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Analysis and homogenisation of a linear Cahn–Larché system with phase separation on the microscale

Lisa Reischmann[1] | **Malte A. Peter**[1,2]

[1]Institute of Mathematics, University of Augsburg, Augsburg, Germany

[2]Centre for Advanced Analytics and Predictive Sciences (CAAPS), University of Augsburg, Augsburg, Germany

Correspondence
Lisa Reischmann, Institute of Test and Simulation for Gas Turbines, German Aerospace Center (DLR e.V.), Am Technologiezentrum 5, 86159 Augsburg, Germany.
Email: Lisa.Reischmann@dlr.de

Present Address
Lisa Reischmann. Institute of Test and Simulation for Gas Turbines, German Aerospace Center (DLR e.V.), Am Technologiezentrum 5, 86159 Augsburg, Germany

We consider the process of phase separation of a binary system under the influence of mechanical stress modelled by the Cahn–Larché system, where the mechanical deformation takes place on a macrosopic scale whereas the phase separation happens on a microscopic level. After linearisation, we prove existence and uniqueness of a weak solution by a Galerkin approach. As discretisation in space leads to a linear differential–algebraic system of equations, we adjust known solution theory for such equations to a weak setting. This approach may be of interest more generally for coupled elliptic–parabolic systems. A-priori estimates enable us to pass to the homogenisation limit of the linear system rigorously using the concept of two-scale convergence. A comparison with the formally homogenised full nonlinear problem shows that both systems lead to models of distributed-microstructure type in the limit and that homogenisation and linearisation commutes.

1 | INTRODUCTION

Phase-separation processes are classically modelled by the Cahn–Hilliard equation. In the case of binary systems (two phases), the nonlinear fourth-order parabolic equation describes the evolution of an order parameter denoting the relative concentration $c = c(x, t)$ of two phases E and C, say, so that $c = 0$ and $c = 1$ in space–time points (x, t) of pure phases E and C, respectively. The Cahn–Hilliard equation describes the H^{-1}-gradient flow of an energy potential incorporating a term associated with the mixture, typically modelled by a double-well potential, and a term penalising phase interfaces.

When elastic effects need to be taken into account as well, the energy functional is extended by an elastic-energy term and the gradient flow is considered in $H^{-1} \times (L^2)^N$ for tuples (c, u), where $u = u(x, t)$ is the displacement vector with values in $\mathbb{R}^N$. The result is the Cahn–Larché system, consisting of an extended Cahn–Hilliard equation coupled with the (linearised) equations of elasticity [1].

In a number of situations, the processes—phase separation and mechanics—happen on different spatial and temporal scales. For example, in phase-separation experiments on Langmuir–Blodgett film balances, monolayers of different lipid phases decompose under mechanical deformations induced by a teflon barrier. The phase separation occurs on length scales of the order of several microns, which differs from the scale of the mechanical deformation induced by the teflon barrier by about five orders of magnitude [2]. We also refer to [3], [4] and [5] for phase separation in lipid membranes and to [6] in particular for modelling of lipid decomposition by the Cahn–Hilliard equation. Similarly, phase separation in alloys, for which the Cahn–Hilliard model was originally derived, also happens on much smaller scales than mechanical deformations of the workpieces made of these alloys [7].

A multiscale Cahn–Larché system taking into account these different length scales was recently derived in the context of phase-separation experiments making use of formal homogenisation techniques [8]. The result was a model of distributed-microstructure type, in which a macroscopic elastic equation is coupled to an extended Cahn–Hilliard equation to be solved in (local) representative unit cells associated with each macroscopic point. The homogenisation process was formal in the sense that asymptotic expansions were assumed valid and the limit system was obtained by matching terms of same order of the (small) homogenisation parameter ϵ. Numerical simulations showcasing typical results demonstrated the applicability of the homogenised model.

It is the aim of this article to make the homogenisation process mentioned above rigorous. As the Cahn–Larché system is highly nonlinear and, more importantly, the ϵ-scale model derived in [8] is degenerate in the homogenisation limit $\epsilon \to 0$, we restrict to a linearised model, where the linearisation is about a given solution of the Cahn–Larché system. For this linearised model, which is a system of a parabolic fourth-order equation and an elliptic second-order equation, we show existence of a weak solution by a Galerkin discretisation in space adapting results from the theory of differential–algebraic equations. The approach is kept general so that it can be easily adapted to other parabolic–elliptic problems. The homogenisation process is carried out in the context of (rigorous) two-scale convergence leading to a linear limit system of distributed-microstructure type. Comparing this limit system with the linearised version of the formally derived limit system of [8] shows that linearisation and homogenisation commute for this problem.

The article is organised as follows: Based on the modelling of phase-separation experiments on Langmuir–Blodgett film balances of [8], we introduce the nonlinear Cahn–Larché system with phase separation on the microscale and linearise it about a given solution in Section 2. The well-posedness of the linear Cahn–Larché system is shown in Section 3 including the ϵ-independent a-priori estimates required for the homogenisation process, where the generalisations of the solution theory for differential–algebraic systems are discussed in Section 3.2.1. The passage to the homogenisation limit is presented in Section 4. The general two-scale compactness results are discussed in Section 4.1 and the final limit system is found in Section 4.2. A short summary and discussion of the results are given in Section 5.

2 | THE MATHEMATICAL MODEL

We give a brief description of the ϵ-scale (microscopic) model derived in [8], which is the starting point for our considerations. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain with Lipschitz continuous boundary $\Gamma := \partial\Omega$ having pairwise disjoint parts Γ_0, Γ_{g} and Γ_{s} such that $\Gamma = \Gamma_0 \cup \Gamma_{\mathrm{g}} \cup \Gamma_{\mathrm{s}}$ and a finite time interval $S = (0, T)$. This choice is motivated by the film-balance experiments: In two dimensions, Ω is a rectangle describing the domain occupied by the lipid monolayer, which is bounded by the lateral boundaries of the film balance Γ_{s}, the movable teflon barrier Γ_{g} and the boundary opposite the teflon barrier Γ_0.

We assume the evolving microstructure of the pattern to have an intrinsic length scale associated with it. For this purpose, we introduce a characteristic macroscopic length scale L, representing the order of magnitude of the size of the film balance and corresponding to the macroscopic process, and a characteristic microscopic length scale l, which corresponds to the order of magnitude of the scale on which the phase separation is observable, and we write

$$\epsilon = \frac{l}{L}. \tag{1}$$

It is clear that it holds $\epsilon \ll 1$. Then, the order parameter, which describes the microstructure, depends on ϵ. We denote this with an ϵ in the index and write c_ϵ and, analogously, u_ϵ for the displacement.

2.1 | The Cahn–Larché system

If only small deformations are considered a linearised theory is applicable so that we only consider infinitesimal strains defined by

$$\mathcal{E}(u_\epsilon) = \frac{1}{2}\left(\nabla u_\epsilon + (\nabla u_\epsilon)^{\mathrm{T}}\right). \tag{2}$$

In general, we have different elastic properties of the two phases. Thus, the elasticity tensor $\mathcal{A}(c_\epsilon)$, characterising the stiffness of the phases, naturally depends on the order parameter c_ϵ. The stress tensor is thus given by

$$\mathcal{S}_\epsilon = \mathcal{A}(c_\epsilon)\big(\mathcal{E}(u_\epsilon) - \bar{\mathcal{E}}(c_\epsilon)\big). \tag{3}$$

More precisely, we assume the two phases to have different elastic properties and hence we denote the elasticity tensor describing the elastic properties of the softer phase by $\mathcal{A}^{\mathrm{E}}$ and the tensor of the harder phase by $\mathcal{A}^{\mathrm{C}}$. (The labelling is motivated by the labelling in [8].) Each of the two pure lipid phases is isotropic, and so are the two component tensors. Then, for the lipid mixture, we consider

$$\mathcal{A}(c) := \mathcal{A}^{\mathrm{E}} + d(c)\big(\mathcal{A}^{\mathrm{C}} - \mathcal{A}^{\mathrm{E}}\big), \tag{4}$$

an elasticity tensor depending on the relative concentration of the mixture, which is simply an interpolation of the two component tensors. The interpolation function $d : [0,1] \to [0,1]$ should be defined such that

$$d(0) = 0, \quad d(1) = 1, \quad d'(0) = 0, \quad d'(1) = 0. \tag{5}$$

With this we have also determined that $c = 0$ corresponds to the elastically softer phase and $c = 1$ corresponds to the elastically stiffer phase. We assume positive definiteness for the individual component tensors, that is, for each $i \in \{\mathrm{E}, \mathrm{C}\}$ we assume the existence of positive numbers $\alpha^i > 0$ such that

$$\alpha^i |X|^2 \leq \mathcal{A}^i X : X, \tag{6}$$

for any symmetric matrix $X \in \mathbb{R}^{N\times N}$. Furthermore, we assume the usual symmetry conditions in linear elasticity theory, that is, for $\mathcal{A}^i = (a^i_{khlm})_{1\leq k,h,l,m\leq N}$, $i \in \{\mathrm{E}, \mathrm{C}\}$, we require

$$a^i_{khlm} = a^i_{khml} = a^i_{hklm} = a^i_{lmkh}. \tag{7}$$

Obviously, the interpolated tensor defined by (4) is also positive definite and fulfils the symmetry condition (7).

By $\bar{\mathcal{E}}(c_\epsilon)$, we denote the eigenstrain. In general, this refers to a strain which is present in the absence of any applied stress. This phenomenon occurs in the presence of inhomogeneities, such as thermal expansions, or as in our case, with phase transitions and leads to self-generated internal stress [9]. The eigenstrain is often referred to as stress-free strain and, just like the elastic material parameters, it may be different for each phase. A natural choice is a multiple of the identity

$$\bar{\mathcal{E}}(c_\epsilon) = e(c_\epsilon)\mathbb{1}, \tag{8}$$

where the scalar-valued function $e(\cdot)$ specifies the eigenstrain behaviour at a particular phase state and $\mathbb{1} \in \mathbb{R}^{N\times N}$ is the second-order identity tensor. According to (8), the eigenstrain is uniform in all directions, which is a common choice, see for example, [9–11].

Assuming that the mechanical equilibrium is reached much faster than the diffusion takes place and using representation (8) for the eigenstrain, then, since

$$\bar{\mathcal{E}}'(c_\epsilon) : \mathcal{S}_\epsilon = e'(c_\epsilon)\mathbb{1} : \mathcal{S}_\epsilon = e'(c_\epsilon)\,\mathrm{tr}(\mathcal{S}_\epsilon),$$

we can write the Cahn–Larché system as follows:

$$\partial_t c_\epsilon = \epsilon^2 \nabla \cdot \Bigg(M\nabla \Bigg(f'(c_\epsilon) - \epsilon^2\lambda\Delta c_\epsilon - e'(c_\epsilon)\,\mathrm{tr}(\mathcal{S}_\epsilon) + \frac{1}{2}(\mathcal{E}(u_\epsilon) - e(c_\epsilon)\mathbb{1}) : \mathcal{A}'(c_\epsilon)(\mathcal{E}(u_\epsilon) - e(c_\epsilon)\mathbb{1}) \Bigg) \Bigg) \quad \text{in } \Omega \times S, \tag{9}$$

$$0 = \nabla \cdot (\mathcal{A}(c_\epsilon)(\mathcal{E}(u_\epsilon) - e(c_\epsilon)\mathbb{1})) \qquad \text{in } \Omega \times S, \tag{10}$$

where $f(c_\epsilon)$ is the free-energy density of the mixture, which we model by a double-well potential,

$$f(c_\epsilon) = c_\epsilon^2(1 - c_\epsilon)^2. \tag{11}$$

The ϵ^2-scaling arises from a nondimensionalisation and we refer to [8] for details.

The boundary conditions are chosen as follows, where n denotes the outer unit normal and τ the unit tangential vector on Γ. At any time, the lipid monolayer remains on the film balance and cannot pass over the edges. Thus, we choose no-flux conditions for the relative concentration c_ϵ and the chemical potential μ_ϵ on the whole boundary Γ,

$$\nabla c_\epsilon \cdot n = 0, \ \ \nabla \mu_\epsilon \cdot n = 0 \quad \text{on } \Gamma \times S, \tag{12}$$

where $\mu_\epsilon = f'(c_\epsilon) - \epsilon^2 \lambda \Delta c_\epsilon - e'(c_\epsilon)\,\mathrm{tr}(\mathcal{S}_\epsilon) + \frac{1}{2}(\mathcal{E}(u_\epsilon) - e(c_\epsilon)\mathbb{1}) : \mathcal{A}'(c_\epsilon)(\mathcal{E}(u_\epsilon) - e(c_\epsilon)\mathbb{1})$. The force applied by the controllable barrier and compressing the lipid monolayer is modelled by applying a boundary force g on Γ_g, hence,

$$\mathcal{S}_\epsilon n = g \quad \text{on } \Gamma_\mathrm{g} \times S. \tag{13}$$

On the opposite boundary part Γ_0, we do not allow for any deformation and hence we require

$$u_\epsilon = 0 \quad \text{on } \Gamma_0 \times S. \tag{14}$$

Furthermore, on the lateral boundary part Γ_s we set

$$u_\epsilon \cdot n = 0 \quad \text{on } \Gamma_\mathrm{s} \times S, \tag{15}$$

and a free-slip condition as well, that is,

$$n \cdot \mathcal{S}_\epsilon \tau = 0 \quad \text{on } \Gamma_\mathrm{s} \times S, \tag{16}$$

These conditions describe that the monolayer cannot expand past the lateral edges and does not adhere there when compressed.

We complete the system with an appropriate initial condition for c_ϵ,

$$c_\epsilon(\,\cdot\,, 0) = c^{\mathrm{in}}(\,\cdot\,) \quad \text{in } \Omega, \tag{17}$$

describing the initial homogeneous relative concentration of the mixture, that is, the initial homogeneous state of the monolayer. Well-posedness is discussed in [8] with reference to [12].

This is the system, which was homogenised formally in [8]. For future reference, we note that the limit system was found to be given by

$$\begin{aligned} \partial_t c_0 = \Delta_y \Big(& f'(c_0) - \lambda \Delta_y c_0 - e'(c_0)\,\mathrm{tr}\,[\mathcal{A}(c_0)(\mathcal{I} + \mathcal{E}_\omega)\,\mathcal{E}_x(u_{n0})] \\ & + \frac{1}{2}(\mathcal{I} + \mathcal{E}_\omega)\,\mathcal{E}_x(u_0) : \mathcal{A}'(c_0)\,(\mathcal{I} + \mathcal{E}_\omega)\,\mathcal{E}_x(u_0) \Big) \quad \text{in } \Omega \times Y \times S, \end{aligned} \tag{18}$$

$$0 = \nabla_x \cdot \left(\mathcal{A}^{\mathrm{hom}}\, \mathcal{E}_x(u_0) \right) \qquad \text{in } \Omega \times S, \tag{19}$$

where Y is the representative unit cell of the microscale and the details of the notation can be found in Section 4.2.

