in our case. After the usual introduction of the ideal elements of the Euclidean plane, we extend the concepts of the affine- and cross ratio to them preserving the previous properties. (For example: if P is an ideal point then $(ABP) := 1$, $(ABCP) := (ABC)$, $(PABC) := (CBA)$, etc.)

In [11] and [12] the concept of isometry is based on the axioms of Reflection related to the primitive concept of “reflection in plane”. We'll make comparison with the equivalents of these axioms related to the concept of reflection in line, therefore we list these theorems (R1-R5). (The theorems which we use are a little bit different from the statements involved in [11] and [12] ([11] pp. 21-22, 25-26; [12] pp. 17-18, 22-23). In [8] we examined the connection between the two ways.)

R1: Any reflection in line is a line-preserving, involutory transformation of the plane with an axis, and the axis separates every other $P - P'$ pair.

R2: For any line there is a unique reflection, whose axis is the given line.

R3: For any two points there is a unique reflection, in which they are corresponding points.

R4: For any two rays, starting from the same point, there is a unique reflection, which transforms the given rays into each other.

R5: If two products of reflections transform a plane-flag into the same one, then the products are equal.

3. Definitions

To construct the concept of the axial affine transformation and the central-axial collineation we connect to the concept of the central dilatation; and to construct the concept of the affine transformation and the collineation we connect to the concept of the similarity. The analogous definition of these concepts can help the understanding of the new concept, it makes them imaginativable and contributes to the developing of a uniform system. It is a further advantage that the proofs of the corresponding properties can be done similarly; it also strengthens unity. It is an evident disadvantage of this way of definition that it is more complicated than the classical one.

Classically central dilatation is defined by giving its center and ratio. One can say that in the definition of the central dilatation we give the way how the position of a point changes in comparison with a fixed point. In the case of the axial affine transformation we compare it with an axis. In our opinion, if we want to make this new concept analogously then it fits best the previous definition of the central dilatation if we define the axial affine transformation by giving its axis, direction and ratio. This way occurs in the secondary school, too, only as supplementary subject, and only with orthogonal direction ([3]).

Definition 3.1.a. By general axial affine transformation we mean the following mapping on the Euclidean plane. Suppose that there are two intersecting lines t and e, and a $\lambda(\neq 0)$ constant. The image of the point P is those P' for which $P_tP' = \lambda P_tP$, where P_t denotes the point on t for which $e \parallel (P_tP)$.

The metrical condition related to P' can be written also in the form of $(P'PP_t) = \lambda$ $(P \notin t)$. In the case of the central dilatation the analogous formulae of this equation is the next one: $(P'PC) = \lambda$ $(P \neq C)$.

Definition 3.1.b. By special axial affine transformation we mean the following mapping on the Euclidean plane. Suppose that there are two parallel lines t and

e, e is directed, a $\lambda(> 0)$ constant, and a halfplane bounded by t is indicated. The image of the point P is those P' for which $d(P, P') = \lambda d(t, P)$, and the PP' segment is either similarly or oppositely directed as e, depending on P whether it is in the indicated halfplane or not.

We use the terms "general" and "special" according to [7]. (For the special axial affine transformation in [1], [5], [10], [13] there is the term "shear". If $e \perp t$, we call it orthogonal axial affine transformation (e.g. [4]), or "strain" (e.g. [10]).)

If we extend the definitions of axial affine collineation, central dilatation and translation to the ideal elements by using the line-preserving property as usual, we can emphasize here the fact that each of them has both axis and center. Furthermore, the next statements are valid: any line-preserving transformation of the Euclidean plane which transforms any line into a parallel one, is either a central dilatation or a translation (e.g. in [1]); any line-preserving transformation of the Euclidean plane which transforms any point P so that the lines (PP') are parallel to each other, is either an axial affine transformation or a translation (e.g. in [13]). They mean that there isn't any other line-preserving transformation that has ideal center or axis. The motivation for making the concept of central-axial collineation is the following: construct a nonidentical line-preserving mapping that has both ordinary center and axis. If we want to preserve the unity, to make the definition in the manner of those of the central dilatation and axial affine transformation, the it is the most natural way if we define the new concept by giving the center, the axis and the ratio, and try to "join" the methods of the previous definitions.