2.2 | Linearisation

For the rigorous homogenisation procedure, we derive a linear (scaled) Cahn–Larché system. Let $c_{n,\epsilon}$ and $u_{n,\epsilon}$ denote general solutions of system (9)–(17), such that

$$c_{n,\epsilon} \in L^\infty(\Omega \times S),\ \ u_{n,\epsilon} \in [L^\infty(\Omega \times S)]^N,\ \ \nabla u_{n,\epsilon} \in [L^\infty(\Omega \times S)]^{N\times N}. \tag{20}$$

Then, we consider $c_{n,\epsilon} + h\,\tilde{c}_\epsilon$ and $u_{n,\epsilon} + h\,\tilde{u}_\epsilon$, for a small $h > 0$ and functions $\tilde{c}_\epsilon,\ \tilde{u}_\epsilon$ having the same multiscale character as described in Section 2.1, to obtain a linear system for $\tilde{c}_\epsilon$ and $\tilde{u}_\epsilon$. Neglecting second-order terms, the linear equations for $\tilde{c}_\epsilon$ and $\tilde{u}_\epsilon$ are as follows.

$$\begin{aligned}\partial_t \tilde{c}_\epsilon = \epsilon^2 \Delta \Big(& f''(c_{n,\epsilon})\tilde{c}_\epsilon - \epsilon^2 \lambda \Delta \tilde{c}_\epsilon - e'(c_{n,\epsilon})\,\mathrm{tr}(\tilde{\mathcal{S}}_\epsilon) - e''(c_{n,\epsilon})\tilde{c}_\epsilon\,\mathrm{tr}(\mathcal{S}_{n,\epsilon}) \\ & + \big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big) : \mathcal{A}'(c_{n,\epsilon})\big(\mathcal{E}(\tilde{u}_\epsilon) - e'(c_{n,\epsilon})\tilde{c}_\epsilon \mathbb{1}\big) \\ & + \frac{1}{2}\big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big) : \mathcal{A}''(c_{n,\epsilon})\tilde{c}_\epsilon\big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big)\Big) \qquad \text{in } \Omega \times S,\end{aligned} \tag{21}$$

$$0 = \nabla \cdot \Big(\mathcal{A}(c_{n,\epsilon})\big(\mathcal{E}(\tilde{u}_\epsilon) - e'(c_{n,\epsilon})\tilde{c}_\epsilon \mathbb{1}\big) + \mathcal{A}'(c_{n,\epsilon})\tilde{c}_\epsilon\big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big)\Big) \qquad \text{in } \Omega \times S, \tag{22}$$

with

$$\mathcal{S}_{n,\epsilon} := \mathcal{A}(c_{n,\epsilon})\big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big)$$

denoting the stress tensor in the nonlinear solutions and

$$\tilde{\mathcal{S}}_\epsilon := \mathcal{A}(c_{n,\epsilon})\big(\mathcal{E}(\tilde{u}_\epsilon) - e'(c_{n,\epsilon})\tilde{c}_\epsilon \mathbb{1}\big) + \mathcal{A}'(c_{n,\epsilon})\tilde{c}_\epsilon\big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big)$$

denoting the linearised stress tensor. The boundary conditions are also linearised, which leads to

$$\begin{aligned} \nabla \tilde{c}_\epsilon \cdot n = 0,\ \ \nabla \tilde{\mu}_\epsilon \cdot n = 0 &\quad \text{on } \Gamma \times S, \\ \tilde{u}_\epsilon = 0 &\quad \text{on } \Gamma_0 \times S, \\ \tilde{\mathcal{S}}_\epsilon\, n = g &\quad \text{on } \Gamma_g \times S, \\ \tilde{u}_\epsilon \cdot n = 0,\ \ \tau \cdot \tilde{\mathcal{S}}_\epsilon\, n = 0 &\quad \text{on } \Gamma_s \times S, \end{aligned} \tag{23}$$

with

$$\begin{aligned}\tilde{\mu}_\epsilon = {} & f''(c_{n,\epsilon})\tilde{c}_\epsilon - \epsilon^2\lambda\Delta\tilde{c}_\epsilon - e'(c_{n,\epsilon})\,\mathrm{tr}(\mathcal{S}_\epsilon) - e''(c_{n,\epsilon})\tilde{c}_\epsilon\,\mathrm{tr}(\mathcal{S}_{n,\epsilon}) + \big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big) : \mathcal{A}'(c_{n,\epsilon})\big(\mathcal{E}(\tilde{u}_\epsilon) - e'(c_{n,\epsilon})\tilde{c}_\epsilon \mathbb{1}\big) \\ & + \frac{1}{2}\big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big) : \mathcal{A}''(c_{n,\epsilon})\tilde{c}_\epsilon\,\big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big)\end{aligned}$$

denoting the linearised chemical potential. In what follows, whenever we talk about the linear Cahn–Larché system, we mean the equations (21) and (22) completed by the boundary conditions (23) and a suitable initial condition for $\tilde{c}_\epsilon$. Furthermore, we drop all tildes for ease of notation and more clarity. From now on, we denote the solutions of the linearised scaled Cahn–Larché system by c_ϵ and u_ϵ, the solutions of the nonlinear scaled Cahn–Larché system (9), (10) completed with the corresponding initial and boundary conditions, are still referred to as $c_{n,\epsilon}$ and $u_{n,\epsilon}$.

Before we analyse the well-posedness of the linearised system, we need to specify further technical details. We consider the interpolated tensor $\mathcal{A}$ defined by (4) with constant tensors $\mathcal{A}^i$, $i \in \{\mathrm{E}, \mathrm{C}\}$ corresponding to two phases C and E, such that there exist $\alpha^i, \beta^i \in \mathbb{R}$, with $0 < \alpha^i < \beta^i$, such that $\mathcal{A}^i \in \mathcal{M}(\alpha^i, \beta^i, Y)$, where $\mathcal{M}(\tilde{\alpha}, \tilde{\beta}, \tilde{\Omega})$ denotes the space of fourth–order tensors, bounded by positive constants $\tilde{\alpha}$ and $\tilde{\beta}$ (see (6)), fulfil the symmetry condition (7) and are periodic on a rectangular domain $\tilde{\Omega} \subset \mathbb{R}^N$. Then, there exist $\alpha, \beta \in \mathbb{R}$, with $0 < \alpha < \beta$, such that

$$\mathcal{A}(c_{n,\epsilon}(\cdot, t)) \in \mathcal{M}(\alpha, \beta, \Omega), \tag{24}$$

for a.e. $t \in S$ and there exist two numbers $\beta', \beta'' > 0$ such that

$$|\mathcal{A}'(c_{n,\epsilon})X| \leq \beta'|X| \quad \text{and} \quad |\mathcal{A}''(c_{n,\epsilon})X| \leq \beta''|X|, \tag{25}$$

for any $X \in \mathbb{R}^{N\times N}$. For the eigenstrain $\bar{\mathcal{E}}(c) = e(c)\mathbb{1}$, we first choose the same type of interpolation as for the elasticity tensor, that is,

$$e(c)\mathbb{1} := (e_{\text{E}} + d(c)(e_{\text{C}} - e_{\text{E}}))\mathbb{1}, \tag{26}$$

with constants $e_{\text{E}}, e_{\text{C}} \in \mathbb{R}$ describing the eigenstrain behaviour of the corresponding lipid phase, and with the interpolation function $d(\cdot)$ defined by (5).

3 | WELL-POSEDNESS OF THE LINEAR CAHN–LARCHÉ SYSTEM

In this section, we examine the linearised Cahn–Larché system (21)–(23). A key point in the analysis, which seems to be of general interest for showing existence of solutions of systems involving equations of different type, is the semidiscretisation in space leading to a system of equations which can be interpreted as a differential–algebraic equation (DAE) in a weak functional-analytical setting. We discuss existence of solutions of DAE systems in this framework before applying it to the linear Cahn–Larché system, which represents a coupled system of partial differential equations of elliptic and parabolic type.

We first fix some assumptions and state the weak formulation. After, we give an a-priori estimate for every $\epsilon > 0$. Furthermore, for every $\epsilon > 0$, we proof the existence and uniqueness of a weak solution using theory about linear differential–algebraic equations. As we want to work in a weak setting, we introduce now some function spaces, specify some assumptions and thus also state the notation we use. Then, we will have all tools available to state the weak formulation of the linear scaled Cahn–Larché system. We denote by

$$(u, v)_\Omega = \int_\Omega u(x)\,v(x)\,\mathrm{d}x \quad \text{and} \quad (u, v)_{\Omega,t} = \int_0^t (u(s)\,, v(s))_\Omega\,\mathrm{d}s$$

the scalar products on $L^2(\Omega)$ and $L^2((0,t), L^2(\Omega))$ for $t \in [0,T]$, respectively, and the abbreviation $\|\cdot\|_\Omega := \|\cdot\|_{L^2(\Omega)}$ for the standard norm on $L^2(\Omega)$ as well as $\|u\|^2_{\Omega,t} := \int_0^t (u(\tau)\,,\,u(\tau))_\Omega\,\mathrm{d}\tau$. Furthermore, we define the function space

$$V(\Omega) := \{v \in H^2(\Omega)\,|\,\nabla v \cdot n = 0 \text{ on } \Gamma\},$$

equipped with the norm

$$\|v\|_{V(\Omega)} := \left(\|v\|^2_\Omega + \|\Delta v\|^2_\Omega\right)^{1/2}, \quad v \in V,$$

which is equivalent to the standard H^2-norm on $V(\Omega)$, and

$$W(\Omega) := \{w \in [H^1(\Omega)]^N \mid w = 0 \text{ on } \Gamma_0,\ w \cdot n = 0 \text{ on } \Gamma_s\},$$

provided with the standard norm on $[H^1(\Omega)]^N$. For the unknown functions, we need the function spaces

$$\mathcal{V}(\Omega) := L^2(S, V(\Omega)) \quad \text{and} \quad \mathcal{W}(\Omega) := L^2(S, W(\Omega)).$$

For matrix-valued functions $A = (a_{ij})_{1\le i,j\le N}, B = (b_{ij})_{1\le i,j\le N} \in [L^2(\Omega)]^{N\times N}$ and $D = (d_{ij})_{1\le i,j\le N} \in [L^\infty(\Omega)]^{N\times N}$, we define the scalar product and the norms

$$(A\,,B)_{F,\Omega} := \int_\Omega A : B\,\mathrm{d}x, \quad \|A\|^2_{F,\Omega} := (A\,,A)_{F,\Omega}, \quad \|D\|^2_{M,\Omega} = N^2 \max_{i,j} \left\|d_{ij}\right\|^2_{L^\infty(\Omega)}.$$

Standard norms of matrix- or vector-valued function are to be understood in an averaged componentwise sense, for example,

$$\|v\|^p_{L^p(\Omega)} = \sum_{i=1}^N \|v_i\|^p_{L^p(\Omega)}, \quad \|w\|^2_{H^1(\Omega)} = \sum_{i=1}^N \|w_i\|^2_{H^1(\Omega)} \quad \text{and} \quad \|M\|^2_\Omega = \sum_{i,j=1}^N \left\|m_{ij}\right\|^2_\Omega,$$

for $v = (v_1, \dots, v_N)^{\mathrm{T}} \in [L^p(\Omega)]^N$ with $p \in [1, \infty)$, $w = (w_1, \dots, w_N)^{\mathrm{T}} \in [H^1(\Omega)]^N$, $M = (m_{ij})_{1\le i,j\le N} \in [L^2(\Omega)]^{N\times N}$.

Assuming for the initial value $c^{\mathrm{in}} \in L^2(\Omega)$ and $g \in L^2(S, [H^{-1/2}(\Gamma_g)]^N)$ for the boundary force, we can state the equations (21) and (22) in their weak form:

Find $(c_\epsilon, u_\epsilon) \in \mathcal{V}(\Omega) \times \mathcal{W}(\Omega)$ with $c_\epsilon(\,\cdot\,, 0) = c^{\mathrm{in}}$, such that

$$\begin{aligned}\langle \partial_t c_\epsilon, \varphi\rangle_{V'(\Omega),V(\Omega)} &- \epsilon^2\big(f''(c_{n,\epsilon})c_\epsilon\,, \Delta\varphi\big)_\Omega + \epsilon^4(\lambda\Delta c_\epsilon\,, \Delta\varphi)_\Omega + \epsilon^2\big(e'(c_{n,\epsilon})\,\mathrm{tr}(\mathcal{S}_\epsilon)\,, \Delta\varphi\big)_\Omega \\ &+ \epsilon^2\big(e''(c_{n,\epsilon})c_\epsilon\,\mathrm{tr}(\mathcal{S}_{n,\epsilon})\,, \Delta\varphi\big)_\Omega - \epsilon^2\big((\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}) : \mathcal{A}'(c_{n,\epsilon})(\mathcal{E}(u_\epsilon) - e'(c_{n,\epsilon})c_\epsilon\mathbb{1})\,, \Delta\varphi\big)_\Omega \\ &- \epsilon^2\frac{1}{2}\big((\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}) : \mathcal{A}''(c_{n,\epsilon})c_\epsilon\,(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1})\,, \Delta\varphi\big)_\Omega = 0, \end{aligned} \tag{27}$$

and

$$\big(\mathcal{A}(c_{n,\epsilon})\,(\mathcal{E}(u_\epsilon) - e'(c_{n,\epsilon})c_\epsilon\mathbb{1}), \mathcal{E}(\psi)\big)_{F,\Omega} + \big(\mathcal{A}'(c_{n,\epsilon})\,c_\epsilon\,(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}), \mathcal{E}(\psi)\big)_{F,\Omega} - \langle g, \psi\rangle_{\Gamma_g} = 0, \tag{28}$$

for any $(\phi, \psi) \in V(\Omega) \times W(\Omega)$ and almost every $t \in S$.

Before beginning with the existence analysis of this system, we show that the time derivative of a function c_ϵ satisfying equation (27) is really an element of $L^2(S, V'(\Omega))$.

Proposition 1. *For every $\epsilon > 0$ and functions $(c_\epsilon, u_\epsilon) \in \mathcal{V}(\Omega) \times \mathcal{W}(\Omega)$ satisfying the equations (27) and (28), it holds that $\partial_t c_\epsilon \in L^2(S, V'(\Omega))$.*

Proof. For almost every $t \in S$ we have

$$\begin{aligned}\|\partial_t c_\epsilon\|_{V'(\Omega)} &= \sup_{\substack{\varphi\in V(\Omega),\\ \|\varphi\|_{V(\Omega)}=1}} \langle \partial_t c_\epsilon, \varphi\rangle_{V'(\Omega),V(\Omega)} \\ &= \sup_{\substack{\varphi\in V(\Omega),\\ \|\varphi\|_{V(\Omega)}=1}} \Big\{\epsilon^2\big(f''(c_{n,\epsilon})\,c_\epsilon, \Delta\varphi\big)_\Omega - \epsilon^4\lambda(\Delta c_\epsilon\,, \Delta\varphi)_\Omega \\ &\quad - \epsilon^2\big(e'(c_{n,\epsilon})\,\mathrm{tr}(\mathcal{S}_\epsilon)\,, \Delta\varphi\big)_\Omega - \epsilon^2\big(e''(c_{n,\epsilon})c_\epsilon\,\mathrm{tr}(\mathcal{S}_{n,\epsilon})\,, \Delta\varphi\big)_\Omega \\ &\quad + \epsilon^2\big((\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}) : \mathcal{A}'(c_{n,\epsilon})(\mathcal{E}(u_\epsilon) - e'(c_{n,\epsilon})c_\epsilon\mathbb{1})\,, \Delta\varphi\big)_\Omega \\ &\quad + \frac{1}{2}\epsilon^2\big((\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}) : \mathcal{A}''(c_{n,\epsilon})c_\epsilon\,(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1})\,, \Delta\varphi\big)_\Omega\Big\}. \end{aligned} \tag{29}$$

First, we take a closer look at the trace terms for which we use the identity $\mathrm{tr}\,\mathcal{S}_\epsilon = \mathcal{S}_\epsilon : \mathbb{1}$. For the norm of the linearised stress tensor, we get

$$\|\mathcal{S}_\epsilon\|_{F,\Omega} \le \beta\Big(\|\mathcal{E}(u_\epsilon)\|_\Omega + N\big\|e'(c_{n,\epsilon})\big\|_{L^\infty(\Omega)}\|c_\epsilon\|_\Omega\Big) + \beta'\big\|\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big\|_{M,\Omega}\|c_\epsilon\|_\Omega, \tag{30}$$

where we used the inequalities of Minkowski and Hölder as well as the boundedness of $\mathcal{A}$ and its first derivative. With (30), we obtain

$$\begin{aligned} \epsilon^2\left(e'(c_{n,\epsilon})\,S_\epsilon\,,\,\mathbb{1}\Delta\varphi\right)_{F,\Omega} \leq \epsilon^2\|e'(c_{n,\epsilon})\|_{L^\infty(\Omega)}\left(\beta\|\mathcal{E}(u_\epsilon)\|_\Omega + N\|e'(c_{n,\epsilon})\|_{L^\infty(\Omega)}\|c_\epsilon\|_\Omega\right)N\|\Delta\varphi\|_\Omega \\ + \epsilon^2\|e'(c_{n,\epsilon})\|_{L^\infty(\Omega)}\beta'\|\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\|_{M,\Omega}\|c_\epsilon\|_\Omega N\|\Delta\varphi\|_\Omega. \end{aligned} \tag{31}$$

Analogous to this, we get

$$\epsilon^2\left(e''(c_{n,\epsilon})\,c_\epsilon\,S_{n,\epsilon}\,,\,\mathbb{1}\Delta\varphi\right)_{F,\Omega} \leq \epsilon^2\|e''(c_{n,\epsilon})\|_{L^\infty(\Omega)}\|c_\epsilon\|_\Omega\beta\|\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1})\|_{M,\Omega}N\|\Delta\varphi\|_\Omega. \tag{32}$$

For the next term, applying the same inequalities as above, we get

$$\begin{aligned} &\epsilon^2\int_\Omega\left(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\right)\,:\,\mathcal{A}'(c_{n,\epsilon})\left(\mathcal{E}(u_\epsilon) - e'(c_{n,\epsilon})c_\epsilon(x,t)\mathbb{1}\right)\Delta\varphi(x)\,\mathrm{d}x \\ &\leq \epsilon^2\|\Delta\varphi\|_\Omega\|\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\|_{M,\Omega}\beta'\left(\|\mathcal{E}(u_\epsilon)\|_\Omega + N\|e'(c_{n,\epsilon})\|_{L^\infty(\Omega)}\|c_\epsilon\|_\Omega\right) \end{aligned} \tag{33}$$

and the last term of the right-hand side of (29) we estimate in an analogous way. The remaining first two terms in (29) are estimated using Hölder's inequality. Altogether we obtain

$$\|\partial_t c_\epsilon\|_{V'(\Omega)} \leq \sup_{\substack{\varphi\in V(\Omega),\\ \|\varphi\|_{V(\Omega)}=1}}\left\{C\left(\epsilon^4\|\Delta c_\epsilon\|_\Omega + \epsilon^2\|\mathcal{E}(u_\epsilon)\|_\Omega + \epsilon^2\|c_\epsilon\|_\Omega\right)\|\varphi\|_{V(\Omega)}\right\},$$

for a constant $C > 0$. Because $c_\epsilon \in \mathcal{V}(\Omega)$, $u_\epsilon \in \mathcal{W}(\Omega)$ and $\|\varphi\|_{V(\Omega)} = 1$, the right-hand side is bounded for almost every $t \in S$. □

3.1 | A-priori estimate

We begin the existence analysis with showing an a-priori estimate necessary for the existence proof. In order to enable the limit passage in the sense of two-scale convergence in Section 4, the constants are carefully tracked in an ϵ-independent way.

Proposition 2 (Boundedness). *There exists a constant $C > 0$, independent of ϵ, such that*

$$\|c_\epsilon\|^2_\Omega + \|\epsilon\nabla c_\epsilon\|^2_{\Omega,t} + \left\|\epsilon^2\Delta c_\epsilon\right\|^2_{\Omega,t} + \|u_\epsilon\|^2_{H^1(\Omega),t} \leq C, \tag{34}$$

for almost every $t \in S$.

Proof. Starting with equation (28), we show first that u_ϵ is bounded in $\mathcal{W}(\Omega)$ if c_ϵ is bounded in $L^2(S, L^2(\Omega))$. Therefore, we use u_ϵ as test function in (28) and after rearranging terms, we get

$$\left(\mathcal{A}(c_{n,\epsilon})\mathcal{E}(u_\epsilon)\,,\,\mathcal{E}(u_\epsilon)\right)_{F,\Omega} = \left(\mathcal{A}(c_{n,\epsilon})e'(c_{n,\epsilon})c_\epsilon\mathbb{1}\,,\,\mathcal{E}(u_\epsilon)\right)_{F,\Omega} + \left(\mathcal{A}'(c_{n,\epsilon})c_\epsilon(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1})\,,\,\mathcal{E}(u_\epsilon)\right)_{F,\Omega} + \langle g\,,u_\epsilon\rangle_{\Gamma_g}. \tag{35}$$

We estimate the left-hand side of (35) by using the positive definiteness of $\mathcal{A}$ as well as Korn's and Poincaré's inequalities, which gives

$$\left(\mathcal{A}(c_{n,\epsilon})\,\mathcal{E}(u_\epsilon), \mathcal{E}(u_\epsilon)\right)_\Omega \geq \alpha\|\mathcal{E}(u_\epsilon)\|^2_{F,\Omega} \geq \frac{\alpha}{2}\|\mathcal{E}(u_\epsilon)\|^2_{F,\Omega} + \frac{\alpha}{2}C\|u_\epsilon\|^2_{H^1(\Omega)}. \tag{36}$$

Next we consider the terms on the right-hand side of (35), which can be estimated by Hölder's and Young's inequalities and using the boundedness of $\mathcal{A}$. We get

$$\big(\mathcal{A}(c_{n,\epsilon})e'(c_{n,\epsilon})c_\epsilon \mathbb{1}\,,\mathcal{E}(u_\epsilon)\big)_{F,\Omega} \leq \frac{1}{2\delta}\beta^2N^2\|e'(c_{n,\epsilon})\|^2_{L^\infty(\Omega)}\|c_\epsilon\|^2_\Omega + \frac{\delta}{2}\|\mathcal{E}(u_\epsilon)\|^2_{F,\Omega} \tag{37}$$

and

$$\big(\mathcal{A}'(c_{n,\epsilon})(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1})c_\epsilon\,,\mathcal{E}(u_\epsilon)\big)_{F,\Omega} \leq \frac{1}{2\delta}(\beta')^2\|\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\|^2_{M,\Omega}\|c_\epsilon\|^2_\Omega + \frac{\delta}{2}\|\mathcal{E}(u_\epsilon)\|^2_{F,\Omega}, \tag{38}$$

for a $\delta > 0$. For the boundary term, we obtain

$$\langle g, u_\epsilon\rangle_{\Gamma_g} \leq c_\gamma\, \|g\|_{H^{-1/2}(\Gamma_g)}\|u_\epsilon\|_{H^1(\Omega)} \leq \frac{c_\gamma}{2\delta}\|g\|^2_{H^{-1/2}(\Gamma_g)} + \frac{c_\gamma\delta}{2}\|u_\epsilon\|^2_{H^1(\Omega)}, \tag{39}$$

where $c_\gamma > 0$ is the constant from the trace inequality and further, we used Young's inequality. Combining now (36) – (39), we absorb the terms with u_ϵ and $\mathcal{E}(u_\epsilon)$ and get

$$\begin{aligned}&(\alpha - 2\delta)\|\mathcal{E}(u_\epsilon)\|^2_{F,\Omega} + (C\alpha - c_\gamma\delta)\|u_\epsilon\|^2_{H^1(\Omega)}\\ &\leq \frac{1}{\delta}\Big(\beta^2N^2\|e'(c_{n,\epsilon})\|^2_{L^\infty(\Omega)} + (\beta')^2\|\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\|^2_{M,\Omega}\Big)\|c_\epsilon\|^2_\Omega + \frac{c_\gamma}{\delta}\|g\|^2_{H^{-1/2}(\Gamma_g)}.\end{aligned} \tag{40}$$

Integration with respect to time from 0 to t, with $t \in (0,T]$ and δ small enough, gives

$$\|\mathcal{E}(u_\epsilon)\|^2_{\Omega,t} + \|u_\epsilon\|^2_{H^1(\Omega),t} \leq C\|c_\epsilon\|^2_{\Omega,t} + C_1\|g\|^2_{H^{-1/2}(\Gamma_g),t}, \tag{41}$$

for some constants $C, C_1 > 0$ independent of ϵ. Hence, $\mathcal{E}(u_\epsilon)$ and u_ϵ are bounded in $L^2(S,[L^2(\Omega)]^{N\times N})$ and $\mathcal{W}(\Omega)$, respectively, if c_ϵ is bounded in $L^2(S,L^2(\Omega))$. To show the latter, we use c_ϵ as test function in (27) and integrate from 0 to t, $t \in (0,T]$, which yields

$$\begin{aligned}\frac{1}{2}\|c_\epsilon\|^2_\Omega + \epsilon^4\lambda\|\Delta c_\epsilon\|^2_{\Omega,t} = {}& \frac{1}{2}\|c_\epsilon(0)\|^2_\Omega + \epsilon^2\big(f''(c_{n,\epsilon})c_\epsilon\,,\Delta c_\epsilon\big)_{\Omega,t}\\ &- \epsilon^2\big(e'(c_{n,\epsilon})\,\mathrm{tr}(\mathcal{S}_\epsilon)\,,\Delta c_\epsilon\big)_{\Omega,t} - \epsilon^2\big(e''(c_{n,\epsilon})c_\epsilon\,\mathrm{tr}(\mathcal{S}_{n,\epsilon})\,,\Delta c_\epsilon\big)_{\Omega,t}\\ &+ \epsilon^2\big((\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1})\,:\,\mathcal{A}'(c_{n,\epsilon})(\mathcal{E}(u_\epsilon) - e'(c_{n,\epsilon})c_\epsilon\mathbb{1})\,,\Delta c_\epsilon\big)_{\Omega,t}\\ &+ \epsilon^2\frac{1}{2}\big((\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1})\,:\,\mathcal{A}''(c_{n,\epsilon})c_\epsilon\,(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1})\,,\Delta c_\epsilon\big)_{\Omega,t}.\end{aligned} \tag{42}$$

We estimate the terms on the right-hand side of (42) successively, using Hölder's and Young's inequalities. The first one gives

$$\epsilon^2\big(f''(c_{n,\epsilon})c_\epsilon\,,\Delta c_\epsilon\big)_{\Omega,t} \leq \frac{1}{2\delta}\|f''(c_{n,\epsilon})\|^2_{L^\infty(\Omega),t}\|c_\epsilon\|^2_{\Omega,t} + \frac{\delta}{2}\epsilon^4\|\Delta c_\epsilon\|^2_{\Omega,t}.$$

Similar to (31), we treat the terms including the traces of the stress tensors. With (30) and Young's inequality, we get

$$\epsilon^2\big(e'(c_{n,\epsilon})\,\mathrm{tr}(\mathcal{S}_\epsilon)\,,\Delta c_\epsilon\big)_{\Omega,t} \leq \frac{1}{2\delta}C\Big(\|\mathcal{E}(u_\epsilon)\|^2_{\Omega,t} + \|c_\epsilon\|^2_{\Omega,t}\Big) + \epsilon^4\frac{\delta}{2}N\|\Delta c_\epsilon\|^2_{\Omega,t}$$

and

$$\epsilon^2\big(e''(c_{n,\epsilon})c_\epsilon\,\mathrm{tr}(\mathcal{S}_{n,\epsilon})\,,\Delta c_\epsilon\big)_{\Omega,t} \leq \frac{2}{\delta}\|e''(c_{n,\epsilon})\|^2_{L^\infty(\Omega),t}\|c_\epsilon\|^2_{\Omega,t}\|\mathcal{S}_{n,\epsilon}\|^2_{M,\Omega,t} + \epsilon^4\frac{\delta}{2}N\|\Delta c_\epsilon\|^2_{\Omega,t}.$$