The first attempt is the following: there is a line t, a point C, $C \notin t$, and a $\lambda(\neq 0)$ constant. The image of the point $P(\neq C, \notin t)$ is those P' for which either $\frac{CP'}{P_tP'} = \lambda \frac{CP}{P_tP}$ or $CP' = \lambda CP$, depending on (CP) whether it intersects t at point P_t or it is parallel to t.

It is not "perfect", because the affine ratio never equals 1 on the Euclidean plane, so there are points without image-point, and there are points without origin-point. Namely, P has not image iff $(CP_tP) = \frac{1}{\lambda}$, and has not origin iff $(CP_tP) = \lambda$. It is obvious that in both cases the locus of these points is a line parallel to the axis. According to [11], we call them the "line of disappearing" and the "line of directions", respectively. This problem is eliminated by using the Euclidean plane extended by ideal elements. According to the above mentioned theorems on the mappings that have ideal center or axis, we can introduce the concept of central-axial collineation by defining separately only the mapping that has both ordinary center and axis, and if one of them is an ideal one, we call the central-axial collineation related to them the appropriate previous mapping. If the axis doesn't contain the center we don't need this way because a unified definition can be given by using the remark after Definition 3.1.a. Unfortunately, if the center is on the axis we can't construct the mapping in such an elegant way.

Definition 3.2.a. By general central-axial collineation we mean the following mapping on the extended Euclidean plane. Suppose that there is a line t, a point C, $C \notin t$, and a $\lambda(\neq 0)$ constant. (P_t will denote the point on t for which $C \in (P_tP)$.) The image of the point $P(\neq C, \notin t)$ is those P' on t for which $(P'PCP_t) = \lambda$. If

$P = C$ or $P \in t$, then $P' = P$. (This cross ratio is called the "characteristical cross ratio" ([6]).) If t is the ideal line we get the central dilatation with center C and ratio λ; if C is an ideal point we get the general axial affine transformation with axis t, ratio $\frac{1}{\lambda}$ and direction C.

Definition 3.2.b. Suppose that there is a line t and a point C on the extended Euclidean plane, $C \in t$. If C and t are both ideal elements by the special central-axial collineations with center C and axis t we mean the translations with direction C. If C is an ideal point and t is an ordinary line by the special central-axial collineations with center C and axis t we mean the special axial affine transformations with axis t. If C and t are both ordinary elements suppose that there is a λ segment, and an (euclidean) halfplane bounded by t is indicated. In this case by special central-axial collineation we mean the following mapping. (T_P will denote the foot of the perpendicular from the point P to t.) The image of the ordinary point $P(\notin t)$ is those P' for which (CPP') equals either $\frac{\lambda}{PT_P}$ or $-\frac{\lambda}{PT_P}$, depending on P whether it is in the indicated halfplane or not. The image of the ideal point $P(\notin t)$ is those P' in the not-indicated halfplane for which $P' \in (CP)$ and $P'T_{P'} = \lambda$. If $P \in t$, then $P' = P$. (The λ and PT_P segments are not directed. We use the terms "general" and "special" again according to [7]. (In [1] for the general case there is the term "homology" and for the special case "elation".)

In the general case the above-mentioned line of disappearing is the origin of the ideal line, and the line of directions is the image of it. In the special case these lines also exist: P has ideal image (origin) iff $PT_P = \lambda$ and P is in the indicated (not-indicated) halfplane.

The following definitions are the analogues of those of isometry and similarity involved in [11], [12].

Definition 3.3. By affine transformation we mean a finite product of axial affine transformations and similarities on the Euclidean plane.

Definition 3.4. By collineation we mean a finite product of central-axial collineations on the extended Euclidean plane.

4. Properties of the axial affine transformation and the central-axial collineation

First we mention properties related to the ratio. Both general axial affine transformation and general central-axial collineation of ratio 1 are the identity, just as the central dilatation with ratio 1. In the general cases we get the inverse mapping simply by changing the ratio to $\frac{1}{\lambda}$, just as in the case of the central dilatation. (In the special cases we have to interchange the roles of the indicated and not-indicated halfplanes.) The general axial affine transformation with ratio -1 and with direction orthogonal to the axis is a reflection in line, just as the central dilatation of ratio -1 is a reflection in point.