For the last two terms from the right-hand side of (42) we obtain

$$\epsilon^2\big((\mathcal{E}(u_{n,\epsilon})-e(c_{n,\epsilon})\mathbb{1}) : \mathcal{A}'(c_{n,\epsilon})(\mathcal{E}(u_\epsilon)-e'(c_{n,\epsilon})c_\epsilon\mathbb{1}), \Delta c_\epsilon\big)_{\Omega,t}$$
$$\leq \frac{1}{2\delta}\|\mathcal{E}(u_{n,\epsilon})-e(c_{n,\epsilon})\mathbb{1}\|^2_{M,\Omega,t}(\beta')^2\, 2\Big(\|\mathcal{E}(u_\epsilon)\|^2_{\Omega,t}+N^2\|e'(c_{n,\epsilon})\|^2_{L^\infty(\Omega),t}\|c_\epsilon\|^2_{\Omega,t}\Big)+\frac{\delta}{2}\epsilon^4\|\Delta c_\epsilon\|^2_{\Omega,t}$$

and

$$\epsilon^2\frac{1}{2}\big((\mathcal{E}(u_{n,\epsilon})-e(c_{n,\epsilon})\mathbb{1}) : \mathcal{A}''(c_{n,\epsilon})c_\epsilon\,(\mathcal{E}(u_{n,\epsilon})-e(c_{n,\epsilon})\mathbb{1}), \Delta c_\epsilon\big)_{\Omega,t}$$
$$\leq \frac{1}{2\delta}(\beta'')^2\|c_\epsilon\|^2_{\Omega,t}\|(\mathcal{E}(u_{n,\epsilon})-e(c_{n,\epsilon})\mathbb{1})\|^4_{M,\Omega,t}+\frac{1}{4}\frac{\delta}{2}\epsilon^4\|\Delta c_\epsilon\|^2_{\Omega,t}.$$

Absorbing the $\epsilon^4\|\Delta c_\epsilon\|_{\Omega,t}$-terms, we get

$$\frac{1}{2}\|c_\epsilon(t)\|^2_\Omega+(\lambda-(9/8+N)\delta)\|\epsilon^2\Delta c_\epsilon\|^2_{\Omega,t}\leq\frac{1}{2}\|c_\epsilon(0)\|^2_\Omega+C\|c_\epsilon\|^2_{\Omega,t}+C_1\|\mathcal{E}(u_\epsilon)\|^2_{\Omega,t} \tag{43}$$

for some constants $C, C_1>0$. For $\delta>0$ small enough, the left-hand side of (43) is positive and with (41) we get

$$\|c_\epsilon(t)\|^2_\Omega+\|\epsilon^2\Delta c_\epsilon\|^2_{\Omega,t}\leq C\|c_\epsilon\|^2_{\Omega,t}+\tilde{C}\Big(\|c_\epsilon(0)\|^2_\Omega+\|g\|^2_{H^{-1/2}(\Gamma_g),t}\Big)$$

for some constants $C,\tilde{C}>0$, which do not depend on ϵ. Now, applying Gronwall's inequality we obtain

$$\|c_\epsilon(t)\|^2_\Omega+\|\epsilon^2\Delta c_\epsilon\|^2_{\Omega,t}\leq C\left(\left\|c^{\text{in}}\right\|^2_\Omega+\|g\|^2_{H^{-1/2}(\Gamma_g),t}\right) \tag{44}$$

for a constant $C>0$ independent of ϵ. Due to the regularity assumptions on g and the initial data c^{in}, the right-hand side of (44) is bounded for a.e. $t\in S$. Since $c_\epsilon(t)$ and $\epsilon^2\Delta c_\epsilon(t)$ are bounded in $L^2(\Omega)$ for a.e. $t\in S$, the scaled gradient $\epsilon\nabla c_\epsilon(t)$ is bounded in $[L^2(\Omega)]^N$, for a.e. $t\in S$, since

$$-\int_\Omega c_\epsilon\,\epsilon^2\Delta c_\epsilon\,dx=\int_\Omega\epsilon^2\,(\nabla c_\epsilon)^2\,dx-\int_{\partial\Omega}c_\epsilon\,\epsilon^2\,\nabla c_\epsilon\cdot n\,d\sigma=\int_\Omega(\epsilon\nabla c_\epsilon)^2\,dx=\|\epsilon\nabla c_\epsilon\|^2_\Omega\geq 0,$$

whereby the boundary integral vanishes because of the no-flux condition $\nabla c_\epsilon\cdot n=0$ on $\partial\Omega$. Then, with

$$\left|\int_\Omega c_\epsilon\,\epsilon^2\Delta c_\epsilon\,dx\right|\leq\|c_\epsilon\|_\Omega\,\|\epsilon^2\Delta c_\epsilon\|_\Omega,$$

it follows

$$\|\epsilon\nabla c_\epsilon\|^2_\Omega\leq\|c_\epsilon\|_\Omega\,\|\epsilon^2\Delta c_\epsilon\|_\Omega. \tag{45}$$

Integration with respect to time then gives the desired result. Estimates (41) and (44) now finally yield the boundedness of u_ϵ in $\mathcal{W}(\Omega)$,

$$\|u_\epsilon\|^2_{H^1(\Omega),t}\leq C\left(\left\|c^{\text{in}}\right\|^2_\Omega+\|g\|^2_{H^{-1/2}(\Gamma_g),t}\right). \tag{46}$$

Altogether we finally obtain

$$\|c_\epsilon\|^2_\Omega+\|\epsilon\nabla c_\epsilon\|^2_{\Omega,t}+\|\epsilon^2\Delta c_\epsilon\|^2_{\Omega,t}+\|\mathcal{E}(u_\epsilon)\|^2_{\Omega,t}+\|u_\epsilon\|^2_{H^1(\Omega),t}\leq C\left(\left\|c^{\text{in}}\right\|^2_\Omega+\|g\|^2_{H^{-1/2}(\Gamma_g),t}\right) \tag{47}$$

for a constant $C>0$, which does not depend on ϵ. □

3.2 | Existence of weak solutions

In this subsection, we show the existence and uniqueness of a weak solution of the considered linear system. The proof is provided by a Galerkin approximation. Since the finite-dimensional system that is created in the course of this represents a linear differential–algebraic equation (DAE), we first introduce some aspects of general theory about solvability of linear DAEs in a weak setting.

3.2.1 | Existence of weak solutions of linear DAEs

We consider now differential–algebraic equations of the form

$$A(t)(D(t)u(t))' + B(t)u(t) = q(t), \tag{48}$$

with matrices $A \in L^\infty((t_0,T),\mathbb{R}^{n\times m})$, $D \in L^\infty((t_0,T),\mathbb{R}^{m\times n})$, $B \in L^\infty((t_0,T),\mathbb{R}^{n\times n})$ and a right-hand side $q \in L^2((t_0,T),\mathbb{R}^n)$. The matrix D specifies the differentiable part of u.

In [13], the author studies coupled systems of partial differential and differential–algebraic equations in Hilbert spaces, so-called abstract differential–algebraic systems of the type (48) but with matrices which are continuous in time. Among other things, the unique solvability is proven by use of a Galerkin method. In what follows, we summarise some results of the theory of linear differential–algebraic equations concerning existence and uniqueness of solutions of linear DAEs according to [13] and extend them to matrices with L^∞ time regularity as in (48). The concept is based on decoupling the DAE into a dynamic part, which represents an ordinary differential equation, and an algebraic part. The first definition tells us when the matrices $A(t)$ and $D(t)$ are well matched in a certain way. This is important when decoupling a system as stated above into a dynamic and an algebraic part.

Definition 1 (Properly stated leading term). A DAE of the form (48) is said to have a *properly stated leading term* if

(i) the coefficient matrices $A(t)$ and $D(t)$ fulfil

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^m \tag{49}$$

for a.e. $t \in (t_0,T)$ and

(ii) there exists a projector

$$R : (t_0,T) \to L(\mathbb{R}^m,\mathbb{R}^m)$$

such that

$$\operatorname{im} R(t) = \operatorname{im} D(t), \quad \ker R(t) = \ker A(t), \tag{50}$$

for a.e. $t \in (t_0,T)$ and its derivative with respect to time, R', exists and is bounded almost everywhere.

Remark 1. The projector function $R(t)$ from the definition above realises the decomposition (49). For the Cahn–Larché system, R is a constant matrix. In general, it holds

$$\operatorname{im} A(t)D(t) = \operatorname{im} A(t)R(t) = \operatorname{im} A(t),$$
$$\ker A(t)D(t) = \ker R(t)D(t) = \ker D(t).$$

Moreover, on the subspace $\operatorname{im} D(t)$ the projector acts like the identity, that is,

$$R(t)D(t)x = D(t)x, \quad x \in \mathbb{R}^n, \tag{51}$$

and further, it holds that

$$A(t)R(t)x = A(t)x, \quad x \in \mathbb{R}^m, \tag{52}$$

since $0 = A(t)x_k = A(t)R(t)x_k$ for all $x_k \in \ker A(t) = \ker R(t)$ and $A(t)R(t)x_i = A(t)x_i$ for all $x_i \in \operatorname{im} D(t)$ due to (51).

Next, we present an index concept for the considered linear DAE. Laxly spoken, the index of a DAE indicates how much it differs from an ordinary differential equation. Following [13, 14], we introduce a projector-based index, which is compatible with working in a weak setting. As the Cahn–Larché system fits this index concept with index $\mu = 1$, we only consider this case and, moreover, we adjust it from [13] to matrices with L^∞ time regularity. As we will see later, the decoupling of the linear DAE, which is based on this index concept, is based on the decomposition of $\mathbb{R}^n$ realised by projectors. For the sake of notational simplicity, from now on we drop the time argument t from the matrices.

Definition 2 (Index $\mu = 1$). A DAE of the form (48) with properly stated leading term has the index $\mu = 1$ if there exists a matrix-valued function $G_1 = G_0 + BQ_0$, where $G_0 = AD$ and Q_0, such that

(i) Q_0 is a projector onto $\ker G_0$ for a.e. $t \in (t_0, T)$,
(ii) G_i has constant rank $r_i > 0$ a.e. on (t_0, T) for $i = 0, 1$, with $r_0 < r_1 = n$.

Let D^- denote the reflexive generalised inverse of D, that is,

$$D^- D D^- = D^-, \quad D D^- D = D.$$

To determine D^- uniquely we set

$$D D^- = R, \quad D^- D = P_0, \tag{53}$$

where R is the projector from Definition 1 and $P_0 = I - Q_0$.

Now we can state the existence result we want to work with, which was proved by [13] for the case of continuous matrices A, D and B. We adapt the existence result to our case closely following the ideas of [13]. For the decoupling of the DAE into its dynamic and its algebraic part, we refer to [15].

Theorem 1 Existence of a unique solution. *An initial-value problem of the form*

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \tag{54}$$

$$D(t_0)x(t_0) = z_0 \in \operatorname{im} D(t_0), \tag{55}$$

with $q \in L^2((t_0, T), \mathbb{R}^n)$ and index $\mu = 1$ has a unique solution $x \in L^2((t_0, T), \mathbb{R}^n)$ such that $Dx \in C([t_0, T], \mathbb{R}^m)$ and Dx is differentiable almost everywhere. The equation (54) holds for a.e. $t \in (t_0, T)$ and there exists a constant $C > 0$ such that

$$\|x\|_{L^2((t_0,T))} + \|Dx\|_{C([t_0,T])} + \left\|(Dx)'\right\|_{L^2((t_0,T))} \leq C\Big(\|z_0\| + \|q\|_{L^2((t_0,T))}\Big). \tag{56}$$

Proof. Due to the index 1 property of (54), the matrix

$$G_1 = AD + BQ_0$$

has constant rank for all $t \in [t_0, T]$ and, hence, its inverse G_1^{-1} exists. For any $x \in \mathbb{R}^m$, it holds that

$$\begin{aligned} G_1 P_0 x =& (AD + BQ_0)P_0 x = (AD + BQ_0)(I - Q_0)x \\ =& ADx + BQ_0 x - ADQ_0 x - BQ_0^2 x = ADx, \end{aligned} \tag{57}$$

since Q_0 is a projection onto $\ker AD = \ker D$ and thus $ADQ_0x = 0$ and $Q_0^2 = Q_0$. With (52) and (57), we write the leading term of (54) as follows

$$A(Dx)' = AR(Dx)' = ADD^-(Dx)' = G_1P_0D^-(Dx)'. \tag{58}$$

Next, we write

$$Bx = BIx = BP_0x + BQ_0x = BP_0x + (AD + BQ_0)Q_0x = BP_0x + G_1Q_0x, \tag{59}$$

for any $x \in \mathbb{R}^m$, since Q_0 is a projection onto $\ker AD$. Hence, using (58) and (59), we write the DAE (54) as

$$G_1P_0D^-(Dx)' + BP_0x + G_1Q_0x = q. \tag{60}$$

Multiplying (60) by G_1^{-1} from the left, we obtain

$$P_0D^-(Dx)' + G_1^{-1}BP_0x + Q_0x = G_1^{-1}q. \tag{61}$$

Now, we multiply (61) with D from the left and get

$$DP_0D^-(Dx)' + DG_1^{-1}BP_0x = DG_1^{-1}q \tag{62}$$

since Q_0 is a projection onto $\ker D$. Due to the identities $P_0 = D^-D$ and $R = DD^-$, we get $DP_0D^- = DD^-DD^- = R^2 = R$, and equation (62) becomes

$$R(Dx)' + DG_1^{-1}BD^-Dx = DG_1^{-1}q. \tag{63}$$

Using $(Dx)' = (RDx)' = R'Dx + R(Dx)'$, from (63) it follows

$$(Dx)' - R'Dx + DG_1^{-1}BD^-Dx = DG_1^{-1}q. \tag{64}$$

We consider again equation (61) and multiply it now with Q_0 from the left and get

$$Q_0P_0D^-(Dx)' + Q_0G_1^{-1}BD^-Dx + Q_0x = Q_0G_1^{-1}q.$$

Since $Q_0P_0 = Q_0(I - Q_0) = 0$, we obtain

$$Q_0x + Q_0G_1^{-1}BD^-Dx = Q_0G_1^{-1}q. \tag{65}$$

With this, the DAE (54) is split into a dynamic part (64) and an algebraic part (65).