To emphasize other analogies with the reflection in line and the central dilatation, we list some other properties of the axial affine transformation (I-IV) and the central-axial collineation (I*-IV*).

I. Any axial affine transformation is a line-preserving transformation of the Euclidean plane with an axis; this line separates every other $P - P'$ pair iff $\lambda < 0$.

II. For any lines t, e and constant $\lambda(\neq 0)$, (if $e \parallel t$, then e is directed, $\lambda > 0$ and a halfplane bounded by t is indicated), there is a unique axial affine transformation with axis t, direction e and ratio λ.

III. For any line t and pair of points $P - P'$ which are off t, there is a unique axial affine transformation with axis t, under which the image of P is P'.

IV. For any lines t, e and pair of lines $a - a'$ which are not parallel to e and differ from t but concurrent with it, there is a unique axial affine transformation with axis t, direction e, under which the image of a is a'.

I*. Any central-axial collineation is a line-preserving transformation of the extended Euclidean plane with center and axis; this point and line separate every other $P - P'$ pair iff $\lambda < 0$.

II*. For any point C, any line t, $C \notin t$ and any constant $\lambda(\neq 0)$, there is a unique central-axial collineation with center C, axis t and ratio λ. (If $C \in t$ then it is quite difficult to formulate this proprety.)

III*. For any point C, any line t and any pair of points $P - P'$, which are off t and differ from C but collinear with it, there is a unique central-axial collineation with center C, axis t, under which the image of P is P'.

IV*. For any point C, any line t and any pair of lines $a - a'$ which are off C and differ from t but concurrent with it, there is a unique central-axial collineation with center C, axis t, under which the image of a is a'.

These properties are just the analogues of the theorems R1-R4 on reflection in line and the corresponding properties of the central dilatation. The analogue of R5 will occur later in Theorems 5.1, 5.3. Statements I and I* contain the most important (non metric) properties of the mappings. The second, third and fourth statements provide techniques to give the mapping. (The only nontrivial proof is that of the line-preserving property: it can be done by the theorems of parallel secants and by the Pappos-Steiner theorem.) Also by using the mentioned theorems we get that the axial affine transformation preserves the affine ratio and the central-axial collineation preserves the cross ratio.

Besides the analogies mentioned, there are further view-points to compare the central dilatation, the axial affine transformation and the central-axial collineation. The first comparison is connected to the properties listed in statements I and in I*. We declared them the most important ones because they determine the mappings. The next two theorems are the analogues of the statement that any line-preserving transformation of the Euclidean plane with a center, is a central dilatation.

Theorem 4.1. *Any line-preserving transformation of the Euclidean plane with an axis, is an axial affine transformation.*

Theorem 4.2. *Any line-preserving transformation of the extended Euclidean plane with a center and an axis, is a central-axial collineation.*

To prove these theorems we can use classical ways. In the case of Theorem 4.1 we get that for the (nonidentical) mapping the lines (PP') are invariant and parallel to each other, and either $(P'PP_t)$ or $\frac{d(P,P')}{d(t,P)}$ is constant, depending on the lines (PP') whether they intersect the axis or not. In the case of Theorem 4.2 due to the previous results we have to prove only in the case of ordinary center and axis. We get that for the (nonidentical) mapping either $(P'PCP_t)$ or the segment $(CPP')PT_P$ is constant depending on the center whether it is off or on the axis. Theorem 4.1 implies that the Definition 3.1 of the axial affine transformation is equivalent to the classical one.

Before the Definition 3.2 we have already mentioned the center and axis of the mappings. According to the emerged conjecture based on the examples above, this is the right time in this treatment to touch upon the fact that the existence of center and axis are inseparable: if a mapping is a line-preserving transformation of the extended Euclidean plane with a center, then it has an axis, too, and conversely. (In the classical projective geometry this theorem is usualy deduced from Desargues's theorem (e.g. in [7], [11]); other, direct type of proof can be found e.g. in [2], [13]. In this treatment we apply the latter one.) By applying the results, it follows:

Theorem 4.3. *Any line-preserving transformation of the extended Euclidean plane with a center or an axis, is a central-axial collineation.*

From this theorem we also get the equivalence of Definition 3.2 and the classical one for the central-axial collineation.