Equation (64), together with the initial condition (55), represents an ordinary differential equation for $y := Dx$ of the form

$$\begin{aligned} y'(t) &= My(t) + b, \quad t \in (t_0, T), \\ y(t_0) &= y_0, \end{aligned} \tag{66}$$

with $M = R' + DG_1^{-1}BD^-$, $M \in L^\infty((t_0, T), \mathbb{R}^{m\times m})$ and $b = DG_1^{-1}q$, $b \in L^2((t_0, T), \mathbb{R}^m)$. The above system satisfies the Carathéodory conditions. Therefore, the initial-value problem (66) has a unique solution $y \in C([t_0, T], \mathbb{R}^m)$ with $y' \in L^2((t_0, T), \mathbb{R}^m)$ such that

$$y(t) = y(t_0) + \int_{t_0}^{T} y'(\tau)\,\mathrm{d}\tau.$$

Furthermore, there exists a constant $C > 0$, such that

$$\|y\|_{C([t_0,T],\mathbb{R}^m)} + \|y'\|_{L^2((t_0,T),\mathbb{R}^m)} \leq C\,(\|y_0\| + \|b\|_{L^2((t_0,T),\mathbb{R}^m)}). \tag{67}$$

Considering the algebraic part, due to the identities $I = Q_0 + P_0$ and $P_0 = D^- D$, from (65) we obtain a representation of a solution of the initial-value problem (54), (55), namely

$$x(t) = D^- y(t) - (Q_0 G_1^{-1} B D^-) y(t) + Q_0 G_1^{-1} q(t), \tag{68}$$

where y is the unique solution of (66). All matrices appearing here are continuous on $[t_0, T]$ and, since $q \in L^2((t_0,T),\mathbb{R}^n)$, we deduce $x \in L^2((t_0,T),\mathbb{R}^n)$. Furthermore, the estimate (56) follows directly from (68) and the estimate (67). □

3.2.2 | Existence of weak solutions of the linear Cahn–Larché system

To prove the existence of a weak solution of the scaled linear Cahn–Larché system, we consider the system in a form, where the influence of the unknowns c_ε and u_ε is separated in both equations. Therefore, we write the equations (27), (28) together with the initial condition as

$$\langle \partial_t c_\varepsilon\,, \varphi\rangle_{V'(\Omega),V(\Omega)} + a_{\mathrm{ch}}(c_\varepsilon\,, \varphi) + b_{\mathrm{ch}}(u_\varepsilon\,, \varphi) = 0, \tag{69}$$

$$a_{\mathrm{m}}(c_\varepsilon\,, \psi) + b_{\mathrm{m}}(u_\varepsilon\,, \psi) = \langle g, \psi\rangle_{\Gamma_{\mathrm{g}}}, \tag{70}$$

$$(c_\varepsilon(\cdot,0), \varphi)_\Omega = (c^{\mathrm{in}}, \varphi)_\Omega, \tag{71}$$

which holds for all $\varphi \in V(\Omega), \psi \in W(\Omega)$ and with bilinear forms $a_{\mathrm{ch}}(\cdot\,,\cdot), b_{\mathrm{ch}}(\cdot\,,\cdot)$ describing the influence of c_ε and u_ε on the extended Cahn–Hilliard equation, respectively, and $a_{\mathrm{m}}(\cdot\,,\cdot), b_{\mathrm{m}}(\cdot\,,\cdot)$ describing the influence of c_ε and u_ε on the mechanical equilibrium equation, respectively.

Theorem 2. *For every fixed $\epsilon > 0$, there exists a unique weak solution*

$$(c_\varepsilon, u_\varepsilon) \in \big(L^\infty(S, L^2(\Omega)) \cap L^2(S, V(\Omega))\big) \times L^2(S, W(\Omega))$$

of system (27), (28), with $\partial_t c_\varepsilon \in L^2(0,T;V(\Omega)')$.

In what follows, we proof this result in four steps using a Galerkin approach and the theory of linear differential–algebraic equations in a weak setting introduced in the previous subsection.

Step 1: Galerkin equations

We consider a Galerkin scheme, that is, finite dimensional subspaces $V_n = \operatorname{span}\{v_1, \dots, v_n\} \subset V$ and $W_m = \operatorname{span}\{w_1, \dots, w_m\} \subset W$, with $\dim V_n = n$, $\dim W_m = m$, such that $\bigcup_{i\in\mathbb{N}} V_i$ and $\bigcup_{j\in\mathbb{N}} W_j$ are dense in V and W, respectively. Furthermore, we choose a sequence c_n^{in} in V_n, which converges strongly to c^{in} in $L^2(\Omega)$. Then, we consider $c_n : [0,T] \to V_n$ and $u_m : [0,T] \to W_m$,

$$c_n(t) = \sum_{i=1}^{n} c_{ni}(t)\, v_i \;\text{ and }\; u_m(t) = \sum_{j=1}^{m} u_{mj}(t)\, w_j, \tag{72}$$

with $v_i \in V_n$, $w_j \in W_m$ for $i \in \{1,\dots,N\}$, $j \in \{1,\dots,M\}$ and coefficient functions c_{ni} and u_{mj}, $1 \leq i \leq n, 1 \leq j \leq m$, to be determined. Using these representations, we consider the Galerkin approximation of system (69), (70),

$$(c_n'(t)\,, v)_\Omega + a_{\mathrm{ch}}(c_n(t)\,, v) + b_{\mathrm{ch}}(u_m(t)\,, v) = 0, \tag{73}$$

$$a_{\mathrm{m}}(c_n(t), w) + b_{\mathrm{m}}(u_m(t), w) = \langle g, w\rangle_{\Gamma_{\mathrm{g}}}, \tag{74}$$

which holds for every $v \in V_n, w \in W_m$ and completed by the initial condition

$$c_n(0) = c_n^{\mathrm{in}}. \tag{75}$$

Let $c_n^{\mathrm{in}} = \sum_{i=1}^n \alpha_{ni} v_i$. Equivalent to this, we consider

$$\begin{aligned} \sum_{i=1}^n c'_{ni}(t)(v_i, v_k)_\Omega + \sum_{i=1}^n c_{ni}(t)\, a_{\mathrm{ch}}(v_i, v_k) + \sum_{j=1}^m u_{mj}(t)\, b_{\mathrm{ch}}(w_j, v_k) = 0, \\ \sum_{i=1}^n c_{ni}(t)\, a_{\mathrm{m}}(v_i, w_l) + \sum_{j=1}^m u_{mj}(t)\, b_{\mathrm{m}}(w_j, w_l) = \langle g, w_l\rangle_{\Gamma_{\mathrm{g}}}, \end{aligned} \tag{76}$$

for $1 \le k \le n$, $1 \le l \le m$ and

$$c_{ni}(0) = \alpha_{ni}, \tag{77}$$

for $1 \le i \le n$.

Proposition 3. *The Galerkin system (73)–(75) has a unique solution (c_n, u_m), where*

$$c_n : [0, T] \to V_n, \quad \textit{and} \quad u_m : [0, T] \to W_m,$$

with

$$c'_n \in L^2(S, V_n) \quad \text{and} \quad c_n(t) = c_n^{\mathrm{in}} + \int_0^T c'_n(s)\,\mathrm{d}s. \tag{78}$$

Proof. System (76), (77) represents a linear differential–algebraic equation of the form (54) with an initial condition (55). According to the previously presented theory about linear DAEs, there exists a unique solution if the differential–algebraic equation has a properly stated leading term and if it has index 1. To show this, we first identify the setting and write the Galerkin equations in the form of an initial-value differential–algebraic system:

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \tag{79}$$

$$D(0)\, x(0) = z_0, \tag{80}$$

with $x \in \mathbb{R}^{n+m}$, $q \in \mathbb{R}^{n+m}$ and $A \in L^\infty(S, \mathbb{R}^{(n+m)\times n})$, $D \in L^\infty(S, \mathbb{R}^{n\times(n+m)})$, $B \in L^\infty(S, \mathbb{R}^{(n+m)\times(n+m)})$. We identify

$$x := (c_{n1}(t), \dots, c_{nn}(t), u_{m1}(t), \dots, u_{mm}(t))^{\mathrm{T}} \in \mathbb{R}^{n+m},$$

and the right-hand side

$$q := (0, \dots 0, q_1(t), \dots, q_m(t))^{\mathrm{T}} \in \mathbb{R}^{n+m},$$

where the non-zero components are defined by $q_l := \langle g, w_l\rangle_{\Gamma_{\mathrm{g}}}$, $1 \le l \le m$. We further identify the matrices

$$A = \left(\frac{\left((v_j, v_i)_\Omega\right)_{1\le i,j\le n}}{0_{m\times n}} \right) \quad \text{and} \quad D = \left(I_n \middle| 0_{n\times m} \right),$$

both constant with respect to time and with $I_n \in \mathbb{R}^{n\times n}$ being the identity matrix and $0_{n\times m} \in \mathbb{R}^{n\times m}$ a matrix only having entries equal to zero. The matrix B corresponds to the elliptic part of the equations and is given by

$$B = \left(\begin{array}{c|c} (a_{ij}^{\mathrm{ch}})_{1\le i,j\le n} & (b_{ij}^{\mathrm{ch}})_{1\le i\le n,1\le j\le m} \\ \hline (a_{ij}^{\mathrm{m}})_{1\le i\le m,1\le j\le n} & (b_{ij}^{\mathrm{m}})_{1\le i,j\le m} \end{array}\right),$$

with $a_{ij}^{\mathrm{ch}} := a_{\mathrm{ch}}(v_j, v_i)$, $b_{ij}^{\mathrm{ch}} := b_{\mathrm{ch}}(w_j, v_i)$, $a_{ij}^{\mathrm{m}} := a_{\mathrm{m}}(v_j, w_i)$ and $b_{ij}^{\mathrm{m}} := b_{\mathrm{m}}(w_j, w_i)$. Finally, we specify the initial value in (80) as

$$z_0 = (\alpha_{n1}, \dots, \alpha_{nn}, 0, \dots, 0) \in \mathbb{R}^{n+m}, \tag{81}$$

Next, we check if the conditions of Theorem 1 are satisfied. Equation (79) has a properly stated leading term. It is $\ker A = \{0\}$, since the stiffness matrix $((v_j, v_i)_\Omega)_{1\le i,j\le n}$ is regular and $\operatorname{im} D = \mathbb{R}^n$. Hence, $\ker A \oplus \operatorname{im} D = \mathbb{R}^n$. Furthermore, we can simply choose $R = I_n$ as constant projector onto $\operatorname{im} D$ along $\ker A$. Notice that $G_0 = AD$ is singular. Let Q_0 be the projection onto the kernel of $G_0 = AD$. If the matrix $G_1 = AD + BQ_0$ is regular the equation (79) has index $\mu = 1$ and hence, there exists a unique solution. We have

$$G_1 = \left(\begin{array}{c|c} \big((v_j, v_i)_\Omega\big)_{1\le i,j\le n} & (b_{ij}^{\mathrm{ch}})_{1\le i\le n,1\le j\le m} \\ \hline 0_{m\times n} & (b_{ij}^{\mathrm{m}})_{1\le i,j\le m} \end{array}\right).$$

Due to the property of the basis functions v_i, $1 \le i \le n$, the matrix $((v_j, v_i)_\Omega)_{1\le i,j\le n}$ is regular. Hence, it is sufficient to show that the matrix $(b_{ij}^{\mathrm{m}})_{1\le i,j\le m}$, which corresponds to the mechanical equation, is regular. This is equivalent to the well-known fact that there exists a unique solution of the Galerkin scheme for the equation of linear elasticity with the applied boundary conditions. Therefore, the differential–algebraic system (79), (80) has index $\mu = 1$ and consequently, according to Theorem 1, there exist a unique solution of the Galerkin equations (76), (77). Thus, there exists a unique solution (c_n, u_m) of the equivalent equations (73), (74), (75), which fulfil (78). □

Step 2: Estimate for approximate solutions

Proposition 4. *There exists a constant $C > 0$, independent of n and m, such that*

$$\|c_n\|_{L^\infty(S,L^2(\Omega))} + \|c_n\|_{L^2(S,V(\Omega))} + \|c_n'\|_{L^2(S,V(\Omega)')} + \|u_m\|_{L^2(S,H^1(\Omega))} \le C. \tag{82}$$

Proof. For fixed $n, m \in \mathbb{N}$ we set $v = c_n$ and $w = u_m$ in (73) and (74). Then the result follows directly from the estimate in § 3.1. □

Step 3: Convergence of approximate solutions

Proposition 5. *There exists a subsequence of the approximated solutions, which converges weakly to a weak solution*

$$(c_\varepsilon, u_\varepsilon) \in L^2(S, V(\Omega)) \times L^2(S, W(\Omega))$$

of (69), (70), (71) with $\partial_t c \in L^2(S, (V(\Omega))')$.

Proof. With the a priori estimates established in Proposition 4, the convergence of the sequence of approximate solutions to the solution of the original system is standard for this linear problem and, thus, further details are omitted. □

Step 4: Uniqueness of the solution

Proposition 6 (Uniqueness). *There exists at most one solution $(c_\varepsilon, u_\varepsilon)$ of system (69), (70), (71).*

Proof. The proof is standard. For two supposedly different pairs of solutions $(c_\epsilon^{(1)}, u_\epsilon^{(1)})$ and $(c_\epsilon^{(2)}, u_\epsilon^{(2)})$ for the same data, the differences $c_\epsilon^{(1)} - c_\epsilon^{(2)}$ and $u_\epsilon^{(1)} - u_\epsilon^{(2)}$ fulfil the equations (69) – (71) with $g \equiv 0$ and $c^{\text{in}} \equiv 0$. Since ϵ is fixed, from (44), we get

$$\left\| c_\epsilon^{(1)}(t) - c_\epsilon^{(2)}(t) \right\|_\Omega^2 + \left\| \Delta c_\epsilon^{(1)} - \Delta c_\epsilon^{(2)} \right\|_{\Omega,t}^2 \leq 0.$$

Therefore, we conclude $\|c_\epsilon^{(1)} - c_\epsilon^{(2)}\|_{V(\Omega),t} = 0$ and hence $c_\epsilon^{(1)} = c_\epsilon^{(2)}$. This, together with (41) implies, that

$$\|u_\epsilon^{(1)} - u_\epsilon^{(2)}\|_{H^1(\Omega),t}^2 \leq \ 0,$$

which provides $u_\epsilon^{(1)} = u_\epsilon^{(2)}$ and which finishes the proof. □

4 | HOMOGENISATION

For the homogenisation process, we use the method of two-scale convergence. We first recall some well-known results and show two extensions necessary for the homogenisation of the linearised Cahn–Larché system. These are then used to upscale the system rigorously.

4.1 | Two-scale convergence

In what follows, we briefly summarise the required essentials of two-scale convergence, going back to Nguetseng [16] and Allaire [17]. Except for the two Propositions proven at the end of this subsection, the results can be found in [17],[18] and [19], which we refer to for more details. Unless stated otherwise, $\Omega \subset \mathbb{R}^N$ is a bounded and open set and $p \in (1, \infty)$ and we set $Y = [0, 1)^N$ as the reference cell for simplicity. Furthermore, whenever we extract a subsequence, for brevity, we always denote it by the same symbol as the sequence itself. We start with the definition of two-scale convergence in $L^p(\Omega)$.

Definition 3 (Two-scale convergence). A function $\phi = \phi(x, y)$ in $L^p(\Omega \times Y)$, which is Y-periodic in y and which satisfies

$$\lim_{\epsilon \to 0} \int_\Omega |\phi(x, \frac{x}{\epsilon})|^p \, \mathrm{d}x = \int_\Omega \int_Y |\phi(x, y)|^p \, \mathrm{d}y \, \mathrm{d}x, \tag{83}$$

is called an admissible test function.

A sequence of functions u_ϵ in $L^p(\Omega)$ is said to two-scale converge to a limit $u_0 \in L^p(\Omega \times Y)$ if

$$\lim_{\epsilon \to 0} \int_\Omega u_\epsilon(x) \phi(x, \frac{x}{\epsilon}) \, \mathrm{d}x = \frac{1}{|Y|} \int_\Omega \int_Y u_0(x, y) \phi(x, y) \, \mathrm{d}y \, \mathrm{d}x$$

for any admissible test function $\phi \in L^q(\Omega \times Y)$, with $1/q + 1/p = 1$. In this case, we write $u_\epsilon \overset{2s.}{\rightharpoonup} u_0$.

A sequence of functions u_ϵ in $L^p(\Omega)$ is said to two-scale converge strongly to a limit $u_0 \in L^p(\Omega \times Y)$ if u_ϵ two-scale converges to u_0 in $L^p(\Omega)$ and

$$\lim_{\epsilon \to 0} \|u_\epsilon\|_{L^p(\Omega)} = \|u_0\|_{L^p(\Omega \times Y)}.$$

Then, we write $u_\epsilon \overset{2s.}{\longrightarrow} u_0$.

Moreover, if the Y-periodic extension of u_0 belongs to $L^p(\Omega, C_\#(Y))$, where the subscript # denotes Y-periodicity, we have

$$\lim_{\epsilon \to 0} \left\| u_\epsilon(x) - u_0(x, \frac{x}{\epsilon}) \right\|_{L^p(\Omega)} = 0.$$

We remark that all admissible test functions two-scale converge strongly by definition. We are interested in criteria which enable us to conclude that a given sequence in $L^p(\Omega)$ is two-scale convergent. The next compactness theorem ensures the existence of a two-scale limit of a sequence bounded in $L^p(\Omega)$ or $W^{1,p}(\Omega)$.

Theorem 3 (Compactness in $L^p(\Omega)$ and $W^{1,p}(\Omega)$). *For each bounded sequence u_ϵ in $L^p(\Omega)$, there exists a subsequence, which two-scale converges to a function $u_0 \in L^p(\Omega \times Y)$.*

Let u_ϵ be a sequence in $W^{1,p}(\Omega)$ such that u_ϵ converges weakly to a limit $u_0 \in W^{1,p}(\Omega)$. Then, u_ϵ two-scale converges to u_0 and there exists a function $u_1 \in L^p(\Omega; W^{1,p}_\#(Y))$ such that, up to a subsequence, ∇u_ϵ two-scale converges to $\nabla u_0 + \nabla_y u_1$.

For dealing with the homogenisation of the linearised Cahn–Larché system, two additional results are required, which we prove in what follows.

Proposition 7 (Compactness of 2nd order derivatives). *Let u_ϵ, $\epsilon\, \partial_{x_i} u_\epsilon$ and $\epsilon^2\, \partial^2_{x_i x_j} u_\epsilon$ all be bounded sequences in $L^p(\Omega)$. Then, there exists a function $u_0 \in L^p(\Omega; W^{2,p}_\#(Y))$, such that, up to a subsequence,*

$$u_\epsilon \overset{2s.}{\rightharpoonup} u_0, \quad \epsilon\, \partial_{x_i} u_\epsilon \overset{2s.}{\rightharpoonup} \partial_{y_i} u_0, \quad \epsilon^2\, \partial^2_{x_i x_j} u_\epsilon \overset{2s.}{\rightharpoonup} \partial^2_{y_i y_j} u_0,$$

for $i, j = 1, \dots, N$.

Proof. From [17] we already know that for sequences u_ϵ and $\epsilon\, \partial_{x_i} u_\epsilon$ bounded in $L^p(\Omega)$, there exists a function $u_0 \in L^p(\Omega; W^{1,p}_\#(Y))$, such that, up to a subsequence, u_ϵ and $\epsilon\, \partial_{x_i} u_\epsilon$ two-scale-converge to u_0 and $\partial_{y_i} u_0$, respectively. Since $\epsilon^2\, \partial^2_{x_i x_j} u_\epsilon$ is also bounded in $L^p(\Omega)$, we can extract a subsequence, still denoted by $\epsilon^2\, \partial^2_{x_i x_j} u_\epsilon$, and there exists a function $w \in L^p(\Omega \times Y)$ such that this subsequence two-scale converges to w, that is,

$$\lim_{\epsilon \to 0} \int_\Omega \epsilon^2\, \partial^2_{x_i x_j} u_\epsilon(x)\, \psi(x, \frac{x}{\epsilon})\, \mathrm{d}x = \frac{1}{|Y|} \int_\Omega \int_Y w(x,y) \psi(x,y)\, \mathrm{d}y\, \mathrm{d}x \tag{84}$$

for any admissible test function ψ. Integrating the left-hand side in (84) by parts and passing to the limit yields

$$\lim_{\epsilon \to 0} - \int_\Omega \epsilon\, \partial_{x_i} u_\epsilon(x)\, [\partial_{y_j} \psi(x, \frac{x}{\epsilon}) + \epsilon \partial_{x_j} \psi(x, \frac{x}{\epsilon})]\, \mathrm{d}x = -\frac{1}{|Y|} \int_\Omega \int_Y \partial_{y_i} u_0(x,y) \partial_{y_j}\, \psi(x,y)\, \mathrm{d}y\, \mathrm{d}x.$$

With (84) we get

$$\lim_{\epsilon \to 0} \int_\Omega \epsilon^2\, \partial^2_{x_i x_j} u_\epsilon(x) \psi(x, \frac{x}{\epsilon})\, \mathrm{d}x = -\frac{1}{|Y|} \int_\Omega \int_Y \partial_{y_i} u_0(x,y) \partial_{y_j} \psi(x,y)\, \mathrm{d}y\, \mathrm{d}x. \tag{85}$$

Integrating the right-hand side of (85) by parts yields the desired result,

$$\lim_{\epsilon \to 0} \int_\Omega \epsilon^2\, \partial^2_{x_i x_j} u_\epsilon(x) \psi(x, \frac{x}{\epsilon})\, \mathrm{d}x = \frac{1}{|Y|} \int_\Omega \int_Y \partial^2_{y_i y_j} u_0(x,y) \psi(x,y)\, \mathrm{d}y\, \mathrm{d}x,$$

and we identify $\partial^2_{y_i y_j} u_0$ as the two-scale limit of $\epsilon^2\, \partial^2_{x_i x_j} u_\epsilon$ and hence, $u_0 \in L^p(\Omega; W^{2,p}_\#(Y))$. □

The next theorem enables to pass to the limit of products of several two-scale convergent sequences. This result is an extension of the well-known result that the product of one strongly two-scale convergent with one weakly two-scale convergent sequence converges towards the product of their two-scale limits in the sense of distributions, see for example, [17, 19].

Proposition 8 (Convergence of products). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $u^{(i)}_\epsilon$ bounded sequences in $L^{p_i}(\Omega)$, for $i \in \{1, \dots, n\}$ and $p^{(i)} \in (1, \infty)$, which two-scale converge strongly to limit functions $u^{(i)}_0 \in L^{p_i}(\Omega \times Y)$, respectively. Then, for* *any bounded sequence w_ϵ in $L^q(\Omega)$, with $q \in (1,\infty)$ such that $\frac{1}{q} + \sum_i \frac{1}{p^{(i)}} \leq 1$, which two-scale converges to $w_0 \in L^q(\Omega \times Y)$, the following convergence holds:*

$$\lim_{\epsilon\to 0} \int_\Omega u_\epsilon^{(1)}(x) \dots u_\epsilon^{(n)}(x)\, w_\epsilon(x)\, \varphi(x)\, \mathrm{d}x = \frac{1}{|Y|} \int_\Omega \int_Y u_0^{(1)}(x,y) \dots u_0^{(n)}(x,y)\, w_0(x,y)\, \varphi(x)\, \mathrm{d}y\, \mathrm{d}x, \tag{86}$$

for every $\varphi \in C^\infty(\Omega)$.

Proof. We show the proof of this result for the product of two strongly and one weakly two-scale converging sequences, that is, $n = 2$. The proof of the general case, for $n > 2$, can be continued successively.

For $i = 1, 2$, let $\varphi_k^{(i)} \in C^\infty(\Omega; C_\#^\infty(Y))$ be sequences such that $\varphi_k^{(i)} \to u_0^{(i)}$ in $L^{p_i}(\Omega \times Y)$ and $\varphi \in C^\infty(\Omega)$. We split the product of the sequences as follows

$$u_\epsilon^{(1)} u_\epsilon^{(2)}\, w_\epsilon = \left(u_\epsilon^{(1)} - \varphi_k^{(1)}\right) u_\epsilon^{(2)} w_\epsilon + \varphi_k^{(1)} u_\epsilon^{(2)} w_\epsilon = \left(u_\epsilon^{(1)} - \varphi_k^{(1)}\right) u_\epsilon^{(2)} w_\epsilon + \varphi_k^{(1)} \left(u_\epsilon^{(2)} - \varphi_k^{(2)}\right) w_\epsilon + \varphi_k^{(1)} \varphi_k^{(2)} w_\epsilon. \tag{87}$$

It is easy to see that this decomposition can be continued successively if further strongly converging consequences are added to the product on left-hand side of (87). We multiply by φ, integrate over Ω and subtract the right-hand side of (86) from both sides and obtain with the triangle inequality

$$\begin{aligned} &\Big| \int_\Omega u_\epsilon^{(1)}(x)\, u_\epsilon^{(2)}(x)\, w_\epsilon(x)\, \varphi(x)\, \mathrm{d}x - \frac{1}{|Y|} \int_\Omega \int_Y u_0^{(1)}(x,y)\, u_0^{(2)}(x,y)\, w_0(x,y)\, \varphi(x)\, \mathrm{d}y\, \mathrm{d}x \Big| \\ &\quad \leq \Big| \int_\Omega \left[u_\epsilon^{(1)}(x) - \varphi_k^{(1)}(x, \tfrac{x}{\epsilon})\right] u_\epsilon^{(2)}(x)\, w_\epsilon(x)\, \varphi(x)\, \mathrm{d}x \Big| + \Big| \int_\Omega \varphi_k^{(1)}(x, \tfrac{x}{\epsilon}) \left[u_\epsilon^{(2)}(x) - \varphi_k^{(2)}(x, \tfrac{x}{\epsilon})\right] w_\epsilon(x)\, \varphi(x)\, \mathrm{d}x \Big| \\ &\qquad + \Big| \int_\Omega \varphi_k^{(1)}(x, \tfrac{x}{\epsilon})\, \varphi_k^{(2)}(x, \tfrac{x}{\epsilon})\, w_\epsilon(x)\, \varphi(x)\, \mathrm{d}x - \frac{1}{|Y|} \int_\Omega \int_Y u_0^{(1)}(x,y)\, u_0^{(2)}(x,y)\, w_0(x,y)\, \varphi(x)\, \mathrm{d}y\, \mathrm{d}x \Big|. \end{aligned} \tag{88}$$

We consider the last term in (88) and pass to the limit, first for $\epsilon \to 0$ and after for $k \to \infty$. We get

$$\lim_{\epsilon\to 0} \int_\Omega \varphi_k^{(1)}(x, \tfrac{x}{\epsilon})\, \varphi_k^{(2)}(x, \tfrac{x}{\epsilon})\, w_\epsilon(x)\, \varphi(x)\, \mathrm{d}x = \frac{1}{|Y|} \int_\Omega \int_Y \varphi_k^{(1)}(x,y)\, \varphi_k^{(2)}(x,y)\, w_0(x,y)\, \varphi(x)\, \mathrm{d}y\, \mathrm{d}x,$$

since w_ϵ two-scale converges to w_0 and $\varphi_k^{(1)} \varphi_k^{(2)} \varphi$ is an admissible test function. Therefore, as k tends to zero, we get

$$\lim_{k\to\infty} \lim_{\epsilon\to 0} \Big| \int_\Omega \varphi_k^{(1)}(x, \tfrac{x}{\epsilon})\, \varphi_k^{(2)}(x, \tfrac{x}{\epsilon})\, w_\epsilon(x)\, \varphi(x)\, \mathrm{d}x - \frac{1}{|Y|} \int_\Omega \int_Y u_0^{(1)}(x,y)\, u_0^{(2)}(x,y)\, w_0(x,y)\, \varphi(x)\, \mathrm{d}y\, \mathrm{d}x \Big| = 0$$

and hence, the third term in (88) vanishes. Using Hölder's inequality, for the first term of the right-hand side we get

$$\begin{aligned} \Big| \int_\Omega \left[u_\epsilon^{(1)}(x) - \varphi_k^{(1)}(x, \tfrac{x}{\epsilon})\right] u_\epsilon^{(2)}(x)\, w_\epsilon(x)\, \varphi(x)\, \mathrm{d}x \Big| &\leq c(\Omega) \sup_{x\in\Omega} |\varphi(x)| \left\| u_\epsilon^{(1)} - \varphi_k^{(1)} \right\|_{L^{p_1}(\Omega)} \left\| u_\epsilon^{(2)} \right\|_{L^{p_2}(\Omega)} \|w_\epsilon\|_{L^q(\Omega)} \\ &\leq C \left\| u_\epsilon^{(1)} - \varphi_k^{(1)} \right\|_{L^{p_1}(\Omega)}, \end{aligned}$$

with constants $c(\Omega) = |\Omega|^{1-\frac{1}{p_1}-\frac{1}{p_2}-\frac{1}{q}}, C > 0$. Here we have used that $u_\epsilon^{(2)}$ and w_ϵ are bounded in $L^{p_2}(\Omega)$ and $L^q(\Omega)$, respectively. For the next term, we get in an analogous way

$$\Big| \int_\Omega \varphi_k^{(1)}(x, \tfrac{x}{\epsilon}) \left[u_\epsilon^{(2)}(x) - \varphi_k^{(2)}(x, \tfrac{x}{\epsilon})\right] w_\epsilon(x)\, \varphi(x)\, \mathrm{d}x \Big| \leq C \left\| u_\epsilon^{(2)} - \varphi_k^{(2)} \right\|_{L^{p_2}(\Omega)}.$$

Since

$$\lim_{k\to\infty}\lim_{\epsilon\to 0}\left\|u_\epsilon^{(1)}-\varphi_k^{(1)}\right\|_{L^{p_1}(\Omega)}=0,\quad \lim_{k\to\infty}\lim_{\epsilon\to 0}\left\|u_\epsilon^{(2)}-\varphi_k^{(2)}\right\|_{L^{p_2}(\Omega)}=0 \tag{89}$$

for which we refer to [19] where (89) is shown in full detail using the Clarkson inequalities, the result is proven. □

For the sake of clarity, we have not included time here, since time only plays the role of a parameter in the homogenisation process. Therefore, all results can be adapted to sequences of functions also depending on time, cf. [18, 20, 21].