5. Fundamental theorems of affine transformations and collineations

The basic properties of the affine transformation and the collineation follow directly from the definitions, as the common properties of the factors of the product (line-preserving transformation of the Euclidean plane, preserves the order and the affine ratio; line-preserving transformation of the extended Euclidean plane, preserves the separatedness and the cross ratio). We also immediately get that the affine transformations – and the collineations as well – form a group. The first group contains the group of the similarities as a subgroup, and the second one similarly contains that of the affine transformations.

Theorem 5.1. *Let us consider on the Euclidean plane two flags, a point on each ray and halfplane. There exists a unique affine transformation, which transforms the first flag to the other one, the points on the first flag to the correspondent points on the other one.*

In the classical treatment the equivalent theorem of Theorem 5.1 is the one, which states that any two triangles are related by a unique affine transformation. (In [11] there is not a theorem like this.) We use Theorem 5.1 instead of this theorem, because this is the analogue of theorem R5 and the fundamental theorems of (plane-) isometries and similarities: there exists a unique isometry (similarity), which transforms a given flag to an other given one (and a given point on the first ray to a given point on the other one).

Proof of Theorem 5.1. The flags are denoted by Z and V, the points on their rays are P and Q, the points on their halfplanes are R and S, respectively. First let us consider the similarity $\mathbf{H}$, which transforms Z to V and P to Q. Then we consider the axial affine transformation whose axis is the line of the ray of V, which transforms $\mathbf{H}(R)$ to S. The product of these transformations has the desired properties. If there is another affine transformation, then it is equal to the first product, due to the line- and affine ratio preserving properties. □

From the construction involved in the previous proof we get the next theorem.

Theorem 5.2. *Any affine transformation can be obtained as a product of a similarity and an axial affine transformation.*

This is the analogue of the "obtaining" theorems for plane-similarities (isometries): any similarity (isometry) can be obtained as a product of an isometry and a central dilatation (at most three reflections in line).

In the sequel we'll draw up the analogues of Theorems 5.1 and 5.2 for collineations. At first, we have to generalize the concepts of ray, halfplane and flag. On the extended Euclidean plane any two points divide the line consisting them into two "segments"; we'll use the term "ray" for the union of the interior of a segment and one of its boundary-points. (We call this point as "starting point", and the other boundary point as "end-point".) Any two lines divide the plane into two parts; we'll use the term "halfplane" for the interior of these parts. On the extended Euclidean plane by "flag" we mean the union of a "halfplane" and a "ray" on its boundary, whose end-point is the point of intersection of the boundary-lines of the "halfplane". (Fig. 1. shows some flag-types.) It is clear that the image of any flag under a collineation is again a flag.

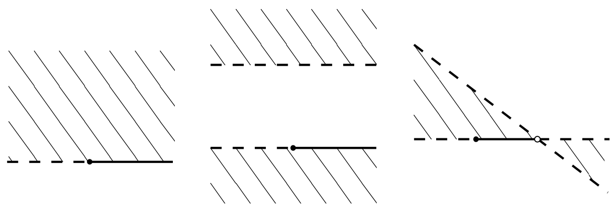

Figure 1.

Theorem 5.3. *Let us consider on the extended Euclidean plane two flags, a point on each ray and halfplane. There exists a unique collineation, which transforms the* *first flag to the other one, the points on the first flag to the correspondent points on the other one.*

This is the analogue of theorem R5 and Theorem 5.1. However, the proof will not be the analogue of that of Theorem 5.1.

Proof of Theorem 5.3. First let us transform each flag by such a central-axial collineations, whose "line of disappearing" is that boundary-line of the flag which doesn't contain its ray. Now, the mentioned boundary-line of the image-flags is the ideal line, hence there is an affine transformation which transforms the first new flag to the second one, and the points on the first one to the correspondent points on the second one. So the original flags and the points are related by a product of two central-axial collineations and an affine transformation. If there is another collineation, then it is equal to the first product, due to the line- and cross ratio preserving properties. □

In the classical treatment its equivalent theorem is this one: any two "quadrangles" on the projective plane are related by a unique collineation. It is obvious from Fig. 2. that the quadrangels $APMK$ and $BQNL$ determine the same collineation as the given flags and points, and conversely.

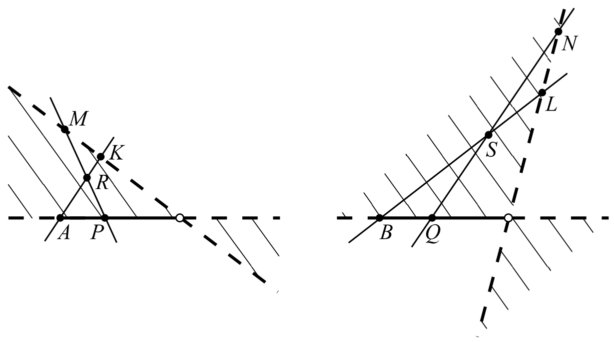

Figure 2.

The next theorem is the analogue of Theorem 5.2. It is not the direct corollary of Theorem 5.3 as Theorem 5.2 was that of Theorem 5.1, here we need a little bit different construction. This will be the proof, which is the analogue of that of Theorem 5.1.

Theorem 5.4. *Any collineation can be obtained as a product of an affine transformation and a central-axial collineation.*

Proof of Theorem 5.4. (Fig. 3.) Let us consider a collineation $\mathbf{K}$, the line f which is the image of the ideal line under $\mathbf{K}$, a flag V whose boundary-lines are

f and an ordinary line which contains the ray and intersects f in an ideal point. Also consider the flag Z which is the origin of V under $\mathbf{K}$, the points P and R on the ray and halfplane of Z, finally the points Q and S which are the image-points of P and R under $\mathbf{K}$. It is easy to see that all these points and the starting points of the rays are ordinary points. (f may be either the ideal line or an ordinary one.) Now let us consider the affine transformation which transforms the ray of Z to that of V, P to Q and R to S. Then we consider the central-axial collineation with center S, whose axis is the line of the ray of V, which transforms the ideal line to f. According to Theorem 5.3 the product of these two transformations is equal to $\mathbf{K}$.

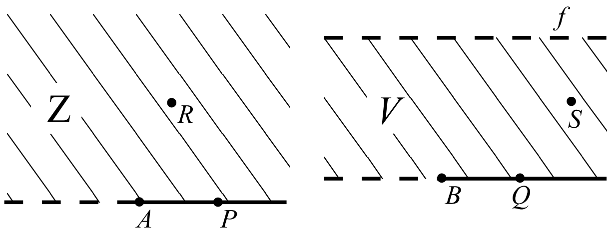

Figure 3.

Now we make a little change on the previous construction. First let us consider a similarity $\mathbf{H}$, which transforms the ray of Z to that of V and P to Q. Then we consider the the central-axial collineation with axis (BQ) which transforms $\mathbf{H}(R)$ to S and the ideal line to f. This concludes that any collineation can be obtained as a product of a similarity and a central-axial collineation. □

Finally there follow the analogues of Theorems 4.1 and 4.4.

Theorem 5.5. *Any line-preserving transformation of the Euclidean plane is an affine transformation.*

Theorem 5.6. *Any line-preserving transformation of the extended Euclidean plane is a collineation.*

Proving the Theorem 5.5 at first we get in the classical way that the mapping preserves the affine ratio. Then, by using the method of the proof of Theorem 5.1, we get that the transformation is a product of a similarity and an axial affine transformation. The proof of Theorem 5.6 has two cases. If the ideal line is invariant, then according to Theorem 5.5 the given transformation is an affine transformation. Otherwise let us consider the product of the given transformation and a central-axial collineation which transforms the image of the ideal line under the given transformation to the ideal line. According to the first case this product is an affine transformation. Thus the given transformation is a collineation. These theorems imply the equivalence of Definitions 3.2, 3.4 and the classical ones, too. Moreover, in our treatment they mean that it is impossible to widen the set of affine transformations and collineations if we demand the properties mentioned in the theorems.