4.2 | Homogenisation limit passage

In order to pass to the limit in each term of the linear system (27), (28), we have to deal with the convergence behaviour of the sequences of the solutions of the nonlinear system. Considering the linearised Cahn–Larché system, there are several products of sequences we have to be aware of. Regarding the eigenstrain, we restrict from now on to a linear interpolation, that is,

$$\bar{\mathcal{E}}(c_{n,\epsilon})=e(c_{n,\epsilon})\mathbb{1}\quad\text{with}\quad e(c_{n,\epsilon})=e_{\mathrm{E}}+c_{n,\epsilon}\,(e_{\mathrm{C}}-e_{\mathrm{E}}). \tag{90}$$

Note, that the derivative $e'(c_{n,\epsilon})=e_{\mathrm{C}}-e_{\mathrm{E}}$ is now only a constant and to emphasise this we write e' instead of $e'(c_{n,\epsilon})$. With regard to (4), (11) and (90), we require the following assumptions concerning the given sequences of the solutions of the nonlinear system, which are the basis of the linearisation in §2.2.

- The initial value is of the form $c^{\mathrm{in}}=c^{\mathrm{in}}(x,y)$, Y-periodic with respect to the second variable.
- There exists a function $c_{n,0}\in L^\infty(\Omega\times Y\times S)$ such that at least a subsequence of $c_{n,\epsilon}$, two-scale converges strongly to $c_{n,0}$ in $L^6(\Omega\times S)$.
- There exist functions $u_{n,0}\in(L^\infty(\Omega\times S))^N$ and $u_{n,1}\in(L^\infty(\Omega\times Y\times S))^N$ such that, up to a subsequence, $(u_{n,\epsilon})_i$ (where $(u_{n,\epsilon})_i$ denotes the i-th component of $u_{n,\epsilon}$) two-scale converges strongly to $(u_{n,0})_i$ in $L^6(\Omega\times S)$ and $\partial_j(u_{n,\epsilon})_i$ two-scale converges strongly to $\partial_{x_j}(u_{n,0})_i+\partial_{y_j}(u_{n,1})_i$ in $L^6(\Omega\times S)$.

The following theorem states the homogenisation result.

Theorem 4. *There exist functions $c_0\in L^2(\Omega\times S;H^2_\#(Y))$, $u_0\in L^2(S;W(\Omega))$ and $u_1\in L^2(\Omega\times S;[H^1_\#(Y)]^N)$ such that the sequences c_ϵ and u_ϵ of the solutions of (21) and (22) two-scale converge to c_0 and u_0, respectively. Furthermore, the sequence $\mathcal{E}(u_\epsilon)$ two-scale converges to $\mathcal{E}_x(u_0)+\mathcal{E}_y(u_1)$ and the sequence $\epsilon^2\Delta c_\epsilon$ two-scale converges to $\Delta_{yy}c_0$. The triple of the limit functions (c_0,u_0,u_1) is the unique solution of the following homogenised system:*

$$\begin{aligned}\partial_t c_0=\Delta_{yy}\Big(&f''(c_{n,0})c_0-\lambda\Delta_{yy}c_0-e'\,\mathrm{tr}(S_0)\\&+\big(\mathcal{E}_x(u_{n,0})+\mathcal{E}_y(u_{n,1})-e(c_{n,0})\mathbb{1}\big):\mathcal{A}'(c_{n,0})\big(\mathcal{E}_x(u_0)+\mathcal{E}_y(u_1)-e'c_0\mathbb{1}\big)\\&+\frac{1}{2}\big(\mathcal{E}_x(u_{n,0})+\mathcal{E}_y(u_{n,1})-e(c_{n,0})\mathbb{1}\big):\mathcal{A}''(c_{n,0})c_0\,\big(\mathcal{E}_x(u_{n,0})+\mathcal{E}_y(u_{n,1})-e(c_{n,0})\mathbb{1}\big)\Big)\quad\text{in }\Omega\times Y\times S,\end{aligned} \tag{91}$$

$$0=-\nabla_y\cdot\big(\mathcal{A}(c_{n,0})\big(\mathcal{E}_x(u_0)+\mathcal{E}_y(u_1)-e'c_0\mathbb{1}\big)+\mathcal{A}'(c_{n,0})c_0\big(\mathcal{E}_x(u_{n,0})+\mathcal{E}_y(u_{n,1})-e(c_{n,0})\mathbb{1}\big)\big)\quad\text{in }\Omega\times Y\times S, \tag{92}$$

$$\begin{aligned}0=-\sum_{j=1}^N\partial_{x_j}\int_Y\sum_{k,h=1}^N\Big(&a_{ijkh}(c_{n,0})\big(e_{khx}(u_0)+e_{khy}(u_1)-e'c_0\delta_{kh}\big)\\&+a'_{ijkh}(c_{n,0})c_0\big(e_{khx}(u_{n,0})+e_{khy}(u_{n,1})-e(c_{n,0})\delta_{kh}\big)\Big)\,\mathrm{d}y\quad\text{in }\Omega\times Y\times S,\end{aligned} \tag{93}$$

for $1 \leq i \leq N$, *where*

$$\mathcal{S}_0 = \mathcal{A}(c_{n,0})\big(\mathcal{E}_x(u_0) + \mathcal{E}_y(u_1) - e' c_0 \mathbb{1}\big) + \mathcal{A}'(c_{n,0}) c_0 \big(\mathcal{E}_x(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big), \tag{94}$$

and with

$$u_0 = 0 \quad \text{on } \Gamma_0 \times S, \tag{95}$$

$$\mathcal{S}_0\, n = g \quad \text{on } \Gamma_g \times S, \tag{96}$$

$$u_0 \cdot n = 0 \quad \text{on } \Gamma_s \times S, \tag{97}$$

$$\tau \cdot \mathcal{S}_0\, n = 0 \quad \text{on } \Gamma_s \times S, \tag{98}$$

and

$$c_0(\,\cdot\,,\,\cdot\,,0) = c^{\text{in}} \quad \text{in } \Omega \times Y. \tag{99}$$

Proof. The proof consists of several steps. First, we pass to the limit in the weak form of the linear Cahn–Larché system. Afterwards, we proof the uniqueness of the solutions of the resulting weak homogenised system and, in a third step, we derive the strong formulation of the homogenised system. □

Homogenisation process

We start by identifying the precise form of the two-scale limits of the sequences of the unknowns. We have already proven that c_ϵ and $\epsilon^2 \Delta c_\epsilon$ are bounded in $L^2(S, L^2(\Omega))$, ∇c_ϵ is bounded in $L^2(S, [L^2(\Omega)]^N)$ and the sequence u_ϵ is bounded in $L^2(S; W(\Omega))$, cf. Proposition 2. Then, from Proposition 7, we know that there exists a function $c_0 \in L^2(\Omega \times S; H^2_\#(Y))$ such that, up to a subsequence,

$$c_\epsilon \overset{2s.}{\rightharpoonup} c_0 \quad \text{and} \quad \epsilon^2 \Delta c_\epsilon \overset{2s.}{\rightharpoonup} \Delta_y c_0. \tag{100}$$

Furthermore, according to Theorem 3, there exists two functions, $u_0 \in L^2(S; W(\Omega))$ and $u_1 \in L^2(S \times \Omega; [H^1_\#(Y)/\mathbb{R}]^N)$ such that, up to a subsequence,

$$u_\epsilon \overset{2s.}{\rightharpoonup} u_0 \quad \text{and} \quad \mathcal{E}(u_\epsilon) \overset{2s.}{\rightharpoonup} \mathcal{E}_x(u_0) + \mathcal{E}_y(u_1). \tag{101}$$

Considering the two-scale limit of u_ϵ, the sequence is expected to behave as $u_0 + \epsilon u_1$. Therefore we choose a test function $\psi \in [C^\infty(\Omega; C^\infty_\#(Y))]^N$ with the same structure, namely $\psi(\,\cdot\,, \frac{\cdot}{\epsilon}) = \psi_0(\,\cdot\,) + \epsilon \psi_1(\,\cdot\,, \frac{\cdot}{\epsilon})$, with $\psi_0 \in [C^\infty(\Omega)]^N$ and $\psi_1 \in [C^\infty(\Omega; C^\infty_\#(Y))]^N$ for the mechanical equation (28). Then, Proposition 8 enables us to pass to the limit. The proof shows that (86) also applies when choosing $\varphi = \varphi(\,\cdot\,, \cdot/\epsilon)$ from $C^\infty(\Omega, C^\infty_\#(Y))$. Considering the cubic interpolation of the elasticity tensor (4), several terms of products of sequences appear. The most critical terms to deal with include products of one weakly two-scale convergent sequence with three strongly two-scale convergent sequences and the required convergences of the sequences $c_{n,\epsilon}$ and $u_{n,\epsilon}$ are sufficient to pass to the limit. Hence, for $\epsilon \to 0$, we get

$$\begin{aligned} &\int_\Omega \int_Y \mathcal{A}(c_{n,0})\big(\mathcal{E}_x(u_0(x,t)) + \mathcal{E}_y(u_1(x,y,t)) - e' c_0(x,y,t)\mathbb{1}\big) : \big(\mathcal{E}_x(\psi_0(x)) + \mathcal{E}_y(\psi_1(x,y))\big)\, \mathrm{d}y\, \mathrm{d}x \\ &+ \int_\Omega \int_Y \mathcal{A}'(c_{n,0}) c_0(x,y,t) \big(\mathcal{E}(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big) : \big(\mathcal{E}_x(\psi_0(x)) + \mathcal{E}_y(\psi_1(x,y))\big)\, \mathrm{d}y\, \mathrm{d}x = \int_{\Gamma_g} g(x,t)\, \psi_0(x)\, d\sigma. \end{aligned} \tag{102}$$

Now, we pass to the limit of the extended Cahn–Hilliard equation. In view of the two-scale limit (100), we choose $\varphi \in C^\infty(\Omega; C^\infty_\#(Y))$ as test function, which reflects the behaviour of c_ϵ. From (27), we get

$$\begin{aligned}\int_\Omega \partial_t c_\epsilon(x,t)\,\varphi(x,\frac{x}{\epsilon})\,\mathrm{d}x &= \epsilon^2 \int_\Omega \big(f''(c_{n,\epsilon})c_\epsilon(x,t) - \epsilon^2\lambda\Delta c_\epsilon(x,t) - e'\,\mathrm{tr}(\mathcal{S}_\epsilon)\big)\,\Delta\varphi(x,\frac{x}{\epsilon})\,\mathrm{d}x \\ &\quad + \epsilon^2 \int_\Omega \big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big) : \mathcal{A}'(c_{n,\epsilon})\big(\mathcal{E}(u_\epsilon(x,t)) - e' c_\epsilon(x,t)\mathbb{1}\big)\,\Delta\varphi(x,\frac{x}{\epsilon})\,\mathrm{d}x \\ &\quad + \frac{1}{2}\epsilon^2 \int_\Omega \big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big) : \mathcal{A}''(c_{n,\epsilon})c_\epsilon(x,t)\big(\mathcal{E}(u_{n,\epsilon}) - e(c_{n,\epsilon})\mathbb{1}\big)\,\Delta\varphi(x,\frac{x}{\epsilon})\,\mathrm{d}x.\end{aligned} \tag{103}$$

Now, we pass to the limit in each term as ϵ tends to zero with Proposition 8. For the first two terms of the right-hand side of (103), for example, the procedure is as follows. We get

$$\begin{aligned}\lim_{\epsilon\to 0} \epsilon^2 \int_\Omega f''(c_{n,\epsilon})c_\epsilon(x,t)\,\Delta\varphi(x,\frac{x}{\epsilon})\,\mathrm{d}x &= \lim_{\epsilon\to 0} \int_\Omega f''(c_{n,\epsilon})c_\epsilon(x,t)\,\big[\epsilon^2\Delta_{xx} + \epsilon\nabla_x\cdot\nabla_y + \epsilon\nabla_y\cdot\nabla_x + \Delta_{yy}\big]\,\varphi(x,\frac{x}{\epsilon})\,\mathrm{d}x \\ &= \int_\Omega\int_Y f''(c_{n,0})c_0(x,y,t)\,\Delta_{yy}\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x\end{aligned} \tag{104}$$

and, with Proposition 7,

$$\lim_{\epsilon\to 0} \epsilon^2 \int_\Omega \epsilon^2\lambda\,\Delta c_\epsilon(x,t)\Delta\varphi(x,\frac{x}{\epsilon})\,\mathrm{d}x = \lim_{\epsilon\to 0} \int_\Omega \epsilon^2\lambda\Delta c_\epsilon(x,t)\,\epsilon^2\Delta\varphi(x,\frac{x}{\epsilon})\,\mathrm{d}x = \int_\Omega\int_Y \lambda\Delta_{yy}c_0(x,y,t)\,\Delta_{yy}\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \tag{105}$$

as the two-scale limit of the Laplacian term. With regard to the limit of the mechanical equation (102), or more precisely to the limit of the sequence of the stress, we get

$$\lim_{\epsilon\to 0} \epsilon^2 \int_\Omega e'\,\mathrm{tr}(\mathcal{S}_\epsilon)\,\Delta\varphi(x,\frac{x}{\epsilon})\,\mathrm{d}x = \int_\Omega\int_Y e'\,\mathrm{tr}(\mathcal{S}_0)\,\Delta_{yy}\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x, \tag{106}$$

where we denote the limit of the stress tensor by

$$\mathcal{S}_0 = \mathcal{A}(c_{n,0})\big(\mathcal{E}_x(u_0) + \mathcal{E}_y(u_1) - e'c_0\mathbb{1}\big) + \mathcal{A}'(c_{n,0})c_0\big(\mathcal{E}_x(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big).$$

The limit passage of the remaining terms of the equation is analogous. Passing to the limit in the term with the time derivative first requires integration by parts with respect to time since $\partial_t c_\epsilon$ is only bounded in $L^2(S,(V(\Omega))')$. Re-integration then results in the limit of the time derivative corresponding to the time derivative of the limit function c_0.

In summary, we can now read off a variational formulation for the two-scale limit functions $(c_0,u_0,u_1) \in L^2(\Omega\times S, H^2_\#(Y)) \times L^2(S,W(\Omega)) \times [L^2(\Omega\times S, H^1_\#(Y)/\mathbb{R})]^N$:

$$\begin{aligned}&\int_\Omega\int_Y \partial_t c_0(x,y,t)\,\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_\Omega\int_Y \big(f''(c_{n,0})c_0(x,y,t) - \lambda\Delta_{yy}c_0(x,y,t) - e'\,\mathrm{tr}(\mathcal{S}_0)\big)\,\Delta_{yy}\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \int_\Omega\int_Y \big(\mathcal{E}_x(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big) : \mathcal{A}'(c_{n,0})\big(\mathcal{E}_x(u_0(x,t)) + \mathcal{E}_y(u_1(x,y,t)) - e'c_0(x,y,t)\mathbb{1}\big)\,\Delta_{yy}\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_\Omega\int_Y \big(\mathcal{E}(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big) : \mathcal{A}''(c_{n,0})c_0(x,y,t)\,\big(\mathcal{E}_x(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big)\,\Delta_{yy}\varphi(x,y)\,\mathrm{d}y\,\mathrm{d}x,\end{aligned} \tag{107}$$

$$\int_\Omega \int_Y \mathcal{A}(c_{n,0})\big(\mathcal{E}_x(u_0(x,t)) + \mathcal{E}_y(u_1(x,y,t)) - e' c_0(x,y,t)\mathbb{1}\big) : \big(\mathcal{E}_x(\psi_0(x)) + \mathcal{E}_y(\psi_1(x,y))\big)\,\mathrm{d}y\,\mathrm{d}x$$
$$+ \int_\Omega \int_Y \mathcal{A}'(c_{n,0})c_0(x,y,t)\left(\mathcal{E}(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\right) : \big(\mathcal{E}_x(\psi_0(x)) + \mathcal{E}_y(\psi_1(x,y))\big)\,\mathrm{d}y\,\mathrm{d}x \tag{108}$$
$$= \int_{\Gamma_g} g(x,t)\,\psi_0(x)\,\mathrm{d}\sigma_x,$$

which holds for all $(\varphi,\psi_0,\psi_1) \in C^\infty(\Omega; C^\infty_\#(Y)) \times [C^\infty(\Omega)]^N \times [C^\infty(\Omega; C^\infty_\#(Y))]^N$. By density, the above equations still hold for all $(\varphi,\psi_0,\psi_1) \in L^2(\Omega, H^2_\#(Y)) \times W(\Omega) \times [L^2(\Omega, H^1_\#(Y)/\mathbb{R})]^N$ and since the limits $c_{n,0}$, $u_{n,0}$ and $u_{n,1}$ are essentially bounded with respect to space and time, the integrals are well-defined.

Uniqueness of the limit solutions

It remains to prove that the solution triple (c_0, u_0, u_1) of (107) and (108) is unique in $L^2(\Omega\times S, H^2_\#(Y)) \times L^2(S, W(\Omega)) \times [L^2(\Omega\times S, H^1_\#(Y)/\mathbb{R})]^N$. To do so, we consider two supposedly different solution triples (c_0, u_0, u_1) and $(\tilde{c}_0, \tilde{u}_0, \tilde{u}_1)$. Their difference fulfils the equations (107) and (108) with $g \equiv 0$ and we also use these as test functions, leading to

$$\left\|\mathcal{E}_x(u_0-\tilde{u}_0) + \mathcal{E}_y(u_1-\tilde{u}_1)\right\|_{\Omega\times Y} \le C\,\|c_0-\tilde{c}_0\|_{L^2(\Omega\times Y)}. \tag{109}$$

Then further, analogous to the a-priori estimate from Section 3, from (107), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c_0-\tilde{c}_0\|^2_{\Omega\times Y} + \left\|\Delta_{yy}(c_0-\tilde{c}_0)\right\|^2_{\Omega\times Y} \le C\Big(\|c_0-\tilde{c}_0\|_{\Omega\times Y} + \left\|\mathcal{E}_x(u_0-\tilde{u}_0)+\mathcal{E}_y(u_1-\tilde{u}_1)\right\|_{\Omega\times Y}\Big)\left\|\Delta_{yy}(c_0-\tilde{c}_0)\right\|_{\Omega\times Y},$$

for a constant $C > 0$ and a.e. $t \in S$. With (109) and Young's inequality, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c_0-\tilde{c}_0\|^2_{\Omega\times Y} + \left\|\Delta_{yy}(c_0-\tilde{c}_0)\right\|^2_{\Omega\times Y} \le \frac{1}{2}C\left(\frac{1}{\delta}\|c_0-\tilde{c}_0\|^2_{\Omega\times Y} + \delta\left\|\Delta_{yy}(c_0-\tilde{c}_0)\right\|^2_{\Omega\times Y}\right),$$

with $0 < \delta < 1$. Absorbing the Laplacian term, integrating with respect to time from 0 to T and applying Gronwall's inequality then yields

$$\|c_0(t)-\tilde{c}_0(t)\|^2_{\Omega\times Y} + \int_0^T \left\|\Delta_{yy}c_0(t) - \Delta_{yy}\tilde{c}_0(t)\right\|^2_{\Omega\times Y}\mathrm{d}t \;\le\; 0. \tag{110}$$

Thus, $c_0 = \tilde{c}_0$ and $\Delta_{yy}c_0 = \Delta_{yy}\tilde{c}_0$, and since

$$\left\|\nabla_y v\right\|_{\Omega\times Y} \le \|v\|_{\Omega\times Y}\left\|\Delta_{yy}v\right\|_{\Omega\times Y}$$

holds for functions $v \in L^2(\Omega, H^2_\#(Y))$, we also get $\nabla_y c_0 = \nabla_y \tilde{c}_0$ and hence, the uniqueness of c_0 in $L^2(\Omega\times S, H^2_\#(Y))$. Furthermore, from (109), it follows

$$0 \ge \left\|\mathcal{E}_x(u_0-\tilde{u}_0) + \mathcal{E}_y(u_1-\tilde{u}_1)\right\|^2_{\Omega\times Y}.$$

It holds

$$\left\|\mathcal{E}_x(u_0-\tilde{u}_0) + \mathcal{E}_y(u_1-\tilde{u}_1)\right\|^2_{\Omega\times Y} = \|\mathcal{E}_x(u_0-\tilde{u}_0)\|^2_\Omega + \left\|\mathcal{E}_y(u_1-\tilde{u}_1)\right\|^2_{\Omega\times Y} + 2\int_\Omega\int_Y \mathcal{E}_x(u_0-\tilde{u}_0) : \mathcal{E}_y(u_1-\tilde{u}_1)\,\mathrm{d}y\,\mathrm{d}x$$
$$\ge C\|u_0-\tilde{u}_0\|^2_{H^1(\Omega)} + \left\|\nabla_y(u_1-\tilde{u}_1)\right\|^2_{\Omega\times Y},$$

where we have applied two variants of Korn's inequalities (standard case and periodic case). The integral term vanishes, which can be seen by applying Gauss' Theorem. Therefore, we get the uniqueness of u_0 in $L^2(S, W(\Omega))$ and u_1 in $[L^2(S \times \Omega; H^1_\#(Y)/\mathbb{R})]^N$. This proves that the entire sequences converge to the respective specified limit.

Strong form of the homogenised system

To finish the proof of Theorem 4, we derive the strong form of the homogenised system above. This is accomplished by choosing special test functions and integration by parts. First, choosing $\psi_0 \equiv 0$ in (108) and integrating by parts with respect to y yields equation (92). Then, choosing $\psi_1 \equiv 0$ in (108) and integrating by parts, we obtain the macroscopic equation (93). At this step, we applied the boundary conditions (95) – (98). Finally, twofold integration by parts of equation (107) leads to equation (91) of the homogenised system and our proof is done.

As it is usual, the unknown u_1 can be eliminated from equations (91), (92), (93) and therefore the homogenised two-scale system can be decoupled into a macroscopic and a microscopic equation by expressing u_1 in terms of u_0. For this purpose we introduce the cell problems:

For each $l, m = 1, \dots, N$, a vector-valued function ω^{lm} is required, which solves

$$-\nabla_y \cdot \big(\mathcal{A}\, \mathcal{E}_y(\omega^{lm})\big) = \nabla_y \cdot \big(\mathcal{A}\, \mathcal{E}_y(\lambda^{lm})\big) \quad \text{in } Y, \tag{111}$$

$$\omega^{lm} \qquad Y\text{-periodic in } y, \tag{112}$$

where $\lambda^{lm} = (\lambda^{lm}_k)_{1\le k\le N} \in \mathbb{R}^N$ is defined by

$$\lambda^{lm}_k(y) := y_m \delta_{kl}, \quad y \in Y, \quad k = 1, \dots, N,$$

with y_m being the m-th component of $y \in Y$.

A comparison of the equations (111) and (93) leads to the representation

$$\mathcal{E}_y(u_1) = \mathcal{E}_\omega\, \mathcal{E}_x(u_0) + e' c_0 \mathbb{1} - \mathcal{A}^{-1}(c_{n,0})\, \mathcal{A}'(c_{n,0})\big(\mathcal{E}_x(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big)c_0, \tag{113}$$

where $\mathcal{E}_\omega = (e^\omega_{lmkh})_{1\le l,m,k,h\le N}$ is a fourth-order tensor with components defined by

$$e^\omega_{lmkh} = e_{lmy}(\omega^{kh}). \tag{114}$$

Note that the inverse tensor $\mathcal{A}^{-1}$ of $\mathcal{A}$ exists due to the positive definiteness of $\mathcal{A}$ and is uniquely defined through $\mathcal{A}\mathcal{A}^{-1} = \mathcal{A}^{-1}\mathcal{A} = \mathcal{I}$. Here $\mathcal{I}$ is the symmetric fourth-order identity tensor with components

$$\mathcal{I}_{lmkh} = \frac{1}{2}(\delta_{lm}\delta_{kh} + \delta_{mh}\delta_{lk}). \tag{115}$$

By using the representation (113) and the notation (114) and (115), system (91)–(93) can be written in the usual decoupled form.

$$\begin{aligned}
\partial_t c_0 = \Delta_{yy} \Bigg(& f''(c_{n,0})c_0 - \lambda \Delta_{yy} c_0 - e' \operatorname{tr}\big[\mathcal{A}(c_{n,0})\,(\mathcal{E}_\omega + \mathcal{I})\mathcal{E}_x(u_0)\big] \\
& - \big(\mathcal{E}_x(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big) : \mathcal{A}'(c_{n,0})\mathcal{A}^{-1}(c_{n,0})\mathcal{A}'(c_{n,0})\big(\mathcal{E}_x(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big)c_0\big) \\
& + \big(\mathcal{E}_x(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big) : \mathcal{A}'(c_{n,0})(\mathcal{E}_\omega + \mathcal{I})\mathcal{E}_x(u_0) \\
& + \frac{1}{2}\big(\mathcal{E}_x(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big) : \mathcal{A}''(c_{n,0})c_0\, \big(\mathcal{E}_x(u_{n,0}) + \mathcal{E}_y(u_{n,1}) - e(c_{n,0})\mathbb{1}\big)\Bigg) \quad \text{in } \Omega \times Y \times S,
\end{aligned} \tag{116}$$

$$0 = \nabla_x \cdot \big(\mathcal{A}^{\text{hom}}\, \mathcal{E}_x(u_0)\big) \quad \text{in } \Omega \times S. \tag{117}$$

In this form of the homogenised system, we now find the effective or homogenised elasticity tensor $\mathcal{A}^{\text{hom}}$, describing the effective stiffness, defined through its components

$$a_{ijkh}^{\text{hom}} = \int_Y \sum_{l,m=1}^{N} a_{ijlm}(c_{n,0})\big(\delta_{kl}\delta_{hm} + e_{lmy}(\omega^{kh})\big)\,\mathrm{d}y, \quad 1 \le i,j,k,h \le N. \tag{118}$$

It can be shown that the effective elasticity tensor has the same properties as $\mathcal{A}$: it is bounded, positive definite and fulfils the symmetry conditions (7). The only unknowns left in this system are c_0, the unknown order parameter of the microscopic equation (116), and the purely macroscopic displacement u_0.

5 | SUMMARY AND CONCLUSIONS

Starting with the nonlinear Cahn–Larché system with phase separation on the microscale, we rigorously passed to the homogenisation limit of the corresponding linearised system in the context two-scale convergence. In passing, a general two-scale-convergence compactness result for second-order derivatives as well as one for products of sequences was established. Moreover, in order to prove existence of the coupled elliptic–parabolic system, results for differential–algebraic equations were generalised to an L^∞-setting allowing to follow a Galerkin approach.

The derived limit system is of the so-called distributed-microstructure type. In such a model, a unit cell is identified for each macroscopic point, on which the local equations are solved, cf. [22–26]. We have a global or macroscopic equation (117) for the global or macroscopic unknown displacement u_0, coupled with a local or microscopic equation (116) for the local or microscopic unknown order parameter c_0. We refer to [8] for prototypical simulation results for the nonlinear limit system.

In ref. [8], the nonlinear Cahn–Larché system (cf. Section 2.1) was homogenised formally using the method of asymptotic expansions. Considering this (nonlinear) limit system and linearising it analogously as done here for the microscopic system in Section 2.2, the same system as derived in Section 4.2 results. Therefore, homogenisation and linearisation (formally) commutate in this case.

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ORCID
Lisa Reischmann https://orcid.org/0000-0003-4682-7153
Malte A. Peter https://orcid.org/0000-0001-6107-9806

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