6. Closing remarks

We think it is important to remark here the following: we don't state that the constructive way is better for studying these mappings than the classical one. The constructive way emphasizes other didactical principles; this has advantageous and disadvantageous consequences, too:

1. The constructive way – as we have mentioned in the Introduction – always progresses from the special case towards the general one. Hence, naturaly, the treatment becomes more lenghty, less "economical", and the main invariants of the mappings appear later, not immeditely in the definitions.

2. The constructive giving of a mapping is in accordance with the most frequent way of giving of the functions: it gives the domain and the rule of the correspondence. Therefore, the definitions are more concrete but also more complicated than in the classical way, where we define classes of functions having certain properties.

3. In order to form a uniform system, we define the new mappings on metrical basis so as to connect strongly with the way of definition of the previous mappings. This is the most natural way for the students, because the metrical thinking about the geometrical mappings is very strong. However, this way may lead to the false idea, that these concepts – and even the whole geometry – can be studied only on metrical basis. It could help if we mention the connection between the classical and constructive ways after the appropriate theorems (4.1, 4.3, 5.5, 5.6) because the classical way loosens the connection to metrical concepts. The study of the possibility of aproaching a domain from an other point of view is also an important principle in teacher training. Another kind of problem arises in connection with the collineations. We defined this mapping on the Euclidean plane extended by ideal elements, which is often called as "projective plane", because from the point of view of incidence, order and continuity it is a modell of the "real (classical) projective plane". However, during our considerations we always made distinction between the "ideal" and "ordinary" elements, moreover we used metrical concepts based on the Euclidean Axioms of Reflection. On the real projective plane there aren't either indicated elements or euclidean based metric. That's why we didn't use the term "projective plane". If the students learn about the axiomatical real projective plane during their further studies, the usage of the term "projective plane" in different ways may cause confusion. The problem would be partly solved, if we defined the central-axial collineation only on the Euclidean plane except the line of disappearing. So we could stay within the frames of the Euclidean geometry. But this change would make the treatment very complicated. Another solution could be given on the basis of the real projective geometry: at first, we study the projective plane and then we make a modell for the Euclidean plane (eg.: [2],

[7]). Clearly, this way can't help us either because it would be a higher course in geometry.

References

[1] Coxeter, H. S. M., A geometriák alapjai, Műszaki Kiadó, Budapest, 1973.

[2] Coxeter, H. S. M., Projektív geometria, Gondolat, Budapest, 1986.

[3] Hajnal, I., Némethy, K., Matematika II. (gimn.) (2. kiadás), Tankönyvkiadó, Budapest, 1990.

[4] Hajós, Gy., Bevezetés a geometriába (8. kiadás), Tankönyvkiadó, Budapest, 1987.

[5] Hollai, M., Horváth, J., A sík és a tér affin transzformációi, ELTE TTK Szakmódszertani közleményei, ELTE, Budapest, 1973.

[6] Horvay, K., Reiman, I., Projektív geometria, Tankönyvkiadó, Budapest, 1980.

[7] Kerékjártó, B., A geometria alapjairól, Második kötet, Projektív geometria, MTA Budapest, 1944.

[8] Krisztin Német, I., Megjegyzések az egybevágóság és a merölegesség fogalmának megalapozásához a föiskolai geometria-oktatásban, *Berzsenyi Dániel Föiskola Tudományos Közleményei XIII. Természettudományok 8.*, (2002) 17–37.

[9] Krisztin Német, I., Remarks on the concept of similarity in teaching geometry in teachers' training college, *Acta Academiae Paedagogicae Agriensis, Sectio Mathematicae*, **31** (2004) 107–118.

[10] Martin, G. E., Transformation Geometry (An Introduction to Symmetry), Springer-Verlag, New York, 1982.

[11] Pelle, B., Geometria, Tankönyvkiadó, Budapest, 1979.

[12] Pelle, B., Geometria (átdolgozott kiadás), EKTF Líceum Kiadó, Eger, 1997.

[13] Reiman, I., A geometria és határterületei, Gondolat, Budapest, 1986.

[14] Szász, G., Geometria, Tankönyvkiadó, Budapest, 1964.

[15] Szász, G., Projektív geometria, Tankönyvkiadó, Budapest, 1964.

István Krisztin Német
Mathematical Department
University of Szeged
Juhász Gyula Teachers' Training College
Boldogasszony sgt. 6.
H-6725 Szeged
Hungary